Applied Metrics for Macro and Finance, Spring Semester, 2023
Midterm Exam, April 12, 2023

1. (20%)
   Ans.

   (a) Yes, ε_t is weak-stationary.

$$\begin{aligned}
&E(\varepsilon_t) = 0 \\
&Var(\varepsilon_t) = \sigma_\varepsilon^2 < \infty \\
&Cov(\varepsilon_t, \varepsilon_{t-k}) = \gamma(k) = \begin{cases} \sigma_\varepsilon^2 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}
\end{aligned}$$

□

   (b) No, Y_t is not weak-stationary.

$$E(Y_t) = \beta_1 + \beta_2 t \text{ (change over } t\text{)}$$

□

   (c) Yes, $\Delta Y_t = Y_t - Y_{t-1}$ is weak-stationary.
   $\Delta Y_t = Y_t - Y_{t-1} = \beta_2 + \varepsilon_t - \varepsilon_{t-1}$

$$\begin{aligned}
\Rightarrow\ &E(\Delta Y_t) = \beta_2 \\
&Var(\Delta Y_t) = 2\sigma_\varepsilon^2 < \infty \\
&Cov(\Delta Y_t, \Delta Y_{t-k}) = \gamma(k) = \begin{cases} 2\sigma_\varepsilon^2 & \text{if } k = 0 \\ -\sigma_\varepsilon^2 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}
\end{aligned}$$

□

   (d) $u_t = \dfrac{1}{2q+1} \displaystyle\sum_{j=-q}^{q} Y_{t-j} = \dfrac{1}{2q+1} \sum_{j=-q}^{q} [\beta_1 + \beta_2(t-j) + \varepsilon_{t-j}] = \beta_1 + \beta_2 t + \dfrac{1}{2q+1} \sum_{j=-q}^{q} \varepsilon_{t-j}$

$$\Rightarrow E(u_t) = \beta_1 + \beta_2 t + \frac{1}{2q+1} \sum_{j=-q}^{q} E(\varepsilon_{t-j}) = \beta_1 + \beta_2 t$$

□

2. (20%)
   Ans.

   (a) $\because$ stationary $\therefore$ all roots of $\beta(z)$ lie outside the unit root circle, so 1 is not a root of $\beta(z)$

$$\Rightarrow \beta(1) \neq 0$$

□

   (b) $\because$ stationary $\therefore E(Y_t) = E(Y_{t-p})\ \forall p$

$$\begin{aligned}
&E(Y_t) = \alpha + \beta_1 E(Y_{t-1}) + \beta_2 E(Y_{t-2}) + \cdots + \beta_p E(Y_{t-p}) + E(e_t) \\
&\Rightarrow E(Y_t) = \frac{\alpha}{1 - \beta_1 - \beta_2 - \cdots - \beta_p}
\end{aligned}$$

□

(c)

$$
\begin{aligned}
\tilde{Y}_t &= Y_t - E(Y_t) \Rightarrow E(\tilde{Y}_t) = \beta(L)^{-1}E(e_t) = 0 \\
\gamma(0) &= Var(Y_t) = E(\tilde{Y}_t^2) = E(\tilde{Y}_t[\beta_1\tilde{Y}_{t-1} + \beta_2\tilde{Y}_{t-2} + \cdots + \beta_p\tilde{Y}_{t-p} + e_t]) \\
&= \beta_1\gamma(1) + \cdots + \beta_p\gamma(p) + \sigma^2 \\
\gamma(1) &= E(\tilde{Y}_t\tilde{Y}_{t-1}) = E([\beta_1\tilde{Y}_{t-1} + \beta_2\tilde{Y}_{t-2} + \cdots + \beta_p\tilde{Y}_{t-p} + e_t]\tilde{Y}_{t-1}) \\
&= \beta_1\gamma(0) + \beta_2\gamma(1) + \cdots + \beta_p\gamma(p-1) \\
\gamma(2) &= E(\tilde{Y}_t\tilde{Y}_{t-2}) = E([\beta_1\tilde{Y}_{t-1} + \beta_2\tilde{Y}_{t-2} + \cdots + \beta_p\tilde{Y}_{t-p} + e_t]\tilde{Y}_{t-2}) \\
&= \beta_1\gamma(1) + \beta_2\gamma(0) + \beta_3\gamma(1) + \cdots + \beta_p\gamma(p-2) \\
\gamma(p) &= E(\tilde{Y}_t\tilde{Y}_{t-p}) = E([\beta_1\tilde{Y}_{t-1} + \beta_2\tilde{Y}_{t-2} + \cdots + \beta_p\tilde{Y}_{t-p} + e_t]\tilde{Y}_{t-p}) \\
&= \beta_1\gamma(p-1) + \cdots + \beta_p\gamma(0) \\
\Rightarrow 1 &= \beta_1\rho(1) + \cdots + \beta_p\rho(p) + \frac{\sigma^2}{\gamma(0)} \\
Var(Y_t) = \gamma(0) &= \frac{\sigma^2}{1 - \beta_1\rho(1) - \beta_2\rho(2) - \cdots - \beta_p\rho(p)}
\end{aligned}
$$

