

# Multiple Rogers–Ramanujan type identities for torus links

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# Motivating Questions

**(Elementary School “College Algebra”) Question.** How many pairs of numbers  $a, b$  among 0 and 1 are there such that  $a \times b$  and  $b \times a$  have the same remainder if divided by 2?

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**Answer:**  $2 \times 2 = 4$ .

# Motivating Questions

**(More Challenging) Question.** Over  $\mathbb{F}_2$ , how many pairs  $(A_2, B_2) \in \text{Mat}_2(\mathbb{F}_2)^2$  of two-by-two matrices are there such that

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```
In[ ]:= entry = {0, 1};  
row = Tuples[{entry, entry}];  
mat = Tuples[{row, row}];  
matpair = Tuples[{mat, mat}];  
commutingmatpair = Select[matpair, Mod[#[[1]].#[[2]], 2] == Mod[#[[2]].#[[1]], 2] &];  
Length@commutingmatpair  
  
Out[ ]:= 88
```



## PAIRS OF COMMUTING MATRICES OVER A FINITE FIELD

BY WALTER FEIT AND N. J. FINE

In this paper, we determine the number of ordered pairs of commuting  $n$  by  $n$  matrices over  $GF(q)$  and give a simple generating function for this number.

### Theorem (Feit–Fine, 1960)

$$\sum_{n \geq 0} \frac{|\{(A, B) \in \text{Mat}_n(\mathbb{F}_q)^2 : AB = BA\}|}{|\text{GL}_n(\mathbb{F}_q)|} t^n = \prod_{i,j \geq 1} \frac{1}{1 - q^{2-j} t^i}.$$

**Note.**  $|\text{GL}_n(\mathbb{F}_q)| = q^{n^2} \prod_{j=1}^n (1 - q^{-j})$ .

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and

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Mutually annihilating matrices, and a  
Cohen–Lenstra series for the nodal singularity



Yifeng Huang

*Dept. of Mathematics, University of British Columbia, Canada*

## Theorem (Huang, 2023)

$$\sum_{n \geq 0} \frac{|\{(A, B) \in \text{Mat}_n(\mathbb{F}_q)^2 : AB = BA \text{ and } f(A, B) = 0\}|}{|\text{GL}_n(\mathbb{F}_q)|} t^n = \widehat{Z}_{\mathbb{F}_q[x, y]/f(x, y)}(t),$$

where  $f$  is a given polynomial over  $\mathbb{F}_q$ .

# Algebraic Geometry

*What is  $\widehat{Z}_{\mathbb{F}_q[x,y]/f(x,y)}(t)$ ? Where does it come from?*

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**(NOT MY) Answer to Question 1:** Let  $R$  be the complete local ring of the germ of a certain  $\mathbb{K}$ -curve at a  $\mathbb{K}$ -point with  $\mathbb{K}$  a fixed field.

By denoting  $\mathrm{Coh}_n(R)$  the stack of  $R$ -modules of  $\mathbb{K}$ -dimension  $n$ , the *motivic Cohen–Lenstra zeta function* is defined by

$$\widehat{Z}_R(t) := \sum_{n \geq 0} [\mathrm{Coh}_n(R)] t^n.$$



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$$\widehat{Z}_R(t) := \sum_{n \geq 0} [\mathrm{Coh}_n(R)] t^n.$$

If we normalize  $R$  as  $\widetilde{R}$ , then we may define the *numerator part*:

$$\widehat{N}_R(t) := \frac{\widehat{Z}_R(t)}{\widehat{Z}_{\widetilde{R}}(t)},$$

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where (with  $\mathbb{L} := [\mathbb{A}^1]$ ) the *Lefschetz motive*; for  $\mathbb{K} = \mathbb{F}_q$ , we have  $\mathbb{L} = q$ )


$$\widehat{Z}_{\widetilde{R}}(t) = \prod_{j \geq 0} \frac{1}{(1 - t\mathbb{L}^{-j-1})^s}.$$

## Interesting Varieties $f(x, y)$ :

- Torus knots  $R^{(2, 2k+1)}$ : the germ of  $y^2 = x^{2k+1}$ ;
- Torus links  $R^{(2, 2k)}$ : the germ of  $y(y - x^k) = 0$

## Interesting Varieties $f(x, y)$ :

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- Torus links  $R^{(2, 2k)}$ : the germ of  $y(y - x^k) = 0$  (if  $\mathbb{K}$  is not of characteristic two, then  $R^{(2, 2k)}$  also admits the variety  $y^2 = x^{2k}$ ).



