# MULTIPLE ROGERS–RAMANUJAN TYPE IDENTITIES FOR TORUS LINKS

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Abstract. In this paper, we establish simple $k$-fold summation expressions for the Quot and motivic Cohen–Lenstra zeta functions associated with the $(2,2k)$ torus links. Such expressions lead us to some multiple Rogers–Ramanujan type identities and their finitizations, thereby confirming a conjecture of Huang and Jiang. Several other properties of the two zeta functions will be examined as well.

## 1. Introduction

The main objective of this paper revolves around some conjectural Rogers–Ramanujan type identities arising from algebraic geometry. To embark on our journey, we let $\mathbb{K}$ be a fixed field. Now given a certain $\mathbb{K}$-curve at a $\mathbb{K}$-point, we let $R$ be the complete local ring of its germ and $\widetilde{R}$ the normalization of $R$, and assume that $E$ is a finitely generated $R$-module; this setting localizes reduced varieties $X$ over $\mathbb{K}$ and coherent sheaves $\mathcal{E}$ on $X$. We further denote by $\mathrm{Quot}_{E,n}$ the Quot scheme parametrizing $R$-submodules of $E$ of $\mathbb{K}$-codimension $n$. What lies at the heart of our work is the *Quot zeta function*:

$$Z_E^R(t) = Z_E(t) := \sum_{n\geq 0}[\mathrm{Quot}_{E,n}]t^n,$$

where the motive $[V]$ denotes the class of $V$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{K}})$ of $\mathbb{K}$-varieties for $V$ a $\mathbb{K}$-scheme.

Investigations on $Z_R^R(t)$ and $Z_{\widetilde{R}}^R(t)$ have been widely performed in the past, and among those the beautiful Hilb-vs-Quot conjecture [12] predicts the connection between $Z_R^R(t)$ and $Z_{\widetilde{R}}^R(t)$. What is then highlighted in a recent work of Huang and Jiang [11] is a high-rank generalization in the sense that $E$ is taken to be a *torsion-free module* of rank $N$ over $R$, meaning that $E$ is

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injective to $E \oplus_R \mathrm{Frac}(R) \simeq \mathrm{Frac}(R)^N$ with $\mathrm{Frac}(R)$ the total fraction ring of $R$.

Notably, the *rationality theorem* of Huang and Jiang [11, Theorem 1.3] asserts that under the assumption that $\widetilde{R} \simeq \mathbb{K}[[T]]^s$ with $s$ the branching number of $R$, we have that $Z_E^R(t)/Z_{\widetilde{R}^{\oplus N}}^{\widetilde{R}}(t)$ is a polynomial in $t$ for any torsion-free module $E$ of rank $N$ over $R$. Here, it is known [6] that

$$Z_{\widetilde{R}^{\oplus N}}^{\widetilde{R}}(t) = \prod_{j=0}^{N-1} \frac{1}{(1-t\mathbb{L}^j)^s}, \tag{1.1}$$

where $\mathbb{L} := [\mathbb{A}^1]$ is the *Lefschetz motive*. This rationality theorem leads one to focus on the *numerator part* of $Z_E^R(t)$:

$$NZ_E^R(t) = NZ_E(t) := \frac{Z_E^R(t)}{Z_{\widetilde{R}^{\oplus N}}^{\widetilde{R}}(t)}. \tag{1.2}$$

In addition, a generalization of the important Cohen–Lenstra zeta function [7] was recently introduced by Huang [10] to the motivic version. Briefly speaking, by denoting $\mathrm{Coh}_n(R)$ the stack of $R$-modules of $\mathbb{K}$-dimension $n$, the *motivic Cohen–Lenstra zeta function* is defined by

$$\widehat{Z}_R(t) := \sum_{n\ge 0} [\mathrm{Coh}_n(R)] t^n.$$

A remarkable result in [11, Theorem 1.12] connects the motivic Cohen–Lenstra zeta functions and the limiting case of the Quot zeta functions. To be specific, if $R$ is a complete local $\mathbb{K}$-algebra of finite type with residue field $\mathbb{K}$, then

$$\widehat{Z}_R(t) = \lim_{N\to\infty} Z_{R^{\oplus N}}(t\mathbb{L}^{-N}). \tag{1.3}$$

Analogous to (1.2), we may also define the *numerator part*:

$$\widehat{NZ}_R(t) := \frac{\widehat{Z}_R(t)}{\widehat{Z}_{\widetilde{R}}(t)}, \tag{1.4}$$

while we note from (1.1) that

$$\widehat{Z}_{\widetilde{R}}(t) = \prod_{j\ge 0} \frac{1}{(1-t\mathbb{L}^{-j-1})^s}. \tag{1.5}$$

The above objects have profound applications to matrix Diophantine equations when the field $\mathbb{K}$ is finite, namely, $\mathbb{K} \simeq \mathbb{F}_q$ for $q$ a prime power. As shown in [10, p. 40, Proposition 4.3], for $R = \mathbb{F}_q[x,y]/f(x,y)$ where $f$ is a given polynomial over $\mathbb{F}_q$,

$$\widehat{Z}_R(t) = \sum_{n\geq 0} \frac{\operatorname{card}\mathscr{M}_n}{\operatorname{card}\mathrm{GL}_n(\mathbb{F}_q)} t^n,$$

where $\mathscr{M}_n$ is the following set of matrix pairs over $\mathbb{F}_q$:

$$\mathscr{M}_n := \big\{(A,B) \in \mathrm{Mat}_n(\mathbb{F}_q)^2 : AB = BA \text{ and } f(A,B) = 0\big\}.$$

In view of this generating series for the enumeration of *commuting* matrix pairs $(A,B)$ over $\mathbb{F}_q$ satisfying the additional restriction that $f(A,B) = 0$, it becomes extremely meaningful to chase nice expressions for motivic Cohen–Lenstra zeta functions.

In [11], planar singularities associated with the $(2,n)$ torus knots and links are particularly considered. That is to say, we define $R^{(2,2k+1)}$ to be the germ of the variety $y^2 = x^{2k+1}$ and $R^{(2,2k)}$ the germ of $y(y-x^k) = 0$; for the latter case, if $\mathbb{K}$ is not of characteristic two, then $R^{(2,2k)}$ also admits the variety $y^2 = x^{2k}$. It is known [11, §8.2] that the branching number of $R^{(2,2k+1)}$ is 1, while for $R^{(2,2k)}$, it is 2.

One important result in [11] is the following formula for the motivic Cohen–Lenstra zeta function $\widehat{NZ}_{R^{(2,2k+1)}}(t)$ [11, Theorem 1.13]:

$$\widehat{NZ}_{R^{(2,2k+1)}}(t) = \sum_{n_1,\ldots,n_k\geq 0} \frac{t^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{\prod_{i=1}^k \prod_{j=1}^{n_i-n_{i-1}}(1-\mathbb{L}^{-j})}, \tag{1.6}$$

where we put $n_0 := 0$. We may further specialize at $t = \pm 1$ [11, eq. (1.22)]:

$$\widehat{NZ}_{R^{(2,2k+1)}}(\pm 1) = \prod_{j\geq 0} \frac{(1-\mathbb{L}^{-(2k+3)j-(k+1)})(1-\mathbb{L}^{-(2k+3)j-(k+2)})(1-\mathbb{L}^{-(2k+3)j-(2k+3)})}{1-\mathbb{L}^{-j-1}}. \tag{1.7}$$

On the other hand, the expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ shown in [11] is unfortunately not satisfactory, as will be seen in (1.10). However, when $t = 1$, the following beautiful equality is given in [11, Theorem 1.16].

**Theorem 1.1** (Huang–Jiang)**.** *For any positive integer $k$,*

$$\widehat{NZ}_{R^{(2,2k)}}(1) = 1. \tag{1.8}$$

Meanwhile, Huang and Jiang [11, Conjecture 1.17] also proposed a neat conjectural evaluation at $t = -1$.

**Conjecture 1.1** (Huang–Jiang)**.** *For any positive integer $k$,*

$$\widehat{NZ}_{R^{(2,2k)}}(-1)=\prod_{j\geq 1}\frac{(1-\mathbb{L}^{-2j})(1-\mathbb{L}^{-(k+1)j})^2}{(1-\mathbb{L}^{-j})^2(1-\mathbb{L}^{-(2k+2)j})}. \tag{1.9}$$

A glimpse at the sum in (1.6) and the product in (1.7) readily reminds one of identities of Rogers–Ramanujan type. Before moving on to this topic, we adopt the conventional *$q$-Pochhammer symbols* for $n\in\mathbb{N}\cup\{\infty\}$:

$$(A;q)_n:=\prod_{j=0}^{n-1}(1-Aq^j),$$
$$(A_1,A_2,\ldots,A_r;q)_n:=(A_1;q)_n(A_2;q)_n\cdots(A_r;q)_n,$$

and the *$q$-binomial coefficients*:

$$\begin{bmatrix}N\\M\end{bmatrix}_q:=\begin{cases}\dfrac{(q;q)_N}{(q;q)_M(q;q)_{N-M}}, & \text{if } 0\leq M\leq N,\\ 0, & \text{otherwise.}\end{cases}$$

The famous *Rogers–Ramanujan identities* refer to the following two $q$-series equalities:

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty},$$
$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

They were first established by Rogers [17] in 1894, and had been unfortunately overlooked until Ramanujan's rediscovery [15] two decades later. Around that time, a new proof was provided jointly by Ramanujan and Rogers [16] and another two fundamentally different proofs were offered by Schur [20]. Since then, we usually refer to $q$-series relations of the form "sum side = product side" as *Roger–Ramanujan type identities*. Now the equality between (1.6) and (1.7) is

$$\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}}=\frac{(q^{k+1},q^{k+2},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}.$$

This identity was first discovered by Andrews [2] in a more general form, which also serves as an analytic counterpart of a partition-theoretic relation due to Gordon [9].

To connect the motivic Cohen–Lenstra zeta functions for the $(2,2k)$ torus links with identities of Rogers–Ramanujan type, it is necessary to find a sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$. Fortunately, this can be achieved by means of the *Hall–Littlewood polynomials*:

$$g_{\boldsymbol{s}}^{\boldsymbol{r}}(q) := q^{\sum_{i=1}^k s_i(r_i-s_i)} \prod_{i=1}^k \begin{bmatrix} r_i - s_{i-1} \\ r_i - s_i \end{bmatrix}_{q^{-1}},$$

where $\boldsymbol{r} = (r_1,\ldots,r_k)$ and $\boldsymbol{s} = (s_1,\ldots,s_k)$ are weakly *increasing* sequences of nonnegative integers, while we assume that $s_0 := 0$. It is notable that in [11], the sequences $\boldsymbol{r}$ and $\boldsymbol{s}$ are weakly *decreasing* so that the top entries of the $q$-binomial coefficients are $r_i - s_{i+1}$, but for our convenience in the present work, we flip them over.

Now [11, Theorem 1.14] asserts that

$$\begin{aligned}\widehat{NZ}_{R^{(2,2k)}}(t) = (t\mathbb{L}^{-1};\mathbb{L}^{-1})_\infty^2 \sum_{\boldsymbol{r},\boldsymbol{s}} &\frac{t^{\sum_{i=1}^k (2r_i - s_i)} \mathbb{L}^{-\sum_{i=1}^k r_i^2} g_{\boldsymbol{s}}^{\boldsymbol{r}}(\mathbb{L})}{(t\mathbb{L}^{-1};\mathbb{L}^{-1})_{r_1}^2 (\mathbb{L}^{-1};\mathbb{L}^{-1})_{s_1}} \\ &\times \frac{1}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{r_k - r_{k-1}} \cdots (\mathbb{L}^{-1};\mathbb{L}^{-1})_{r_2 - r_1}}. \end{aligned} \tag{1.10}$$

To facilitate our analysis, we define

$$\begin{aligned}\mathcal{Z}_k(t,q) := \sum_{\substack{r_k \ge \cdots \ge r_1 \ge 0 \\ s_k \ge \cdots \ge s_1 \ge 0}} &\frac{t^{\sum_{i=1}^k (2r_i - s_i)} q^{\sum_{i=1}^k (r_i^2 - r_i s_i + s_i^2)}}{(q;q)_{r_k - r_{k-1}} \cdots (q;q)_{r_2 - r_1} (tq;q)_{r_1}^2 (q;q)_{s_1}} \\ &\times \begin{bmatrix} r_k - s_{k-1} \\ r_k - s_k \end{bmatrix}_q \begin{bmatrix} r_{k-1} - s_{k-2} \\ r_{k-1} - s_{k-1} \end{bmatrix}_q \cdots \begin{bmatrix} r_2 - s_1 \\ r_2 - s_2 \end{bmatrix}_q \begin{bmatrix} r_1 \\ r_1 - s_1 \end{bmatrix}_q. \end{aligned} \tag{1.11}$$

It is then clear that

$$\widehat{NZ}_{R^{(2,2k)}}(t)|_{\mathbb{L}\mapsto q^{-1}} = (tq;q)_\infty^2 \mathcal{Z}_k(t,q). \tag{1.12}$$

Note that the proof of (1.8) in [11] relies heavily on hardcore techniques in algebraic geometry. Recently, in a private communication with Yifeng Huang, one of the authors of [11], a purely $q$-theoretic proof of (1.8) was requested. This is the starting point of our work.

