Asymptotics for moments of the minimal partition excludant in congruence classes

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Abstract. The minimal excludant statistic, which denotes the smallest positive integer that is not a part of an integer partition, has received great interest in recent years. In this paper, we move on to the smallest positive integer whose frequency is less than a given number. We establish an asymptotic formula for the moments of such generalized minimal excludants that fall in a specific congruence class. In particular, our estimation reveals that the moments associated with a fixed modulus are asymptotically "equal".

Keywords. Integer partition, minimal excludant, congruence class, asymptotic formula.

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1. Introduction

For a given natural number n, we call a collection of weakly decreasing positive integers $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ a *partition* of n if $|\pi| := \sum \pi_i = n$, while the numbers π_i are called *parts* of this partition. We use the notation $\pi \vdash n$ to represent that π is a partition of n.

The study of integer partitions and their counting traces back to Leibniz in his correspondence with Jacob Bernoulli [16, pp. 740–755], and stays vibrant in modern number theory and combinatorics, partially through various statistics associated with partitions. One recent example is the *minimal excludant* statistic, abbreviated *mex*, which was named by Andrews and Newman [2] based on a terminology in combinatorial game theory [8].

Definition 1.1. The *minimal excludant* of an integer partition π, denoted by $\mathrm{mex}(\pi)$, is the smallest positive integer such that it is not a part of π.

Notably, the study of this statistic was launched earlier by Andrews himself in [1] under the name of "the smallest part that is not a summand," which is semantically more accessible, and even earlier by Grabner and Knopfmacher [9], who called it "the least gap." In recent years, the mex statistic has been found to exhibit connections with a handful of objects such as Dyson's crank [12, 13], Schmidt-type partitions [18, 19], unrefinable partitions [4, 5], and the super Yang–Mills theory [11].

Recall that the *frequency* of a part π_i is the number of its occurrence in the partition π. Hence, the mex of π can be paraphrased as the smallest positive integer whose frequency is less than one. Along this line, we may generalize the mex statistic as follows.

Definition 1.2. Letting s be a positive integer, we define the *minimal excludant of frequency s* (abbr. *s-mex*) of an integer partition π, denoted by $\mathrm{mex}^{[s]}(\pi)$, as the smallest positive integer such that its frequency in π is less than s.

With this definition, the 1-mex reduces to the usual mex statistic. It is also notable that consideration along this line is not fresh; for example, Wagner [17] considered probabilistic aspects of this family of statistics, and Ballantine and Merca [6] entered this topic from a combinatorial perspective.

Example 1.1. Considering the partition $\pi = (6, 4, 3, 3, 2, 2, 2, 1, 1)$, we have

$$\begin{aligned}
\operatorname{mex}^{[1]}(\pi) &= 5,\\
\operatorname{mex}^{[2]}(\pi) &= 4,\\
\operatorname{mex}^{[s]}(\pi) &= 1 \quad (s \geq 3).
\end{aligned}$$

Given a combinatorial statistic, it is of general interest to consider its arithmetic behavior. Therefore, our first objective revolves around partitions whose s-mex falls in a specific congruence class. In particular, letting A and M with $0 < A \leq M$ be integers, we are interested in the following moments:

$$\sigma^{[s]}_{M,A}(r;n) := \sum_{\substack{\pi \vdash n \\ \operatorname{mex}^{[s]}(\pi) \equiv A \bmod M}} \big(\operatorname{mex}^{[s]}(\pi)\big)^r, \tag{1.1}$$

where r is a nonnegative integer.

