Chapter 1

Mean-variance analysis

1.1 Two risky assets

Suppose we can choose to invest in two risky assets with returns R_1 and R_2. If we invest a proportion w_i in asset i, then we must have $w_1 + w_2 = 1$. We can also call w_i the *portfolio share* invested in asset i.

The portfolio return is

$$R_p = w_1 R_1 + w_2 R_2.$$

The mean portfolio return is

$$\overline{R}_p = w_1 \overline{R}_1 + w_2 \overline{R}_2,$$

and the variance of the portfolio return is

$$\begin{aligned}
\sigma_p^2 &= \operatorname{var}\left[w_1 R_1 + w_2 R_2\right] \\
&= w_1^2 \operatorname{var} R_1 + 2 w_1 w_2 \operatorname{cov}(R_1, R_2) + w_2^2 \operatorname{var} R_2 \\
&= w_1^2 \sigma_1^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2 + w_2^2 \sigma_2^2
\end{aligned}$$

The last line defines some notation: σ_i is the standard deviation of the return on asset i, and ρ is the correlation between R_1 and R_2. (Remember that $\operatorname{cov}(X, Y) = \operatorname{corr}(X, Y)\sigma(X)\sigma(Y)$.)

With only two assets, we have $w_2 = 1 - w_1$, so

$$\overline{R}_p = \overline{R}_2 + w_1 \left(\overline{R}_1 - \overline{R}_2\right).$$

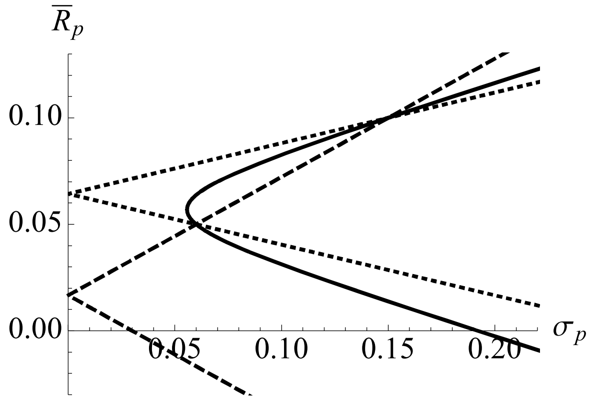

Figure 1.1: Three possible mean-standard deviation frontiers

Thus specifying a target mean portfolio return pins down

$$w_1 = \frac{\overline{R}_p - \overline{R}_2}{\overline{R}_1 - \overline{R}_2}.$$

The variance of the portfolio return is

$$\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1(1-w_1)\rho\sigma_1\sigma_2 + (1-w_1)^2\sigma_2^2.$$

This is a quadratic in w_1, and hence in the mean portfolio return $\overline{R}_p$. This means that if we plot mean against variance, the resulting curve is a *parabola.* If we plot mean against standard deviation, the resulting curve is a *hyperbola.*

Suppose $\overline{R}_1 = 10\%$, $\sigma_1 = 15\%$, $\overline{R}_2 = 5\%$ and $\sigma_2 = 6\%$. Figure 1.1 plots the attainable values of $(\sigma_p, \overline{R}_p)$ for $\rho = 1$ (dashed), $\rho = 0$ (solid), and $\rho = -1$ (dotted).

- Consider the case $\rho = 0$, and suppose you start with a portfolio that is fully invested in the less risky asset: $w_2 = 1$
- What happens to the portfolio mean if you shift a very small amount of wealth into the riskier asset? What happens to the portfolio variance?

If $\rho = \pm 1$, so the two assets are perfectly positively or negatively correlated, then the expression $\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2$ is a perfect square. For example, if $\rho = -1$, then

$$\sigma_p^2 = w_1^2\sigma_1^2 - 2w_1w_2\sigma_1\sigma_2 + w_2^2\sigma_2^2 = (w_1\sigma_1 - w_2\sigma_2)^2\,,$$

and we can choose w_1 and w_2 so that this variance is zero. (We *can*; an investor will not necessarily want to do so.)

