Leading coefficient in the Hankel determinants related to binomial and q-binomial transforms

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Abstract. It is a standard result that the Hankel determinants for a sequence stay invariant after performing the binomial transform on this sequence. In this work, we extend the scenario to q-binomial transforms and study the behavior of the leading coefficient in such Hankel determinants. We also investigate the leading coefficient in the Hankel determinants for even-indexed Bernoulli polynomials with recourse to a curious binomial transform. In particular, the degrees of these Hankel determinants share the same nature as those in one of the q-binomial cases.

Keywords. Hankel determinant, Bernoulli polynomial, Bernoulli umbra, q-binomial transform, leading coefficient.

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1. Introduction

Given a sequence $(a_k)_{k\geq 0}$, its n-th *Hankel determinant* is the determinant of the *Hankel matrix*

$$H_n(a_k) := \det_{0\leq i,j\leq n}(a_{i+j}) = \det\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}.$$

The study of Hankel determinants has been extensively developed, mainly based on the close relationship to classical orthogonal polynomials and continued fractions; see, e.g., [15, Chapter 2].

The choice of $(a_k)_{k\geq 0}$ relies on various reasons, but it is usually of general interest to consider classical number-theoretic sequences. For example, the *Bernoulli numbers* B_k and *Bernoulli polynomials* $B_k(x)$ are defined by their exponential generating functions:

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \qquad \text{and} \qquad \frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},$$

with a simple connection that

$$B_k(x) = \sum_{\ell=0}^{k}\binom{k}{\ell} B_{k-\ell}x^{\ell}.$$

Al-Salam and Carlitz [1, p. 93, Eq. (3.1)] discovered that

$$H_n(B_k) = (-1)^{\binom{n+1}{2}}\prod_{j=1}^{n}\frac{(j!)^6}{(2j)!(2j+1)!}. \tag{1.1}$$

TABLE 1. $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ for $n = 1, 2, 3$

$H_1\big(B_{2k}(\frac{1+x}{2})\big)$	$-\frac{1}{12}x^2 + \frac{1}{45}$
$H_2\big(B_{2k}(\frac{1+x}{2})\big)$	$-\frac{1}{540}x^6 + \frac{97}{18900}x^4 - \frac{11}{4725}x^2 + \frac{16}{55125}$
$H_3\big(B_{2k}(\frac{1+x}{2})\big)$	$\frac{1}{42000}x^{12} - \frac{121}{441000}x^{10} + \frac{153}{154000}x^8 - \frac{17441}{12262250}x^6 + \frac{8369}{11036025}x^4 - \frac{1632}{9634625}x^2 + \frac{256}{18883865}$

For the Bernoulli polynomials $B_k(x)$, it is surprising that their Hankel determinants stay invariant for any choice of the argument x:

$$H_n(B_k(x)) = H_n(B_k). \tag{1.2}$$

In general, there is a classical result on the invariance of Hankel determinants under the binomial transform

$$a_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell} x^\ell \tag{1.3}$$

for a generic sequence $(a_k)_{k\geq 0}$. See [21, p. 419, Entry 445] for a direct proof, or an alternative proof in [17, p. 393, Eq. (10)] for the scenario wherein a_k is understood as the k-th moment of a certain random variable.

Theorem 1.1 (Invariance of Hankel determinants under the binomial transform)**.** *For every $n \geq 0$, define $a_k(x)$ as in (1.3). Then,*

$$H_n(a_k(x)) = H_n(a_k). \tag{1.4}$$

The story becomes much more intricate when considering non-consecutive terms in sequences of polynomials as in (1.3). In early work, Dilcher and Jiu [11, p. 3, Theorem 1.1] evaluated the Hankel determinants for odd-indexed Bernoulli polynomials. Namely,

$$H_n\left(B_{2k+1}\left(\frac{1+x}{2}\right)\right) = (-1)^{\binom{n+1}{2}} \left(\frac{x}{2}\right)^{n+1} \prod_{j=1}^{n} \left(\frac{j^4(x^2-j^2)}{4(2j+1)(2j-1)}\right)^{n+1-j}.$$

Here, the argument is chosen as $(1+x)/2$ instead of x, to obtain the better factorization of these Hankel determinants. On the other hand, the even-indexed case remains open [11, pp. 10–11]. It is notable that in view of the first few evaluations as shown in Table 1, these Hankel determinants neither factor nor exhibit any obvious pattern. In fact, there are merely some partial results on special choices of the argument x. For instance, Chen [9, p. 390, Corollary 5.6] obtained for the case $x = 0$:

$$H_n\big(B_{2k}\big(\tfrac{1}{2}\big)\big) = \prod_{j=1}^{n} \frac{\big((2j)!\big)^6}{(4j)!(4j+1)!}. \tag{1.5}$$

Although it seems that a closed expression for $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is out of reach, we shall show in this paper that the degree and leading coefficient of each Hankel determinant for even-indexed Bernoulli polynomials can be characterized, via a curious binomial transform (3.1).

Theorem 1.2. *For every $n \geq 0$, $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is a polynomial in x of degree $n(n+1)$ with leading coefficient*

$$\big[x^{n(n+1)}\big] H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right) = (-1)^{\binom{n+1}{2}} \prod_{j=1}^{n} \frac{(j!)^6}{(2j)!(2j+1)!}. \tag{1.6}$$

Remark 1.1. It is notable that the Hankel determinant $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is a linear combination of the terms

$$B_{2j_0}\left(\frac{1+x}{2}\right) B_{2(j_1+1)}\left(\frac{1+x}{2}\right)\cdots B_{2(j_n+n)}\left(\frac{1+x}{2}\right)$$

with $j_0, j_1, \ldots, j_n$ a permutation of $0, 1, \ldots, n$. Here, each term is of degree $\sum_{i=0}^{n} 2(j_i+i) = 2n(n+1)$. However, Theorem 1.2 states that the degree of the Hankel determinant $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is $n(n+1)$, which is only **half** of the above terms, thereby indicating abundant cancelations of higher powers of x in this determinant expansion.

