Empirical Macroeconomics and Finance

Structural Vector Autoregressive Models

Shiu-Sheng Chen

Department of Economics
National Taiwan University

Section 1

Structural VAR (SVAR) Models

Structural VAR (SVAR)

- The SVAR is

$$y_t = D_0 y_t + D_1 y_{t-1} + \cdots + D_p y_{t-p} + B u_t$$

  - where $u_t \sim^{i.i.d.} (0, I)$, and $Bu_t \sim^{i.i.d.} (0, BB')$
  - Bu_t is called the structural shock.

- We can rewrite it as:

$$y_t = (I - D_0)^{-1} D_1 y_{t-1} + \cdots + (I - D_0)^{-1} D_p y_{t-p} + (I - D_0)^{-1} B u_t$$

- That is,

$$y_t = \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \varepsilon_t$$

  - where $\Phi_j = (I - D_0)^{-1} D_j$, and $\varepsilon_t = (I - D_0)^{-1} B u_t$

Identification

- As $D_j = (I - D_0)\Phi_j$, we can obtain D_j from D_0.
- Hence, according to

$$\varepsilon_t = (I - D_0)^{-1} B u_t$$

  the identification can be achieved by

$$\Sigma_\varepsilon = (I - D_0)^{-1} B B' (I - D_0)^{-1'}$$

  - $\frac{k(k-1)}{2} + k$ parameters can be identified from Σ_ε.
  - On the other hand, we need to identify $2k^2$ parameters in D_0 and B.
  - Thus, the difference is $\frac{k(3k-1)}{2}$.

Standard Assumptions

(a) B is diagonal. (Structural shocks are uncorrelated to each other.)

(b) Standardization: $D_{jj,0} = 0, \;\; j = 1, ...k$ or $[D_0]_{jj} = 0$

- These conditions imply

$$\underbrace{k^2 - k}_{\text{by}(a)} + \underbrace{k}_{\text{by}(b)}$$

- We still need to identify

$$\frac{k(3k-1)}{2} - (k^2 - k) - k = \frac{k(k-1)}{2}$$

Identification

- How to obtain $\frac{k(k-1)}{2}$ conditions?
  - Short-run restriction
    - ◇ Recursive (semi-structural)
    - ◇ Economic theory (structural)
  - Long-run restriction
- See Sims (1980), Bernanke (1986), and Blanchard and Quah (1989).

Short-Run Recursive Restriction on D_0

- For instance, k=4, we need $\frac{4(4-1)}{2} = 6$ restrictions.

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ y_{4t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ D_{21,0} & 0 & 0 & 0 \\ D_{31,0} & D_{32,0} & 0 & 0 \\ D_{41,0} & D_{42,0} & D_{43,0} & 0 \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ y_{4t} \end{bmatrix} + D_1 y_{t-1} + \cdots + D_p y_{t-p} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{bmatrix}$$

  - Semi-structural
  - Use your economic knowledge

How to Implement Recursive Restrictions

- Recall that

$$\Sigma_\varepsilon = (I - D_0)^{-1} B B' (I - D_0)^{-1'}$$

- D_0 is lower triangular by the recursive restriction
  - $(I - D_0)^{-1} B$ is lower triangular
  - The diagonal elements on $(I - D_0)^{-1} B$ are exactly the diagonal elements on B.

How to Implement Recursive Restrictions

- By Choleski decomposition.

$$\overbrace{(I-D_0)^{-1}BB'(I-D_0)^{-1'} = \Sigma_\varepsilon}^{\text{VAR model}} = CC'$$

$$\underbrace{\Sigma_\varepsilon = CC'}_{\text{Choleski}}$$

  where C is a lower triangular matrix.

