Lab: VAR

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目錄

1 VAR Slides

$$E[\epsilon_t] = E\begin{bmatrix}\epsilon_{1t}\\ \epsilon_{2t}\\ \vdots \\ \epsilon_{kt}\end{bmatrix} = \begin{bmatrix}E\epsilon_{1t}\\ E\epsilon_{2t}\\ \vdots \\ E\epsilon_{kt}\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \vdots \\ 0\end{bmatrix} \tag{1}$$

$$\Sigma_\epsilon = Var(\epsilon_t) = E[(\epsilon_t - E[\epsilon_t])(\epsilon_t - E[\epsilon_t])'] = E[\epsilon_t \epsilon_t'] \tag{2}$$

$$= \begin{bmatrix} Var(\epsilon_{1t}) & Cov(\epsilon_{1t}, \epsilon_{2t}) & \dots & Cov(\epsilon_{1t}, \epsilon_{kt}) \\ & Var(\epsilon_{2t}) & \dots & \\ \vdots & & \ddots & \\ Cov(\epsilon_{kt}, \epsilon_{1t}) & & & Var(\epsilon_{kt}) \end{bmatrix} \tag{3}$$

$Cov(\epsilon_{it}, \epsilon_{jt}) \neq 0$

$$y_t = \beta_1 y_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.(0, \sigma^2) \tag{4}$$

$$y_{t+1} = \beta_1 y_t + \epsilon_{t+1} \tag{5}$$

$$E_t y_{t+1} = \beta_1 E_t y_t + E_t \epsilon_{t+1} = \beta_1 y_t \tag{6}$$

Notice that $E_t \epsilon_{t+1} \underbrace{=}_{indep} E(\epsilon_{t+1}) = 0$

2 Some Derivations

Consider AR(p):

$$y_t = \beta_1 y_{t-1} + ... + \beta_p y_{t-p} + \epsilon_t, y_t \in \mathcal{R} \tag{7}$$

In particular AR(1)

$$y_t = \beta_1 y_{t-1} + \epsilon_t, y_t \in \mathcal{R} \tag{8}$$

Consider VAR(p):

$$y_t = \Phi_1 y_{t-1} + ... + \Phi_p y_{t-p} + \epsilon_t, y_t \in \mathcal{R}^k \tag{9}$$

$$y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{bmatrix} \tag{10}$$

$$\Phi_j = \begin{bmatrix} \Phi_j^{11} & \Phi_j^{12} & ... & \Phi_j^{1k} \\ \Phi_j^{21} & \Phi_j^{22} & ... & \Phi_j^{2k} \\ \vdots \\ \Phi_j^{k1} & \Phi_j^{k2} & ... & \Phi_j^{kk} \end{bmatrix} \tag{11}$$

$$\epsilon_t = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \vdots \\ \epsilon_{kt} \end{bmatrix} \tag{12}$$

where $y_{jt} \in \mathcal{R}, \Phi_j^{mn} \in \mathcal{R}, \epsilon_{jt} \in \mathcal{R}$

Particularly, consider Bivariate VAR(1)

$$y_t = \begin{bmatrix} q_t \\ m_t \end{bmatrix}, y_t \in \mathcal{R}^2 \tag{13}$$

$$\underbrace{\begin{bmatrix} q_t \\ m_t \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} \Phi_1^{11} & \Phi_1^{12} \\ \Phi_1^{21} & \Phi_1^{22} \end{bmatrix}}_{\Phi_1} \underbrace{\begin{bmatrix} q_{t-1} \\ m_{t-1} \end{bmatrix}}_{y_{t-1}} + \underbrace{\begin{bmatrix} \epsilon_{qt} \\ \epsilon_{mt} \end{bmatrix}}_{\epsilon_t} \tag{14}$$

Write out the matrix, we have

$$\begin{cases} q_t = \Phi_1^{11} q_{t-1} + \Phi_1^{12} m_{t-1} + \epsilon_{qt} \\ m_t = \Phi_1^{21} q_{t-1} + \Phi_1^{22} m_{t-1} + \epsilon_{mt} \end{cases}$$

$\frac{\partial q_{t+s}}{\partial \epsilon_{qt}} = ?$ $\frac{\partial m_{t+s}}{\partial \epsilon_{mt}} = ?$ $\frac{\partial q_{t+s}}{\partial \epsilon_{mt}} = ?$ $\frac{\partial m_{t+s}}{\partial \epsilon_{qt}} = ?$

