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APPM 4650
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HW2

1. *Show that repeated application of* $P(x)=(x-\alpha)Q(x)+R$ *will eventually produce the Taylor series expansion of the polynomial function* $f(x)$ *. That is, show that* $R_k=\frac{P^{(k)}(\alpha)}{k!}=\frac{f^{(k)}(\alpha)}{k!}$ *. What is the center of this expansion?*

   Recall that $P(x)=R_0+Q_0(x)(x-\alpha)$
   If we let $x=\alpha\Rightarrow P(\alpha)=R_0$
   Therefore, $P(x)=P(\alpha)+Q_0(x)(x-\alpha)$
   Thus

$$\begin{aligned} P(x) &= P(\alpha)+(Q_1(x)(x-\alpha)+R_1)(x-\alpha) \\ P(x) &= P(\alpha)+R_1(x-\alpha)+Q_1(x)(x-\alpha)^2 \end{aligned}$$

   Now consider

$$\begin{aligned} P'(x) &= R_1+\frac{d}{dx}[Q_1(x)(x-\alpha)^2] \\ P'(\alpha) &= R_1+0 \end{aligned}$$

   Now we have

$$\begin{aligned} P(x) &= P(\alpha)+P'(\alpha)(x-\alpha)+Q_1(x)(x-\alpha)^2 \\ P(x) &= P(\alpha)+P'(\alpha)(x-\alpha)+(Q_2(x)(x-\alpha)+R_2)(x-\alpha)^2 \\ P(x) &= P(\alpha)+P'(\alpha)(x-\alpha)+R_2(x-\alpha)^2+Q_2(x)(x-\alpha)^3 \\ \Rightarrow P'(x) &= P'(\alpha)+2*R_2(x-\alpha)+\frac{d}{dx}[Q_2(x)(x-\alpha)^3] \\ P''(x) &= 2*R_2+\frac{d^2}{dx^2}[Q_2(x)(x-\alpha)^3] \\ P''(\alpha) &= 2*R_2\Rightarrow R_2=\frac{P''(\alpha)}{2} \end{aligned}$$

   As a result, we have

$$P(x)=P(\alpha)+P'(\alpha)(x-\alpha)+\frac{P''(\alpha)}{2}(x-\alpha)^2+Q_2(x)(x-\alpha)^3$$

   Continuing with this pattern, we find that $P(x)$ expands like a taylor polynomial centered at α, suggesting that

$$\frac{P^{(k)}(\alpha)}{k!}=\frac{f^{(k)}(\alpha)}{k!}$$

2. *Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate* $f(8.4)$ *if* $f(8.1)=16.94410, f(8.3)=17.56492, f(8.6)=18.50515,$ *and* $f(8.7)=18.82091$.

Degree 1:

Points used: $f(8.3)$ and $f(8.6)$

Reason: The closest known values that surround $f(8.4)$

Result:

$$\begin{aligned}
f(\alpha=8.4) &\approx \frac{\alpha-x_1}{x_0-x_1} f(x_0)+\frac{\alpha-x_0}{x_1-x_0} f(x_1) \\
&\approx \frac{8.4-8.6}{8.3-8.6} f(8.3)+\frac{8.4-8.3}{8.6-8.3} f(8.6) \\
&\approx 11.7099+6.16838 \\
&\approx 17.87833
\end{aligned}$$

Degree 2:

Note: there are two equally valid ways to do this

Method 1:

Points used: $f(8.1), f(8.3), f(8.6)$

Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.1)$ is very close on the left.

Result:

$$\begin{aligned}
f(\alpha=8.4) &\approx \frac{(\alpha-x_1)(\alpha-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0)+\frac{(\alpha-x_0)(\alpha-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)+\frac{(\alpha-x_0)(\alpha-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \\
&\approx \frac{(8.4-8.3)(8.4-8.6)}{(8.1-8.3)(8.1-8.6)} f(8.1)+\frac{(8.4-8.1)(8.4-8.6)}{(8.3-8.1)(8.3-8.6)} f(8.3)+\frac{(8.4-8.1)(8.4-8.3)}{(8.6-8.1)(8.6-8.3)} f(8.6) \\
&\approx -3.38882+17.5649+3.70103 \\
&\approx 17.87713
\end{aligned}$$

Method 2:

Points used: $f(8.3), f(8.6), f(8.7)$

Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.7)$ is very close on the right.