**Theorem 1.2.** *For any positive integer $k$,*

$$\mathcal{Z}_k(1,q) = \frac{1}{(q;q)_\infty^2}. \tag{1.13}$$

*Consequently, (1.8) is true.*

We will show that this relation is indeed a consequence of the following multiple Rogers–Ramanujan type identity.

**Theorem 1.3.** *For any nonnegative integers $d_1,\ldots,d_k$,*

$$\frac{1}{(aq;q)_\infty}=\sum_{n_1,\ldots,n_k\geq 0}\frac{a^{\sum_{i=1}^k n_i}q^{\sum_{i=1}^k n_i^2+\sum_{i=1}^k (d_1+\cdots+d_i)n_i}}{(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}}\times\frac{(q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{n_k}\cdots(q;q)_{n_1}(aq;q)_{n_1+d_1}}. \tag{1.14}$$

Notably, letting $d_1=\cdots=d_k=0$, the above becomes

$$\frac{1}{(aq;q)_\infty}=\sum_{n_1,\ldots,n_k\geq 0}\frac{a^{\sum_{i=1}^k n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(aq;q)_{n_1}}. \tag{1.15}$$

This is an instance of [3, p. 30, eq. (3.44)] by choosing the following *Bailey pair* relative to $(a,q)$ [3, p. 25, eq. (3.27)]:

$$\alpha_n=\begin{cases}1, & n=0,\\ 0, & n\geq 1,\end{cases}\qquad\text{and}\qquad \beta_n=\frac{1}{(q;q)_n(aq;q)_n}.$$

One may wonder if the same method works for the evaluation of $\widehat{NZ}_{R^{(2,2k)}}(t)$ at $t=-1$ so as to attack Conjecture 1.1. Sadly, this is not the case. Now a natural idea is to figure out a simpler expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$, by reducing the number of summation folds from $2k$ to $k$, thereby yielding an analog to the case of $(2,2k+1)$ torus knots in (1.6).

**Theorem 1.4.** *For any positive integer $k$,*

$$\mathscr{Z}_k(t,q)=\frac{1}{(tq;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(tq;q)_{n_1}}. \tag{1.16}$$

*Consequently,*

$$\widehat{NZ}_{R^{(2,2k)}}(t)=(t\mathbb{L}^{-1};\mathbb{L}^{-1})_\infty\sum_{n_1,\ldots,n_k\geq 0}\frac{t^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{(t\mathbb{L}^{-1};\mathbb{L}^{-1})_{n_1}}\times\frac{1}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{n_k-n_{k-1}}\cdots(\mathbb{L}^{-1};\mathbb{L}^{-1})_{n_2-n_1}(\mathbb{L}^{-1};\mathbb{L}^{-1})_{n_1}}. \tag{1.17}$$

Now the evaluation at $t=-1$ becomes immediate.

**Theorem 1.5.** *For any positive integer $k$,*

$$\mathcal{Z}_k(-1,q) = \frac{(q^{k+1};q^{k+1})_\infty^2}{(q^2;q^2)_\infty (q^{2k+2};q^{2k+2})_\infty}. \tag{1.18}$$

*Consequently, (1.9) in Huang–Jiang's Conjecture 1.1 is true.*

It is remarkable that as a middle step in our proof of Theorems 1.4 and 1.5, we observe that (1.18) is closely tied with a more surprising identity.

**Theorem 1.6.** *For any positive integer $k$,*

$$\begin{aligned}\frac{(q^2;q^2)_\infty (q^{k+1};q^{k+1})_\infty^2}{(q;q)_\infty^3 (q^{2k+2};q^{2k+2})_\infty} &= \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{(-1)^{\sum_{i=1}^k m_i} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (-1;q)_{n_1}^2}{(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}\\ &\quad\times \begin{bmatrix} m_k\\ m_{k-1}\end{bmatrix}_q \begin{bmatrix} m_{k-1}\\ m_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} m_2\\ m_1\end{bmatrix}_q \begin{bmatrix} n_1\\ n_2\end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2}\\ n_{k-1}\end{bmatrix}_q \begin{bmatrix} n_{k-1}\\ n_k\end{bmatrix}_q. \end{aligned} \tag{1.19}$$

It is an easy observation that the format of (1.19) resembles $\mathrm{A}_2$ Rogers–Ramanujan type identities introduced by Andrews, Schilling and Warnaar such as [5, p. 694, eq. (5.22)]:

$$\begin{aligned}&\frac{(q^{k+1},q^{k+1},q^{k+2},q^{2k+2},q^{2k+3},q^{2k+3},q^{3k+4},q^{3k+4};q^{3k+4})_\infty}{(q;q)_\infty^3}\\ &\qquad= \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{q^{\sum_{i=1}^k (m_i^2-m_in_i+n_i^2)}(1-q^{m_1+n_1+1})}{(q;q)_{m_k}(q;q)_{n_k}(q;q)_{m_1+n_1+1}}\\ &\qquad\quad\times \begin{bmatrix} m_k\\ m_{k-1}\end{bmatrix}_q \begin{bmatrix} m_{k-1}\\ m_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} m_2\\ m_1\end{bmatrix}_q \begin{bmatrix} n_k\\ n_{k-1}\end{bmatrix}_q \begin{bmatrix} n_{k-1}\\ n_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} n_2\\ n_1\end{bmatrix}_q. \end{aligned}$$

However, the fact that the summation indices $m_i$ and $n_i$ in (1.19) are in reverse order makes it fundamentally different from the above $\mathrm{A}_2$ Rogers–Ramanujan type identity. This is also to some extent a hint for (1.19) lacking the usual "triple the number of folds" phenomenon (coming from the $\mathrm{A}_2$ Macdonald identity [13]) for the modulus.

Next, let us recall from (1.3) that the motivic Cohen–Lenstra zeta functions are the limiting case of Quot zeta functions. It turns out that for the $(2,2k)$ torus links, the expression of the Quot zeta function $Z_{R^{(2,2k)}\oplus N}(t)$, or equivalently its numerator part $NZ_{R^{(2,2k)}\oplus N}(t)$, is also known, as given in

[11, Theorem 1.10]:

$$
\begin{aligned}
&NZ_{R^{(2,2k)\oplus N}}(t)\\
&\quad = t^{2N}\mathbb{L}^{N^2-N}(\mathbb{L}^{-1};\mathbb{L}^{-1})_N(t^{-1};\mathbb{L}^{-1})_N^2\\
&\qquad \times \sum_{\substack{r_k\ge\cdots\ge r_1\ge 0\\ s_k\ge\cdots\ge s_1\ge 0}} \frac{(t\mathbb{L}^N)^{\sum_{i=1}^k(2r_i-s_i)}\mathbb{L}^{-\sum_{i=1}^k(r_i^2-r_is_i+s_i^2)}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{N-r_k}(\mathbb{L}^{-1};\mathbb{L}^{-1})_{r_k-r_{k-1}}\cdots(\mathbb{L}^{-1};\mathbb{L}^{-1})_{r_2-r_1}}\\
&\qquad \times \frac{1}{(t\mathbb{L}^{N-1};\mathbb{L}^{-1})_{r_1}^2(\mathbb{L}^{-1};\mathbb{L}^{-1})_{s_1}} \begin{bmatrix} r_k-s_{k-1}\\ r_k-s_k\end{bmatrix}_{\mathbb{L}^{-1}}\cdots \begin{bmatrix} r_2-s_1\\ r_2-s_2\end{bmatrix}_{\mathbb{L}^{-1}} \begin{bmatrix} r_1\\ r_1-s_1\end{bmatrix}_{\mathbb{L}^{-1}},
\end{aligned}
$$

where we have used the relation:

$$
(t;\mathbb{L})_{N-r_1} = (-t)^N\mathbb{L}^{\binom{N}{2}}\frac{(t^{-1};\mathbb{L}^{-1})_N^2}{(t\mathbb{L}^{N-1};\mathbb{L}^{-1})_{r_1}}.
$$

Hence, we are strongly suggested to consider the following truncation of $\mathcal{Z}_k(t,q)$:

$$
\begin{aligned}
\mathcal{Z}_k(N;t,q) := \sum_{\substack{r_k\ge\cdots\ge r_1\ge 0\\ s_k\ge\cdots\ge s_1\ge 0}} &\frac{t^{\sum_{i=1}^k(2r_i-s_i)}q^{\sum_{i=1}^k(r_i^2-r_is_i+s_i^2)}}{(q;q)_{N-r_k}(q;q)_{r_2-r_1}\cdots(q;q)_{r_k-r_{k-1}}(tq;q)_{r_1}^2(q;q)_{s_1}}\\
&\times \begin{bmatrix} r_k-s_{k-1}\\ r_k-s_k\end{bmatrix}_q \begin{bmatrix} r_{k-1}-s_{k-2}\\ r_{k-1}-s_{k-1}\end{bmatrix}_q\cdots \begin{bmatrix} r_2-s_1\\ r_2-s_2\end{bmatrix}_q \begin{bmatrix} r_1\\ r_1-s_1\end{bmatrix}_q. \qquad (1.20)
\end{aligned}
$$

It is notable that this sum is *finite* as the factor $1/(q;q)_{N-r_k}$ requires $r_k\le N$ to ensure its nonvanishing. Also, at the limit $N\to\infty$,

$$
\lim_{N\to\infty}\mathcal{Z}_k(N;t,q) = \frac{\mathcal{Z}_k(t,q)}{(q;q)_\infty}. \qquad (1.21)
$$

Furthermore,

$$
NZ_{R^{(2,2k)\oplus N}}(t) = t^{2N}\mathbb{L}^{N^2-N}(\mathbb{L}^{-1};\mathbb{L}^{-1})_N(t^{-1};\mathbb{L}^{-1})_N^2\mathcal{Z}_k(N;t\mathbb{L}^N,\mathbb{L}^{-1}). \qquad (1.22)
$$

Now our objective is to show that Theorem 1.4 can be finitized as follows.

**Theorem 1.7.** *For any nonnegative integer $N$,*

$$
\begin{aligned}
\mathcal{Z}_k(N;t,q) = \frac{1}{(tq;q)_N}\sum_{n_1,\ldots,n_k\ge 0} &\frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{N-n_k}(q;q)_{n_k}(tq;q)_{n_1}}\\
&\times \begin{bmatrix} n_k\\ n_{k-1}\end{bmatrix}_q \begin{bmatrix} n_{k-1}\\ n_{k-2}\end{bmatrix}_q\cdots \begin{bmatrix} n_2\\ n_1\end{bmatrix}_q. \qquad (1.23)
\end{aligned}
$$

*Consequently,*

$$NZ_{R^{(2,2k)\oplus N}}(t) = (t\mathbb{L}^{N-1};\mathbb{L}^{-1})_N \sum_{n_1,\ldots,n_k\geq 0} \frac{(t\mathbb{L}^N)^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{(t\mathbb{L}^{N-1};\mathbb{L}^{-1})_{n_1}} \times \begin{bmatrix} N \\ n_k \end{bmatrix}_{\mathbb{L}^{-1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{\mathbb{L}^{-1}} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_{\mathbb{L}^{-1}} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{\mathbb{L}^{-1}}. \tag{1.24}$$

Here (1.24) follows from (1.22) with an application of the relation:

$$(t\mathbb{L}^{N-1};\mathbb{L}^{-1})_N = (-1)^N t^N \mathbb{L}^{\binom{N}{2}} (t^{-1};\mathbb{L}^{-1})_N.$$

Finally, to close this section, we present two implications of our prior results on $\widehat{NZ}_{R^{(2,2k)}}(t)$ and $NZ_{R^{(2,2k)\oplus N}}(t)$.

The first one concerns a remarkable *reflection formula* of Huang and Jiang [11, Conjecture 1.6 and Theorem 1.7]. To state this formula, we let $E = \Omega^{\oplus N}$ with $\Omega$ the dualizing module of $R$ under the assumption that $\widetilde{R} \simeq \mathbb{K}[[T]]^s$. Then [11, Conjecture 1.6] predicates that

$$NZ_E(t) \stackrel{?}{=} (t^{2N}\mathbb{L}^{N^2})^\delta NZ_E(t^{-1}\mathbb{L}^{-N}),$$

where $\delta := \dim_{\mathbb{K}} \widetilde{R}/R$ is the Serre invariant. This formula remains conjectural but Huang and Jiang proved its point-counting version in [11, Theorem 1.7] with recourse to deep techniques in harmonic analysis. Specializing to the case of torus links $R^{(2,2k)}$ and recalling (1.22), it is clear that the reflection

$$NZ_{R^{(2,2k)\oplus N}}(t) = (t^{2N}\mathbb{L}^{N^2})^k NZ_{R^{(2,2k)\oplus N}}(t^{-1}\mathbb{L}^{-N}) \tag{1.25}$$

is equivalent to the following relation, for which we shall offer a purely $q$-theoretic proof.

