Bayesian linear regression

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MATH 347 Bayesian Statistics

Outline

Outline

Review: the Normal model

- When you have continuous outcomes, you can use a Normal model:
$$Y_i \mid \mu, \sigma \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma), \quad i = 1, \cdots, n. \tag{1}$$
- This model assumes each observation follows the same Normal density with mean μ and standard deviation σ.
- Suppose now you have another continuous variable available, x_i. And you want to use the information in x_i to learn about Y_i.
    1. Y_i is the log of expenditure of CU's
    2. x_i is the log of total income of CU's
- Is the model in Equation (1) flexible to include x_i?

An observation specific mean

- We can adjust the model in Equation (1) to Equation (2), where the common mean μ is replaced by an observation specific mean μ_i:
$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma), \quad i = 1, \cdots, n. \tag{2}$$
- How to link μ_i and x_i?

Linear relationship between the mean and the predictor

- One basic approach: use a linear relationship:
$$\mu_i = \beta_0 + \beta_1 x_i, \quad i = 1, \cdots, n. \tag{3}$$
- x_i's are known constants.
- β_0 and β_1 are unknown parameters.
- Interpretation:
  1. the linear function $\beta_0 + \beta_1 x_i$ is the expected outcome with x_i
  2. β_0 is the **intercept**: then expected outcome when $x_i = 0$
  3. β_1 is the **slope**: the increase in the expected outcome when x_i increases by 1 unit

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- Bayesian approach:

Linear relationship between the mean and the predictor

- One basic approach: use a linear relationship:

$$\mu_i = \beta_0 + \beta_1 x_i, \quad i = 1, \cdots, n. \tag{3}$$

- x_i's are known constants. beta_0, beta_1 are random, x_i is fixed, so mu_i is random
- β_0 and β_1 are unknown parameters.
- Interpretation:
  1. the linear function $\beta_0 + \beta_1 x_i$ is the expected outcome with x_i
  2. β_0 is the **intercept**: then expected outcome when $x_i = 0$
  3. β_1 is the **slope**: the increase in the expected outcome when x_i increases by 1 unit
- Bayesian approach:
  1. assign a prior distribution to $(\beta_0, \beta_1, \sigma)$
  2. perform inference
  3. summarize posterior distribution of these parameters

random

don't need to assign for mu_i, as mu_i has been specified a linear relationship

The simple linear regression model

- To put everything together, a linear regression model:

$$Y_i \mid x_i, \beta_0, \beta_1, \sigma \overset{ind}{\sim} \text{Normal}(\beta_0 + \beta_1 x_i, \sigma), \quad i = 1, \cdots, n. \tag{4}$$

- Alternatively:

$$\begin{aligned} Y_i &= \mu_i + \epsilon_i, & (5) \\ \mu_i &= \beta_0 + \beta_1 x_i, & (6) \\ \epsilon_i &\overset{i.i.d.}{\sim} \text{Normal}(0, \sigma) \;\; i = 1, \cdots, n. & (7) \end{aligned}$$

- What assumptions does this model make?

The simple linear regression model

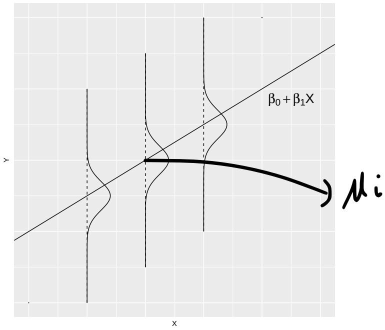
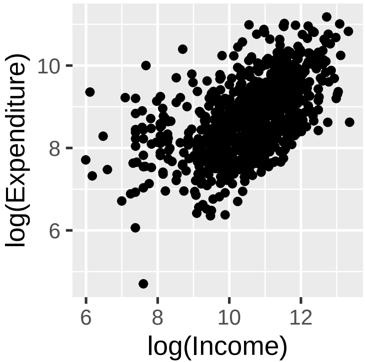


Figure 1: Display of linear regression model. The line represents the unknown regression line $\beta_0 + \beta_1 x$ and the normal curves represent the distribution of the response Y about the line.

The simple linear regression model cont'd

```r
CEData <- read.csv("CEsample.csv", header = T, sep = ",")
g1 <- ggplot(CEData, aes(x = log_TotalIncome, y = log_TotalExp)) +
  geom_point(size=1) +
  labs(x = "log(Income)", y = "log(Expenditure)") +
  theme_grey(base_size = 10, base_family = "")
g1
```



Outline

The CE sample

The CE sample comes from the 2017 Q1 CE PUMD: 4 variables, 994 observations.

