

# Asymptotics for the enumeration of commuting matrices over finite fields

Kathrin Bringmann, Shane Chern, Johann Franke, and Bernhard Heim

*In honor of James Lepowsky's 80-th birthday.*

**Abstract.** We give asymptotic expressions for the number of commuting matrices over finite fields. For this we use product expansions for the corresponding generating functions.

## 1. Introduction and statement of results

Let  $\mathbb{F}_{p^r}$  be the finite field with  $p^r$  elements, where  $p$  is a prime and  $r \in \mathbb{N}$ . The study of the enumeration of commuting square matrix pairs over  $\mathbb{F}_{p^r}$  was first launched by Feit and Fine [5]. Let  $Q_{p^r}(n)$  denote the number of ordered pairs of (not necessarily distinct) commuting  $n \times n$  matrices with coefficients in  $\mathbb{F}_{p^r}$ . Feit and Fine proved that  $Q_{p^r}(n)$  (re-normalized) admits the following product-like generating function.

**Theorem 1.1** (Feit–Fine, [5, p. 91]). *We have*

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell}, \quad (1.1)$$

where

$$f_{p^r}(n) := \prod_{j=1}^n (1 - p^{-rj}). \quad (1.2)$$

**Remarks.**

- (1) The product in (1.1) converges for  $w \in \mathbb{C}$  with  $|w| < 1$  not being a pole and continues to a meromorphic function for  $|w| < 1$ . This is due to the absolute and uniform convergence on compact subsets of the double series

$$\sum_{\substack{\ell \geq 1 \\ j \geq 0}} p^{r(1-j)} w^\ell$$

away from a pole.

- (2) Recall that (see [8, (2.1)])

$$|\mathrm{GL}_n(\mathbb{F}_{p^r})| = p^{rn^2} f_{p^r}(n). \quad (1.3)$$

Therefore, (1.1) can be written as

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell}.$$

(3) By [5, (1)] and (1.3), we have

$$Q_{p^r}(n) = |\mathrm{GL}_n(\mathbb{F}_{p^r})| \sum_{\lambda \vdash n} \frac{p^{r \sum_{k=1}^n b_k}}{\prod_{k=1}^n f_{p^r}(b_k)}, \quad (1.4)$$

where the sum runs over all partitions of  $n$  and  $\lambda$  has the frequencies  $b_k$  given by  $n = \sum_{k=1}^n b_k k$ .

A natural question is to approximate  $Q_{p^r}(n)$ , as  $n \rightarrow \infty$ . It was shown by Motzkin–Tausky [12] (see also [9, p. 71, Theorem 2]) that the space of  $n \times n$  commuting matrices over an algebraically closed field has dimension  $n^2 + n$ . Hence, heuristically,  $Q_{p^r}(n)$  should be proportional to  $p^{r(n^2+n)}$  for  $n$  sufficiently large. This observation was rigorously proved by Fulman and Guralnick [7, Theorem 2.6] as a byproduct of their new proof of the result of Feit and Fine. More precisely, Fulman and Guralnick established that

$$\lim_{n \rightarrow \infty} \frac{Q_{p^r}(n)}{p^{r(n^2+n)}} = \prod_{j \geq 1} (1 - p^{-rj})^{-j}. \quad (1.5)$$

In this paper, we prove more precise asymptotics for  $Q_{p^r}(n)$ . To start with, we write, for  $w \in \mathbb{C}$  with  $|w| < 1$ ,

$$\prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell} = P_{0,p^r}(w) F_{p^r}(w), \quad (1.6)$$

where for  $x > 0$

$$P_{m,x}(w) := \prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \frac{1}{1 - x w^\ell}, \quad F_{p^r}(w) := \prod_{\ell, j \geq 1} \frac{1}{1 - p^{r(1-j)} w^\ell}. \quad (1.7)$$

We show the following asymptotic behavior of  $Q_{p^r}(n)$ , as  $n \rightarrow \infty$ . Throughout, we use the notation  $\zeta_m := e^{\frac{2\pi i}{m}}$ .

