Solutions_to_Gamma_Poisson

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- Prior $Gamma(a, b)$: complete the prior density

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} exp(-b\theta) \tag{1}$$

- Likelihood: complete the joint pmf

$$\begin{align} Pr(Y_1 = y_1, ..., Y_n = y_n|\theta) &= \prod_{i=1}^{n} p(y_i|\theta) \tag{2} \\ &= \prod_{i=1}^{n} \frac{\theta^{y_i} exp(-\theta)}{y_i!} = \frac{\theta^{\sum y_i} exp(-n\theta)}{\prod_{i=1}^{n} y_i!} \tag{3} \end{align}$$

- Posterior $Gamma(a + \sum_{i=1}^{n} y_i, b + n)$: derive and recognize the hyper-parameters

$$\begin{align} \pi(\theta|y_1, ..., y_n) &\propto \pi(\theta)L(\theta) \tag{4} \\ &= \theta^{a+\sum y_i - 1} exp(-(b + n)\theta) \tag{5} \end{align}$$

- Prediction

$$\begin{align} Step1 &: \tilde{\theta} \sim Gamma(a + \sum y_i, b + n) \tag{6} \\ Step2 &: \tilde{y} \sim Poisson(\tilde{\theta}) \tag{7} \end{align}$$

Note: In Binomial case, we have

$$\begin{align} Step1 &: \tilde{p} \sim Beta(a + y, b + n - y) \tag{8} \\ Step2 &: \tilde{y} \sim Binomial(n, \tilde{p}) \tag{9} \end{align}$$