**Theorem 1.8.** *For any nonnegative integer $N$,*

$$\mathscr{Z}_k(N;t,q) = \frac{(1-t)^2 q^N (t^{2N}q^{N^2})^{k-1}}{(1-tq^N)^2} \mathscr{Z}_k(N;t^{-1}q^{-N},q). \tag{1.26}$$

Our second interest revolves around a *nonnegativity conjecture* in [11, Conjecture 9.13].

**Conjecture 1.2** (Huang–Jiang, Nonnegativity Conjecture)**.** *The zeta functions $NZ_{R^{(2,2k)\oplus N}}(-t)$ and $\widehat{NZ}_{R^{(2,2k)}}(-t)$, as series in $t$ and $\mathbb{L}$, have nonnegative coefficients.*

We shall answer it in the affirmative.

**Theorem 1.9.** *Conjecture 1.2 is true.*

*Proof.* We only need to recall (1.17) and (1.24), and notice the trivial fact that for any nonnegative integer $n$, $(-t\mathbb{L}^{-1};\mathbb{L}^{-1})_N/(-t\mathbb{L}^{-1};\mathbb{L}^{-1})_n$ is a nonnegative bivariate series whenever $N \geq n$ or $N \to \infty$. $\square$

## 2. $q$-Series prerequisites

In this section, we collect some preliminary results on $q$-series. First, we recall *Jacobi's triple product identity* [4, p. 21, eq. (2.2.10)]:

**Lemma 2.1** (Jacobi's triple product)**.**

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty. \tag{2.1}$$

Next, let the *$q$-hypergeometric function* ${}_r\phi_s$ be defined by

$${}_r\phi_s\left(\begin{matrix}A_1, A_2, \ldots, A_r \\ B_1, B_2, \ldots, B_s\end{matrix}; q, z\right) := \sum_{n\geq 0} \frac{(A_1, A_2, \ldots, A_r; q)_n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n}{(q, B_1, B_2, \ldots, B_s; q)_n}.$$

The *$q$-binomial theorem* [8, p. 354, eq. (II.3)] is as follows:

**Lemma 2.2** ($q$-Binomial theorem)**.**

$${}_1\phi_0\left(\begin{matrix}a \\ -\end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.2}$$

We also require the *$q$-Gauß sum* [8, p. 354, eq. (II.8)]:

**Lemma 2.3** ($q$-Gauß sum)**.**

$${}_2\phi_1\left(\begin{matrix}a, b \\ c\end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty}. \tag{2.3}$$

The *first $q$-Chu–Vandermonde sum* [8, p. 354, eq. (II.7)] is a specialization:

**Lemma 2.4** (First $q$-Chu–Vandermonde sum)**.** *For any nonnegative integer $N$,*

$${}_2\phi_1\left(\begin{matrix}a, q^{-N} \\ c\end{matrix}; q, \frac{cq^N}{a}\right) = \frac{(c/a;q)_N}{(c;q)_N}. \tag{2.4}$$

We then recall *Heine's three transformations* [8, p. 359, eqs. (III.1–3)] for ${}_2\phi_1$ series:

**Lemma 2.5** (Heine's transformations)**.**

$$ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)=\frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/b,z\\az\end{matrix};q,b\right), \tag{2.5}$$

$$ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)=\frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}abz/c,b\\bz\end{matrix};q,\frac{c}{b}\right), \tag{2.6}$$

$$ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)=\frac{(abz/c;q)_\infty}{(z;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right). \tag{2.7}$$

Finally, the following transform for ${}_3\phi_2$ series [8, p. 359, eq. (III.9)] is necessary:

**Lemma 2.6.**

$$ {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,\frac{de}{abc}\right)=\frac{(e/a,de/(bc);q)_\infty}{(e,de/(abc);q)_\infty}{}_3\phi_2\left(\begin{matrix}a,d/b,d/c\\d,de/(bc)\end{matrix};q,\frac{e}{a}\right). \tag{2.8}$$

## 3. Iteration seed toward Theorem 1.3

Our objective here is to prove Theorem 1.3 by offering its finitization. To begin with, we establish a simple $q$-hypergeometric transform.

**Lemma 3.1.** *For any nonnegative integers $M$ and $N$,*

$$\sum_{n\ge 0}\frac{a^n q^{n^2+Mn}}{(q;q)_{N-n}(q;q)_n(aq;q)_{M+n}}=\frac{1}{(q;q)_N(aq;q)_{M+N}}. \tag{3.1}$$

*Proof.* We have

$$\begin{aligned}
\text{LHS (3.1)} &= \frac{1}{(aq;q)_M}\sum_{n=0}^{N}\frac{a^n q^{n^2+Mn}}{(q;q)_{N-n}(q;q)_n(aq^{M+1};q)_n}\\
&= \frac{1}{(q;q)_N(aq;q)_M}\sum_{n=0}^{N}\frac{a^n q^{n^2+Mn}\cdot(-1)^n q^{-\binom{n}{2}+Nn}(q^{-N};q)_n}{(q;q)_n(aq^{M+1};q)_n}\\
&= \frac{1}{(q;q)_N(aq;q)_M}\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}1/\tau,q^{-N}\\aq^{M+1}\end{matrix};q,aq^{M+N+1}\tau\right)\\
\text{(by (2.4))}\quad &= \frac{1}{(q;q)_N(aq;q)_M}\lim_{\tau\to 0}\frac{(aq^{M+1}\tau;q)_N}{(aq^{M+1};q)_N}\\
&= \frac{1}{(q;q)_N(aq;q)_{M+N}},
\end{aligned}$$

as claimed. ∎

Now we show that (3.1) serves as an iteration seed. Let us start by reformulating it as

$$\frac{1}{(q;q)_N(aq;q)_{(M'+M'')+N}}=\sum_{L\geq 0}\frac{a^Lq^{L^2+(M'+M'')L}(q;q)_{L+M'}}{(q;q)_{N-L}(q;q)_L}\times\frac{1}{(q;q)_{L+M'}(aq;q)_{M''+(L+M')}}.$$

Then we may as well apply (3.1) to

$$\frac{1}{(q;q)_{L+M'}(aq;q)_{M''+(L+M')}}$$

in the summand. Repeating this process $k$ times, we arrive at the following finite version of Theorem 1.3.

**Theorem 3.2.** *For any nonnegative integers $d_1,\ldots,d_k$ and $N$,*

$$\begin{aligned}\frac{1}{(q;q)_N(aq;q)_{N+d_1+\cdots+d_k}}&=\sum_{n_k=0}^{N}\sum_{n_{k-1}=0}^{n_k+d_k}\cdots\sum_{n_1=0}^{n_2+d_2}\frac{a^{\sum_{i=1}^k n_i}q^{\sum_{i=1}^k n_i^2+\sum_{i=1}^k(d_1+\cdots+d_i)n_i}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}}\\&\quad\times\frac{(q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{n_k}\cdots(q;q)_{n_1}(aq;q)_{n_1+d_1}}.\end{aligned}\tag{3.2}$$

Finally, we are in a position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let $N\to\infty$ in (3.2). Noting the fact that $1/(q;q)_n=0$ whenever $n<0$, we may loosen the conditions of the indices in (3.2) and see that

$$\lim_{N\to\infty}\mathrm{RHS}\,(3.2)=\frac{1}{(q;q)_\infty}\,\mathrm{RHS}\,(1.14).$$

Meanwhile,

$$\lim_{N\to\infty}\mathrm{LHS}\,(3.2)=\frac{1}{(q;q)_\infty(aq;q)_\infty},$$

thereby implying the desired result. $\square$

## 4. Theorem 1.2 and its finitization

We warm up with a proof of Theorem 1.2 by means of Theorem 1.3.

*Proof of Theorem 1.2.* We open the $q$-binomial coefficients and see that

$$\begin{aligned}\mathcal{Z}_k(1,q) &= \sum_{\substack{r_k\geq\cdots\geq r_1\geq 0\\ s_k\geq\cdots\geq s_1\geq 0}} \frac{q^{\sum_{i=1}^k (r_i^2-r_is_i+s_i^2)}}{(q;q)_{r_k-r_{k-1}}\cdots(q;q)_{r_2-r_1}(q;q)_{r_1}^2(q;q)_{s_1}}\\ &\qquad\times\frac{(q;q)_{r_k-s_{k-1}}\cdots(q;q)_{r_2-s_1}(q;q)_{r_1}}{(q;q)_{r_k-s_k}\cdots(q;q)_{r_1-s_1}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}}\\ &= \sum_{\substack{r_k\geq\cdots\geq r_1\geq 0\\ s_k\geq\cdots\geq s_1\geq 0}} \frac{q^{\sum_{i=1}^k (r_i^2-r_is_i+s_i^2)}}{(q;q)_{r_k-r_{k-1}}\cdots(q;q)_{r_2-r_1}(q;q)_{r_1}}\\ &\qquad\times\frac{(q;q)_{(r_k-s_k)+(s_k-s_{k-1})}\cdots(q;q)_{(r_2-s_2)+(s_2-s_1)}}{(q;q)_{r_k-s_k}\cdots(q;q)_{r_1-s_1}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}^2}.\end{aligned}$$

Now for $1\leq i\leq k$, we put

$$d_i := \begin{cases} s_1, & i=1,\\ s_i-s_{i-1}, & i\geq 2.\end{cases} \tag{4.1}$$

Making the change of variables for each $1\leq j\leq k$:

$$n_j := r_j - s_j, \tag{4.2}$$

we find that $\mathcal{Z}_k(1,q)$ equals

$$\begin{aligned}&\sum_{s_1,\ldots,s_k\geq 0}\frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{s_2-s_1}\cdots(q;q)_{s_k-s_{k-1}}(q;q)_{s_1}^2}\\ &\times\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2+\sum_{i=1}^k (d_1+\cdots+d_i)n_i}(q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}(q;q)_{n_k}\cdots(q;q)_{n_1}(q;q)_{n_1+d_1}},\end{aligned}$$

where we have loosened the conditions for the sums by using the vanishing of $1/(q;q)_n$ whenever $n<0$. Applying (1.14) with $a=1$ to the inner sum gives

$$\mathcal{Z}_k(1,q) = \frac{1}{(q;q)_\infty}\sum_{s_1,\ldots,s_k\geq 0}\frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}^2},$$

which further yields (1.13) in view of the same reasoning. □

In addition, it is notable that the finite version of Theorem 1.3, namely, the identity (3.2), at the same time implies a finitization of Theorem 1.2.

