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HW 4

1. *Find the values of* a *and* b *in the expression* $\int_0^1 f(x)\,dx \approx af(\frac{1}{3})+bf(\frac{2}{3})$ *by using the "direct method."*

Note that we want to find a and b such that

$$\int_0^1 f(x)\,dx \approx af(\frac{1}{3})+bf(\frac{2}{3})$$

Using the direct method, we first let $f(x)=1$ (linear combination allows this)

$$\begin{aligned}\int_0^1 1\,dx &= a(1)+b(1)\\ 1 &= a+b\end{aligned}$$

Similarly, we let $f(x)=x$

$$\begin{aligned}\int_0^1 x\,dx &= a(x=\frac{1}{3})+b(x=\frac{2}{3})\\ \frac{1}{2} &= \frac{1}{3}a+\frac{2}{3}b\end{aligned}$$

Now with this system of equations we solve for a and b

$$\begin{aligned}\frac{1}{2} &= \frac{1}{3}a+\frac{2}{3}b, & 1 &= a+b\\ 3 &= 2a+4b, & 1-b &= a\\ 3 &= 2(1-b)+4b, & 1-b &= a\\ 3 &= 2-2b+4b, & 1-b &= a\\ 1 &= 2b, & 1-b &= a\\ \frac{1}{2} &= b, & \frac{1}{2} &= a\end{aligned}$$

Thus we have

$$\int_0^1 f(x) \approx \frac{1}{2}f(\frac{1}{3})+\frac{1}{2}f(\frac{2}{3})$$

2. *Use the 4th order Runge-Kutta method for systems to approximate the solution of the following system of first-order differential equations for* $0 \le t \le 1$ *, and* $h = 0.2$ *, and compare the results to the actual solution.*

$$u'_1 = 3u_1 + 2u_2 - (2t^2 + 1)e^{2t}$$
$$u'_2 = 4u_1 + u_2 + (t^2 + 2t - 4)e^{2t}$$

The initial conditions are $u_1(0) = 1$ *and* $u_2(0) = 1$ *. For comparison, the exact solutions are* $u_1(t) = \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} + e^{2t}$ *and* $u_2(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} + t^2 e^{2t}$.

I used the following set of equations in the implementation of RK4 for the system outlined above:

$$k_0 = hf_1(t_i, u_{1,i}, u_{2,i}) \qquad l_0 = hf_2(t_i, u_{1,i}, u_{2,i})$$
$$k_1 = hf_1(t_i + \frac{h}{2,} u_{1,i} + \frac{k_0}{2,} u_{2,i} + \frac{l_0}{2}) \qquad l_1 = hf_2(t_i + \frac{h}{2,} u_{1,i} + \frac{k_0}{2,} u_{2,i} + \frac{l_0}{2})$$
$$k_2 = hf_1(t_i + \frac{h}{2,} u_{1,i} + \frac{k_1}{2,} u_{2,i} + \frac{l_1}{2}) \qquad l_2 = hf_2(t_i + \frac{h}{2,} u_{1,i} + \frac{k_1}{2,} u_{2,i} + \frac{l_1}{2})$$
$$k_3 = hf_1(t_i + h, u_{1,i} + k_{2,} u_{2,i} + l_2) \qquad l_3 = hf_2(t_i + h, u_{1,i} + k_{2,} u_{2,i} + l_2)$$
$$u_{1,i+1} = u_{1,i} + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) \qquad u_{2,i+1} = u_{2,i} + \frac{1}{6}(l_0 + 2l_1 + 2l_2 + l_3)$$

where

$$f_1(t_i, u_{1,i}, u_{2,i}) = 3u_1 + 2u_2 - (2t^2 + 1)e^{2t}$$
$$f_2(t_i, u_{1,i}, u_{2,i}) = 4u_1 + u_2 + (t^2 + 2t - 4)e^{2t}$$

Using this set-up I was able to generate the following table:

t	My u_1	True u_1	My u_2	True u_2
0	1	1	1	1
0.2	2.12037	2.12501	1.50699	1.51159
0.4	4.44123	4.46512	3.24224	3.26599
0.6	9.73913	9.83236	8.16342	8.2563
0.8	22.6766	23.0026	21.3435	21.6689
1.0	55.6612	56.7375	56.0305	57.1054

Based on the table above, it seems that the RK4 method I developed performed well as all of the values my method produced were within 0.1 of the actual value.

