SIGNED COUNTING OF PARTITION MATRICES

SHANE CHERN AND SHISHUO FU

Abstract. We prove that the signed counting (with respect to the parity of the "inv" statistic) of partition matrices equals the cardinality of a subclass of inversion sequences. In the course of establishing this result, we introduce an interesting class of partition matrices called improper partition matrices. We further show that a subset of improper partition matrices is equinumerous with the set of Motzkin paths. Such an equidistribution is established both analytically and bijectively.

1. Introduction

A *Fishburn matrix* is an upper-triangular square matrix over nonnegative integers such that each row and column contains at least one nonzero entry. The *weight* of such a matrix M, denoted by $w(M)$, is defined as the sum of all its entries. Also, the *dimension* of M, denoted by $\dim(M)$, is the number of rows (or equivalently, the number of columns) in this square matrix. For example, the following Fishburn matrix

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ & 2 & 1 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix} \tag{1.1}$$

has weight 8 and dimension 4. In this work, we denote by FM_n the set of Fishburn matrices of weight n and write $\mathrm{FM} := \cup_{n\geq 1} \mathrm{FM}_n$.

Historically, this concept was first considered by Fishburn [9, 10], arising naturally from his earlier study of interval orders in [8]. In recent years, Fishburn matrices have been connected to other combinatorial objects such as ascent sequences [3], pattern-avoiding permutations [3, 6], and pattern-avoiding insertion tables [16]. In addition, their counting function, known

2020 *Mathematics Subject Classification.* 05A05, 05A15, 05A19.

Key words and phrases. Partition matrix, Fishburn matrix, inversion sequence, Stirling number of the second kind, Motzkin path, generating function, bijection.

as the *Fishburn number* [18, A022493], plays a motivating role in topics including q-hypergeometric series [1], quantum modular forms [21], and Vassiliev invariants [19, 20].

In 2011, Claesson, Dukes, and Kubitzke [7] introduced two matrix analogs for set partitions, and the one of our interest in the present work is the partition matrix. Let $[n] := \{1, 2, \dots, n\}$. A *partition matrix* P on $[n]$ is an upper-triangular square matrix over the power set of $[n]$ such that

- each row and column contains at least one nonempty subset of $[n]$;
- these nonempty subsets partition $[n]$;
- for any $i, j \in [n]$, if $\mathrm{col}(i) < \mathrm{col}(j)$, then we must have $i < j$, where $\mathrm{col}(k) = \mathrm{col}_P(k)$ is the *column index* of k (i.e., the column that contains the subset including k) in P for $k \in [n]$.

For a partition matrix P on $[n]$, we define its *weight* $w(P)$ as n and its *dimension* $\dim(P)$ as the number of rows. As an example, the following matrix

$$\begin{pmatrix} \{1\} & \varnothing & \{4,5\} & \varnothing \\ & \{2,3\} & \{6\} & \varnothing \\ & & \varnothing & \{8\} \\ & & & \{7\} \end{pmatrix} \tag{1.2}$$

is a partition matrix on $\{1, 2, \dots, 8\}$, and has weight 8 and dimension 4. Throughout, let PM denote the set of all partition matrices and PM_n the set of partition matrices on $[n]$.

Notably, partition matrices have a natural connection with Fishburn matrices. To be precise, given a partition matrix on $[n]$, if we switch its entries from the subsets of $[n]$ to the cardinalities of these subsets, then the resulting matrix is a Fishburn matrix of the same weight and dimension, just like the matrix in (1.2) reduces to that in (1.1). Such a matrix is called the *natural Fishburn matrix induced by* our partition matrix.

On the other hand, unlike Fishburn numbers, which have no simple closed expression[1], the number of partition matrices of weight n is the factorial of n. This relation is a consequence of a bijection between partition matrices and inversion sequences discovered by Claesson, Dukes, and Kubitzke [7, p. 1626,

[1]However, as shown by Zagier [20, p. 948, Theorem 1], the generating function for Fishburn numbers can still be simply expressed as

$$\sum_{n\geq 0}\prod_{i=1}^{n}\big(1-(1-t)^i\big);$$

this is a typical example of quantum modular forms (after replacing $1-t$ with t).

Theorem 3]. Here, an *inversion sequence* is of the form

$$e := (e_1, e_2, \ldots, e_n) \text{ with } 0 \le e_i < i \text{ for every } i \text{ with } 1 \le i \le n;$$

we call n the *length* of this inversion sequence.

To more closely tie partition matrices and Fishburn matrices, as well as the counting of them, we consider the following polynomial in $\mathbb{N}[q]$:

$$S_n(q) := \sum_{M \in \mathrm{FM}_n} \prod_{1 \le j \le \dim(M)} \begin{bmatrix} m_{1,j} + m_{2,j} + \cdots + m_{j,j} \\ m_{1,j}, m_{2,j}, \ldots, m_{j,j} \end{bmatrix}_q, \tag{1.3}$$

where the sum runs over all Fishburn matrices $M := (m_{i,j})_{1 \le i,j \le \dim(M)}$ of weight n. Here the *q-multinomial coefficients* are defined by

$$\begin{bmatrix} a_1 + \cdots + a_k \\ a_1, \ldots, a_k \end{bmatrix}_q := \frac{[a_1 + \cdots + a_k]_q!}{[a_1]_q! \cdots [a_k]_q!} \in \mathbb{N}[q],$$

with $[0]_q! := 1$ and $[a]_q! := \prod_{i=1}^{a} \frac{1-q^i}{1-q}$ for any positive integer a. Meanwhile, given a partition matrix P on $[n]$, a pair of numbers (i, j) over $[n]$ is said to be an *inversion* in P if

- $i > j$,
- $\mathrm{col}(i) = \mathrm{col}(j)$,
- $\mathrm{row}(i) < \mathrm{row}(j)$,

where we similarly define the *row index* $\mathrm{row}(k) = \mathrm{row}_P(k)$ for each $k \in [n]$ as the row that contains the subset including k. We denote by $\mathrm{inv}(P)$ the number of inversions in P. According to a standard result on q-multinomial coefficients [13, Theorem 5.1], it is true that

$$S_n(q) = \sum_{P \in \mathrm{PM}_n} q^{\mathrm{inv}(P)}. \tag{1.4}$$

Note that for q-multinomial coefficients $\begin{bmatrix} a_1 + \cdots + a_k \\ a_1, \ldots, a_k \end{bmatrix}_q$, it is true that

$$\begin{bmatrix} a_1 + \cdots + a_k \\ a_1, \ldots, a_k \end{bmatrix}_0 = 1, \qquad \begin{bmatrix} a_1 + \cdots + a_k \\ a_1, \ldots, a_k \end{bmatrix}_1 = \binom{a_1 + \cdots + a_k}{a_1, \ldots, a_k}.$$

Therefore, we have the relations

$$S_n(0) = |\,\mathrm{FM}_n\,| = \mathrm{Fis}_n, \tag{1.5}$$

$$S_n(1) = |\,\mathrm{PM}_n\,| = n!. \tag{1.6}$$

Here, Fis_n is the n-th Fishburn number. It is then natural to ask for an explicit expression of $S_n(q)$, at least at some typical values of q other than zero and one.

The above discussion motivates us to consider the following *signed* enumeration of partition matrices:

$$S_n(-1) = \sum_{P\in \mathrm{PM}_n} (-1)^{\mathrm{inv}(P)}. \tag{1.7}$$

The initial values of this counting function are

$$1, 2, 4, 10, 28, 88, 304, 1144, \ldots.$$

After searching it in the On-Line Encyclopedia of Integer Sequences (OEIS, [18]), we find that this sequence coincides with [18, A229046]. This observation leads us to discover one of the main results in the present work.

Theorem 1.1. *For every $n \geq 1$, the number $S_n(-1)$ equals the number of inversion sequences in $\mathbf{I}_n(-,-,=)$, that is, inversion sequences $(e_1,\ldots,e_n)$ of length n such that there is no triple $i<j<k$ with $e_i = e_k$.*

Remark 1.2. The numbers $|\mathbf{I}_n(-,-,=)|$ give one of the combinatorial explanations of the sequence [18, A229046]; Martinez and Savage proposed this relation as an open problem in [17, Section 2.13]. In the same work, Martinez and Savage also considered the "dist" statistic for sequences in $\mathbf{I}_n(-,-,=)$ counting distinct elements in a given inversion sequence, and obtained a recurrence for the corresponding bi-parametric counting function. Later on, Cao, Jin, and Lin [5, p. 96, Proposition 7.2][2] confirmed Martinez and Savage's hunch by establishing the following relation:

$$\sum_{e\in \mathbf{I}_n(-,-,=)} z^{\mathrm{dist}(e)} = \sum_{j=1}^{n+1} (j-1)! \left\{ {n+2-j \atop j} \right\} z^{n+1-j}, \tag{1.8}$$

where $\left\{ {n \atop m} \right\}$ are the *Stirling numbers of the second kind*. Alternatively, as claimed by Heinz [18, A229046] and confirmed by Cao, Jin, and Lin [5, p. 96, Theorem 7.1], the number $|\mathbf{I}_n(-,-,=)|$ also equals the number of set partitions of $[n+1]$ such that the absolute difference between the smallest elements of consecutive blocks is always greater than 1.

We will take two steps to establish Theorem 1.1:

(1) Find a subset of PM_n such that its cardinality is given by $S_n(-1)$;

(2) Show that this subset is equinumerous with $\mathbf{I}_n(-,-,=)$.

[2]In [5], the "dist" statistic only counts distinct *positive* elements, and hence it is one less than that of Martinez and Savage because 0 always appears in an inversion sequence. Throughout this work, we adopt the convention of Martinez and Savage and therefore modify (1.8) from its original form in [5] accordingly.