## q-series from counting matrix points

Yifeng Huang University of British Columbia [huangyf@math.ubc.ca](mailto:huangyf@math.ubc.ca)

### INTRODUCTION

An (affine) variety over  $\mathbb{Z}$  is a system of polynomial equations  $f_1(T_1, \dots, T_n) = \dots = f_r(T_1, \dots, T_n) = 0$  with integer coefficients. Counting finite field points on varieties is of fundamental importance in number theory and arithmetic geometry.

**Question.** Can we count matrix points?

**Definition** (Matrix point). Given  $n \in \mathbb{Z}_{\geq 1}$  and finite field  $\mathbb{F}_q$ , a  $\text{Mat}_n(\mathbb{F}_q)$  point on the said variety is a tuple of pairwise commuting matrices  $A = (A_1, \dots, A_m)$  in  $\text{Mat}_n(\mathbb{F}_q)$  such that  $f_i(A) = O_{n \times n}$  for all  $i$ .

**Short answer.** Yes, for smooth curves, smooth surfaces, and some singular curves!!! And partitions appear in the formulas.

EXAMPLE: MATRIX POINTS ON PLANE

**Theorem** (Folt-Fine, '60).

$$\sum_{n \geq 0} \frac{|\{(A, B) \in \text{Mat}_n(\mathbb{F}_q)^2 : AB = BA\}|}{|\text{GL}_n(\mathbb{F}_q)|} q^n = \prod_{i \geq 1} \frac{1}{1 - q^{i-1}}$$

SMOOTH VARIETIES AND SATO-TATE

Using geometric argument, one can bootstrap from Folt-Fine and get

**Theorem 1** (H.). If the variety is smooth of  $\dim \leq 2$  over  $\mathbb{F}_q$ , then its matrix point counts are determined by (usual) point counts over finite extensions of  $\mathbb{F}_q$ .  
More precisely, the analogous generating function counting its matrix points is an explicit infinite product in its zeta function.

This allows to lift deep theorems about finite field point counts to matrix point counts. For example, consider the Legendre elliptic curve



$$E_\lambda : y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1$$

and the Alghero-Cro-Perridon K3 surface

$$X_\lambda : z^2 = x(x+1)(y+1)(z+\lambda y), \quad \lambda \neq 0, -1.$$

They are special, for their finite field point counts are given by Andrei hypergeometric functions  ${}_2F_1$  and  ${}_2F_2$  respectively, and as  $\lambda \in \mathbb{F}_q$  varies, the normalized error terms in the point counts follow Sato-Tate type distributions (Dro-Saad-Sepke).

**Theorem 2** (H.-Dro-Saad). For any fixed  $n \in \mathbb{Z}_{\geq 1}$ , analogous statements hold for  $\text{Mat}_n(\mathbb{F}_q)$  points on  $E_\lambda$  and  $X_\lambda$ . In addition, the explicit formulas involve partitions of size up to  $n$ .

SINGULAR CURVES AND ROGERS-RAMANUJAN

Consider the singular curve  $C_k : Y^2 = X^k$  for  $k \in \mathbb{Z}_{\geq 2}$

**Theorem 3** (H.-Jiang). A series  $f_k(q, t) \in \mathbb{Z}[q^{-1}, t]$  encoding matrix point counts on  $C_k$  satisfies (i)  $f_k(q, t)$  converges for  $|q| > 1, |t| \leq 1$ ; (ii)  $f_k(q, 1)$  is a modular function in  $t$  with  $q^{-1} = e^{2\pi i \tau}$ .

The content is two-fold. On one hand, by giving  $f_k(q, 1)$ , we explicitly answer a wild binary algebra question: count solutions to  $A^2 = B^2$ ,  $B^2 = A^k$  for  $A, B \in \text{Mat}_n(\mathbb{F}_q)$ . On the other hand, the assertion that  $f_k(q, 1)$  equals a modular function amounts to a Rogers-Ramanujan type identity.