**Theorem 4.1.** *For any nonnegative integer $N$,*

$$\mathcal{Z}_k(N;1,q) = \frac{1}{(q;q)_N^3}. \tag{4.3}$$

*Proof.* Similar to how the proof of Theorem 1.2 has been proceeded, we have the simplification:

$$\begin{aligned}\mathcal{Z}_k(N;1,q) &= \sum_{s_1,\ldots,s_k\geq 0} \frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}^2}\\ &\quad\times \sum_{n_1,\ldots,n_k\geq 0} \frac{q^{\sum_{i=1}^k n_i^2+\sum_{i=1}^k (d_1+\cdots+d_i)n_i}}{(q;q)_{(N-s_k)-n_k}(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}}\\ &\quad\times \frac{(q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{n_k}\cdots(q;q)_{n_1}(q;q)_{n_1+d_1}},\end{aligned}$$

where we have still used the substitutions (4.1) and (4.2). Noting that $d_1+\cdots+d_k=s_k$, we apply (3.2) to simplify the inner sum over $n_1,\ldots,n_k$ as

$$\frac{1}{(q;q)_N(q;q)_{N-s_k}}.$$

It follows that

$$\mathcal{Z}_k(N;1,q) = \frac{1}{(q;q)_N}\sum_{s_1,\ldots,s_k\geq 0}\frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{N-s_k}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}^2}.$$

Applying (3.2) with $d_1=\cdots=d_k=0$ further gives

$$\mathcal{Z}_k(N;1,q) = \frac{1}{(q;q)_N}\cdot\frac{1}{(q;q)_N^2},$$

which is as desired. □

## 5. Reformulating $\mathcal{Z}_k(N;t,q)$

To achieve the $k$-fold sum for $\mathcal{Z}_k(N;t,q)$ in (1.23), our first step is to reformulate it to a form that aligns with the $2k$-fold sum in (1.19). We begin with

$$\begin{aligned}\mathcal{Z}_k(N;t,q) &= \sum_{\substack{r_k\geq\cdots\geq r_1\geq 0\\ s_k\geq\cdots\geq s_1\geq 0}} \frac{t^{\sum_{i=1}^k(2r_i-s_i)}q^{\sum_{i=1}^k(r_i^2-r_is_i+s_i^2)}}{(q;q)_{N-r_k}(q;q)_{r_k-r_{k-1}}\cdots(q;q)_{r_2-r_1}(tq;q)_{r_1}^2(q;q)_{s_1}}\\ &\quad\times \begin{bmatrix} r_k-s_{k-1}\\ r_k-s_k\end{bmatrix}_q \begin{bmatrix} r_{k-1}-s_{k-2}\\ r_{k-1}-s_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} r_2-s_1\\ r_2-s_2\end{bmatrix}_q \begin{bmatrix} r_1\\ r_1-s_1\end{bmatrix}_q.\end{aligned}$$

By opening the $q$-binomial coefficients and reorganizing the $q$-factorials, the above can be reformulated as

$$\mathcal{Z}_k(N;t,q) = \sum_{\substack{r_k\ge\cdots\ge r_1\ge 0\\ s_k\ge\cdots\ge s_1\ge 0}} \frac{t^{\sum_{i=1}^k(2r_i-s_i)}q^{\sum_{i=1}^k(r_i^2-r_is_i+s_i^2)}(q;q)_{r_1}}{(q;q)_{N-r_k}(q;q)_{r_k-s_k}(q;q)_{s_k}(tq;q)_{r_1}^2(q;q)_{s_1}} \\ \times \begin{bmatrix} s_k \\ s_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} s_2 \\ s_1\end{bmatrix}_q \begin{bmatrix} r_k-s_{k-1} \\ r_{k-1}-s_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} r_2-s_1 \\ r_1-s_1\end{bmatrix}_q.$$

Invoking the substitutions for $1\le j\le k$:

$$n_j := r_j - s_j,$$

we further have

$$\mathcal{Z}_k(N;t,q) = \sum_{\substack{s_1,\ldots,s_k\ge 0\\ n_1,\ldots,n_k\ge 0}} \frac{t^{\sum_{i=1}^k(s_i+2n_i)}q^{\sum_{i=1}^k(s_i^2+s_in_i+n_i^2)}(q;q)_{n_1+s_1}}{(q;q)_{N-s_k-n_k}(q;q)_{s_k}(q;q)_{n_k}(q;q)_{s_1}(tq;q)_{n_1+s_1}^2} \\ \times \begin{bmatrix} s_k \\ s_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} s_2 \\ s_1\end{bmatrix}_q \begin{bmatrix} n_k+s_k-s_{k-1} \\ n_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} n_2+s_2-s_1 \\ n_1\end{bmatrix}_q. \tag{5.1}$$

Now we work on the sums over $n_1,\ldots,n_k$:

$$\Sigma := \sum_{n_1,\ldots,n_k\ge 0} \frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k(n_i^2+s_in_i)}(q;q)_{n_1+s_1}}{(q;q)_{(N-s_k)-n_k}(q;q)_{n_k}(tq;q)_{n_1+s_1}^2} \\ \times \begin{bmatrix} n_k+s_k-s_{k-1} \\ n_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} n_2+s_2-s_1 \\ n_1\end{bmatrix}_q.$$

Let us single out the sum over $n_1$:

$$\Sigma = \frac{1}{(q;q)_{N-s_k}} \sum_{n_2,\ldots,n_k\ge 0} t^{\sum_{i=2}^k 2n_i}q^{\sum_{i=2}^k(n_i^2+s_in_i)} \\ \times \begin{bmatrix} N-s_k \\ n_k\end{bmatrix}_q \begin{bmatrix} n_k+s_k-s_{k-1} \\ n_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} n_3+s_3-s_2 \\ n_2\end{bmatrix}_q \\ \times (q;q)_{n_2+s_2-s_1} \sum_{n_1\ge 0} \frac{t^{2n_1}q^{n_1^2+s_1n_1}(q;q)_{n_1+s_1}}{(q;q)_{(n_2+s_2-s_1)-n_1}(q;q)_{n_1}(tq;q)_{n_1+s_1}^2}.$$

To simplify this sum over $n_1$, we require a basic hypergeometric transform.

**Lemma 5.1.** *For any nonnegative integers $M$ and $N$,*

$$\sum_{n\geq 0}\frac{a^{2n}q^{n^2+Mn}(q;q)_{M+n}}{(q;q)_{N-n}(q;q)_n(aq;q)_{M+n}^2}=\frac{(q;q)_\infty(a^2q;q)_\infty}{(aq;q)_\infty^2(q;q)_N}\sum_{n\geq 0}\frac{q^{(M+1)n}(a;q)_n^2}{(q;q)_n(a^2q;q)_{M+N+n}}. \tag{5.2}$$

*Proof.* We have

$$\begin{aligned}
\text{LHS (5.2)} &= \frac{(q;q)_M}{(q;q)_N(aq;q)_M^2}\sum_{n\geq 0}\frac{(-1)^n a^{2n}q^{\binom{n}{2}+(M+N+1)n}(q^{-N};q)_n(q^{M+1};q)_n}{(q;q)_n(aq^{M+1};q)_n^2}\\
&= \frac{(q;q)_M}{(q;q)_N(aq;q)_M^2}\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q^{-N},1/\tau,q^{M+1}\\ aq^{M+1},aq^{M+1}\end{matrix};q,a^2q^{M+N+1}\tau\right)\\
\text{(by (2.8))} &= \frac{(q;q)_M}{(q;q)_N(aq;q)_M^2}\lim_{\tau\to 0}\frac{(aq^{M+N+1},a^2q^{M+1}\tau;q)_\infty}{(aq^{M+1},a^2q^{M+N+1}\tau;q)_\infty}\\
&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-N},aq^{M+1}\tau,a\\ aq^{M+1},a^2q^{M+1}\tau\end{matrix};q,aq^{M+N+1}\right)\\
&= \frac{(q;q)_M}{(q;q)_N(aq;q)_M(aq;q)_{M+N}}\,{}_2\phi_1\left(\begin{matrix}q^{-N},a\\ aq^{M+1}\end{matrix};q,aq^{M+N+1}\right)\\
\text{(by (2.6))} &= \frac{(q;q)_M}{(q;q)_N(aq;q)_M(aq;q)_{M+N}}\frac{(q^{M+1},a^2q^{M+N+1};q)_\infty}{(aq^{M+1},aq^{M+N+1};q)_\infty}\\
&\quad\times{}_2\phi_1\left(\begin{matrix}a,a\\ a^2q^{M+N+1}\end{matrix};q,q^{M+1}\right)\\
&= \frac{(q;q)_\infty(a^2q;q)_\infty}{(aq;q)_\infty^2(q;q)_N(a^2q;q)_{M+N}}\sum_{n\geq 0}\frac{q^{(M+1)n}(a;q)_n^2}{(q;q)_n(a^2q^{M+N+1};q)_n}\\
&= \frac{(q;q)_\infty(a^2q;q)_\infty}{(aq;q)_\infty^2(q;q)_N}\sum_{n\geq 0}\frac{q^{(M+1)n}(a;q)_n^2}{(q;q)_n(a^2q;q)_{M+N+n}},
\end{aligned}$$

as claimed. □

It follows by applying (5.2) to the previous sum over $n_1$ that

$$\begin{aligned}
\Sigma &= \frac{(q;q)_\infty(t^2q;q)_\infty}{(tq;q)_\infty^2(q;q)_{N-s_k}}\sum_{n_2,\ldots,n_k\geq 0}t^{\sum_{i=2}^k 2n_i}q^{\sum_{i=2}^k(n_i^2+s_in_i)}\\
&\quad\times\begin{bmatrix}N-s_k\\ n_k\end{bmatrix}_q\begin{bmatrix}n_k+s_k-s_{k-1}\\ n_{k-1}\end{bmatrix}_q\cdots\begin{bmatrix}n_3+s_3-s_2\\ n_2\end{bmatrix}_q
\end{aligned}$$

$$\times \sum_{n_1\geq 0} \frac{q^{n_1+s_1n_1}(t;q)_{n_1}^2}{(q;q)_{n_1}(t^2q;q)_{n_1+n_2+s_2}}.$$

Interchanging the sum over $n_1$ and the remaining sums gives

$$\Sigma = \frac{(q;q)_\infty(t^2q;q)_\infty}{(tq;q)_\infty^2(q;q)_{N-s_k}} \sum_{n_1\geq 0} \frac{q^{n_1+s_1n_1}(t;q)_{n_1}^2}{(q;q)_{n_1}} \sum_{n_2,\ldots,n_k\geq 0} \frac{t^{\sum_{i=2}^k 2n_i} q^{\sum_{i=2}^k (n_i^2+s_in_i)}}{(t^2q;q)_{n_1+n_2+s_2}}$$
$$\times \begin{bmatrix} N-s_k \\ n_k \end{bmatrix}_q \begin{bmatrix} n_k+s_k-s_{k-1} \\ n_{k-1} \end{bmatrix}_q \cdots \begin{bmatrix} n_3+s_3-s_2 \\ n_2 \end{bmatrix}_q.$$

Our next trick relies on a slight extension of a transform due to Warnaar [22, p. 746, Lemma 7.2].

**Lemma 5.2.** *Let $m_0$ be a nonnegative integer and let $u_1 \leq u_2 \leq \cdots \leq u_{k+1}$ be integers. We have, for any $\ell \in \{0,1,\ldots,k\}$,*

$$\sum_{m_1,\ldots,m_k\geq 0} \frac{t^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k m_i(m_i+u_i)}}{(tq;q)_{m_k+u_{k+1}}} \prod_{i=1}^k \begin{bmatrix} m_{i-1} \\ m_i \end{bmatrix}_q$$
$$= \sum_{m_1,\ldots,m_k\geq 0} \frac{t^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k m_i(m_i+u_i)}}{(tq;q)_{m_\ell+m_{\ell+1}+u_{\ell+1}}} \prod_{i=1}^{\ell} \begin{bmatrix} m_{i-1} \\ m_i \end{bmatrix}_q \prod_{i=\ell+1}^{k} \begin{bmatrix} m_{i+1}+u_{i+1}-u_i \\ m_i \end{bmatrix}_q, \tag{5.3}$$

*where $m_{k+1} := 0$.*

*Proof.* The proof is almost identical to that for [22, p. 746, Lemma 7.2]. The only modification is that in the following identity [22, p. 746, above eq. (7.6)]:

$$\sum_{k\geq 0}(-z)^k q^{\binom{k}{2}} \frac{(a;q)_k(cq^k;q)_\infty}{(q;q)_k} = \sum_{k\geq 0}(-c)^k q^{\binom{k}{2}} \frac{(az/c;q)_k(zq^k;q)_\infty}{(q;q)_k},$$

we instead set $(a,c,z) \mapsto (q^{-(n_2-p)}, tq^{n_1+1}, tq^{n_2+1})$ so as to extend [22, p. 747, eq. (7.7)] as

$$\sum_{m\geq 0} \frac{t^m q^{m(m+p)}}{(tq;q)_{m+n_1}} \begin{bmatrix} n_2-p \\ m \end{bmatrix}_q = \sum_{m\geq 0} \frac{t^m q^{m(m+p)}}{(tq;q)_{m+n_2}} \begin{bmatrix} n_1-p \\ m \end{bmatrix}_q.$$

The rest follows by the same induction argument. ∎

The above lemma tells us that

$$\sum_{n_2,\ldots,n_k\geq 0} \frac{t^{\sum_{i=2}^k 2n_i} q^{\sum_{i=2}^k (n_i^2+s_in_i)}}{(t^2q;q)_{n_1+n_2+s_2}} \begin{bmatrix} N-s_k \\ n_k \end{bmatrix}_q \begin{bmatrix} n_k+s_k-s_{k-1} \\ n_{k-1} \end{bmatrix}_q \cdots \begin{bmatrix} n_3+s_3-s_2 \\ n_2 \end{bmatrix}_q$$

$$= \sum_{n_2,\ldots,n_k\geq 0} \frac{t^{\sum_{i=2}^k 2n_i} q^{\sum_{i=2}^k (n_i^2+s_in_i)}}{(t^2q;q)_{N+n_k}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

As a consequence,

$$\Sigma = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2 (q;q)_{N-s_k}} \sum_{n_1,\ldots,n_k\geq 0} t^{\sum_{i=2}^k 2n_i} q^{n_1+s_1n_1+\sum_{i=2}^k (n_i^2+s_in_i)}$$
$$\times \frac{(t;q)_{n_1}^2}{(q;q)_{n_1}(t^2q;q)_{N+n_k}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

Finally, substituting the above into (5.1) and renaming $s_i$ by $m_i$, we are led to the following reformulation of $\mathcal{Z}_k(N;t,q)$.