In between, $-1 < \rho < 1$, the portfolio variance will be strictly positive, and if the weights w_1 and w_2 lie between 0 and 1 then $\sigma_p^2 < (w_1\sigma_1 + w_2\sigma_2)^2$. We also have

$$\frac{d\sigma_p^2}{dw_1} = 2w_1 \underbrace{\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)}_{\geq 0} - 2\left(\sigma_2^2 - \rho\sigma_1\sigma_2\right).$$

We can find the minimum variance portfolio by setting this to zero:

$$\frac{d\sigma_p^2}{dw_1} = 0 \implies w_{MV,1} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$

- If $\sigma_1 = \sigma_2$, then $w_{MV,1} = 1/2$
- If $\rho = 0$, then $w_{MV,1} = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$

If asset 2 is riskless, $\sigma_2 = 0$, then we write $R_2 = R_f$, and we have $\overline{R}_p = R_f + w_1(\overline{R}_1 - R_f)$ and $\sigma_p^2 = w_1^2\sigma_1^2$. So in the case $w_1 > 0$, we have $w_1 = \sigma_p/\sigma_1$, and

$$\overline{R}_p = R_f + \sigma_p \left(\frac{\overline{R}_1 - R_f}{\sigma_1}\right).$$

This defines a line in the mean-standard deviation diagram that is known as the *capital allocation line*. Its slope,

$$S_1 = \frac{\overline{R}_1 - R_f}{\sigma_1},$$

is the *Sharpe ratio* of the risky asset.

1.2 N risky assets

Suppose we have N risky assets to choose from with returns $R_1, R_2, \ldots, R_N$. Write $\overline{\boldsymbol{R}}$ for the vector of mean returns, $(\overline{R}_1, \ldots, \overline{R}_N)'$, and $\boldsymbol{\Sigma}$ for the variance-covariance matrix of returns, which is assumed to be non-singular. The ij-th entry of $\boldsymbol{\Sigma}$ is $\sigma_{ij} \equiv \text{cov}(R_i, R_j)$.

If we write $\boldsymbol{w} = (w_1, \ldots, w_N)'$ for the vector of portfolio weights on the N assets, then the overall mean portfolio return is

$$\overline{R}_p = w_1\overline{R}_1 + \cdots + w_N\overline{R}_N = \boldsymbol{w}'\overline{\boldsymbol{R}}$$

and the variance of the portfolio return is

$$\sigma_p = w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + \cdots + w_N^2\sigma_{NN} = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}.$$

When $N = 2$, the mean portfolio return uniquely identifies the portfolio weights and hence the portfolio variance. When $N > 2$, the portfolio choice problem is to solve

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

$$\begin{aligned} \text{s.t. } \boldsymbol{w}'\overline{\boldsymbol{R}} &= \overline{R}_p \\ \boldsymbol{w}'\boldsymbol{e} &= 1. \end{aligned}$$

The notation $\boldsymbol{e}$ means the vector of ones, $\boldsymbol{e} = (1, \ldots, 1)'$. Portfolios that solve this problem for some $\overline{R}_p$ are said to lie on the *minimum-variance frontier*. Portfolios on the "upper half" of the minimum-variance frontier are said to be *mean-variance efficient*.

This is a standard constrained optimization problem. Form the Lagrangian

$$\mathscr{L}(\boldsymbol{w}, \lambda_1, \lambda_2) = \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \lambda_1(\overline{R}_p - \boldsymbol{w}'\overline{\boldsymbol{R}}) + \lambda_2(1 - \boldsymbol{w}'\boldsymbol{e}).$$

The first-order conditions are

$$\boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1\overline{\boldsymbol{R}} - \lambda_2\boldsymbol{e} = 0,$$

which implies that

$$\boldsymbol{w} = \lambda_1\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{\Sigma}^{-1}\boldsymbol{e}. \tag{1.1}$$

- Don't be misled by the notation: there are N first-order conditions here

To solve for the Lagrange multipliers λ_1 and λ_2, we use the two constraints:

$$\begin{aligned} \overline{R}_p &= \lambda_1\overline{\boldsymbol{R}}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}} = \lambda_1 A + \lambda_2 B \\ 1 &= \lambda_1\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e} = \lambda_1 B + \lambda_2 C \end{aligned} \tag{1.2}$$

where $A \equiv \overline{\boldsymbol{R}}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}$, $B \equiv \boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}$ and $C \equiv \boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}$. These imply that

$$\begin{aligned} \lambda_1 &= \frac{C\overline{R}_p - B}{D} \\ \lambda_2 &= \frac{A - B\overline{R}_p}{D} \end{aligned}$$

where $D \equiv AC - B^2$. Note that while A, C and D are all mathematically guaranteed to be positive,[1] we cannot say the same about B. Below, though, we will suggest that for economic reasons it is likely that $B > 0$.

The optimized variance is

$$\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \boldsymbol{w}'\left(\lambda_1\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{e}\right) = \lambda_1\overline{R}_p + \lambda_2 = \frac{C\overline{R}_p^2 - 2B\overline{R}_p + A}{D}$$

which, as before, is quadratic in $\overline{R}_p$.

