Applied Econometrics for Macro and Finance

Unit-Root Econometrics

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Nonstationarity

Two important types of nonstationarity are:

- Trends
  - Deterministic Trend
  - Stochastic Trend (unit-root nonstationary)
- Structural breaks (parameter instability)
  - Threshold Model
  - Markov Switching Model

We will focus on Unit-Root Nonstationarity in this lecture.

- Nelson and Plosser (1982) found that the null hypothesis of unit root nonstationarity was not rejected for many macroeconomic series.

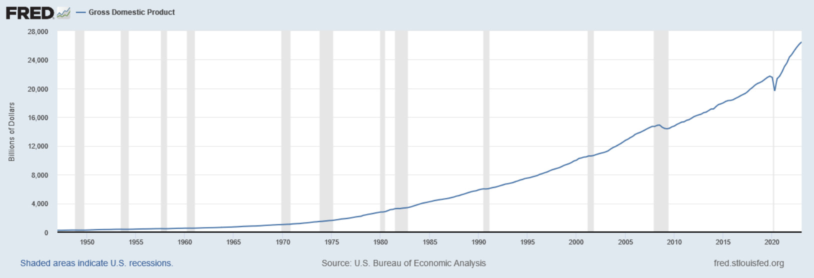
Outline of Discussion of Trends in Time Series Data

- What is a trend?
- What problems are caused by trends?
- How to address problems raised by trends
- How do you detect stochastic trends (statistical tests)?

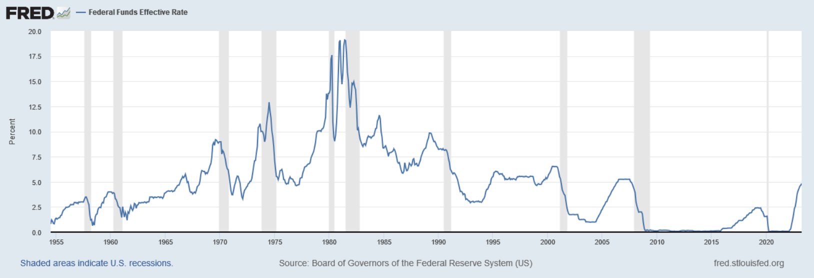
What is a Trend?

- A trend is a persistent, long-term movement or tendency in the data.
- Trends need not be just a straight line!
- Let's check the following three series.

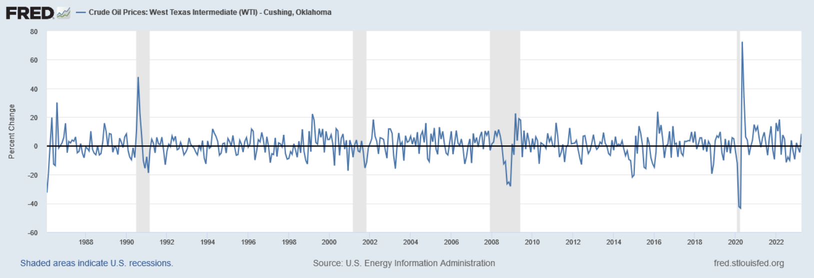
U.S. GDP


FRED
Gross Domestic Product
Billions of Dollars
28,000
24,000
20,000
16,000
12,000
8,000
4,000
0
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020
Shaded areas indicate U.S. recessions.
Source: U.S. Bureau of Economic Analysis
fred.stlouisfed.org

Federal Funds Rate



Monthly Changes in WTI Prices



What is a Trend?

- Different types of a trend
  - US GDP clearly has a long-run upward trend.
  - Federal funds rate has long-term swings, periods in which it is persistently high for many years (1970s/early 1980s) and periods in which it is persistently low. Maybe it has a trend: hard to tell.
  - The changes in WTI prices has no apparent trend.
- Trending time series
  - Trend stationary
  - Difference stationary

Deterministic and Stochastic Trends

- A deterministic trend is a nonrandom function of time.
    - For instance, consider a linear trend model

$$y_t = \alpha + \delta t + u_t, \;\; u_t \sim^{i.i.d.} (0, \sigma^2)$$

- A stochastic process that is stationary around a deterministic trend is called a trend stationary (TS) process.

Deterministic and Stochastic Trends

- A stochastic trend is random and varies over time.
- An important example of a stochastic trend is a random walk:
  - Driftless

$$y_t = y_{t-1} + u_t, \quad u_t \sim^{i.i.d.} (0, \sigma^2)$$

  - With drift

$$y_t = \mu + y_{t-1} + u_t, \quad u_t \sim^{i.i.d.} (0, \sigma^2)$$

- If y_t follows a random walk, then the value of y tomorrow is the value of y today, plus an unpredictable disturbance.