Result:

$$\begin{aligned}
f(\alpha=8.4) &\approx \frac{(\alpha-x_1)(\alpha-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0)+\frac{(\alpha-x_0)(\alpha-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)+\frac{(\alpha-x_0)(\alpha-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \\
&\approx \frac{(8.4-8.6)(8.4-8.7)}{(8.3-8.6)(8.1-8.7)} f(8.3)+\frac{(8.4-8.3)(8.4-8.7)}{(8.6-8.3)(8.6-8.7)} f(8.6)+\frac{(8.4-8.3)(8.4-8.6)}{(8.7-8.3)(8.7-8.6)} f(8.7) \\
&\approx -8.78246+18.5052-9.41046 \\
&\approx 17.877155
\end{aligned}$$

Degree 3:

Points used: $f(8.1), f(8.3), f(8.6), f(8.7)$

Reason: All available points

Result:

$$
\begin{aligned}
f(\alpha=8.4) \quad &\approx \quad \frac{(\alpha-x_1)(\alpha-x_2)(\alpha-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(\alpha-x_0)(\alpha-x_2)(\alpha-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \\
&\qquad \frac{(\alpha-x_0)(\alpha-x_1)(\alpha-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(\alpha-x_0)(\alpha-x_1)(\alpha-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\
&\approx -1.69441 + 13.1737 + 11.1031 - 4.70523 \\
&\approx \quad 17.877143
\end{aligned}
$$

3. *Use Neville's method for the previous problem.*

make

Degree 1:

Points used: $f(8.3), f(8.6)$

Reason: The two points are the closest to $x=8.4$ and are on each side of it.

Result:

$$\begin{aligned} f(\alpha=8.4) &\approx Q_{0,0}(8.4) &=& \quad f(8.3)=17.56492 \\ &\approx Q_{1,0}(8.4) &=& \quad f(8.6)=18.50515 \\ &\approx Q_{1,1}(8.4) &=& \quad \frac{(\alpha-x_0)Q_{1,0}-(\alpha-x_1)Q_{0,0}}{x1-x0} \\ && =& \quad 17.8783 \end{aligned}$$

Degree 2:

Method 1:

Points used: $f(8.1), f(8.3), f(8.6)$

Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.1)$ is very close on the left.

Result:

$$\begin{aligned} f(\alpha=8.4) &\approx Q_{0,0}(8.4) &=& \quad f(8.1)=16.94410 && \text{degree 0} \\ &\approx Q_{1,0}(8.4) &=& \quad f(8.3)=17.56492 && \text{degree 0} \\ &\approx Q_{2,0}(8.4) &=& \quad f(8.6)=18.50515 && \text{degree 0} \\ &\approx Q_{1,1}(8.4) &=& \quad \frac{(\alpha-x_0)Q_{1,0}(\alpha)}{x_1-x_0}+\frac{(\alpha-x_1)Q_{0,0}(8.4)}{x_0-x_1} && \text{degree 1} \\ && =& \quad 17.8753 \\ &\approx Q_{2,1}(8.4) &=& \quad \frac{(\alpha-x_1)Q_{2,0}(\alpha)}{x_2-x_1}+\frac{(\alpha-x_2)Q_{1,0}(\alpha)}{x_1-x_2} && \text{degree 1} \\ && =& \quad 17.8783 \\ &\approx Q_{2,2}(8.4) &=& \quad \frac{(\alpha-x_0)Q_{2,1}(\alpha)}{x_2-x_0}+\frac{(\alpha-x_2)Q_{1,1}(\alpha)}{x_0-x_2} && \text{degree 2} \\ && =& \quad 17.8763 \end{aligned}$$

Method 2:

Points used: $f(8.3), f(8.6), f(8.7)$

Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.7)$ is very close on the right.

Result:

$$
\begin{aligned}
f(\alpha=8.4) &\approx Q_{0,0}(8.4) = f(8.3)=17.56492 && \text{degree 0}\\
&\approx Q_{1,0}(8.4) = f(8.6)=18.50515 && \text{¿ degree 0}\\
&\approx Q_{2,0}(8.4) = f(8.7)=18.82091 && \text{degree 0}\\
&\approx Q_{1,1}(8.4) = \frac{(\alpha-x_0)Q_{1,0}(\alpha)}{x_1-x_0}+\frac{(\alpha-x_1)Q_{0,0}(8.4)}{x_0-x_1} && \text{degree 1}\\
&= 17.8783\\
&\approx Q_{2,1}(8.4) = \frac{(\alpha-x_1)Q_{2,0}(\alpha)}{x_2-x_1}+\frac{(\alpha-x_2)Q_{1,0}(\alpha)}{x_1-x_2} && \text{degree 1}\\
&= 17.8736\\
&\approx Q_{2,2}(8.4) = \frac{(\alpha-x_0)Q_{2,1}(\alpha)}{x_2-x_0}+\frac{(\alpha-x_2)Q_{1,1}(\alpha)}{x_0-x_2} && \text{degree 2}\\
&= 17.8877
\end{aligned}
$$

