Bayesian Inference for a Proportion

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MATH 347 Bayesian Statistics

Outline

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Tokyo Express customers' dining preference

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- Suppose the restaurant owner wants to improve the business even more, especially for dinner.
- The owner plans to conduct a survey by asking their customers: "what is your favorite day to eat out for dinner?"
- The owner wants to find out how popular is choice of Friday.

Outline

Bayesian inference with discrete priors

- Three general steps of Bayesian inference:
  - Step 1: express an opinion about the location of the proportion p before sampling (prior).
  - Step 2: take the sample and record the observed proportion of preferring Friday (data/likelihood).
  - Step 3: use Bayes' rule to sharpen and update the previous opinion about p given the information from the sample (posterior).

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- Bayesian inference with discrete priors:
  - Step 1 on priors: list the finite number of possible values for the proportion p, and assign probability to each value.
  - Step 2 on data/likelihood: Binomial distribution.
  - Step 3 on posterior: use the discrete version of the Bayes' rule (summation Σ) for to sharpen and update the probability of each specified possible values of p.

Step 1: Prior distribution

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- Before giving out the survey, let's consider:
  - the possible value(s) of proportion p;
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- Suppose p can take 6 possible values

$$p = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}. \tag{1}$$

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$$p = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}. \tag{1}$$

- Further suppose Tokyo Express owner believes that some values are more likely than the others, specifically, a prior distribution:

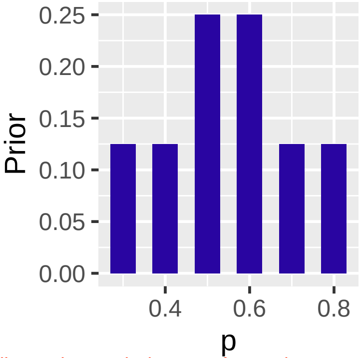
$$\pi_{owner}(p) = (0.125, 0.125, 0.250, 0.250, 0.125, 0.125). \tag{2}$$

- Exercise: Is the prior distribution $\pi_{owner}(p)$ reasonable? (Hint: 3 axioms of probability)

There are 6 associated probabilities, and they add up to 1. So it is reasonable.

Using R/RStudio to express and plot the prior $\pi_{owner}(p)$

```
bayes_table <- data.frame(p = seq(.3, .8, by=.1),
                          Prior = c(0.125, 0.125, 0.250,
                                    0.250, 0.125, 0.125))
ggplot(data=bayes_table, aes(x=p, y=Prior)) +
  geom_bar(stat="identity", fill=crcblue, width = 0.06)
```



Note that if the owner instead believes that each day out of a week are equally likely, then he can assign 100% on 1/7.

Step 2: Data/likelihood of proportion p

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Step 2: Data/likelihood of proportion p

- Now the Tokyo Express owner gives a survey to 20 customers.
- Out of the 20 responses, 12 say that their favorite day for eating out for dinner is Friday.
- Quantity of interest: p, the proportion of customers prefer eating out for dinner on Friday.
- The data/likelihood is a function of the quantity of interest.
- What would be the function of 12 out of 20 preferring Friday, in terms of the proportion p?

The Binomial distribution

- A Binomial experiment:
  1. One is repeating the same basic task or trial many times – let the number of trials be denoted by n.
  2. On each trial, there are two possible outcomes that are called success" orfailure".
  3. The probability of a success, denoted by p, is the same for each trial.
  4. The results of outcomes from different trials are independent.
- Do you think the survey is a Binomial experiment?

Well, we can think of answering Friday as a success and answering any other day as a failure, so indeed we can have two possible outcomes.

But 3. and 4. are questionable: for number 3, we need to assume that every customer shares the same probability of success, i.e. answering Friday. As for number 4, we need to assume that one person's response shouldn't affect the other

But, we will still go with Binomial experiment anyway.

The Binomial distribution

- A Binomial experiment:
  1. One is repeating the same basic task or trial many times – let the number of trials be denoted by n.
  2. On each trial, there are two possible outcomes that are called `success" orfailure".`
  3. The probability of a success, denoted by p, is the same for each trial.
  4. The results of outcomes from different trials are independent.
- Do you think the survey is a Binomial experiment?
- The probability of y successes in a Binomial experiment is given by

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, y = 0, \cdots, n, \tag{3}$$

where n is the number of trials and p is the success probability.

The likelihood function

- The probability of y successes in a binomial experiment is given by
$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, y = 0, \cdots, n, \tag{4}$$
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The likelihood function

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$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, y = 0, \cdots, n, \tag{4}$$
where n is the number of trials and p is the success probability.
- The likelihood is the chance of 12 successes in 20 trials viewed as a function of the probability of success is p:
$$\text{Likelihood} = L(p) = \binom{20}{12} p^{12} (1-p)^8. \tag{5}$$
  - L is a function of p.
  - n is fixed and known.
  - Y is the random variable.
  - p is the quantity of interest, also the unknown parameter in the Binomial distribution.