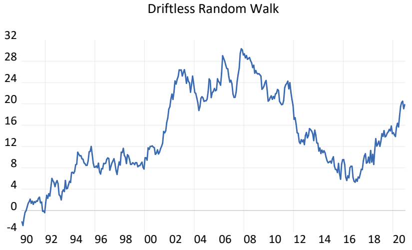
Deterministic and Stochastic Trends

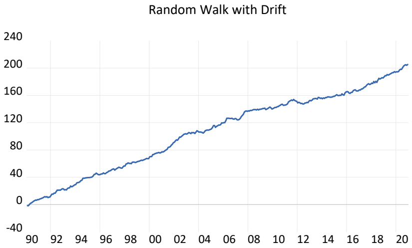
- Note that given a random walk with drift

$$
\begin{aligned}
y_t &= \mu + y_{t-1} + u_t \\
&= \underbrace{\mu t}_{\text{deterministic trend}} + \underbrace{\sum_{s=1}^{t} u_s}_{\text{stochastic trend}} + y_0
\end{aligned}
$$

- The drift is μ: y_t follows a random walk around a linear trend.

Random Walk with Drift vs. Driftless Random Walk





Key Features of a Driftless Random Walk

- Martingale

$$E(y_{T+h}|\Omega_T) = E_T(y_{T+h}) = y_T$$

    - Your best prediction of the value of y in the future is the value of y today
    - To a first approximation, log stock prices follow a random walk (more precisely, stock returns are unpredictable)

- Suppose $y_0 = 0$, then

$$Var(y_t) = t\sigma^2$$

This variance depends on t (increases linearly with t), so y_t isn't stationary.

Unit Roots

- The random walk model (with or without drift) is a good description of stochastic trends in many economic time series.
  - Random walk process is an example of a unit root process.
- Consider an AR(1) process

$$\beta(L)y_t = u_t,$$

which has a unit root if $\beta(1) = 0$.

- That is,

$$\beta(L) = 1 - L,$$

then

beta(L) = 1-beta_1 L = 1 - L, since beta_1 = 1

$$(1 - L)y_t = u_t \quad \text{or} \quad y_t = y_{t-1} + u_t$$

A driftless random walk process!

AR(2) model and Unit Root

- Given an AR(2) process

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + u_t$$

In lag operator

$$(1 - \beta_1 L - \beta_2 L^2) y_t = \beta(L) y_t = \beta_0 + u_t$$

- If $\beta(Z) = 0$ has a unit root,

LHS=因式分解

$$(1 - Z)(1 - \theta Z) = 0 = 1 - \beta_1 Z - \beta_2 Z^2$$

Hence, the necessary condition for a unit root in AR(2) process is

$$\beta_1 + \beta_2 = 1$$

AR(2) model and Dickey-Fuller Reparameterization

- Given AR(2)

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + u_t$$

- Dickey-Fuller reparameterization:

$$\Delta y_t = \beta_0 + \delta y_{t-1} + \alpha_1 \Delta y_{t-1} + u_t,$$

where $\delta = \beta_1 + \beta_2 - 1$, $\alpha_1 = -\beta_2$.

  - Recall the condition for AR(2) process with a unit root:

$$\beta_1 + \beta_2 = 1$$

  - So if there is a unit root, then $\delta = 0$.

Unit Roots in the AR(p) Model

- The Dickey-Fuller reparameterization of AR(p) model:

$$\Delta y_t = \beta_0 + \delta y_{t-1} + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + \cdots + \alpha_k \Delta y_{t-p+1} + u_t$$

where

$$\delta = \beta_1 + \beta_2 + \cdots + \beta_p - 1,$$

$$\alpha_j = -\sum_{s=j+1}^{p} \beta_s, \quad j = 1, \ldots, p-1$$

- If there is a unit root in the AR(p) model, then $\delta = 0$

What Problems are Caused by Stochastic Trends?

- If y and x both have stochastic trends then they can look related even if they are not: spurious regression
- AR coefficients are strongly downward biased. This leads to poor forecasts.
- Some t-statistics don't have a standard normal distribution, even in large samples (more on this later).

