Optimal Diagonal Width for a Given Sparsity

Donnelly Shane

Variable Definitions

$$
\begin{aligned}
\text{sparsity} \in [0,1] &\quad : \text{sparsity parameter} \\
n &\quad : \text{matrix length (assuming square matrix)} \\
\text{diagonal\_width} &\quad : \text{width of the diagonal band (odd number)} \\
\text{density} = 1 - \text{sparsity} &\quad : \text{matrix density} \\
\text{sdw} = \frac{\text{diagonal\_width} - 1}{2} &\quad : \text{semi-diagonal width}
\end{aligned}
$$

Conditions

- `diagonal_width` is an odd number between 0 and $2n - 1$
- If `diagonal_width = 0`, then density $= 0$ and sparsity $= 1$
- If `diagonal_width` $\geq 2n - 1$, then density $= 1$ and sparsity $= 0$

Diagonal Area Calculation

The diagonal area represents the number of elements covered by the diagonal band.

$$
\begin{aligned}
\text{diagonal\_area} &= n + 2\sum_{i=1}^{sdw}(n - i) \\
&= n(1 + 2sdw) - 2\sum_{i=1}^{sdw} i \\
&= n(1 + 2sdw) - sdw(sdw + 1)
\end{aligned}
$$

Diagonal Width from Given Sparsity

To find the optimal diagonal width for a given sparsity and matrix length (n):

$$\begin{aligned}
\text{density} &= 1 - \text{sparsity} \\
d &= 1 - s \\
\text{target_diagonal_area} &= n^2 \cdot d \\
\text{target_diagonal_area} &= n(1 + 2sdw) - sdw(sdw + 1) \\
\text{(Using the formula above)} &
\end{aligned}$$

Now we just rename $sdw \rightarrow x$ and $target\_diagonal\_area \rightarrow da$:

$$\begin{aligned}
da &= n(1 + 2x) - x(x + 1) \\
&\Rightarrow n + 2nx - x^2 - x - da = 0 \\
&\Rightarrow -x^2 + (2n - 1)x + (n - da) = 0
\end{aligned}$$

This gives a quadratic equation:

$$a = -1, \quad b = 2n - 1, \quad c = n - da \text{ (with } da = n(1 - s))$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{(the real solution)}$$

The chosen sdw is then:

$$sdw = \text{round}(x)$$

Final Formula

By adding everything together, we get:

$$\text{best\_diagonal\_width}(n, s) = 2 \cdot \left(\text{round} \left(\frac{2n - 1 - \sqrt{(2n - 1)^2 + 4(n - n^2(1 - s))}}{2} \right) \right) + 1$$

which is odd as expected.