Multiple Rogers–Ramanujan type identities for torus links

Shane Chern

Universität Wien

xiaohangc92@univie.ac.at

Séminaire Lotharingien de Combinatoire 93

Pocinho, Portugal

Mar 25, 2024

Supported by the Austrian Science Fund (No. 10.55776/F1002)

Motivating Questions

(~~Elementary School~~ "College Algebra") Question. How many pairs of numbers a, b among 0 and 1 are there such that $a \times b$ and $b \times a$ have the same remainder if divided by 2?

Motivating Questions

(~~Elementary School~~ "College Algebra") Question. How many pairs of numbers a, b among 0 and 1 are there such that $a \times b$ and $b \times a$ have the same remainder if divided by 2?

TRIVIAL(!) **Answer:** 4: $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

Motivating Questions

(~~Elementary School~~ "College Algebra") Question. How many pairs of numbers a, b among 0 and 1 are there such that $a \times b$ and $b \times a$ have the same remainder if divided by 2?

TRIVIAL(!) **Answer:** 4: $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.

("Abstract Algebra" Week 1(?)) Question. In $\mathbb{F}_2$, how many pairs $(A_1, B_1) \in \mathbb{F}_2^2$ are there such that

$$A_1 B_1 = B_1 A_1.$$

Motivating Questions

(~~Elementary School~~ “College Algebra”) Question. How many pairs of numbers a, b among 0 and 1 are there such that $a \times b$ and $b \times a$ have the same remainder if divided by 2?

TRIVIAL(!) Answer: 4: $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.

(“Abstract Algebra” Week 1(?)) Question. In $\mathbb{F}_2$, how many pairs $(A_1, B_1) \in \mathbb{F}_2^2$ are there such that

$$A_1 B_1 = B_1 A_1.$$

Answer: $2 \times 2 = 4$.

Motivating Questions

(More Challenging) Question. Over $\mathbb{F}_2$, how many pairs $(A_2, B_2) \in \mathrm{Mat}_2(\mathbb{F}_2)^2$ of two-by-two matrices are there such that

$$A_2 B_2 = B_2 A_2.$$

Motivating Questions

(More Challenging) Question. Over $\mathbb{F}_2$, how many pairs $(A_2, B_2) \in \mathrm{Mat}_2(\mathbb{F}_2)^2$ of two-by-two matrices are there such that

$$A_2 B_2 = B_2 A_2.$$

NON(!)-Trivial Answer: 88.

(More Challenging) Question. Over $\mathbb{F}_2$, how many pairs $(A_2, B_2) \in \mathrm{Mat}_2(\mathbb{F}_2)^2$ of two-by-two matrices are there such that

$$A_2 B_2 = B_2 A_2.$$

NON(!)-Trivial Answer: 88.

```
In[ ]:= entry = {0, 1};
        row = Tuples[{entry, entry}];
        mat = Tuples[{row, row}];
        matpair = Tuples[{mat, mat}];
        commutingmatpair = Select[matpair, Mod[#[[1]].#[[2]], 2] == Mod[#[[2]].#[[1]], 2] &];
        Length@commutingmatpair
Out[ ]= 88
```

Motivating Questions

PAIRS OF COMMUTING MATRICES OVER A FINITE FIELD

By Walter Feit and N. J. Fine

In this paper, we determine the number of ordered pairs of commuting n by n matrices over $GF(q)$ and give a simple generating function for this number.

Theorem (Feit–Fine, 1960)

$$\sum_{n\geq 0} \frac{|\{(A,B)\in \mathrm{Mat}_n(\mathbb{F}_q)^2 : AB = BA\}|}{|\,\mathrm{GL_n}(\mathbb{F}_q)\,|} t^n = \prod_{i,j\geq 1} \frac{1}{1-q^{2-j}t^i}.$$

Note. $|\,\mathrm{GL_n}(\mathbb{F}_q)\,| = q^{n^2}\prod_{j=1}^{n}(1-q^{-j})$.