□

(d)

$$
\begin{aligned}
Y_t &= \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} + e_t \\
E(Y_t) &= \alpha + \beta_1 E(Y_{t-1}) + \beta_2 E(Y_{t-2}) + \cdots + \beta_p E(Y_{t-p}) + E(e_t) \\
\Rightarrow\ Y_t - E(Y_t) &= \beta_1[Y_{t-1} - E(Y_{t-1})] + \beta_2[Y_{t-2} - E(Y_{t-2})] + \cdots + \beta_p[Y_{t-p} - E(Y_{t-p})] + e_t \\
\tilde{Y}_t &= \beta_1\tilde{Y}_{t-1} + \beta_2\tilde{Y}_{t-2} + \cdots + \beta_p\tilde{Y}_{t-p} + e_t \\
&\Rightarrow \beta(L)\tilde{Y}_t = e_t
\end{aligned}
$$

□

3. (10%)
 Ans.

(a)

$$
\begin{aligned}
E(e_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots) &= E(\varepsilon_t + \varepsilon_{t-1}\varepsilon_{t-2}|\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots) \\
&= E(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots) + \varepsilon_{t-1}\varepsilon_{t-2} \\
&= \varepsilon_{t-1}\varepsilon_{t-2} \neq 0 \\
&\Rightarrow e_t \text{ is not a MDS}
\end{aligned}
$$

□

(b)

$$
\begin{aligned}
&E(e_t) = E(\varepsilon_t) + E(\varepsilon_{t-1})E(\varepsilon_{t-2}) = 0 \neq \varepsilon_{t-1}\varepsilon_{t-2} = E(e_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots) \\
&\Rightarrow e_t \text{ is not an i.i.d. process}
\end{aligned}
$$

□

4. (15%)
 Ans.

(a) Y_t is an infinite order MA(∞) process,

$$\sum_{j=0}^{\infty} |\theta^j| = \frac{1}{1-|\theta|} < \infty \text{ (absolutely summable)}$$
$$\Rightarrow Y_t \text{ is weak-stationary}$$

$\square$

(b)

$$Var(Y_t) = \sum_{j=0}^{\infty} (\theta^j)^2 Var(e_{t-j}) = \sum_{j=0}^{\infty} \theta^{2j}\sigma^2 = \frac{\sigma^2}{1-\theta^2}$$

$\square$

(c)

$$\Psi(k) = \frac{\partial Y_{t+k}}{\partial e_t} = \theta^k = 0.5$$
$$\Rightarrow k = \frac{\log(0.5)}{\log(\theta)} \text{ is the half-life of the impulse responses}$$

$\square$

5. (20%)
Ans.

(a)

$$E(Y_t) = E(X_t) + E(e_t) = \mu$$
$$Var(Y_t) = \sigma_X^2 + \sigma_e^2 < \infty$$
$$Cov(Y_t, Y_{t-k}) = Cov(X_t + e_t, X_{t-k} + e_{t-k}) = \gamma(k) = \begin{cases} \sigma_X^2 + \sigma_e^2 & \text{if } k = 0 \\ \sigma_X^2 \rho_X(k) & \text{if } k > 0 \end{cases}$$
$$\Rightarrow Y_t \text{ is also weakly stationary}$$

$\square$

(b)

$$Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} = \begin{cases} 1 & \text{if } k = 0 \\ \frac{\sigma_X^2 \rho_X(k)}{\sigma_X^2 + \sigma_e^2} & \text{if } k > 0 \end{cases}$$

$\square$

(c) $\eta = \frac{\sigma_X^2}{\sigma_e^2}$,

$$\rho_Y(k) = \frac{\sigma_X^2 \rho_X(k)}{\sigma_X^2 + \sigma_e^2} = \frac{\rho_X(k)}{1 + \frac{\sigma_e^2}{\sigma_X^2}} = \frac{\rho_X(k)}{1 + \frac{1}{\eta}}$$
$$\Rightarrow \rho_Y(k) \longrightarrow \rho_X(k) \text{ as } \eta \longrightarrow \infty$$

$\square$

(d) When the signal variation (σ_X^2) is sufficiently larger than the measurement noise variation (σ_e^2), the $\{Y_t\}$ process approaches the $\{X_t\}$ process.

$\square$

6. (10%, 5%)

Ans.

(a) $\theta = \frac{1}{\phi}$, $\sigma_u^2 = \phi^2 \sigma_\varepsilon^2$,

$$Cov(Y_t, Y_{t-k}) = \gamma(k) = \begin{cases} (1+\phi^2)\sigma_\varepsilon^2 & \text{if } k = 0 \\ \phi\sigma_\varepsilon^2 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

$$Cov(X_t, X_{t-k}) = \gamma(k) = \begin{cases} (1+\theta^2)\sigma_u^2 & \text{if } k = 0 \\ \theta\sigma_u^2 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

$$\Rightarrow \phi\sigma_\varepsilon^2 = \frac{1}{\phi}\sigma_u^2 = \theta\sigma_u^2$$

$$(1+\phi^2)\sigma_\varepsilon^2 = (1+\phi^2)\frac{1}{\phi^2}\sigma_u^2 = (1+\frac{1}{\phi^2})\sigma_u^2 = (1+\theta^2)\sigma_u^2$$

$\square$

(b) Any invertible MA process has a non-invertible MA counterpart.

$\square$