Use R/RStudio to compute the likelihood function

$$\text{Likelihood} = L(p) = \binom{20}{12} p^{12}(1-p)^8 \tag{6}$$

- Need: sample size n (20), number of successes k (12), and possible values of proportion p ($\{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$).
- Do not need: the assigned probabilities (0.125, 0.125, 0.250, 0.250, 0.125, 0.125) in the prior distribution $\pi_{owner}(p)$.

Use R/RStudio to compute the likelihood function

```
bayes_table$Likelihood <- dbinom(12, size=20,
                                 prob=bayes_table$p)
bayes_table
```

```
##     p Prior  Likelihood
## 1 0.3 0.125 0.003859282
## 2 0.4 0.125 0.035497440
## 3 0.5 0.250 0.120134354
## 4 0.6 0.250 0.179705788
## 5 0.7 0.125 0.114396740
## 6 0.8 0.125 0.022160877
```

Step 3: Posterior distribution

- Notations:
  - $\pi(p)$ the prior distribution of p.
  - $L(p)$ is the likelihood function.
  - $\pi(p \mid y)$ the posterior distribution of p after observing the number of successes y.

Step 3: Posterior distribution

- Notations:
  - $\pi(p)$ the prior distribution of p.
  - $L(p)$ is the likelihood function.
  - $\pi(p \mid y)$ the posterior distribution of p after observing the number of successes y.
- The Bayes' rule for a discrete parameter has the form

$$\pi(p_i \mid y) = \frac{\pi(p_i) \times L(p_i)}{\sum_j \pi(p_j) \times L(p_j)}, \tag{7}$$

  - $\pi(p_i)$ the prior probability of $p = p_i$.
  - $L(p_i)$ the likelihood function evaluated at $p = p_i$.
  - $\pi(p_i \mid y)$ the posterior probability of $p = p_i$ given the number of successes y.
  - the denominator gives the marginal distribution of the observation y (by the Law of Total Probability).

Remember that L(p_i) = p(y l p_i), and
\pi(p_j) x L(p_j) = p(y)

Use R/RStudio to compute and plot the posterior

```r
bayesian_crank(bayes_table) -> bayes_table
bayes_table
```

```
##     p Prior  Likelihood      Product   Posterior
## 1 0.3 0.125 0.003859282 0.0004824102 0.004975901
## 2 0.4 0.125 0.035497440 0.0044371799 0.045768032
## 3 0.5 0.250 0.120134354 0.0300335884 0.309786454
## 4 0.6 0.250 0.179705788 0.0449264469 0.463401326
## 5 0.7 0.125 0.114396740 0.0142995925 0.147495530
## 6 0.8 0.125 0.022160877 0.0027701096 0.028572757
```

- Inference question: What is the posterior probability that over half of the customers prefer eating out on Friday for dinner?

Use R/RStudio to compute and plot the posterior

```
bayesian_crank(bayes_table) -> bayes_table
bayes_table
```

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##     p Prior  Likelihood      Product   Posterior
## 1 0.3 0.125 0.003859282 0.0004824102 0.004975901
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```

- Inference question: What is the posterior probability that over half of the customers prefer eating out on Friday for dinner?

$$Prob(p > 0.5) = 0.463 + 0.147 + 0.029 = 0.639. \quad (8)$$

Use R/RStudio to compute and plot the posterior

```
bayesian_crank(bayes_table) -> bayes_table
sum(bayes_table$Posterior[bayes_table$p > 0.5])
```

```
## [1] 0.6394696
```

Use R/RStudio to compute and plot the posterior

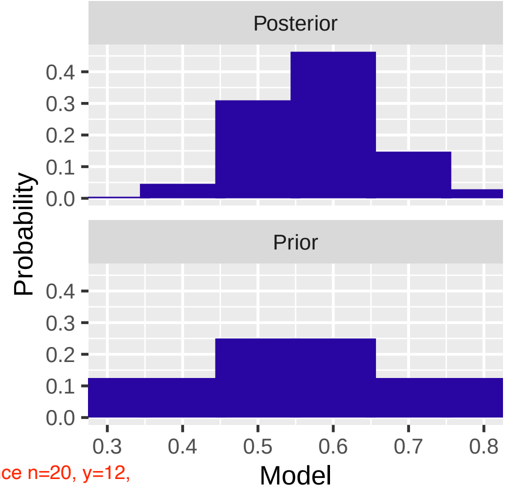
```r
bayesian_crank(bayes_table) -> bayes_table
sum(bayes_table$Posterior[bayes_table$p > 0.5])
```

```
## [1] 0.6394696
```

- Exercise: What is the posterior probability that less than 40% of the customers prefer eating out on Friday for dinner?