Degree 3:
Points used: $f(8.1), f(8.3), f(8.6), f(8.7)$
Reason: All available points
Result:

$$
\begin{aligned}
f(\alpha=8.4) &\approx Q_{0,0}(8.4) = f(8.1)=16.94410 && \text{degree 0}\\
&\approx Q_{1,0}(8.4) = f(8.3)=17.56492 && \text{degree 0}\\
&\approx Q_{2,0}(8.4) = f(8.6)=18.50515 && \text{degree 0}\\
&\approx Q_{3,0}(8.4) = f(8.7)=18.82091 && \text{degree 0}\\
&\approx Q_{1,1}(8.4) = \frac{(\alpha-x_0)Q_{1,0}(\alpha)}{x_1-x_0}+\frac{(\alpha-x_1)Q_{0,0}(8.4)}{x_0-x_1} && \text{degree 1}\\
&= 17.8753\\
&\approx Q_{2,1}(8.4) = \frac{(\alpha-x_1)Q_{2,0}(\alpha)}{x_2-x_1}+\frac{(\alpha-x_2)Q_{1,0}(\alpha)}{x_1-x_2} && \text{degree 1}\\
&= 17.8783\\
&\approx Q_{3,1}(8.4) = \frac{(\alpha-x_2)Q_{3,0}(\alpha)}{x_3-x_2}+\frac{(\alpha-x_3)Q_{2,0}(\alpha)}{x_2-x_3} && \text{degree 1}\\
&= 17.8736\\
&\approx Q_{2,2}(8.4) = \frac{(\alpha-x_0)Q_{2,1}(\alpha)}{x_2-x_0}+\frac{(\alpha-x_2)Q_{1,1}(\alpha)}{x_0-x_2} && \text{degree 2}\\
&= 17.8763\\
&\approx Q_{3,2}(8.4) = \frac{(\alpha-x_1)Q_{3,1}(\alpha)}{x_3-x_1}+\frac{(\alpha-x_3)Q_{1,1}(\alpha)}{x_1-x_3} && \text{degree 2}\\
&= 17.8877\\
&\approx Q_{3,3}(8.4) = \frac{(\alpha-x_0)Q_{3,2}(\alpha)}{x_3-x_0}+\frac{(\alpha-x_3)Q_{1,1}(\alpha)}{x_0-x_3} && \text{degree 3}\\
&= 17.8535
\end{aligned}
$$

4. *Use Newton's interpolating polynomials of degrees one, two, and three to approximate* $f(8.4)$ *if* $f(8.1)=16.94410, f(8.3)=17.56492, f(8.6)=18.50515,$ *and* $f(8.7)=18.82091$.

Degree 1:
Points used: $f(8.3), f(8.6)$
Reason: The two points are the closest to $x=8.4$ and are on each side of it.
Result:

Note that:

$$\begin{aligned} f[x_0](8.4) &= f(8.3)=17.56492 \\ f[x_1](8.4) &= f(8.6)=18.50515 \\ f[x_0 x_1](8.4) &= \frac{f[x_1]-f[x_0]}{x1-x0} \\ &= 3.1341 \end{aligned}$$

thus

$$\begin{aligned} f(8.4) &\approx f[x_0]+f[x_0 x_1](x-x_0) \\ &= 17.8783 \end{aligned}$$

Degree 2:

Method 1:
Points used: $f(8.1), f(8.3), f(8.6)$
Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.1)$ is very close on the left.
Result:

Note that:

$$\begin{aligned} f[x_0](8.4) &= f(8.1)=16.94410 \\ f[x_1](8.4) &= f(8.3)=17.56492 \\ f[x_2](8.4) &= f(8.6)=18.50515 \\ f[x_0 x_1](8.4) &= \frac{f[x_1]-f[x_0]}{x_1-x_0} \\ f[x_1 x_2](8.4) &= \frac{f[x_2]-f[x_1]}{x_2-x_1} \\ &= 3.1041 \\ f[x_0 x_1 x_2](8.4) &= \frac{f[x_1 x_2]-f[x_0 x_1]}{x_2-x_0} \\ &= 0.06 \end{aligned}$$

thus

$$\begin{aligned} f(8.4) &\approx f[x_0]+f[x_0 x_1](x-x_0)+f[x_0 x_1 x_2](x-x_0)(x-x_1) \\ &= 17.8771 \end{aligned}$$

Method 2:

Points used: $f(8.3), f(8.6), f(8.7)$

Reason: $f(8.3)$ and $f(8.6)$ surround $f(8.4)$ and $f(8.7)$ is very close on the right.