3. *Consider the initial value, ordinary differential equation,* $y' = x + y$ *with* $y(0) = 0$. *Find* $y(x)$ *for* $0 \le x \le 0.5$ *with a step size of* $h = 0.1$ *using the following methods:*

(a) Euler's Method

Recall that the Euler's Method is written as

$$y_{i+1} = y_i + h f(x_i, y_i)$$

In this problem, $f(x, y) = x + y$ and $h = 0.1$.

Using this, we derive the following table:

i	x_i	y_i
0	0	0
1	0.1	$0+0.1(0.1+0)=0.01$
2	0.2	$0.01+0.1(0.2+0.01)=0.031$
3	0.3	$0.031+0.1(0.3+0.031)=0.0641$
4	0.4	$0.0641+0.1(0.4+0.0641)=0.11051$
5	0.5	$0.11051+0.1(0.5+0.11051)=0.171561$

(b) *4*[th] *order Runge-Kutta*

Recall that the 4th order Runge-Kutta method is written as

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Using this for the ODE $y' = x + y, y(0) = 0$

We have the following table:

i	x_i	y_i	k_1	k_2	k_3	k_4	y_{i+1}
0	0	0	0	0	0	0	0
1	0.1	0	0	0.005	0.00525	0.010525	0.00517083
2	0.2	0.00517083	0.0105171	0.0160429	0.0163192	0.022149	0.0214026
3	0.3	0.0214026	0.0221403	0.0282473	0.0285526	0.0349955	0.0498585
4	0.4	0.0498585	0.0349858	0.0417351	0.0420726	0.0491931	0.0918242
5	0.5	0.0918242	0.0491824	0.0566415	0.0570145	0.0648839	0.148721

(c) Adams-Bashforth 2-step

Recall that the Adams-Bashforth 2-step is represented by:

$$y_{i+1}=y_i+\frac{h}{2}[3f_i-f_{i-1}]+O(h^2)$$

$$y_{i+1}=y_i+\frac{h}{2}[3f(x_i,y_i)-f(x_{i-1},y_{i-1})]+O(h^2)$$

Using this, we get the following table:

i	x_i	y_i	x_{i-1}	$f(x_i,y_i)$	$f(x_{i-1},y_{i-1})$	y_{i+1}
0	0	0	NA	NA	NA	0
1	0.1	0	0	0.105	0	0.005
2	0.2	0.005	0.1	0.12075	0.105	0.02075
3	0.3	0.02075	0.2	0.2486125	0.12075	0.0486125
4	0.4	0.0486125	0.3	0.3486125	0.2486125	0.0898669
5	0.5	0.0898669	0.4	0.4898669	0.3486125	0.145916

Note that to find $y(0.1)$ we needed to use an alternate method since there were not two known points to use. Instead, we estimated $y(0.1)$ to be

$$y(0.1)\approx\frac{(0.1)^2}{2}=0.005$$

(d) Adams-Moulton 2-step

Recall the general formula for the Adams-Moulton 2-step:

$$y_{i+1}=y_i+\frac{h}{12}(5f(x_{i+1},y_{i+1})+8f(x_i,y_i)-f(x_{i-1},y_{i-1}))$$

Note that since this method implicitely finds y_{i+1}, the method is only useful if we can solve for y_{i+1}.

Now recall that $y'=y+x$, thus we have:

$$\begin{aligned}
y_{i+1} &= y_i+\frac{h}{12}[5(y_{i+1}+x_{i+1})+8f_i+f_{i-1}] \\
y_{i+1} &= y_i+5\frac{h}{12}[(y_{i+1}+x_{i+1})]+\frac{h}{12}[8f_i+f_{i-1}] \\
y_{i+1} &= y_i+5\frac{h}{12}y_{i+1}+5\frac{h}{12}x_{i+1}+\frac{h}{12}[8f_i+f_{i-1}] \\
y_{i+1}-5\frac{h}{12}y_{i+1} &= y_i+5\frac{h}{12}x_{i+1}+\frac{h}{12}[8f_i+f_{i-1}] \\
y_{i+1}(1-5\frac{h}{12}) &= y_i+5\frac{h}{12}x_{i+1}+\frac{h}{12}[8f_i+f_{i-1}] \\
y_{i+1} &= \frac{y_i+5\frac{h}{12}x_{i+1}+\frac{h}{12}[8f_i+f_{i-1}]}{1-5\frac{h}{12}}
\end{aligned}$$