As for generalizations, it is always of great interest and importance to consider the q-analogue of a number or polynomial sequence. For example, Carlitz [7] introduced the q-Bernoulli numbers, denoted by β_k, as

$$\beta_k := \frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{j+1}{[j+1]_q}.$$

In 2017, Chapoton and Zeng [8, p. 359, Eq. (4.7)] showed that

$$H_n(\beta_k) = (-1)^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{j=1}^{n} \frac{\big([j]_q!\big)^6}{[2j]_q![2j+1]_q!}.$$

Here, the *q-integers* for $m \in \mathbb{Z}$ and the *q-factorials* for $M \in \mathbb{N}$ are respectively defined by

$$[m]_q := \frac{1-q^m}{1-q} \qquad \text{and} \qquad [M]_q! := \prod_{m=1}^{M}[m]_q.$$

More recently, a similar study [10] was executed on the q-Euler numbers.

Instead of seeking new q-analogues of known evaluations of Hankel determinants, in this work, we keep an eye on q-analogues of the binomial transform (1.3). Recall that the *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise,} \end{cases}$$

where the *q-Pochhammer symbols* for $N \in \mathbb{N} \cup \{\infty\}$ are defined by

$$(A;q)_N := \prod_{k=0}^{N-1}(1 - Aq^k),$$

with the compact notation

$$(A_1, A_2, \ldots, A_\ell; q)_N := (A_1;q)_N (A_2;q)_N \cdots (A_\ell;q)_N.$$

It is notable that when the ordinary binomial coefficients $\binom{n}{k}$ in (1.3) are replaced with the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, the invariance (1.4) is no longer preserved. Now our objective is to characterize the leading coefficient for this q-binomial scenario.

Throughout, we assume that $(\alpha_k)_{k\geq 0}$ is an arbitrary sequence. Define the polynomial sequence

$$\alpha_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^{\ell}. \tag{1.7}$$

Theorem 1.3. *For every $n \geq 0$, the Hankel determinant $H_n(\alpha_k(x))$ is a polynomial in x of degree $n(n+1)$ with leading coefficient*

$$\left[x^{n(n+1)}\right] H_n(\alpha_k(x)) = \alpha_0^{n+1} (-1)^{\binom{n+1}{2}} q^{3\binom{n+1}{3}} \prod_{j=1}^{n} (1-q^j)^{n+1-j}. \tag{1.8}$$

Clearly, the binomial transform (1.3) is symmetric in the sense that

$$a_k(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell} x^{\ell} = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{\ell} x^{k-\ell}.$$

However, due to the prefactor $q^{\binom{\ell}{2}}$ (whose occurrence would be explained in Section 4) in the summand of the q-binomial transform (1.7), such a symmetry disappears. Hence, one may also want to consider

$$\widetilde{\alpha}_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{\ell} x^{k-\ell}. \tag{1.9}$$

Theorem 1.4. *For every $n \geq 0$, the Hankel determinant $H_n(\widetilde{\alpha}_k(x))$ is a polynomial in x of degree $\frac{n(n+1)}{2}$ with leading coefficient*

$$\left[x^{\frac{n(n+1)}{2}}\right] H_n(\widetilde{\alpha}_k(x)) = \alpha_0\alpha_1\cdots\alpha_n (-1)^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n} (1-q^j)^{n+1-j}. \tag{1.10}$$

Remark 1.2. The degree of $H_n(\alpha_k(x))$ is *not* unexpected as in its determinant expansion, terms are of the form $\alpha_{j_0+0}(x)\alpha_{j_1+1}(x)\cdots\alpha_{j_n+n}(x)$ with $j_0, j_1, \ldots, j_n$ a permutation of $0, 1, \ldots, n$, and each is of degree $n(n+1)$. Hence, $H_n(\alpha_k(x))$ has the **same** degree as these terms. However, the situation becomes different for $H_n(\widetilde{\alpha}_k(x))$ as its degree is only $\frac{n(n+1)}{2}$, being **half** of any term in its expansion. This fact places Theorem 1.4 in the same boat as Theorem 1.2, and similar cancelations of higher powers of x should happen.

This paper is organized as follows. In Section 2, we introduce some preliminary notation and lemmas for our subsequent use. In Section 3, we establish a general transform for Bernoulli polynomials by means of the theory of umbral calculus, thereby offering a proof of Theorem 1.2. In Section 4, we characterize the leading coefficient in the Hankel determinants related to the q-binomial transforms (1.7) and (1.9) by proving Theorems 1.3 and 1.4. Meanwhile, we specialize our sequence $(\alpha_k)_{k\geq 0}$ and provide an example with explicit Hankel determinant expressions.

2. Preliminaries

In this section, we shall collect some preliminaries for later use.

2.1. Determinants. We need the following two basic relations on Hankel determinants.

Lemma 2.1. *Let $(a_k)_{k\geq 0}$ be a sequence and x an indeterminate. Then for $n \geq 0$,*

$$H_n(xa_k) = x^{n+1}H_n(a_k), \tag{2.1}$$

$$H_n(x^k a_k) = x^{n(n+1)}H_n(a_k). \tag{2.2}$$

These can be proved easily by extracting powers of x from rows or columns. In particular, the relation (2.2) appeared as [11, p. 4, Lemma 2.1].