**Theorem 1.2.** *We have, as  $n \rightarrow \infty$ ,*

$$Q_{p^r}(n) \sim p^{rn^2} f_{p^r}(n) \sum_{m \geq 1} C_{m,p^r}(n) p^{\frac{rn}{m}}, \quad (1.8)$$

where

$$C_{m,p^r}(n) := \frac{1}{m} \sum_{j=0}^{m-1} P_{m,p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}) F_{p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}) \zeta_m^{nj}.$$

In particular, we have, as  $n \rightarrow \infty$ ,

$$Q_{p^r}(n) = p^{r(n^2+n)} \prod_{j \geq 1} (1 - p^{-rj})^{-j} + O\left(p^{rn^2 + \frac{rn}{2}}\right).$$

**Remark 1.1.** The series in (1.8) does not converge, but should be viewed as an asymptotic expansion. Also note that  $C_{m,p^r}(n)$  only depends on  $n \pmod{m}$ . In particular,  $C_{1,p^r}(n)$  does not depend on  $n$ . Furthermore, we have

$$|C_{m,p^r}(n)| \leq \max_{0 \leq j \leq m-1} |P_{m,p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}) F_{p^r}(\zeta_m^{-j} p^{-\frac{r}{m}})| = |P_{m,p^r}(p^{-\frac{r}{m}}) F_{p^r}(p^{-\frac{r}{m}})|.$$

As a result, when truncated at  $m = N$ , the error in (1.8) is  $O(p^{\frac{rn}{N+1}})$ .

**Example.** Let  $p = 2$  and  $r = 1$ . By (1.1), we have

$$(1 - 2w) \sum_{n \geq 0} \frac{Q_2(n)}{2^{n^2} f_2(n)} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0 \\ (\ell, j) \neq (1, 0)}} \frac{1}{1 - 2^{1-j} w^\ell}. \quad (1.9)$$

While (1.1) has radius of convergence  $\frac{1}{2}$ , the right-hand side of (1.9) has radius of convergence  $\frac{1}{\sqrt{2}}$ . Hence, we obtain

$$C_{1,2}(1) = \lim_{n \rightarrow \infty} \frac{Q_2(n)}{2^{n^2+n} f_2(n)} = \prod_{j \geq 1} \frac{1}{(1 - 2^{-j})^{j+1}} = 34.738723457 \dots$$

Further values are given by

$$\begin{aligned} C_{2,2}(1) &= -11716.7651425569 \dots & C_{2,2}(2) &= -11716.3960075313 \dots \\ C_{3,2}(1) &= 7970793.64416118 \dots & C_{3,2}(2) &= 7970793.59033743 \dots \\ C_{3,2}(3) &= 7970793.67801128 \dots \end{aligned}$$

Looking at these first values one notes that they are very close to each other for  $m$  fixed. A further investigation of their behavior would be interesting.

The paper is organized as follows. First we recall some basic definitions. Then we give a proof of Theorem 1.2, and generalize these ideas to another Cohen–Lenstra series in Theorem 4.1. In the final section, we propose problems for further research.

## 2. Preliminaries

In the following, we recall some required results from complex analysis. We start with a well-known and simple observation that is still very useful for many aspects in the theory of  $q$ -series, recurrences, and combinatorics. For  $\varrho \in \mathbb{R}^+$  let  $B_\varrho(a) := \{z \in \mathbb{C} : |z - a| < \varrho\}$ .

**Proposition 2.1.** *Let  $a \in \mathbb{C}$ ,  $\varrho \in \mathbb{R}^+$ , and  $f$  be a holomorphic function defined in a neighborhood of  $a$ . Then the following are equivalent:*

- (1) *The function  $f$  extends to an analytic function on  $B_\varrho(a)$ , but not on  $B_{\varrho+\varepsilon}(a)$  for any  $\varepsilon > 0$ .*
- (2) *The power series  $\sum_{n \geq 0} \frac{f^{(n)}(a)}{n!} (z - a)^n$  has radius of convergence  $\varrho$ .*