- We thus identify D_0 and B from

$$C = (I-D_0)^{-1}B$$

Nonrecursive Structural VAR: The AB Model

- Given the SVAR(p)

$$y_t = D_0 y_t + D_1 y_{t-1} + \cdots + D_p y_{t-p} + Bu_t$$

- Let $A = (I - D_0)$,

$$Ay_t = D_1 y_{t-1} + \cdots + D_p y_{t-p} + Bu_t$$

- The likelihood function is

$$\begin{aligned}\mathcal{L} = \text{constant} + \frac{T}{2}\log|A|^2 - \frac{T}{2}\log|B|^2 \\ - \frac{T}{2} trace(A'B'^{-1}B^{-1}A\hat{\Sigma}_{\varepsilon}).\end{aligned}$$

- For more details, see Lutkepohl (2005), New Introduction to Multiple Time Series Analysis, pp. 372–373.

Examples of Studies Using Nonrecursive Identification

- Bernanke, Ben S. and Mihov, Ilian (1998), "Measuring monetary policy", Quarterly Journal of Economics, 113(3), 869–902.
- Kim, Soyoung and Roubini, Nouriel (2000), "Exchange rate anomalies in the industrial countries: A solution with a structural VAR approach", Journal of Monetary Economics, 45(3), 561–586.

Section 2

Structural VAR Tools

Structural VAR Tools

- Impulse Response Functions
- Variance Decomposition
- Historical Decomposition

Impulse Response Functions

- In State Space Form

$$\underbrace{\begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ \vdots \\ y_{t-p+1} \end{bmatrix}}_{kp\times 1} = \underbrace{\begin{bmatrix} \Phi_1 & \Phi_2 & \dots & \dots & \Phi_p \\ I & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & 0 & I & 0 \end{bmatrix}}_{kp\times kp} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ \vdots \\ y_{t-p} \end{bmatrix}}_{kp\times 1} + \underbrace{\begin{bmatrix} C \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{kp\times k} \underbrace{u_t}_{k\times 1}$$

  where $C = (I - D_0)^{-1}B$

- Let $F = (C\ 0 \cdots 0)'$,

$$Y_t = AY_{t-1} + Fu_t$$

Impulse Response Functions

- The impulse response function (IRF) is

$$\Psi(s)_{ij} = [\Psi(s)]_{ij} = \frac{\partial y_{it+s}}{\partial u_{jt}} = \frac{\partial y_{it}}{\partial u_{jt-s}}$$

$$= \underbrace{\begin{bmatrix} 0 & \cdots & 0 \; 1 \; 0 & \cdots & 0 \end{bmatrix}}_{1\times kp} A^s F \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{k\times 1}$$

Impulse Response Functions

- In practice, $\Psi(s)$ is estimated by

$$\hat{\Psi}(s)_{ij} = \underbrace{[0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]}_{1\times kp} \hat{A}^s \hat{F} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{k\times 1}$$

Confidence Interval for Impulse Response Functions

- When you estimate a VAR model such as $Y_t = AY_{t-1} + \epsilon_t$ via OLS to obtain estimates $\hat{A}$, you can use the estimated coefficients to construct estimates of impulse responses $\{\hat{A}^0\hat{F}, \hat{A}^1\hat{F}, \hat{A}^2\hat{F}, \hat{A}^3\hat{F}, \ldots\}$
- As with the underlying coefficient estimates, the estimated IRFs are just our best guesses. We would like to know how confident we can be about them. For example, how sure can we be that the response of a variable to a shock is always positive?
- For this reason, researchers often calculate confidence intervals with 90% or 95% significance level for each response in an impulse response graph.

Bootstrapping Confidence Interval for IRFs

Bootstrap methods are now commonly to derive the confidence bands for IRFs:

1. Estimate the VAR: $Y_t = AY_{t-1} + \epsilon_t$, and save the errors $\hat{\epsilon}_t$.
2. Resample from these errors to create, say 10,000 new error series ϵ_t^*, and 10,000 bootstrap data series via $Z_t^* = \hat{A}Z_{t-1}^* + \epsilon_t^*$.
3. Estimate a VAR model on the bootstrap data and save the 10,000 different IRFs associated with these estimates.
4. Calculate quantiles of the simulated IRFs at each horizon.
5. Use the n-th and $(100 - n)$-th quantiles of the simulated IRFs as confidence intervals.