2.2. The Bernoulli umbra $\boldsymbol{\mathcal{B}}$. The *umbral calculus* (see, e.g., [22]) associates each term of a sequence $(a_k)_{k\geq 0}$ with the corresponding power $\mathcal{A}^k$ of an *umbra* $\mathcal{A}$. Along this line, we may define the *Bernoulli umbra* $\mathcal{B}$ via a simple evaluation map:

$$(\mathcal{B}+x)^k = \mathsf{Eval}\left[(\mathcal{B}+x)^k\right] = B_k(x).$$

For simplicity, the functional "Eval", short for evaluation, is usually omitted. Its probabilistic formalism [12, Theorem 2.3] begins with a random variable L_B subject to the density function $\frac{\pi}{2}\operatorname{sech}^2(\pi x)$. Then $\mathcal{B} = iL_B - \frac{1}{2}$, where $i = \sqrt{-1}$ is the imaginary unit. That is,

$$B_k(x) = \mathbb{E}\left[(\mathcal{B}+x)^k\right] = \mathsf{Eval}\left[(\mathcal{B}+x)^k\right] = \frac{\pi}{2}\int_{\mathbb{R}}\left(x + i\,t - \frac{1}{2}\right)^k \operatorname{sech}^2(\pi t)\,dt,$$

which interprets the evaluation as the expectation operator $\mathbb{E}$, and $B_k(x)$ as the k-th moment of the random variable $iL_B - \frac{1}{2} + x$. For simplification, we shall just write

$$\mathcal{B}^k = B_k \quad \text{and} \quad (\mathcal{B}+x)^k = B_k(x). \tag{2.3}$$

Remark 2.1. The above random variable interpretation triggers the probabilistic proof in [17] for the invariance (1.4) of Hankel determinants under the binomial transform.

2.3. Orthogonal polynomials. Orthogonal polynomials play a crucial role in the evaluation of Hankel determinants. See Ismail's monograph [15, Chapter 2] or Krattenthaler's surveys [19, Section 2.7] and [20, Section 5.4] for a comprehensive account. Here we only collect some basics.

Definition 2.1. Given a sequence $(a_n)_{n\geq 0}$, the family of (monic) polynomials $(p_n(z))_{n\geq 0}$ with $\deg p_n = n$ is called *orthogonal with respect to* $(a_n)_{n\geq 0}$, if there exists a linear functional L on the space of polynomials in z such that

$$L(z^n) = a_n \quad \text{and} \quad L\big(p_n(z)p_m(z)\big) = \delta_{n,m}\sigma_n,$$

where $\delta_{n,m}$ is the *Kronecker delta* and $(\sigma_n)_{n\geq 0}$ is a sequence of nonzero constants.

In view of *Favard's Theorem* [19, p. 21, Theorem 12], there is a one-to-one correspondence between orthogonal polynomials and three-term recurrences.

Lemma 2.2 (Favard's Theorem)**.** *Let $(p_n(z))_{n\geq 0}$ be a family of monic polynomials with $p_n(z)$ of degree n. Then they are orthogonal if and only if there exist sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 1}$ with v_n nonzero such that $p_0(z) = 1$, $p_1(z) = u_0 + z$, and for $n \geq 1$,*

$$p_{n+1}(z) = (u_n + z)p_n(z) - v_n p_{n-1}(z). \tag{2.4}$$

The following relation [23, p. 197, Theorem 51.1], which is attributed to Heilermann [13], further connects the Hankel determinants for a sequence with its associated orthogonal polynomials.

Lemma 2.3. *Given any sequence $(a_n)_{n\geq 0}$, let the monic polynomials $(p_n(z))_{n\geq 0}$ be orthogonal with respect to $(a_n)_{n\geq 0}$. Assume further that $(p_n(z))_{n\geq 0}$ satisfies the recurrence in* (2.4)*. Then*

$$H_n(a_k) = a_0^{n+1} v_1^n v_2^{n-1} \cdots v_n. \tag{2.5}$$

Finally, we recall the *big q-Jacobi polynomials* introduced by Andrews and Askey [5], which form a family of basic hypergeometric orthogonal polynomials in the basic Askey scheme. To begin with, we need the *q-hypergeometric series*:

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, t\right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n t^n}{(q, b_1, b_2, \ldots, b_r; q)_n}.$$

Definition 2.2. The *big q-Jacobi polynomials* are defined by

$$P_n(z; a, b, c; q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, z \\ aq, cq \end{matrix}; q, q\right). \tag{2.6}$$

The associated three-term recurrence for the big q-Jacobi polynomials is well-known; see, e.g., [18, p. 438, Eq. (14.5.3)].

Lemma 2.4. *For $n \geq 1$,*

$$\begin{aligned}(z-1)P_n(z; a, b, c; q) = A_n P_{n+1}(z; a, b, c; q) - (A_n + C_n) P_n(z; a, b, c; q) \\ + C_n P_{n-1}(z; a, b, c; q),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A_n &= \frac{(1-aq^{n+1})(1-cq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \\ C_n &= -\,acq^{n+1}\frac{(1-q^n)(1-bq^n)(1-abc^{-1}q^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \end{aligned}$$

From [18, p. 438, Eq. (14.5.4)], we find the recurrence for the normalized sequence

$$Q_n(z; a, b, c; q) := \frac{(aq, cq; q)_n}{(abq^{n+1}; q)_n} P_n(z; a, b, c; q).$$

Corollary 2.5. *For $n \geq 1$,*

$$\begin{aligned} Q_{n+1}(z; a, b, c; q) = (z + A_n + C_n - 1) Q_n(z; a, b, c; q) \\ - A_{n-1} C_n Q_{n-1}(z; a, b, c; q). \end{aligned} \tag{2.8}$$

In view of the orthogonality of the big q-Jacobi polynomials, we define the following sequence.

Definition 2.3. Let $(\mu_n(a, b, c))_{n\geq 0}$ with $\mu_0(a, b, c) = 1$ be a sequence with respect to which $P_n(z; a, b, c; q)$ is orthogonal.

Remark 2.2. By definition, the normalized polynomials $Q_n(z; a, b, c; q)$ are also orthogonal with respect to $(\mu_n(a, b, c))_{n\geq 0}$.

To close this section, we note from [16, p. 529, Theorem 3.1] that $\mu_n(a, b, c)$ can be expressed in terms of the *Al-Salam–Chihara polynomials* [18, Section 14.8].