Back to VAR(1),

$$\begin{aligned} y_t &= \Phi_1 y_{t-1} + \epsilon_t, y_t \in \mathcal{R}^k \\ &= \Phi_1[\Phi_1 y_{t-2} + \epsilon_{t-1}] + \epsilon_t \\ &= \epsilon_t + \Phi_1 \epsilon_{t-1} + \Phi_1^2 y_{t-2} \\ &= \epsilon_t + \Phi_1 \epsilon_{t-1} + \Phi_1^2 \epsilon_{t-2} + \cdots + \Phi_1^s \epsilon_{t-s} + \Phi_1^{s+1} y_{t-s-1} \end{aligned}$$

$$y_{t+s} = \epsilon_{t+s} + \Phi_1 \epsilon_{t+s-1} + \Phi_1^2 \epsilon_{t+s-2} + \cdots + \Phi_1^s \epsilon_t + \Phi_1^{s+1} y_{t-1}$$

$$\frac{\partial y_{1t+s}}{\partial \epsilon_{2t}} = \begin{bmatrix} \underbrace{1}_{1st} & 0 & \dots & 0 \end{bmatrix} \Phi_1^s \begin{bmatrix} 0 \\ \underbrace{1}_{2nd} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

In general,

$$\frac{\partial y_{jt+s}}{\partial \epsilon_{it}} = \begin{bmatrix} 0 & \dots & 0 & \underbrace{1}_{jth} & 0 & \dots & 0 \end{bmatrix} \Phi_1^s \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \underbrace{1}_{ith} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{16}$$

Back to example,

$$\underbrace{\begin{bmatrix} q_t \\ m_t \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} \Phi_1^{11} & \Phi_1^{12} \\ \Phi_1^{21} & \Phi_1^{22} \end{bmatrix}}_{\Phi_1} \underbrace{\begin{bmatrix} q_{t-1} \\ m_{t-1} \end{bmatrix}}_{y_{t-1}} + \underbrace{\begin{bmatrix} \epsilon_{qt} \\ \epsilon_{mt} \end{bmatrix}}_{\epsilon_t} \tag{17}$$

$\frac{\partial q_{t+s}}{\partial \epsilon_{mt}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Phi_1^s \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{\partial m_{t+s}}{\partial \epsilon_{qt}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \Phi_1^s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

However, there exists a problem (correlation !!!).

$$\epsilon_t \sim (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_\epsilon) \tag{18}$$

Notice $E(\epsilon_{qt}) = E(\epsilon_{mt}) = 0$

$$\begin{aligned}
\Sigma_\epsilon &= E(\epsilon_t \epsilon_t') && (19)\\
&= E[\begin{bmatrix} \epsilon_{qt} \\ \epsilon_{mt} \end{bmatrix} \begin{bmatrix} \epsilon_{qt} & \epsilon_{mt} \end{bmatrix}] && (20)\\
&= E \begin{bmatrix} \epsilon_{qt}\epsilon_{qt} & \epsilon_{qt}\epsilon_{mt} \\ \epsilon_{mt}\epsilon_{qt} & \epsilon_{mt}\epsilon_{mt} \end{bmatrix} && (21)\\
&= \begin{bmatrix} E(\epsilon_{qt}^2) & E(\epsilon_{qt}\epsilon_{mt}) \\ E(\epsilon_{mt}\epsilon_{qt}) & E(\epsilon_{mt}^2) \end{bmatrix} && (22)\\
&= \begin{bmatrix} Var(\epsilon_{qt}) & Cov(\epsilon_{qt}, \epsilon_{mt}) \\ Cov(\epsilon_{mt}, \epsilon_{qt}) & Var(\epsilon_{mt}) \end{bmatrix} && (23)
\end{aligned}$$

Notice that $Cov(\epsilon_{qt}, \epsilon_{mt}) = Cov(\epsilon_{mt}, \epsilon_{qt}) \neq 0$. Hence $\frac{\partial q_{t+s}}{\partial \epsilon_{mt}}$ becomes meaningless as the condition "other things being equal" fails.

Therefore, our goal is

$$\underbrace{Cov(\epsilon_{qt}, \epsilon_{mt}) \neq 0}_{(reduced-form)VAR} \rightarrow \underbrace{Cov(e_{qt}, e_{mt}) = 0}_{SVAR}$$

where e_t is structural shock. In such cases, $\frac{\partial q_{t+s}}{\partial e_{mt}}$ possesses economic meanings.