THE KKS DETERMINANTS AND THEIR COMBINATORICS

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KKS determinants

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Determinant evaluations inspired by Di Francesco's determinant for twenty-vertex configurations ☆

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KKS determinants

$$\det_{0\le i,j\le n-1}\left(l^{i+b}\binom{mj+i+c}{mj+a}+\binom{mj-i+d}{mj+a}\right)$$

with $a, b, c, d, l, m \in \mathbb{Z}$ free parameters.

KKS determinants

Motivating Example: $$\det_{0\le i,j\le n-1}\left(2^i\binom{2j+i+1}{2j+1}+\binom{2j-i+1}{2j+1}\right)$$

- $\det\begin{pmatrix}2\end{pmatrix}=2$
- $\det\begin{pmatrix}2 & 2\\ 4 & 8\end{pmatrix}=8$
- $\det\begin{pmatrix}2 & 2 & 2\\ 4 & 8 & 12\\ 11 & 40 & 84\end{pmatrix}=120$
- $\det\begin{pmatrix}2 & 2 & 2 & 2\\ 4 & 8 & 12 & 16\\ 11 & 40 & 84 & 144\\ 30 & 160 & 448 & 960\end{pmatrix}=6656$
- $\det\begin{pmatrix}2 & 2 & 2 & 2 & 2\\ 4 & 8 & 12 & 16 & 20\\ 11 & 40 & 84 & 144 & 220\\ 30 & 160 & 448 & 960 & 1760\\ 77 & 559 & 2016 & 5280 & 11440\end{pmatrix}=1357824$

KKS determinants

Motivating Example: $\det\limits_{0\le i,j\le n-1}\left(2^i\binom{2j+i+1}{2j+1}+\binom{2j-i+1}{2j+1}\right)$

- $\det\begin{pmatrix}2\end{pmatrix}=2=2^1$
- $\det\begin{pmatrix}2 & 2\\ 4 & 8\end{pmatrix}=8=2^3$
- $\det\begin{pmatrix}2 & 2 & 2\\ 4 & 8 & 12\\ 11 & 40 & 84\end{pmatrix}=120=2^3*3^1*5^1$
- $\det\begin{pmatrix}2 & 2 & 2 & 2\\ 4 & 8 & 12 & 16\\ 11 & 40 & 84 & 144\\ 30 & 160 & 448 & 960\end{pmatrix}=6656=2^9*13^1$
- $\det\begin{pmatrix}2 & 2 & 2 & 2 & 2\\ 4 & 8 & 12 & 16 & 20\\ 11 & 40 & 84 & 144 & 220\\ 30 & 160 & 448 & 960 & 1760\\ 77 & 559 & 2016 & 5280 & 11440\end{pmatrix}=1357824=2^{11}*3^1*13^1*17^1$

KKS determinants

$$\det_{0\le i,j\le n-1}\left(2^i\binom{2j+i+1}{2j+1}+\binom{2j-i+1}{2j+1}\right)=2^{\frac{n(n-1)}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}$$

Important Note:

For the KKS determinants

$$\det_{0\le i,j\le n-1}\left(l^{i+b}\binom{mj+i+c}{mj+a}+\binom{mj-i+d}{mj+a}\right),$$

we are indeed computing the determinants of the **leading principal submatrices** of the *infinite* matrix

$$\left(l^{i+b}\binom{mj+i+c}{mj+a}+\binom{mj-i+d}{mj+a}\right)_{i,j\ge 0}.$$

Holonomic Ansatz

Setting:

Fix a (nice!) *infinite* matrix $A := (a_{i,j})_{i,j \geq 0}$ whose entries $a_{i,j}$ form holonomic sequences in the indices i and j.

We want to evaluate the determinants $\det(A_n)$ where the square matrices $A_n := (a_{i,j})_{0 \leq i,j < n}$ come from the leading principal submatrices of A.

Holonomic Ansatz

STEP 1:

Define the $(n-1, j)$-*th normalized cofactors* of A_n:

$$c_{n,j} := (-1)^{n-1+j} \frac{M_{n-1,j}}{M_{n-1,n-1}} \qquad (0 \leq j \leq n-1),$$

where $M_{i,j}$ is the (i,j)-*th minor* of A_n.

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Why called **"Holonomic Ansatz"**? — We *often* have, although *not universally* true, that the bivariate sequence $c_{n,j}$ is *holonomic*.

Holonomic Ansatz

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Why called **"Holonomic Ansatz"**? — We *often* have, although *not universally* true, that the bivariate sequence $c_{n,j}$ is *holonomic*.

In practice, we can compute a dataset of $c_{n,j}$ from the infinite matrix A, and take advantage of Kauers' *Mathematica* package `Guess` to predict a set of recurrences satisfied by $c_{n,j}$, which in turn, produces a *(left) Gröbner basis* "annc" of the *annihilators* for $c_{n,j}$.

Holonomic Ansatz

In[ ]:=
```
data = Table[2^k * Factorial[n], {n, 0, 5}, {k, 0, 5}];
TableForm@data
```

Out[ ]//TableForm=

1	2	4	8	16	32
1	2	4	8	16	32
2	4	8	16	32	64
6	12	24	48	96	192
24	48	96	192	384	768
120	240	480	960	1920	3840

In[ ]:=
```
GuessMultRE[data, {f[n, k], f[n + 1, k], f[n, k + 1]}, {n, k}, 1]
```

Out[ ]=
```
{-(1 + n) f[n, k] + f[1 + n, k], -2 n f[n, k] + n f[n, 1 + k], -2 k f[n, k] + k f[n, 1 + k], -2 f[n, k] + f[n, 1 + k]}
```

Holonomic Ansatz

```
In[•]:= start = CurrentDate[];

        MAX = 60;

        ClearAll[cc, n, j];

        guess = GuessMultRE[Table[Piecewise[{{matc[n, j], j ≤ n - 1}}, 0], {n, 1, MAX}, {j, 0, MAX - 1}],
           Flatten[Table[cc[n + l1, j + l2], {l1, 0, 3}, {l2, 0, 3}]], {n, j}, 5, StartPoint → {1, 0}, Constraints → (j < n)];
        Export["guess.txt", {guess}]

        Print["Time used: ", CurrentDate[] - start];

Out[•]= guess.txt

        Time used:  11.3257 s

In[•]:= guess = ToExpression[Import["guess.txt"]];

        start = CurrentDate[];

        annc = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[guess, cc[n, j]]];
        AnnInfo[annc]
        Export["annc.txt", {annc}]

        Print["Time used: ", CurrentDate[] - start];
```

ByteCount: 164120

Support: $\{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}$

degree $\{n, j\}$: $\{\{6, 10\}, \{5, 8\}, \{11, 11\}\}$

Standard Monomials: $\{1, S_j, S_n\}$

Holonomic Rank: 3

```
Out[•]= annc.txt

        Time used:  2.33624 min
```

Holonomic Ansatz

STEP 2:

It remains to show $c_{n,j}$ is indeed annihilated by this **guessed** "annc".