Result:

Note that:

$$f[x_0](8.4) = f(8.3) = 17.56492$$

$$f[x_1](8.4) = f(8.6) = 18.50515$$

$$f[x_2](8.4) = f(8.7) = 18.82091$$

$$f[x_0 x_1](8.4) = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$= 3.1341$$

$$f[x_0 x_1 x_2](8.4) = \frac{f[x_2] - f[x_0]}{x_1 - x_0}$$

$$= 3.1576$$

$$f[x_0 x_1 x_2](8.4) = \frac{f[x_1 x_2] - f[x_0 x_1]}{x_2 - x_0}$$

$$= 0.05875$$

thus

$$f(8.4) \approx f[x_0] + f[x_0 x_1](x - x_0) + f[x_0 x_1 x_2](x - x_0)(x - x_1)$$

$$= 17.8772$$

Degree 3:

Points used: $f(8.1), f(8.3), f(8.6), f(8.7)$

Reason: All available points

Result:

Note that:

$$
\begin{aligned}
f[x_0](8.4) &= f(8.1)=16.94410 \\
f[x_1](8.4) &= f(8.3)=17.56492 \\
f[x_2](8.4) &= f(8.6)=18.50515 \\
f[x_3](8.4) &= f(8.6)=18.82091 \\
f[x_0 x_1](8.4) &= \frac{f[x_1]-f[x_0]}{x_1-x_0} \\
&= 3.1041 \\
f[x_1 x_2](8.4) &= \frac{f[x_2]-f[x_1]}{x_2-x_1} \\
&= 3.1341 \\
f[x_2 x_3](8.4) &= \frac{f[x_3]-f[x_2]}{x_3-x_2} \\
&= 3.1576 \\
f[x_0 x_1 x_2](8.4) &= \frac{f[x_1 x_2]-f[x_0 x_1]}{x_2-x_0} \\
&= 0.06 \\
f[x_1 x_2 x_3](8.4) &= \frac{f[x_2 x_3]-f[x_1 x_2]}{x_3-x_1} \\
&= 0.05875 \\
f[x_0 x_1 x_2 x_3](8.4) &= \frac{f[x_1 x_2 x_3]-f[x_0 x_1 x_2]}{x_3-x_0} \\
&= -0.00208333
\end{aligned}
$$

thus

$$
\begin{aligned}
f(8.4) &\approx f[x_0]+f[x_0 x_1](x-x_0)+f[x_0 x_1 x_2](x-x_0)(x-x_1)+ \\
&\quad f[x_0 x_1 x_2 x_3](x-x_0)(x-x_1)(x-x_2) \\
&= 17.8771
\end{aligned}
$$

5. *Use cubic splines to approximate the function* $f(x)=x^4$ *on the interval* $-3\le x\le 3$ *. Use the values of the function* $f(x)$ *at* $x=-3,-1,1,3$ *. Match the slope of the spline to the slope of the function at the locations* $x=-3$ *and* $x=3$ *.*

Note that for this problem, I used the algorithm provided in the book to find all of the coefficients. The problem statement isn't clear about what is expected, so here are my coefficient results:

For $x\in[-3,-1]: S_0(x)=81-26(x+3)-16.7066(x+3)^2+4.85331(x+3)^3$
For $x\in[-1,1]: S_1(x)=1-34.5868(x+1)+12.4132(x+1)^2+2.44008(x+1)^3$
For $x\in[1,3]: S_2(x)=1+44.3471(x-1)+27.0537(x-1)^2-14.6136(x-1)^3$