For a sequence  $a_m$  of non-zero complex numbers

$$F(w) := \prod_{m \geq 1} \left(1 - \frac{w}{a_m}\right). \quad (2.1)$$

**Proposition 2.2** (Wang, [13, Theorem 1]). *Let  $a_m$  be a sequence of non-zero complex numbers such that  $a_m \neq a_n$  for  $m \neq n$  and such that  $\sum_{m \geq 1} |a_m|^{-1}$  converges. Then the function  $\frac{1}{F}$  is holomorphic in a neighborhood of  $w = 0$  and we have there*

$$\frac{1}{F(w)} = - \sum_{k \geq 0} \sum_{m \geq 1} \frac{1}{F'(a_m) a_m^{k+1}} w^k.$$

### 3. Proof of Theorem 1.2

To prove Theorem 1.2, we require the following lemma.

**Lemma 3.1.** *Let  $x > 1$ ,  $n \in \mathbb{N}$ ,  $0 < \varepsilon < x^{-\frac{1}{n}}$ , and  $f$  be a holomorphic function in some region  $\mathcal{R}_{x,n,\varepsilon} := \{w \in \mathbb{C} : x^{-\frac{1}{n}} - \varepsilon < |w| < x^{-\frac{1}{n}} + \varepsilon\}$ . Define*

$$g(w) := \frac{f(w)}{1 - xw^n}.$$

*Then*

$$g(w) - \frac{1}{n} \sum_{j=0}^{n-1} \frac{f(\zeta_n^{-j} x^{-\frac{1}{n}})}{1 - \zeta_n^j x^{\frac{1}{n}} w}$$

*continues holomorphically to  $\mathcal{R}_{x,n,\varepsilon}$ .*

*Proof.* The only poles of  $g$  in  $\mathcal{R}_{x,n,\varepsilon}$  lie in  $\{\zeta_n^{-j} x^{-\frac{1}{n}} : 0 \leq j \leq n-1\}$ . These are simple. We compute, using l'Hospital,

$$\lim_{w \rightarrow \zeta_n^{-j} x^{-\frac{1}{n}}} \frac{(1 - \zeta_n^j x^{\frac{1}{n}} w) f(w)}{1 - xw^n} = \frac{1}{n} f(\zeta_n^{-j} x^{-\frac{1}{n}}).$$

The claim follows.  $\square$

We have the following corollary to Lemma 3.1.

**Corollary 3.2.** *Let  $f$  be a function holomorphic inside the unit disk and  $N \in \mathbb{N}$ . Then*

$$P_{0,p^r}(w) f(w) - \sum_{m=1}^N \frac{1}{m} \sum_{j=0}^{m-1} \frac{P_{m,p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}) f(\zeta_m^{-j} p^{-\frac{r}{m}})}{1 - \zeta_m^j p^{\frac{r}{m}} w}$$

*continues holomorphically to the region  $\{w \in \mathbb{C} : |w| < p^{-\frac{r}{N+1}}\}$ .*

*Proof.* By (1.7), we have, for  $m \in \mathbb{N}$ ,

$$P_{0,p^r}(w) f(w) = \frac{P_{m,p^r}(w) f(w)}{1 - p^r w^m}.$$

Note that the numerator is holomorphic in the region

$$\{w \in \mathbb{C} : p^{-\frac{r}{m-1}} < |w| < p^{-\frac{r}{m+1}}\}$$

(where  $p^{-\frac{r}{0}} := 0$ ). By applying Lemma 3.1 iteratively, the claim follows.  $\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $N \in \mathbb{N}$  be arbitrary. We abbreviate

$$c_{m,j} := P_{m,p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}) F_{p^r}(\zeta_m^{-j} p^{-\frac{r}{m}}).$$

By (1.1), (1.6), and Corollary 3.2 we conclude that

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} w^n - \sum_{m=1}^N \frac{1}{m} \sum_{j=0}^{m-1} \frac{c_{m,j}}{1 - \zeta_m^j p^{\frac{r}{m}} w}$$

continues holomorphically to  $\{w \in \mathbb{C} : |w| < p^{-\frac{r}{N+1}}\}$ . We can then write

$$\sum_{m=1}^N \frac{1}{m} \sum_{j=0}^{m-1} \frac{c_{m,j}}{1 - \zeta_m^j p^{\frac{r}{m}} w} = \sum_{n \geq 0} \sum_{m=1}^N \frac{p^{\frac{rn}{m}} w^n}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn}.$$

