Bayesian Inference for a Mean

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MATH 347 Bayesian Statistics

Outline

1. Example: Expenditures in the Consumer Expenditure Surveys
2. Prior and posterior distributions for mean and standard deviation
3. Bayesian inference for unknown mean μ (Lab 2)
4. Recap

Outline

The Consumer Expenditure Surveys Data (CE)

- Conducted by the U.S. Census Bureau for the BLS.
- Contains data on expenditures, income, and tax statistics about consumer units (CU) across the country.
- Provides information on the buying habits of U.S. consumers.

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  - Tabular data (aggregated)
  - Micro-level data: public-use microdata PUMD (CU-level)

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  - Tabular data (aggregated)
  - Micro-level data: public-use microdata PUMD (CU-level)
- We work with PUMD micro-level data, with the continuous variable TOTEXPPQ: CU total expenditures last quarter.
- We work with Q1 2017 sample: $n = 6,208$.

The TOTEXPPQ variable

```
CEsample = read_csv("CEsample1.csv")

summary(CEsample$TotalExpLastQ)
```

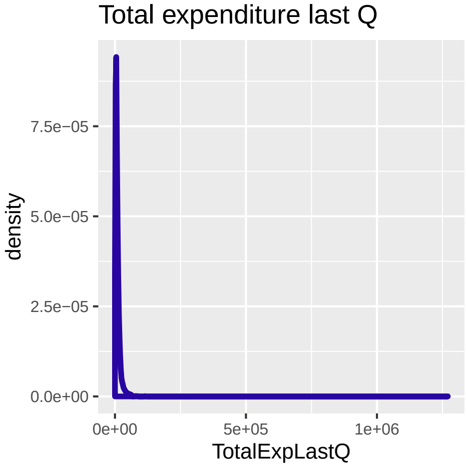
```
##     Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
##       30     3522     6417     9513    11450  1270598
```

```
sd(CEsample$TotalExpLastQ)
```

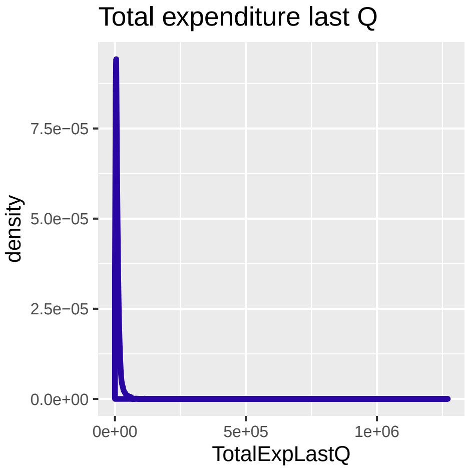
```
## [1] 19341.25
```

The TOTEXPPQ variable cont'd

```
ggplot(data = CEsample, aes(TotalExpLastQ)) +
  geom_density(color = crcblue, size = 1) +
  labs(title = "Total expenditure last Q") +
  theme_grey(base_size = 8, base_family = "")
```



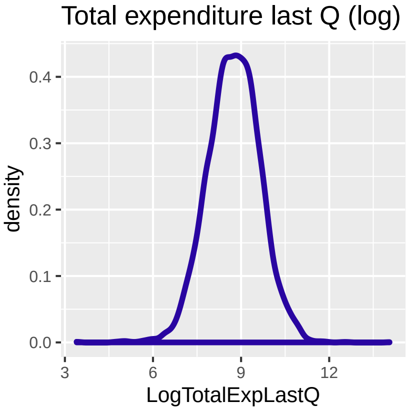
The TOTEXPPQ variable cont'd



- Very skewed to the right.
- Take log and transform it to the log scale.