Using R/RStudio to compute and plot the posterior

```
prior_post_plot(bayes_table, Color = crcblue) +
  theme(text=element_text(size=10))
```



In frequentist's view, since n=20, y=12,
so \hat p = 12/20 = 0.6

Outline

Step 1: Prior distribution

- Before giving out the survey, we need to specify a prior distribution for unknown parameter p.
- Previously, p can take 6 possible values

$$p = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}. \tag{9}$$

- And a discrete prior distribution:

$$\pi_{owner}(p) = (0.125, 0.125, 0.250, 0.250, 0.125, 0.125). \tag{10}$$

Step 1: Prior distribution

```
bayes_table
```

```
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- Anything unsatisfactory?

Move to continuous priors

- A limitation of specifying a discrete prior for p
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Move to continuous priors

- A limitation of specifying a discrete prior for p
  - If a plausible value is not specified in the prior distribution (e.g. $p = 0.2$), it will be assigned a 0 probability in the posterior distribution.
- Ideally, we want a distribution that allows p to be any value in [0, 1].
- The continuous Uniform distribution:
  - Any value of p is equally likely.
  - The probability density function of the continuous Uniform on the interval $[a, b]$ is

$$\pi(p) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq p \leq b, \\ 0 & \text{for } p < a \text{ or } p > b. \end{cases} \tag{11}$$

  - $p \sim \mathrm{Uniform}(0, 1)$: a very special case of p.
- The Beta distribution!

The Beta distribution

- Notation: $\mathrm{Beta}(a, b)$.
- For a random variable falling between 0 and 1, suitable for proportion p.
- Beta distribution has two shape parameters a and b.
- Probability density function (pdf) is:

$$\pi(p) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1}, \ 0 \leq p \leq 1. \tag{12}$$

  - $B(a, b)$ is the Beta function $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.
  - Γ is the Gamma function.

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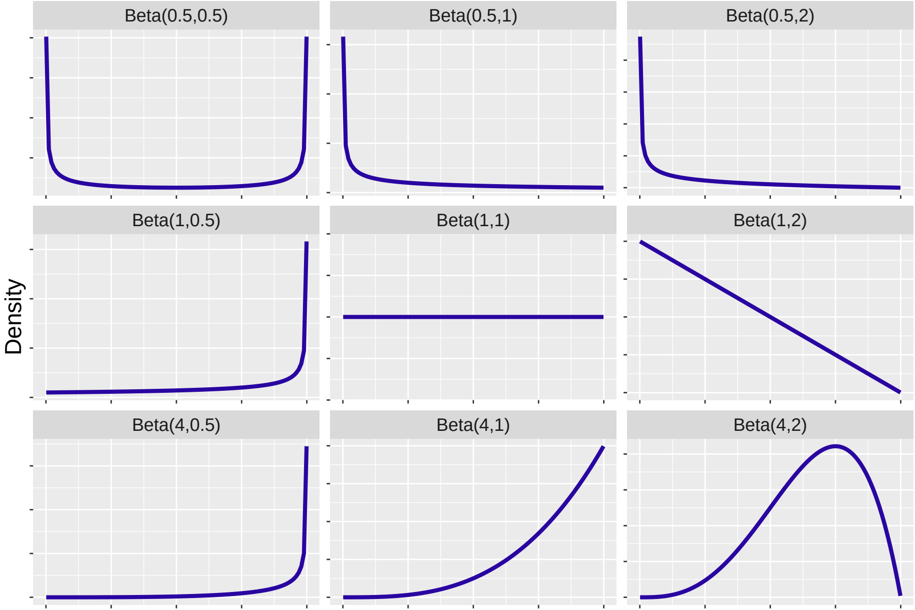
  - Γ is the Gamma function.
  - Continuous Uniform on [0, 1] is a special case of Beta with $a = b = 1$: $\mathrm{Uniform}(0,1) = \mathrm{Beta}(1,1)$

In terms of derivation, using Beta distribution is still much better even in the case of Beta(1,1)

weakly informative prior: I don't know what the prior should be at all

Examples of Beta curves



Choose a Beta curve to represent prior opinion

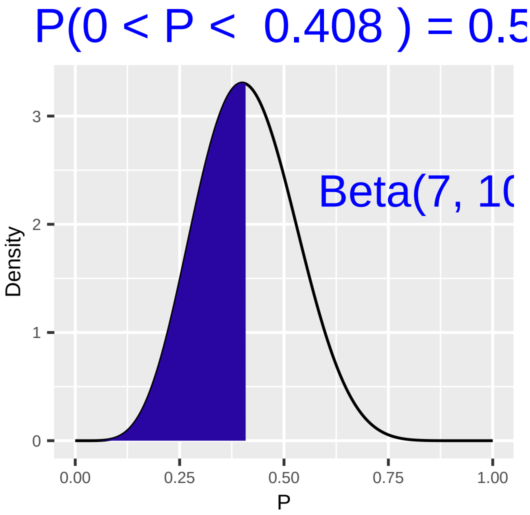
- Prior opinion: values of p and associated probabilities.
- Difficult to guess values of a and b in $\mathrm{Beta}(a, b)$.
- Solution: specifying a Beta prior by specification of quantiles of the distribution.

Choose a Beta curve to represent prior opinion

- Prior opinion: values of p and associated probabilities.
- Difficult to guess values of a and b in $\mathrm{Beta}(a, b)$.
- Solution: specifying a Beta prior by specification of quantiles of the distribution.
    - Quantiles are about rank order of values.
    - e.g. Middle quantile/50-th percentile: median.
    - `beta_quantile()` function in R: inputs a probability measure p and outputs the value of x such that $Prob(X <= x) = p$. (e.g. $x = 0.408$ when $p = 0.5$ for $\mathrm{Beta}(7, 10)$)

Choose a Beta curve to represent prior opinion