Theorem 1.1. *For integers A and M with $0 < A \leq M$, nonnegative integers r and positive integers s, we have, as $n \to \infty$,*

$$\sigma^{[s]}_{M,A}(r;n) \sim \begin{cases} 2^{-2} 3^{-\frac{1}{2}} M^{-1} n^{-1} e^{\pi\sqrt{\frac{2n}{3}}}, & \text{if } r = 0,\\ 2^{\frac{3r-12}{4}} 3^{\frac{r-2}{4}} \pi^{-\frac{r}{2}} M^{-1} s^{-\frac{r}{2}} r\, \Gamma\big(\frac{r}{2}\big)\, n^{\frac{r-4}{4}} e^{\pi\sqrt{\frac{2n}{3}}}, & \text{if } r \geq 1, \end{cases} \tag{1.2}$$

where $\Gamma(z)$ is Euler's gamma function.

As a corollary, we notice that the moments associated with a fixed modulus are asymptotically "equal".

Corollary 1.2. *For integers A, A' and M with $0 < A, A' \leq M$, nonnegative integers r and positive integers s,*

$$\lim_{n\to\infty} \frac{\sigma^{[s]}_{M,A}(r;n)}{\sigma^{[s]}_{M,A'}(r;n)} = 1. \tag{1.3}$$

In [2], Andrews and Newman also defined the *minimal odd excludant* to be the smallest odd positive integer that is not a part of a partition π. It is notable that this terminology should not be confused with the minimal excludant that is odd. In a subsequent paper [3], Andrews and Newman further extended this idea to arbitrary congruence classes, by defining $\operatorname{mex}_{M,A}(\pi)$ as the smallest positive integer congruent to A modulo M such that it is not a part of π. Now we may similarly introduce restrictions on the frequency of parts.

Definition 1.3. Letting s be a positive integer and A and M with $0 < A \leq M$ be integers, we define the *minimal excludant of frequency s in the congruence class A modulo M* (abbr. *$s_{M,A}$-mex*) of an integer partition π, denoted by $\operatorname{mex}^{[s]}_{M,A}(\pi)$, as the smallest positive integer congruent to A modulo M such that its frequency in π is less than s.

Thus, $\mathrm{mex}^{[1]}_{M,A}(\pi) = \mathrm{mex}_{M,A}(\pi)$. It is also notable that the $s_{M,A}$-mex reduces to the s-mex when the pair (M, A) takes $(1, 1)$.

Example 1.2. Considering the partition $\pi = (6, 4, 3, 3, 2, 2, 2, 1, 1)$, we have

$$\begin{aligned} \mathrm{mex}^{[1]}_{2,1}(\pi) &= 5, & \mathrm{mex}^{[1]}_{2,2}(\pi) &= 8,\\ \mathrm{mex}^{[2]}_{2,1}(\pi) &= 5, & \mathrm{mex}^{[2]}_{2,2}(\pi) &= 4,\\ \mathrm{mex}^{[3]}_{2,1}(\pi) &= 1, & \mathrm{mex}^{[3]}_{2,2}(\pi) &= 4,\\ \mathrm{mex}^{[s]}_{2,1}(\pi) &= 1, & \mathrm{mex}^{[s]}_{2,2}(\pi) &= 2 \quad (s \geq 4). \end{aligned}$$

Our second objective focuses on the moments of the $s_{M,A}$-mex:

$$\varsigma^{[s]}_{M,A}(r; n) := \sum_{\pi \vdash n} \big(\mathrm{mex}^{[s]}_{M,A}(\pi)\big)^r, \tag{1.4}$$

where r is a nonnegative integer. It is clear that when $r = 0$,

$$\varsigma^{[s]}_{M,A}(0; n) = \sum_{\pi \vdash n} 1 = p(n), \tag{1.5}$$

where the *partition function* $p(n)$ counts the number of partitions of n. It is a standard result due to Hardy and Ramanujan [10, p. 79, eq. (1.41)] that

$$p(n) \sim 2^{-2} 3^{-\frac{1}{2}} n^{-1} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Hence, for any choice of A, M and s,

$$\varsigma^{[s]}_{M,A}(0; n) \sim 2^{-2} 3^{-\frac{1}{2}} n^{-1} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1.6}$$

In what follows, we focus on the cases where $r \geq 1$.