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- By definition,

$$c_{n,n-1} = (-1)^{n-1+n-1}\frac{M_{n-1,n-1}}{M_{n-1,n-1}} = 1 \qquad (n \geq 1); \tag{H1}$$

- By the Laplace expansion with respect to the i-th row of A_n,

$$\Sigma_{i,n} := \sum_{j=0}^{n-1} a_{i,j}c_{n,j} = 0 \qquad (0 \leq i < n-1). \tag{H2}$$

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Important Note: If a bivariate sequence $\tilde{c}_{n,j}$ satisfies the relations (H1) and (H2), then it is uniquely determined under the *a priori* assumption that each A_n has full rank, or equivalently, $\det(A_n) \neq 0$.

STEP 2 (continued...):

Let $c'_{n,j}$ be produced under the annihilation of the guessed "annc" together with the initial values from $c_{n,j}$. We have to show that $c'_{n,j}$ satisfies the relations (H1) and (H2). Once this is done, we can safely argue that

$$c_{n,j} = c'_{n,j}$$

so that $c_{n,j}$ is annihilated by the basis "annc".

Holonomic Ansatz

STEP 2(H1):

We produce an annihilator for $c'_{n,n-1}$ from "annc". Then we examine that this annihilator admits a constant sequence as its solution and then check certain initial values.

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Koutschan's *Mathematica* package `HolonomicFunctions`:

```
In[ ]:= start = CurrentDate[];

        ClearAll[n, j];
        cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]];

        Print["Time used: ", CurrentDate[] - start];

        Time used:  2.04081 s
```

STEP 2(H2):

We apply the method of *creative telescoping* to the summand $a_{i,j}c'_{n,j}$ to get an annihilating basis for the summation

$$\Sigma'_{i,n} := \sum_{j=0}^{n-1} a_{i,j}c'_{n,j}.$$

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$$\Sigma'_{i,n} := \sum_{j=0}^{n-1} a_{i,j}c'_{n,j}.$$

Then we check certain initial values.

```
ClearAll[n, j, i];

annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
annH2Smnd1 = DFiniteTimesHyper[annci, mata1[i, j]];
annH2CT1No1 = FindCreativeTelescoping[annH2Smnd1, S[j] - 1];
```

STEP 3:

Once we have proven the annihilators annc for $c_{n,j}$, we apply creative telescoping to get a recurrence for the univariate sequence

$$\Sigma_{n-1,n} := \sum_{j=0}^{n-1} a_{n-1,j} c_{n,j} \qquad (n \geq 1).$$

Holonomic Ansatz

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BUT wait... What's $\Sigma_{n-1,n}$?

By the Laplace expansion with respect to the last row of A_n,

$$\sum_{j=0}^{n-1} a_{n-1,j} c_{n,j} = \frac{\det(A_n)}{\det(A_{n-1})} \qquad (n \geq 1). \tag{H3}$$

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We finally check that the potential *product formula* for $\det(A_n)$ (or more precisely, the ratio $\det(A_n)/\det(A_{n-1})$ from the product formula) meets the recurrences for $\Sigma_{n-1,n}$.

Two conjectures in KKS

Let $G_{a,b,c,d}(n) := \det_{0\le i,j\le n-1}\left(2^{i+b}\binom{4j+i+c}{4j+a}+\binom{4j-i+d}{4j+a}\right).$

Conjecture 23. *The following determinant evaluations hold for all $n \geq 1$:*

$$G_{0,2,3,-1}(n)=\prod_{i=1}^{n}\frac{(2i-1)(4i-3)(4i-1)\,\Gamma(6i)\,\Gamma\left(\frac{i+3}{4}\right)}{i(i+1)(i+2)(3i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i+3}{4}\right)},\tag{10.5}$$

$$G_{1,3,6,0}(n)=\prod_{i=1}^{n}\frac{8(2i-1)(2i+1)^2(4i-1)(4i+1)\,\Gamma(6i+2)\,\Gamma\left(\frac{i+2}{4}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(5i+2)\,\Gamma\left(\frac{5i+6}{4}\right)},\tag{10.6}$$

$$G_{1,1,0,-2}(n)=-4\prod_{i=1}^{n}\frac{(3i-2)\,\Gamma(6i-5)\,\Gamma\left(\frac{i}{4}\right)}{8\,\Gamma(5i-4)\,\Gamma\left(\frac{5i}{4}\right)},\tag{10.7}$$

$$G_{3,0,3,3}(n)=2\prod_{i=1}^{n}\frac{\Gamma(6i-1)\,\Gamma\left(\frac{i+3}{4}\right)}{\Gamma(5i)\,\Gamma\left(\frac{5i-1}{4}\right)},\tag{10.8}$$

$$G_{2,1,2,0}(n)=\prod_{i=1}^{n}\frac{\Gamma(6i-1)\,\Gamma\left(\frac{i+2}{4}\right)}{2(2i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i-2}{4}\right)}.\tag{10.9}$$

Moreover, the following identities are conjectured to hold for all $n \geq 3$:

$$\begin{aligned}G_{3,0,3,3}(n)&=\frac{2}{3}G_{0,1,-2,-4}(n+1)=-\frac{1}{672}G_{1,3,-2,-8}(n+1)\\&=\frac{1}{63}G_{5,4,3,-5}(n)=\frac{4}{1002001}G_{6,6,3,-9}(n)=-\frac{8}{5}G_{9,5,8,-2}(n-1),\end{aligned}\tag{10.10}$$

$$G_{1,1,0,-2}(n)=-\frac{1}{49}G_{2,3,0,-6}(n)=-\frac{2}{7}G_{6,4,5,-3}(n-1)=-\frac{4}{5577}G_{7,6,5,-7}(n-1),\tag{10.11}$$

$$G_{2,1,2,0}(n)=2\,G_{7,4,7,-1}(n-1).\tag{10.12}$$

Two conjectures in KKS

Theorem (Chen–C.–Yoshida, 2025)

For all $n \geq 1$,

$$\det_{0\leq i,j\leq n-1}\left(2^{i+1}\binom{4j+i+2}{4j+2}+\binom{4j-i}{4j+2}\right)=\prod_{i=1}^{n}\frac{\Gamma(6i-1)\Gamma(\frac{i+2}{4})}{2(2i-1)\Gamma(5i-1)\Gamma(\frac{5i-2}{4})},$$

and

$$\det_{0\leq i,j\leq n-1}\left(2^{i}\binom{4j+i+3}{4j+3}+\binom{4j-i+3}{4j+3}\right)=2\prod_{i=1}^{n}\frac{\Gamma(6i-1)\Gamma(\frac{i+3}{4})}{\Gamma(5i)\Gamma(\frac{5i-1}{4})}.$$

Two conjectures in KKS

KKS, p. 18:

Table 1
Computational data from the proofs by holonomic Ansatz; if there is more than a single determinant in a theorem, the range of values over all instances is displayed; if in such a situation only a single value appears, it means that all instances had the same value. `ByteCount` refers to the homonymous *Mathematica* command (applied to the final annihilator, not the intermediate creative telescoping results) and the values are given in MB. All timings are given in hours.