Motivating Questions

Diophantine Equations: Solutions in $\mathbb{F}_q^2$:

$$B^2 = A^3.$$

Motivating Questions

Diophantine Equations: Solutions in $\mathbb{F}_q^2$:

$$B^2 = A^3.$$

Matrix Diophantine Equations: Solutions in $\mathrm{Mat}_n(\mathbb{F}_q)^2$:

$$B^2 = A^3$$

Motivating Questions

Diophantine Equations: Solutions in $\mathbb{F}_q^2$:

$$B^2 = A^3.$$

Matrix Diophantine Equations: Solutions in $\mathrm{Mat}_n(\mathbb{F}_q)^2$:

$$B^2 = A^3$$

and

$$AB = BA.$$

Algebraic Geometry

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity

Yifeng Huang

Dept. of Mathematics, University of British Columbia, Canada

Theorem (Huang, 2023)

$$\sum_{n\geq 0} \frac{\left|\left\{(A,B)\in \mathrm{Mat}_n(\mathbb{F}_q)^2 : AB = BA \text{ and } f(A,B)=0\right\}\right|}{|\mathrm{GL}_n(\mathbb{F}_q)|} t^n = \widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t),$$

where f is a given polynomial over $\mathbb{F}_q$.

Algebraic Geometry

What is $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$*? Where does it come from?*

Algebraic Geometry

What is $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$*? Where does it come from?*

Answer to Question 2: Algebraic Geometry!

Algebraic Geometry

What is $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$*? Where does it come from?*

Answer to Question 2: Algebraic Geometry!

(NOT MY) Answer to Question 1: Let R be the complete local ring of the germ of a certain $\mathbb{K}$-curve at a $\mathbb{K}$-point with $\mathbb{K}$ a fixed field.

By denoting $\mathrm{Coh}_n(R)$ the stack of R-modules of $\mathbb{K}$-dimension n, the *motivic Cohen–Lenstra zeta function* is defined by

$$\widehat{Z}_R(t) := \sum_{n\geq 0} [\mathrm{Coh}_n(R)] t^n.$$

Algebraic Geometry

What is $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$*? Where does it come from?*

Answer to Question 2: Algebraic Geometry!

(NOT MY) Answer to Question 1: Let R be the complete local ring of the germ of a certain $\mathbb{K}$-curve at a $\mathbb{K}$-point with $\mathbb{K}$ a fixed field.

By denoting $\mathrm{Coh}_n(R)$ the stack of R-modules of $\mathbb{K}$-dimension n, the *motivic Cohen–Lenstra zeta function* is defined by

$$\widehat{Z}_R(t) := \sum_{n\geq 0}[\mathrm{Coh}_n(R)]t^n.$$

If we normalize R as $\widetilde{R}$, then we may define the *numerator part*:

$$\widehat{NZ}_R(t) := \frac{\widehat{Z}_R(t)}{\widehat{Z}_{\widetilde{R}}(t)},$$

Algebraic Geometry

What is $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$*? Where does it come from?*

Answer to Question 2: Algebraic Geometry!

(NOT MY) Answer to Question 1: Let R be the complete local ring of the germ of a certain $\mathbb{K}$-curve at a $\mathbb{K}$-point with $\mathbb{K}$ a fixed field.

By denoting $\mathrm{Coh}_n(R)$ the stack of R-modules of $\mathbb{K}$-dimension n, the *motivic Cohen–Lenstra zeta function* is defined by

$$\widehat{Z}_R(t) := \sum_{n\geq 0} [\mathrm{Coh}_n(R)] t^n.$$

If we normalize R as $\widetilde{R}$, then we may define the *numerator part*:

$$\widehat{NZ}_R(t) := \frac{\widehat{Z}_R(t)}{\widehat{Z}_{\widetilde{R}}(t)},$$

where (with $\mathbb{L} := [\mathbb{A}^1]$ the *Lefschetz motive*; for $\mathbb{K} = \mathbb{F}_q$, we have $\mathbb{L} = q$)

$$\widehat{Z}_{\widetilde{R}}(t) = \prod_{j\geq 0} \frac{1}{(1 - t\mathbb{L}^{-j-1})^s}.$$