**Theorem 5.3.** *For any nonnegative integer $N$,*

$$\mathcal{Z}_k(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2}$$
$$\times \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(t^2q;q)_{N+n_k}(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q. \tag{5.4}$$

## 6. A semi-truncation

We move on to the following multisum:

$$\mathcal{V}_k(N;t,q) := \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q. \tag{6.1}$$

It is notable that only the sums over $m_1,\ldots,m_k$ are finite.

Let us assume that $k\geq 2$.

We start by opening the $q$-binomial coefficients:

$$\mathcal{V}_k(N;t,q) = \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(q;q)_{n_k}(q;q)_{m_1}^2}$$

$$\times \frac{1}{(q;q)_{m_k-m_{k-1}}\cdots(q;q)_{m_2-m_1}(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}-n_k}}.$$

Singling out the sums over $m_1,\dots,m_{k-1}$ and $n_1,\dots,n_{k-1}$ then gives

$$\begin{aligned}\mathcal{V}_k(N;t,q) = \sum_{m_k,n_k\geq 0} & \frac{t^{m_k+2n_k}q^{m_k^2+m_kn_k+n_k^2}}{(q;q)_{N-m_k}(q;q)_{n_k}} \\ \times \sum_{\substack{m_1,\dots,m_{k-1}\geq 0\\ n_1,\dots,n_{k-1}\geq n_k}} & \frac{t^{-2n_1+\sum_{i=1}^{k-1}(m_i+2n_i)}q^{-n_1^2+n_1+\sum_{i=1}^{k-1}(m_i^2+m_in_i+n_i^2)}(t;q)_{n_1}^2}{(q;q)_{m_k-m_{k-1}}(q;q)_{n_{k-1}-n_k}(q;q)_{m_1}^2} \\ \times & \frac{1}{(q;q)_{m_{k-1}-m_{k-2}}\cdots(q;q)_{m_2-m_1}(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-2}-n_{k-1}}}.\end{aligned}$$

Now we make the substitutions for $1\leq i\leq k-1$:

$$n_i \mapsto n_i+n_k.$$

Then,

$$\begin{aligned}&\mathcal{V}_k(N;t,q) \\ &\quad = \sum_{m_k,n_k\geq 0} \frac{t^{m_k+2(k-1)n_k}q^{m_k^2+(m_k+1)n_k+(k-1)n_k^2}(t;q)_{n_k}^2}{(q;q)_{N-m_k}(q;q)_{n_k}} \\ &\quad \times \sum_{\substack{m_1,\dots,m_{k-1}\geq 0\\ n_1,\dots,n_{k-1}\geq 0}} \frac{(tq^{n_k})^{-2n_1+\sum_{i=1}^{k-1}(m_i+2n_i)}q^{-n_1^2+n_1+\sum_{i=1}^{k-1}(m_i^2+m_in_i+n_i^2)}(tq^{n_k};q)_{n_1}^2}{(q;q)_{m_k-m_{k-1}}(q;q)_{n_{k-1}}(q;q)_{m_1}^2} \\ &\quad \times \frac{1}{(q;q)_{m_{k-1}-m_{k-2}}\cdots(q;q)_{m_2-m_1}(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-2}-n_{k-1}}}.\end{aligned}$$

In other words,

$$\mathcal{V}_k(N;t,q) = \sum_{m,n\geq 0} \frac{t^{m+2(k-1)n}q^{m^2+(m+1)n+(k-1)n^2}(t;q)_n^2}{(q;q)_{N-m}(q;q)_n}\mathcal{V}_{k-1}(m;tq^n,q). \tag{6.2}$$

Now we simplify $\mathcal{V}_k(N;t,q)$ to a great extent as follows.

**Theorem 6.1.** *For any nonnegative integer $N$,*

$$\begin{aligned}\mathcal{V}_k(N;t,q) = \frac{(tq;q)_\infty}{(q;q)_\infty(q;q)_N} & \sum_{n_1,\dots,n_k\geq 0} (-1)^{n_k} t^{-n_k+\sum_{i=1}^k 2n_i} q^{-\binom{n_k}{2}+\sum_{i=1}^k n_i^2} \\ & \times \frac{(t;q)_{n_k}}{(q;q)_{n_k}(tq;q)_{N+n_1}} \begin{bmatrix} n_k \\ n_{k-1}\end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1\end{bmatrix}_q.\end{aligned} \tag{6.3}$$

Our strategy is to apply induction on $k$ by means of (6.2). Here we work on the base case at $k=1$ and the inductive step separately.

*Proof of the base case.* Recall that

$$\mathcal{V}_1(N;t,q)=\sum_{m_1,n_1\ge 0}\frac{t^{m_1}q^{m_1^2+m_1n_1+n_1}(t;q)_{n_1}^2}{(q;q)_{N-m_1}(q;q)_{m_1}^2(q;q)_{n_1}}.$$

We first focus on the sum over $m_1$:

$$\mathcal{V}_1(N;t,q)=\sum_{n_1\ge 0}\frac{q^{n_1}(t;q)_{n_1}^2}{(q;q)_{n_1}}\sum_{m_1\ge 0}\frac{t^{m_1}q^{m_1^2+n_1m_1}}{(q;q)_{N-m_1}(q;q)_{m_1}^2}.$$

It is clear that

$$\begin{aligned}\sum_{m_1\ge 0}\frac{t^{m_1}q^{m_1^2+m_1n_1}}{(q;q)_{N-m_1}(q;q)_{m_1}^2}&=\frac{1}{(q;q)_N}\sum_{m_1\ge 0}\frac{(-1)^{m_1}t^{m_1}q^{\binom{m_1}{2}+(N+n_1+1)m_1}(q^{-N};q)_{m_1}}{(q;q)_{n_1}^2}\\&=\frac{1}{(q;q)_N}\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}1/\tau,q^{-N}\\ q\end{matrix};q,tq^{N+n_1+1}\tau\right).\end{aligned}$$

We temporally assume that $|t|<1$ to ensure the convergence condition for the application of Heine's third transformation (2.7) to the ${}_2\phi_1$ series especially when $n_1=0$. Then,

$$\begin{aligned}\sum_{m_1\ge 0}\frac{t^{m_1}q^{m_1^2+m_1n_1}}{(q;q)_{N-m_1}(q;q)_{m_1}^2}&=\frac{1}{(q;q)_N}\cdot(tq^{n_1};q)_\infty{}_2\phi_1\left(\begin{matrix}0,q^{N+1}\\ q\end{matrix};q,tq^{n_1}\tau\right)\\&=\frac{(t;q)_\infty}{(q;q)_N(t;q)_{n_1}}\sum_{m_1\ge 0}\frac{t^{m_1}q^{n_1m_1}(q^{N+1};q)_{m_1}}{(q;q)_{m_1}^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{V}_1(N;t,q)&=\frac{(t;q)_\infty}{(q;q)_N}\sum_{m_1\ge 0}\frac{t^{m_1}(q^{N+1};q)_{m_1}}{(q;q)_{m_1}^2}\sum_{n_1\ge 0}\frac{q^{(m_1+1)n_1}(t;q)_{n_1}}{(q;q)_{n_1}}\\ \text{(by (2.2))}\quad&=\frac{(t;q)_\infty}{(q;q)_N}\sum_{m_1\ge 0}\frac{t^{m_1}(q^{N+1};q)_{m_1}}{(q;q)_{m_1}^2}\frac{(tq^{m_1+1};q)_\infty}{(q^{m_1+1};q)_\infty}\\&=\frac{(t;q)_\infty(tq;q)_\infty}{(q;q)_\infty(q;q)_N}\sum_{m_1\ge 0}\frac{t^{m_1}(q^{N+1};q)_{m_1}}{(q;q)_{m_1}(tq;q)_{m_1}}\\&=\frac{(t;q)_\infty(tq;q)_\infty}{(q;q)_\infty(q;q)_N}\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}q^{N+1},tq\tau\\ tq\end{matrix};q,t\right)\end{aligned}$$

$$\text{(by (2.5))} = \frac{(t;q)_\infty (tq;q)_\infty}{(q;q)_\infty (q;q)_N} \lim_{\tau\to 0} \frac{(tq\tau, tq^{N+1};q)_\infty}{(tq,t;q)_\infty} {}_2\phi_1\left(\begin{matrix}1/\tau, t\\ tq^{N+1}\end{matrix}; q, tq\tau\right)$$
$$= \frac{(tq;q)_\infty}{(q;q)_\infty (q;q)_N} \sum_{n_1\geq 0} \frac{(-1)^{n_1} t^{n_1} q^{\binom{n_1+1}{2}} (t;q)_{n_1}}{(q;q)_{n_1} (tq;q)_{N+n_1}}.$$

It is notable that this relation can be analytically continued from $|t| < 1$, which has been assumed earlier, to $t \in \mathbb{C}$. Hence, we arrive at (6.3) for $k = 1$. □

*Proof of the inductive step.* Assume that (6.3) is valid for some $k-1$ with $k \geq 2$. Thus,

$$\mathcal{V}_{k-1}(m; tq^n, q) = \frac{(tq^{n+1};q)_\infty}{(q;q)_\infty (q;q)_m} \sum_{n_1,\ldots,n_{k-1}\geq 0} (-1)^{n_{k-1}} t^{-n_{k-1}+\sum_{i=1}^{k-1} 2n_i}$$
$$\times \frac{q^{-\binom{n_{k-1}}{2} - nn_{k-1} + \sum_{i=1}^{k-1}(n_i^2+2nn_i)} (tq^n;q)_{n_{k-1}}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1} (q;q)_{n_1} (tq^{n+1};q)_{m+n_1}}.$$

Invoking (6.2),

$$\mathcal{V}_k(N;t,q) = \sum_{m,n\geq 0} \frac{(-1)^n t^{m+n} q^{m^2+mn+\binom{n+1}{2}} (t;q)_n^2}{(q;q)_{N-m} (q;q)_n} \frac{(tq^{n+1};q)_\infty}{(q;q)_\infty (q;q)_m}$$
$$\times \sum_{n_1,\ldots,n_{k-1}\geq 0} (-1)^{n_{k-1}+n} t^{-(n_{k-1}+n)+\sum_{i=1}^{k-1} 2(n_i+n)}$$
$$\times \frac{q^{-\binom{n_{k-1}+n}{2} + \sum_{i=1}^{k-1}(n_i+n)^2} (tq^n;q)_{n_{k-1}}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1} (q;q)_{n_1} (tq^{n+1};q)_{m+n_1}}$$
$$= \frac{(tq;q)_\infty}{(q;q)_\infty} \sum_{m,n\geq 0} \frac{(-1)^n t^{m+n} q^{m^2+mn+\binom{n+1}{2}} (t;q)_n}{(q;q)_{N-m} (q;q)_m (q;q)_n}$$
$$\times \sum_{n_1,\ldots,n_{k-1}\geq 0} (-1)^{n_{k-1}+n} t^{-(n_{k-1}+n)+\sum_{i=1}^{k-1} 2(n_i+n)}$$
$$\times \frac{q^{-\binom{n_{k-1}+n}{2} + \sum_{i=1}^{k-1}(n_i+n)^2} (t;q)_{n_{k-1}+n}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1} (q;q)_{n_1} (tq;q)_{m+(n_1+n)}}.$$

We put, for each $1 \leq i \leq k-1$:

$$l_i := n_i + n,$$

and interchange the sums over $m,n$ and the rest. Then,

$$\mathcal{V}_k(N;t,q)=\frac{(tq;q)_\infty}{(q;q)_\infty}\sum_{l_1,\ldots,l_{k-1}\geq 0}\frac{(-1)^{l_{k-1}}t^{-l_{k-1}+\sum_{i=1}^{k-1}2l_i}q^{-\binom{l_{k-1}}{2}+\sum_{i=1}^{k-1}l_i^2}(t;q)_{l_{k-1}}}{(q;q)_{l_{k-1}-l_{k-2}}\cdots(q;q)_{l_2-l_1}}$$

$$\times\sum_{m,n\geq 0}\frac{(-1)^n t^{m+n}q^{m^2+mn+\binom{n+1}{2}}(t;q)_n}{(q;q)_{N-m}(q;q)_{l_1-n}(tq;q)_{l_1+m}(q;q)_m(q;q)_n}.$$