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)
Rural	Binary; the urban/rural status of CU: 0 = Urban, 1 = Rural
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race

Outline

A SLR for the CE sample

- For now, we focus on a simple linear regression:

$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \text{Normal}(\mu_i, \sigma), \tag{8}$$
$$\mu_i = \beta_0 + \beta_1 x_i. \tag{9}$$

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)

- Remarks:
  1. Y_i is log(Expenditure), and x_i is log(Income).
  2. The intercept β_0: the expected log(Expenditure) μ_i for a CU i that has zero log(Income) (i.e. $x_i = 0$).
  3. The slope β_1: the change in the expected log(Expenditure) μ_i when the log(Income) of CU i increases by 1 unit.'

A weakly informative prior

- Sometimes one has limited prior information about the regression parameters β_0 and β_1 and/or the standard deviation σ.

- Assuming independence:
$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0, \beta_1)\pi(\sigma). \tag{10}$$

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1. Assuming independence between β_0 and β_1:

$$\begin{aligned} \pi(\beta_0, \beta_1) &= \pi(\beta_0)\pi(\beta_1), & (11) \\ \beta_0 &\sim \mathrm{Normal}(\mu_0, s_0), & (12) \\ \beta_1 &\sim \mathrm{Normal}(\mu_1, s_1). & (13) \end{aligned}$$

e.g. $\mathrm{Normal}(0, 100)$.

A weakly informative prior

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   e.g. $\mathrm{Normal}(0, 100)$.

2. Assigning a weakly informative prior for the standard deviation σ:

$$1/\sigma^2 \sim \mathrm{Gamma}(a, b). \tag{14}$$

   e.g. $\mathrm{Gamma}(1, 1)$.

use these distributions for conjugacy

Full conditional derivation?

- The sampling model:

$$Y_i \mid x_i, \beta_0, \beta_1, \sigma \overset{ind}{\sim} \mathrm{Normal}(\beta_0 + \beta_1 x_i, \sigma). \tag{15}$$

- The joint likelihood:

$$\begin{aligned} L(\beta_0, \beta_1, \sigma) &= \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right\} \right] \\ &\propto (1/\sigma^2)^{n/2} \exp\left\{ -\frac{1/\sigma^2}{2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right\} \end{aligned}$$

Full conditional derivation? cont'd

- The joint posterior:

$$
\begin{aligned}
\pi(\beta_0, \beta_1, 1/\sigma^2 | y) \quad &\propto \quad (1/\sigma^2)^{n/2} \exp\left\{-\frac{1/\sigma^2}{2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2\right\} \\
&\times \quad \exp\left\{-\frac{1}{2s_0^2}(\beta_0 - \mu_0)^2\right\} \exp\left\{-\frac{1}{2s_1^2}(\beta_1 - \mu_1)^2\right\} \\
&\times \quad (1/\sigma^2)^{a-1} \exp(-b(1/\sigma^2))
\end{aligned}
$$

- While it is possible to further simplify and recognize full conditional posterior distribution, we will rely on JAGS to perform the MCMC for us.

Outline

JAGS script for the SLR model

```
modelString <-"
model {
## sampling
for (i in 1:N){
y[i] ~ dnorm(beta0 + beta1*x[i], invsigma2)
}

## priors
beta0 ~ dnorm(mu0, g0)
beta1 ~ dnorm(mu1, g1)
invsigma2 ~ dgamma(a, b)
sigma <- sqrt(pow(invsigma2, -1))
}
"
```

you need to pass precision for JAGS syntax

JAGS script for the SLR model cont'd

0.0001=1/(100)^2

- Pass the data and hyperparameter values to JAGS:

```
y <- as.vector(CEData$log_TotalExp)
x <- as.vector(CEData$log_TotalIncome)
N <- length(y)
the_data <- list("y" = y, "x" = x, "N" = N,
                 "mu0" = 0, "g0" = 0.0001,
                 "mu1" = 0, "g1" = 0.0001,
                 "a" = 1, "b" = 1)

initsfunction <- function(chain){
  .RNG.seed <- c(1,2)[chain]
  .RNG.name <- c("base::Super-Duper",
                 "base::Wichmann-Hill")[chain]
  return(list(.RNG.seed=.RNG.seed,
              .RNG.name=.RNG.name))
}
```

beta_0~N(0, 100)
beta_1~N(0, 100)
1/sigma^2~Gamma(1,1)