Using this formula, we can determine pretty good estimates ($O(h^3)$) for each y_{i+1} by using the two previous points. Note that, like the Adams-Bashforth method, we must determine two initial points to build off of.

Recall that we are provided the initial condition $y(0)=0$, giving us our first point.

Also, we are told that $y \approx \frac{x^2}{2}$ when $x \ll 1$, so when $x=0.1, y(0.1) \approx \frac{(0.1)^2}{2} = 0.005$

Using these values, we can complete the following table for $x \in [0, 0.5]$ and $h = 0.1$

i	x_{i-1}	y_{i-1}	x_i	y_i	y_{i+1}
0	NA	NA	0	0	NA
1	0	0	0.1	0.005	0.0212174
2	0.1	0.005	0.2	0.0212174	0.0496594
3	0.2	0.0212174	0.3	0.0496594	0.0916103
4	0.3	0.0496594	0.4	0.0916103	0.148491

This gives us the following summary:

$$\begin{aligned} y(0) &= 0 \\ y(0.1) &= 0.005 \\ y(0.2) &= 0.0212174 \\ y(0.3) &= 0.0496594 \\ y(0.4) &= 0.0916103 \\ y(0.5) &= 0.148491 \end{aligned}$$

(e) Improved Euler

For the Improved Euler Method, we have

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})\right)$$

where we find the $\bar{y}_{i+1}$ by using the basic Euler Method to predict a reasonable value for y_{i+1}:

$$\bar{y}_{i+1} = y_i + hf(x_i, y_i)$$

Using this formula, we can predict y_{i+1} fairly well, as depicted in the following table:

i	x_i	y_i	x_{i+1}	$\bar{y}_{i+1}$	y_{i+1}
0	0	0	0.1	0	0.005
1	0.1	0.005	0.2	0.0155	0.021025
2	0.2	0.021025	0.3	0.0431275	0.0492326
3	0.3	0.0492326	0.4	0.0841559	0.0909021
4	0.4	0.0909021	0.5	0.139992	0.147447

This gives us the following summary:

$$\begin{aligned} y(0) &= 0 \\ y(0.1) &= 0.005 \\ y(0.2) &= 0.021025 \\ y(0.3) &= 0.0492326 \\ y(0.4) &= 0.0909021 \\ y(0.5) &= 0.147447 \end{aligned}$$

(f) *Predictor Corrector using the Adams-Bashforth 4-step for the predictor and the Adams-Moulton 3-step as the corrector.*

Note that to solve this problem, we needed to use two methods: Adams-Bashforth 4-step and Adams-Moulton 3-step. The Adams-Bashforth 4-step algorithm acts as the predictor and the Adams-Moulton 3-step will act as the corrector in the following equations:

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{24}\left(9f\left(x_{i+1}, \bar{y}_{i+1}\right) + 19f_i - 5f_{i-1} + f_{i-2}\right) \quad \text{(Corrector)} \\ \bar{y}_{i+1} &= y_i + \frac{h}{24}\left(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right) \quad \text{(Predictor)} \end{aligned}$$

In order to use these methods, three points must be already known, so for this problem, they will be estimated using the equation $y \approx \frac{x^2}{2,}$ giving us

$$\begin{aligned} y(0) &= 0 \\ y(0.1) &= 0.005 \\ y(0.2) &= 0.02 \end{aligned}$$

Since these values are being estimated using a truncated Taylor Series, the error will likely be larger than some of the other methods. If we instead used the results from the last couple of parts as seeding values, the results would likely be better. However, using the values above with the equations above, we get the following results for the remaining x values:

$$\begin{aligned} y(0.4) &= 0.0863979 \\ y(0.4) &= 0.142747 \end{aligned}$$