Hence, as long as we can show

$$\sum_{m,n\geq 0}\frac{(-1)^n t^{m+n}q^{m^2+mn+\binom{n+1}{2}}(t;q)_n}{(q;q)_{N-m}(q;q)_{l_1-n}(tq;q)_{l_1+m}(q;q)_m(q;q)_n}$$
$$=\frac{1}{(q;q)_N}\sum_{l_0\geq 0}\frac{t^{2l_0}q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{N+l_0}},\quad(6.4)$$

then (6.3) holds for $k$ by renaming the indices with $l_i\mapsto n_{i+1}$ for each $0\leq i\leq k-1$. To acquire this last ingredient in the recipe, we single out the sum over $n$:

$$\text{LHS (6.4)}=\sum_{m\geq 0}\frac{t^m q^{m^2}}{(q;q)_{N-m}(q;q)_m(tq;q)_{l_1+m}}\sum_{n\geq 0}\frac{(-1)^n t^n q^{\binom{n}{2}+(m+1)n}(t;q)_n}{(q;q)_{l_1-n}(q;q)_n}.$$

Note that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{(-1)^n t^n q^{\binom{n}{2}+(m+1)n}(t;q)_n}{(q;q)_{l_1-n}(q;q)_n}\\
&\quad=\frac{1}{(q;q)_{l_1}}\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}t,q^{-l_1}\\ t^2q^{m+1}\tau\end{matrix};q,tq^{l_1+m+1}\right)\\
&\quad=\frac{1}{(q;q)_{l_1}}\lim_{\tau\to 0}\frac{(t^2q^{l_1+m+1}\tau,tq^{m+1};q)_\infty}{(t^2q^{m+1}\tau,tq^{l_1+m+1};q)_\infty}{}_2\phi_1\left(\begin{matrix}1/\tau,q^{-l_1}\\ tq^{m+1}\end{matrix};q,t^2q^{l_1+m+1}\tau\right)\\
&\quad=\frac{(tq^{m+1};q)_\infty}{(tq^{l_1+m+1};q)_\infty}\sum_{l_0\geq 0}\frac{t^{2l_0}q^{l_0^2+ml_0}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq^{m+1};q)_{l_0}},
\end{aligned}$$

where we have applied Heine's second transform (2.6). Hence,

$$\begin{aligned}
&\text{LHS (6.4)}\\
&\quad=\sum_{l_0\geq 0}\frac{t^{2l_0}q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}}\sum_{m\geq 0}\frac{t^m q^{m^2+l_0m}}{(q;q)_{N-m}(q;q)_m(tq^{l_0+1};q)_m}
\end{aligned}$$

$$= \frac{1}{(q;q)_N} \sum_{l_0 \geq 0} \frac{t^{2l_0} q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}} \lim_{\tau \to 0} {}_2\phi_1 \left( \begin{matrix} 1/\tau, q^{-N} \\ tq^{l_0+1} \end{matrix} ; q, tq^{N+l_0+1}\tau \right).$$

Applying the first $q$-Chu–Vandermonde sum (2.4) yields

$$\mathrm{LHS}\,(6.4) = \frac{1}{(q;q)_N} \sum_{l_0 \geq 0} \frac{t^{2l_0} q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}} \frac{1}{(tq^{l_0+1};q)_N},$$

which is exactly what we need. □

## 7. $q$-Lebesgue identity

Recall from (5.4) that

$$\mathcal{Z}_1(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2} \sum_{m_1,n_1 \geq 0} \frac{t^{m_1} q^{m_1^2+m_1n_1+n_1} (t;q)_{n_1}^2}{(q;q)_{N-m_1}(t^2q;q)_{N+n_1}(q;q)_{m_1}^2 (q;q)_{n_1}}.$$

Unlike how we treat $\mathcal{V}_1(N;t,q)$, this time we focus on the sum over $n_1$ at first:

$$\mathcal{Z}_1(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2 (t^2q;q)_N} \sum_{m_1 \geq 0} \frac{t^{m_1} q^{m_1^2}}{(q;q)_{N-m_1}(q;q)_{m_1}^2} \\ \times \sum_{n_1 \geq 0} \frac{q^{(m_1+1)n_1}(t;q)_{n_1}^2}{(q;q)_{n_1}(t^2q^{N+1};q)_{n_1}}.$$

Note that

$$\begin{aligned} \sum_{n_1 \geq 0} \frac{q^{(m_1+1)n_1}(t;q)_{n_1}^2}{(q;q)_{n_1}(t^2q^{N+1};q)_{n_1}} &= {}_2\phi_1 \left( \begin{matrix} t, t \\ t^2q^{N+1} \end{matrix} ; q, q^{m_1+1} \right) \\ \text{(by (2.6))} \quad &= \frac{(tq^{N+1}, tq^{m_1+1};q)_\infty}{(t^2q^{N+1}, q^{m_1+1};q)_\infty} {}_2\phi_1 \left( \begin{matrix} q^{m_1-N}, t \\ tq^{m_1+1} \end{matrix} ; q, tq^{N+1} \right) \\ &= \frac{(tq^{N+1}, tq^{m_1+1};q)_\infty}{(t^2q^{N+1}, q^{m_1+1};q)_\infty} \sum_{n_1 \geq 0} \frac{t^{n_1} q^{(N+1)n_1}(q^{m_1-N}, t;q)_{n_1}}{(q, tq^{m_1+1};q)_{n_1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{Z}_1(N;t,q) &= \frac{1}{(tq;q)_N} \sum_{m_1,n_1 \geq 0} \frac{t^{m_1+n_1} q^{m_1^2+(N+1)n_1}(t;q)_{n_1}(q^{m_1-N};q)_{n_1}}{(q;q)_{N-m_1}(q;q)_{m_1}(q;q)_{n_1}(tq;q)_{m_1+n_1}} \\ &= \frac{1}{(tq;q)_N} \sum_{n_1 \geq 0} \frac{(-1)^{n_1} t^{n_1} q^{\binom{n_1+1}{2}}(t;q)_{n_1}}{(q;q)_{n_1}(tq;q)_{n_1}} \end{aligned}$$

$$\times \sum_{m_1\geq 0} \frac{t^{m_1} q^{m_1^2+n_1m_1}}{(q;q)_{(N-n_1)-m_1}(q;q)_{m_1}(tq^{n_1+1};q)_{m_1}}.$$

For the inner sum over $m_1$, we have

$$\begin{aligned}\sum_{m_1\geq 0} &\frac{t^{m_1} q^{m_1^2+n_1m_1}}{(q;q)_{(N-n_1)-m_1}(q;q)_{m_1}(tq^{n_1+1};q)_{m_1}}\\ &= \frac{1}{(q;q)_{N-n_1}} \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix}1/\tau, q^{-N+n_1}\\ tq^{n_1+1}\end{matrix}; q, tq^{N+1}\tau\right)\\ \text{(by (2.3))} &= \frac{1}{(q;q)_{N-n_1}} \frac{(tq^{N+1};q)_\infty}{(tq^{n_1+1};q)_\infty}.\end{aligned}$$

It follows that

$$\mathcal{Z}_1(N;t,q) = \frac{1}{(tq;q)_N^2} \sum_{n_1\geq 0} \frac{(-1)^{n_1} t^{n_1} q^{\binom{n_1+1}{2}} (t;q)_{n_1}}{(q;q)_{N-n_1}(q;q)_{n_1}}. \tag{7.1}$$

We may further rewrite the above as

$$\begin{aligned}\mathcal{Z}_1(N;t,q) &= \frac{1}{(q;q)_N(tq;q)_N^2} \sum_{n_1\geq 0} \frac{t^{n_1} q^{(N+1)n_1} (t;q)_{n_1} (q^{-N};q)_{n_1}}{(q;q)_{n_1}}\\ &= \frac{1}{(q;q)_N(tq;q)_N^2} \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix}t, q^{-N}\\ t^2q\tau\end{matrix}; q, tq^{N+1}\right)\\ \text{(by (2.6))} &= \frac{1}{(q;q)_N(tq;q)_N^2} \lim_{\tau\to 0} \frac{(t^2q^{N+1}\tau, tq;q)_\infty}{(t^2q\tau, tq^{N+1};q)_\infty} {}_2\phi_1\left(\begin{matrix}1/\tau, q^{-N}\\ tq\end{matrix}; q, t^2q^{N+1}\tau\right)\\ &= \frac{1}{(q;q)_N(tq;q)_N} \sum_{n_1\geq 0} \frac{(-1)^{n_1} t^{2n_1} q^{\binom{n_1}{2}+(N+1)n_1} (q^{-N};q)_{n_1}}{(q;q)_{n_1}(tq;q)_{n_1}}.\end{aligned}$$

Consequently,

$$\mathcal{Z}_1(N;t,q) = \frac{1}{(tq;q)_N} \sum_{n_1\geq 0} \frac{t^{2n_1} q^{n_1^2}}{(q;q)_{N-n_1}(q;q)_{n_1}(tq;q)_{n_1}}. \tag{7.2}$$

Now recall a polynomial identity discovered by Paule [14, p. 272, eq. (43)]:

$$\sum_{n=-N}^{N} (-1)^n q^{2n^2} \begin{bmatrix}2N\\ N-n\end{bmatrix}_q = \frac{(q;q)_{2N}}{(q;q)_N} \sum_{n=0}^{N} \frac{q^{n^2}}{(-q;q)_n} \begin{bmatrix}N\\ n\end{bmatrix}_q.$$

Invoking (7.1) and (7.2) with $t=-1$, we have the following identity.

**Theorem 7.1.** *For any nonnegative integer $N$,*

$$\sum_{n\ge 0} q^{\binom{n+1}{2}}(-1;q)_n \begin{bmatrix} N \\ n \end{bmatrix}_q = \frac{1}{(q;q^2)_N} \sum_{n=-N}^{N} (-1)^n q^{2n^2} \begin{bmatrix} 2N \\ N-n \end{bmatrix}_q . \tag{7.3}$$

Remarkably, the above serves as a new finitization of a special case of the $q$-*Lebesgue sum* [4, p. 21, Corollary 2.7 with $a=-1$]:

$$\sum_{n\ge 0} \frac{q^{\binom{n+1}{2}}(-1;q)_n}{(q;q)_n} = (-q;q)_\infty(-q;q^2)_\infty .$$

Another finitization of this identity was discovered by Santos and Sills [19, p. 128, eq. (3.1)], while for the generic $q$-Lebesgue sum, we have witnessed finite analogs derived by Alladi and Berkovich [1, p. 803, eq. (1.15)] and Rowell [18, p. 786, eq. (1.5)].

## 8. Toward the $\mathrm{A}_1$-type sum in Theorem 1.7

Our objective in this part is to reduce $\mathcal{Z}_k(N;t,q)$ to the $\mathrm{A}_1$-type sum as recorded in Theorem 1.7. We start with the following result.

**Theorem 8.1.** *When $k\ge 2$, for any nonnegative integer $N$,*

$$\begin{aligned} &\mathcal{Z}_k(N;t,q) \\ &\quad = \frac{(t^2q;q)_\infty}{(tq;q)_\infty (q;q)_N (tq;q)_N} \sum_{n_1,\ldots,n_k\ge 0} (-1)^{n_k} t^{-n_k+\sum_{i=1}^k 2n_i} q^{-\binom{n_k}{2}+\sum_{i=1}^k n_i^2} \\ &\quad\quad \times \frac{(t;q)_{n_k}}{(q;q)_{n_k}(t^2q;q)_{N+n_1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_q . \end{aligned} \tag{8.1}$$

*Proof.* We recall (5.4) and mimic how (6.2) is derived so as to get the relation:

$$\begin{aligned} \mathcal{Z}_k(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2} &\sum_{m,n\ge 0} t^{m+2(k-1)n} q^{m^2+(m+1)n+(k-1)n^2} \\ &\times \frac{(t;q)_n^2}{(q;q)_{N-m}(t^2q;q)_{N+n}(q;q)_n} \mathcal{V}_{k-1}(m;tq^n,q). \end{aligned} \tag{8.2}$$

Now the term $\mathcal{V}_{k-1}(m;tq^n,q)$ can be replaced by means of (6.3). Then,

$$\mathcal{Z}_k(N;t,q) = \frac{(t^2q;q)_\infty}{(tq;q)_\infty} \sum_{m,n\ge 0} \frac{(-1)^n t^{m+n} q^{m^2+mn+\binom{n+1}{2}}(t;q)_n}{(q;q)_{N-m}(t^2q;q)_{N+n}(q;q)_m(q;q)_n}$$

$$\times \sum_{n_1,\ldots,n_{k-1}\geq 0} (-1)^{n_{k-1}+n} t^{-(n_{k-1}+n)+\sum_{i=1}^{k-1} 2(n_i+n)}$$
$$\times \frac{q^{-\binom{n_{k-1}+n}{2}+\sum_{i=1}^{k-1}(n_i+n)^2}(t;q)_{n_{k-1}+n}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(tq;q)_{m+(n_1+n)}}$$
$$= \frac{(t^2q;q)_\infty}{(tq;q)_\infty} \sum_{l_1,\ldots,l_{k-1}\geq 0} \frac{(-1)^{l_{k-1}} t^{-l_{k-1}+\sum_{i=1}^{k-1} 2l_i} q^{-\binom{l_{k-1}}{2}+\sum_{i=1}^{k-1} l_i^2}(t;q)_{l_{k-1}}}{(q;q)_{l_{k-1}-l_{k-2}}\cdots(q;q)_{l_2-l_1}}$$
$$\times \sum_{m,n\geq 0} \frac{(-1)^n t^{m+n} q^{m^2+mn+\binom{n+1}{2}}(t;q)_n}{(q;q)_{N-m}(t^2q;q)_{N+n}(q;q)_{l_1-n}(tq;q)_{l_1+m}(q;q)_m(q;q)_n}.$$