Theorem 1.3. *For integers A and M with $0 < A \leq M$,* ***positive*** *integers r and positive integers s, we have, as $n \to \infty$,*

$$\varsigma^{[s]}_{M,A}(r; n) \sim 2^{\frac{3r-12}{4}} 3^{\frac{r-2}{4}} \pi^{-\frac{r}{2}} M^{\frac{r}{2}} s^{-\frac{r}{2}} r\, \Gamma(\tfrac{r}{2})\, n^{\frac{r-4}{4}} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1.7}$$

We have similar "equalities" of these moments in the asymptotic sense.

Corollary 1.4. *For integers A, A' and M with $0 < A, A' \leq M$, positive integers r and positive integers s,*

$$\lim_{n\to\infty} \frac{\varsigma^{[s]}_{M,A}(r; n)}{\varsigma^{[s]}_{M,A'}(r; n)} = 1. \tag{1.8}$$

2. Generating functions

It is known, according to Euler [7, p. 260, Ch. XVI], that the generating function of the partition function $p(n)$ is

$$\sum_{n\geq 0} p(n) q^n = \prod_{k\geq 1} \frac{1}{1-q^k} = \frac{1}{(q;q)_\infty},$$

wherein we adopt the *q-Pochhammer symbol* for complex q with $|q| < 1$:

$$(a; q)_\infty := \prod_{k\geq 0} (1 - aq^k).$$

Now we look into the generating functions of $\sigma_{M,A}^{[s]}(r;n)$ and $\varsigma_{M,A}^{[s]}(r;n)$ à la Andrews and Newman [2,3]. Throughout this section, we assume that A and M with $0<A\le M$ are integers, r is a nonnegative integer and s is a positive integer.

We first work on $\sigma_{M,A}^{[s]}(r;n)$. It is clear that

$$\begin{aligned}\sum_{n\ge 0}&\sigma_{M,A}^{[s]}(r;n)q^n\\ &=\sum_{n\ge 0}(Mn+A)^r\big(1+q^{Mn+A}+\cdots+q^{(s-1)(Mn+A)}\big)\\ &\quad\times\left(\prod_{k>Mn+A}\frac{1}{1-q^k}\right)\left(\prod_{1\le k<Mn+A}\frac{q^{sk}}{1-q^k}\right)\\ &=\sum_{n\ge 0}(Mn+A)^r\frac{1-q^{s(Mn+A)}}{1-q^{Mn+A}}\cdot q^{s(1+2+\cdots(Mn+A-1))}\prod_{\substack{k\ge 1\\ k\ne Mn+A}}\frac{1}{1-q^k}.\end{aligned}$$

Thus the generating function of $\sigma_{M,A}^{[s]}(r;n)$ can be formulated as follows.

Proposition 2.1. *We have*

$$\sum_{n\ge 0}\sigma_{M,A}^{[s]}(r;n)q^n=\frac{1}{(q;q)_\infty}\sum_{n\ge 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)}(1-q^{s(Mn+A)}). \tag{2.1}$$

For $\varsigma_{M,A}^{[s]}(r;n)$, we have

$$\begin{aligned}\sum_{n\ge 0}&\varsigma_{M,A}^{[s]}(r;n)q^n\\ &=\prod_{\substack{k\ge 1\\ k\not\equiv A \bmod M}}\frac{1}{1-q^k}\cdot\sum_{n\ge 0}(Mn+A)^r\big(1+q^{Mn+A}+\cdots+q^{(s-1)(Mn+A)}\big)\\ &\quad\times\left(\prod_{k>n}\frac{1}{1-q^{Mk+A}}\right)\left(\prod_{0\le k<n}\frac{q^{s(Mk+A)}}{1-q^{Mk+A}}\right)\\ &=\prod_{\substack{k\ge 1\\ k\not\equiv A \bmod M}}\frac{1}{1-q^k}\cdot\sum_{n\ge 0}(Mn+A)^r\frac{1-q^{s(Mn+A)}}{1-q^{Mn+A}}\\ &\quad\times q^{s(A+(M+A)+\cdots+(M(n-1)+A))}\prod_{\substack{k\ge 0\\ k\ne n}}\frac{1}{1-q^{Mk+A}}.\end{aligned}$$