- A, B, C, and D are quantities that can in principle be estimated empirically. They represent all relevant information about the investment opportunity set
- It is up to an individual investor to decide what $\overline{R}_p$ she wants, and hence what λ_1 and λ_2 are

The portfolio choice problem is a convex optimization problem (because we have a quadratic objective subject to linear constraints) so equation (1.1) is necessary *and sufficient* for a portfolio to be mean-variance efficient. Because we have both necessity and sufficiency, we can say that the mean-variance efficient portfolios are precisely those portfolios $\boldsymbol{w}$ such that $\boldsymbol{w} = \lambda_1\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{\Sigma}^{-1}\boldsymbol{e}$ for some λ_1 and λ_2. Moreover, given any two distinct mean-variance efficient portfolios, we can construct any other mean-variance efficient portfolio by combining these two.

It is also informative to see this sufficiency—that any portfolio such that $\boldsymbol{\Sigma}\boldsymbol{w} = \lambda_1\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{e}$ is mean-variance efficient—directly by a different argument. Suppose that we have such a portfolio $\boldsymbol{w}$. We aim to show that any other portfolio $\boldsymbol{v}$ with the same mean has a variance that is at least as large. So, suppose $\boldsymbol{v}'\overline{\boldsymbol{R}} = \boldsymbol{w}'\overline{\boldsymbol{R}}$. Because $\boldsymbol{w}$ and

[1] This follows for A and C because $\boldsymbol{\Sigma}$ is positive definite, and hence $\boldsymbol{\Sigma}^{-1}$ is too (why?). To see that it holds for D, notice that $(B\overline{\boldsymbol{R}} - A\boldsymbol{e})'\boldsymbol{\Sigma}^{-1}(B\overline{\boldsymbol{R}} - A\boldsymbol{e}) > 0$. But expanding out the left-hand side, this is equivalent to $A^2C - AB^2 > 0$. Since A is positive, this implies that $AC > B^2$ as required.

$\boldsymbol{v}$ are both portfolios, we know that $\boldsymbol{v}'\boldsymbol{e} = \boldsymbol{w}'\boldsymbol{e} = 1$. Let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{w}$. So, $\boldsymbol{\eta}'\overline{\boldsymbol{R}} = 0$ and $\boldsymbol{\eta}'\boldsymbol{e} = 0$. The variance of the candidate portfolio $\boldsymbol{v}$ is

$$\begin{aligned}
\boldsymbol{v}'\boldsymbol{\Sigma}\boldsymbol{v} &= (\boldsymbol{w}+\boldsymbol{\eta})'\boldsymbol{\Sigma}(\boldsymbol{w}+\boldsymbol{\eta}) \\
&= \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + 2\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{w} + \boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta} \\
&= \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + 2\boldsymbol{\eta}'(\lambda_1\overline{\boldsymbol{R}} + \lambda_2\boldsymbol{e}) + \boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta} \\
&= \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta} \\
&\geq \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}.
\end{aligned}$$

So $\boldsymbol{w}$ was, indeed, mean-variance efficient.

1.2.1 The minimum-variance portfolio

Think about an investor who wants to minimize variance at all costs. Such an investor will choose the *minimum variance portfolio.* To find this portfolio, we drop the constraint that $\boldsymbol{w}'\overline{\boldsymbol{R}} = \overline{R}_p$, and find

$$\boldsymbol{w}_{MV} = \lambda\boldsymbol{\Sigma}^{-1}\boldsymbol{e} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}.$$

The expected return on this portfolio is

$$\overline{R}_{MV} = \boldsymbol{w}_{MV}'\overline{\boldsymbol{R}} = \frac{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}} = \frac{B}{C}$$

Empirically, the minimum-variance portfolio has a positive expected return, so this suggests that $B > 0$.

The variance that this portfolio achieves is

$$\boldsymbol{w}_{MV}'\boldsymbol{\Sigma}\boldsymbol{w}_{MV} = \frac{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}{\left(\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}\right)^2} = \frac{1}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}.$$

To get an intuitive sense of what this means in the case in which all assets have equal variance σ^2 and correlation ρ between any two assets, note that the minimum variance portfolio puts equal weight on each asset: $\boldsymbol{w}_{MV} = \boldsymbol{e}/N$. Then, $\boldsymbol{w}_{MV}'\boldsymbol{\Sigma}\boldsymbol{w}_{MV} = (1/N^2)\left[N\sigma^2 + (N^2 - N)\rho\sigma^2\right] = \rho\sigma^2 + (1-\rho)\sigma^2/N$. When N is very large, this is very close to $\rho\sigma^2$.