In Section 2, we construct a subclass of partition matrices called *improper partition matrices* in Definition 2.1 to fulfill our objective in Step (1). Then in Section 3, a bivariate generating function is established for our improper partition matrices. As shown in Theorem 3.3, this generating function is identical to a bivariate generating function for the desired inversion sequences in $\mathbf{I}_n(-,-,=)$, thereby completing Step (2).

It is notable that improper partition matrices themselves are of independent research interest. Therefore, Section 4 is devoted to a further subset of improper partition matrices. This subset, surprisingly, is equinumerous with the set of Motzkin paths. In particular, such an equidistribution can be strengthened by accommodating two extra statistics on both sides; this subtle relation is presented in Theorem 4.4. Moreover, two proofs are provided, with one analytic and the other bijective.

We close this paper with some questions for future research in Section 5.

2. Improper partition matrices

As we have promised, in this section, we are going to determine a subset of partition matrices, the cardinality of which is counted by $S_n(-1)$. To begin with, we introduce further statistics on partition matrices. Then such a subset is precisely the set of fixed points under an involution Θ that we are going to construct.

Given a partition matrix P on $[n]$, for $1 < i < n$, the element i is called a *descent* (resp. an *ascent*) in P if the pair $(i, i+1)$ satisfies that

- $\mathrm{col}(i) = \mathrm{col}(i+1)$,
- $\mathrm{row}(i) > \mathrm{row}(i+1)$ (resp. $\mathrm{row}(i) < \mathrm{row}(i+1)$).

And $(i, i+1)$ is said to form a *descent pair* (resp. an *ascent pair*) in P.

Definition 2.1. Given a partition matrix $P \in \mathrm{PM}$, a descent or an ascent i of P is said to be *proper* if $i \equiv j \pmod 2$, where j is the minimal element of the $\mathrm{col}(i)$-th column of P. Otherwise, it is said to be *improper*. Let IPPM denote the subset of PM consisting of all *improper partition matrices*[3], which are partition matrices with every descent and ascent (if any) improper; we also denote by IPPM_n the subset of matrices in IPPM of weight n.

For instance, in the earlier example, the matrix in (1.2) has an improper ascent at 5 and a proper descent at 7. As another example, among the six

[3]This acronymic format of nomenclature, which is adopted throughout this work, is somewhat influenced by the names of different classes of alternating sign matrices; see, for example, [2, Section 2].

matrices in PM_3 listed below, only the four in the first line are improper:

$$\begin{pmatrix}\{1,2,3\}\end{pmatrix}, \quad \begin{pmatrix}\{1,2\} & \varnothing \\ & \{3\}\end{pmatrix}, \quad \begin{pmatrix}\{1\} & \varnothing \\ & \{2,3\}\end{pmatrix}, \quad \begin{pmatrix}\{1\} & \varnothing & \varnothing \\ & \{2\} & \varnothing \\ & & \{3\}\end{pmatrix},$$

and

$$\begin{pmatrix}\{1\} & \{2\} \\ & \{3\}\end{pmatrix}, \quad \begin{pmatrix}\{1\} & \{3\} \\ & \{2\}\end{pmatrix}.$$

Definition 2.2. For every $n \geq 1$, we let Θ be a mapping from PM to itself, with the image $\Theta(P)$ constructed for any $P \in \mathrm{PM}$ as follows:

- if $P \in \mathrm{IPPM}$, then simply set $\Theta(P) := P$;
- if $P \in \mathrm{PM} \setminus \mathrm{IPPM}$, find the smallest proper ascent or descent, say i, in P, and switch i and $i+1$ to obtain $\Theta(P)$.

The following theorem indicates that IPPM is our desired subset.

Theorem 2.3. *The mapping* $\Theta : \mathrm{PM} \to \mathrm{PM}$ *is a weight-preserving involution that fixes* IPPM *and for any* $P \in \mathrm{PM} \setminus \mathrm{IPPM}$, *we have*

$$|\operatorname{inv}(P) - \operatorname{inv}(\Theta(P))| = 1. \tag{2.1}$$

Moreover, for every $P \in \mathrm{IPPM}$,

$$\operatorname{inv}(P) \equiv 0 \pmod 2. \tag{2.2}$$

Consequently, we see that for every $n \geq 1$,

$$|\,\mathrm{IPPM}_n\,| = S_n(-1). \tag{2.3}$$

Proof. It is trivial that for $P \in \mathrm{IPPM}$, the map Θ is a weight-preserving involution because $\Theta(P)$ is P itself. Meanwhile, we note that for any $P \in \mathrm{PM} \setminus \mathrm{IPPM}$, the smallest proper descent (resp. ascent) in P becomes the smallest proper ascent (resp. descent) in $\Theta(P)$. It should then be clear from the construction of Θ that

$$w(P) = w(\Theta(P))$$

and

$$\Theta(\Theta(P)) = P.$$

As such, Θ preserves weight and is an involution in both cases.

Now we examine the pair $(P, \Theta(P))$ for $P \in \mathrm{PM} \setminus \mathrm{IPPM}$. According to the definition of Θ, the only difference between P and $\Theta(P)$ is that there exists a unique pair $(i, i+1)$ so that their positions in the matrices are switched going from P to $\Theta(P)$. We first suppose that i is a proper ascent in P;

then it becomes a proper descent in $\Theta(P)$. Moreover, for any third element j with $j < i$ in the $\mathrm{col}_P(i)$-th column such that $\mathrm{row}_P(j) < \mathrm{row}_P(i)$ and $\mathrm{row}_P(j) < \mathrm{row}_P(i+1)$, we also have $\mathrm{row}_{\Theta(P)}(j) < \mathrm{row}_{\Theta(P)}(i)$ and $\mathrm{row}_{\Theta(P)}(j) < \mathrm{row}_{\Theta(P)}(i+1)$ because $\mathrm{row}_{\Theta(P)}(j) = \mathrm{row}_P(j)$ while $\mathrm{row}_{\Theta(P)}(i) = \mathrm{row}_P(i+1)$ and $\mathrm{row}_{\Theta(P)}(i+1) = \mathrm{row}_P(i)$ by our construction. Hence, neither of the pairs (i, j) and $(i+1, j)$ form an inversion in P and $\Theta(P)$. Other possibilities of elements $j \notin \{i, i+1\}$ in the $\mathrm{col}_P(i)$-th column and possibilities of the row index $\mathrm{row}_P(j)$ can be individually checked in the same way. We conclude that $\mathrm{inv}(P) + 1 = \mathrm{inv}(\Theta(P))$ in this scenario. The case of $(i, i+1)$ being a proper descent in P can be argued in a similar fashion, and the relation $\mathrm{inv}(P) - 1 = \mathrm{inv}(\Theta(P))$ follows. We have proven (2.1).

Next, take any improper matrix P. If $\mathrm{inv}(P) = 0$, then (2.2) trivially holds true. Otherwise, suppose (i, j) is an inversion in the c-th column of P so that $i > j$ and $\mathrm{row}_P(i) < \mathrm{row}_P(j)$. Let S be the subset that contains j. We further assume that

$$(j - \alpha, \ldots, j - 1, j, j + 1, \ldots, j + \beta)$$

is the maximal sequence of consecutive integers containing j inside S for certain $\alpha, \beta \geq 0$. On one hand, $j-\alpha$ must have the same parity as the smallest element, say $c_{\min}$, in the c-th column; otherwise, $j-\alpha-1$, which shares the same parity with $c_{\min}$, also lies in the c-th column and becomes a proper ascent or descent, thereby resulting in a contradiction. On the other hand, $j+\beta$ is not the largest element in the c-th column; otherwise, we cannot find the element i to create our inversion (i, j). Thus, $j+\beta+1$ is also in the c-th column, and in particular, $\mathrm{row}_P(j+\beta) \neq \mathrm{row}_P(j+\beta+1)$, implying that $j+\beta$ is either an ascent or a descent. Now since P is improper, $j+\beta$ must be improper, thereby having a different parity to $c_{\min}$, and hence to $j-\alpha$. We are then led to the fact that

$$|\{j - \alpha, \ldots, j, \ldots, j + \beta\}| \equiv 0 \pmod 2.$$

Consequently, each of $(i, j - \alpha), \ldots, (i, j), \ldots, (i, j + \beta)$ is an inversion pair, contributing collectively an even count to $\mathrm{inv}(P)$, so we obtain (2.2) accordingly.

Finally, we note that due to (2.1), each pair $(P, \Theta(P))$ jointly contributes zero in the signed counting $S_n(-1)$ provided that $P \in \mathrm{PM} \setminus \mathrm{IPPM}$, while (2.2) tells us that every improper matrix in IPPM contributes one in the calculation of $S_n(-1)$. These two facts imply (2.3) and the proof is now completed. $\square$

3. Proof of Theorem 1.1

Our proof of Theorem 1.1 is built upon a bivariate generating function for improper partition matrices, in which we also include the semi-weight statistic introduced below.

Definition 3.1. The *semi-weight* of a partition matrix $P \in \mathrm{PM}$ is defined as

$$v(P) := \sum_{d=1}^{D} \left\lceil \frac{n_d}{2} \right\rceil,$$

where D is the dimension of P, and each n_d counts the number of elements in the d-th column.