JAGS script for the SLR model cont'd

- Run the JAGS code for this model:

```
posterior <- run.jags(modelString,
                      n.chains = 1,
                      data = the_data,
                      monitor = c("beta0", "beta1", "sigma"),
                      adapt = 1000,
                      burnin = 5000,
                      sample = 5000,
                      thin = 1,
                      inits = initsfunction)
```

JAGS output for the SLR model

- Obtain posterior summaries of all parameters:

```
summary(posterior)
```

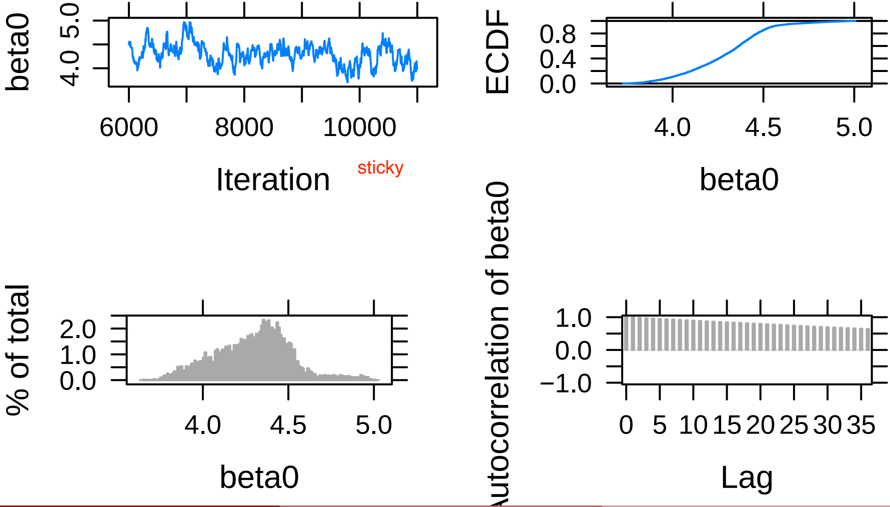
```
##        Lower95    Median  Upper95      Mean         SD Mode        MCerr
## beta0 3.771420 4.3136450 4.655760 4.2947731 0.22275382   NA 0.0421194486
## beta1 0.388552 0.4218125 0.471380 0.4237176 0.02091679   NA 0.0039634093
## sigma 0.694060 0.7249680 0.757081 0.7251875 0.01624723   NA 0.0002175128
##       MC%ofSD SSeff      AC.10 psrf
## beta0    18.9    28 0.89361268   NA
## beta1    18.9    28 0.89351502   NA
## sigma     1.3  5579 0.02058294   NA
```