Definition 2.4. The *Al-Salam–Chihara polynomials* are defined by

$$p_n(\theta, t_1, t_2) := {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1e^{i\theta}, t_2e^{-i\theta} \\ t_1t_2, 0 \end{matrix}; q, q\right). \tag{2.9}$$

Lemma 2.6. *Let $a = t_1e^{i\theta}/q$, $b = t_2e^{-i\theta}/q$, and $c = t_1e^{-i\theta}/q$. Then*

$$\mu_n(a, b, c) = p_n(\theta, t_1, t_2). \tag{2.10}$$

3. Even-indexed Bernoulli polynomials

Our starting point is the following identity for Bernoulli polynomials.

Lemma 3.1. *For any indeterminates a and b,*

$$\sum_{k=0}^{n} \binom{n}{k} B_{2k}(ax+b)(-b^2)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}(ax)(2b)^{n-k}. \tag{3.1}$$

Proof. We make use of the Bernoulli umbra $\mathcal{B}$ and derive that

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} B_{2k}(ax+b)(-b^2)^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} (\mathcal{B}+ax+b)^{2k}(-b^2)^{n-k} \\
&= \big((\mathcal{B}+ax+b)^2 - b^2\big)^n \\
&= (\mathcal{B}+ax)^n(\mathcal{B}+ax+2b)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (\mathcal{B}+ax)^{n+k}(2b)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} B_{n+k}(ax)(2b)^{n-k}.
\end{aligned}$$

In particular, for the first equality, $B_{2k}(ax+b)$ is replaced with $(\mathcal{B}+ax+b)^{2k}$, and for the last equality, $(\mathcal{B}+ax)^{n+k}$ is replaced with $B_{n+k}(ax)$, both by means of (2.3). □

In [9, p. 390, Eq. (42)], Chen established the following relation on the Hankel determinants for the *median Bernoulli numbers* K_k,

$$H_n(K_k) = \left(-\tfrac{1}{2}\right)^{n+1} H_n\big(B_{2k}\left(\tfrac{1}{2}\right)\big), \tag{3.2}$$

wherein

$$K_k := -\frac{1}{2}\sum_{\ell=0}^{k} \binom{k}{\ell} B_{k+\ell}.$$

Now, (3.1) gives a generalization of Chen's relation (3.2) as follows.

Corollary 3.2. *For any indeterminates a and b, define*

$$K_k^{a,b}(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} B_{k+\ell}(ax)(2b)^{k-\ell}.$$

Then

$$H_n(K_k^{a,b}(x)) = H_n(B_{2k}(ax+b)). \tag{3.3}$$

Proof. Note that (3.1) allows us to rewrite $K_k^{a,b}(x)$ as

$$K_k^{a,b}(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} B_{2\ell}(ax+b)(-b^2)^{k-\ell}.$$

Applying Theorem 1.1 with the specialization $a_k \mapsto B_{2k}(ax+b)$ and $x \mapsto -b^2$ yields the desired evaluation. $\square$

It is an easy observation that to show Theorem 1.2, especially (1.6), it suffices to confirm that the leading coefficient of $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ equals $H_n(B_k)$ by recalling (1.1). Now our objective is to prove this claim, thereby offering an assertion of Theorem 1.2.

Proof of Theorem 1.2. Letting $a \mapsto \frac{1}{2x}$ and $b \mapsto \frac{1}{2x}$ in (3.3), we have

$$H_n\left(B_{2k}\left(\frac{1+x^{-1}}{2}\right)\right) = H_n\left(\sum_{\ell=0}^{k}\binom{k}{\ell}B_{k+\ell}\big(\tfrac{1}{2}\big)\left(\frac{1}{x}\right)^{k-\ell}\right).$$

Note that for the two sequences

$$\left(B_{2k}\left(\frac{1+x^{-1}}{2}\right)\right)_{k\geq 0} \qquad \text{and} \qquad \left(\sum_{\ell=0}^{k}\binom{k}{\ell}B_{k+\ell}\big(\tfrac{1}{2}\big)\left(\frac{1}{x}\right)^{k-\ell}\right)_{k\geq 0},$$

which share the same Hankel determinants, if we simultaneously multiply the k-th term of them by x^k, then (2.2) asserts that the new Hankel determinants are still identical. Thus,

$$\begin{aligned} H_n\left(\sum_{\ell=0}^{k}\binom{k}{\ell}B_{k+\ell}\left(\frac{1}{2}\right)x^{\ell}\right) &= H_n\left(x^k B_{2k}\left(\frac{1+x^{-1}}{2}\right)\right) \\ &= x^{n(n+1)}H_n\left(B_{2k}\left(\frac{1+x^{-1}}{2}\right)\right). \end{aligned}$$

We then observe that $H_n\big(B_{2k}(\frac{1+x^{-1}}{2})\big)$ are *polynomials* in x^{-1}. Assuming that the degree of $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is $d > n(n+1)$, then in $x^{n(n+1)}H_n\big(B_{2k}(\frac{1+x^{-1}}{2})\big)$, there exists a non-vanishing *negative* power of x, that is, $x^{n(n+1)-d}$. However, the Hankel determinants $H_n\big(\sum_{j=0}^{k}\binom{k}{j}B_{k+j}(\frac{1}{2})x^j\big)$ are *polynomials* in x, and so are $x^{n(n+1)}H_n\big(B_{2k}(\frac{1+x^{-1}}{2})\big)$, which should *not* contain any negative power of x. We are led to a contradiction, and hence, the degree of $H_n\big(B_{2k}(\frac{1+x}{2})\big)$ is *at most* $n(n+1)$. Finally, we compute the coefficient at $x^{n(n+1)}$ as follows:

$$\begin{aligned} \big[x^{n(n+1)}\big]H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right) &= \big[x^0\big]x^{n(n+1)}H_n\left(B_{2k}\left(\frac{1+x^{-1}}{2}\right)\right) \\ &= \big[x^0\big]H_n\left(\sum_{\ell=0}^{k}\binom{k}{\ell}B_{k+\ell}\big(\tfrac{1}{2}\big)\,x^{\ell}\right) \\ &= H_n\left(\sum_{\ell=0}^{k}\binom{k}{\ell}B_{k+\ell}\big(\tfrac{1}{2}\big)\,x^{\ell}\bigg|_{x=0}\right). \end{aligned}$$