Hence we have, for  $\varepsilon > 0$ ,

$$\frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} - \sum_{m=1}^N \frac{1}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn} p^{\frac{rn}{m}} = O_\varepsilon \left( p^{\left(\frac{r}{N+1} + \varepsilon\right)n} \right) \quad (n \rightarrow \infty).$$

Equivalently, we have, as  $n \rightarrow \infty$ ,

$$Q_{p^r}(n) = p^{rn^2} f_{p^r}(n) \sum_{m=1}^N \frac{1}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn} p^{\frac{rn}{m}} + O_\varepsilon \left( p^{rn^2 + \left(\frac{r}{N+1} + \varepsilon\right)n} \right).$$

The result follows after a straightforward calculation.  $\square$

Finally, we give an upper bound for the coefficients in the asymptotic expansion in Theorem 1.2.

**Proposition 3.3.** *We have, for  $\varepsilon > 0$ , uniformly in  $m$ ,  $p^r$ , and  $n$*

$$C_{m,p^r}(n) \ll_\varepsilon \exp \left( \frac{\zeta(3) + \varepsilon}{\left(1 - p^{-\frac{r}{m}}\right)^2} \right),$$

where  $\zeta$  denotes the Riemann zeta function.

*Proof.* By (1.7) we have

$$|P_{m,p^r}(p^{-\frac{r}{m}}) F_{p^r}(p^{-\frac{r}{m}})| = \prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \left| \frac{1}{1 - p^{r - \frac{r\ell}{m}}} \right| \prod_{\ell, j \geq 1} \left| \frac{1}{1 - p^{r(1-j)} p^{-\frac{r\ell}{m}}} \right|.$$

We estimate both products separately. First, we have

$$\begin{aligned} \prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \frac{1}{1 - p^{r - \frac{r\ell}{m}}} &= (-1)^{m+1} p^{-\frac{rm(m+1)}{2m}} P_{0,1}(p^{-\frac{r}{m}}) \prod_{\ell=1}^{m-1} \frac{1}{1 - p^{-\frac{r\ell}{m}}} \\ &\ll p^{-\frac{r(m-1)}{2}} P_{0,1}^2(p^{-\frac{r}{m}}). \end{aligned}$$

For the second product, we find

$$\prod_{\ell, j \geq 1} \frac{1}{1 - p^{r(1-j)} p^{-\frac{r\ell}{m}}} = \prod_{\substack{j \geq 0 \\ \ell \geq 1}} \frac{1}{1 - p^{-\frac{r}{m}(mj + \ell)}} \ll \prod_{\substack{j \geq 0 \\ \ell \geq 1}} \frac{1}{1 - p^{-\frac{r}{m}(j + \ell)}} = \prod_{j \geq 1} \left( 1 - p^{-\frac{rj}{m}} \right)^{-j}.$$

Applying [2, Lemma 3.2], we have, as  $w \rightarrow 1^-$ ,

$$P_{0,1}^2(w) \ll_\varepsilon e^{\frac{\zeta(2) + \varepsilon}{|w-1|}}, \quad \prod_{j \geq 1} \frac{1}{(1 - w^j)^j} \ll_\varepsilon e^{\frac{\zeta(3) + \varepsilon}{|w-1|^2}}$$

for all  $\varepsilon > 0$ . As a result, for all  $\varepsilon > 0$ , we find the claimed bound.  $\square$