Lemma 3.2. *We have*

$$\sum_{Q \in \mathrm{IPPM}} z^{v(Q)} t^{w(Q)} = \sum_{P \in \mathrm{PM}} z^{w(P)} t^{2w(P)} (1+t^{-1})^{\dim(P)}. \tag{3.1}$$

To make preparations for the proof of this lemma, we define the *reduction* of partition matrices $P \in \mathrm{PM}$ in the following way. As before, we assume that D is the dimension of P, and each n_d counts the number of elements in the d-th column. It is clear that the d-th column gives a set partition for integers in the interval $[\sum_{i=1}^{d-1} n_i + 1, \sum_{i=1}^{d-1} n_i + n_d]$. Now we create a new square matrix by subtracting $\sum_{i=1}^{d-1} n_i$ from every element in the d-th column for every d with $1 \le d \le D$; by doing so, the d-th column simply becomes a set partition of $[n_d]$. Such a new matrix, denoted by P^-, is called the *reduced matrix* of P. For example, the matrix in (1.2) is reduced to

$$\begin{pmatrix} \{1\} & \varnothing & \{1,2\} & \varnothing \\ & \{1,2\} & \{3\} & \varnothing \\ & & \varnothing & \{2\} \\ & & & \{1\} \end{pmatrix}. \tag{3.2}$$

Clearly, there is a one-to-one correspondence between partition matrices and their reductions.

Proof of Lemma 3.2. It has been indicated in the proof of Theorem 2.3 that given any $Q \in \mathrm{IPPM} \subset \mathrm{PM}$, in every column of the reduced matrix Q^-, all even elements $2i$ must be placed in the same row as $2i-1$.

Now we map every $P \in \mathrm{PM}$ to $2^{\dim(P)}$ matrices in IPPM as follows. First, in every column of the reduction P^-, we replace each number k by two numbers

$2k-1$ and $2k$. For example, the reduced matrix in (3.2) becomes

$$\begin{pmatrix} \{1,2\} & \varnothing & \{1,2,3,4\} & \varnothing \\ & \{1,2,3,4\} & \{5,6\} & \varnothing \\ & & \varnothing & \{3,4\} \\ & & & \{1,2\} \end{pmatrix}$$

after this procedure. It is notable that in the new matrix, there are an even number of elements in each column. Next, in each column, we may either preserve the largest element or remove it; in total, there are $2^{\dim(P)}$ ways to do so. More importantly, every matrix derived in this way is the reduction of a matrix in IPPM. Also, the reduction of every matrix in IPPM can be uniquely created as above.

After recovering the corresponding $2^{\dim(P)}$ matrices in IPPM, it is clear that their semi-weights are all equal to the weight of P. Therefore,

$$\sum_{Q\in\mathrm{IPPM}} z^{v(Q)}t^{w(Q)} = \sum_{P\in\mathrm{PM}} z^{w(P)}t^{2w(P)} \sum_{i=0}^{\dim(P)} \binom{\dim(P)}{i} t^{-i},$$

which implies the desired relation. $\square$

It has been shown by Claesson, Dukes, and Kubitzke [7, p. 1627, Corollary 5] that the statistic "dim" on PM_n is Eulerian[4]. In other words,

$$\sum_{P\in\mathrm{PM}} x^{\dim(P)}t^{w(P)} = \sum_{n\geq 1} xE_n(x)t^n, \tag{3.3}$$

where $E_n(x)$ is the n-th *Eulerian polynomial*. In addition, according to an identity of Frobenius [12] (see also [14, p. 1, eq. (3)]), we have

$$E_n(x) = \sum_{j=1}^{n} j! \left\{ {n \atop j} \right\} (x-1)^{n-j}. \tag{3.4}$$

Therefore,

$$\sum_{P\in\mathrm{PM}} x^{\dim(P)}t^{w(P)} = \sum_{n\geq 1}\sum_{j=1}^{n} j! \left\{ {n \atop j} \right\} x(x-1)^{n-j}t^n. \tag{3.5}$$

Toward a proof of Theorem 1.1, we need the following equidistribution between IPPM_n and $\mathbf{I}_n(-,-,=)$.

[4]Note that $\dim(P)$ for $P\in\mathrm{PM}_n$ is supported on $[1,n]$ instead of $[0,n-1]$. This explains the extra factor x on the right-hand side of (3.3).

Theorem 3.3. *For every $n \geq 1$,*

$$\sum_{Q \in \mathrm{IPPM}_n} z^{v(Q)} = \sum_{e \in \mathbf{I}_n(-,-,=)} z^{\mathrm{dist}(e)}. \tag{3.6}$$

Proof. It follows from (3.1) and (3.5) that

$$\sum_{Q \in \mathrm{IPPM}} z^{v(Q)} t^{w(Q)} = \sum_{n \geq 1} \sum_{j=1}^{n} j! \left\{ {n \atop j} \right\} z^n (1+t^{-1}) t^{n+j}.$$

Hence,

$$\sum_{Q \in \mathrm{IPPM}_N} z^{v(Q)} = \sum_{j=1}^{N-1} j! \left\{ {N-j \atop j} \right\} z^{N-j} + \sum_{j=1}^{N} j! \left\{ {N+1-j \atop j} \right\} z^{N+1-j}.$$

In view of (1.8), it suffices to show that for every $N \geq 1$,

$$\begin{aligned} &\sum_{j=1}^{N+1} (j-1)! \left\{ {N+2-j \atop j} \right\} z^{N+1-j} \\ &\quad = \sum_{j=1}^{N-1} j! \left\{ {N-j \atop j} \right\} z^{N-j} + \sum_{j=1}^{N} j! \left\{ {N+1-j \atop j} \right\} z^{N+1-j}. \end{aligned}$$

Recall the following recurrence for the Stirling numbers of the second kind [4, p. 625, eq. (26.8.22)]:

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}.$$

Thus,

$$\begin{aligned} &\sum_{j=1}^{N+1} (j-1)! \left\{ {N+2-j \atop j} \right\} z^{N+1-j} \\ &\quad = N! \left\{ {1 \atop N+1} \right\} + \sum_{j=1}^{N} (j-1)! \left(j \left\{ {N+1-j \atop j} \right\} + \left\{ {N+1-j \atop j-1} \right\} \right) z^{N+1-j} \\ &\quad = \sum_{j=1}^{N} j! \left\{ {N+1-j \atop j} \right\} z^{N+1-j} + \sum_{j=1}^{N} (j-1)! \left\{ {N+1-j \atop j-1} \right\} z^{N+1-j} \\ &\quad = \sum_{j=1}^{N} j! \left\{ {N+1-j \atop j} \right\} z^{N+1-j} + \sum_{j=1}^{N-1} j! \left\{ {N-j \atop j} \right\} z^{N-j}, \end{aligned}$$

where we have used the vanishing of $\left\{ {n \atop m} \right\}$ when $m = 0$ or $m > n$. Our proof is therefore completed. ☐

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. In (3.6), we take $z = 1$ to get $|\mathbf{I}_n(-,-,=)| = |\operatorname{IPPM}_n|$. Now Theorem 1.1 is a direct consequence of (2.3). ☐

4. Nondecreasing improper partition matrices

This section is devoted to an interesting subset of improper partition matrices. To begin with, we introduce partition matrices that are nondecreasing.

Definition 4.1. Given a partition matrix P on $[n]$, we say it is *nondecreasing* if for any two elements $i, j \in [n]$ with $i < j$, we always have $\operatorname{row}_P(i) \leq \operatorname{row}_P(j)$ and $\operatorname{col}_P(i) \leq \operatorname{col}_P(j)$.

The motivation for studying such a property is as follows. Recall that Claesson, Dukes, and Kubitzke [7, p. 1626, Theorem 3] provided a natural bijection between partition matrices of weight n and inversion sequences of length n. Briefly speaking, the matrix $P \in \operatorname{PM}_n$ corresponds to the inversion sequence $e = (e_1, \ldots, e_n)$ such that for each j with $1 \leq j \leq n$,

$$e_j := \sum_{d=1}^{\operatorname{row}(j)-1} n_d,$$

where n_d counts the number of elements in the d-th column, and the empty summation (i.e., when $\operatorname{row}(j) = 1$) is understood to equal 0 as usual. This inversion sequence e is called the *induced inversion sequence* of P.

Fact 4.2. *For every nondecreasing partition matrix, its induced inversion sequence is nondecreasing.*

Let NDPM_n denote the set of nondecreasing partition matrices of weight n, and write $\operatorname{NDPM} := \cup_{n\geq 1} \operatorname{NDPM}_n$.

It is clear that for any matrix $A \in \operatorname{NDPM}_n$, every nonempty entry is a set of consecutive integers in $[n]$. Another notable feature is that a partition matrix is nondecreasing if and only if every diagonal block in its irreducible diagonal block form (i.e., partitioning of the matrix into irreducible diagonal blocks) is still a nondecreasing partition matrix after normalizing the elements so as to start from 1. Let $\operatorname{blk}(A)$ count the number of irreducible diagonal blocks in such a representation of A.

For example, there are five nondecreasing partition matrices of weight 3:

$$\begin{pmatrix} \{1,2,3\} \end{pmatrix}, \quad \begin{pmatrix} \{1\} & \{2\} \\ & \{3\} \end{pmatrix}, \quad \begin{pmatrix} \{1,2\} & \varnothing \\ & \{3\} \end{pmatrix}, \quad \begin{pmatrix} \{1\} & \varnothing \\ & \{2,3\} \end{pmatrix}, \quad \begin{pmatrix} \{1\} & \varnothing & \varnothing \\ & \{2\} & \varnothing \\ & & \{3\} \end{pmatrix},$$

and they consist of 1, 1, 2, 2, 3 irreducible blocks, respectively.

Now we further restrict our attention to improper partition matrices that are nondecreasing, the set of which is written as NDIPPM. Furthermore, let NDIPPM_n be the set of such matrices of weight n.

For any matrix $B \in \mathrm{NDIPPM}_n$, we note that in each column, only the last nonempty entry (i.e., the nonempty entry with the largest row index) may have an odd cardinality, while all other nonempty entries must have even cardinalities. Let us denote by $\mathrm{odd}(B)$ the number of nonempty entries in B of an odd cardinality; this value also equals the number of columns in B collectively containing an odd number of elements.