Theorem		3	4	6	8	10	12	13	14	15	23
$\mathfrak{I}$	hol. rank	2	3	3	3	3	4	4	3	3	5
	degree in n	2	11	14	14	10–14	7–8	7	5	5	14–19
	degree in j	4	11	12	12	7–11	8–17	9–11	5–8	9	10–15
	`ByteCount`	0.03	0.16	2.12	1.94	0.08–0.17	0.06–0.23	0.26–0.51	0.03–0.05	0.26	0.24–0.52
(H1)	order of rec.	2	3	3	3	3	4	4	3	3	5
	degree in n	3	15	25	25	12–21	18–26	21–23	6–10	11	49–59
(H2)	time 1. sum	0.006	0.67	-	-	0.18–0.74	2.26–24.6	-	0.01–0.03	2.54	-
	time 2. sum	0.009	0.71	-	-	0.19–0.83	3.36–20.8	-	0.01–0.04	7.18	-
	hol. rank	3	6	-	-	6	10	-	7	7	-
	`ByteCount`	0.03	1.32	-	-	0.71–1.62	0.47–2.51	-	0.04–0.19	2.08	-
(H3)	time 1. sum	0.005	0.77	21.7	17.0	0.47–0.92	7.39–12.2	-	0.02–0.04	0.35	-
	time 2. sum	0.011	0.6	-	65.7	0.35–0.71	3.81–8.42	-	0.01–0.03	0.97	-
	order of rec.	2	6	-	6	6	10	-	5	5	-
	degree in n	1	52	-	75	45–57	74–93	-	14–22	24	-

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	`ByteCount`	0.03	0.16	2.12	1.94	0.08–0.17	0.06–0.23	0.26–0.51	0.03–0.05	0.26	0.24–0.52
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	degree in n	3	15	25	25	12–21	18–26	21–23	6–10	11	49–59
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	hol. rank	3	6	-	-	6	10	-	7	7	-
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	order of rec.	2	6	-	6	6	10	-	5	5	-
	degree in n	1	52	-	75	45–57	74–93	-	14–22	24	-

In *practice*, we are not able to compute these annihilators *directly* by the "FindCreativeTelescoping" command offered in the `HolonomicFunctions` package within a reasonable time frame.

Modular reduction

Random Walk on multiheaded lattices:

IOP Publishing

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **57** (2024) 465204 (21pp)

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Multi-headed lattices and Green functions

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Abstract

Lattice geometries and random walks on them are of great interest for their applications in different fields such as physics, chemistry, and computer science. In this work, we focus on multi-headed lattices and study properties of the Green functions for these lattices such as the associated differential equations and the Pólya numbers. In particular, we complete the analysis of three missing cases in dimensions no larger than five. Our results are built upon an automatic machinery of creative telescoping.

Modular reduction

Random Walk on multiheaded lattices and ... **Random Work** on clam collecting:

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **57** (2024) 465204 (21pp) https://doi.org/10.1088/1751-8121/ad8791

Multi-headed lattices and Green functions

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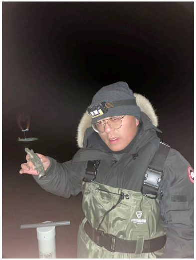

Modular reduction

(H2):

Creative telescoping: Goal is to find annihilators for the bivariate sequence

$$\Sigma_{i,n} = \sum_{j=0}^{n-1} a_{i,j} c_{n,j}.$$

Modular reduction

Step 1: For the summand $a_{i,j}c_{n,j}$, find annihilators of the form:

$$\boldsymbol{P} + (S_j - 1)\boldsymbol{Q} := \underbrace{\sum P_{\alpha_i,\alpha_n}(i,n) S_i^{\alpha_i} S_n^{\alpha_n}}_{\text{telescoper part}} + (S_j - 1) \underbrace{\sum \frac{\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)\, j^{\mu}}{D_{\beta_i,\beta_n,\nu}(i,n;j)} S_i^{\beta_i} S_n^{\beta_n} S_j^{\nu}}_{\text{delta part}}.$$

Modular reduction

Step 1: For the summand $a_{i,j}c_{n,j}$, find annihilators of the form:

$$\boldsymbol{P} + (S_j - 1)\boldsymbol{Q} := \underbrace{\sum P_{\alpha_i,\alpha_n}(i,n) S_i^{\alpha_i} S_n^{\alpha_n}}_{\text{telescoper part}} + (S_j - 1) \underbrace{\sum \frac{\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)\, j^{\mu}}{D_{\beta_i,\beta_n,\nu}(i,n;j)} S_i^{\beta_i} S_n^{\beta_n} S_j^{\nu}}_{\text{delta part}}.$$

WHY $\boldsymbol{P} + (S_j - 1)\boldsymbol{Q}$?

Modular reduction

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WHY $\boldsymbol{P} + (S_j - 1)\boldsymbol{Q}$?

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1} \big(\boldsymbol{P} + (S_j - 1)\boldsymbol{Q}\big) \circ a_{i,j}c_{n,j} \\ &= \boldsymbol{P} \circ \sum_{j=0}^{n-1} a_{n-1,j}c_{n,j} + \underbrace{\left(\Big[\boldsymbol{Q} \circ a^{(1)}_{n-1,j}c_{n,j}\Big]_{j=n} - \Big[\boldsymbol{Q} \circ a^{(1)}_{n-1,j}c_{n,j}\Big]_{j=0} \right)}_{\text{usually equals } 0} \end{aligned}$$

$\Rightarrow$ The telescoper part $\boldsymbol{P}$ usually annihilates the sum $\Sigma_{i,n}$.

Modular reduction

Step 1 (continued...): Some facts about Koutschan's "FindCreativeTelescoping" command:

- The denominators $D_{\beta_i,\beta_n,\nu}(i,n;j)$ in the delta part can be automatically predicted and computed;
- KKS also computed $P_{\alpha_i,\alpha_n}(i,n)$ and $\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)$ automatically.

Modular reduction

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What if the summand $a_{i,j}c_{n,j}$ is so complicated that the automatic computation for $P_{\alpha_i,\alpha_n}(i,n)$ and $\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)$ fails to work?