A nice property of the minimum-variance portfolio is that *it has the same covariance with every portfolio.* To see this, note that the covariance of $\boldsymbol{w}_{MV}$ with some arbitrary portfolio $\boldsymbol{w}$ is

$$\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}_{MV} = \frac{\boldsymbol{w}'\boldsymbol{e}}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}} = \frac{1}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}} > 0.$$

- Intuition: Suppose you had two portfolios $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ that had different covariances with $\boldsymbol{w}_{MV}$. Can you find a portfolio with lower variance than $\boldsymbol{w}_{MV}$?
- In particular, the global minimum-variance portfolio is positively correlated with every asset

The minimum-variance portfolio, which is proportional to $\boldsymbol{\Sigma}^{-1}\boldsymbol{e}$, is a natural portfolio to be interested in. In view of equation (1.1), another natural portfolio to look at is $\widetilde{\boldsymbol{w}}$,

$$\widetilde{\boldsymbol{w}} \equiv \frac{\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}}{\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}}.$$

The denominator of this is a scalar which normalizes $\widetilde{\boldsymbol{w}}$ so that its entries sum to 1. We can use this portfolio together with the minimum variance portfolio to rewrite (1.1) as

$$\boldsymbol{w} = \lambda_1 \boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\overline{\boldsymbol{R}}\widetilde{\boldsymbol{w}} + \lambda_2 \boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e}\boldsymbol{w}_{MV} = \lambda_1 B\widetilde{\boldsymbol{w}} + \lambda_2 C\boldsymbol{w}_{MV}.$$

Now, from (1.2), $\lambda_1 B + \lambda_2 C = 1$. This shows that any portfolio on the minimum-variance frontier can be expressed as a linear combination of the two portfolios $\widetilde{\boldsymbol{w}}$ and $\boldsymbol{w}_{MV}$. We have reduced the dimensionality of the problem from N risky assets to two portfolios. Any mean-variance investor would be equally happy whether (i) choosing between the menu of N assets, or (ii) choosing between just two mutual funds, one of which holds portfolio $\widetilde{\boldsymbol{w}}$ and the other of which holds portfolio $\boldsymbol{w}_{MV}$.

1.2.2 N risky assets and a riskless asset

We now add to the previous analysis a riskless asset with return R_f. By definition, R_f is not a random variable: it's a known *constant*, because the asset is riskless. Let the portfolio weights on the risky assets be $w_1, \ldots, w_N$ as before, and let the weight on the riskless asset be w_0. So we want to solve

$$\min_{w_0,\ldots,w_N} \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

$$\begin{aligned} \text{s.t. } w_0 R_f + \boldsymbol{w}'\overline{\boldsymbol{R}} &= \overline{R}_p \\ w_0 + \boldsymbol{w}'\boldsymbol{e} &= 1. \end{aligned}$$

We can use the second constraint to substitute out for w_0. Then the problem becomes

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

$$\text{s.t. } \boldsymbol{w}'(\overline{\boldsymbol{R}} - R_f\boldsymbol{e}) = \overline{R}_p - R_f$$

The vector $\overline{\boldsymbol{R}} - R_f\boldsymbol{e}$ summarizes the *excess returns* on the N risky assets. The first-order condition for this problem is

$$\boldsymbol{\Sigma}\boldsymbol{w} = \lambda(\overline{\boldsymbol{R}} - R_f\boldsymbol{e}),$$

so the optimal choice of risky assets is

$$\boldsymbol{w} = \lambda\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{R}} - R_f\boldsymbol{e})$$

for some λ chosen so that the constraint $\boldsymbol{w}'(\overline{\boldsymbol{R}} - R_f\boldsymbol{e}) = \overline{R}_p - R_f$ is satisfied, i.e.

$$\lambda = \frac{\overline{R}_p - R_f}{E}$$

where $E \equiv (\overline{\boldsymbol{R}} - R_f\boldsymbol{e})'\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{R}} - R_f\boldsymbol{e})$. With this choice of λ, the portfolio variance is

$$\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = (\overline{\boldsymbol{R}} - R_f\boldsymbol{e})'\boldsymbol{\Sigma}^{-1}\lambda\boldsymbol{\Sigma}\lambda\boldsymbol{\Sigma}^{-1}(\overline{\boldsymbol{R}} - R_f\boldsymbol{e}) = \lambda^2 E = \frac{(\overline{R}_p - R_f)^2}{E},$$

so the portfolio standard deviation satisfies

$$\left|\overline{R}_p - R_f\right| = \sigma_p\sqrt{E}.$$

This is a straight line on a plot of mean against standard deviation. The line is tangent to the minimum-variance frontier achievable by trading only in risky assets. The slope of the line is the Sharpe ratio of the *tangency portfolio*, which has the highest Sharpe ratio of any portfolio of risky assets. (See Exercise 6.1.9.)