Proposition 2.2. *We have*

$$\sum_{n\ge 0}\varsigma_{M,A}^{[s]}(r;n)q^n=\frac{1}{(q;q)_\infty}\sum_{n\ge 0}(Mn+A)^r q^{s(\frac{1}{2}Mn(n-1)+An)}(1-q^{s(Mn+A)}). \tag{2.2}$$

Remark 2.1. It is clear that when $r=0$,

$$\sum_{n\ge 0}q^{s(\frac{1}{2}Mn(n-1)+An)}(1-q^{s(Mn+A)})$$

$$= 1 + \sum_{n\geq 1} q^{s(\frac{1}{2}Mn(n-1)+An)} - \sum_{n\geq 0} q^{s(\frac{1}{2}Mn(n+1)+A(n+1))} = 1.$$

Hence, the relation $\varsigma^{[s]}_{M,A}(0;n) = p(n)$ as given in (1.5) is recovered. In general, the sum on the right-hand side of (2.2) can be rewritten as

$$\begin{aligned}\sum_{n\geq 0}&(Mn+A)^r q^{s(\frac{1}{2}Mn(n-1)+An)}(1-q^{s(Mn+A)})\\ &= A^r + \sum_{n\geq 0}(M(n+1)+A)^r q^{s(\frac{1}{2}Mn(n+1)+A(n+1))}\\ &\quad - \sum_{n\geq 0}(Mn+A)^r q^{s(\frac{1}{2}Mn(n+1)+A(n+1))}. \end{aligned}\tag{2.3}$$

This reformulation will facilitate our proof of Theorem 1.3.

3. Lemmas

Here we collect some lemmas for our asymptotic analysis in the next section. Throughout, the *Bernoulli polynomials* $B_n(x)$ are defined by the exponential generating function:

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1}.$$

We begin with an asymptotic estimation for weighted partial theta functions.

Lemma 3.1. *Let u be a positive real number and let r be a nonnegative integer. As $t\to 0^+$, it is true for any $N\geq 1$ that*

$$\sum_{n\geq 0}(n+u)^r e^{-(n+u)^2t^2} = \frac{\Gamma(\frac{r+1}{2})}{2t^{r+1}} - \sum_{n=0}^{N-1}\frac{(-1)^n B_{2n+r+1}(u)}{(2n+r+1)n!}t^{2n} + O(t^{2N}). \tag{3.1}$$

Proof. Asymptotic relations of this type were systematically analyzed in the work of Zagier [20], who had utilized the Euler–Maclaurin summation formula. In particular, in [20, p. 321, eq. (6.76)][1], we choose

$$f(x) = x^r e^{-x^2} = \sum_{n\geq 0}\frac{(-1)^n}{n!}x^{2n+r},$$

so that

$$\int_0^\infty f(x)dx = \int_0^\infty x^r e^{-x^2}dx = \frac{\Gamma(\frac{r+1}{2})}{2}.$$

Noting the fact that

$$\sum_{n\geq 0}(n+u)^r e^{-(n+u)^2t^2} = \frac{1}{t^r}\sum_{n\geq 0}\big((n+u)t\big)e^{-((n+u)t)^2},$$

[1]This equation should be corrected as

$$\sum_{m=0}^\infty f\big((m+a)x\big) \sim \frac{I_f}{x} - \sum_{n=0}^\infty b_n\frac{B_{n+1}(a)}{n+1}x^n,$$

since [20, p. 314, eq. (6.55)] reads $\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$ so that the sign before the summation on the right-hand side of the above should be negative.

we arrive at (3.1) by invoking Zagier's result. □

Next, we recall Ingham's Tauberian theorem [14].