**Example 4** (Andrews-Gordon from old K3). For  $k = 2m+1$ ,

$$f_k(q, 1) = \sum_{\lambda \vdash n} q^{|\lambda|} \prod_{i \geq 1} \frac{q^{2i\lambda_i}}{(1 - q^{2i})^{2\lambda_i}}$$

so by an Andrews-Gordon identity

$$f_k(q, 1) = \prod_{n \in \mathbb{Z}_{\geq 1} (2m+1)} (1 - q^{-n})^{-1}$$

**Example 5** (New identity from even  $k$ ). Let  $\mathcal{G}_k^{\text{even}}(q)$  be the Hall polynomial counting type  $\mu$  subgroups of an abelian group of type  $\lambda$ .

$$f_{2m}(q, 1) = \left( q^{1/2} \prod_{i \geq 1} \frac{q^{2i\lambda_i}}{(1 - q^{2i})^{2\lambda_i}} \right) \left( q^{1/2} \prod_{i \geq 1} \frac{q^{2i\lambda_i}}{(1 - q^{2i})^{2\lambda_i}} \right)$$

and we prove the identity (indirectly, no direct proof yet)

$$f_{2m}(q, 1) = 1. \quad (1)$$

**Conjecture.** Analogous of Theorem 3 holds for all plane curves.

If true, the conjecture would produce a new framework for Rogers-Ramanujan identities: take a plane singularity, find the series  $f_k(q, t)$  by counting matrix points, and get

sum side =  $f_k(q, 1)$  = product side = modular function.

Andrews-Berndt Conference at Penn State, June 6-9, 2024

# Algebraic Geometry

- Torus knots  $R^{(2,2k+1)}$ ?

Theorem (Huang–Jiang, 2023)

$$\widehat{\mathcal{N}}_{R^{(2,2k+1)}}(t) = \sum_{n_1, \dots, n_k \geq 0} \frac{t^{\sum_{i=1}^k 2n_i \mathbb{L} - \sum_{i=1}^k n_i^2}}{\prod_{i=1}^k \prod_{j=1}^{n_i - n_{i-1}} (1 - \mathbb{L}^{-j})}.$$

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*In particular,*

$$\begin{aligned} & \widehat{\mathcal{N}}_{R^{(2,2k+1)}}(\pm 1) \\ &= \prod_{j \geq 0} \frac{(1 - \mathbb{L}^{-(2k+3)j - (k+1)})(1 - \mathbb{L}^{-(2k+3)j - (k+2)})(1 - \mathbb{L}^{-(2k+3)j - (2k+3)})}{1 - \mathbb{L}^{-j-1}}. \end{aligned}$$

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**Andrews' multiple Roger–Ramanujan type identity:**

$$\sum_{n_1, \dots, n_k \geq 0} \frac{q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q; q)_{n_1}} = \frac{(q^{k+1}, q^{k+2}, q^{2k+3}; q^{2k+3})_{\infty}}{(q; q)_{\infty}}.$$



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**GOOD news:** There is one proved Roger–Ramanujan type identity and another conjectural one.

Theorem/Conjecture (Huang–Jiang, 2023)

$$\widehat{\mathcal{NZ}}_{R^{(2,2k)}}(1) = 1$$

and

$$\widehat{\mathcal{NZ}}_{R^{(2,2k)}}(-1) \stackrel{?}{=} \prod_{j \geq 1} \frac{(1 - \mathbb{L}^{-2j})(1 - \mathbb{L}^{-(k+1)j})^2}{(1 - \mathbb{L}^{-j})^2(1 - \mathbb{L}^{-(2k+2)j})}.$$

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**BAD news AGAIN:**  $\widehat{\mathcal{NZ}}_{R^{(2,2k)}}(1) = 1$  can ONLY be proved by hardcore techniques in algebraic geometry.