We will show that nondecreasing improper partition matrices are equidistributed with Motzkin paths.

Definition 4.3. A *Motzkin path* $\mathscr{M}$ is a lattice path starting from $(0,0)$, ending on the right half of the x-axis (i.e., a certain $(n,0)$ with $n \geq 1$), and never falling below the x-axis such that only *up* ($\nearrow$) steps $(1,1)$, *down* ($\searrow$) steps $(1,-1)$, and *level* ($\rightarrow$) steps $(1,0)$ are used.

Let $\mathbb{M}$ denote the set of Motzkin paths. Furthermore, the x-coordinate of the ending point (i.e., the number n) in the above definition is called the *length* of this Motzkin path $\mathscr{M}$, and we write it as $\mathrm{len}(\mathscr{M})$. Accordingly, we denote by $\mathbb{M}_n$ the set of Motzkin paths of length n. It is known that the number of Motzkin paths of length n is given by the n-th *Motzkin number*, listed in the sequence A001006 in OEIS [18].

We denote by $\#_{\mathscr{M}}(\nearrow)$, $\#_{\mathscr{M}}(\searrow)$, and $\#_{\mathscr{M}}(\rightarrow)$ the number of up, down, and level steps on $\mathscr{M}$, respectively. In addition, we also use $\mathrm{level}(\mathscr{M})$ to denote $\#_{\mathscr{M}}(\rightarrow)$, especially as a statistic. It is clear that

$$\mathrm{len}(\mathscr{M}) = \#_{\mathscr{M}}(\nearrow) + \#_{\mathscr{M}}(\searrow) + \#_{\mathscr{M}}(\rightarrow).$$

Finally, we denote by $\mathrm{comp}(\mathscr{M})$ the number of times a step on $\mathscr{M}$ ending on the x-axis. Note that a level step lying on the x-axis is also viewed as "ending on the x-axis." Also, we denote by $\#^{\mathrm{x}}_{\mathscr{M}}(\searrow)$ and $\#^{\mathrm{x}}_{\mathscr{M}}(\rightarrow)$ the number of down and level steps ending on the x-axis, respectively. Then,

$$\mathrm{comp}(\mathscr{M}) = \#^{\mathrm{x}}_{\mathscr{M}}(\searrow) + \#^{\mathrm{x}}_{\mathscr{M}}(\rightarrow).$$

Now we state our equidistribution result as follows.

Theorem 4.4. *For every $n \geq 1$, the pair of statistics* $(\mathrm{blk}, \mathrm{odd})$ *on* NDIPPM_n *is equidistributed with the pair of statistics* $(\mathrm{comp}, \mathrm{level})$ *on* $\mathbb{M}_n$.

We will provide two proofs for this result with one based on generating functions and the other bijective. Among the two proofs, a commonly used ingredient is the induced lattice paths for nondecreasing partition matrices, which we shall introduce in the upcoming subsection.

4.1. **Induced lattice paths.** Throughout this subsection, let us assume that $A \in \mathrm{NDPM}_n$ has dimension D. We also assign the following *total order* to indices in $[D]^2$:

$$\begin{array}{ccccccccc} & (1,1) & < & (1,2) & < & \cdots & < & (1,D) \\ < & (2,1) & < & (2,2) & < & \cdots & < & (2,D) \\ < & \cdots & & & & & & \\ < & (D,1) & < & (D,2) & < & \cdots & < & (D,D) \end{array}. \tag{4.1}$$

As we have pointed out, every nonempty entry in A is a set of consecutive integers in $[n]$. Moreover, if two elements $i < j$ in $[n]$ are in different entries, we must have $(\mathrm{row}(i), \mathrm{col}(i)) < (\mathrm{row}(j), \mathrm{col}(j))$ by the definition of nodecreasing partition matrices. These observations lead us to the first feature of A.

Fact 4.5. *A is uniquely determined by the natural Fishburn matrix induced by A.*

Also, to ensure that A is a partition matrix so that each row and column contains at least one nonempty subset of $[n]$, the following statement must be true.

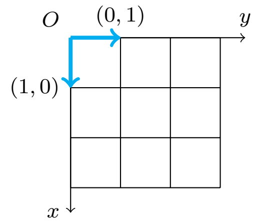
Fact 4.6. *Letting (r, c) be the index of a nonempty entry in A, then the next nonempty entry in A (according to the total order (4.1)), if existing, must lie in one of the three positions: $(r+1, c)$, $(r, c+1)$, or $(r+1, c+1)$.*

The two facts immediately imply a correspondence between nondecreasing partition matrices and certain lattice paths.

To describe our desired lattice paths, we first create a lattice such that the vector $(1, 0)$ goes toward *south* and the vector $(0, 1)$ goes toward *east*[5]; such a lattice may easily mimic the grid of indices of a matrix, as illustrated in (4.2)

[5]Note that this lattice is different from the usual lattice in which the vector $(1, 0)$ goes toward east and the vector $(0, 1)$ goes toward north.

below:



. (4.2)

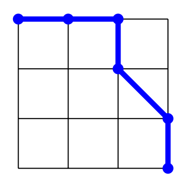
The lattice paths of our interest are those starting at $(1,1)$, ending at a certain (D,D) with D a positive integer, and restricted within the region $\{(x,y)\in\mathbb{Z}^2 : y\geq x\geq 1\}$ such that the *south* $(\downarrow)$ steps $(1,0)$, *east* $(\rightarrow)$ steps $(0,1)$ and *southeast* $(\searrow)$ steps $(1,1)$ are used. Let $\mathbb{P}_D$ denote the set of such lattice paths, and write $\mathbb{P} := \cup_{D\geq 1}\mathbb{P}_D$.

As before, for $\mathscr{P}\in\mathbb{P}$, we denote by $\#_{\mathscr{P}}(\downarrow)$, $\#_{\mathscr{P}}(\rightarrow)$, and $\#_{\mathscr{P}}(\searrow)$ the number of south, east, and southeast steps on $\mathscr{P}$, respectively, and by $\#_{\mathscr{P}}^{\mathrm{diag}}(\downarrow)$ and $\#_{\mathscr{P}}^{\mathrm{diag}}(\searrow)$ the number of south and southeast steps ending on the diagonal $\{(x,x)\in\mathbb{Z}^2 : x\geq 1\}$, respectively.

Definition 4.7. The *induced lattice path* of A, denoted by $\mathrm{Path}(A)$, is constructed in such a way that for every index in $[D]^2$ at which the entry in A is nonempty, we add a step from this index to the next index (according to the total order (4.1)) featuring a nonempty entry in A.

The following example illustrates this construction:

$$\begin{pmatrix} \{1\} & \{2,3\} & \{4,5\} & \varnothing \\ & \varnothing & \{6\} & \varnothing \\ & & \varnothing & \{7,8\} \\ & & & \{9,10\} \end{pmatrix} \quad \longrightarrow$$



. (4.3)

It is clear that in A, the two entries with indices $(1,1)$ and (D,D) are nonempty. Also, by Fact 4.6, the steps on $\mathrm{Path}(A)$ are south, east, or southeast. In addition, $\mathrm{Path}(A)$ is restricted within the region $\{(x,y)\in\mathbb{Z}^2 : y\geq x\geq 1\}$ because A is upper-triangular. Hence, $\mathrm{Path}(A)\in\mathbb{P}_D$. In the meantime, the following relation is obvious:

$$\mathrm{blk}(A) = \#_{\mathrm{Path}(A)}^{\mathrm{diag}}(\searrow) + 1. \tag{4.4}$$

4.2. **Analytic proof of Theorem 4.4.** Our first proof of Theorem 4.4 relies on a multivariate generating function for Motzkin paths. Let

$$F(\mu_1,\mu_2,\mu_3,\nu,\omega) := 1 + \sum_{\mathscr{M}\in\mathbb{M}} \mu_1^{\#_{\mathscr{M}}(\nearrow)}\mu_2^{\#_{\mathscr{M}}(\searrow)}\mu_3^{\#_{\mathscr{M}}(\rightarrow)}\nu^{\#_{\mathscr{M}}^{\mathrm{x}}(\searrow)}\omega^{\#_{\mathscr{M}}^{\mathrm{x}}(\rightarrow)}. \tag{4.5}$$

Lemma 4.8. *We have*

$$F(\mu_1,\mu_2,\mu_3,\nu,\omega)=\frac{2}{2(1-\mu_3\omega)-\nu\big(1-\mu_3-\sqrt{(1-\mu_3)^2-4\mu_1\mu_2}\big)}. \tag{4.6}$$

Proof. In view of a result of Flajolet [11, p. 129, Theorem 1] (see also [15, p. 527, Theorem A1]), the generating function $F(1,\mu_2,\mu_3,\nu,\omega)$ can be expressed as a continued fraction:

$$F(1,\mu_2,\mu_3,\nu,\omega)=\cfrac{1}{1-\mu_3\omega-\cfrac{\mu_2\nu}{1-\mu_3-\cfrac{\mu_2}{1-\mu_3-\cfrac{\mu_2}{1-\mu_3-\ddots}}}}.$$

In particular,

$$F(1,\mu_2,\mu_3,1,1)=\frac{1}{1-\mu_3-\mu_2F(1,\mu_2,\mu_3,1,1)},$$

from which we solve that

$$F(1,\mu_2,\mu_3,1,1)=\frac{1-\mu_3-\sqrt{(1-\mu_3)^2-4\mu_2}}{2\mu_2}.$$

Thus,

$$\begin{aligned}F(1,\mu_2,\mu_3,\nu,\omega)&=\frac{1}{1-\mu_3\omega-\mu_2\nu F(1,\mu_2,\mu_3,1,1)}\\&=\frac{2}{2(1-\mu_3\omega)-\nu\big(1-\mu_3-\sqrt{(1-\mu_3)^2-4\mu_2}\big)}.\end{aligned}$$

