Financial Econometrics

Volatility Modeling

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Predictability of Return Volatility

- Consider a standard ARMA(1,1) model for an asset return Y.

$$Y_{t+1} = \phi_0 + \phi_1 Y_t + \epsilon_{t+1} + \theta\epsilon_t, \quad \epsilon_{t+1} \sim WN(0, \sigma^2)$$

- Now, let's turn our focus on conditional variance, which is standing at time t and predicting the variance at time $t+1$.

$$\begin{aligned} var_t[Y_{t+1}] &= var_t[\phi_0 + \phi_1 Y_t + \epsilon_{t+1} + \theta\epsilon_t] \\ &= var_t[\epsilon_{t+1}] = E_t[\epsilon_{t+1}^2] = \sigma^2 \end{aligned}$$

- In any model, as long as $\epsilon_{t+1} \sim WN(0, \sigma^2)$, asset returns would have a constant conditional variance.

Predictability of Return Volatility (cont.)

- Now, here's the question:
  Is the conditional variance of returns really constant?

- To access this empirically, note that

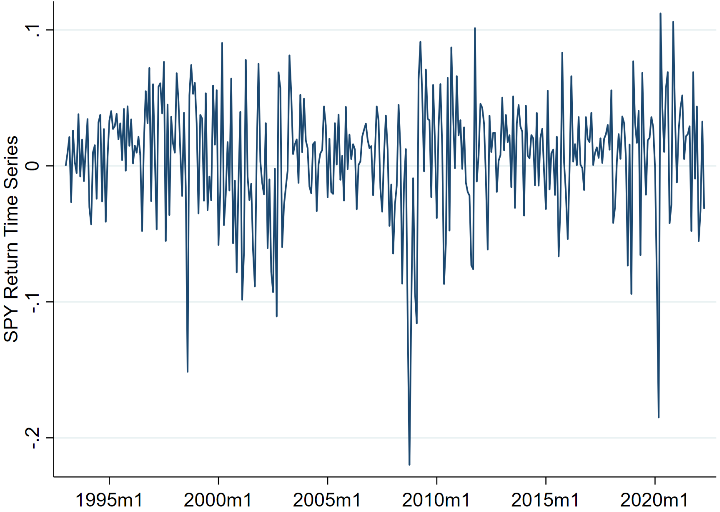
$$\begin{aligned}\epsilon_{t+1}^2 &= E_t[\epsilon_{t+1}^2] + \eta_{t+1}, \quad \eta_{t+1} \sim WN(0) \\ &= \sigma^2 + \eta_{t+1}, \quad \text{since } \epsilon_{t+1} \sim WN(0, \sigma^2)\end{aligned}$$

- If the conditional variance is truly constant, what would the ACF of ϵ_{t+1}^2 look like?

$$\begin{aligned}\gamma_j &= cov[\epsilon_{t+1}^2, \epsilon_{t+1-j}^2] \\ &= E[(\epsilon_{t+1}^2 - \sigma^2)(\epsilon_{t+1-j}^2 - \sigma^2)] \\ &= E[\eta_{t+1}\eta_{t+1-j}] = 0, \quad \forall j \neq 0\end{aligned}$$

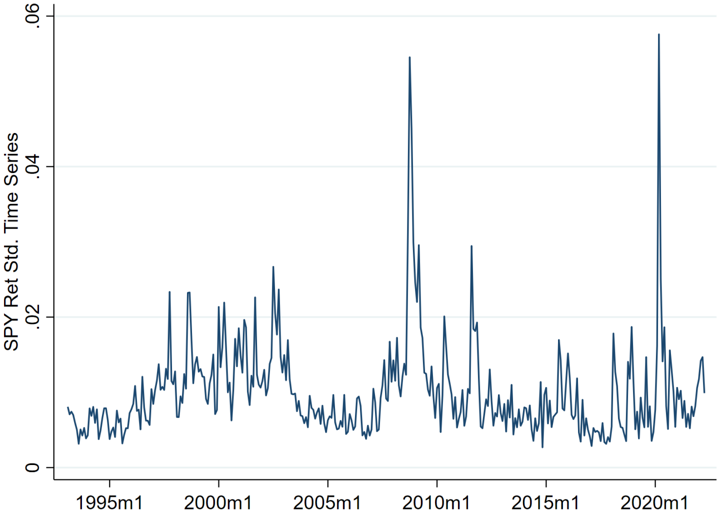
Example: SPY Return

- Monthly return, 1993 Jan. - 2022 Apr.

