Introduction - linear oscillators

Depending on what classes you have taken, some of this material may be familiar or little rusty. We will work through these notes and the associated problems in class. The notes are not complete and in many cases may assume things that you do not know. We will just use these notes as a guide and will fill in many gaps in class. Depending on what courses you have taken, you may need to fill in a few more gaps by consulting us or your peers. These notes and problems are meant to guide you through the exploration of some key ideas. Many of the problems are written for you to explore some ideas as opposed to us telling you precisely what to do step by step. Thus many times you will read something and have questions. If you have trouble understanding the problem, just ask.

1 Concepts from Linearity I or QEA

Let's start by reviewing a few key ideas you should have seen prior to this course. Hopefully, as you work through this first section some of these ideas will come back to you. If they don't we can spend a little more time here as eigenvalues and eigenvectors will be central to much of our analysis throughout the course.

1.1 Solutions to ODEs

You may recall that when we have an equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{1}$$

the solutions can be written as the linear combination,

$$\mathbf{x}(t) = c_1\mathbf{v}_1\mathrm{e}^{\lambda_1 t} + c_2\mathbf{v}_2\mathrm{e}^{\lambda_2 t} + c_3\mathbf{v}_3\mathrm{e}^{\lambda_3 t} + ... \tag{2}$$

Here $\mathbf{v}_k$ and λ_k are the k^{th} eigenvectors and eigenvalues of the matrix A. The sum is taken over all the eigenvalues/vectors of A. When A is a 2x2 matrix the sum is taken over two eigenvectors, when A is 3x3 the sum is taken over three eigenvectors, and so on.

Recall that the eigenvalues λ and eigenvectors $\mathbf{v}$ of an $N \times N$ matrix A satisfy $A\mathbf{v} = \lambda\mathbf{v}$. On rearrangement we seek solutions of $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Non-trivial solutions exist if and only if $\det(A - \lambda I) = 0$. The resulting characteristic equation is an Nth order polynomial in λ with precisely N roots (counting multiplicities). If the eigenvalues are all distinct then there are precisely N corresponding eigenvectors.

1.2 Checking form of the solution

It is easy to confirm that this form of the solution satisfies the differential equation by simply plugging it in;

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \mathbf{v}_3 e^{\lambda_3 t} + ...\right) = c_1\lambda_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\lambda_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\lambda_3\mathbf{v}_3 e^{\lambda_3 t} + ...$$

Since $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, and so on the right side of the above equation becomes,

$$= c_1 A\mathbf{v}_1 e^{\lambda_1 t} + c_2 A\mathbf{v}_2 e^{\lambda_2 t} + c_3 A\mathbf{v}_3 e^{\lambda_3 t} + ... = A\left(c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}\right) = A\mathbf{x}.$$

Therefore, we can see that the equation is satisfied.

The constants $c_1, c_2, c_3...$ are found from the initial condition. If we take $\mathbf{x}(t = 0)$ as the known, then

$$\mathbf{x}(0) = c_1\mathbf{v}_1 e^{\lambda_1 0} + c_2\mathbf{v}_2 e^{\lambda_2 0} + c_3\mathbf{v}_3 e^{\lambda_3 0} + ... = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + ...$$

If we define P as the matrix of eigenvectors as columns, then the previous equation is equivalent to

$$\mathbf{x}(0) = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|...]\,[c_1, c_2, c_3, ...]^T = P\mathbf{c}$$

where c is the vector of coefficients, $\mathbf{c} = [c_1, c_2, c_3, ..]^T$. Note that everything here is assuming that A diagonalizes. We have not discussed cases where A does not diagonalize, but let's ignore this for now.

1.3 Deriving the form of the solution

Sometimes, just checking that a solution works is less than satisfying. Let's try an alternate derivation of the form of the solution. If A diagonalizes, then it's eigenvectors are linearly independent. Therefore, we can define $\mathbf{x}(t)$ at any instant in time as a linear combination of the eigenvectors of A,

$$\mathbf{x}(t) = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2 + c_3(t)\mathbf{v}_3 + ...$$

Note that since the eigenvectors of A are fixed, the constants $c_1(t)$ must be functions of time. Substituting this form into the original differential equation $d\mathbf{x}/dt = A\mathbf{x}$ we have,

$$\frac{dc_1(t)}{dt}\mathbf{v}_1 + \frac{dc_2(t)}{dt}\mathbf{v}_2 + \frac{dc_3(t)}{dt}\mathbf{v}_3 + ... = c_1(t)A\mathbf{v}_1 + c_2(t)A\mathbf{v}_2 + c_3(t)A\mathbf{v}_3 + ...$$

