Problem 1 (20%) Given a stationary AR(p) process

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim^{i.i.d.} (0, \sigma^2)$$

In lag operator form,

$$\beta(L) Y_t = \alpha + \varepsilon_t,$$

1. Prove that $\beta(1) \neq 0$.
2. Show that $E(Y_t) = \beta(1)^{-1}\alpha$.

Problem 2 (10%) Consider the following MA(∞)

$$Y_t = \sum_{j=0}^{\infty} \theta^j \varepsilon_{t-j},$$

where $\varepsilon_t \sim^{i.i.d.} (0, \sigma^2)$ and $|\theta| < 1$. Show that Y_t is weak-stationary.

Problem 3 (20%) Consider following two MA(1) models:

$$Y_t = \varepsilon_t + \phi \varepsilon_{t-1},$$

and

$$x_t = u_t + \theta u_{t-1},$$

where, $|\phi| < 1$, $\varepsilon_t \sim^{i.i.d.} (0, \sigma_\varepsilon^2)$ and $u_t \sim^{i.i.d.} (0, \sigma_u^2)$.

1. Show that given $\theta = 1/\phi$ and $\sigma_u^2 = \phi^2 \sigma_\varepsilon^2$, the two models produce same autocovariance: $\gamma(0), \gamma(1), \dots$.
2. What does the above result imply?

Problem 4 (20%) Consider the stochastic process

$$Y_t = 2k\left(\varepsilon_t - \frac{1}{2}\right), \quad k > 0$$

where

$$\varepsilon_t \sim^{i.i.d.} \text{Bernoulli}\left(\frac{1}{2}\right)$$

1. Is the process Y_t strict-stationary?
2. Is the process Y_t weak-stationary?

Problem 5 (20%) Given that Y_t is a random walk process:

$$Y_t = \sum_{i=1}^{t} \varepsilon_i$$

with $Y_0 = 0$.

1. Show that Y_t can be rewritten as

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim^{i.i.d.} (0, 1)$$

2. Show that Y_t is NOT a strictly stationary process.

Problem 6 (10%) Given the following AR(1) process:

$$y_t = c + \beta y_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \sim WN(0, \sigma^2)$ and $|\beta| < 1$. Now suppose that this process starts at $t = 0$ with the initial value of y_0, which is a constant. Show that y_t is NOT covariance-stationary.