high correlation in
beta_0, beta_1

JAGS output for the SLR model cont'd

```
plot(posterior, vars = "beta0")
```

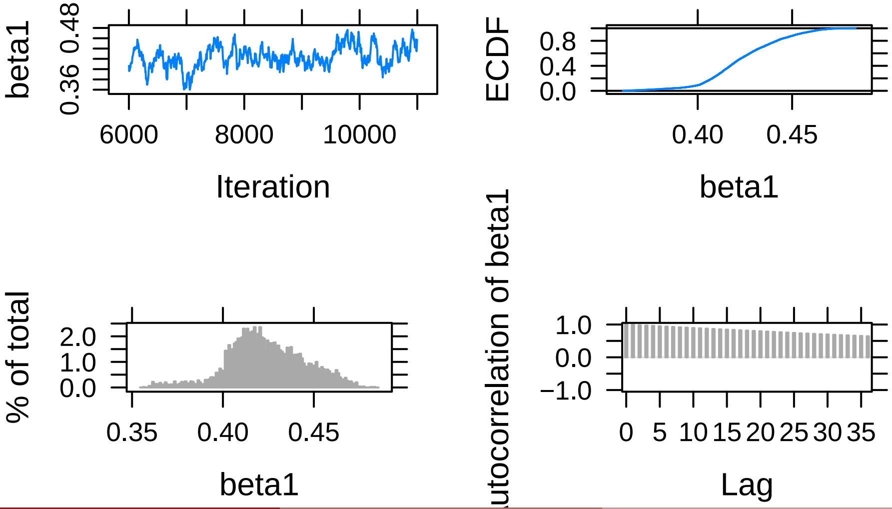
```
## Generating plots...
```



JAGS output for the SLR model cont'd

```
plot(posterior, vars = "beta1")
```

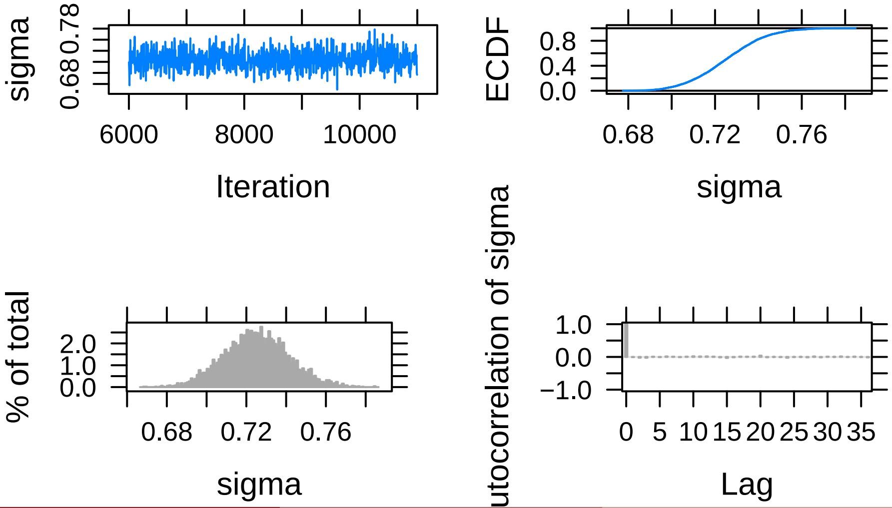
```
## Generating plots...
```



JAGS output for the SLR model cont'd

```
plot(posterior, vars = "sigma")
```

```
## Generating plots...
```



New JAGS script for the SLR model

Setting `thin = 50`, to get rid of the stickiness in β_0 and β_1.

```
posterior_new <- run.jags(modelString,
                          n.chains = 1,
                          data = the_data,
                          monitor = c("beta0", "beta1", "sigma"),
                          adapt = 1000,
                          burnin = 5000,
                          sample = 5000,
                          thin = 50,
                          inits = initsfunction)
```

New JAGS output for the SLR model

- Obtain posterior summaries of all parameters:

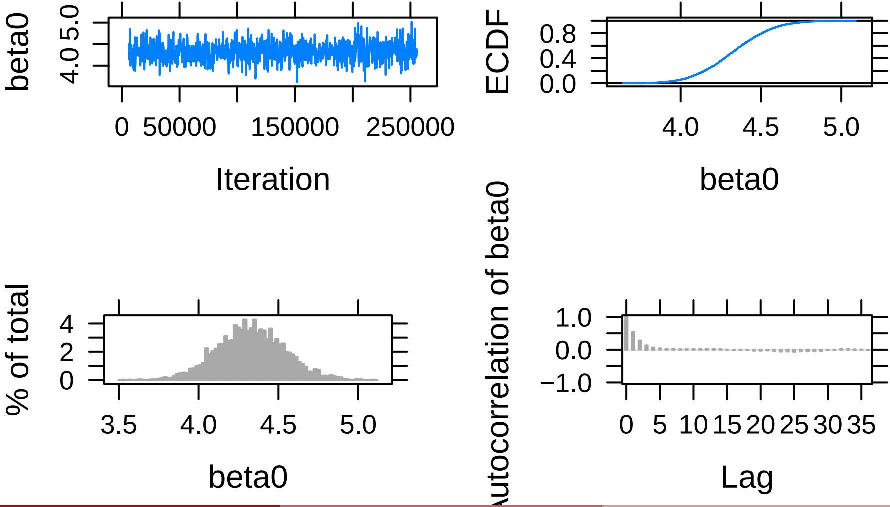
```
summary(posterior_new)
```

```
##         Lower95    Median  Upper95      Mean         SD Mode        MCerr
## beta0 3.926200 4.3251800 4.753090 4.3269816 0.21092686   NA 0.0054707075
## beta1 0.381057 0.4208825 0.458968 0.4207384 0.01977946   NA 0.0004930936
## sigma 0.693623 0.7254665 0.757397 0.7255765 0.01624891   NA 0.0002297942
##       MC%ofSD SSeff       AC.500 psrf
## beta0     2.6  1487 0.018912301   NA
## beta1     2.5  1609 0.018555972   NA
## sigma     1.4  5000 0.003076245   NA
```

New JAGS output for the SLR model cont'd

```
plot(posterior_new, vars = "beta0")
```