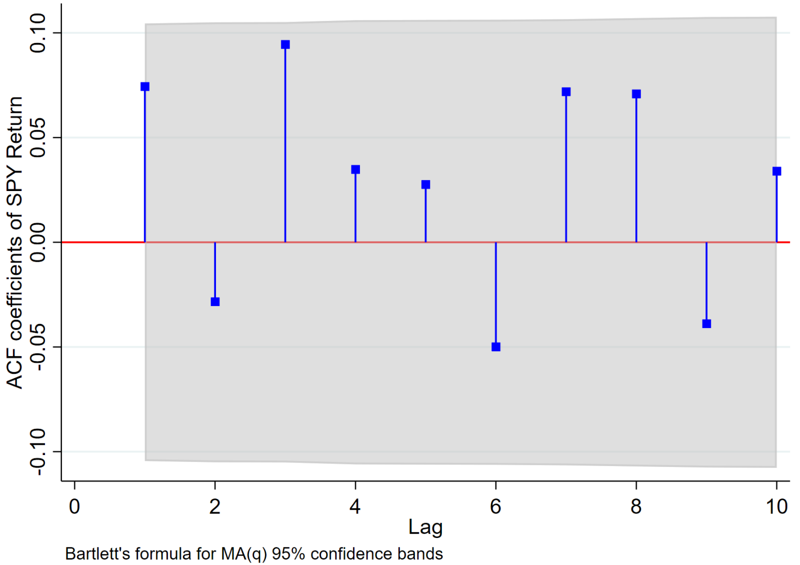


Example: SPY Return Std.

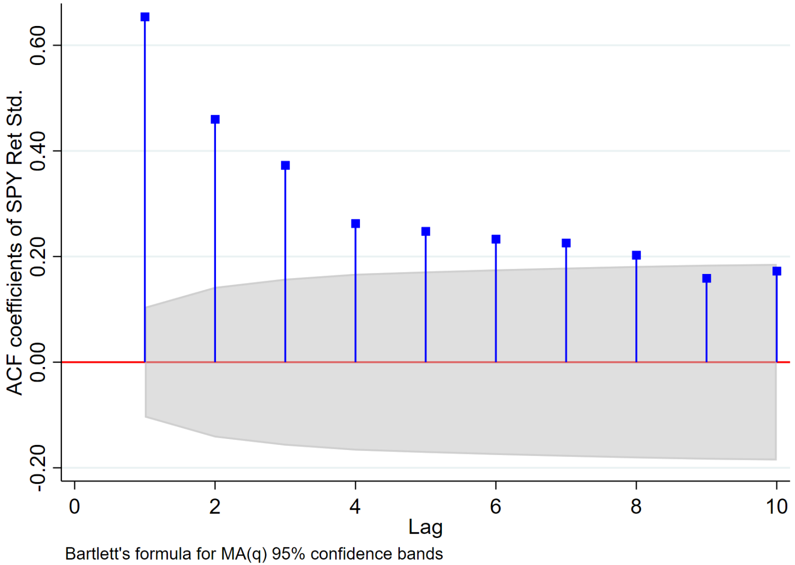
- Monthly return standard deviation, 1993 Jan. - 2022 Apr.



Example: SPY Return ACF



Example: SPY Return Std. ACF



Predictability of Return Volatility (cont.)

- What do you see?
  $\rightarrow$ there is substantial predictability in volatility!

- If we can capture the predictability in volatility, we may be able to improve our portfolio decisions, risk management decisions, option pricing, etc.

ARCH

- **Autoregressive conditional heteroscedasticity (ARCH) processes**

- If the conditional variance is predictable and not constant, how might we model it?
- A place to start might be an AR model for ϵ_{t+1}^2:

$$\epsilon_{t+1}^2 = \omega + \alpha\epsilon_t^2 + \eta_{t+1}, \quad \eta_{t+1} \sim WN(0)$$

- For the conditional variance of returns, this imply:

$$V_t[Y_{t+1}] = E_t[\epsilon_{t+1}^2] \equiv \sigma_{t+1}^2, \text{ and}$$

$$\sigma_{t+1}^2 = E_t[\omega + \alpha\epsilon_t^2 + \eta_{t+1}] = \omega + \alpha\epsilon_t^2$$

ARCH (Cont.)

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$$\sigma_{t+1}^2 = \omega + \alpha\epsilon_t^2$$

- This equation is the famous ARCH(1) from Engle (1982, *ECMA*)

  $\rightarrow$ The conditional variance of tomorrow's return is equal to a constant, plus today's residual squared.