Modular reduction

Step 2: Search for a possible support for the telescoper part:

$$\mathrm{Supp}^{\dagger}_{\mathrm{H}_2} := \{S_i^5, S_i^4 S_n, S_i^3 S_n^2, S_i^2 S_n^3, S_i S_n^4, S_n^5, S_i^4, S_i^3 S_n, S_i^2 S_n^2, S_i S_n^3, S_n^4, S_i^3, S_i^2 S_n, S_i S_n^2, S_n^3, S_i^2, S_i S_n, S_n^2, S_i, S_n, 1\}.$$

Under this *Ansatz*, the shape of the delta part and the associated denominator polynomials D can be decided.

Modular reduction

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Under this *Ansatz*, the shape of the delta part and the associated denominator polynomials D can be decided.

It remains to formulate each $P_{\alpha_i,\alpha_n}(i,n)$ and $\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)$.

Modular reduction

Step 2 (continued...): We may still easily use Koutschan's "FindCreativeTelescoping" command to compute

$$P(i,n) \text{ or } \tilde{Q}(i,n) \bmod p$$

for **fixed** i and n, and primes p.

Modular reduction

Step 2 (continued...): We may still easily use Koutschan's "FindCreativeTelescoping" command to compute

$$P(i, n) \text{ or } \tilde{Q}(i, n) \bmod p$$

for **fixed** i and n, and primes p.

Choose a set of i-values, say $\{i_1, \dots, i_K\}$, a set of n-values, say $\{n_1, \dots, n_L\}$, and a set of *large* primes, say $\{p_1, \dots, p_M\}$, and compute a dataset of $K \times L \times M$ interpolated values:

$$P(i_k, n_l) \text{ or } \tilde{Q}(i_k, n_l) \bmod p_m.$$

Modular reduction

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(In practice, we choose $K = 160$ different i-values, $L = 528$ different n-values, and $M = 40$ different large primes of size around 2^{32}. So over 3 million interpolations!)

Modular reduction

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(In practice, we choose $K=160$ different i-values, $L=528$ different n-values, and $M=40$ different large primes of size around 2^{32}. So over 3 million interpolations!)

We finally reconstruct the rational functions $P_{\alpha_i,\alpha_n}(i,n)$ and $\tilde{Q}_{\beta_i,\beta_n,\mu,\nu}(i,n)$ from the datasets.

Modular reduction

Parallel Computation: Amazon cluster (AWS EC2, instance type: m5.24xlarge — 96 CPUs, 384 GB Memory)

Modular reduction

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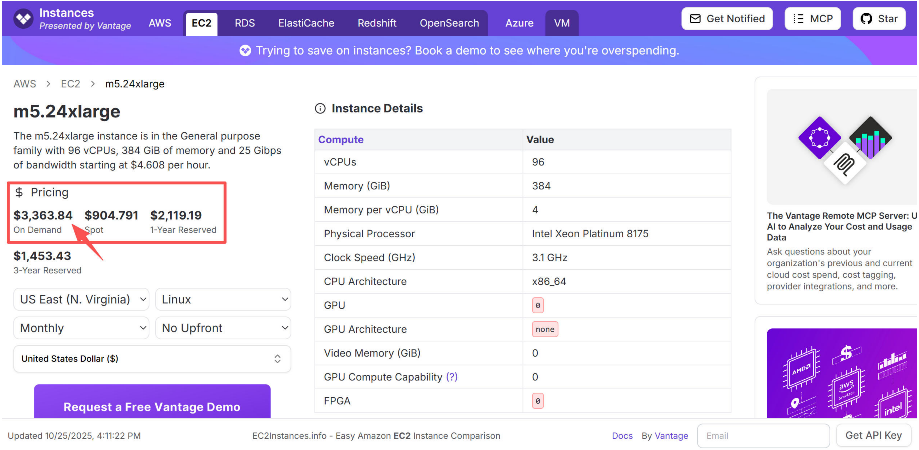

Modular reduction

Det. (5.1)	(H2), #1		(H2), #2		(H3)	
	Smnd 1	Smnd 2	Smnd 1	Smnd 2	Smnd 1	Smnd 2
Telescoper part						
Support	$\mathrm{Supp}^{\dagger}_{\mathrm{H}_2}$ in (5.10)		$\mathrm{Supp}^{\ddagger}_{\mathrm{H}_2}$ in (5.11)		$\mathrm{Supp}_{\mathrm{H}_3}$ in (5.8)	
# of paras.	21	21	21	21	21	21
ByteCount	27 393 808	27 393 808	21 764 496	20 869 424	6 102 528	5 638 048
Delta part						
# of paras.	148	166	123	137	691	595
ByteCount	1 777 395 424	2 657 465 696	1 033 241 824	1 558 607 768	3 182 669 576	4 665 533 376

Modular reduction

Det. (5.1)	(H2), #1		(H2), #2		(H3)	
	Smnd 1	Smnd 2	Smnd 1	Smnd 2	Smnd 1	Smnd 2
Telescoper part						
Support	$\mathrm{Supp}^{\dagger}_{\mathrm{H}_2}$ in (5.10)		$\mathrm{Supp}^{\ddagger}_{\mathrm{H}_2}$ in (5.11)		$\mathrm{Supp}_{\mathrm{H}_3}$ in (5.8)	
# of paras.	21	21	21	21	21	21
ByteCount	27 393 808	27 393 808	21 764 496	20 869 424	6 102 528	5 638 048
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Q.E.D.!

BUT WAIT

$$\vdots$$

BUT WAIT

$\vdots$

This is

AG
Diskrete Mathematik

Di Francesco's observation

Lattice Paths, Combinatorics and Interactions
Marches aléatoires, combinatoire et interactions

21 - 25 June 2021 (Hybrid conference)

KKS, p. 3:

C. Koutschan, C. Krattenthaler and M.J. Schlosser *Journal of Symbolic Computation 127 (2025) 102352*

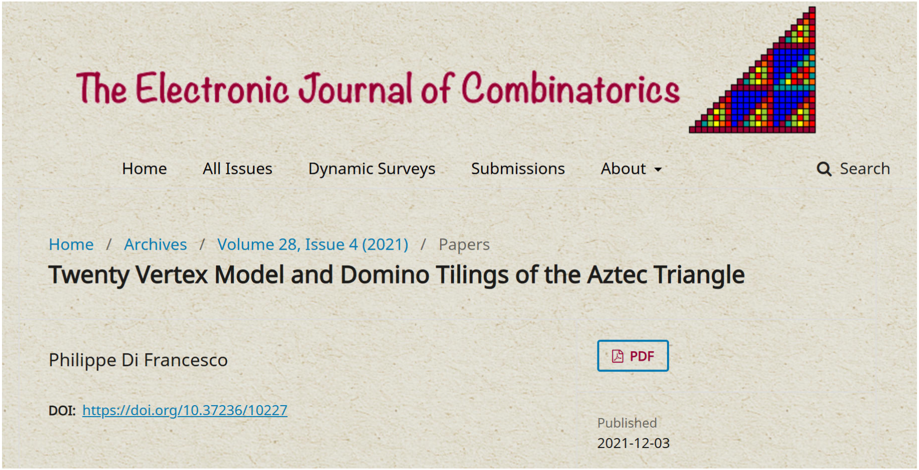
```
Dear Christoph,

Philippe Di Francesco just gave a great talk at the Lattice path con-
ference mentioning, inter alia, a certain conjectured determinant. It
is
Conj. 8.1 (combined with Th. 8.2) in
https://arxiv.org/pdf/2102.02920.pdf

I am curious if you can prove it by the Koutschan-Zeilberger-Aek holo-
nomic ansatz method. If you can do it before Friday, June 25, 2021,
17:00 Paris time, I will mention it in my talk in that conference.

Best wishes
        Doron
```

Di Francesco's observation



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Twenty Vertex Model and Domino Tilings of the Aztec Triangle

Philippe Di Francesco

PDF

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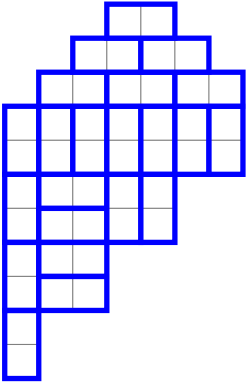
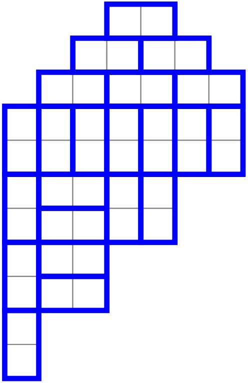
Published
2021-12-03

Di Francesco's observation



A domino tiling of the Aztec triangle $\mathfrak{A}(4)$

Di Francesco's observation



A domino tiling of the Aztec triangle $\mathfrak{A}(4)$

Let $\text{Д}(n)$ count the number of domino tilings of $\mathfrak{A}(n)$.

Di Francesco observed that

$$\text{Д}(n) \stackrel{?}{=} 2^{\frac{n(n-1)}{2}} \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!}.$$

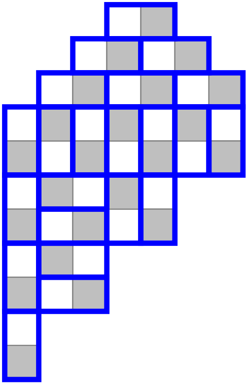
What Di Francesco was able to show is that

$$\text{Д}(n) = \frac{1}{2} \det_{0\le i,j<n} \left(2^i \binom{2j+i+1}{2j+1} - \binom{i-1}{2j+1} \right).$$

Meanwhile, noting that $\binom{2j-i+1}{2j+1} = -\binom{i-1}{2j+1}$,

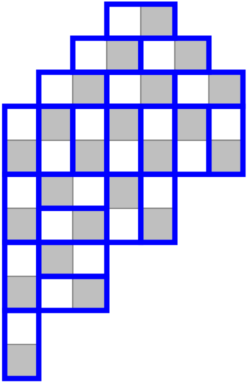
$$\text{Д}(n) = \frac{1}{2} \det_{0\le i,j<n} \left(2^i \binom{2j+i+1}{2j+1} + \binom{2j-i+1}{2j+1} \right).$$

Weighted counting

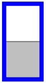


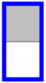
A domino tiling of the Aztec triangle $\mathfrak{A}(4)$

Weighted counting



A domino tiling of the Aztec triangle $\mathfrak{A}(4)$

$\mathfrak{D}_1 =$  $\dashrightarrow w_1$

$\mathfrak{D}_2 =$  $\dashrightarrow w_2$

$\mathfrak{D}_3 =$  $\dashrightarrow w_3$

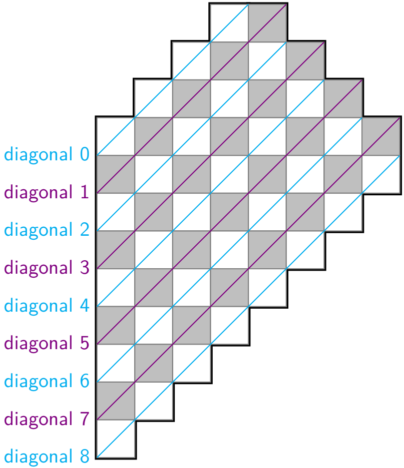
$\mathfrak{D}_4 =$  $\dashrightarrow 1$

$$\boxed{\sum_{T} w_1^{\#_T(\mathfrak{D}_1)} w_2^{\#_T(\mathfrak{D}_2)} w_3^{\#_T(\mathfrak{D}_3)}}$$

Weighted counting

Prototypical domains $\mathfrak{A}_{\mathrm{D}}(M; N)$:

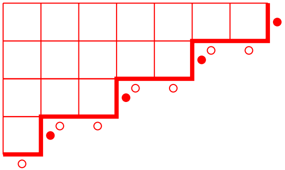
$\mathfrak{A}_{\mathrm{D}}(4; 8)$:



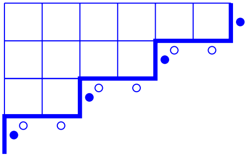
Weighted counting

Boundary encoding of an integer partition λ:

$(7,5,3,1)$



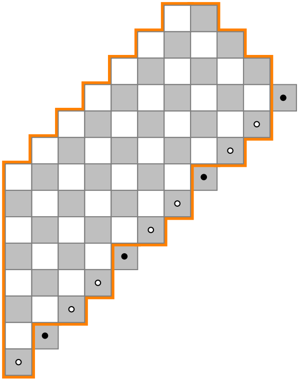
$(6,4,2,0)$



$(7,5,3,1) \xrightarrow[\text{encoding}]{\text{boundary}}$ ○ ● ○ ○ ● ○ ○ ● ○ ○ ● $(6,4,2,0) \xrightarrow[\text{encoding}]{\text{boundary}}$ ● ○ ○ ● ○ ○ ● ○ ○ ●

Weighted counting

Aztec-type domain $\mathfrak{A}_\lambda(n)$ (of Type 1) marked by λ:



$\mathfrak{A}_{(7,5,3,1)}(4)$

Weighted counting

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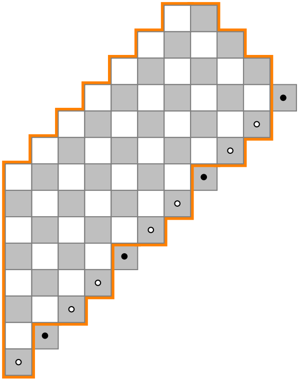

$\mathfrak{A}_{(7,5,3,1)}(4)$

- Start with the prototypical domain $\mathfrak{A}_{\mathrm{D}}(\lambda_1; 2n-1)$.