Algebraic Geometry

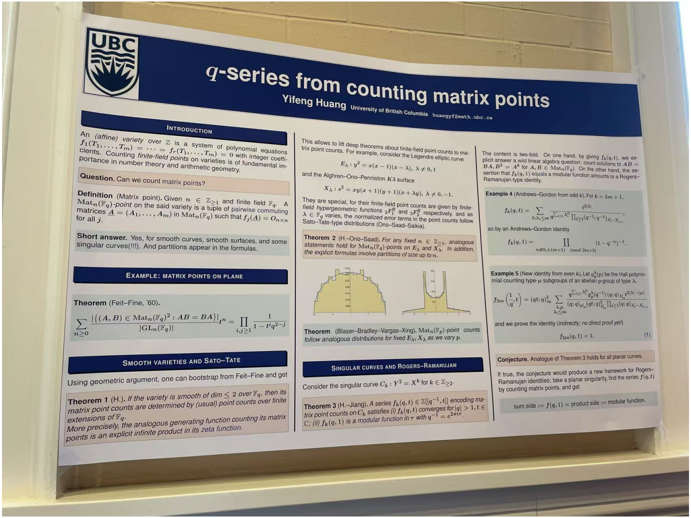
Interesting Varieties $f(x, y)$:

- Torus knots $R^{(2,2k+1)}$: the germ of $y^2 = x^{2k+1}$;
- Torus links $R^{(2,2k)}$: the germ of $y(y - x^k) = 0$

Algebraic Geometry

Interesting Varieties $f(x, y)$:

- Torus knots $R^{(2,2k+1)}$: the germ of $y^2 = x^{2k+1}$;
- Torus links $R^{(2,2k)}$: the germ of $y(y - x^k) = 0$ (if $\mathbb{K}$ is not of characteristic two, then $R^{(2,2k)}$ also admits the variety $y^2 = x^{2k}$).



Andrews–Berndt Conference at Penn State, June 6–9, 2024

Algebraic Geometry

- Torus knots $R^{(2,2k+1)}$?

Theorem (Huang–Jiang, 2023)

$$\widehat{NZ}_{R^{(2,2k+1)}}(t) = \sum_{n_1,\ldots,n_k\geq 0} \frac{t^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{\prod_{i=1}^k \prod_{j=1}^{n_i-n_{i-1}}(1-\mathbb{L}^{-j})}.$$

Algebraic Geometry

- Torus knots $R^{(2,2k+1)}$?

Theorem (Huang–Jiang, 2023)

$$\widehat{NZ}_{R^{(2,2k+1)}}(t) = \sum_{n_1,\ldots,n_k\geq 0} \frac{t^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{\prod_{i=1}^k \prod_{j=1}^{n_i-n_{i-1}} (1-\mathbb{L}^{-j})}.$$

In particular,

$$\widehat{NZ}_{R^{(2,2k+1)}}(\pm 1) = \prod_{j\geq 0} \frac{(1-\mathbb{L}^{-(2k+3)j-(k+1)})(1-\mathbb{L}^{-(2k+3)j-(k+2)})(1-\mathbb{L}^{-(2k+3)j-(2k+3)})}{1-\mathbb{L}^{-j-1}}.$$

Algebraic Geometry

- Torus knots $R^{(2,2k+1)}$?

Theorem (Huang–Jiang, 2023)

$$\widehat{NZ}_{R^{(2,2k+1)}}(t) = \sum_{n_1,\ldots,n_k\geq 0} \frac{t^{\sum_{i=1}^k 2n_i}\mathbb{L}^{-\sum_{i=1}^k n_i^2}}{\prod_{i=1}^k \prod_{j=1}^{n_i-n_{i-1}}(1-\mathbb{L}^{-j})}.$$

In particular,

$$\widehat{NZ}_{R^{(2,2k+1)}}(\pm 1) = \prod_{j\geq 0} \frac{(1-\mathbb{L}^{-(2k+3)j-(k+1)})(1-\mathbb{L}^{-(2k+3)j-(k+2)})(1-\mathbb{L}^{-(2k+3)j-(2k+3)})}{1-\mathbb{L}^{-j-1}}.$$

Andrews' *multiple Roger–Ramanujan type identity*:

$$\sum_{n_1,\ldots,n_k\geq 0} \frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}} = \frac{(q^{k+1},q^{k+2},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}.$$

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

BAD news: There is only an ugly $2k$-fold sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ in terms of the *Hall–Littlewood polynomials*.