Since $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ (and so on for all subscripts),

$$\frac{dc_1(t)}{dt}\mathbf{v}_1 + \frac{dc_2(t)}{dt}\mathbf{v}_2 + \frac{dc_3(t)}{dt}\mathbf{v}_3 + ... = c_1(t)\lambda_1\mathbf{v}_1 + c_2(t)\lambda\mathbf{v}_2 + c_3(t)\lambda\mathbf{v}_3 + ...$$

Collecting terms for each eigenvector yields,

$$\left(\frac{dc_1(t)}{dt} - \lambda_1 c_1(t)\right)\mathbf{v}_1 + \left(\frac{dc_2(t)}{dt} - \lambda_2 c_2(t)\right)\mathbf{v}_2 + \left(\frac{dc_3(t)}{dt} - \lambda_3 c_3(t)\right)\mathbf{v}_3 + ... = 0.$$

Since the eigenvectors are linearly independent, then ALL the terms in parentheses must equal zero. Therefore, each coefficient must satisfy,

$$\frac{dc_1(t)}{dt} - \lambda_1 c_1(t) = 0.$$

Separating variables for this equation,

$$\frac{dc_1(t)}{c_1} = \lambda_1 dt,$$

integrating,

$$\ln(c_1) = \lambda_1 t + C,$$

taking the exponential of the result,

$$c_1 = \mathrm{e}^{\lambda_1 t + C} = \mathrm{e}^C \mathrm{e}^{\lambda_1 t},$$

and applying the initial condition yields the expression for the time dependent coefficients,

$$c_1 = c_1(t = 0)\mathrm{e}^{\lambda_1 t}.$$

Going back to the expression for $\mathbf{x}$ we have,

$$\mathbf{x}(t) = c_1(0)\mathbf{v}_1\mathrm{e}^{\lambda_1 t} + c_2(0)\mathbf{v}_2\mathrm{e}^{\lambda_1 t} + c_3(0)\mathbf{v}_3\mathrm{e}^{\lambda_1 t} + ...$$

Now, using the fact that the initial condition $\mathbf{x}(t = 0)$ is known, we see that the coefficients, $c_1(0)$, $c_2(0)$, and so on have the same definition as in our "proof" by checking. Note that the proof in this section has only assumed that A diagonalizes, thus we derived the form of the solution for $\mathbf{x}$, we did not "guess" it.

1.4 Putting it together

Putting these pieces together we have a simple procedure for solving Equation 1,

- Given A, compute P and D, where P is the matrix with eigenvectors as the columns and D is the matrix with corresponding eigenvalues λ_k along the diagonal. Depending on the problem and the goal you may choose to do this symbolically or for particular numbers in A you might use MATLAB as your calculator.

- Look at the eigenvalues so you can imagine what your solution looks like. Recall that if you see all negative real numbers the solutions will decay, if you see any positive real numbers the solutions will grow with time, if you see imaginary parts to the eigenvalues the solutions will oscillate. Remember that just knowing about your eigenvalues gives you immediate qualitative information about the nature of the solution. If you have some physical intuition about your system then hopefully the nature of the eigenvalues fits with that intuition.

- Compute the coefficients, $\mathbf{c}$, using $P^{-1}\mathbf{x}(0) = \mathbf{c}$.

- Now you have all the values to plot $\mathbf{x}(t) = c_1\mathbf{v}_1\mathrm{e}^{\lambda_1 t} + c_2\mathbf{v}_2\mathrm{e}^{\lambda_2 t} + c_3\mathbf{v}_3\mathrm{e}^{\lambda_3 t} + ...$

To implement this in MATLAB, you could do something like this example for a 3x3:

```
A = [ a11 a12 a12; a21 a22 a23; a31 a32 a33];   %%% put actual numbers here.
x0 = [b1;b2;b3];                                %%% put actual numbers here

[P, D] = eig(A);
lam = diag(D);                            %% creates a vector  - for convenience
v1 = P(:,1); v2 = P(:,2); v2 = P(:,2)  %% redefine to match our notation
```

```
c = inv(P)*x0;  %% constants for the eigenvector combinations

t = linspace(0,50,1000);
x = c(1)*v1*exp(lam(1)*t) + c(2)*v2*exp(lam(2)*t) +  c(3)*v3*exp(lam(3)*t);
plot(t,x);  %%% plots all solutions as a function of time
```