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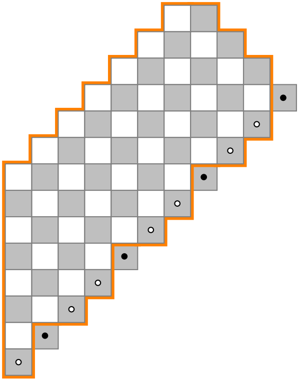

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- Start with the prototypical domain $\mathfrak{A}_{\mathrm{D}}(\lambda_1; 2n-1)$.
- There are $\lambda_1 + n$ cells in the last diagonal;

Weighted counting

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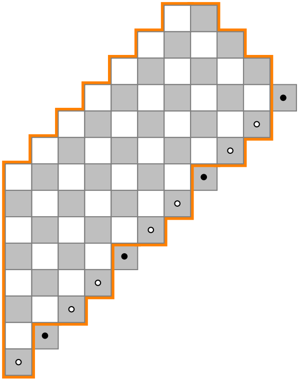

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- Start with the prototypical domain $\mathfrak{A}_{\mathrm{D}}(\lambda_1; 2n-1)$.
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  There are $\lambda_1 + n$ circles/bullets in the boundary encoding of λ.

Weighted counting

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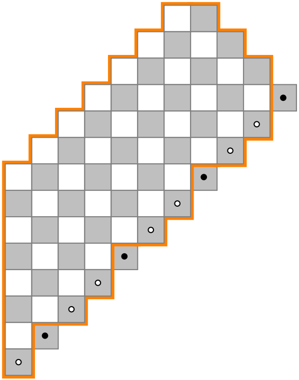

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- There are $\lambda_1 + n$ cells in the last diagonal;
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  Fill these circles and bullets into the last diagonal of the prototypical domain from left-bottom to right-top.

Weighted counting

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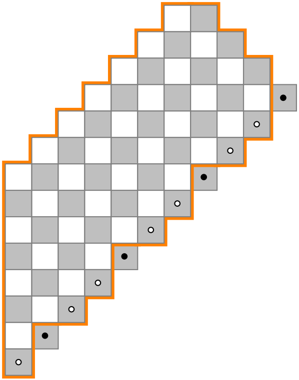
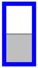
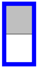


$\mathfrak{A}_{(7,5,3,1)}(4)$

- Start with the prototypical domain $\mathfrak{A}_{\mathrm{D}}(\lambda_1; 2n-1)$.
- There are $\lambda_1 + n$ cells in the last diagonal;
  There are $\lambda_1 + n$ circles/bullets in the boundary encoding of λ.
  Fill these circles and bullets into the last diagonal of the prototypical domain from left-bottom to right-top.
- Remove the cells marked by a **bullet**.

Weighted counting

Recall

$\mathfrak{D}_1 =$ , $\mathfrak{D}_2 =$ , $\mathfrak{D}_3 =$ , $\mathfrak{D}_4 =$ .

Consider the weighted enumeration

$$\mathit{Д}^{\lambda}_{w_1,w_2,w_3}(n) := \sum_{T\in\mathcal{D}_\lambda(n)} w_1^{\#_T(\mathfrak{D}_1)} w_2^{\#_T(\mathfrak{D}_2)} w_3^{\#_T(\mathfrak{D}_3)}.$$

Weighted counting

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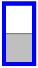
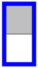
$\mathfrak{D}_1 =$ , $\mathfrak{D}_2 =$ , $\mathfrak{D}_3 =$ , $\mathfrak{D}_4 =$ .

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$$\text{Д}^{\lambda}_{w_1,w_2,w_3}(n) := \sum_{T \in \mathscr{D}_{\lambda}(n)} w_1^{\#_T(\mathfrak{D}_1)} w_2^{\#_T(\mathfrak{D}_2)} w_3^{\#_T(\mathfrak{D}_3)}.$$

We are in particular interested in partitions that form an **arithmetic progression**:

$$\lambda^{(s,r)} := (s(n-1)+r, s(n-2)+r, \ldots, r),$$

Write

$$\text{Д}^{(s,r)}_{w_1,w_2,w_3}(n) := \text{Д}^{\lambda^{(s,r)}}_{w_1,w_2,w_3}(n) = \sum_{T \in \mathscr{D}^{(s,r)}(n)} w_1^{\#_T(\mathfrak{D}_1)} w_2^{\#_T(\mathfrak{D}_2)} w_3^{\#_T(\mathfrak{D}_3)}.$$

Weighted counting

Theorem (Chen–C.–Yoshida, 2025)

Let

$$\text{Б}^{(m,l)}_{a,b,c,d}(n) := \det_{0\le i,j\le n-1}\left(l^{j+b}\binom{mi+j+c}{mi+a}+\binom{mi-j+d}{mi+a}\right).$$

Then for all $n \geq 1$,

$$\text{Д}^{(m-1,a)}_{1,l-1,1}(n) = \tfrac{1}{2}\text{Б}^{(m,l)}_{a,0,a,a}(n).$$

Weighted counting

Corollary (Chen–C.–Yoshida, 2025)

$$Д_{1,1,1}^{(1,0)}(n) = \tfrac{1}{2}Б_{0,0,0,0}^{(2,2)}(n) = 2^{n(n-1)} \prod_{i=1}^{n} \frac{\Gamma(i)\Gamma(2i)\Gamma(4i-3)\Gamma(\frac{3i-1}{2})}{\Gamma(2i-1)\Gamma(3i-1)\Gamma(3i-2)\Gamma(\frac{i+1}{2})},$$

$$Д_{1,1,1}^{(1,1)}(n) = \tfrac{1}{2}Б_{1,0,1,1}^{(2,2)}(n) = 2^{n(n-1)} \prod_{i=1}^{n} \frac{\Gamma(i)\Gamma(4i-1)\Gamma(\frac{3i-2}{2})}{\Gamma(3i)\Gamma(3i-2)\Gamma(\frac{i}{2})},$$

$$Д_{1,3,1}^{(1,1)}(n) = \tfrac{1}{2}Б_{1,0,1,1}^{(2,4)}(n) = 3^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n} \frac{\Gamma(3i-1)\Gamma(\frac{i+1}{2})}{\Gamma(2i)\Gamma(\frac{3i-1}{2})},$$