```
beta_quantile(0.5, c(7, 10), Color = crcblue) +
  theme(text=element_text(size=8))
```



Use `beta.select()` to choose a Beta curve

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- Example:
  - Suppose we believe that $p_{50} = 0.55$ (50-th quantile)
  - Suppose we also believe that $p_{90} = 0.8$ (90-th quantile)
  - Input these two sets of values into beta.select()

```
beta.select(list(x = 0.55, p = 0.5),
            list(x = 0.80, p = 0.9))
```

```
## [1] 3.06 2.56
```

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- Exercise 1: Verify that $\text{Beta}(3.06, 2.56)$ has $p_{50} = 0.55$ and $p_{90} = 0.8$. (Hint: use the beta_quantile() function)
- Exercise 2: Come up with your own Beta prior distribution, and share it with your neighbors.

Outline

Step 2: Data/likelihood of proportion p

- Recall that
  - The Tokyo Express owner gives a survey to 20 customers, and 12 respond that their favorite day is Friday.
  - The data/likelihood is a function of the quantity of interest, p.
  - It is a Binomial experiment, and

$$P(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}, y = 0, \cdots, n, \tag{13}$$

where n is the number of trials and p is the success probability.

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- Exercise: Write out the likelihood function for $n = 20$ and $y = 12$.

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- Exercise: Write out the likelihood function for $n = 20$ and $y = 12$.
- Solution:

$$\text{Likelihood} = L(p) = \binom{20}{12} p^{12} (1-p)^8. \tag{14}$$

Bayes' rule for continuous priors

- The likelihood function is the same, regardless of the prior distribution.
- Recall: the Bayes' rule for a discrete parameter has the form
$$\pi(p_i \mid y) = \frac{\pi(p_i) \times L(p_i)}{\sum_j \pi(p_j) \times L(p_j)} \tag{15}$$
- What about for continuous p? Unfortunately we can not list each value of p anymore.

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- What about for continuous p? Unfortunately we can not list each value of p anymore.
- Solution: With continuous p, the denominator changes from summation Σ to integration $\int$.
- Since $\int \pi(p) \times L(p)dp = f(y)$ is fixed, we can write the Bayes' rule for continuous p in proportional sign:
$$\pi(p \mid y) \propto \pi(p) \times L(p). \tag{16}$$

\pi(p) x L(p) = \pi(p) x f(y I p) = f(y, p)

Step 3: Derive the posterior

$$\pi(p \mid y) \propto \pi(p) \times L(p) \tag{17}$$

- For prior $\pi(p)$, we have $p \sim \mathrm{Beta}(3.06, 2.56)$.
- For data/likelihood $L(p)$, we have $Y \sim \mathrm{Binomial}(20, p)$.
- We need to derive $\pi(p \mid y)$.

Step 3: Derive the posterior

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- For prior $\pi(p)$, we have $p \sim \mathrm{Beta}(3.06, 2.56)$.
- For data/likelihood $L(p)$, we have $Y \sim \mathrm{Binomial}(20, p)$.
- We need to derive $\pi(p \mid y)$.
- Exercise: Derive $\pi(p \mid y)$ with the setup below.
  - The prior distribution:
    $$\pi(p) = \frac{1}{B(3.06, 2.56)} p^{3.06-1}(1-p)^{2.56-1}.$$
  - The likelihood:
    $$f(Y = 12 \mid p) = L(p) = \binom{20}{12} p^{12}(1-p)^{8}.$$
  - The posterior?

Step 3: Derive the posterior cont'd

- The posterior:

$$\pi(p \mid Y = 12) \quad \propto \quad \pi(p) \times f(Y = 12 \mid p)$$

Step 3: Derive the posterior cont'd

- The posterior:

$$\begin{aligned}\pi(p \mid Y = 12) &\propto \pi(p) \times f(Y = 12 \mid p) \\ &= \frac{1}{B(3.06, 2.56)} p^{3.06-1}(1-p)^{2.56-1} \times \\ &\quad \binom{20}{12} p^{12}(1-p)^{8}\end{aligned}$$

Step 3: Derive the posterior cont'd

- The posterior:

$$\begin{aligned}
\pi(p \mid Y = 12) &\propto \pi(p) \times f(Y = 12 \mid p) \\
&= \frac{1}{B(3.06, 2.56)} p^{3.06-1}(1-p)^{2.56-1} \times \\
&\quad \binom{20}{12} p^{12}(1-p)^{8} \\
\texttt{[drop the constants]} &\propto p^{12}(1-p)^{8} p^{3.06-1}(1-p)^{2.56-1} \\
\texttt{[combine the powers]} &= p^{15.06-1}(1-p)^{10.56-1}. \qquad (18)
\end{aligned}$$

- That is,

$$\pi(p \mid Y = 12) \propto p^{15.06-1}(1-p)^{10.56-1},$$

which means

Step 3: Derive the posterior cont'd

- The posterior:

$$\begin{aligned}
\pi(p \mid Y = 12) &\propto \pi(p) \times f(Y = 12 \mid p) \\
&= \frac{1}{B(3.06, 2.56)} p^{3.06-1}(1-p)^{2.56-1} \times \\
& \quad \binom{20}{12} p^{12}(1-p)^{8} \\
\texttt{[drop the constants]} &\propto p^{12}(1-p)^{8} p^{3.06-1}(1-p)^{2.56-1} \\
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\end{aligned}$$

- That is,

$$\pi(p \mid Y = 12) \propto p^{15.06-1}(1-p)^{10.56-1},$$

which means

$$p \mid Y = 12 \sim \mathrm{Beta}(15.06, 10.56).$$

From Beta prior to Beta posterior: conjugate priors

- The prior distribution:
$$p \sim \mathrm{Beta}(a, b)$$
- The sampling density:
$$Y \sim \mathrm{Binomial}(n, p)$$
- The posterior distribution:
$$p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$$
- Conjugate priors: from Beta prior to Beta posterior.

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$$p \sim \mathrm{Beta}(a, b)$$
- The sampling density:
$$Y \sim \mathrm{Binomial}(n, p)$$
- The posterior distribution:
$$p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$$
- Conjugate priors: from Beta prior to Beta posterior.

Source	Successes	Failures
Prior	a	b
Data/Likelihood	y	$n - y$
Posterior	$a + y$	$b + n - y$

The result in this table is quite intuitive!!

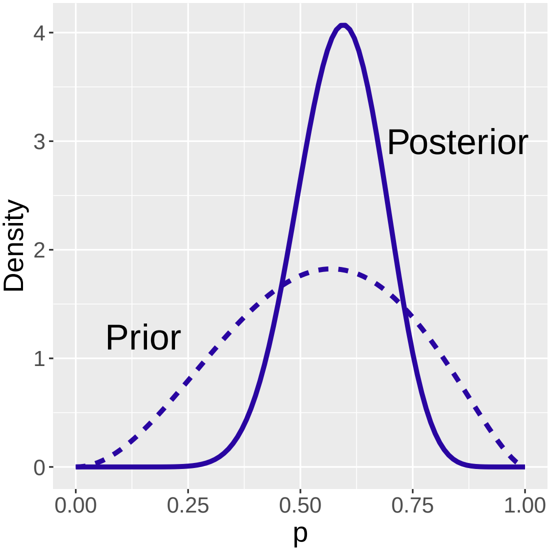
Use R/RStudio to compute and plot the posterior

$$p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$$

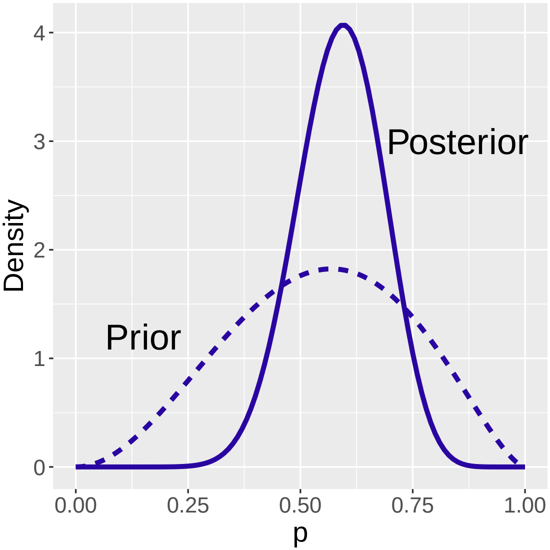
```
ab <- c(3.06, 2.56)
yny <- c(12, 8)
(ab_new <- ab + yny)
```