#### 4. Cohen–Lenstra series and the nilpotent classes

Since its discovery, Feit and Fine’s generating function (1.1) has been reproved by other means such as the motivic Donaldson–Thomas theory [3]. More recently, Huang [10] placed such counting problems into the framework of Cohen–Lenstra series [4]. Simply speaking, Huang considered *(generalized) Cohen–Lenstra series*, which are defined as

$$Z_R(w) = Z_{R/\mathbb{F}_{p^r}}(w) := \sum_M \frac{1}{|\mathrm{Aut}(M)|} w^{\dim_{\mathbb{F}_{p^r}}(M)},$$

where  $M$  runs over all isomorphism classes of finite  $R$ -modules and  $R$  is a commutative ring containing the finite field  $\mathbb{F}_{p^r}$  such that the above series is well-defined<sup>1</sup>. A more general setting was later provided by Huang in [11]. Note that, by (1.3), the relation (1.1) corresponds to the Cohen–Lenstra series (see also item 3 in Huang’s list in [10, p. 27]):

$$Z_{\mathbb{F}_{p^r}[u,v]}(w) = \sum_{n \geq 0} \frac{|\{(A, B) \in \mathrm{Mat}_n(\mathbb{F}_{p^r})^2 : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell},$$

where, as usual,  $\mathrm{Mat}_n(\mathbb{F}_{p^r})$  denotes the set of  $n \times n$  matrices over  $\mathbb{F}_{p^r}$ . In this setting, we may generalize  $R$  to count commuting varieties (i.e., subvarieties of  $XY - YX = 0$ ) over a finite field; see [10, p. 40, Proposition 4.3]. However, these Cohen–Lenstra series in general do not admit a product form and mostly behave multisum-like, see [10, (1.14) and (1.15)]. Therefore, our method of proof used for Theorem 1.2 does not apply. To work out new methods to deal with these products would be worthwhile.

In the meantime, we can restrict the enumeration of commuting square matrix pairs to nilpotent classes. Let

$$\mathrm{Nilp}_n(\mathbb{F}_{p^r}) := \{A \in \mathrm{Mat}_n(\mathbb{F}_{p^r}) : A \text{ is nilpotent}\}.$$

In the simplest cases, due to Fine and Herstein [6] and Fulman and Guralnick [7], we still have product-like generating functions similar to (1.1). In terms of the Cohen–Lenstra series, the two cases may be written as follows:

- $R = \mathbb{F}_{p^r}[[u]]$  – Fine–Herstein [6, p. 499, Theorem 1] (see also item 2 on the list of Huang [10, p. 27]):

$$Z_{\mathbb{F}_{p^r}[[u]]}(w) = \sum_{n \geq 0} \frac{|\mathrm{Nilp}_n(\mathbb{F}_{p^r})|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{j \geq 1} \frac{1}{1 - p^{-rj} w}, \quad (4.1)$$

- $R = \mathbb{F}_{p^r}[[u, v]]$  – Fulman–Guralnick [7, p. 301, Theorem 2.9] (see also item 4 on the list of Huang [10, p. 27]):

$$\begin{aligned} Z_{\mathbb{F}_{p^r}[[u,v]]}(w) &= \sum_{n \geq 0} \frac{|\{(A, B) \in \mathrm{Nilp}_n(\mathbb{F}_{p^r})^2 : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n \\ &= \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{-r(1+j)} w^\ell}. \end{aligned} \quad (4.2)$$

<sup>1</sup>The original Cohen–Lenstra series in [4] are defined for  $R$  a Dedekind domain over  $\mathbb{F}_{p^r}$ .

The structure of (4.1) is particularly simple, which allows us to use Proposition 2.2 to derive not only an asymptotic expression but even a closed-form series representation for its coefficients. For  $w \in \mathbb{C}$ , we define

$$Z_{m, \mathbb{F}_{p^r}[[u]]}(w) := \prod_{\substack{j \geq 1 \\ j \neq m}} \frac{1}{1 - p^{-rj}w}.$$

**Theorem 4.1.** *Let  $p$  be a prime and  $r \in \mathbb{N}$ . Then we have, for  $n \in \mathbb{N}_0$ ,*

$$\frac{|\text{Nilp}_n(\mathbb{F}_{p^r})|}{|\text{GL}_n(\mathbb{F}_{p^r})|} = \sum_{m \geq 1} \frac{Z_{m, \mathbb{F}_{p^r}[[u]]}(p^{rm})}{p^{rnm}}.$$