Log transformation of the TOTEXPPQ variable

```
CEsample$LogTotalExpLastQ <- log(CEsample$TotalExpLastQ)
ggplot(data = CEsample, aes(LogTotalExpLastQ)) +
  geom_density(color = crcblue, size = 1) +
  labs(title = "Total expenditure last Q (log)") +
  theme_grey(base_size = 8, base_family = "")
```



The Normal distribution

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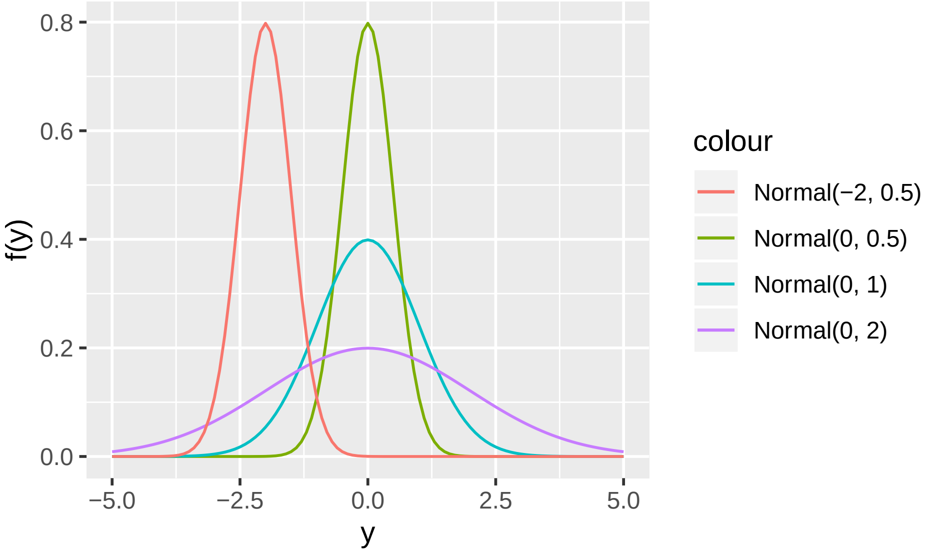
- The probability density function (pdf) of $\mathrm{Normal}(\mu, \sigma)$ is:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right), -\infty < y < \infty.$$

The Normal distribution cont'd



i.i.d. Normals

- Suppose there are a sequence of n responses: $Y_1, Y_2, \cdots, Y_n$.
- Further suppose each response independently and identically follows a Normal distribution:

$$Y_i \overset{i.i.d.}{\sim} \textrm{Normal}(\mu, \sigma).$$

- Then the joint probability density function (joint pdf) of $y_1, \cdots, y_n$ is:

$$f(y_1, \cdots, y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_i - \mu)^2}{2\sigma^2}\right), -\infty < y_i < \infty. \quad (1)$$

Outline

Recap from proportion lecture

- Bayesian inference procedure:
  - Step 1: express an opinion about the location of the proportion p before sampling (prior).
  - Step 2: take the sample and record the observed proportion (data/likelihood).
  - Step 3: use Bayes' rule to sharpen and update the previous opinion about p given the information from the sample (posterior).

Recap from proportion lecture

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- For Binomial data/likelihood, the Beta distributions are conjugate priors.
  - The prior distribution: $p \sim \mathrm{Beta}(a, b)$
  - The sampling density: $Y \sim \mathrm{Binomial}(n, p)$
  - The posterior distribution: $p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$

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- What to do for a Normal model $Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma)$?
  - Data model/sampling density is chosen: Normal.
  - What to do with two parameters μ and σ?
  - How to specify priors? Conjugate priors exist?