Thus,

$$\left[x^{n(n+1)}\right] H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right) = H_n\left(B_k\left(\tfrac{1}{2}\right)\right) \overset{(1.2)}{=} H_n(B_k).$$

In particular, this value is non-vanishing in view of (1.1), thereby implying that the degree of $H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$ is *exactly* $n(n+1)$. □

4. *q*-Binomial transforms

To understand the underlying motivation of our q-binomial transform (1.7), we recall the theory of umbral calculus and define an umbra $\mathcal{A}$ with evaluation $\mathcal{A}^k = a_k$ in order to rewrite the binomial transform (1.3) as

$$a_k(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{k-\ell} x^\ell = \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{A}^{k-\ell} x^\ell = \prod_{j=0}^{k-1} (\mathcal{A} + x),$$

wherein the last equality follows from the *binomial theorem*, which can be further generalized as the *q-binomial theorem* [4, p. 421, Eq. (17.2.35)]:

$$\prod_{k=0}^{n-1} (q^k \alpha + x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha^{n-k} x^k.$$

Thus, we similarly define an umbra $\boldsymbol{\alpha}$ with evaluation $\boldsymbol{\alpha}^k = \alpha_k$ for $(\alpha_k)_{k\geq 0}$. Then the q-binomial transform (1.7) is reformulated as

$$\alpha_k(x) = \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^\ell = \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \boldsymbol{\alpha}^{k-\ell} x^\ell = \prod_{j=0}^{k-1} (q^j \boldsymbol{\alpha} + x).$$

The above discussion explains the occurrence of the prefactor $q^{\binom{\ell}{2}}$ in the summand.

4.1. Leading coefficient. In this section, our main objective is to analyze the Hankel determinants $H_n(\alpha_k(x))$ and $H_n(\widetilde{\alpha}_k(x))$, especially the leading coefficient in them as stated in Theorems 1.3 and 1.4.

4.1.1. *The Hankel determinants $H_n(\alpha_k(x))$.* It is clear that our q-binomial transform (1.7) reduces to the binomial transform (1.3) by taking the $q \to 1$ limit. However, unlike the invariance relation (1.4) for the binomial case, there is no obvious pattern connecting $H_n(\alpha_k(x))$ and $H_n(\alpha_k)$. In Table 2, we have computed the polynomial expansion of $H_2(\alpha_k(x))$, which is of degree 6, and one may see the chaos among these coefficients.

In this subsection, we explore the behavior of the leading coefficient in $H_n(\alpha_k(x))$, and prove Theorem 1.3. To begin with, we need the Hankel determinant evaluation for the sequence $\left(q^{\binom{k}{2}}\right)_{k\geq 0}$.

Lemma 4.1. *For $n \geq 0$,*

$$H_n\left(q^{\binom{k}{2}}\right) = (-1)^{\binom{n+1}{2}} q^{3\binom{n+1}{3}} \prod_{j=1}^{n} (1-q^j)^{n+1-j}. \tag{4.1}$$

TABLE 2. Coefficients in $H_2(\alpha_k(x))$

x^6	$-q^3(1-q)^3(1+q)\alpha_0^3$
x^5	$-q^2(1-q)^3(1+q)(1+2q)\alpha_0^2\alpha_1$
x^4	$-q(1-q)^2(1+q)\big[(1+2q+q^2-q^3)\alpha_0\alpha_1^2-(1+2q^2)\alpha_0^2\alpha_2\big]$
x^3	$(1-q)^2(1+q)\big[(2+q+q^2+2q^3)\alpha_0\alpha_1\alpha_2-(1-q)\alpha_0^2\alpha_3-(1+2q+2q^2+q^3)\alpha_1^3\big]$
x^2	$(1-q)\big[(1+3q+q^2-q^3)\alpha_0\alpha_1\alpha_3+(1-q-q^2-q^3-q^4)\alpha_0\alpha_2^2-\alpha_0^2\alpha_4-(1+2q-2q^3-q^4)\alpha_1^2\alpha_2\big]$
x^1	$-(1-q)\big[\alpha_0\alpha_1\alpha_4-(1-q^2)\alpha_0\alpha_2\alpha_3+(1+2q)\alpha_1\alpha_2^2-(1+2q+q^2)\alpha_1^2\alpha_3\big]$
x^0	$\alpha_0\alpha_2\alpha_4-\alpha_0\alpha_3^2+2\alpha_1\alpha_2\alpha_3-\alpha_1^2\alpha_4-\alpha_2^3$

Proof. Note that $H_n\big(q^{\binom{k}{2}}\big)$ is the determinant of the Hankel matrix $\big(q^{\binom{i+j}{2}}\big)_{i,j=0}^n$. For its (i,j)-th entry, we may rewrite it as

$$q^{\binom{i+j}{2}} = q^{\binom{i}{2}}\cdot q^{\binom{j}{2}}\cdot q^{ij}.$$

Hence, we first factor out $q^{\binom{i}{2}}$ from the i-th row for each i, and then $q^{\binom{j}{2}}$ from the j-th column for each j. Then

$$H_n\big(q^{\binom{k}{2}}\big) = q^{\sum_{i=0}^n \binom{i}{2}}\cdot q^{\sum_{j=0}^n \binom{j}{2}}\cdot \det_{0\le i,j\le n}(q^{ij}) = q^{2\binom{n+1}{3}}\cdot \prod_{0\le i<j\le n}(q^j-q^i),$$

where we have applied the evaluation for the Vandermonde determinant [6, p. 366, Theorem 9.67] in the last equality. Finally, we note that