Similarly to the previous section, this says that mean-variance investors would be indifferent between trading all $N + 1$ assets, and trading just the riskless asset and tangency portfolio. This is sometimes called Tobin's mutual fund theorem.

This does *not* imply that all investors should hold the same portfolio.[2] Extremely risk-averse investors may want almost all of their wealth in the riskless asset. Conversely, investors with high risk tolerance may want to hold much of their wealth in the tangency portfolio, or to lever up, shorting the riskless asset in order to buy even more of the tangency portfolio.

In practice, to implement mean-variance analysis, we need to come up with estimates of $\overline{\boldsymbol{R}}$ and $\boldsymbol{\Sigma}$. It may be hard to do so over a relatively short sample. This issue motivates a search for ways of identifying optimal portfolios that are based on further theoretical considerations. The capital asset pricing model (CAPM) is a equilibrium model that does exactly this.

1.2.3 Covariance properties of efficient portfolios

Consider a mean-variance efficient portfolio p, and suppose we are considering increasing slightly the weight w_i on one of the assets, i, financed by a decrease in the weight on the riskless asset. This will affect the mean and variance of the portfolio return as follows:

$$\begin{aligned} \frac{d\overline{R}_p}{dw_i} &= \overline{R}_i - R_f \\ \frac{d \operatorname{var} R_p}{dw_i} &= 2\operatorname{cov}(R_i, R_p), \end{aligned}$$

because the terms in $\operatorname{var} R_p$ that involve w_i are

$$2w_1 w_i \operatorname{cov}(R_1, R_i) + \cdots + w_i^2 \operatorname{var} R_i + \cdots + 2w_N w_i \operatorname{cov}(R_N, R_i),$$

so the derivative

$$\frac{d \operatorname{var} R_p}{dw_i} = 2w_1 \operatorname{cov}(R_1, R_i) + \cdots + 2w_i \operatorname{var} R_i + \cdots + 2w_N \operatorname{cov}(R_N, R_i) = 2\operatorname{cov}(R_p, R_i).$$

The ratio of the effects on mean and on variance is

$$\frac{d\overline{R}_p/dw_i}{d \operatorname{var} R_p/dw_i} = \frac{\overline{R}_i - R_f}{2\operatorname{cov}(R_p, R_i)}.$$

[2]However, it *does* imply that if you only look at the risky assets held by investors, all investors should hold risky assets in the same proportions. This contradicts standard investment advice, as pointed out by Canner, Mankiw and Weil (1997).

Claim: If portfolio p is efficient, this ratio should be the same for all assets. Why? Consider adjusting two different portfolio weights w_i and w_j (financed in each case by changes in holdings of the riskless asset). The effects on mean and variance of R_p are

$$\begin{aligned} d\overline{R}_p &= (\overline{R}_i - R_f)dw_i + (\overline{R}_j - R_f)dw_j \\ d \operatorname{var} R_p &= 2\operatorname{cov}(R_p, R_i)dw_i + 2\operatorname{cov}(R_p, R_j)dw_j \end{aligned}$$

We can choose dw_j so that the mean portfolio return $d\overline{R}_p = 0$:

$$dw_j = -\frac{\overline{R}_i - R_f}{\overline{R}_j - R_f} dw_i.$$

If we do so, the portfolio variance must also be unchanged—otherwise we could achieve a lower variance with the same mean, contradicting the assumption that p is efficient. So we must have

$$d \operatorname{var} R_p = \left[2\operatorname{cov}(R_p, R_i) - 2\operatorname{cov}(R_p, R_j) \frac{\overline{R}_i - R_f}{\overline{R}_j - R_f} \right] dw_i = 0.$$

This implies that

$$\frac{\overline{R}_i - R_f}{\operatorname{cov}(R_p, R_i)} = \frac{\overline{R}_j - R_f}{\operatorname{cov}(R_p, R_j)}$$

for *any* assets i and j. This must also hold, therefore, for the portfolio p itself:

$$\frac{\overline{R}_i - R_f}{\operatorname{cov}(R_p, R_i)} = \frac{\overline{R}_p - R_f}{\operatorname{var} R_p}.$$

Rearranging,

$$\overline{R}_i - R_f = \frac{\operatorname{cov}(R_i, R_p)}{\operatorname{var} R_p} \left(\overline{R}_p - R_f \right) = \beta_{ip} \left(\overline{R}_p - R_f \right). \tag{1.3}$$