```
## [1] 15.06 10.56
```

Use R/RStudio to compute and plot the posterior cont'd



Use R/RStudio to compute and plot the posterior cont'd



- Exercise 1: Compare prior mean 0.544 and posterior mean 0.588. (Recall that sample mean is 0.6.)
- Exercise 2: Compare the spreads of the two curves.

Outline

Bayesian hypothesis testing

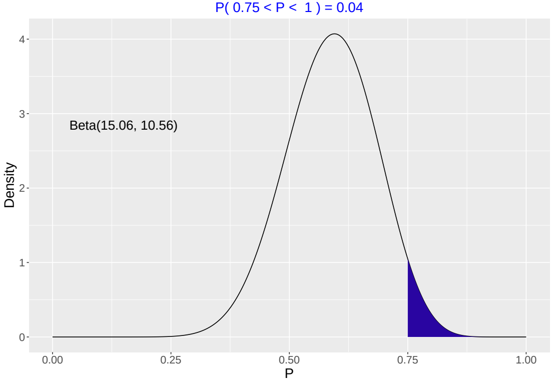
- Suppose one of the Tokyo Express's workers claims that at least 75% of the customers prefer Friday. Is this a reasonable claim?

Bayesian hypothesis testing

- Suppose one of the Tokyo Express's workers claims that at least 75% of the customers prefer Friday. Is this a reasonable claim?
- From a Bayesian viewpoint,
  - Find the posterior probability that $p >= 0.75$.
  - Make a decision based on the value of the posterior probability.
  - If the probability is small, we can reject this claim.

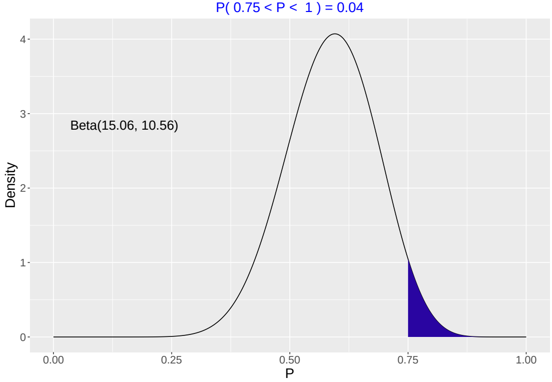
Bayesian hypothesis testing cont'd