Lemma 3.2. *Let $F(q)=\sum_{n\ge 0} f(n)q^n$ be a power series with eventually nondecreasing and nonnegative coefficients such that its radius of convergence equals 1. If there are constants $A>0$ and $\lambda,\alpha\in\mathbb{R}$ such that*

$$F(e^{-t})\sim \lambda\, t^{\alpha} e^{\frac{A}{t}}$$

as $t\to 0^+$, then

$$f(n)\sim \frac{\lambda}{2\sqrt{\pi}}\frac{A^{\frac{\alpha}{2}+\frac14}}{n^{\frac{\alpha}{2}+\frac34}}\, e^{2\sqrt{An}}$$

as $n\to\infty$.

4. Asymptotic analysis

In this section, we first prove Theorem 1.1.

Proof of Theorem 1.1. We first examine that the moments $\sigma^{[s]}_{M,A}(r;n)$ are eventually nondecreasing and nonnegative. The nonnegativity is apparent as

$$\sum_{n\ge 0}\sigma^{[s]}_{M,A}(r;n)q^n=\sum_{n\ge 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)}\prod_{\substack{k\ge 1\\ k\ne s(Mn+A)}}\frac{1}{1-q^k},$$

wherein each summand is a series with nonnegative coefficients. To see the eventual monotonicity, we notice that when $s\ge 2$, the number $s(Mn+A)$ cannot be 1 for any nonnegative n, thereby implying that

$$\begin{aligned}(1-q)\cdot&\sum_{n\ge 0}\sigma^{[s]}_{M,A}(r;n)q^n\\&=\sum_{n\ge 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)}\prod_{\substack{k\ge 2\\ k\ne s(Mn+A)}}\frac{1}{1-q^k}\end{aligned}$$

is a nonnegative series, and hence that the numbers $\sigma^{[s]}_{M,A}(r;n)$ are nondecreasing in this circumstance. Now we consider the case where $s=1$. Note that

$$\begin{aligned}&\sum_{n\ge 0}\sigma^{[1]}_{M,A}(r;n)q^n\\&=\frac{A^r q^{\frac12 A(A-1)}(1-q^A)}{(q;q)_\infty}+\frac{1}{(q;q)_\infty}\sum_{n\ge 1}(Mn+A)^r q^{\frac12(Mn+A)(Mn+A-1)}(1-q^{Mn+A}).\end{aligned}$$

Clearly, $Mn+A\ne 1$ when $n\ge 1$ because A and M are positive integers. Thus, the latter sum in the above is a series with nondecreasing coefficients since

$$\begin{aligned}(1-q)\cdot\frac{1}{(q;q)_\infty}&\sum_{n\ge 1}(Mn+A)^r q^{\frac12(Mn+A)(Mn+A-1)}(1-q^{Mn+A})\\&=\sum_{n\ge 1}(Mn+A)^r q^{\frac12(Mn+A)(Mn+A-1)}\prod_{\substack{k\ge 2\\ k\ne Mn+A}}\frac{1}{1-q^k}\end{aligned}$$

is a series with nonnegative coefficients. For the remaining term $A^r q^{\frac{1}{2}A(A-1)}(1-q^A)(q;q)_\infty^{-1}$, we see that when $A \geq 2$,

$$(1-q)\cdot\frac{A^r q^{\frac{1}{2}A(A-1)}(1-q^A)}{(q;q)_\infty} = A^r q^{\frac{1}{2}A(A-1)} \prod_{\substack{k\geq 2\\ k\neq A}} \frac{1}{1-q^k}$$

is a nonnegative series so that $A^r q^{\frac{1}{2}A(A-1)}(1-q^A)(q;q)_\infty^{-1}$ itself is a series with nondecreasing coefficients. When $A=1$,

$$\frac{A^r q^{\frac{1}{2}A(A-1)}(1-q^A)}{(q;q)_\infty} = \frac{1}{(q^2;q)_\infty} = \sum_{n\geq 0} p_{>1}(n) q^n,$$

where $p_{>1}(n)$ enumerates the number of *non-unitary partitions* (i.e., partitions with no part equal to one) of n. It is notable that $p_{>1}(n+1) \geq p_{>1}(n)$ with only one exception that $p_{>1}(1) = 0 < 1 = p_{>1}(0)$ since for $n \geq 1$ there is a natural injection from non-unitary partitions of n to non-unitary partitions of $n+1$ by adding 1 to the largest part of the former. Hence, we also have the eventual monotonicity when $A=1$, thereby completing the discussion for all circumstances.