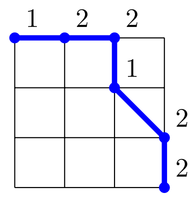
We then note that on a Motzkin path, up steps are always equinumerous with down steps. Hence, $F(\mu_1,\mu_2,\mu_3,\nu,\omega)=F(1,\mu_1\mu_2,\mu_3,\nu,\omega)$, which gives the desired expression. □

It is notable that the lattice paths in $\mathbb{P}_D$ are isomorphic to Motzkin paths of length $D-1$ for every $D\geq 2$. In particular, we have

$$\sum_{\mathscr{P}\in\mathbb{P}}\mu_1^{\#_{\mathscr{P}}(\downarrow)}\mu_2^{\#_{\mathscr{P}}(\rightarrow)}\mu_3^{\#_{\mathscr{P}}(\searrow)}\nu^{\#_{\mathscr{P}}^{\mathrm{diag}}(\downarrow)}\omega^{\#_{\mathscr{P}}^{\mathrm{diag}}(\searrow)}=F(\mu_1,\mu_2,\mu_3,\nu,\omega). \tag{4.7}$$

For the moment, let us assume that $B\in\mathrm{NDIPPM}_n$ has dimension D.

Since NDIPPM is a subset of NDPM, we also have an induced lattice path $\mathrm{Path}(B)\in\mathbb{P}_D$ for B. However, given $\mathrm{Path}(B)\in\mathbb{P}_D$, we are not yet able to recover B because we have no information about the cardinalities of the nonempty entries in B. To resolve this problem, we may assign a *positive* integral weight to each lattice point on $\mathrm{Path}(B)$ to describe the corresponding cardinality of nonempty entries in B. By doing so, we may first recover the natural Fishburn matrix induced by B, and then B itself according to Fact 4.5. For example, the lattice path in (4.3) becomes the following weighted one after our procedure:



. (4.8)

Now it is simple to see that $w(B)$ is exactly the sum of the weights of all lattice points on $\mathrm{Path}(B)$.

To further ensure that B is improper, we note that the weight for the starting point of every south step must be even, while there is no restriction on the starting points of east and southeast steps because the corresponding entry in B is the last nonempty one in its column. Also, for the index (D, D), which is not the starting point of any step on the lattice path, we have no restriction on the parity of its weight.

Putting the above arguments together, we arrive at

$$\sum_{B\in\mathrm{NDIPPM}} x^{\mathrm{blk}(B)}y^{\mathrm{odd}(B)}t^{w(B)} = \left(\frac{yt+t^2}{1-t^2}\right)\sum_{\mathscr{P}\in\mathbb{P}}\left(\frac{t^2}{1-t^2}\right)^{\#_{\mathscr{P}}(\downarrow)}\left(\frac{yt+t^2}{1-t^2}\right)^{\#_{\mathscr{P}}(\rightarrow)+\#_{\mathscr{P}}(\searrow)} x^{\#_{\mathscr{P}}^{\mathrm{diag}}(\searrow)+1}. \tag{4.9}$$

Here $\frac{t^2}{1-t^2} = t^2 + t^4 + \cdots$ generates the possible weights for the starting points of south steps, while $\frac{yt+t^2}{1-t^2} = yt + t^2 + yt^3 + t^4 + \cdots$ corresponds to other lattice points on the path.

In view of (4.7), we may write (4.9) in terms of the series F in (4.5):

$$\sum_{B\in\mathrm{NDIPPM}} x^{\mathrm{blk}(B)}y^{\mathrm{odd}(B)}t^{w(B)} = \left(x\cdot\frac{yt+t^2}{1-t^2}\right)F\left(\frac{t^2}{1-t^2},\frac{yt+t^2}{1-t^2},\frac{yt+t^2}{1-t^2},1,x\right).$$

Meanwhile,

$$\sum_{\mathscr{M}\in\mathbb{M}} x^{\mathrm{comp}(\mathscr{M})}y^{\mathrm{level}(\mathscr{M})}t^{\mathrm{len}(\mathscr{M})} = F(t,t,yt,x,x)-1.$$

A direct simplification reveals that the two series above are identical, and hence demonstrates the desired equidistribution in Theorem 4.4.

Corollary 4.9. *We have*

$$\sum_{B\in\mathrm{NDIPPM}} x^{\mathrm{blk}(B)} y^{\mathrm{odd}(B)} t^{w(B)} = \frac{x + xyt - 2x^2yt - 2x^2t^2 - x\sqrt{(1-yt)^2 - 4t^2}}{2 - 2x - 2xyt + 2x^2yt + 2x^2t^2}. \tag{4.10}$$

Remark 4.10. We may also derive the generating function for NDPM in the same way as we treat NDIPPM. In particular, when we assign weights to the lattice points on Path(A) for $A \in$ NDPM, we should remove the restriction that "the weight for the starting point of every south step must be even," which brings about the improperness. Hence,

$$\sum_{A\in\mathrm{NDPM}} x^{\mathrm{blk}(A)} t^{w(A)} = \left(x \cdot \frac{t}{1-t}\right) F\left(\frac{t}{1-t}, \frac{t}{1-t}, \frac{t}{1-t}, 1, x\right).$$

On the other hand, a *Dyck path* $\mathscr{D}$ of *semilength* n is a staircase walk from $(0,0)$ to (n,n) that lies strictly below (but may touch) the diagonal $y = x$; we write semilen$(\mathscr{D}) = n$. Such paths are enumerated by the n-th *Catalan number* $\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$, listed in the sequence A000108 in OEIS [18]. In addition, Dyck paths of semilength n are isomorphic to Motzkin paths of length $2n$ with level steps forbidden. Let $\mathbb{D}_n$ denote the set of Dyck paths of semilength n, and write $\mathbb{D} := \cup_{n\geq 1}\mathbb{D}_n$. Let touch$(\mathscr{D})$ denote the number of touches of the Dyck path $\mathscr{D}$ at the diagonal, with the starting point $(0,0)$ excluded. Then,

$$\sum_{\mathscr{D}\in\mathbb{D}} x^{\mathrm{touch}(\mathscr{D})} t^{\mathrm{semilen}(\mathscr{D})} = F(t,1,0,x,1) - 1.$$

A direct verification shows that

$$F(t,1,0,x,1) - 1 = \left(x \cdot \frac{t}{1-t}\right) F\left(\frac{t}{1-t}, \frac{t}{1-t}, \frac{t}{1-t}, 1, x\right).$$

Therefore, the following result is also true.

Theorem 4.11. *For every $n \geq 1$, the statistic* blk *on* NDPM_n *is equidistributed with the statistic* touch *on* $\mathbb{D}_n$*. In particular,*

$$\sum_{A\in\mathrm{NDPM}} x^{\mathrm{blk}(A)} t^{w(A)} = \frac{x - 2x^2t - x\sqrt{1-4t}}{2 - 2x + 2x^2t}. \tag{4.11}$$

Remark 4.12. Recall that the sets of nondecreasing partition matrices of weight n and nondecreasing inversion sequences of length n are equinumerous.

Thus, the $x = 1$ specialization of (4.11), namely, there being Cat_n nondecreasing partition matrices of weight n, recovers a result of Martinez and Savage [17, p. 18, Theorem 27] that nondecreasing inversion sequences of length n are counted by Cat_n.

4.3. **Bijective proof of Theorem 4.4.** To establish Theorem 4.4 in a bijective manner, we aim to *recursively* construct a bijection

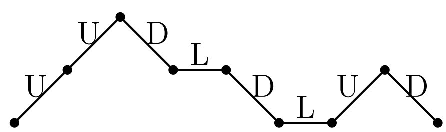
$$\begin{array}{rccc} \Phi: & \mathrm{NDIPPM} & \to & \mathbb{M} \\ & B & \mapsto & \mathscr{M} \end{array}$$

such that

$$\{w, \mathrm{blk}, \mathrm{odd}\}(B) = \{\mathrm{len}, \mathrm{comp}, \mathrm{level}\}(\mathscr{M}).$$

We also occasionally use the notation Φ_n to restrict the map Φ from NDIPPM_n to $\mathbb{M}_n$.

Recall that each Motzkin path can be represented as a *word* in the following way. First, for each of the three types of steps on a Motzkin path, we assign a *letter*: **u**p steps $\mathrm{U} := (1, 1)$, **d**own steps $\mathrm{D} := (1, -1)$, and **l**evel steps $\mathrm{L} := (1, 0)$. Then each Motzkin path $\mathscr{M} \in \mathbb{M}_n$ can be translated into a word of length n consisting of the letters U, D, and L by scanning the steps from left to right; one concrete example is depicted below:



,

which corresponds to the word U U D L D L U D. In this subsection, we will use such a word representation by default.

Let us start with a *trivial* subclass for which the construction of Φ is most clear. Let $B = ([n])$ be the unique one-dimensional matrix in NDIPPM_n. Define its image

$$\mathscr{M} = \Phi(B) := \begin{cases} \mathrm{U}^m\,\mathrm{D}^m, & \text{if } n = 2m, \\ \mathrm{U}^m\,\mathrm{L}\,\mathrm{D}^m, & \text{if } n = 2m+1, \end{cases} \tag{4.12}$$

where X^k stands for k consecutive copies of $\mathrm{X} \in \{\mathrm{U}, \mathrm{D}, \mathrm{L}\}$. It is easy to verify that $\mathrm{blk}(B) = \mathrm{comp}(\mathscr{M}) = 1$, while $\mathrm{odd}(B) = \mathrm{level}(\mathscr{M}) = 0$ if n is even, and $\mathrm{odd}(B) = \mathrm{level}(\mathscr{M}) = 1$ if n is odd.