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

BAD news: There is only an ugly $2k$-fold sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ in terms of the *Hall–Littlewood polynomials*.

GOOD news: There is one proved Roger–Ramanujan type identity and another conjectural one.

Theorem/Conjecture (Huang–Jiang, 2023)

$$\widehat{NZ}_{R^{(2,2k)}}(1) = 1$$

and

$$\widehat{NZ}_{R^{(2,2k)}}(-1) \overset{?}{=} \prod_{j\geq 1} \frac{(1-\mathbb{L}^{-2j})(1-\mathbb{L}^{-(k+1)j})^2}{(1-\mathbb{L}^{-j})^2(1-\mathbb{L}^{-(2k+2)j})}.$$

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

BAD news: There is only an ugly $2k$-fold sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ in terms of the *Hall–Littlewood polynomials*.

GOOD news: There is one proved Roger–Ramanujan type identity and another conjectural one.

Theorem/Conjecture (Huang–Jiang, 2023)

$$\widehat{NZ}_{R^{(2,2k)}}(1) = 1$$

and

$$\widehat{NZ}_{R^{(2,2k)}}(-1) \stackrel{?}{=} \prod_{j\geq 1} \frac{(1-\mathbb{L}^{-2j})(1-\mathbb{L}^{-(k+1)j})^2}{(1-\mathbb{L}^{-j})^2(1-\mathbb{L}^{-(2k+2)j})}.$$

BAD news AGAIN: $\widehat{NZ}_{R^{(2,2k)}}(1) = 1$ can ONLY be proved by hardcore techniques in algebraic geometry.

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

Algebraic Geometry

- Torus links $R^{(2,2k)}$?

NOT BAD news (to me): There is ~~only an ugly~~ still a $2k$-fold sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ in terms of the *Hall–Littlewood polynomials*.

- Torus links $R^{(2,2k)}$?

NOT BAD news (to me): There is ~~only an ugly~~ still a $2k$-fold sum-like expression for $\widehat{NZ}_{R^{(2,2k)}}(t)$ in terms of the *Hall–Littlewood polynomials*.

$$\widehat{NZ}_{R^{(2,2k)}}(t)|_{\mathbb{L}\mapsto q^{-1}} = (tq;q)_\infty^2 \mathcal{Z}_k(t,q),$$

where

$$\mathcal{Z}_k(t,q) := \sum_{\substack{r_k\geq\cdots\geq r_1\geq 0\\ s_k\geq\cdots\geq s_1\geq 0}} \frac{t^{\sum_{i=1}^k (2r_i-s_i)} q^{\sum_{i=1}^k (r_i^2-r_is_i+s_i^2)}}{(q;q)_{r_k-r_{k-1}}\cdots(q;q)_{r_2-r_1}(tq;q)_{r_1}^2(q;q)_{s_1}} \times \begin{bmatrix} r_k-s_{k-1}\\ r_k-s_k\end{bmatrix}_q \begin{bmatrix} r_{k-1}-s_{k-2}\\ r_{k-1}-s_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} r_2-s_1\\ r_2-s_2\end{bmatrix}_q \begin{bmatrix} r_1\\ r_1-s_1\end{bmatrix}_q.$$