1.5 Problems

Implement a MATLAB program to generate solutions for a 2x2 system following the guidance above. Take the system to be

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where $\mathbf{x} = [x, v]^T$. and the matrix A to be that from the mass-spring-damper system,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2b \end{bmatrix}.$$

The eigenvalue for this system are,

$$\lambda = b \pm \sqrt{b^2 - 1}.$$

If you have forgotten about eigenvalues, derive this result.

Now, for distinct values of $b = -1.5, -1.05, -0.95, -.5, -.1, 0, 0.1, 0.5, 0.95, 1.05, 1.5$:

1. Compute the eigenvalues for each value of b. Guess what the solution will look like qualitatively based on this information.

2. Take an initial condition of $\mathbf{x} = [1, 0]^T$. Plot the solutions as a function of time, i.e $x(t)$ and $v(t)$. Explore how the solution looks for different values of b carefully paying attention when the magnitude of b crosses 1.

2 RLC circuit experiment

Let's take and resistor, inductor, and capacitor in series. You should recall from your ISIM days that the fundamental laws for a resistor and capacitor are

$$\Delta V = IR$$

$$I = C\frac{d\Delta V}{dt}$$

where I is the current through the part in amps, ΔV is the voltage across the part in volts, R is the resistance in Ohms and C is the capacitance in farads. An inductor may be less familiar to you, but it's law is

$$\Delta V = L\frac{dI}{dt}$$

where L is the inductance measured in Henry's. The product RC has units of time, while the product of LC has units of time squared. If we connect the parts in series, the total voltage, V, across the three parts is,

$$\Delta V = \Delta V_R + \Delta V_L + \Delta V_C = IR + L\frac{dI}{dt} + \Delta V_C$$

Taking the time derivative yields

$$\frac{dV}{dt} = R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{I}{C}$$

Let's imagine a case where we hold the voltage across the three components constant. In this case we have a final equation for the current.

$$0 = RC\frac{dI}{dt} + LC\frac{d^2I}{dt^2} + I.$$

This equation should suspiciously look like the mass-spring-damper equation.

1. Build the physical RLC circuit on a breadboard, starting with a values of $R = 0$, $C = 1\ \mu$F, and $L = 1$ mH. We will use the equipment for your ISIM days to control the voltage across the circuit with the waveform output and measure the response of the circuit with the scope. Set the voltage across the circuit to be the waveform output. Set the output to square wave with an amplitude 0.5 V and frequency 100 Hz. Monitor the input voltage with one measurement channel and the voltage across the capacitor with another measurement channel. Observe the behavior. If you have forgotten how to build circuits, use the software, or use the analog discovery - just ask for help.

2. Make the equation above dimensionless using the systems natural frequency as the time scale. What is the solution to this equation when $R = 0$. Why is the predicted behavior for the circuit different than what you measured in the previous part? What initial conditions are needed for the equations to represent what you are doing experimentally?

3. In the experiment put a resistor in the circuit. Start with a 10 ohm potentiometer. Watch the output of the circuit as you adjust the potentiometer. Does the result make sense to you?

4. Cast the dimensionless governing equation to 2x2 matrix form. Compute the eignevalues. Compute the value of resistance where the eigenvalues transition from have an imaginary component to have just real components. Test this idea in the experiment by changing R so you are a little above and a little below this criteria and see if the response makes sense to you.

5. Create a Bode plot to test the circuit's response to a sinusoidal driving with $R = 0$. Use the network feature in Waveforms to generate the Bode plot quickly. If you have forgotten how to generate and interpret Bode plots ask for help. Put the 10 ohm potentiometer into the circuit and generate a Bode plot with a few different values of R. Observe the behavior. Create a Bode plot where your value of R is right around the critical point where the eigenvalues transition to having only a real part.