- Estimation of ARCH model:
  1. Build some model for the return series, say RW, AR, ARMA models.
  2. Estimate residuals, $\widehat{\epsilon_t}$, from the model.
  3. Run the following regression:

$$\widehat{\epsilon_{t+1}}^2 = a_0 + a_1\widehat{\epsilon_t}^2 + u_{t+1}$$

- If there's no ARCH effect, $\widehat{a_1}$ should be 0. We can conduct a t-test.
- We will discuss about Lagrange Multiplier test later.

Generalized ARCH

- An AR(1) process for ϵ^2_{t+1} leads to ARCH(1). What about a ARMA(1,1) process for ϵ^2_{t+1}?

$$\epsilon^2_{t+1} = \omega + \gamma\epsilon^2_t + \eta_{t+1} + \lambda\eta_t, \quad \eta_{t+1} \sim WN(0)$$

- What would it imply for $V_t[Y_{t+1}] = E_t[\epsilon^2_{t+1}] \equiv \sigma^2_{t+1}$?

$$\begin{aligned}
\sigma^2_{t+1} &= E_t[\omega + \gamma\epsilon^2_t + \eta_{t+1} + \lambda\eta_t] \\
&= \omega + \gamma\epsilon^2_t + \lambda(\epsilon^2_t - E_{t-1}[\epsilon^2_t]), \quad \text{substituting in for } \eta_t \\
&= \omega + \gamma\epsilon^2_t + \lambda(\epsilon^2_t - \sigma^2_t) \\
&= \omega + \alpha\epsilon^2_t + \beta\sigma^2_t
\end{aligned}$$

  where we set $\alpha = (\gamma + \lambda)$ and $\beta = -\lambda$.

- This is the famous GARCH(1,1) of Bollerslev (1986, *J.Econometrics*)

ARMA(1,1)-GARCH(1,1) Model

- The assumption of ARMA(1,1) returns and GARCH(1,1) is the workhorse of financial times series analysis.

- The model can be written as:

$$
\begin{aligned}
Y_{t+1} &= \mu_{t+1} + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim WN(0, \sigma_{t+1}^2) \\
\mu_{t+1} &= E_t[Y_{t+1}] = \phi_0 + \phi_1 Y_t + \lambda \epsilon_t \\
\sigma_{t+1}^2 &= V_t[Y_{t+1}] = \omega + \alpha \epsilon_t^2 + \beta \sigma_t^2
\end{aligned}
$$

- Since we cannot observe conditional variance, we cannot estimate a GARCH model using OLS. We could only specify its distribution and estimate it using MLE.

Covariance Stationarity and Parameter Restrictions

- A variable Y is covaraince stationary iff:
  1. $E[Y_t] = \mu \ \forall t$
  2. $V[Y_t] = \bar{\sigma}_Y^2 \ \forall t$
  3. $cov(Y_t, Y_{t-j}) = \gamma_j \ \forall t$

  i.e. unconditional moments are constants!

- Note: We will generally focus only on processes that satisfy this property.

- **But**, GARCH processes may violate the above conditions if we don't impose parameter restrictions on the process!

- Example: for a GARCH(1,1) defined as:

$$
\begin{aligned}
Y_{t+1} &= \mu_{t+1} + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim WN(0, \sigma_{t+1}^2) \\
\epsilon_{t+1} &= \sigma_{t+1} v_{t+1}, \quad v_{t+1} | F_t \sim F(0,1) \\
\sigma_{t+1}^2 &= V_t[Y_{t+1}] = \omega + \alpha \epsilon_t^2 + \beta \sigma_t^2
\end{aligned}
$$

where $F(0,1)$ is some mean 0 and variance 1 distribution.

Condition 1 $\omega > 0$, $\alpha, \beta \geq 0$ for positive variance.
Condition 2 $\beta = 0$ if $\alpha = 0$ for identification.
Condition 3 $\alpha + \beta < 1$ for covariance stationary.

- If Condition 3 holds, we have:

$$
V[\epsilon_t] = E[\epsilon_t^2] = E[E_{t-1}[\epsilon_t^2]] = E[\sigma_t^2] = \frac{\omega}{1 - \alpha - \beta} := \sigma_Y^2
$$

Covariance Stationarity and Restrictions (Cont.)