$$\begin{aligned}\prod_{0\le i<j\le n}(q^j-q^i) &= (-1)^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(q^i-q^j)\\ &= (-1)^{\binom{n+1}{2}}\left(\prod_{i=0}^n (q^i)^{n-i}\right)\left(\prod_{0\le i<j\le n}(1-q^{j-i})\right)\\ (\text{with } \ell := j-i) &= (-1)^{\binom{n+1}{2}}q^{\binom{n+1}{3}}\prod_{\ell=1}^n (1-q^\ell)^{n+1-\ell},\end{aligned}$$

thereby confirming the required identity. □

We are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We reflect the polynomials $\alpha_k(x)$ by defining for each $k\ge 0$,

$$\alpha'_k(x) := x^k\alpha_k(x^{-1}) = \sum_{\ell=0}^k q^{\binom{\ell}{2}}\begin{bmatrix}k\\ \ell\end{bmatrix}_q \alpha_{k-\ell}x^{k-\ell}. \tag{4.2}$$

In particular, by (2.2),

$$H_n(\alpha'_k(x)) = x^{n(n+1)}H_n(\alpha_k(x^{-1})). \tag{4.3}$$

Note that $\alpha_k(x)$ and $\alpha'_k(x)$ are *polynomials* in x, and so are $H_n(\alpha_k(x))$ and $H_n(\alpha'_k(x))$. Hence, the degree of $H_n(\alpha_k(x))$ is *at most* $n(n+1)$; otherwise, there

TABLE 3. Coefficients in $H_2(\widetilde{\alpha}_k(x))$

x^3	$-q^2(1-q)^3(1+q)\alpha_0\alpha_1\alpha_2$
x^2	$q^2(1-q)^2(1+q)\big[(q+q^2)\alpha_0\alpha_1\alpha_3 - \alpha_0\alpha_2^2 - \alpha_1^2\alpha_2\big]$
x^1	$-q^2(1-q)\big[q^4\alpha_0\alpha_1\alpha_4 - (q^2-q^4)\alpha_0\alpha_2\alpha_3 + (1+2q)\alpha_1\alpha_2^2 - (q+2q^2+q^3)\alpha_1^2\alpha_3\big]$
x^0	$q^3\big[q^4\alpha_0\alpha_2\alpha_4 - q^3\alpha_0\alpha_3^2 + 2q\alpha_1\alpha_2\alpha_3 - q^3\alpha_1^2\alpha_4 - \alpha_2^3\big]$

exists a *negative* power of x in $x^{n(n+1)}H_n(\alpha_k(x^{-1})) = H_n(\alpha'_k(x))$, which is invalid. Finally, (4.3) implies that $\big[x^{n(n+1)}\big]H_n(\alpha_k(x))$ equals

$$\big[x^0\big]H_n(\alpha'_k(x)) = H_n(\alpha'_k(0)) = H_n\big(\alpha_0 q^{\binom{k}{2}}\big) \overset{(2.1)}{=} \alpha_0^{n+1}H_n\big(q^{\binom{k}{2}}\big).$$

Invoking (4.1) gives (1.8). Also, this value is non-vanishing, and hence, the degree of $H_n(\alpha_k(x))$ is *exactly* $n(n+1)$. □

4.1.2. *The Hankel determinants $H_n(\widetilde{\alpha}_k(x))$.* Now, we move on to the second q-binomial transform (1.9). It can be similarly seen from Table 3 that the polynomial expansions of the Hankel determinants $H_n(\widetilde{\alpha}_k(x))$ behave chaotically.

As in (4.2), to prove Theorem 1.4, we also need to reflect the polynomials $\widetilde{\alpha}_k(x)$ for each $k \geq 0$ as

$$\widetilde{\alpha}'_k(x) := x^k\widetilde{\alpha}_k(x^{-1}) = \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_\ell x^\ell. \tag{4.4}$$

In light of (2.2),

$$H_n(\widetilde{\alpha}'_k(x)) = x^{n(n+1)}H_n(\widetilde{\alpha}_k(x^{-1})). \tag{4.5}$$

Now we require some extra effort to proceed with our proof. To begin with, we define a family of modified difference operators Δ_i for arbitrary sequences $(s_k)_{k\geq 0}$ as follows:

$$\Delta_i(s_k) := \begin{cases} s_k, & \text{if } i = 0, \\ s_k - q^{i-1}s_{k-1}, & \text{if } i \geq 1. \end{cases}$$

Lemma 4.2. *For every $m \geq 0$,*

$$\Delta_m\Delta_{m-1}\cdots\Delta_0(\widetilde{\alpha}'_k(x)) = \sum_{\ell=m}^{k} q^{(k-\ell)m+\binom{\ell}{2}} \begin{bmatrix} k-m \\ \ell-m \end{bmatrix}_q \alpha_\ell x^\ell. \tag{4.6}$$

Proof. We prove this lemma by induction on m. For $m = 0$, the relation (4.6) simply reduces to (4.4). Suppose that (4.6) holds for a certain integer $m \geq 0$. Then

$$\begin{aligned} &\Delta_{m+1}\big(\Delta_m\cdots\Delta_1(\widetilde{\alpha}'_k(x))\big) \\ &\quad = q^{\binom{k}{2}}\alpha_k x^k + \sum_{\ell=m}^{k-1} q^{(k-\ell)m+\binom{\ell}{2}} \begin{bmatrix} k-m \\ \ell-m \end{bmatrix}_q \alpha_\ell x^\ell \\ &\quad\quad - q^m \sum_{\ell=m}^{k-1} q^{(k-1-\ell)m+\binom{\ell}{2}} \begin{bmatrix} k-1-m \\ \ell-m \end{bmatrix}_q \alpha_\ell x^\ell \end{aligned}$$

$$= q^{\binom{k}{2}}\alpha_k x^k + \sum_{\ell=m}^{k-1} q^{(k-\ell)m+\binom{\ell}{2}} \left(\begin{bmatrix} k-m \\ \ell-m \end{bmatrix}_q - \begin{bmatrix} k-1-m \\ \ell-m \end{bmatrix}_q \right) \alpha_\ell x^\ell$$
$$= q^{\binom{k}{2}}\alpha_k x^k + \sum_{\ell=m+1}^{k-1} q^{(k-\ell)(m+1)+\binom{\ell}{2}} \begin{bmatrix} k-1-m \\ \ell-1-m \end{bmatrix}_q \alpha_\ell x^\ell,$$

where we have applied the following Pascal-type relation [3, p. 35, Eq. (3.3.3)] for the last equality:

$$\begin{bmatrix} N \\ M \end{bmatrix}_q - \begin{bmatrix} N-1 \\ M \end{bmatrix}_q = q^{N-M} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}_q.$$