Theorem (C., 2024, reproving H–J's $\widehat{N\!Z}_{R(2,2k)}(1) = 1$)

$$\mathcal{Z}_k(1, q) = \frac{1}{(q; q)_\infty^2}.$$

Rogers–Ramanujan

Theorem (C., 2024, reproving H–J's $\widehat{NZ}_{R(2,2k)}(1)=1$)

$$\mathcal{Z}_k(1,q)=\frac{1}{(q;q)_\infty^2}.$$

Opening the q-binomial coefficients and making the substitutions

$$d_i:=\begin{cases} s_1, & i=1,\\ s_i-s_{i-1}, & i\geq 2., \end{cases} \qquad \text{and} \qquad n_j:=r_j-s_j,$$

we find that $\mathcal{Z}_k(1,q)$ equals

$$\sum_{s_1,\ldots,s_k\geq 0}\frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{s_2-s_1}\cdots(q;q)_{s_k-s_{k-1}}(q;q)_{s_1}^2}$$
$$\times\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2+\sum_{i=1}^k(d_1+\cdots+d_i)n_i}(q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}(q;q)_{n_k}\cdots(q;q)_{n_1}(q;q)_{n_1+d_1}}.$$

Believe it or not — But at least trust **W. N. Bailey** — the $d_1 = \cdots = d_k = 0$ specialization is an instance of Bailey pairs:

$$\frac{1}{(q;q)_N(aq;q)_{N+d_1+\cdots+d_k}}$$
$$= \sum_{n_1,\ldots,n_k\geq 0} \frac{a^{\sum_{i=1}^k n_i} q^{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (d_1+\cdots+d_i)n_i} (q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}(q;q)_{n_k}\cdots(q;q)_{n_1}(aq;q)_{n_1+d_1}}.$$

Believe it or not — But at least trust **W. N. Bailey** — the $d_1 = \cdots = d_k = 0$ specialization is an instance of Bailey pairs:

$$\frac{1}{(q;q)_N(aq;q)_{N+d_1+\cdots+d_k}} = \sum_{n_1,\ldots,n_k\geq 0} \frac{a^{\sum_{i=1}^k n_i} q^{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (d_1+\cdots+d_i)n_i} (q;q)_{n_k+d_k}\cdots(q;q)_{n_2+d_2}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}+d_k}\cdots(q;q)_{n_2-n_1+d_2}(q;q)_{n_k}\cdots(q;q)_{n_1}(aq;q)_{n_1+d_1}}.$$

What does the above imply?

$$\begin{aligned}\mathcal{Z}_k(1,q) &= \sum_{s_1,\ldots,s_k\geq 0} \frac{q^{\sum_{i=1}^k s_i^2}}{(q;q)_{s_2-s_1}\cdots(q;q)_{s_k-s_{k-1}}(q;q)_{s_1}^2} \times \frac{1}{(q;q)_\infty} \\ &= \frac{1}{(q;q)_\infty} \times \frac{1}{(q;q)_\infty}!\end{aligned}$$

Rogers–Ramanujan

Iterate the basic hypergeometric transform:

$$\sum_{n\geq 0} \frac{a^n q^{n^2+Mn}}{(q;q)_{N-n}(q;q)_n(aq;q)_{M+n}} = \frac{1}{(q;q)_N(aq;q)_{M+N}}.$$

Iterate the basic hypergeometric transform:

$$\sum_{n\geq 0} \frac{a^n q^{n^2+Mn}}{(q;q)_{N-n}(q;q)_n(aq;q)_{M+n}} = \frac{1}{(q;q)_N(aq;q)_{M+N}}.$$

Step 1. Make the reformulation

$$\frac{1}{(q;q)_N(aq;q)_{(M'+M'')+N}} = \sum_{L\geq 0} \frac{a^L q^{L^2+(M'+M'')L}(q;q)_{L+M'}}{(q;q)_{N-L}(q;q)_L} \times \frac{1}{(q;q)_{L+M'}(aq;q)_{M''+(L+M')}}.$$

Rogers–Ramanujan

Iterate the basic hypergeometric transform:

$$\sum_{n\geq 0} \frac{a^n q^{n^2+Mn}}{(q;q)_{N-n}(q;q)_n(aq;q)_{M+n}} = \frac{1}{(q;q)_N(aq;q)_{M+N}}.$$

Step 1. Make the reformulation

$$\frac{1}{(q;q)_N(aq;q)_{(M'+M'')+N}} = \sum_{L\geq 0} \frac{a^L q^{L^2+(M'+M'')L}(q;q)_{L+M'}}{(q;q)_{N-L}(q;q)_L} \times \frac{1}{(q;q)_{L+M'}(aq;q)_{M''+(L+M')}}.$$

Step 2. Apply the same transform to

$$\frac{1}{(q;q)_{L+M'}(aq;q)_{M''+(L+M')}}$$

in the summand.