Step 1: Prior distributions

- The data model/sampling density for n observations:
$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$
- There are two parameters μ and σ in the Normal model.
- Therefore, the likelihood is in terms of both unknown parameters:
$$f(y_1, \cdots, y_n) = L(\mu, \sigma). \tag{2}$$

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$$f(y_1, \cdots, y_n) = L(\mu, \sigma). \tag{2}$$
- Need a joint prior distribution:
$$\pi(\mu, \sigma). \tag{3}$$
- Bayes' rule will help us derive a joint posterior:
$$\pi(\mu, \sigma \mid y_1, \cdots, y_n). \tag{4}$$

If only mean μ is unknown

- Special case: μ is unknown, σ is known.
- There is only one parameter μ in $Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma)$.
- The Bayesian inference procedure simplifies to:
  - The data model for n observations with σ known:
$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$
  - The likelihood is in terms of unknown parameter μ:
$$f(y_1, \cdots, y_n) = L(\mu). \tag{5}$$
  - Need a prior distribution for μ:
$$\pi(\mu \mid \sigma). \tag{6}$$
  - Bayes' rule will help us derive a posterior for μ:
$$\pi(\mu \mid y_1, \cdots, y_n, \sigma). \tag{7}$$

If only mean μ is unknown: Normal conjugate prior

- For this special case, Normal prior for μ is a conjugate prior:
  - The prior distribution:

$$\mu \mid \sigma \sim \text{Normal}(\mu_0, \sigma_0). \tag{8}$$

  - The sampling density:

$$y_1, \cdots, y_n \mid \mu, \sigma \overset{i.i.d.}{\sim} \text{Normal}(\mu, \sigma). \tag{9}$$

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  - The posterior distribution:

$$\mu \mid y_1, \cdots, y_n, \phi \sim \mathrm{Normal}\left(\frac{\phi_0\mu_0 + n\phi\bar{y}}{\phi_0 + n\phi}, \sqrt{\frac{1}{\phi_0 + n\phi}}\right), \tag{10}$$

where $\phi = \frac{1}{\sigma^2}$ (and $\phi_0 = \frac{1}{\sigma_0^2}$), the precision. Since σ (and σ_0) is known, ϕ (and ϕ_0) is known too.

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This is \mu 's own Normal distrubution

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Recall: \pi(\mu l y_1,...,y_n, \sigma) \propto \pi(\mu l \sigma)x L(\mu)

$$\mu \mid y_1, \cdots, y_n, \phi \sim \text{Normal}\left(\frac{\phi_0\mu_0 + n\phi\bar{y}}{\phi_0 + n\phi}, \sqrt{\frac{1}{\phi_0 + n\phi}}\right), \tag{10}$$

where $\phi = \frac{1}{\sigma^2}$ (and $\phi_0 = \frac{1}{\sigma_0^2}$), the precision. Since σ (and σ_0) is known, ϕ (and ϕ_0) is known too.

Note that again, the posterior mean is the weighted average of prior mean \mu_0 (with weight \phi_0) and sample mean \bar y (with weight n\phi)

- We can then use the `rnorm()` R function to sample posterior draws of μ from Equation (10). Known quantities: $\phi_0, \mu_0, n, \bar{y}, \phi$

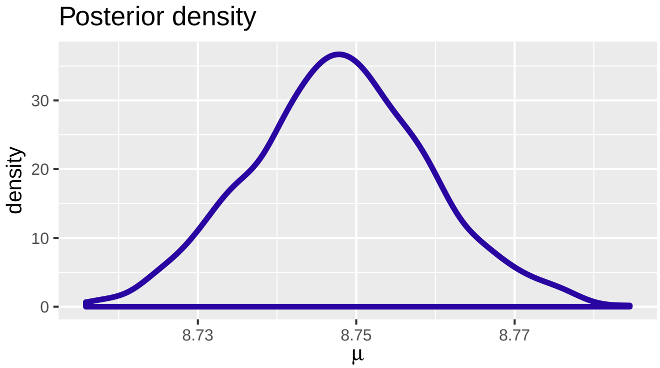
Simulate posterior draws of μ

```
mu_0 <- 5
sigma_0 <- 1
phi_0 <- 1/sigma_0^2
ybar <- mean(CEsample$LogTotalExpLastQ)
phi <- 1.25
n <- dim(CEsample)[1]
mu_n <- (phi_0*mu_0+n*ybar*phi)/(phi_0+n*phi)
sd_n <- sqrt(1/(phi_0+n*phi))

set.seed(123)
S <- 1000
mu_post <- rnorm(S, mean = mu_n, sd = sd_n)
df <- as.data.frame(mu_post)
```