In view of Ingham's Tauberian theorem, it suffices to estimate the right-hand side of (2.1) with $q = e^{-t}$ when $t \to 0^+$. To fulfill this goal, we start with a reformulation of the sum on the right-hand side of (2.1) as

$$\sum_{n\geq 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)}(1-q^{s(Mn+A)}) = S_1(q) - S_2(q),$$

where

$$\begin{aligned} S_1(q) &:= \sum_{n\geq 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)},\\ S_2(q) &:= \sum_{n\geq 0}(Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A+1)}. \end{aligned}$$

When $r=0$, we see that

$$\begin{aligned} S_1(e^{-t}) &= e^{-\frac{s}{2}A(A-1)t} + e^{\frac{st}{8}} \sum_{n\geq 1} e^{-\frac{s}{2}(Mn+A-\frac{1}{2})^2 t}\\ &= e^{-\frac{s}{2}A(A-1)t} + e^{\frac{st}{8}} \sum_{n\geq 0} e^{-\frac{sM^2}{2}(n+1+\frac{A}{M}-\frac{1}{2M})^2 t}\\ &= \tfrac{1}{M}\sqrt{\tfrac{\pi}{2}}\, s^{-\frac{1}{2}} t^{-\frac{1}{2}} + 1 - B_1\big(1+\tfrac{A}{M}-\tfrac{1}{2M}\big) + O(t^{\frac{1}{2}}), \end{aligned}$$

and that

$$\begin{aligned} S_2(e^{-t}) &= e^{\frac{st}{8}} \sum_{n\geq 0} e^{-\frac{sM^2}{2}(n+\frac{A}{M}+\frac{1}{2M})^2 t}\\ &= \tfrac{1}{M}\sqrt{\tfrac{\pi}{2}}\, s^{-\frac{1}{2}} t^{-\frac{1}{2}} - B_1\big(\tfrac{A}{M}+\tfrac{1}{2M}\big) + O(t^{\frac{1}{2}}). \end{aligned}$$

We have applied Lemma 3.1 for both asymptotic relations above as $t \to 0^+$. Hence,

$$\sum_{n\geq 0} q^{\frac{s}{2}(Mn+A)(Mn+A-1)}(1-q^{s(Mn+A)})\Big|_{q=e^{-t}} \sim M^{-1},$$

where we have used the fact that $B_1(x) = x - \frac{1}{2}$. Finally, we recall the modular inversion formula for Dedekind's eta function (see, for example, [15, p. 121, Proposition 14]), which implies that as $t \to 0^+$,

$$(e^{-t}; e^{-t})_\infty \sim \sqrt{2\pi}\, t^{-\frac{1}{2}} e^{-\frac{\pi^2}{6t}}.$$

Hence,

$$\sum_{n\geq 0} \sigma_{M,A}^{[s]}(0;n) e^{-nt} \sim \tfrac{1}{\sqrt{2\pi}}\, M^{-1} t^{\frac{1}{2}} e^{\frac{\pi^2}{6t}} \qquad (t \to 0^+). \tag{4.1}$$