BAD news AGAIN: This strategy does NOT work for the evaluation of $\widehat{NZ}_{R^{(2,2k)}}(-1)$...

Rogers–Ramanujan

BAD news AGAIN: This strategy does NOT work for the evaluation of $\widehat{NZ}_{R^{(2,2k)}}(-1)$...

Recall that

$$\widehat{NZ}_{R^{(2,2k+1)}}(t)\big|_{\mathbb{L}\mapsto q^{-1}} = \sum_{n_1,\ldots,n_k\geq 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}}.$$

Rogers–Ramanujan

Theorem (C., 2024)

$$\mathcal{Z}_k(t,q) = \frac{1}{(tq;q)_\infty} \sum_{n_1,\ldots,n_k \geq 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k - n_{k-1}} \cdots (q;q)_{n_2 - n_1} (q;q)_{n_1} (tq;q)_{n_1}}.$$

Theorem (C., 2024)

$$\mathcal{Z}_k(t,q)=\frac{1}{(tq;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(tq;q)_{n_1}}.$$

Case $t=1$:

$$\mathcal{Z}_k(1,q)=\frac{1}{(q;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}^2}=\frac{1}{(q;q)_\infty^2}.$$

Rogers–Ramanujan

Theorem (C., 2024)

$$\mathcal{Z}_k(t,q)=\frac{1}{(tq;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{t^{\sum_{i=1}^k 2n_i}q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(tq;q)_{n_1}}.$$

Case $t=1$:

$$\mathcal{Z}_k(1,q)=\frac{1}{(q;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}^2}=\frac{1}{(q;q)_\infty^2}.$$

Case $t=-1$ (Thanks to David Bressoud's discovery in 1980!):

$$\mathcal{Z}_k(-1,q)=\frac{1}{(q;q)_\infty}\sum_{n_1,\ldots,n_k\geq 0}\frac{q^{\sum_{i=1}^k n_i^2}}{(q;q)_{n_k-n_{k-1}}\cdots(q;q)_{n_2-n_1}(q^2;q^2)_{n_1}}$$
$$=\frac{(q^{k+1};q^{k+1})_\infty^2}{(q^2;q^2)_\infty(q^{2k+2};q^{2k+2})_\infty}.$$

So Huang–Jiang's *Conjectural* evaluation of $\widehat{NZ}_{R(2,2k+1)}(-1)$ is CORRECT!

Ole Warnaar made an exposition of David Bressoud's identity and other similar Rogers–Ramanujan type identities at this **SLC '42 — The Andrews Festschrift**!

Séminaire Lotharingien de Combinatoire, B42n (1999), 22 pp.

S. Ole Warnaar

Supernomial Coefficients, Bailey's Lemma and Rogers-Ramanujan-Type Identities. A Survey of Results and Open Problems

Abstract. An elementary introduction to the recently introduced A_2 Bailey lemma for supernomial coefficients is presented. As illustration, some A_2 q-series identities are (re)derived which are natural analogues of the classical (A_1) Rogers-Ramanujan identities and their generalizations of Andrews and Bressoud. The intimately related, but unsolved problems of supernomial inversion, A_{n-1} and higher level extensions are also discussed. This yields new results and conjectures involving A_{n-1} basic hypergeometric series, string functions and cylindric partitions.

Received: December 17, 1998; Accepted: May 10, 1999.

How to achieve that k-fold sum for $\mathcal{Z}_k(t,q)$?

