Hypothesis Testing

29.11.23

H_0: statement 1
H_1: statement 2

真相只有一個, which one?

We have data X: $X_1, X_2, X_3, \dots X_n$

A test is data-driven decision rule $D(x)$

$$D(x) \in \{\text{Reject } H_0, \text{ Accept } H_0\}$$

A good test

Reject H_0 when H_0 false, accept H_0 when H_0 true

Example: A fair coin

X: outcome of coin toss $\quad X \sim Ber(p), \ p \in [0, 1]$

H_0: $p = 0.5$
H_1: $p \neq 0.5$

How to construt $D(x)$?
Suppose we have $X_1, X_2, \dots X_n \sim Ber(p)$
An intuitive test

$$D(X) = \begin{cases} \text{reject} & |\bar{X}_n - 0.5| > c \\ \text{accept } H_0 & \text{otherwise} \end{cases}$$

What $c = ?$

If c too small, likely to reject H_0 when H_0 is true. type-I error
If c too large, likely to accept H_0 when H_0 is false. type-II error

It's impossible to get rid of type-I and type-II error, but we can calculate the probability of error.

> Recall: size = P(type-I error)
> power = $1 - P$(type-II error)

Next step:
given c, calculate size & power

$n = 10$, $c = 0.15$, $D(x) = \begin{cases} \text{reject} & \text{if } |\bar{X}_n - 0.5| > 0.15 \\ \text{accept} & \text{otherwise} \end{cases}$

P(type-I error): $\quad p = 0.5 \quad p(D(X) = \text{rejects})$

$$\bar{X}_n \in \{0, 0.1, 0.2, 0.3, \dots, 1\}$$

$$\begin{aligned} P(\text{reject}) = P(|\bar{X}_n - 0.5| > 0.15) &= P(\bar{X}_n = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}) \\ &= 1 - P(\bar{X}_n = \{0.4, 0.5, 0.6\}) \\ &= 1 - P(\bar{X}_n = 0.4) - P(\bar{X}_n = 0.5) - P(\bar{X}_n = 0.6) \\ &= 1 - C_4^{10}\, 0.5^4 \cdot 0.5^6 - C_5^{10}\, 0.5^5 \cdot 0.5^5 - C_6^{10}\, 0.5^6 \cdot 0.5^4 \end{aligned}$$

Remark:

$P(\text{type-I error})$ depends on c.

e.g. $c = 0.08$ $(n=10)$ $\quad |\bar{X}_n - 0.5| > 0.08$ rejects

when $n=10$ $\quad \{|\bar{X}_n - 0.5| < 0.08\} = \{\bar{X}_n = 0.5\}$

size $= 1 - P(\bar{X}_n = 0.5)$ $\quad c \downarrow$ size $\uparrow$

$P(\text{type-II error})$

$p \neq 0.5 \quad n=10, \quad c = 0.5$

$$P(D(X) \text{ accepts}) = P(\bar{X}_n = 0.6) + P(\bar{X}_n = 0.5) + P(\bar{X}_n = 0.4)$$
$$= C^{10}_6 p^6 (1-p)^4 + C^{10}_5 p^5 (1-p)^5 + C^{10}_4 p^4 (1-p^4)$$

Remark:

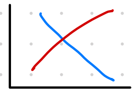
$P(\text{type-II error})$ depends on c $\quad c\uparrow \quad P(\text{type-II error}) \uparrow$

Remark:

$P(\text{type-II error})$ depends on p

Remark:

There's a trade-off between $\boxed{P(\text{type-I})}$ & $\boxed{P(\text{type-II})}$



Problem of exact calculations:

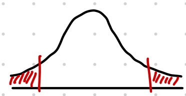
when $n = 1000$ $\quad \binom{1000}{500} 0.5^{500} \cdot 0.5^{500}$ $\quad$ such term will appear

Error probability approximation with CLT

$$\sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, p(1-p)) \Rightarrow \frac{\sqrt{n}(\bar{X}_n - p)}{\sqrt{p(1-p)}} \approx N(0,1) \quad \text{by CLT} \quad n \to \infty$$

$$P(\text{type-I error}) \quad P(D(X) = \text{reject}) = P\left(\frac{\sqrt{n}\,|\bar{X}_n - 0.5|}{\sqrt{0.5(1-0.5)}} > \frac{\sqrt{n}\,c}{\sqrt{0.5(1-0.5)}}\right)$$

$$\text{approximately} \quad \tilde{=}\ P\Big(|N(0,1)| > \underbrace{\frac{\sqrt{n}\,c}{0.5}}_{=\tilde{c}}\Big)$$



$$P(|N(0,1)| > \tilde{c}) = 2P(N(0,1) > \tilde{c}) = 2\Phi(-\tilde{c})$$

Φ: $N(0,1)$ CDF

$$= 2\Phi\left(\frac{-\sqrt{n}\,c}{0.5}\right)$$

When n large, given n, c

$$P(\text{type-I error}) = 2\Phi\left(\frac{-\sqrt{n}\,c}{0.5}\right)$$

When $n = 1000$ want $P(\text{type-I}) = 0.05$

$\Rightarrow$ solve $\quad 2\Phi(-2\sqrt{1000}\cdot c) = 0.05$

$\Phi(-2\sqrt{1000}\cdot c) = 0.025$

take $\Phi^{-1}(\cdot)$ $\Rightarrow -2\sqrt{1000}\cdot c = \Phi^{-1}(0.025)$

$$\Rightarrow c = \frac{\Phi^{-1}(-0.025)}{-2\sqrt{1000}}$$

$$= \frac{1.96}{2\sqrt{1000}} = \frac{1.96}{64} = 0.03 = 3\%$$