```
beta_area(lo = 0.75, hi = 1.0, shape_par = c(15.06, 10.56),
          Color = crcblue) +
  theme(text=element_text(size=18))
```



Bayesian hypothesis testing cont'd

```
beta_area(lo = 0.75, hi = 1.0, shape_par = c(15.06, 10.56),
          Color = crcblue) +
  theme(text=element_text(size=18))
```



- Posterior probability is only 4%, reject the claim.
- Exercise: What about a claim “at most 30% of the customers prefer Friday"?

Bayesian hypothesis testing cont'd

When the posterior distribution is known...

Bayesian hypothesis testing cont'd

When the posterior distribution is known...

- Exact solution: use the `beta_area()` function in the `ProbBayes` package and/or the `pbeta()` R function

```
beta_area(lo = 0.75, hi = 1.0, shape_par = c(15.06, 10.56))
```

```
pbeta(1, 15.06, 10.56) - pbeta(0.75, 15.06, 10.56)
```

```
## [1] 0.03973022
```

Bayesian hypothesis testing cont'd

When the posterior distribution is known...

- Exact solution: use the `beta_area()` function in the `ProbBayes` package and/or the `pbeta()` R function

```
beta_area(lo = 0.75, hi = 1.0, shape_par = c(15.06, 10.56))
```

```
pbeta(1, 15.06, 10.56) - pbeta(0.75, 15.06, 10.56)
```

```
## [1] 0.03973022
```

- Approximation through Monte Carlo simulation using the `rbeta()` R function

```
S <- 1000
BetaSamples <- rbeta(S, 15.06, 10.56)
sum(BetaSamples >= 0.75)/S
```

```
## [1] 0.037
```

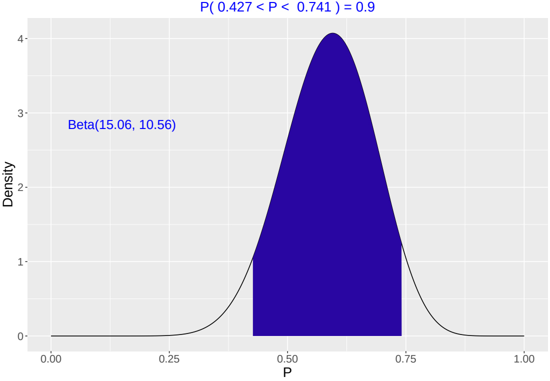
Bayesian credible intervals

- Consider an interval that we are confident contains p.
- Such an interval provides an uncertainty estimate for our parameter p.
- A 90% Bayesian credible interval is an interval contains 90% of the posterior probability.

This is so different from the confidence intervals in Frequentist's view!!

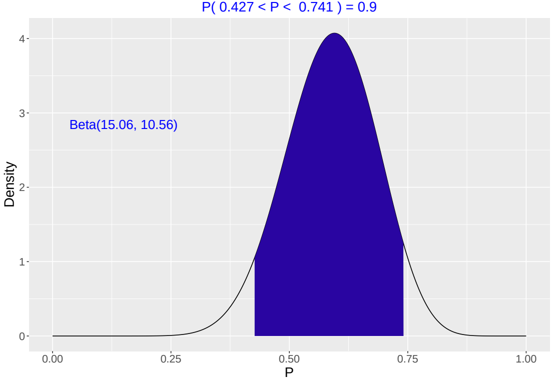
Bayesian credible intervals cont'd

```
beta_interval(0.9, c(15.06, 10.56), Color = crcblue) +
  theme(text=element_text(size=18))
```



Bayesian credible intervals cont'd