Now we apply induction on n, the weight of the matrix, to construct the desired map Φ. For $n = 1$, the only matrix of this weight is $(\{1\})$, and the correspondence

$$\big(\{1\}\big) \overset{\Phi}{\mapsto} \mathrm{L}$$

is already covered by the trivial subclass discussed earlier. For $n = 2$, we also have the simple correspondences

$$\big(\{1,2\}\big) \overset{\Phi}{\mapsto} \mathrm{U}\,\mathrm{D}, \qquad\qquad \begin{pmatrix} \{1\} & \\ & \{2\} \end{pmatrix} \overset{\Phi}{\mapsto} \mathrm{L}\,\mathrm{L}\,.$$

Next, suppose that a statistics-preserving bijection $\Phi_n : \mathrm{NDIPPM}_n \to \mathbb{M}_n$ has been constructed for every $n < N$ for a certain $N \geq 3$, and we are about to consider the case of $n = N$.

Take any partition matrix $B \in \mathrm{NDIPPM}_N$ with $\mathrm{blk}(B) > 1$. In this case, the matrix B can be *uniquely* partitioned as

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where both B_1 and B_2 are nondecreasing improper partition matrices of smaller weight than B, and in addition, the number of irreducible diagonal blocks in B_2 is exactly one[6]. Hence, we can use the inductive hypothesis to derive two Motzkin paths and then concatenate them to obtain the image of B under Φ. Namely, we let

$$\mathscr{M} = \Phi(B) := \Phi(B_1)\Phi(B_2).$$

The statistics are transformed as shown below:

$$\begin{aligned} \mathrm{comp}(\mathscr{M}) &= \mathrm{comp}(\Phi(B_1)) + \mathrm{comp}(\Phi(B_2)) \\ &= \mathrm{blk}(B_1) + \mathrm{blk}(B_2) \\ &= \mathrm{blk}(B), \end{aligned}$$

and

$$\begin{aligned} \mathrm{level}(\mathscr{M}) &= \mathrm{level}(\Phi(B_1)) + \mathrm{level}(\Phi(B_2)) \\ &= \mathrm{odd}(B_1) + \mathrm{odd}(B_2) \\ &= \mathrm{odd}(B). \end{aligned}$$

It then remains to deal with the case where $\mathrm{blk}(B) = 1$. Suppose $B = (b_{i,j})_{1 \leq i,j \leq D}$ is of dimension $D \geq 2$ (recall that the one-dimensional case belongs to the trivial subclass).

Our task is to construct a new matrix $B^* \in \mathrm{NDIPPM}_{N-2}$ from the given $B \in \mathrm{NDIPPM}_N$ and then define the image Motzkin path as the concatenation

[6]Strictly speaking, we should subtract each element in B_2 by $w(B_1)$ so that the numbers become $1, \ldots, w(B_2)$. From now on, we tacitly carry out such a reduction for each submatrix when needed without further declaration.

$\mathscr{M} = \Phi(B) := \mathrm{U}\,\Phi(B^*)\,\mathrm{D}$, where $\Phi(B^*) \in \mathbb{M}_{N-2}$ has already been constructed according to our inductive hypothesis. Clearly,

$$\mathrm{comp}(\mathscr{M}) = 1 = \mathrm{blk}(B),$$

where the only step ending on the x-axis is the last step. Also,

$$\mathrm{level}(\mathscr{M}) = \mathrm{level}(\Phi(B^*)) = \mathrm{odd}(B^*),$$

so that to ensure $\mathrm{level}(\mathscr{M}) = \mathrm{odd}(B)$, it suffices to require

$$\mathrm{odd}(B^*) = \mathrm{odd}(B). \tag{4.13}$$

Before proceeding to our construction, we recall that, as illustrated around (4.8), each nondecreasing improper partition matrix B is in one-to-one correspondence with the weighted version of its induced lattice path, where the weight of each lattice point on the path equals the cardinality of the corresponding entry in B. For the moment, we still use $\mathrm{Path}(B)$ to call such a weighted lattice path by abuse of notation, and further write the weight of the point (x,y) on $\mathrm{Path}(B)$ as $w_{\mathrm{Path}(B)}(x,y)$. Also, we denote by $\mathrm{odd}(\mathrm{Path}(B))$ the number of lattice points of an odd weight on $\mathrm{Path}(B)$ so that

$$\mathrm{odd}(\mathrm{Path}(B)) = \mathrm{odd}(B).$$

Hence, the requirement (4.13) becomes

$$\mathrm{odd}(\mathrm{Path}(B^*)) = \mathrm{odd}(\mathrm{Path}(B)). \tag{4.14}$$

In what follows, we will describe our construction from $\mathrm{Path}(B)$ to $\mathrm{Path}(B^*)$, rather than from B to B^*.

As a final preparation, we note that $b_{D-1,D} \neq \varnothing$ since otherwise $b_{D,D}$ is the only nonempty set in the D-th column, causing $\mathrm{blk}(B) \geq 2$, a contradiction. Moreover, $b_{D,D} \neq \varnothing$ so $b_{D-1,D}$ must contain an even number of elements. Thus, the lattice point $(D-1,D)$ lies on $\mathrm{Path}(B)$ and $w_{\mathrm{Path}(B)}(D-1,D)$ is even. Let us denote by $\mathrm{Pre}(D-1,D)$ the preceding node of $(D-1,D)$ on $\mathrm{Path}(B)$. It is clear that

$$\mathrm{Pre}(D-1,D) \in \{(D-1,D-1),(D-2,D-1),(D-2,D)\}.$$

(1) $w_{\mathrm{Path}(B)}(D-1,D) \geq 4$. We preserve the weighted path $\mathrm{Path}(B)$ with the only exception that $w_{\mathrm{Path}(B)}(D-1,D)$ is decreased by 2. The new path is the desired $\mathrm{Path}(B^*)$ corresponding to a certain $B^* \in \mathrm{NDIPPM}_{N-2}$. Also,

$$\mathrm{odd}(\mathrm{Path}(B^*)) = \mathrm{odd}(\mathrm{Path}(B)).$$

Furthermore, an application of (4.4) yields

$$\#^{\mathrm{diag}}_{\mathrm{Path}(B^*)}(\searrow) = \#^{\mathrm{diag}}_{\mathrm{Path}(B)}(\searrow) = \mathrm{blk}(B) - 1 = 0.$$

(2) $w_{\mathrm{Path}(B)}(D-1,D)=2$ and $\mathrm{Pre}(D-1,D)=(D-1,D-1)$. Note that $\mathrm{Path}(B)$ ends with an east step from $(D-1,D-1)$ to $(D-1,D)$ and then a south step from $(D-1,D)$ to (D,D). Now we first remove the lattice point $(D-1,D)$, which has weight 2, from $\mathrm{Path}(B)$, and then reconnect $(D-1,D-1)$ to (D,D) by a direct southeast step; this process is illustrated in Fig. 1. After preserving all other weights on the path, we obtain the desired $\mathrm{Path}(B^*)$ corresponding to a certain $B^*\in\mathrm{NDIPPM}_{N-2}$. Clearly,

$$\mathrm{odd}(\mathrm{Path}(B^*))=\mathrm{odd}(\mathrm{Path}(B)).$$

Furthermore, in this case, we have

$$\#^{\mathrm{diag}}_{\mathrm{Path}(B^*)}(\searrow)=1,$$

with this single southeast step on the diagonal from $(D-1,D-1)$ to (D,D).

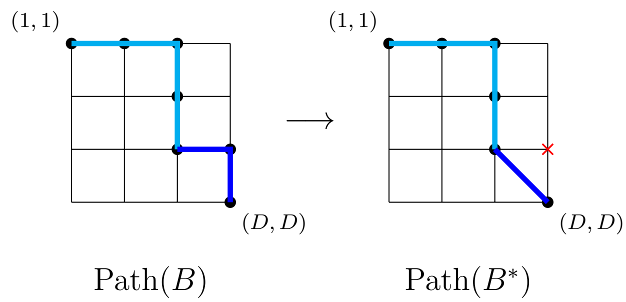


FIGURE 1. The construction of B^* in Case (2)