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$$\widehat{\mathcal{N}}_{R^{(2,2k)}}(t)|_{\mathbb{L} \mapsto q^{-1}} = (tq; q)_{\infty}^2 \mathcal{Z}_k(t, q),$$

where

$$\begin{aligned} \mathcal{Z}_k(t, q) := & \sum_{\substack{r_k \geq \dots \geq r_1 \geq 0 \\ s_k \geq \dots \geq s_1 \geq 0}} \frac{t^{\sum_{i=1}^k (2r_i - s_i)} q^{\sum_{i=1}^k (r_i^2 - r_i s_i + s_i^2)}}{(q; q)_{r_k - r_{k-1}} \cdots (q; q)_{r_2 - r_1} (tq; q)_{r_1}^2 (q; q)_{s_1}} \\ & \times \begin{bmatrix} r_k - s_{k-1} \\ r_k - s_k \end{bmatrix}_q \begin{bmatrix} r_{k-1} - s_{k-2} \\ r_{k-1} - s_{k-1} \end{bmatrix}_q \cdots \begin{bmatrix} r_2 - s_1 \\ r_2 - s_2 \end{bmatrix}_q \begin{bmatrix} r_1 \\ r_1 - s_1 \end{bmatrix}_q. \end{aligned}$$

Theorem (C., 2024, reproving H–J's  $\widehat{\mathcal{NZ}}_{R(2,2k)}(1) = 1$ )

$$\mathcal{Z}_k(1, q) = \frac{1}{(q; q)_{\infty}^2}.$$



Theorem (C., 2024, reproving H–J's  $\widehat{\mathcal{NZ}}_{R^{(2,2k)}}(1) = 1$ )

$$\mathcal{Z}_k(1, q) = \frac{1}{(q; q)_{\infty}^2}.$$

Opening the  $q$ -binomial coefficients and making the substitutions

$$d_i := \begin{cases} s_1, & i = 1, \\ s_i - s_{i-1}, & i \geq 2, \end{cases} \quad \text{and} \quad n_j := r_j - s_j,$$

we find that  $\mathcal{Z}_k(1, q)$  equals

$$\sum_{s_1, \dots, s_k \geq 0} \frac{q^{\sum_{i=1}^k s_i^2}}{(q; q)_{s_2-s_1} \cdots (q; q)_{s_k-s_{k-1}} (q; q)_{s_1}^2} \\ \times \sum_{n_1, \dots, n_k \geq 0} \frac{q^{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (d_1 + \cdots + d_i) n_i} (q; q)_{n_k+d_k} \cdots (q; q)_{n_2+d_2}}{(q; q)_{n_k-n_{k-1}+d_k} \cdots (q; q)_{n_2-n_1+d_2} (q; q)_{n_k} \cdots (q; q)_{n_1} (q; q)_{n_1+d_1}}.$$

# Rogers–Ramanujan

Believe it or not — But at least trust **W. N. Bailey** — the  $d_1 = \dots = d_k = 0$  specialization is an instance of Bailey pairs:

$$\frac{1}{(q; q)_N (aq; q)_{N+d_1+\dots+d_k}} = \sum_{n_1, \dots, n_k \geq 0} \frac{a^{\sum_{i=1}^k n_i} q^{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (d_1 + \dots + d_i) n_i} (q; q)_{n_k+d_k} \cdots (q; q)_{n_2+d_2}}{(q; q)_{N-n_k} (q; q)_{n_k-n_{k-1}+d_k} \cdots (q; q)_{n_2-n_1+d_2} (q; q)_{n_k} \cdots (q; q)_{n_1} (aq; q)_{n_1+d_1}}.$$

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What does the above imply?

$$\begin{aligned} \mathcal{Z}_k(1, q) &= \sum_{s_1, \dots, s_k \geq 0} \frac{q^{\sum_{i=1}^k s_i^2}}{(q; q)_{s_2-s_1} \cdots (q; q)_{s_k-s_{k-1}} (q; q)_{s_1}^2} \times \frac{1}{(q; q)_\infty} \\ &= \frac{1}{(q; q)_\infty} \times \frac{1}{(q; q)_\infty}! \end{aligned}$$

# Rogers–Ramanujan

Iterate the basic hypergeometric transform:

$$\sum_{n \geq 0} \frac{a^n q^{n^2 + Mn}}{(q; q)_{N-n} (q; q)_n (aq; q)_{M+n}} = \frac{1}{(q; q)_N (aq; q)_{M+N}}.$$

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**Step 1.** Make the reformulation

$$\begin{aligned} \frac{1}{(q; q)_N (aq; q)_{(M'+M'')+N}} &= \sum_{L \geq 0} \frac{a^L q^{L^2 + (M'+M'')L} (q; q)_{L+M'}}{(q; q)_{N-L} (q; q)_L} \\ &\quad \times \frac{1}{(q; q)_{L+M'} (aq; q)_{M''+(L+M')}}. \end{aligned}$$

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**Step 2.** Apply the same transform to

$$\frac{1}{(q; q)_{L+M'} (aq; q)_{M''+(L+M')}}.$$

in the summand.