Loose Monetary Policy Shocks in Bernanke and Mihov (1998)

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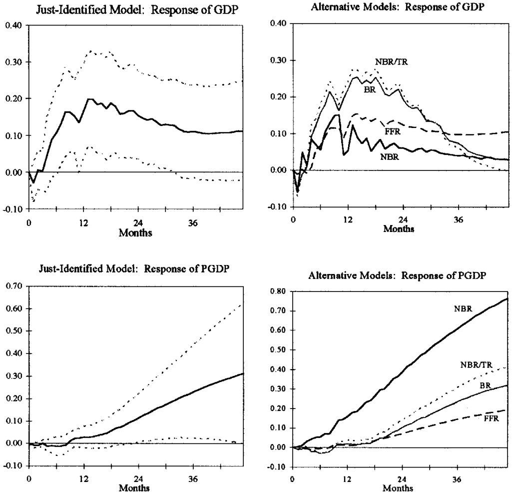

Variance Decomposition

- Accounting of forecast error variance.
- Idea:
  - Find out Var(Forecast Error of Variable i)
  - Compute

$$R^2_{ji,h} = \frac{\text{Var(Forecast Error of Variable } i \text{ due to Shock } j)}{\text{Var(Forecast Error of Variable } i)}$$

- $R^2_{ji,h}$ indicates how much variation of the i-th variable at the h step is due to the j-th shock.

Variance Decomposition

- Given the forecast error

$$y_{t+h} - E_t(y_{t+h}),$$

- We want to decompose the variance

$$Var(y_{t+h} - E_t(y_{t+h}))$$

  into different components that can be attributed to different structural shocks.

- For instance, want to know the contributions of monetary and real shocks on the volatility of real exchange rates.

Variance Decomposition

- Given SVAR, $y_t \in \mathbb{R}^k$

$$D(L)y_t = Bu_t,$$

we can rewrite it as SVMA(∞)

$$y_t = D(L)^{-1}Bu_t = A(L)Bu_t = A_0Bu_t + A_1Bu_{t-1} + A_2Bu_{t-2} + \ldots$$

- Therefore,

$$\begin{aligned} y_{t+h} &= A_0Bu_{t+h} + A_1Bu_{t+h-1} + A_2Bu_{t+h-2} + \ldots \\ &= \sum_{s=0}^{h-1} A_sBu_{t+h-s} + \sum_{s=h}^{\infty} A_sBu_{t+h-s} \end{aligned}$$

Variance Decomposition

- Clearly, for the first term,

$$
\begin{aligned}
E\left(\sum_{s=0}^{h-1} A_s B u_{t+h-s} \middle| \Omega_t\right) &= E_t\left(\sum_{s=0}^{h-1} A_s B u_{t+h-s}\right) \\
&= E_t(A_0 B u_{t+h}) + E_t(A_1 B u_{t+h-1}) \\
&\quad + \cdots + E_t(A_{h-1} B u_{t+2}) + E_t(A_{h-1} B u_{t+1}) \\
&= 0 + 0 + \cdots + 0 \\
&= 0
\end{aligned}
$$

Variance Decomposition

- Hence,

$$E_t(y_{t+h}) = 0 + \sum_{s=h}^{\infty} A_s B u_{t+h-s} = \sum_{s=h}^{\infty} A_s B u_{t+h-s}$$

and

$$y_{t+h} = \sum_{s=0}^{h-1} A_s B u_{t+h-s} + \underbrace{\sum_{s=h}^{\infty} A_s B u_{t+h-s}}_{E_t(y_{t+h})}$$