Therefore, we have shown the $m+1$ case, and hence the relation (4.6) in general. □

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We first show that the degree of $H_n(\widetilde{\alpha}_k(x))$ is *at most* $\frac{n(n+1)}{2}$. Note that in view of (4.5), it is sufficient to prove that the degree of $H_n(\widetilde{\alpha}'_k(x))$ is *at least* $\frac{n(n+1)}{2}$. To see this, we apply n steps of row transformations to the determinant. Specifically speaking, for each $k = 1, \ldots, n$, in the k-th step we apply the row transformation $r_i - q^{k-1} r_{i-1}$ to the i-th row for each $i = n, n-1, \ldots, k$. Here, we should recall that our Hankel matrix is indexed from the *zeroth* row. Now, the determinant becomes

$$H_n(\widetilde{\alpha}'_k(x)) = \det_{0\le i,j\le n} \big(\Delta_i \Delta_{i-1} \cdots \Delta_0(\widetilde{\alpha}'_{i+j}(x))\big),$$

whose (i,j)-entry can be further evaluated by (4.6),

$$\Delta_i \Delta_{i-1} \cdots \Delta_0(\widetilde{\alpha}'_{i+j}(x)) = \sum_{\ell=i}^{i+j} q^{(i+j-\ell)i+\binom{\ell}{2}} \begin{bmatrix} j \\ \ell-i \end{bmatrix}_q \alpha_\ell x^\ell$$
$$= q^{ij+\binom{i}{2}}\alpha_i x^i + x^{i+1} R_{i,j}(x),$$

where each $R_{i,j}(x)$ is a *polynomial* in x whose explicit form is not needed in our arguments. It turns out that entries in the i-th row have a common factor x^i. Extracting the factor x^i from the i-th row for each $i = 0, 1, \ldots, n$, we deduce that

$$H_n(\widetilde{\alpha}'_k(x)) = x^{\frac{n(n+1)}{2}} \det_{0\le i,j\le n} \big(q^{ij+\binom{i}{2}}\alpha_i + x R_{i,j}(x)\big), \tag{4.7}$$

which implies that $\deg H_n(\widetilde{\alpha}'_k(x)) \ge \frac{n(n+1)}{2}$, as requested.

Finally, to show that the degree of $H_n(\widetilde{\alpha}_k(x))$ is *exactly* $\frac{n(n+1)}{2}$, we only need to confirm that the corresponding coefficient is non-vanishing, as given in (1.10):

$$\big[x^{\frac{n(n+1)}{2}}\big] H_n(\widetilde{\alpha}_k(x)) \overset{(4.5)}{=} \big[x^{\frac{n(n+1)}{2}}\big] H_n(\widetilde{\alpha}'_k(x)) \overset{(4.7)}{=} \det_{0\le i,j\le n} \big(q^{ij+\binom{i}{2}}\alpha_i\big),$$

which equals

$$\alpha_0\alpha_1 \cdots \alpha_n q^{\binom{n+1}{3}} \det_{0\le i,j\le n}\big(q^{ij}\big) = \alpha_0\alpha_1\cdots\alpha_n (-1)^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n} (1-q^j)^{n+1-j},$$

wherein the Vandermonde determinant has been evaluated in the proof of Lemma 4.1. □

4.2. An example admitting explicit Hankel determinant expressions. For q-binomial transforms (1.7) and (1.9), it is in general out of reach to establish closed expressions for the related Hankel determinants. However, some specific sequences $(\alpha_k)_{k\geq 0}$ still admit a neat Hankel determinant evaluation. In this section, we provide such an instance.

Theorem 4.3. *Choose the sequence $\alpha_k^{u,v} := q^{-\binom{k}{2}}(u;q)_k\, v^k$ with u and v indeterminates in* (1.9)*, and let*

$$\widetilde{\alpha}_k^{u,v}(x) := \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_\ell\, v^\ell x^{k-\ell} = \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_{k-\ell}\, v^{k-\ell} x^\ell.$$

Then

$$H_n(\widetilde{\alpha}_k^{u,v}(x)) = v^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{j=1}^{n} (uvq^{j-1} - x)^{n+1-j}(u,q;q)_{n+1-j}. \tag{4.8}$$

Proof. Recall that the *second family of Al-Salam–Carlitz polynomials* [2, p. 53, Eq. (4.8)]:

$$V_k^{(a)}(x;q) := (-1)^k q^{-\binom{k}{2}} \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q a^\ell (x;q)_{k-\ell}$$

admits the following generating function identity [16, p. 531, Eq. (3.22)]:

$$\sum_{k=0}^{\infty} V_k^{(a)}(x;q) \frac{q^{\binom{k}{2}}}{(q;q)_k}(-t)^k = \frac{(xt;q)_\infty}{(t,at;q)_\infty}.$$

Now letting $t \mapsto vw$, $a \mapsto x/v$, and $x \mapsto u$, we have

$$\frac{(uvw;q)_\infty}{(vw,xw;q)_\infty} = \sum_{k=0}^{\infty} V_k^{(x/v)}(u;q) \frac{q^{\binom{k}{2}}}{(q;q)_k}(-vw)^k,$$

wherein it is further notable that

$$V_k^{(x/v)}(u;q) q^{\binom{k}{2}} (-v)^k = \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \frac{x^\ell}{v^\ell}(u;q)_{k-\ell} v^k = \widetilde{\alpha}_k^{u,v}(x).$$