Relevant Code:

```
vector<vector<double>> Interpolation::clampedCubicSpline(vector<Data_Point> data, double fp0,
double fpn) {
	int num_points = data.size();
	double a_n[num_points];
	double h_n[num_points];
	double alpha_n[num_points];

	for (int i = 0; i < num_points; i++) {
		a_n[i] = data[i].getY();
	}

	for (int i = 0; i < num_points; i++) {
		h_n[i] = data[i+1].getX() - data[i].getX();
	}

	alpha_n[0] = 3*(a_n[1] - a_n[0])/ h_n[0] - 3 - fp0;
	alpha_n[num_points-1] = 3*fpn - 3*(a_n[num_points-1] - a_n[num_points - 2])/
h_n[num_points - 2];

	for (int i = 1; i < num_points; i++) {
		alpha_n[i] = (3 / h_n[i])*(a_n[i + 1] - a_n[i]) - (3 / h_n[i-1])*(a_n[i] - a_n[i - 1]);
	}

	double l_n[num_points + 1];
	double mu_n[num_points + 1];
	double z_n[num_points + 1];
	double c_n[num_points + 1];
	double b_n[num_points];
	double d_n[num_points];

	l_n[0] = 2*h_n[0];
	mu_n[0] = 0.5;
	z_n[0] = alpha_n[0] / l_n[0];
```

```
        for (int i = 1; i < num_points; i++) {
                l_n[i] = 2*(data[i+1].getX() - data[i - 1].getX()) - h_n[i - 1]*mu_n[i-1];
                mu_n[i] = h_n[i] / l_n[i];
                z_n[i] = (alpha_n[i] - h_n[i-1]*z_n[i-1]) / l_n[i];
        }

        l_n[num_points] = h_n[num_points-1] * (2 - mu_n[num_points - 1]);
        z_n[num_points] = (alpha_n[num_points] - h_n[num_points - 1] * z_n[num_points - 1]);
        c_n[num_points] = z_n[num_points];

        for (int j = num_points - 1; j >= 0; j--) {
                c_n[j] = z_n[j] - mu_n[j]*c_n[j+1];
                b_n[j] = (a_n[j+1] - a_n[j])/h_n[j] - h_n[j] * (c_n[j+1] + 2*c_n[j]) / 3;
                d_n[j] = (c_n[j+1] - c_n[j]) / (3*h_n[j]);
        }

        vector<double> a_list;
        vector<double> b_list;
        vector<double> c_list;
        vector<double> d_list;


        for (int i = 0; i < num_points; i++) {
                a_list.push_back(a_n[i]);
                b_list.push_back(b_n[i]);
                c_list.push_back(c_n[i]);
                d_list.push_back(d_n[i]);

                cout << "a" << i << " = " << a_n[i] << endl;
                cout << "b" << i << " = " << b_n[i] << endl;
                cout << "c" << i << " = " << c_n[i] << endl;
                cout << "d" << i << " = " << d_n[i] << endl;
        }

        vector<vector<double>> coefficients;
        coefficients.push_back(a_list);
        coefficients.push_back(b_list);
        coefficients.push_back(c_list);
        coefficients.push_back(d_list);

        return coefficients;
}
```

6. *Use cubic splines to approximate the function* $f(x)=x^4$ *on the interval* $-1\le x\le 1$ *by using the values of the function* $f(x)$ *at* $x=-1,0,1$ *. Match the slope of the spline to the forward and backward approximations of* $f'(x)$ *,*

$$f'(x)\approx\frac{f_1-f_0}{h} \quad \textit{and} \quad f'(x)\approx\frac{f_2-f_1}{h}$$

at locations $x=-1$ *and* $x=1$ *. Let* P_0 *be valid on* $-1\le x\le 0$ *and let* P_1 *be valid on* $0\le x\le 1$ *.*

Note that, similar to problem 5, the question is not particularly clear about what the expected response should be. As a result, I will again provide the equations outputted by my program:

For $x\in[-1,0]:S_0(x)=1-3.75(x+1)^2+2.75(x+1)^3$

For $x\in[0,1]:S_1(x)=0+0.75x+4.5x^2-4.25x^3$

7. *Use cubic splines to approximate the function* $f(x)=x^4$ *on the interval* $-1\le x\le 1$ *by using the values of the function* $f(x)$ *at* $x=-1,0,1$. *Use "free boundary conditions" at locations* $x=-1$ *and* $x=1$. *Let* P_0 *be valid on* $-1\le x\le 0$ *and let* P_1 *be valid on* $0\le x\le 1$.

Using the natural cubic splines algorithm provided in the book, I built the equations to approximate the given function to be:

For $x\in[-1,0]:S_0(x)=1-(x+1)=-x$

For $x\in[0,1]:S_1(x)=-x+2x^3$