- Note that $V[\epsilon_t] = E[\sigma_t^2]$ is only part of the unconditional variance of Y

$$\begin{aligned}\bar{\sigma}_Y^2 &= V[Y_t] \\ &= V[\mu_t + \epsilon_t] \\ &= V[\mu_t] + V[\epsilon_t] + 2cov(\mu_t, \epsilon_t) \\ &= V[\mu_t] + V[\epsilon_t]\end{aligned}$$

- Thus, the unconditional variance of returns is the sum of:
  - the unconditional variance of the conditional mean
  - the unconditional variance of the innovation term

Memory of GARCH

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- How long does a shock to the volatility of the process take to die out?

- Consider the one step ahead forecast of the GARCH(1,1)

$$\sigma^2_{t+1,t} \equiv E_t[\epsilon^2_{t+1}] \equiv \sigma^2_{t+1} = \omega + \alpha\epsilon^2_t + \beta\sigma^2_{t,t-1}$$

  Adding and subtracting σ^2_Y, and recall that $\sigma^2_Y = \frac{\omega}{1-\alpha-\beta}$

$$\begin{aligned}\sigma^2_{t+1,t} &= \omega + (\alpha+\beta)\sigma^2_Y + \alpha(\epsilon^2_t - \sigma^2_Y) + \beta(\sigma^2_{t,t-1} - \sigma^2_Y)\\ &= \sigma^2_Y + \alpha(\epsilon^2_t - \sigma^2_Y) + \beta(\sigma^2_{t,t-1} - \sigma^2_Y)\end{aligned}$$

- The GARCH(1,1) forecast is the weighted average of:
  i) unconditional variance
  ii) deviation of last period's forecast from the unconditional variance
  iii) deviation of last period's squared residual from the unconditional variance

Memory of GARCH (Cont.)

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- Now consider the two steps ahead forecast:

$$\begin{aligned}\sigma^2_{t+2,t} &\equiv E_t[\epsilon^2_{t+2}] = E_t[E_{t+1}[\epsilon^2_{t+2}]], \quad \text{by LIE} \\ &= E_t[\sigma^2_Y + \alpha(\epsilon^2_{t+1} - \sigma^2_Y) + \beta(\sigma^2_{t+1,t} - \sigma^2_Y)] \\ &= \sigma^2_Y + (\alpha+\beta)(\sigma^2_{t+1,t} - \sigma^2_Y)\end{aligned}$$

- Similarly, the 3-step ahead forecast:

$$\sigma^2_{t+3,t} \equiv E_t[\epsilon^2_{t+3}] = \sigma^2_Y + (\alpha+\beta)^2(\sigma^2_{t+1,t} - \sigma^2_Y)$$

- And the h-step ahead forecast:

$$\sigma^2_{t+h,t} \equiv E_t[\epsilon^2_{t+h}] = \sigma^2_Y + (\alpha+\beta)^{h-1}(\sigma^2_{t+1,t} - \sigma^2_Y)$$

- If $\alpha + \beta < 1$, the longer the forecasting horizon, the closer the forecast will get to the unconditional forecast. The smaller $\alpha + \beta$, the quicker the predictability will die out.

Memory of GARCH (Cont.)

- A common way to quantify the "memory" process is the half time.
- Half time := the number of periods, h^*, it takes for the conditional variance to revert back half-way towards the unconditional variance. i.e.

$$\sigma^2_{t+h^*,t} - \sigma^2_Y = \frac{1}{2}(\sigma^2_{t+1,t} - \sigma^2_Y)$$

- For our GARCH(1,1),

$$(\alpha + \beta)^{h^*-1}(\sigma^2_{t+1,t} - \sigma^2_Y) = \frac{1}{2}(\sigma^2_{t+1,t} - \sigma^2_Y)$$

$$h^* = 1 + \frac{log(1/2)}{log(\alpha + \beta)}$$

- Empirically, in daily and monthly financial data, $(\alpha + \beta)$ usually range between 0.8-0.99, implying h^* between 4.11-69.97.

Testing for Volatility Clustering

- A simple way to check for volatility clustering is to test whether the squared residuals are autocorrelated.
- There are two very simple ways to do it:
  1. Applying the Ljung-Box (LB) test to the squared residuals (or to the squared returns, if they have conditional mean zero).
  2. Using the the ARCH Lagrange Multiplier test of Engle (1982).

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Ljung-Box Test

- The LB test is a joint test that all autocorrelation coefficients of a particular series up to lag L are zero. It is an improved version of the Box-Pierce test.

- The LB test has the following null and alternative hypothesis:

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_L = 0$$
$$H_1 : \rho_j \neq 0 \text{ for some } j = 1, 2, \cdots, L$$

  where ρ_j is the j^{th} autocorrelation.