6. Analyze the circuit's response to sinusoidal driving using the ideas of complex impedance. Create a dimensionless theoretical Bode and compare to what you observe in the experiment. If you have forgotten about complex impedance, you can review chapter 6: http://faculty.olin.edu/bstorey/isi

3 Discrete time integration

We have seen that when we have a system of first-order differential equations, we can write the solution to that system based on the eigenvalues and eigenvectors. This method works when we have linear equations and can write the system in matrix-vector form. In many situations, we might have some more complicated right hand side. Many physical systems are non-linear and thus the mathematical tools we have developed thus far are not always adequate. Often in cases where the right side is non-linear, we have to resort to numerical techniques to find the solution. Also, you may have noticed that sometimes seemingly simple changes to the boundary conditions make the analytical solutions more difficult but are trivial to change in our numerical approximations. As a way of studying the basics of numerical techniques for integrating in time, let's explore numerical solutions to our system of linear differential equations.

We can numerically approximate the solution to a differential equation using an approximation for the time derivative. The simplest approximation method is known as Euler's method. Euler's method relies on the basic definition of the derivative. Using Euler's method our matrix equation becomes,

$$\frac{d\mathbf{x}}{dt} \approx \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = A\mathbf{x}(t).$$

Note that the approximation of the derivative becomes precise in the limit that Δt goes to zero. We can re-arrange the above equation to obtain,

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \Delta t A\mathbf{x}(t),$$

or

$$\mathbf{x}(t+\Delta t) = (I + \Delta t A)\,\mathbf{x}(t) \qquad \text{(Forward Euler).}$$

Note that the above equation is a discrete iteration equation. The new value of $\mathbf{x}$ is given as a matrix, $I + \Delta t A$, times the current value of $\mathbf{x}$. This approximation is known as forward Euler since the right hand side of the equation is evaluated using the current state at time, t, and thus the solution can be marched forward in time.

An approximation known as backward Euler uses the future value at time $t + \Delta t$ to evaluate the right hand side of the equation, namely,

$$\frac{d\mathbf{x}}{dt} \approx \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = A\mathbf{x}(t+\Delta t).$$

With a little re-arranging this approximation can be written as,

$$\mathbf{x}(t+\Delta t) = (I - \Delta t A)^{-1}\,\mathbf{x}(t) \qquad \text{(Backward Euler).}$$

The basic structure of forward Euler and backward Euler for linear equations is the same, only we use a different matrix with which to iterate on our vector $\mathbf{x}$. A further approximation would be to evaluate the right side at the average between time t and $t + \Delta t$.

$$\frac{d\mathbf{x}}{dt} \approx \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = A\frac{\mathbf{x}(t+\Delta t) + \mathbf{x}(t)}{2}$$

which can be re-arranged to give,

$$\mathbf{x}(t+\Delta t) = \left(I - \frac{\Delta t}{2}A\right)^{-1}\left(I + \frac{\Delta t}{2}A\right)\mathbf{x}(t) \qquad \text{(Midpoint method).}$$

The three methods presented above are just a small subset of the different numerical approximations that are used to numerically integrate differential equations.

Now you may have forgotten, but if we have a discrete iteration equation of the form,

$$\mathbf{x}_{j+1} = M\mathbf{x}_j$$

where the subscript denotes the iteration number, the eigenvalues of M tell you about the systems long term behavior. If the magnitude of any of the eigenvalues is greater than 1, then the solution will grow exponentially. In order to have numerically stable iteration schemes, we will want the magnitude of the eignevalues of M to be less than 1.

3.1 Problems

Start with the mass spring system with no damping,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Our vector is $\mathbf{x} = [x, v]^T$ where x is position and v is velocity. We will consider initial conditions $\mathbf{x}(0) = [1, 0]^T$. You have already seen that the solution to this problem is $x(t) = \cos(t)$ and $v(t) = -\sin(t)$.

1. Generate the numerical solution (in MATLAB) to this problem using Forward Euler, Backward Euler, and the Midpoint Method. Try using values of $\Delta t = 0.1$ and 0.01. Iterate for the equivalent to 50 units of time (500 iterations for $\Delta t = 0.1$ and 5000 for $\Delta t = 0.01$). Plot the solution for x and note whether the amplitude of oscillations in the numerical solution grow, decay, or stay the same. Compare the numerical behavior to the true mathematical behavior.

   Compute the eigenvalues of the discrete matrix and connect these computed eigenvalues to the observed numerical solutions.

2. Look up the help menu for MATLAB's ODE45 command. This is a command which automatically integrates your equations in time numerically. Conceptually, the algorithm is similar to Euler's method, but it is a little fancier and adapts to the equations. Try and implement the dimensionless RLC equations in MATLAB using ODE45.