```
beta_interval(0.9, c(15.06, 10.56), Color = crcblue) +
  theme(text=element_text(size=18))
```



- The middle 90%.
- Interpretation: different from a confidence interval.
- Exercise: Construct a middle 98% credible interval for p.

Bayesian credible intervals cont'd

When the posterior distribution is known...

Bayesian credible intervals cont'd

When the posterior distribution is known...

- Exact solution: use the `beta_interval()` function in the `ProbBayes` package (only the middle 90%) and or the `qbeta()` R function (not necessarily the middle 90%)

```
beta_interval(0.9, c(15.06, 10.56), Color = crcblue)
```

```
c(qbeta(0.05, 15.06, 10.56), qbeta(0.95, 15.06, 10.56))
```

```
## [1] 0.4266788 0.7410141
```

Bayesian credible intervals cont'd

When the posterior distribution is known...

- Exact solution: use the `beta_interval()` function in the `ProbBayes` package (only the middle 90%) and or the `qbeta()` R function (not necessarily the middle 90%)

```
beta_interval(0.9, c(15.06, 10.56), Color = crcblue)
```

```
c(qbeta(0.05, 15.06, 10.56), qbeta(0.95, 15.06, 10.56))
```

```
## [1] 0.4266788 0.7410141
```

- Approximation through Monte Carlo simulation using the `rbeta()` R function (not necessarily the middle 90%)

```
S <- 1000; BetaSamples <- rbeta(S, 15.06, 10.56)
quantile(BetaSamples, c(0.05, 0.95))
```

```
##        5%       95%
## 0.4024251 0.7425349
```

Bayesian prediction

- Suppose the Tokyo Express owner gives out another survey to m customers, how many would prefer Friday?
- The predictive distribution: $\tilde{Y} \mid Y = y$.

Bayesian prediction

- Suppose the Tokyo Express owner gives out another survey to m customers, how many would prefer Friday?
- The predictive distribution: $\tilde{Y} \mid Y = y$.
  - The exact prediction:

$$\tilde{Y} \mid Y = y \sim \mathrm{Beta\text{-}Binomial}(m, a + y, b + n - y).$$

  - Prediction through simulation (our focus):

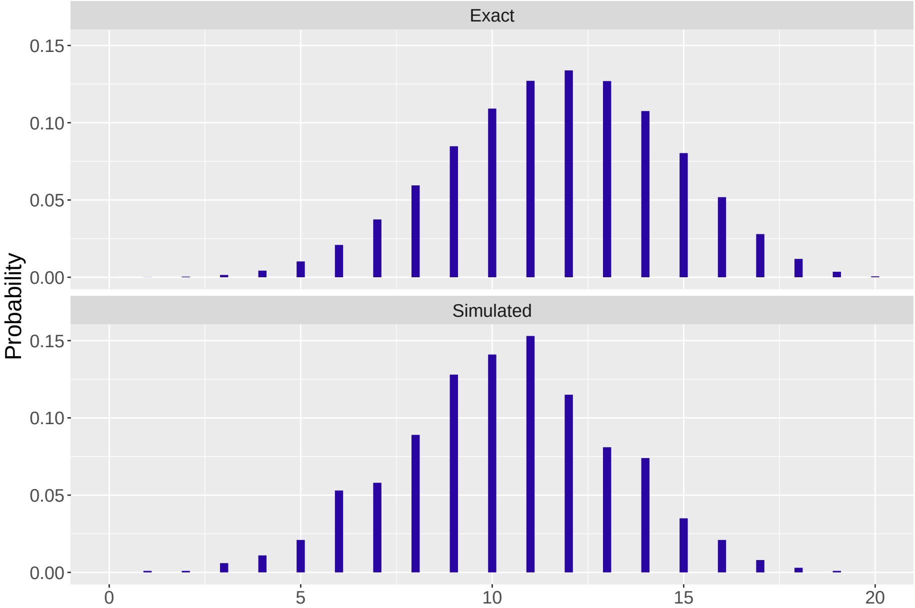
$$\text{sample } p \sim \mathrm{Beta}(a + y, b + n - y) \quad \rightarrow \quad \text{sample } \tilde{y} \sim \mathrm{Binomial}(m, p)$$

Use R/RStudio to make Bayesian predictions

```r
S <- 1000
a <- 3.06; b <- 2.56
n <- 20; y <- 12
m <- 20
pred_p_sim <- rbeta(S, a + y, b + n - y)
pred_y_sim <- rbinom(S, m, pred_p_sim)
sum(pred_y_sim >=5 & pred_y_sim <= 15)/S
```