Thus, we have the generating function identity

$$\sum_{k=0}^{\infty} \frac{\widetilde{\alpha}_k^{u,v}(x)}{(q;q)_k} w^k = \frac{(uvw;q)_\infty}{(vw,xw;q)_\infty}. \tag{4.9}$$

In the meantime, note that the generating function for the Al-Salam–Chiahara polynomials (2.9) can be found in [16, p. 531, Eq. (3.18)]:

$$\sum_{k=0}^{\infty} \frac{(t_1t_2;q)_k\, p_k(\theta;t_1,t_2)}{(q;q)_k t_1^k} t^k = \frac{(t_1t, t_2t;q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}.$$

In light of Lemma 2.6, we know that $p_k(\theta, t_1, t_2) = \mu_k(a,b,c)$ by taking $a = t_1e^{i\theta}/q$, $b = t_2e^{-i\theta}/q$, and $c = t_1e^{-i\theta}/q$. We further let $e^{i\theta} \mapsto \sqrt{v/x}$, $t_1 \mapsto u\sqrt{v/x}$, and $t_2 \mapsto 0$, which leads to

$$(a,b,c) = \left(\frac{uv}{xq}, 0, \frac{u}{q}\right). \tag{4.10}$$

For this choice of a, b, and c,

$$\sum_{k=0}^{\infty} \frac{\mu_k(a,b,c)}{(q;q)_k} \left(\frac{t}{u\sqrt{v/x}}\right)^k = \frac{(ut\sqrt{v/x};q)_\infty}{(t\sqrt{v/x}, t\sqrt{x/v};q)_\infty}.$$

Now, making the substitution $w = t/\sqrt{vx}$ in the above yields

$$\sum_{k=0}^{\infty} \frac{\mu_k(a,b,c)}{(q;q)_k} \left(\frac{wx}{u}\right)^k = \frac{(uvw;q)_\infty}{(vw, xw;q)_\infty} \overset{(4.9)}{=} \sum_{k=0}^{\infty} \frac{\widetilde{\alpha}_k^{u,v}(x)}{(q;q)_k} w^k.$$

This shows that for every $k \geq 0$,

$$\widetilde{\alpha}_k^{u,v}(x) = \left(\frac{x}{u}\right)^k \mu_k(a,b,c). \tag{4.11}$$

Now it remains to evaluate the Hankel determinants $H_n(\mu_k(a,b,c))$. In view of Remark 2.2, we combine (2.4), (2.5), and (2.8) to get

$$H_n(\mu_k(a,b,c)) = \prod_{j=1}^{n} (A_{j-1}C_j)^{n+1-j},$$

wherein the substitution of variables in (4.10) should be executed on the sequences A_j and C_j in Lemma 2.4:

$$\begin{aligned}
A_j =& \frac{(1-aq^{j+1})(1-cq^{j+1})(1-abq^{j+1})}{(1-abq^{2j+1})(1-abq^{2j+2})} = \left(1 - \frac{uvq^j}{x}\right)(1-uq^j),\\
C_j =& -\frac{(1-q^j)(1-bq^j)(1-abc^{-1}q^j)}{(1-abq^{2j})(1-abq^{2j+1})} acq^{j+1} = -\frac{u^2vq^{j-1}(1-q^j)}{x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_n(\mu_k(a,b,c)) &= \prod_{j=1}^{n} \left(\frac{u^2v}{x} \cdot q^{j-1} \cdot \left(\frac{uvq^{j-1}}{x} - 1\right) \cdot (1-q^j)(1-uq^{j-1})\right)^{n+1-j}\\
&= \left(\frac{u^2v}{x^2}\right)^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{j=1}^{n} (uvq^{j-1} - x)^{n+1-j} (u,q;q)_{n+1-j}.
\end{aligned}$$

Finally, we apply (2.2) to (4.11), and derive that

$$H_n(\widetilde{\alpha}_k^{u,v}(x)) = \left(\frac{x}{u}\right)^{n(n+1)} H_n(\mu_k(a,b,c)),$$

which yields (4.8) by recalling the above evaluation of $H_n(\mu_k(a,b,c))$. $\square$

Remark 4.1. From [14, p. 21, Eq. (1.13)], we know that for the *q-generalized Catalan numbers*:

$$\lambda_k := \frac{(uq;q)_k}{(uvq^2;q)_k},$$

their Hankel determinants are given by

$$H_n(\lambda_k) = u^{\binom{n+1}{2}} q^{\frac{n(n+1)(2n+1)}{6}} \prod_{j=1}^{n+1} \frac{(q,uq,vq;q)_{n+1-j}}{(uvq^{n+2-j};q)_{n+1-j}(uvq^2;q)_{2(n+1-j)}}.$$

Letting $u \mapsto u/q$ and $v \mapsto 0$ gives

$$H_n((u;q)_k) = u^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n+1} (q,u;q)_{n+1-j}.$$

Therefore, by (2.2),

$$H_n\big(q^{\binom{k}{2}} \alpha_k^{u,v}\big) = H_n((u;q)_k v^k) = u^{\binom{n+1}{2}} v^{n(n+1)} q^{2\binom{n+1}{3}} \prod_{\ell=1}^{n+1} (u,q;q)_{n+1-\ell}.$$

Hence, (4.8) indicates that

$$\frac{H_n(\widetilde{\alpha}_k^{u,v}(x))}{H_n\big(q^{\binom{k}{2}} \alpha_k^{u,v}\big)} = \frac{\prod_{j=0}^{n} (uvq^j - x)^{n-j}}{(uv)^{\binom{n+1}{2}} q^{\binom{n+1}{3}}}.$$

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