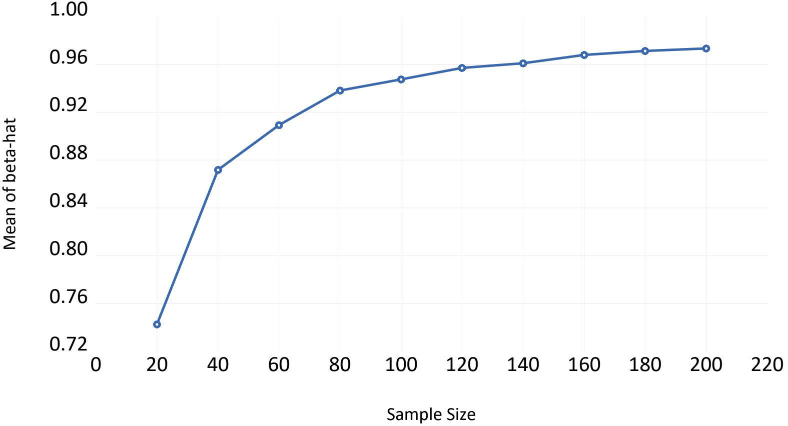
1. Spurious Regression

- Consider

$$y_t = \alpha + \beta z_t + e_t$$

- Granger and Newbold (1974) illustrate that if variables y_t and z_t are independent but contain stochastic trends,
  - the null hypothesis $\beta = 0$ is rejected
  - the regressions usually have very high R^2 values.
- When a regression model appears to find relationships between y_t and z_t that do not really exist, it is called a spurious regression.

2. Small Sample Downward Bias



3. Spurious Detrending and Inference Problem

Spurious Detrending

- The presence of stochastic trend implies the effects of shocks persist forever and the cyclical fluctuations cannot be studied by simply removing the fixed time trend.

Inference Problem

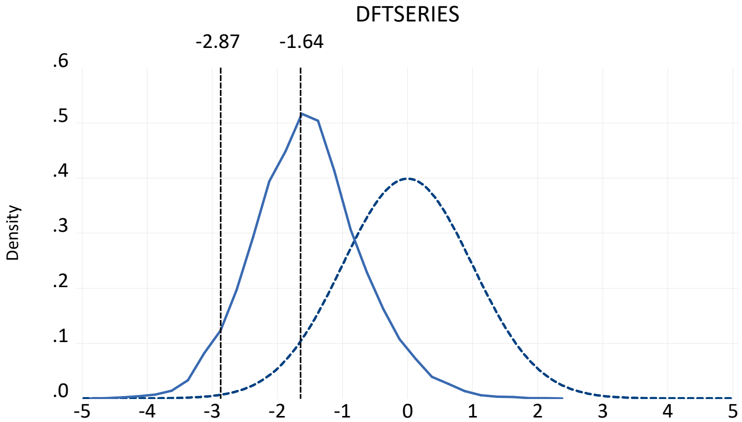
- Consider the following regression

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

- x_t contain stochastic trend: t-ratio for β doe not have standard asymptotic distribution

Inference Problem

Figure: 模擬在虛無假設 $\beta_1 = 1$ 下 t-統計量之抽樣分配



How To Remove the Trend

Differencing for stochastic trend (difference stationary)

Detrending for deterministic trend (trend stationary)

1. Differencing

$$\Delta y_t = y_t - y_{t-1}$$

2. Detrending

$$y_t - \mu_t$$

where

$$\mu_t = a_0 + a_1 t$$

Differencing vs. Detrending

y_t=beta*t+epsilon_t
y_t-y_{t-1}=beta+epsilon_t-epsilon_{t-1}

- First-differencing a trend-stationary (TS) process has introduced a noninvertible unit root process into the MA component of the model.
- Substracting a deterministic time trend from a difference-stationary (DS) process does not necessary result in a stationary series.
- Hence, we need a tool to determine whether the series is TS or DS. This requires a formal test of stationarity: tests for a unit root.

Differencing vs. Detrending: Nelson and Plosser (1982)

- Before the 1980's, it is believed that macroeconomic variables grow at a constant trend rate.
- Hence, it was a common practice of detrending macroeconomic data using a linear (or polynomial) deterministic trend.
- Nelson and Plosser (1982) challenge the traditional view by demonstrating that important macroeconomic variables tend to be DS rather than TS.