$$Д_{1,2,1}^{(2,0)}(n) = \tfrac{1}{2}Б_{0,0,0,0}^{(3,3)}(n) = 2^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n} \frac{\Gamma(4i-3)\Gamma(\frac{i+1}{3})}{\Gamma(3i-2)\Gamma(\frac{4i-2}{3})},$$

$$Д_{1,2,1}^{(2,1)}(n) = \tfrac{1}{2}Б_{1,0,1,1}^{(3,3)}(n) = 2^{\frac{1}{2}n(n-3)}3^{-n} \prod_{i=1}^{n} \frac{\Gamma(4i-1)\Gamma(\frac{i}{3})}{\Gamma(3i-1)\Gamma(\frac{4i}{3})},$$

$$Д_{1,2,1}^{(2,2)}(n) = \tfrac{1}{2}Б_{2,0,2,2}^{(3,3)}(n) = 2^{\frac{1}{2}n(n-5)} \prod_{i=1}^{n} \frac{\Gamma(4i+1)\Gamma(\frac{i+2}{3})}{\Gamma(3i+1)\Gamma(\frac{4i+2}{3})},$$

$$Д_{1,1,1}^{(3,3)}(n) = \tfrac{1}{2}Б_{3,0,3,3}^{(4,2)}(n) = \prod_{i=1}^{n} \frac{\Gamma(6i-1)\Gamma(\frac{i+3}{4})}{\Gamma(5i)\Gamma(\frac{5i-1}{4})}.$$

Weighted counting

Conjecture 23. *The following determinant evaluations hold for all $n \geq 1$:*

$$G_{0,2,3,-1}(n) = \prod_{i=1}^{n} \frac{(2i-1)(4i-3)(4i-1)\,\Gamma(6i)\,\Gamma\left(\frac{i+3}{4}\right)}{i(i+1)(i+2)(3i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i+3}{4}\right)}, \tag{10.5}$$

$$G_{1,3,6,0}(n) = \prod_{i=1}^{n} \frac{8(2i-1)(2i+1)^2(4i-1)(4i+1)\,\Gamma(6i+2)\,\Gamma\left(\frac{i+2}{4}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(5i+2)\,\Gamma\left(\frac{5i+6}{4}\right)}, \tag{10.6}$$

$$G_{1,1,0,-2}(n) = -4\prod_{i=1}^{n} \frac{(3i-2)\,\Gamma(6i-5)\,\Gamma\left(\frac{i}{4}\right)}{8\,\Gamma(5i-4)\,\Gamma\left(\frac{5i}{4}\right)}, \tag{10.7}$$

$$G_{3,0,3,3}(n) = 2\prod_{i=1}^{n} \frac{\Gamma(6i-1)\,\Gamma\left(\frac{i+3}{4}\right)}{\Gamma(5i)\,\Gamma\left(\frac{5i-1}{4}\right)}, \tag{10.8}$$

$$G_{2,1,2,0}(n) = \prod_{i=1}^{n} \frac{\Gamma(6i-1)\,\Gamma\left(\frac{i+2}{4}\right)}{2(2i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i-2}{4}\right)}. \tag{10.9}$$

Moreover, the following identities are conjectured to hold for all $n \geq 3$:

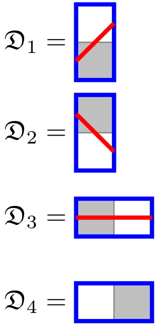
$$\begin{aligned} G_{3,0,3,3}(n) &= \frac{2}{3}\,G_{0,1,-2,-4}(n+1) = -\frac{1}{672}\,G_{1,3,-2,-8}(n+1) \\ &= \frac{1}{63}\,G_{5,4,3,-5}(n) = \frac{4}{1002001}\,G_{6,6,3,-9}(n) = -\frac{8}{5}\,G_{9,5,8,-2}(n-1), \end{aligned} \tag{10.10}$$

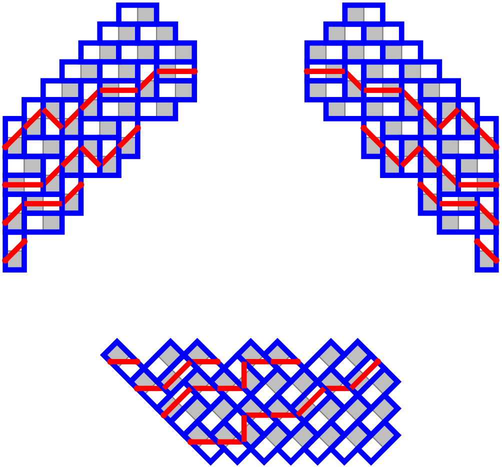
$$G_{1,1,0,-2}(n) = -\frac{1}{49}\,G_{2,3,0,-6}(n) = -\frac{2}{7}\,G_{6,4,5,-3}(n-1) = -\frac{4}{5577}\,G_{7,6,5,-7}(n-1), \tag{10.11}$$

$$G_{2,1,2,0}(n) = 2\,G_{7,4,7,-1}(n-1). \tag{10.12}$$

Weighted counting

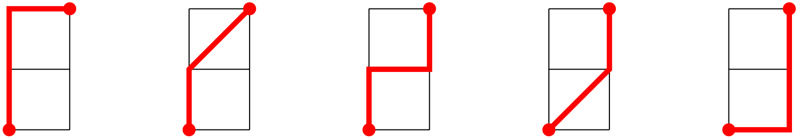
Corteel–Huang–Krattenthaler's bijection:





Weighted counting

Delannoy paths: Lattice path using only east ($\rightarrow$), north ($\uparrow$) or northeast ($\nearrow$) steps.



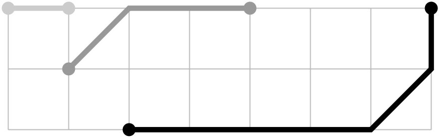
Weighted counting

Now let us fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length n.

A *system of nonintersecting Delannoy paths marked by* λ is an n-tuple $p = (p_1, \ldots, p_n)$ of Delannoy paths such that p_j moves from $(-j, j)$ to $(\lambda_j - j, n)$ for $1 \le j \le n$ and that these Delannoy paths are pairwise *nonintersecting* in the sense that every two different paths do not have any lattice point in common.

We denote by $\mathcal{L}_\lambda(n)$ the set of such nonintersecting Delannoy paths.