Step 1. Prove the reformulation:

$$\mathcal{Z}_k(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2}$$
$$\times \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(t^2q;q)_{N+n_k}(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

How to achieve that k-fold sum for $\mathcal{Z}_k(t,q)$?

Step 1 (BONUS!) — A Fake A_2 Rogers–Ramanujan type identity:

$$\frac{(q^2;q^2)_\infty (q^{k+1};q^{k+1})_\infty^2}{(q;q)_\infty^3 (q^{2k+2};q^{\color{red}{2k+2}})_\infty} = \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{(-1)^{\sum_{i=1}^k m_i} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)}(-1;q)_{n_1}^2}{(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$

$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

How to achieve that k-fold sum for $\mathcal{Z}_k(t,q)$?

Step 1 (BONUS!) — A Fake A_2 Rogers–Ramanujan type identity:

$$\frac{(q^2;q^2)_\infty (q^{k+1};q^{k+1})_\infty^2}{(q;q)_\infty^3 (q^{2k+2};q^{2k+2})_\infty} = \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{(-1)^{\sum_{i=1}^k m_i} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (-1;q)_{n_1}^2}{(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q .$$

Andrews–Schilling–Warnaar's Authentic A_2 type identity:

$$\frac{(q^{k+1},q^{k+1},q^{k+2},q^{2k+2},q^{2k+3},q^{2k+3},q^{3k+4},q^{3k+4};q^{3k+4})_\infty}{(q;q)_\infty^3}$$
$$= \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{q^{\sum_{i=1}^k (m_i^2-m_in_i+n_i^2)}(1-q^{m_1+n_1+1})}{(q;q)_{m_k}(q;q)_{n_k}(q;q)_{m_1+n_1+1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_q .$$

How to achieve that k-fold sum for $\mathcal{Z}_k(t,q)$?

Step 2. Evaluate the semi-truncation (as a k-fold sum):

$$\mathcal{V}_k(N;t,q) := \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}} \\ \times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

How to achieve that k-fold sum for $\mathcal{Z}_k(t,q)$?

Step 2. Evaluate the semi-truncation (as a k-fold sum):

$$\mathcal{V}_k(N;t,q) := \sum_{\substack{m_1,\ldots,m_k\geq 0\\ n_1,\ldots,n_k\geq 0}} \frac{t^{-2n_1+\sum_{i=1}^k (m_i+2n_i)} q^{-n_1^2+n_1+\sum_{i=1}^k (m_i^2+m_in_i+n_i^2)} (t;q)_{n_1}^2}{(q;q)_{N-m_k}(q;q)_{m_k}(q;q)_{m_1}(q;q)_{n_1}}$$
$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q .$$

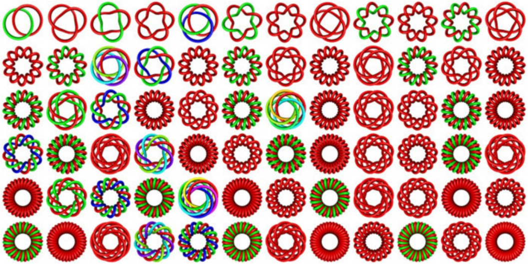
Step 3. Use the relation and reduce $\mathcal{Z}_k(N;t,q)$ to the desired k-fold sum:

$$\mathcal{Z}_k(N;t,q) = \frac{(q;q)_\infty (t^2q;q)_\infty}{(tq;q)_\infty^2} \sum_{m,n\geq 0} t^{m+2(k-1)n} q^{m^2+(m+1)n+(k-1)n^2}$$
$$\times \frac{(t;q)_n^2}{(q;q)_{N-m}(t^2q;q)_{N+n}(q;q)_n} \mathcal{V}_{k-1}(m;tq^n,q).$$

Finitization

All results in this talk can be **finitized** — and their finitizations, from the view of Algebraic Geometry, align with the **Quot zeta functions**.

S. Chern, Multiple Rogers–Ramanujan type identities for torus links, preprint, https://arxiv.org/abs/2411.07198.



(Taken from https://knotplot.com/knot-theory/torus_xing.html)

Obrigado!