(3) $w_{\mathrm{Path}(B)}(D-1,D)=2$ and $\mathrm{Pre}(D-1,D)\in\{(D-2,D-1),(D-2,D)\}$. Note that this case can only happen when $D\geq 3$. Also, $\#^{\mathrm{diag}}_{\mathrm{Path}(B)}(\searrow)=\mathrm{blk}(B)-1=0$ so that the first step on $\mathrm{Path}(B)$ must be an east step. Hence, we may define a nonempty set

$$S:=\{i:1\leq i\leq D-2 \text{ and both } (i,i),(i,i+1) \text{ are on } \mathrm{Path}(B)\},$$

which at least contains 1. Let

$$s:=\max S.$$

Since $\mathrm{Pre}(D-1,D)\neq(D-1,D-1)$ so that the lattice point $(D-1,D-1)$ is not on $\mathrm{Path}(B)$, we have $s<D-1$. Now we *slide down* the subpath of $\mathrm{Path}(B)$ between $(s,s+1)$ and $\mathrm{Pre}(D-1,D)$ one unit to the south, with weights also transported. This new subpath starts at $(s+1,s+1)$ and ends at either $(D-1,D-1)$ or $(D-1,D)$. We then reconnect the two endpoints

to (s,s) and (D,D), respectively, and preserve all remaining weights. See Fig. 2 for an illustration. By doing so, we essentially remove the original lattice point $(D-1,D)$ on $\mathrm{Path}(B)$, which has weight 2. Also, the new path is still composed of south, east, and southeast steps. Finally, we argue that it does not fall below the diagonal. Equivalently, we show that the subpath of $\mathrm{Path}(B)$ between $(s,s+1)$ and $\mathrm{Pre}(D-1,D)$ never touches the diagonal. Otherwise, supposing that a certain diagonal point (α,α) with $\alpha > s$ lies on this subpath and noting that it is not the ending point $\mathrm{Pre}(D-1,D)$ of the subpath, then there should exist a step starting from this (α,α), which is either an east or a southeast one. However, for the former case, we have $(\alpha,\alpha+1)$ on $\mathrm{Path}(B)$ so that $\alpha \in S$, violating the maximality of s; for the latter case, we have $\#^{\mathrm{diag}}_{\mathrm{Path}(B)}(\searrow) \geq 1$ so that $\mathrm{blk}(B) \geq 2$, violating the assumption that $\mathrm{blk}(B) = 1$. In aggregate, we have a weighted path $\mathrm{Path}(B^*)$, which corresponds to a certain $B^* \in \mathrm{NDIPPM}_{N-2}$. Also, it is clear that

$$\mathrm{odd}(\mathrm{Path}(B^*)) = \mathrm{odd}(\mathrm{Path}(B)).$$

Furthermore, in this case, we have

$$\#^{\mathrm{diag}}_{\mathrm{Path}(B^*)}(\searrow) \geq 1.$$

In particular, since on $\mathrm{Path}(B)$ and hence on $\mathrm{Path}(B^*)$, there is no southeast step between $(1,1)$ and (s,s) lying on the diagonal (because $\mathrm{blk}(B) = 1$), the first southeast step on the diagonal is from (s,s) to $(s+1,s+1)$. Noting that we have argued that $s < D-1$, this southeast step is different from that in Case (2).

The injectivity of the map Φ can be deduced from our discussions at the end of each case, since the three image sets are mutually disjoint from each other.

Finally, for the surjectivity of Φ, it suffices to show that Φ is invertible in the $\mathrm{comp}(\mathscr{M}) = 1$ case. To this end, we briefly explain how to recover its preimage for each Motzkin path $\mathscr{M} \in \mathbb{M}_N$. Since $\mathrm{comp}(\mathscr{M}) = 1$ and $N \geq 3$, we can write $\mathscr{M}$ as $\mathrm{U}\,\mathscr{M}^*\,\mathrm{D}$, where $\mathscr{M}^* \in \mathbb{M}_{N-2}$. We then find the unique matrix $B^* = (b^*_{i,j})_{1\leq i,j\leq D} := \Phi^{-1}(\mathscr{M}^*)$ with $\dim(B^*) = D$ by our inductive hypothesis.

Now we aim to construct a new matrix $B = (b_{i,j})_{1\leq i,j\leq D} \in \mathrm{NDIPPM}_N$ from the above $B^* \in \mathrm{NDIPPM}_{N-2}$ and take this B as the preimage $\Phi^{-1}(\mathscr{M})$. For convenience, we still describe our construction from $\mathrm{Path}(B^*)$ to $\mathrm{Path}(B)$. Also, write the preceding node of the lattice point (D,D) on $\mathrm{Path}(B^*)$ as $\mathrm{Pre}(D,D)$.

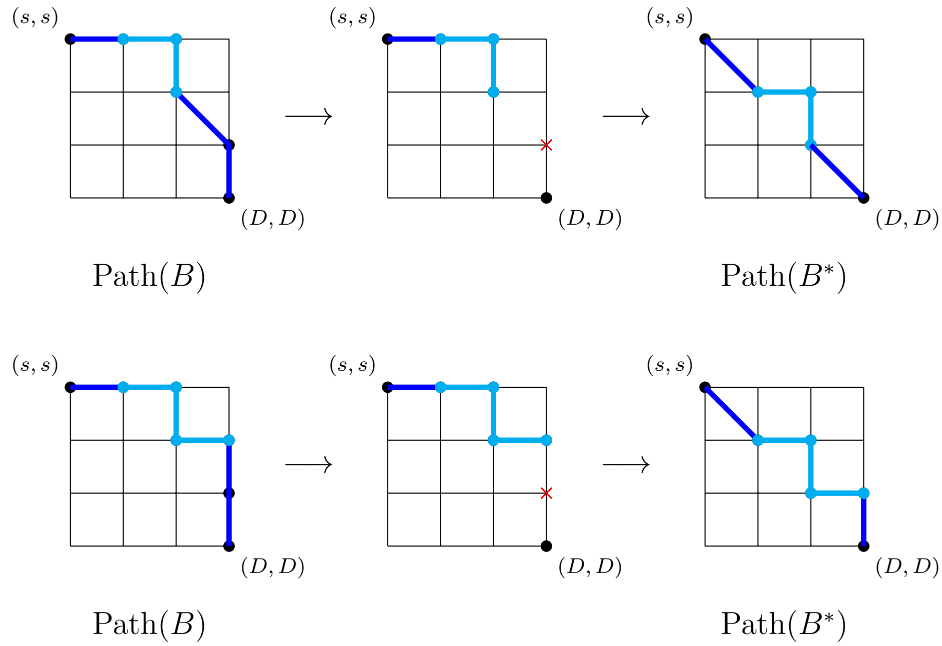


FIGURE 2. The construction of B^* in Case (3)

We have the following three cases.

(1') $\#^{\text{diag}}_{\text{Path}(B^*)}(\searrow) = 0$. This case corresponds to the previous Case (1). Note that $\text{Pre}(D, D) = (D-1, D)$. Now we preserve the weighted path $\text{Path}(B^*)$ with the only exception that the weight at $(D-1, D)$ is increased by 2. The new weighted path is $\text{Path}(B)$.

(2') $\#^{\text{diag}}_{\text{Path}(B^*)}(\searrow) = 1$ and the only southeast step on the diagonal starts at $(D-1, D-1)$. This case corresponds to the previous Case (2). Note that $\text{Pre}(D, D) = (D-1, D-1)$. Now we remove the southeast step from $(D-1, D-1)$ to (D, D), and add an east step from $(D-1, D-1)$ to $(D-1, D)$ and a south step from $(D-1, D)$ to (D, D). By doing so, we have an extra lattice point $(D-1, D)$ inserted on the new path. We assign it a weight of 2, and preserve all other weights. The new weighted path is $\text{Path}(B)$.

(3') $\#^{\text{diag}}_{\text{Path}(B^*)}(\searrow) \geq 1$ and the first southeast step on the diagonal starts at a certain (s, s) with $s < D-1$. This case corresponds to the previous Case (3). Note that $\text{Pre}(D, D) \in \{(D-1, D-1), (D-1, D)\}$. Now we *slide up* the subpath from $(s+1, s+1)$ (which is the succeding node of

$(s,s))$ to $\mathrm{Pre}(D,D)$ one unit to the north, with weights also transported; this new subpath then starts at $(s,s+1)$ and ends at $(D-2,D-1)$ or $(D-2,D)$. We then reconnect its left end to (s,s) and right end to a new lattice point $(D-1,D)$, and add a further south step from $(D-1,D)$ to (D,D). Finally, we assign the newly added point $(D-1,D)$ a weight of 2 and preserve all other weights. The new weighted path is $\mathrm{Path}(B)$.

We have constructed a bijective mapping $\Phi : \mathrm{NDIPPM}_N \to \mathbb{M}_N$ such that for every $B \in \mathrm{NDIPPM}_N$, there holds that $\mathrm{comp}(\Phi(B)) = \mathrm{blk}(B)$ and $\mathrm{level}(\Phi(B)) = \mathrm{odd}(B)$. The proof is now completed by induction.

5. Outlook

In this work, our attention is paid to the evaluation of the polynomials $S_n(q)$ at $q=-1$. Meanwhile, we have computed the exact expressions of $S_n(q)$ for some small n:

$$\begin{aligned}
S_1(q) &= 1,\\
S_2(q) &= 2,\\
S_3(q) &= 5+q,\\
S_4(q) &= 15+7q+2q^2,\\
S_5(q) &= 53+41q+20q^2+5q^3+q^4,\\
S_6(q) &= 217+240q+161q^2+68q^3+24q^4+8q^5+2q^6,\\
S_7(q) &= 1014+1475q+1253q^2+716q^3+334q^4+154q^5+62q^6\\
&\quad +22q^7+9q^8+q^9,\\
S_8(q) &= 5335+9677q+9950q^2+7066q^3+4034q^4+2192q^5+1098q^6\\
&\quad +527q^7+271q^8+108q^9+40q^{10}+18q^{11}+4q^{12}.
\end{aligned}$$

The following two questions are then natural.

Question 5.1. *Is there a closed expression for the generating series* $\sum_{n\geq 1} S_n(q)t^n$*?*

Question 5.2. *For each $n \geq 1$, are the coefficients in the polynomial $S_n(q) \in \mathbb{N}[q]$ unimodal? In addition, do these coefficients satisfy a certain distributional law when n is sufficiently large?*

Besides studying $S_n(q)$ for its own sake, connecting it to other objects might be interesting as well. Let us call any set of combinatorial objects enumerated by the Fishburn number Fis_n (resp. the factorial $n!$) a *Fishburn structure* (resp. a *factorial structure*). As witnessed by (1.5) and (1.6), our polynomial $S_n(q)$ interpolates between a Fishburn structure (namely, FM_n) and a factorial structure (namely, PM_n). Recall that there are at least four other Fishburn structures that have been well studied, namely, the $(\mathbf{2}+\mathbf{2})$-free unlabeled posets (also known as the interval orders), the ascent sequences, the Fishburn permutations, and the Stoimenow matchings; see [3] for undefined terms. More notably, each of these Fishburn structures admits a superset that is a factorial structure. Now the following question seems pertinent.