Testing Unit Root Hypothesis: AR(1) Model

- $y_t = \alpha y_{t-1} + u_t,\ u_t \sim^{i.i.d.} (0, \sigma^2)$
  - Null Hypothesis: $\alpha = 1$
- Dickey-Fuller Tests (the t-ratio)

$$\tau = \frac{\hat{\alpha} - 1}{se(\hat{\alpha})} \xrightarrow{d} DF$$

alpha

Testing for a Unit Root in the AR(1) Model

- Most of the time, we write the DF regression into

$$\Delta y_t = (\alpha - 1) y_{t-1} + u_t = \delta y_{t-1} + u_t$$

- Then the hypothesis becomes

$$H_0 : \delta = 0 \text{ vs. } H_1 : \delta < 0$$

- The t-ratio is

$$\tau = \frac{\hat{\delta}}{se(\hat{\delta})} \xrightarrow{d} DF_\tau$$

Remarks

- We have assumed that $u_t \sim^{i.i.d.} (0, \sigma^2)$. Suppose that u_t is a dependent process, we need to modify the asymptotic distributions.

u_t i.i.d.	u_t serially correlated
DF_α test$/DF_\tau$ test	ADF test

- Let's see how the ADF test works.

Testing Unit Root Hypothesis: AR(k) Model

- Consider the AR(k) model

$$\varphi(L)y_t = \mu + \varepsilon_t$$

where

$$\varphi(L) = 1 - \varphi_1 L - \cdots - \varphi_k L^k, \quad \varepsilon_t \sim^{i.i.d.} (0, \sigma^2)$$

- We say that y_t has a unit root if $\varphi(z) = 0$ has a root on the unit circle: $\varphi(1) = 0$.

Testing Unit Root Hypothesis: AR(k) Model

- Let $p = k - 1$.
- It can be shown that

$$\varphi(L) = (1 - L) - \delta\, L - \alpha_1(L - L^2) - \cdots - \alpha_p(L^p - L^{p+1})$$

- We thus have the following Dickey-Fuller reparameterization of $\varphi(L)y_t = \mu + \varepsilon_t$:

$$\Delta y_t = \mu + \delta\, y_{t-1} + \alpha_1\, \Delta y_{t-1} + \ldots + \alpha_p\, \Delta y_{t-p} + \varepsilon_t$$

- Hence, $\varphi(1) = -\delta$, that is , the parameter δ summarizes the information about the unit root.

Testing Unit Root Hypothesis: AR(k) Model

- Dickey-Fuller reparameterization

$$\Delta y_t = \mu + \delta\, y_{t-1} + \alpha_1\, \Delta y_{t-1} + \ldots + \alpha_p\, \Delta y_{t-p} + \varepsilon_t$$

- Therefore, the hypothesis of a unit root in y_t can be stated as

$$H_0 : \delta = 0 \quad \text{versus} \quad H_1 : \delta < 0$$

  - This is the most popular unit root test, and is called Augmented Dickey-Fuller (ADF) test.
  - Under H_0, we can not assess the significance of the ADF statistic using the normal table.

Simulating the Dickey-Fuller Distribution

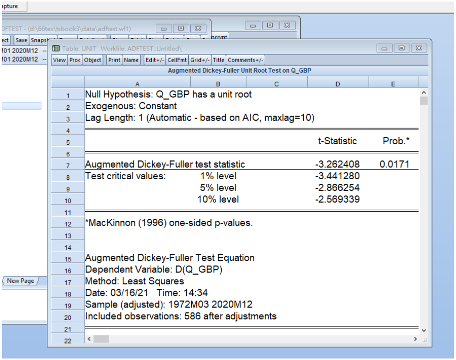
- Given $\delta = 0$. (under H_0),

$$\text{ADF}_t = \frac{\hat{\delta}}{se(\hat{\delta})} \xrightarrow{d} DF_\tau^\mu$$

- $T = 300$, $B = 10000$ replications
  - -3.44980 (1%), -2.86484 (5%), -2.55646 (10%)

Example: Does US-UK (log) Real Exchange Rate have a Unit Root?

- $ADF\text{-}t = -3.2624$, which can reject a unit root at 5% significance level.
- The long-run relative PPP holds.



Augmented Dickey-Fuller Unit Root Test on Q_GBP

Null Hypothesis: Q_GBP has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on AIC, maxlag=10)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-3.262408	0.0171
Test critical values:	1% level	-3.441280	
	5% level	-2.866254	
	10% level	-2.569339	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(Q_GBP)
Method: Least Squares
Date: 03/16/21 Time: 14:34
Sample (adjusted): 1972M03 2020M12
Included observations: 586 after adjustments