As long as we can show

$$\sum_{m,n\geq 0} \frac{(-1)^n t^{m+n} q^{m^2+mn+\binom{n+1}{2}}(t;q)_n}{(q;q)_{N-m}(t^2q;q)_{N+n}(q;q)_{l_1-n}(tq;q)_{l_1+m}(q;q)_m(q;q)_n}$$
$$= \frac{1}{(q;q)_N(tq;q)_N} \sum_{l_0\geq 0} \frac{t^{2l_0} q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(t^2q;q)_{N+l_0}}, \tag{8.3}$$

then (8.1) becomes valid. Note that

$$\text{LHS}\,(8.3) = \sum_{m\geq 0} \frac{t^m q^{m^2}}{(q;q)_{N-m}(q;q)_m(tq;q)_{l_1+m}} \sum_{n\geq 0} \frac{(-1)^n t^n q^{\binom{n}{2}+(m+1)n}(t;q)_n}{(q;q)_{l_1-n}(q;q)_n(t^2q;q)_{N+n}}.$$

For the inner sum over $n$, we have

$$\sum_{n\geq 0} \frac{(-1)^n t^n q^{\binom{n}{2}+(m+1)n}(t;q)_n}{(q;q)_{l_1-n}(q;q)_n(t^2q;q)_{N+n}}$$
$$= \frac{1}{(q;q)_{l_1}(t^2q;q)_N}\, {}_2\phi_1\left(\begin{matrix} t, q^{-l_1} \\ t^2q^{N+1} \end{matrix}; q, tq^{l_1+m+1}\right)$$
$$= \frac{1}{(q;q)_{l_1}(t^2q;q)_N} \frac{(t^2q^{N+l_1+1}, tq^{m+1};q)_\infty}{(t^2q^{N+1}, tq^{l_1+m+1};q)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{-(N-m)}, q^{-l_1} \\ tq^{m+1} \end{matrix}; q, t^2q^{N+l_1+1}\right)$$
$$= \frac{(t^2q^{N+l_1+1}, tq^{m+1};q)_\infty (q;q)_{N-m}}{(t^2q, tq^{l_1+m+1};q)_\infty}$$
$$\times \sum_{l_0\geq 0} \frac{t^{2l_0} q^{l_0^2+ml_0}}{(q;q)_{l_1-l_0}(q;q)_{N-m-l_0}(q;q)_{l_0}(tq^{m+1};q)_{l_0}},$$

where we have applied Heine's second transform (2.6). Hence,

$$
\begin{aligned}
\mathrm{LHS}\,(8.3) &= \frac{1}{(t^2q;q)_{N+l_1}} \sum_{l_0\ge 0} \frac{t^{2l_0}q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}} \\
&\quad \times \sum_{m\ge 0} \frac{t^m q^{m^2+l_0 m}}{(q;q)_{(N-l_0)-m}(q;q)_m(tq^{l_0+1};q)_m} \\
&= \frac{1}{(t^2q;q)_{N+l_1}} \sum_{l_0\ge 0} \frac{t^{2l_0}q^{l_0^2}}{(q;q)_{N-l_0}(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}} \\
&\quad \times \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix} 1/\tau, q^{-(N-l_0)} \\ tq^{l_0+1} \end{matrix}; q, tq^{N+1}\tau\right) \\
\text{(by (2.4))} &= \frac{1}{(t^2q;q)_{N+l_1}} \sum_{l_0\ge 0} \frac{t^{2l_0}q^{l_0^2}}{(q;q)_{N-l_0}(q;q)_{l_1-l_0}(q;q)_{l_0}(tq;q)_{l_0}} \frac{1}{(tq^{l_0+1};q)_{N-l_0}} \\
&= \frac{1}{(tq;q)_N(t^2q;q)_{N+l_1}} \frac{1}{(q;q)_N(q;q)_{l_1}} \\
&\quad \times \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix} q^{-N}, q^{-l_1} \\ t^2q\tau \end{matrix}; q, t^2q^{N+l_1+1}\right) \\
\text{(by (2.6))} &= \frac{1}{(tq;q)_N(t^2q;q)_{N+l_1}} \frac{1}{(q;q)_N(q;q)_{l_1}} \\
&\quad \times \lim_{\tau\to 0} \frac{(t^2q^{l_1+1}\tau, t^2q^{N+1};q)_\infty}{(t^2q\tau, t^2q^{N+l_1+1};q)_\infty} {}_2\phi_1\left(\begin{matrix} 1/\tau, q^{-l_1} \\ t^2q^{N+1} \end{matrix}; q, t^2q^{l_1+1}\tau\right) \\
&= \frac{1}{(q;q)_N(tq;q)_N} \sum_{l_0\ge 0} \frac{t^{2l_0}q^{l_0^2}}{(q;q)_{l_1-l_0}(q;q)_{l_0}(t^2q;q)_{N+l_0}},
\end{aligned}
$$

as requested. □

To relate the sum in Theorem 8.1 to that in Theorem 1.7, we require the following general result.

**Theorem 8.2.** *For any nonnegative integer $N$,*

$$
\begin{aligned}
&\sum_{n_1,\dots,n_k\ge 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(aq;q)_{N-n_k}(q;q)_{n_k}(tq;q)_{n_1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_q \\
&\qquad = \frac{(at^2q;q)_\infty}{(tq;q)_\infty(aq;q)_N} \sum_{n_1,\dots,n_k\ge 0} (-1)^{n_k} t^{-n_k+\sum_{i=1}^k 2n_i} q^{-\binom{n_k}{2}+\sum_{i=1}^k n_i^2}
\end{aligned}
$$

$$\times \frac{(t;q)_{n_k}}{(q;q)_{n_k}(at^2q;q)_{N+n_1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_q . \tag{8.4}$$

Before providing its proof, we refresh our memory of the connection between (7.1) and (7.2). What we have done is the identity

$$\frac{1}{(tq;q)_N} \sum_{n\ge 0} \frac{(-1)^n t^n q^{\binom{n+1}{2}} (t;q)_n}{(q;q)_{N-n}(q;q)_n} = \sum_{n\ge 0} \frac{t^{2n} q^{n^2}}{(q;q)_{N-n}(q;q)_n(tq;q)_n}.$$

Now we shall go slightly further.

**Lemma 8.3.** *For any nonnegative integers $L$, $M$ and $N$,*

$$\begin{aligned} \sum_{n\ge L} & \frac{(-1)^n t^{2n} q^{\binom{n+1}{2}} b^{-n} (b;q)_n}{(q;q)_{M-n}(q;q)_{n-L}(at^2q;q)_{N+n}} \\ &= \frac{(-1)^L q^{-\binom{L}{2}} b^{-L} (b;q)_L (aq;q)_{N-L} (b^{-1}t^2q;q)_M}{(at^2q;q)_{N+M}} \\ &\quad \times \sum_{n\ge L} \frac{t^{2n}q^{n^2}}{(aq;q)_{N-n}(q;q)_{M-n}(q;q)_{n-L}(b^{-1}t^2q;q)_n}. \end{aligned} \tag{8.5}$$

In (8.5), we may put $b = 1/\tau$ and take the limit at $\tau \to 0$:

$$\begin{aligned} \sum_{n\ge L} & \frac{t^{2n}q^{n^2}}{(q;q)_{M-n}(q;q)_{n-L}(at^2q;q)_{N+n}} \\ &= \frac{(aq;q)_{N-L}}{(at^2q;q)_{N+M}} \sum_{n\ge L} \frac{t^{2n}q^{n^2}}{(aq;q)_{N-n}(q;q)_{M-n}(q;q)_{n-L}}. \end{aligned}$$

We shall refer to this process as "*taking* $b = \infty$." Meanwhile, we may also take the limit at $M \to \infty$ in (8.5):

$$\begin{aligned} \sum_{n\ge L} & \frac{(-1)^n t^{2n} q^{\binom{n+1}{2}} b^{-n} (b;q)_n}{(q;q)_{n-L}(at^2q;q)_{N+n}} \\ &= \frac{(-1)^L q^{-\binom{L}{2}} b^{-L} (b;q)_L (aq;q)_{N-L} (b^{-1}t^2q;q)_\infty}{(at^2q;q)_\infty} \\ &\quad \times \sum_{n\ge L} \frac{t^{2n}q^{n^2}}{(aq;q)_{N-n}(q;q)_{n-L}(b^{-1}t^2q;q)_n}. \end{aligned}$$

This process will be read as "*taking* $M = \infty$."

*Proof of Lemma 8.3.* We have

$$
\begin{aligned}
&\mathrm{LHS}\,(8.5)\\
&= \frac{(-1)^L t^{2L} q^{\binom{L+1}{2}} b^{-L} (b;q)_L}{(at^2q;q)_{N+L}} \sum_{n\ge 0} \frac{(-1)^n t^{2n} q^{\binom{n+1}{2}+Ln} b^{-n} (bq^L;q)_n}{(q;q)_{(M-L)-n}(q;q)_n (at^2q^{N+L+1};q)_n}\\
&= \frac{(-1)^L t^{2L} q^{\binom{L+1}{2}} b^{-L} (b;q)_L}{(at^2q;q)_{N+L}(q;q)_{M-L}} {}_2\phi_1\left(\begin{matrix} bq^L, q^{-(M-L)} \\ at^2q^{N+L+1}\end{matrix}; q, b^{-1}t^2q^{M+1}\right)\\
\text{(by (2.6))} &= \frac{(-1)^L t^{2L} q^{\binom{L+1}{2}} b^{-L} (b;q)_L}{(at^2q;q)_{N+L}(q;q)_{M-L}} \frac{(at^2q^{N+M+1}, b^{-1}t^2q^{L+1};q)_\infty}{(at^2q^{N+L+1}, b^{-1}t^2q^{M+1};q)_\infty}\\
&\quad\times {}_2\phi_1\left(\begin{matrix} a^{-1}q^{-(N-L)}, q^{-(M-L)} \\ b^{-1}t^2q^{L+1}\end{matrix}; q, at^2q^{N+M+1}\right)\\
&= \frac{(-1)^L t^{2L} q^{\binom{L+1}{2}} b^{-L} (b;q)_L (aq;q)_{N-L} (b^{-1}t^2q;q)_M}{(at^2q;q)_{N+M}}\\
&\quad\times \sum_{n\ge 0} \frac{t^{2n} q^{n^2+2Ln}}{(aq;q)_{N-L-n}(q;q)_{M-L-n}(q;q)_n (b^{-1}t^2q;q)_{L+n}}\\
&= \frac{(-1)^L q^{-\binom{L}{2}} b^{-L} (b;q)_L (aq;q)_{N-L} (b^{-1}t^2q;q)_M}{(at^2q;q)_{N+M}}\\
&\quad\times \sum_{n\ge L} \frac{t^{2n} q^{n^2}}{(aq;q)_{N-n}(q;q)_{M-n}(q;q)_{n-L} (b^{-1}t^2q;q)_n},
\end{aligned}
$$

as desired. □

We are then in a position to prove Theorem 8.2.