Question 5.3. *Does there exist a suitably chosen statistic on any of the four remaining factorial structures, such that the refined enumeration according to this statistic gives rise to an alternative interpretation of $S_n(q)$? Or from a different perspective, do any of these other four structures possess natural interweaving q-enumeration such that the signed (i.e., $q=-1$) counting leads to further findings?*

Lastly, recall that our proof of Theorem 3.3 builds on several known results such as Frobenius' identity (3.4) and is analytic in nature. It remains an open problem to find a direct bijective proof[7]. To that end, we make some further observations here, which hopefully would motivate the interested reader to pursue this question. First, let us split both IPPM_n and $\mathbf{I}_n(-,-,=)$ into two complementary subsets respectively. For every $n \geq 1$, we decompose IPPM_n as the *disjoint union* $\mathrm{IPPM}_n^+ \sqcup \mathrm{IPPM}_n^-$, where

$$\mathrm{IPPM}_n^+ := \{Q \in \mathrm{IPPM}_n : n \text{ belongs to a subset in } Q \text{ of an even cardinality}\},$$
$$\mathrm{IPPM}_n^- := \{Q \in \mathrm{IPPM}_n : n \text{ belongs to a subset in } Q \text{ of an odd cardinality}\}.$$

In a similar vein, we split $\mathbf{I}_n(-,-,=)$ into two disjoint subsets $\mathbf{I}_n^+(-,-,=)$ and $\mathbf{I}_n^-(-,-,=)$, where

$$\mathbf{I}_n^+(-,-,=) := \{e \in \mathbf{I}_n(-,-,=) : e_{n-1} = e_n\},$$
$$\mathbf{I}_n^-(-,-,=) := \{e \in \mathbf{I}_n(-,-,=) : e_{n-1} \neq e_n\}.$$

We observe the following correlations between the $+/-$ subsets.

Lemma 5.4. *For every $n \geq 1$, there exist two bijections*

$$\phi_n : \mathrm{IPPM}_n^- \to \mathrm{IPPM}_{n+1}^+,$$

[7] As mentioned earlier, Claesson, Dukes, and Kubitzke [7, p. 1626, Theorem 3] constructed a bijection $\eta : \mathrm{PM}_n \to \mathbf{I}_n$. When we restrict their mapping to the subclass IPPM_n for $n \geq 3$ however, its image is not $\mathbf{I}_n(-,-,=)$. For instance, we see that $\eta((\{1,2,3\})) = (0,0,0) \notin \mathbf{I}_3(-,-,=)$.

and

$$\psi_n : \mathbf{I}_n^-(-,-,=) \to \mathbf{I}_{n+1}^+(-,-,=),$$

such that for every matrix $Q \in \mathrm{IPPM}_n^-$*, we have*

$$v(\phi_n(Q)) = v(Q), \tag{5.1}$$

while for every inversion sequence $e \in \mathbf{I}_n^-(-,-,=)$*, we have*

$$\mathrm{dist}(\psi_n(e)) = \mathrm{dist}(e). \tag{5.2}$$

Proof. Given a matrix $Q \in \mathrm{IPPM}_n^-$, we can append $n+1$ to the subset containing n to get a matrix $\widetilde{Q} \in \mathrm{IPPM}_{n+1}^+$, which we take to be the image $\phi_n(Q)$. Conversely, given a matrix $\widetilde{Q} \in \mathrm{IPPM}_{n+1}^+$, we can remove its maximal letter $n+1$ to recover its preimage $Q \in \mathrm{IPPM}_n^-$ under ϕ_n. These two operations are clearly inverses of each other, and (5.1) follows from the fact that $\lceil \frac{m}{2} \rceil = \lceil \frac{m+1}{2} \rceil$ for any odd integer m.

To construct ψ_n, we note that for any inversion sequence $e \in \mathbf{I}_n^-(-,-,=)$, its last entry e_n is not repeated so appending another copy of e_n to its end produces an inversion sequence $\tilde{e} \in \mathbf{I}_{n+1}^+(-,-,=)$, which we take to be the image $\psi_n(e)$. Conversely, removing the last entry of an inversion sequence $\tilde{e} \in \mathbf{I}_{n+1}^+(-,-,=)$ does recover its preimage $e \in \mathbf{I}_n^-(-,-,=)$ under ψ_n. Defined in this way, ψ_n is seen to be a bijection and the relation (5.2) follows directly. $\square$

Combining Lemma 5.4 with the two decompositions presented earlier, we obtain the following relations between the bivariate generating functions:

$$\sum_{n\geq 1} t^n \sum_{Q\in \mathrm{IPPM}_n} z^{v(Q)} = (1+t) \sum_{n\geq 1} t^n \sum_{Q\in \mathrm{IPPM}_n^-} z^{v(Q)},$$

and

$$\sum_{n\geq 1} t^n \sum_{e\in \mathbf{I}_n(-,-,=)} z^{\mathrm{dist}(e)} = (1+t) \sum_{n\geq 1} t^n \sum_{e\in \mathbf{I}_n^-(-,-,=)} z^{\mathrm{dist}(e)}.$$

Consequently, our inquiry for a direct bijective proof of Theorem 3.3 can be equivalently rephrased as follows.

Question 5.5. *For every* $n \geq 1$*, can one find a direct bijection, say* ρ_n*, from* IPPM_n^- *to* $\mathbf{I}_n^-(-,-,=)$ *such that for each matrix* $Q \in \mathrm{IPPM}_n^-$*, it is true that*

$$v(Q) = \mathrm{dist}(\rho_n(Q))?$$

Acknowledgements. Shane Chern was supported by the Austrian Science Fund (No. 10.55776/F1002). Shishuo Fu was partially supported by the National Natural Science Foundation of China grant (No. 12171059).

References

1. G. E. Andrews and V. Jelínek, On q-series identities related to interval orders, *European J. Combin.* **39** (2014), 178–187. 2
2. R. E. Behrend, I. Fischer, and C. Koutschan, Diagonally symmetric alternating sign matrices, preprint, available at arXiv:2309.08446. 5
3. M. Bousquet-Mélou, A. Claesson, M. Dukes, and S. Kitaev, $(2+2)$-free posets, ascent sequences and pattern avoiding permutations, *J. Combin. Theory Ser. A* **117** (2010), no. 7, 884–909. 1, 25
4. D. M. Bressoud, Combinatorial analysis, in: *NIST Handbook of Mathematical Functions*, 617–636, U.S. Dept. Commerce, Washington, DC, 2010. 10
5. W. Cao, E. Y. Jin, and Z. Lin, Enumeration of inversion sequences avoiding triples of relations, *Discrete Appl. Math.* **260** (2019), 86–97. 4
6. D. Chen, S. H. F. Yan, and R. D. P. Zhou, Equidistributed statistics on Fishburn matrices and permutations, *Electron. J. Combin.* **26** (2019), no. 1, Paper No. 1.11, 14 pp. 1
7. A. Claesson, M. Dukes, and M. Kubitzke, Partition and composition matrices, *J. Combin. Theory Ser. A* **118** (2011), no. 5, 1624–1637. 2, 9, 11, 25
8. P. C. Fishburn, Intransitive indifference with unequal indifference intervals, *J. Mathematical Psychology* **7** (1970), 144–149. 1
9. P. C. Fishburn, Interval lengths for interval orders: a minimization problem, *Discrete Math.* **47** (1983), no. 1, 63–82. 1
10. P. C. Fishburn, Interval Orders and Interval Graphs. A Study of Partially Ordered Sets, John Wiley & Sons, Ltd., Chichester, 1985. 1
11. P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Math.* **32** (1980), no. 2, 125–161. 15
12. G. Frobenius, Über die Bernoullischen Zahlen und die Eulerschen Polynome, *Sitze Berichte Preuss. Akad. Wiss.* (1910), 808–847. 9
13. D. Foata and G.-N. Han, q-Series in Combinatorics; Permutation Statistics (Lecture Notes), Preliminary Edition, private notes, 2011. 3
14. D. Foata and M.-P. Schützenberger, Théorie Géométrique des Polynômes Eulériens, Springer-Verlag, Berlin-New York, 1970. 9
15. C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. in Appl. Math.* **27** (2001), no. 2-3, 510–530. 15
16. P. Levande, Fishburn diagrams, Fishburn numbers and their refined generating functions, *J. Combin. Theory Ser. A* **120** (2013), no. 1, 194–217. 1
17. M. Martinez and C. D. Savage, Patterns in inversion sequences II: inversion sequences avoiding triples of relations, *J. Integer Seq.* **21** (2018), no. 2, Art. 18.2.2, 44 pp. 4, 18
18. N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, available at `http://oeis.org`. 2, 4, 12, 17
19. A. Stoimenow, Enumeration of chord diagrams and an upper bound for Vassiliev invariants, *J. Knot Theory Ramifications* **7** (1998), no. 1, 93–114. 2
20. D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, *Topology* **40** (2001), no. 5, 945–960. 2
21. D. Zagier, Quantum modular forms, in: *Quanta of Maths*, 659–675, Amer. Math. Soc., Providence, RI, 2010. 2

(S. Chern) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, Wien 1090, Austria

Email address: chenxiaohang92@gmail.com, xiaohangc92@univie.ac.at

(S. Fu) College of Mathematics and Statistics, Chongqing University & Key Laboratory of Nonlinear Analysis and its Applications (Chongqing University), Ministry of Education, Chongqing 401331, P.R. China

Email address: fsshuo@cqu.edu.cn