**BAD news AGAIN:** This strategy does NOT work for the evaluation of  $\widehat{N}_{R^{(2,2k)}}(-1)\dots$

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Recall that

$$\widehat{\mathcal{N}}_{R^{(2,2k+1)}}(t)|_{\mathbb{L}_t \rightarrow q^{-1}} = \sum_{n_1, \dots, n_k \geq 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q; q)_{n_1}}.$$



## Theorem (C., 2024)

$$\mathcal{Z}_k(t, q) = \frac{1}{(tq; q)_\infty} \sum_{n_1, \dots, n_k \geq 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q; q)_{n_1} (tq; q)_{n_1}}.$$

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Case  $t = 1$ :

$$\mathcal{Z}_k(1, q) = \frac{1}{(q; q)_\infty} \sum_{n_1, \dots, n_k \geq 0} \frac{q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q; q)_{n_1}^2} = \frac{1}{(q; q)_\infty^2}.$$

## Theorem (C., 2024)

$$\mathcal{Z}_k(t, q) = \frac{1}{(tq; q)_\infty} \sum_{n_1, \dots, n_k \geq 0} \frac{t^{\sum_{i=1}^k 2n_i} q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q; q)_{n_1} (tq; q)_{n_1}}.$$

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Case  $t = -1$  (Thanks to David Bressoud's discovery in 1980!):

$$\begin{aligned} \mathcal{Z}_k(-1, q) &= \frac{1}{(q; q)_\infty} \sum_{n_1, \dots, n_k \geq 0} \frac{q^{\sum_{i=1}^k n_i^2}}{(q; q)_{n_k - n_{k-1}} \cdots (q; q)_{n_2 - n_1} (q^2; q^2)_{n_1}} \\ &= \frac{(q^{k+1}; q^{k+1})_\infty^2}{(q^2; q^2)_\infty (q^{2k+2}; q^{2k+2})_\infty}. \end{aligned}$$

**So Huang–Jiang's Conjectural evaluation of  $\widehat{\mathcal{NZ}}_{R(2,2k+1)}(-1)$  is CORRECT!**

Ole Warnaar made an exposition of David Bressoud's identity and other similar Rogers–Ramanujan type identities at this **SLC '42 — The Andrews Festschrift!**

Séminaire Lotharingien de Combinatoire, B42n (1999), 22 pp.

## S. Ole Warnaar

### Supernomial Coefficients, Bailey's Lemma and Rogers-Ramanujan-Type Identities. A Survey of Results and Open Problems

**Abstract.** An elementary introduction to the recently introduced  $A_2$  Bailey lemma for supernomial coefficients is presented. As illustration, some  $A_2$   $q$ -series identities are (re)derived which are natural analogues of the classical ( $A_1$ ) Rogers-Ramanujan identities and their generalizations of Andrews and Bressoud. The intimately related, but unsolved problems of supernomial inversion,  $A_{n-1}$  and higher level extensions are also discussed. This yields new results and conjectures involving  $A_{n-1}$  basic hypergeometric series, string functions and cylindric partitions.

Received: December 17, 1998; Accepted: May 10, 1999.

# How to achieve that $k$ -fold sum for $\mathcal{Z}_k(t, q)$ ?

**Step 1.** Prove the reformulation:

$$\begin{aligned} \mathcal{Z}_k(N; t, q) &= \frac{(q; q)_\infty (t^2 q; q)_\infty}{(tq; q)_\infty^2} \\ &\times \sum_{\substack{m_1, \dots, m_k \geq 0 \\ n_1, \dots, n_k \geq 0}} \frac{t^{-2n_1 + \sum_{i=1}^k (m_i + 2n_i)} q^{-n_1^2 + n_1 + \sum_{i=1}^k (m_i^2 + m_i n_i + n_i^2)} (t; q)_{n_1}^2}{(q; q)_{N-m_k} (t^2 q; q)_{N+n_k} (q; q)_{m_k} (q; q)_{m_1} (q; q)_{n_1}} \\ &\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q. \end{aligned}$$