*Proof of Theorem 8.2.* It is clear that the $k=1$ case of (8.4) is

$$
\begin{aligned}
&\sum_{n_1\ge 0} \frac{t^{2n_1} q^{n_1^2}}{(aq;q)_{N-n_1}(q;q)_{n_1}(tq;q)_{n_1}}\\
&\qquad = \frac{(at^2q;q)_\infty}{(tq;q)_\infty (aq;q)_N} \sum_{n_1\ge 0} \frac{(-1)^{n_1} t^{n_1} q^{\binom{n_1+1}{2}} (t;q)_{n_1}}{(q;q)_{n_1}(at^2q;q)_{N+n_1}};
\end{aligned}
$$

this is exactly (8.5) with $(b,L,M)=(t,0,\infty)$. Now we assume that $k\ge 2$ and begin with the right-hand side of (8.4) by singling out the sum over $n_1$:

RHS (8.4)

$$= \frac{(at^2q;q)_\infty}{(tq;q)_\infty (aq;q)_N} \sum_{n_2,\ldots,n_k\geq 0} \frac{(-1)^{n_k} t^{-n_k+\sum_{i=2}^k 2n_i} q^{-\binom{n_k}{2}+\sum_{i=2}^k n_i^2} (t;q)_{n_k}}{(q;q)_{n_k-n_{k-1}} \cdots (q;q)_{n_3-n_2}}$$
$$\times \sum_{n_1\geq 0} \frac{t^{2n_1} q^{n_1^2}}{(q;q)_{n_2-n_1}(q;q)_{n_1}(at^2q;q)_{N+n_1}}.$$

We then apply (8.5) with $(b,L,M)=(\infty,0,n_2)$ to this sum over $n_1$ to derive

$$\mathrm{RHS}\,(8.4)$$
$$= \frac{(at^2q;q)_\infty}{(tq;q)_\infty} \sum_{n_1\geq 0} \frac{t^{2n_1}q^{n_1^2}}{(q;q)_{n_1}} \sum_{n_3,\ldots,n_k\geq 0} \frac{(-1)^{n_k} t^{-n_k+\sum_{i=3}^k 2n_i} q^{-\binom{n_k}{2}+\sum_{i=3}^k n_i^2} (t;q)_{n_k}}{(q;q)_{n_k-n_{k-1}} \cdots (q;q)_{n_4-n_3}}$$
$$\times \frac{1}{(aq;q)_{N-n_1}} \sum_{n_2\geq n_1} \frac{t^{2n_2}q^{n_2^2}}{(q;q)_{n_3-n_2}(q;q)_{n_2-n_1}(at^2q;q)_{N+n_2}}.$$

We continue to use (8.5) with $(b,L,M)=(\infty,n_1,n_3)$ to this sum over $n_2$. In general, we sequentially apply (8.5) with $(b,L,M)=(\infty,n_{i-1},n_{i+1})$ to the sum over $n_i$ for $i=2,\ldots,k-1$. Thus,

$$\mathrm{RHS}\,(8.4) = \frac{(at^2q;q)_\infty}{(tq;q)_\infty} \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{t^{\sum_{i=1}^{k-1} 2n_i} q^{\sum_{i=1}^{k-1} n_i^2}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1}(q;q)_{n_1}}$$
$$\times \frac{1}{(aq;q)_{N-n_{k-1}}} \sum_{n_k\geq n_{k-1}} \frac{(-1)^{n_k} t^{n_k} q^{\binom{n_k+1}{2}} (t;q)_{n_k}}{(q;q)_{n_k-n_{k-1}}(at^2q;q)_{N+n_k}}.$$

For the sum over $n_k$, we apply (8.5) with $(b,L,M)=(t,n_{k-1},\infty)$ and get

$$\mathrm{RHS}\,(8.4) = \sum_{n_k\geq 0} \frac{t^{n_k}q^{n_k^2}}{(aq;q)_{N-n_k}(tq;q)_{n_k}}$$
$$\times \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{(-1)^{n_{k-1}} t^{-n_{k-1}+\sum_{i=1}^{k-1} 2n_i} q^{-\binom{n_{k-1}}{2}+\sum_{i=1}^{k-1} n_i^2} (t;q)_{n_{k-1}}}{(q;q)_{n_k-n_{k-1}} \cdots (q;q)_{n_2-n_1}(q;q)_{n_1}}. \tag{8.6}$$

Now we single out the sum over $n_{k-1}$:

$$\mathrm{RHS}\,(8.4) = \sum_{n_k\geq 0} \frac{t^{n_k}q^{n_k^2}}{(aq;q)_{N-n_k}} \sum_{n_1,\ldots,n_{k-2}\geq 0} \frac{t^{\sum_{i=1}^{k-2} 2n_i} q^{\sum_{i=1}^{k-2} n_i^2}}{(q;q)_{n_{k-2}-n_{k-3}} \cdots (q;q)_{n_2-n_1}(q;q)_{n_1}}$$
$$\times \frac{1}{(tq;q)_{n_k}} \sum_{n_{k-1}\geq n_{k-2}} \frac{(-1)^{n_{k-1}} t^{n_{k-1}} q^{\binom{n_{k-1}+1}{2}} (t;q)_{n_{k-1}}}{(q;q)_{n_k-n_{k-1}}(q;q)_{n_{k-1}-n_{k-2}}}.$$

We then utilize (8.5) with $(a,b,L,M)=(0,t,n_{k-2},n_k)$ to this sum over $n_{k-1}$. In general, we take turns applying (8.5) with $(a,b,L,M)=(0,t,n_{i-1},n_{i+1})$ to the sum over $n_i$ for $i=k-1,\dots,2$. Hence,

$$\begin{aligned}\text{RHS (8.4)} = \sum_{n_2,\dots,n_k\ge 0} &\frac{t^{\sum_{i=2}^k 2n_i} q^{\sum_{i=2}^k n_i^2}}{(aq;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_3-n_2}}\\ &\times \frac{1}{(tq;q)_{n_2}}\sum_{n_1\ge 0}\frac{(-1)^{n_1}t^{n_1}q^{\binom{n_1+1}{2}}(t;q)_{n_1}}{(q;q)_{n_2-n_1}(q;q)_{n_1}}.\end{aligned}$$

Finally, applying (8.5) with $(a,b,L,M)=(0,t,0,n_2)$ to the sum over $n_1$ yields the left-hand side of (8.4). □

Now Theorem 1.7 becomes clear.

*Proof of Theorem 1.7.* The $k=1$ case has been shown in (7.2). For $k\ge 2$, we recall (8.1) and use (8.4) with $a=1$. □

It is also notable that from (6.3), we may apply (8.4) with $a=t^{-1}$ to derive the following alternative expression for $\mathcal{V}_k(N;t,q)$.

**Theorem 8.4.** *For any nonnegative integer $N$,*

$$\begin{aligned}\mathcal{V}_k(N;t,q) = \frac{(tq;q)_\infty(t^{-1}q;q)_N}{(q;q)_\infty(q;q)_N} &\sum_{n_1,\dots,n_k\ge 0} t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}\\ &\times \frac{1}{(t^{-1}q;q)_{N-n_k}(q;q)_{n_k}(tq;q)_{n_1}} \begin{bmatrix} n_k\\ n_{k-1}\end{bmatrix}_q \begin{bmatrix} n_{k-1}\\ n_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} n_2\\ n_1\end{bmatrix}_q. \qquad (8.7)\end{aligned}$$

## 9. Theorem 1.2 revisited

As the first application of (1.23), we revisit Theorem 1.2, or more precisely, its finitization Theorem 4.1 concerning $\mathcal{Z}_k(N;1,q)$, and give an alternative proof.

*Second proof of Theorem 4.1.* It follows from (1.23) that

$$\mathcal{Z}_k(N;1,q) = \frac{1}{(q;q)_N}\sum_{n_1,\dots,n_k\ge 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}^2}.$$

Then we only need to apply (3.2) with $d_1=\cdots=d_k=0$ and $a=1$ to arrive at (4.3). □

## 10. Finitization of Theorems 1.5 and 1.6

For the second application of (1.23), we complete the proof of Theorems 1.5 and 1.6. To begin with, we need the following single-sum expression for the finite multisum $\mathcal{Z}_k(N;-1,q)$.

**Theorem 10.1.** *For any nonnegative integer $N$,*

$$\mathcal{Z}_k(N;-1,q)=\frac{1}{(q;q)_{2N}(-q;q)_N}\sum_{n=-N}^{N}(-1)^n q^{(k+1)n^2}\begin{bmatrix}2N\\N-n\end{bmatrix}_q. \tag{10.1}$$

*Proof.* In light of (1.23),

$$\begin{aligned}&\mathcal{Z}_k(N;-1,q)\\&\quad=\frac{1}{(-q;q)_N}\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q^2;q^2)_{n_1}}.\end{aligned}$$

Meanwhile, it is a standard result on $\mathrm{A}_1$ Rogers–Ramanujan type identities [21, p. 3] that

$$\begin{aligned}&\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q^2;q^2)_{n_1}}\\&\qquad=\frac{1}{(q;q)_{2N}}\sum_{n=-N}^{N}(-1)^n q^{(k+1)n^2}\begin{bmatrix}2N\\N-n\end{bmatrix}_q,\end{aligned}$$

which leads us to the claimed equality. □

The limiting case at $N\to\infty$ fills in the last piece of the puzzle.

*Proof of Theorems 1.5 and 1.6.* Recalling (5.4), we have

$$\begin{aligned}\mathrm{RHS}\,(1.19)&=(-q;q)_\infty^2\lim_{N\to\infty}\mathcal{Z}_k(N;-1,q)\\ \text{(by (10.1))}&=(-q;q)_\infty^2\cdot\frac{1}{(q;q)_\infty^2(-q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(k+1)n^2}\\ \text{(by (2.1))}&=\frac{(q^2;q^2)_\infty(q^{k+1};q^{k+1})_\infty^2}{(q;q)_\infty^3(q^{2k+2};q^{2k+2})_\infty};\end{aligned}$$

this is the left-hand side of (1.19). In the meantime, we know from (1.21) that

$$\mathcal{Z}_k(-1,q)=(q;q)_\infty\lim_{N\to\infty}\mathcal{Z}_k(N;-1,q),$$

and hence arrive at (1.18). □

## 11. Huang and Jiang's reflection formula

Our last episode revolves around Huang and Jiang's reflection formula in Theorem 1.8.

*Proof of Theorem 1.8.* In view of (1.23),

$$\begin{aligned}
&\mathcal{Z}_k(N;t^{-1}q^{-N},q)\\
&\quad=\frac{1}{(t^{-1}q^{1-N};q)_N}\sum_{n_1,\ldots,n_k\geq 0} t^{-\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k (n_i^2-2Nn_i)}\\
&\qquad\times\frac{1}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(t^{-1}q^{1-N};q)_{n_1}}.
\end{aligned}$$

Now we change the variables for $1\leq i\leq k$:

$$n_i\mapsto N-n_{k+1-i}.$$

Then,

$$\begin{aligned}
&\mathcal{Z}_k(N;t^{-1}q^{-N},q)\\
&\quad=\frac{t^{-2kN}q^{-kN^2}}{(t^{-1}q^{1-N};q)_N}\sum_{n_1,\ldots,n_k\geq 0} t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}\\
&\qquad\times\frac{1}{(t^{-1}q^{1-N};q)_{N-n_k}(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}},
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{(1-t)^2q^N(t^{2N}q^{N^2})^{k-1}}{(1-tq^N)^2}\mathcal{Z}_k(N;t^{-1}q^{-N},q)\\
&\quad=\frac{1}{(tq;q)_N^2}\sum_{n_1,\ldots,n_k\geq 0}\frac{(-1)^{n_k}t^{-n_k+\sum_{i=1}^k 2n_i}q^{-\binom{n_k}{2}+\sum_{i=1}^k n_i^2}(t;q)_{n_k}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}}.
\end{aligned}$$

Hence, our task becomes to show that

$$\begin{aligned}
&\frac{1}{(tq;q)_N}\sum_{n_1,\ldots,n_k\geq 0}\frac{(-1)^{n_k}t^{-n_k+\sum_{i=1}^k 2n_i}q^{-\binom{n_k}{2}+\sum_{i=1}^k n_i^2}(t;q)_{n_k}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}}\\
&\quad=\sum_{n_1,\ldots,n_k\geq 0}\frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(tq;q)_{n_1}}. \qquad (11.1)
\end{aligned}$$

For the left-hand side of (11.1), we single out the sum over $n_k$:

$$\mathrm{LHS}\,(11.1) = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{t^{\sum_{i=1}^{k-1} 2n_i} q^{\sum_{i=1}^{k-1} n_i^2}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots (q;q)_{n_2-n_1}(q;q)_{n_1}} \times \frac{1}{(tq;q)_N} \sum_{n_k\geq n_{k-1}} \frac{(-1)^{n_k} t^{n_k} q^{\binom{n_k+1}{2}} (t;q)_{n_k}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}}}.$$

Using (8.5) with $(a,b,L,M) = (0,t,n_{k-1},N)$ to this sum over $n_k$ implies that

$$\mathrm{LHS}\,(11.1) = \sum_{n_k\geq 0} \frac{t^{n_k} q^{n_k^2}}{(q;q)_{N-n_k}(tq;q)_{n_k}} \times \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{(-1)^{n_{k-1}} t^{-n_{k-1}+\sum_{i=1}^{k-1} 2n_i} q^{-\binom{n_{k-1}}{2}+\sum_{i=1}^{k-1} n_i^2} (t;q)_{n_{k-1}}}{(q;q)_{n_k-n_{k-1}}\cdots (q;q)_{n_2-n_1}(q;q)_{n_1}},$$

which is exactly the right-hand side of (8.6) with $a = 1$. Due to the equality between (8.6) and both sides of (8.4), the right-hand side of (11.1) becomes the final output. $\square$

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