- The Ljung-Box Q statistic is:

$$Q_{LB}(L) = T(T+2) \sum_{j=1}^{L} \left(\frac{1}{T-j} \right) \widehat{\rho_j}^2 \sim \chi^2_L$$

The ARCH LM Test

- The ARCH LM test involves:
  1. using the estimated residuals to run the regression

$$\widehat{\epsilon_t^2} = a_0 + a_1 \widehat{\epsilon_{t-1}^2} + \cdots + a_L \widehat{\epsilon_{t-L}^2} + u_t$$

  2. then perform a χ_L^2 test (or an F-test) of the following hypothesis:

$$H_0 : a_1 = a_2 = \cdots = a_L = 0$$

- Remarks: The ARCH LM test has the advantage that robust standard errors can be used to test the above null hypothesis, whereas the Ljung-Box test is based on the assumption that the process is *iid*.

Modeling the Leverage Effect

- In the data, volatility tends to rise more following a bad news (a negative return) than following a good one (a positive return).
- This is the so-called **leverage effect** of Black (1976): stock returns are negatively correlated with changes is volatility (and leverage is a partial explanation of this feature).
- **But**, this behaviour cannot be captured by a standard GARCH:

$$\sigma_{t+1}^2 = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2$$

(sign of the shock does not matter in this formulation)

Threshold-GARCH (GJR-GARCH)

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- The simplest extension to accommodate this relation is the model of Glosten, Jagannathan and Runkle (1993, *JF*) (GJR-GARCH):

$$GJR - GARCH : \sigma_{t+1}^2 = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2 + \delta\epsilon_t^2\mathbf{1}\{\epsilon_t < 0\}$$

- If $\delta > 0$ then the impact on tomorrow's volatility of today's return is greater if today's return is negative.

Exponential GARCH (EGARCH)

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- Nelson (1991, *ECMA*), developed this specification to:
  1. capture the leverage effect
  2. avoid the need to impose parameter restrictions to have positive volatility (as we have to do in the vanilla GARCH)

$$EGARCH : ln\sigma_{t+1}^2 = \omega + \beta ln\sigma_t^2 + \alpha|\frac{\epsilon_t}{\sigma_t}| + \gamma\frac{\epsilon_t}{\sigma_t}$$

- $ln\sigma_{t+1}^2$ ensures positivity of the variance.
- If γ is different from zero, we have the leverage effect.

- Beside the extensions we've just seen, there are MANY more alternative ARCH type specifications (and more come out literally every year).
- We'll quickly list the main ones as an addition to our toolset.
- As usual the notation we'll use is

$$
\begin{aligned}
Y_{t+1} &= \mu_{t+1} + \epsilon_{t+1} \\
\epsilon_{t+1} &= \sigma_{t+1} v_{t+1} \\
v_{t+1}|F_t &\sim iid\ F(0,1)
\end{aligned}
$$

ARCH/GARCH Family (Cont.)

- **IGARCH**(Engle and Bollerslev, 1986)

$$\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + (1-\beta)\epsilon_t^2$$

- **PARCH**(Ding, Granger and Engle, 1993)

$$\sigma_{t+1}^\gamma = \omega + \beta\sigma_t^\gamma + \alpha\epsilon_t^{2\gamma}$$

- **APARCH**(Ding, Granger and Engle, 1993)

$$\sigma_{t+1}^\gamma = \omega + \beta\sigma_t^\gamma + \alpha\gamma(|\epsilon_t| - \delta\epsilon_t)^\gamma$$

- **QGARCH**(Sentana, 1991)

$$\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\epsilon_t^2 + \delta\epsilon_t$$

- **SQARCH**(Ishida and Engle, 2001)

$$\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\sigma_t(v_t^2 - 1)$$

Choosing Volatility Model

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- Choosing a volatility model should depend on what we intend to do
  i.e. If you plan to use it for out-of-sample forecasting of volatility, pick according to some measure of its out-of-sample forecast performance.

- Comparing nested models via tests on parameters
  i.e. δ in GJR-GARCH against the vanilla GARCH.

- Use information criteria: AIC, BIC, HQIC, etc.

- Use statistic goodness-of-fit: MSE, MAE, R^2, etc.

- Use economic goodness-of-fit:
  If a conditional variance forecast is going to be used to make some economic decision, such as an investment decision, then the right way to compare the performance of competing models is to compare the profits that are generated by each model.