# How to achieve that $k$ -fold sum for $\mathcal{Z}_k(t, q)$ ?

Step 1 (BONUS!) — A **Fake**  $A_2$  Rogers–Ramanujan type identity:

$$\frac{(q^2; q^2)_\infty (q^{k+1}; q^{k+1})_\infty^2}{(q; q)_\infty^3 (q^{2k+2}; q^{2k+2})_\infty} = \sum_{\substack{m_1, \dots, m_k \geq 0 \\ n_1, \dots, n_k \geq 0}} \frac{(-1)^{\sum_{i=1}^k m_i} q^{-n_1^2 + n_1 + \sum_{i=1}^k (m_i^2 + m_i n_i + n_i^2)} (-1; q)_{n_1}^2}{(q; q)_{m_k} (q; q)_{m_1} (q; q)_{n_1}}$$

$$\times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$

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Andrews–Schilling–Warnaar's **Authentic**  $A_2$  type identity:

$$\frac{(q^{k+1}, q^{k+1}, q^{k+2}, q^{2k+2}, q^{2k+3}, q^{2k+3}, q^{3k+4}, q^{3k+4}; q^{3k+4})_\infty}{(q; q)_\infty^3} \\ = \sum_{\substack{m_1, \dots, m_k \geq 0 \\ n_1, \dots, n_k \geq 0}} \frac{q^{\sum_{i=1}^k (m_i^2 - m_i n_i + n_i^2)} (1 - q^{m_1 + n_1 + 1})}{(q; q)_{m_k} (q; q)_{n_k} (q; q)_{m_1 + n_1 + 1}} \\ \times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_q.$$

# How to achieve that $k$ -fold sum for $\mathcal{Z}_k(t, q)$ ?

**Step 2.** Evaluate the semi-truncation (as a  $k$ -fold sum):

$$\mathcal{V}_k(N; t, q) := \sum_{\substack{m_1, \dots, m_k \geq 0 \\ n_1, \dots, n_k \geq 0}} \frac{t^{-2n_1 + \sum_{i=1}^k (m_i + 2n_i)} q^{-n_1^2 + n_1 + \sum_{i=1}^k (m_i^2 + m_i n_i + n_i^2)} (t, q)_{n_1}^2}{(q; q)_{N-m_k} (q; q)_{m_k} (q; q)_{m_1} (q; q)_{n_1}} \\ \times \begin{bmatrix} m_k \\ m_{k-1} \end{bmatrix}_q \begin{bmatrix} m_{k-1} \\ m_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \begin{bmatrix} n_{k-1} \\ n_k \end{bmatrix}_q.$$



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**Step 3.** Use the relation and reduce  $\mathcal{Z}_k(N; t, q)$  to the desired  $k$ -fold sum:

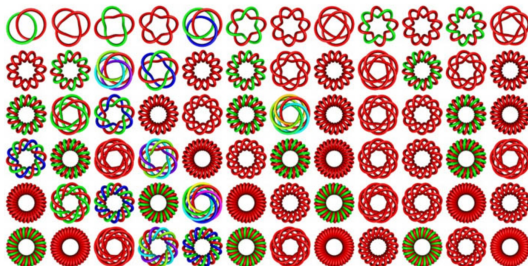
$$\begin{aligned} \mathcal{Z}_k(N; t, q) = & \frac{(q; q)_\infty (t^2 q; q)_\infty}{(tq; q)_\infty^2} \sum_{m, n \geq 0} t^{m+2(k-1)n} q^{m^2 + (m+1)n + (k-1)n^2} \\ & \times \frac{(t; q)_n^2}{(q; q)_{N-m} (t^2 q; q)_{N+n} (q; q)_n} \mathcal{V}_{k-1}(m; tq^n, q). \end{aligned}$$

# Finitization

All results in this talk can be **finitized** — and their finitizations, from the view of Algebraic Geometry, align with the **Quot zeta functions**.



S. Chern, Multiple Rogers–Ramanujan type identities for torus links, preprint, <https://arxiv.org/abs/2411.07198>.



(Taken from [https://knotplot.com/knot-theory/torus\\_xing.html](https://knotplot.com/knot-theory/torus_xing.html))

# Obrigado!