Example: A system of nonintersecting Delannoy paths marked by the partition $(5, 3, 1)$



Weighted counting

Consider the weighted enumeration

$$\text{Л}^{\lambda}_{w_1,w_2,w_3}(n) := \sum_{p\in\mathcal{L}_\lambda(n)} w_1^{\#_p(\rightarrow)} w_2^{\#_p(\uparrow)} w_3^{\#_p(\nearrow)}.$$

Weighted counting

Consider the weighted enumeration

$$\text{Л}^{\lambda}_{w_1,w_2,w_3}(n) := \sum_{p \in \mathcal{L}_\lambda(n)} w_1^{\#_p(\rightarrow)} w_2^{\#_p(\uparrow)} w_3^{\#_p(\nearrow)}.$$

We also look at partitions that form an **arithmetic progression**:

$$\lambda^{(s,r)} := (s(n-1)+r, s(n-2)+r, \ldots, r),$$

Write

$$\text{Л}^{(s,r)}_{w_1,w_2,w_3}(n) := \text{Л}^{\lambda^{(s,r)}}_{w_1,w_2,w_3}(n) = \sum_{p \in \mathcal{L}_{\lambda^{(s,r)}}(n)} w_1^{\#_p(\rightarrow)} w_2^{\#_p(\uparrow)} w_3^{\#_p(\nearrow)}.$$

Weighted counting

Lemma (Corteel–Huang–Krattenthaler)

For any $s, r \in \mathbb{N}$, there is a bijection

$$\begin{array}{rccc} \phi : & \mathscr{D}^{(s,r)}(n) & \to & \mathscr{L}^{(s,r)}(n) \\ & T & \mapsto & p \end{array}$$

such that

$$\#_T(\mathfrak{D}_1) = \#_p(\rightarrow), \qquad \#_T(\mathfrak{D}_2) = \#_p(\uparrow), \qquad \#_T(\mathfrak{D}_3) = \#_p(\nearrow).$$

Weighted counting

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Consequently, for any $s, r \in \mathbb{N}$,

$$Д^{(s,r)}_{w_1,w_2,w_3}(n) = Л^{(s,r)}_{w_1,w_2,w_3}(n).$$

Weighted counting

Lindström–Gessel–Viennot lemma: In $p = (p_1, \ldots, p_n)$,

$$\begin{array}{cccc} p_j : & (-j, j) & \to & (s(n-j) + r - j, n) \\ & \mathrm{start}_j & \to & \mathrm{end}_j \end{array}$$

Then

$$Я^{(s,r)}_{w_1,w_2,w_3}(n) = \det_{1\le i,j\le n}\left(\sum_{p_0 : \mathrm{start}_j \xrightarrow{D} \mathrm{end}_i} w_1^{\#_{p_0}(\to)} w_2^{\#_{p_0}(\uparrow)} w_3^{\#_{p_0}(\nearrow)} \right).$$

Weighted counting

Define the *weighted Delannoy numbers* by

$$D_{w_1,w_2,w_3}(i,j) := \sum_{p_0:(0,0)\xrightarrow{D}(i,j)} w_1^{\#_{p_0}(\rightarrow)} w_2^{\#_{p_0}(\uparrow)} w_3^{\#_{p_0}(\nearrow)}.$$

It is true that

$$\sum_{i,j\geq 0} D_{w_1,w_2,w_3}(i,j) u^i v^j = \frac{1}{1 - w_1 u - w_2 v - w_3 uv}.$$

Weighted counting

Define the *weighted Delannoy numbers* by

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We have

$$\begin{aligned} \text{Л}^{(s,r)}_{w_1,w_2,w_3}(n) &= \det_{1\leq i,j\leq n}\Big(D_{w_1,w_2,w_3}(s(n-i)-i+j+r, n-j)\Big) \\ &= \det_{0\leq i,j\leq n-1}\Big(D_{w_1,w_2,w_3}(s(n-i-1)-i+j+r, n-j-1)\Big) \\ {\scriptstyle ((i,j)\mapsto(n-1-i,n-1-j))} &= \det_{0\leq i,j\leq n-1}\Big(D_{w_1,w_2,w_3}((s+1)i-j+r, j)\Big). \end{aligned}$$

Weighted counting

$$\text{Д}_{1,l-1,1}^{(m-1,a)}(n) = \text{Л}_{1,l-1,1}^{(m-1,a)}(n) = \det_{0\le i,j\le n-1}\big(D_{1,l-1,1}(mi-j+a,j)\big)$$
$$\stackrel{?}{=} \frac{1}{2}\cdot \det_{0\le i,j\le n-1}\big(\text{б}_{a,0,a,a}^{(m,l)}(i,j)\big),$$

where

$$\text{б}_{a,b,c,d}^{(m,l)}(i,j) := l^{j+b}\binom{mi+j+c}{mi+a} + \binom{mi-j+d}{mi+a}.$$

Weighted counting

Lemma

Let $F(u,v) \in \mathbb{C}[[u,v]]$. *Suppose that* $\alpha(v)$ *and* $\beta(v)$ *are both in* $\mathbb{C}[[v]]$ *with constant term equal to* 1. *Then*

$$\det_{0\le i,j\le n-1}\big([u^i v^j]F(u,v)\big) = \det_{0\le i,j\le n-1}\big([u^i v^j]F^\dagger(u,v)\big),$$

where

$$F^\dagger(u,v) = \alpha(v)\cdot F\big(u, v\beta(v)\big).$$

Weighted counting

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$$\big([u^i v^j]F^\dagger(u,v)\big)_{i,j\ge 0} = \big([u^i v^j]F(u,v)\big)_{i,j\ge 0} \underbrace{\Big([u^i v^j]\tfrac{\alpha(v)}{1-uv\beta(v)}\Big)_{i,j\ge 0}}_{\substack{\text{upper triangular}\\ \text{diagonal entries equal to } 1}}.$$

Weighted counting

Write

$$P^{(s,r)}_{w_1,w_2,w_3}(u,v) := \sum_{i,j\geq 0} D_{w_1,w_2,w_3}((s+1)i-j+r,j)u^i v^j,$$

$$B^{(m,l)}_{a,b,c,d}(u,v) := \sum_{i,j\geq 0} \text{б}^{(m,l)}_{a,b,c,d}(i,j)u^i v^j.$$

It is true that

$$\underbrace{\frac{1-lv^2}{(1-v)(1-lv)} P^{(m-1,a)}_{1,l-1,1}\left(u, \frac{v}{(1-v)(1-lv)}\right)}_{\det\limits_{0\leq i,j\leq n-1}\left(D_{1,l-1,1}(mi-j+a,j)\right)} = \underbrace{B^{(m,l)}_{a,0,a,a}(u,v) - \frac{1}{1-u}}_{\frac{1}{2}\cdot\det\limits_{0\leq i,j\leq n-1}\left(\text{б}^{(m,l)}_{a,0,a,a}(i,j)\right)}.$$

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Q.E.D.!

Epilogue

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If so, I invite you to check the remaining conjectures of Koutschan–Krattenthaler–Schlosser,

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If so, I invite you to check the remaining conjectures of Koutschan–Krattenthaler–Schlosser, and to suffer from the pain I had experienced!

Thank You!