- The forecast error is

$$y_{t+h} - E_t(y_{t+h}) = \sum_{s=0}^{h-1} A_s B u_{t+h-s}$$

Variance Decomposition

$$
\begin{aligned}
&Var(\text{Forecast Error}) \\
&= Var\left(y_{t+h} - E_t(y_{t+h})\right) = Var\left(\sum_{s=0}^{h-1} A_s B u_{t+h-s}\right) \\
&= Var(A_0 B u_{t+h} + A_1 B u_{t+h-1} + \cdots + A_{h-1} B u_{t+1}) \\
&= E\left[\left(A_0 B u_{t+h} + A_1 B u_{t+h-1} + \cdots + A_{h-1} B u_{t+1}\right)\left(A_0 B u_{t+h}\right.\right. \\
&\quad \left.\left. + A_1 B u_{t+h-1} + \cdots + A_{h-1} B u_{t+1}\right)'\right] \\
&= A_0 B E\left[u_{t+h} u_{t+h}'\right] B' A_0' + A_1 B E\left[u_{t+h-1} u_{t+h-1}'\right] B' A_1' \\
&\quad + \cdots + A_{h-1} B E\left[u_{t+1} u_{t+1}'\right] B' A_{h-1}'
\end{aligned}
$$

Variance Decomposition

- Since $u_t \sim^{i.i.d.} (0, I)$, we have

$$
\begin{aligned}
& Var(\text{Forecast Error}) \\
& = A_0 B B' A_0' + A_1 B B' A_1' + \cdots + A_{h-1} B B' A_{h-1}' \\
& = \sum_{s=0}^{h-1} A_s B B' A_s'
\end{aligned}
$$

- This is a variance-covariance matrix!

Variance Decomposition

For the i-th element of y_t,

$$
\begin{aligned}
&Var\left(y_{i,t+h} - E_t(y_{i,t+h})\right) \\
&= i\text{-th element on the diagonal of } \left[\sum_{s=0}^{h-1} A_s B B' A_s'\right] \\
&= \sum_{s=0}^{h-1} \left(i\text{-th element on the diagonal of } [A_s B B' A_s']\right) \\
&= \sum_{s=0}^{h-1} \left[\sum_{j=1}^{k} A_{ij,s}^2 \sigma_j^2\right] = \sum_{j=1}^{k} \sum_{s=0}^{h-1} A_{ij,s}^2 \sigma_j^2,
\end{aligned}
$$

where $A_{ij,s}$ is the (i,j)-th element of A_s.

Variance Decomposition

In general, we will report

$$R^2_{ji,h} \equiv \frac{\sum_{s=0}^{h-1} A^2_{ij,s}\sigma^2_j}{\sum_{j=1}^{k}\sum_{s=0}^{h-1} A^2_{ij,s}\sigma^2_j}$$

to demonstrate the contribution of shock $u_{j,t}$ on the forecast error variance of $y_{i,t}$.

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Table 2
Forecast error variance decomposition.

Variable	*Horizon (quarters)*	*Proportion of variance explained by shocks to:*					
		Supply	Demand	Monetary policy	Exchange rate	Persistent global shocks	Transitory global shocks
GDP	1	0.50	0.08	0.04	0.06	0.14	0.17
	20	0.47	0.05	0.03	0.04	0.28	0.13
CPI	1	0.14	0.15	0.17	0.07	0.33	0.13
	20	0.15	0.12	0.16	0.07	0.36	0.15
Shadow BR	1	0.22	0.10	0.07	0.12	0.25	0.25
	20	0.21	0.09	0.08	0.05	0.29	0.28
Exchange rate	1	0.09	0.28	0.18	0.22	0.12	0.11
	20	0.11	0.23	0.15	0.19	0.17	0.15
Import prices	1	0.08	0.11	0.22	0.12	0.23	0.24
	20	0.08	0.10	0.19	0.12	0.26	0.26
Foreign export	1	0.00	0.00	0.00	0.00	0.48	0.52
prices	20	0.01	0.01	0.01	0.00	0.46	0.51

Note: The forecast error variance decomposition is the average of the 1000 variance decompositions obtained from the saved iterations of the estimation algorithm. See Appendix A for further detail on the estimation methodology.