*Proof.* We apply Proposition 2.2 with  $a_j := p^{rj}$ . As  $p^r > 1$ , the series  $\sum_{m \geq 1} p^{-rm}$  converges and  $a_m \neq a_n$  for all  $m \neq n$ . By the product rule (see (2.1) for the definition of  $F$ )

$$F'(p^{rm}) = -\frac{1}{p^{rm} Z_{m, \mathbb{F}_{p^r}[[u]]}(p^{rm})}.$$

Using this, we obtain the theorem.  $\square$

**Example.** The series in Theorem 4.1 is actually convergent. We choose  $p = 2$  and  $r = 1$  in this example. And let

$$Z_{m, \mathbb{F}_2[[u]], N}(w) := \prod_{\substack{N \geq j \geq 1 \\ j \neq m}} \frac{1}{1 - 2^{-j}w}.$$

We have

$$Z_{\mathbb{F}_2[[u]]}(w) = \prod_{j \geq 1} \frac{1}{1 - 2^{-j}w} = 1 + w + \frac{2}{3}w^2 + \frac{4}{11}w^3 + \frac{64}{315}w^4 + \dots$$

On the other hand, for  $N = 100$ , one finds numerically

$$\begin{aligned} \sum_{1 \leq m \leq 100} Z_{m, \mathbb{F}_2[[u]], 100}(2^m) &= 0.999999999999999667\dots, \\ \sum_{1 \leq m \leq 100} \frac{Z_{m, \mathbb{F}_2[[u]], 100}(2^m)}{2^m} &= 0.999999999999999998\dots, \\ \sum_{1 \leq m \leq 100} \frac{Z_{m, \mathbb{F}_2[[u]], 100}(2^m)}{2^{2m}} &= 0.666666666666666666\dots, \\ \sum_{1 \leq m \leq 100} \frac{Z_{m, \mathbb{F}_2[[u]], 100}(2^m)}{2^{3m}} &= 0.380952380952380952\dots, \\ \sum_{1 \leq m \leq 100} \frac{Z_{m, \mathbb{F}_2[[u]], 100}(2^m)}{2^{4m}} &= 0.203174603174603174\dots \end{aligned}$$

These sums represent the cases  $0 \leq n \leq 4$ . Note that also the degenerate case  $n = 0$  is included.

## 5. Questions for future research

We propose the following research questions.

- (1) Is there any recursive formula for the  $Q_{p^r}(n)$  (or quotients of it by normalization factors) that is analogous to the divisor recurrence formula/the Euler Pentagonal Theorem [1, (1.3.1)]:

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}?$$

In the second scenario there is no exact identity but rather an error term smaller than all exponential functions in the main series. Is there a combinatorial interpretation for these recurrences and the error term?

- (2) Can we improve the bounds for  $C_{m,p^r}(n)$ ? What can we say about these coefficients for fixed  $m$ ?
- (3) How could the methods in this paper be extended to deal with multisum-like products as (4.2)?

**Acknowledgements.** The first author has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No. 101001179). The second author has been supported by the Austrian Science Fund (grant agreement No. 10.55776/F1002). The authors are grateful to Yifeng Huang for helpful conversations regarding the background of Cohen–Lenstra series.

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(K. Bringmann) UNIVERSITY OF COLOGNE, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, WEYERTAL 86-90, 50931 COLOGNE, GERMANY

*Email address:* `kbringma@uni-koeln.de`

(S. Chern) FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, WIEN 1090, AUSTRIA

*Email address:* `chenxiaohang92@gmail.com`, `xiaohangc92@univie.ac.at`

(J. Franke) UNIVERSITY OF COLOGNE, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, WEYERTAL 86-90, 50931 COLOGNE, GERMANY

*Email address:* `jfrank12@uni-koeln.de`

(B. Heim) UNIVERSITY OF COLOGNE, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, WEYERTAL 86-90, 50931 COLOGNE, GERMANY

*Email address:* `bheim@uni-koeln.de`