When $r \geq 1$, we have

$$\begin{aligned} &S_1(e^{-t}) \\ &\quad = A^r e^{-\frac{s}{2}A(A-1)t} + e^{\frac{st}{8}} \sum_{n\geq 1} (Mn+A)^r e^{-\frac{s}{2}(Mn+A-\frac{1}{2})^2 t} \\ &\quad = A^r e^{-\frac{s}{2}A(A-1)t} + e^{\frac{st}{8}} \sum_{n\geq 0} \big[M(n+1+\tfrac{A}{M}-\tfrac{1}{2M}) + \tfrac{1}{2}\big]^r e^{-\frac{sM^2}{2}(n+1+\frac{A}{M}-\frac{1}{2M})^2 t} \\ &\quad = 2^{\frac{r-1}{2}} M^{-1} s^{-\frac{r+1}{2}}\, \Gamma(\tfrac{r+1}{2})\, t^{-\frac{r+1}{2}} + 2^{\frac{r-4}{2}} M^{-1} s^{-\frac{r}{2}} r\, \Gamma(\tfrac{r}{2})\, t^{-\frac{r}{2}} + O(t^{-\frac{r-1}{2}}). \end{aligned}$$

Meanwhile,

$$\begin{aligned} S_2(e^{-t}) &= e^{\frac{st}{8}} \sum_{n\geq 0} \big[M(n+\tfrac{A}{M}+\tfrac{1}{2M}) - \tfrac{1}{2}\big]^r e^{-\frac{sM^2}{2}(n+\frac{A}{M}+\frac{1}{2M})^2 t} \\ &\quad = 2^{\frac{r-1}{2}} M^{-1} s^{-\frac{r+1}{2}}\, \Gamma(\tfrac{r+1}{2})\, t^{-\frac{r+1}{2}} - 2^{\frac{r-4}{2}} M^{-1} s^{-\frac{r}{2}} r\, \Gamma(\tfrac{r}{2})\, t^{-\frac{r}{2}} + O(t^{-\frac{r-1}{2}}). \end{aligned}$$

Therefore,

$$\sum_{n\geq 0} (Mn+A)^r q^{\frac{s}{2}(Mn+A)(Mn+A-1)} (1 - q^{s(Mn+A)}) \Big|_{q=e^{-t}} \sim 2^{\frac{r-2}{2}} M^{-1} s^{-\frac{r}{2}} r\, \Gamma(\tfrac{r}{2})\, t^{-\frac{r}{2}}.$$

Now, invoking the asymptotic formula for $(e^{-t}; e^{-t})_\infty$, it follows that for $r \geq 1$,

$$\sum_{n\geq 0} \sigma_{M,A}^{[s]}(r;n) e^{-nt} \sim 2^{\frac{r-3}{2}} \pi^{-\frac{1}{2}} M^{-1} s^{-\frac{r}{2}} r\, \Gamma(\tfrac{r}{2})\, t^{\frac{1-r}{2}} e^{\frac{\pi^2}{6t}} \qquad (t \to 0^+). \tag{4.2}$$

Finally, with a direct application of Ingham's Tauberian theorem to (4.1) and (4.2), we are led to the desired relation (1.2). □

Next, we move on to the proof of Theorem 1.3.

Proof of Theorem 1.3. In light of (2.3), we see that

$$\begin{aligned} &\sum_{n\geq 0} \varsigma_{M,A}^{[s]}(r;n) q^n \\ &\qquad = \frac{1}{(q;q)_\infty} \left(A^r + \sum_{n\geq 0} \big[(Mn+M+A)^r - (Mn+A)^r\big] q^{s(\frac{1}{2}Mn(n+1)+A(n+1))} \right) \end{aligned}$$

is a nonnegative series since $(Mn+M+A)^r - (Mn+A)^r$ is nonnegative for any $n \geq 0$. Meanwhile, $(1-q) \cdot \sum_{n\geq 0} \varsigma_{M,A}^{[s]}(r;n) q^n$ is also a nonnegative series because we only need to replace $(q;q)_\infty^{-1}$ with $(q^2;q)_\infty^{-1}$ in the above. Thus, the moments $\varsigma_{M,A}^{[s]}(r;n)$ are nondecreasing and nonnegative.