```
## [1] 0.902
```

Use R/RStudio to make Bayesian predictions



Bayesian predictive checking

Check how good is our prediction

- Bayesian prediction is posterior predictive

$$\text{sample } p \sim \text{Beta}(a+y, b+n-y) \quad \rightarrow \quad \text{sample } \tilde{y} \sim \text{Binomial}(m, p)$$

When doing prediction, we use m samples

- We can perform posterior predictive checking through simulation

$$\begin{aligned}
\text{sample } p^{(1)} &\sim \text{Beta}(a+y, b+n-y) \quad \rightarrow \quad \text{sample } \tilde{y}^{(1)} \sim \text{Binomial}(n, p^{(1)}) \\
\text{sample } p^{(2)} &\sim \text{Beta}(a+y, b+n-y) \quad \rightarrow \quad \text{sample } \tilde{y}^{(2)} \sim \text{Binomial}(n, p^{(2)}) \\
&\vdots \\
\text{sample } p^{(S)} &\sim \text{Beta}(a+y, b+n-y) \quad \rightarrow \quad \text{sample } \tilde{y}^{(S)} \sim \text{Binomial}(n, p^{(S)})
\end{aligned}$$

When checking prediction, we use n samples,
since we want to see how our prediction performs on observed data

The sample $\{\tilde{y}^{(1)}, ..., \tilde{y}^{(S)}\}$ is an approximation to the posterior predictive distribution that can be used for model checking.

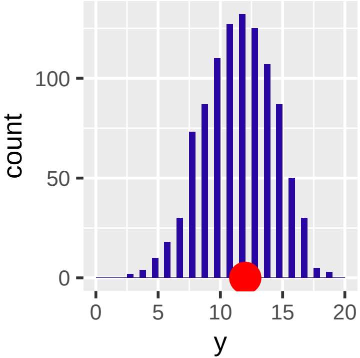
Use R/Rstudio to perform posterior predictive checking

```
S <- 1000
a <- 3.06; b <- 2.56
n <- 20; y <- 12
newy = as.data.frame(rep(NA, S))
names(newy) = c("y")

set.seed(123)
for (s in 1:S){
  pred_p_sim <- rbeta(1, a + y, b + n - y)
  pred_y_sim <- rbinom(1, n, pred_p_sim)
  newy[s,] = pred_y_sim
}
```

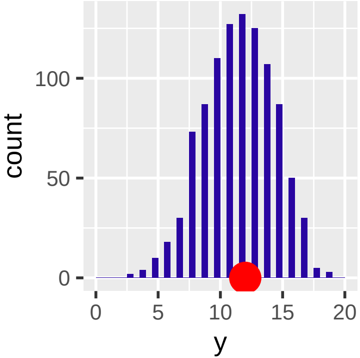
Use R/Rstudio to perform posterior predictive checking cont'd

```
ggplot(data=newy, aes(newy$y)) +
  geom_histogram(breaks=seq(0, 20, by=0.5), fill = crcblue) +
  annotate("point", x = 12, y = 0, colour = "red", size = 5) +
  xlab("y") + theme(text=element_text(size=10))
```



Use R/Rstudio to perform posterior predictive checking cont'd

```
ggplot(data=newy, aes(newy$y)) +
  geom_histogram(breaks=seq(0, 20, by=0.5), fill = crcblue) +
  annotate("point", x = 12, y = 0, colour = "red", size = 5) +
  xlab("y") + theme(text=element_text(size=10))
```



- The observed $y = 12$ is plotted as a red dot. The observed value of y is consistent with simulations of replicated data from this predictive distribution.

Use R/Rstudio to perform posterior predictive checking cont'd

- More formally, one can calculate the following probabilities:

$$Prob(y > \tilde{y} \mid y), \text{ or } 1 - Prob(y > \tilde{y} \mid y). \tag{19}$$

- If either probability is small, it suggests the model does not describe y very well.

Use R/Rstudio to perform posterior predictive checking cont'd

- More formally, one can calculate the following probabilities:

$$Prob(y > \tilde{y} \mid y), \text{ or } 1 - Prob(y > \tilde{y} \mid y). \tag{19}$$

- If either probability is small, it suggests the model does not describe y very well.

```
sum(newy > y)/S
```

```
## [1] 0.407
```

```
1 - sum(newy > y)/S
```

```
## [1] 0.593
```

- Since 0.407 and 0.593 are not small, it suggests the model describe y well. The inference passes the posterior predictive checking.

Outline

Recap

- Bayesian inference procedure:
  - Step 1: express an opinion about the location of the proportion p before sampling (prior).
  - Step 2: take the sample and record the observed proportion (data/likelihood).
  - Step 3: use Bayes' rule to sharpen and update the previous opinion about p given the information from the sample (posterior).

Recap

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- For Binomial data/likelihood, the Beta distributions are conjugate priors.
  - The prior distribution: $p \sim \mathrm{Beta}(a, b)$.
  - The sampling density: $Y \sim \mathrm{Binomial}(n, p)$.
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Recap

- Bayesian inference procedure:
  - Step 1: express an opinion about the location of the proportion p before sampling (prior).
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- For Binomial data/likelihood, the Beta distributions are conjugate priors.
  - The prior distribution: $p \sim \mathrm{Beta}(a, b)$.
  - The sampling density: $Y \sim \mathrm{Binomial}(n, p)$.
  - The posterior distribution: $p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$.
- Bayesian inferences (exact vs simulated)
  - Bayesian hypothesis testing & Bayesian credible intervals
  - Bayesian predictions
  - Bayesian posterior predictive checking