Simulate posterior draws of μ cont'd

```
ggplot(data = df, aes(mu_post)) +
  geom_density(color = crcblue, size = 1) +
  labs(title = "Posterior density") +
  xlab(expression(mu)) +
  theme_grey(base_size = 8, base_family = "")
```



If only standard deviation σ is unknown

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  - The data model/sampling density for n observations with μ known:
  $$Y_i \stackrel{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$
  - The likelihood is in terms of unknown parameter σ:
  $$f(y_1, \cdots, y_n) = L(\sigma). \tag{11}$$
  - Need a prior distribution for σ:
  $$\pi(\sigma \mid \mu). \tag{12}$$
  - Bayes' rule will help us derive a posterior for σ:
  $$\pi(\sigma \mid y_1, \cdots, y_n, \mu). \tag{13}$$

If only standard deviation σ is unknown: Gamma conjugate prior for $1/\sigma^2$

- For this special case, Gamma prior for $1/\sigma^2$ is a conjugate prior:
  - The prior distribution:

$$1/\sigma^2 \mid \mu \sim \text{Gamma}(\alpha, \beta). \tag{14}$$

  - The sampling density:

$$y_1, \cdots, y_n \mid \mu, \sigma \overset{i.i.d.}{\sim} \text{Normal}(\mu, \sigma). \tag{15}$$

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  - The posterior distribution:

$$1/\sigma^2 \mid y_1, \cdots, y_n, \mu \sim \text{Gamma}\left(\alpha + \frac{n}{2}, \beta + \frac{1}{2}\sum_{i=1}^{n}(y_i - \mu)^2\right) \tag{16}$$

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- We can then use `rgamma()` R function to sample posterior draws of σ from Equation (16). Known quantities: $\alpha, n, \beta, \{y_i\}, \mu$

Simulate posterior draws of σ

```
alpha <- 1
beta <- 1
mu <- 8
n <- dim(CEsample)[1]
alpha_n <- alpha+n/2
beta_n <- beta+1/2*sum((CEsample$LogTotalExpLastQ-mu)^2)

set.seed(123)
S <- 1000
invsigma2_post <- rgamma(S, shape=alpha_n, rate=beta_n)
df <- as.data.frame(invsigma2_post)
```

Simulate posterior draws of σ cont'd

```
ggplot(data = df, aes(invsigma2_post)) +
  geom_density(color = crcblue, size = 1) +
  labs(title = "Posterior density") +
  xlab(expression(1/sigma^2)) +
  theme_grey(base_size = 8, base_family = "")
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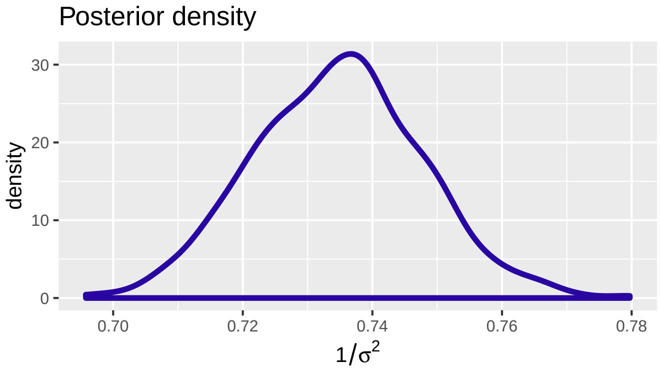

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- Bayesian hypothesis testing
  - exact solution: conjugacy
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- Posterior predictive checking
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  - Posterior predictive checking
- What if we want to use a different prior for μ? What if both μ and σ are unknown?