Now we are left to evaluate $\sum_{n\geq 0} \varsigma^{[s]}_{M,A}(r;n)e^{-nt}$ as $t\to 0^+$. It is clear from (2.3) that

$$\begin{aligned}\sum_{n\geq 0}&(Mn+A)^r q^{s(\frac{1}{2}Mn(n-1)+An)}(1-q^{s(Mn+A)})\\ &= A^r + q^{-\frac{sM}{2}(\frac{A}{M}-\frac{1}{2})^2}\sum_{n\geq 0}\big[M(n+\tfrac{A}{M}+\tfrac{1}{2})+\tfrac{M}{2}\big]^r q^{\frac{sM}{2}(n+\frac{A}{M}+\frac{1}{2})^2}\\ &\quad - q^{-\frac{sM}{2}(\frac{A}{M}-\frac{1}{2})^2}\sum_{n\geq 0}\big[M(n+\tfrac{A}{M}+\tfrac{1}{2})-\tfrac{M}{2}\big]^r q^{\frac{sM}{2}(n+\frac{A}{M}+\frac{1}{2})^2}.\end{aligned}$$

Hence, an application of Lemma 3.1 gives

$$\sum_{n\geq 0}(Mn+A)^r q^{s(\frac{1}{2}Mn(n-1)+An)}(1-q^{s(Mn+A)})\Big|_{q=e^{-t}} \sim 2^{\frac{r-2}{2}} M^{\frac{r}{2}} s^{-\frac{r}{2}} r\,\Gamma(\tfrac{r}{2})\,t^{-\frac{r}{2}}.$$

We further invoke the contribution from $(e^{-t};e^{-t})_\infty^{-1}$ and get

$$\sum_{n\geq 0}\varsigma_r(a,b;n)e^{-nt} \sim 2^{\frac{r-3}{2}}\pi^{-\frac{1}{2}} M^{\frac{r}{2}} s^{-\frac{r}{2}} r\,\Gamma(\tfrac{r}{2})\,t^{\frac{1-r}{2}}\,e^{\frac{\pi^2}{6t}} \qquad (t\to 0^+). \tag{4.3}$$

Finally, (1.7) follows by applying Ingham's Tauberian theorem to (4.3). □

5. Conclusion

It should be noted that Ingham's Tauberian theorem only gives the dominant term of the asymptotic estimation in question, while it talks nothing about the errors. Such a type of estimation is sufficient to demonstrate asymptotic "equalities" like our Corollaries 1.2 and 1.4. However, when it comes to asymptotic "inequalities" such as the eventual log-concavity or bias of $\sigma^{[s]}_{M,A}(r;n)$ and $\varsigma^{[s]}_{M,A}(r;n)$ as inquired below, a more precise asymptotic expansion becomes necessary. For this purpose, we expect a delicate application of the circle method, which will be left for future research.

Problem 5.1 (Log-concavity)**.** *Fix r, s, A and M. Is it true for all sufficiently large n that*

$$\sigma^{[s]}_{M,A}(r;n)^2 > \sigma^{[s]}_{M,A}(r;n-1)\sigma^{[s]}_{M,A}(r;n+1), \tag{5.1}$$

$$\varsigma^{[s]}_{M,A}(r;n)^2 > \varsigma^{[s]}_{M,A}(r;n-1)\varsigma^{[s]}_{M,A}(r;n+1). \tag{5.2}$$

Problem 5.2 (Bias)**.** *Fix r, s and M. Is there a reordering $A_1, A_2, \ldots, A_M$ of $1, 2, \ldots, M$ such that for all sufficiently large n,*

$$\sigma^{[s]}_{M,A_1}(r;n) \leq \sigma^{[s]}_{M,A_2}(r;n) \leq \cdots \leq \sigma^{[s]}_{M,A_M}(r;n). \tag{5.3}$$

The same question may also be asked for $\varsigma^{[s]}_{M,A}(r;n)$.

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