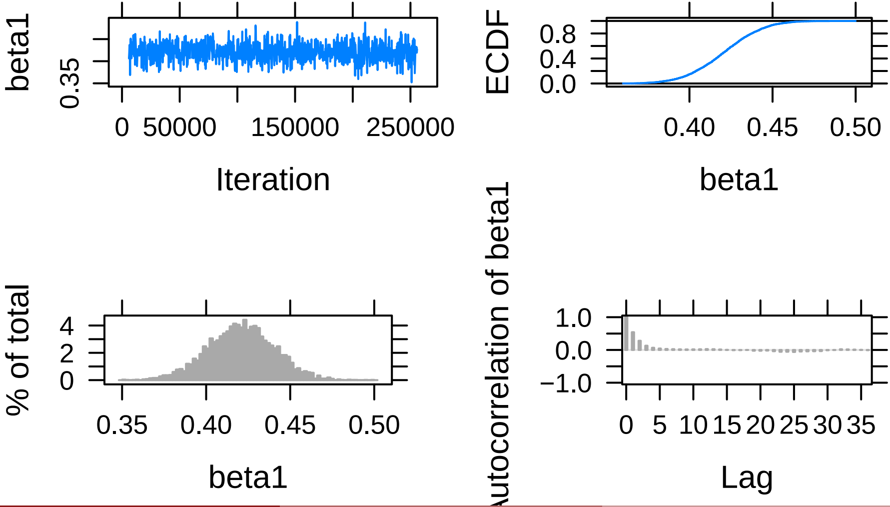
```
## Generating plots...
```



New JAGS output for the SLR model cont'd

```
plot(posterior_new, vars = "beta1")
```

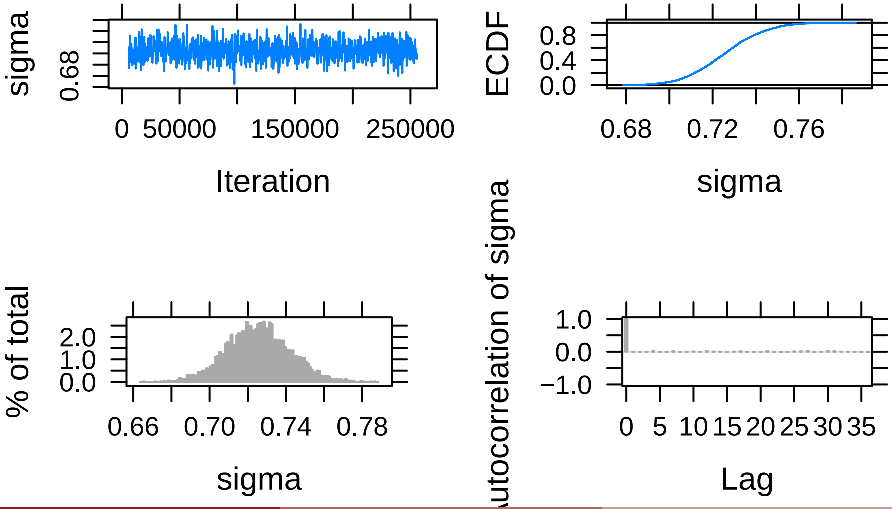
```
## Generating plots...
```



New JAGS output for the SLR model cont'd

```
plot(posterior_new, vars = "sigma")
```

```
## Generating plots...
```



Interpretation of regression coefficients

- The intercept β_0:

For a CU with log(Income) of \$0 (i.e. Income of \$1), the expected log(Expenditure) is \$4.327 (the median), and it falls in the interval (\$3.926, \$4.753) with 90% posterior probability.

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- The slope β_1:

With every \$1 increase in the log(Income) of a CU, its log(Expenditure) increases by \$0.421 (the median). In addition, this increase in the log(Expenditure) falls in the interval (\$0.381, \$0.459) with 90% posterior probability.

Outline

Simulate fits from the regression model

- The SLR model assumes
$$E(Y) = \beta_0 + \beta_1 x. \tag{16}$$
- Each pair of values (β_0, β_1) corresponds to a line $\beta_0 + \beta_1 x$ in the space of values of x and y.

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- Using posterior mean $\tilde{\beta}_0$ and $\tilde{\beta}_1$, one can find a "best" line of fit through the data
$$y = \tilde{\beta}_0 + \tilde{\beta}_1 x. \tag{17}$$

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$$y = \tilde{\beta}_0 + \tilde{\beta}_1 x. \tag{17}$$
- What about the uncertainty of the line estimate?

Simulate fits from the regression model cont'd

- To learn about the uncertainty of the line estimate, one can draw a sample of J rows from the matrix of posterior draws of (β_0, β_1) and collect the line estimates

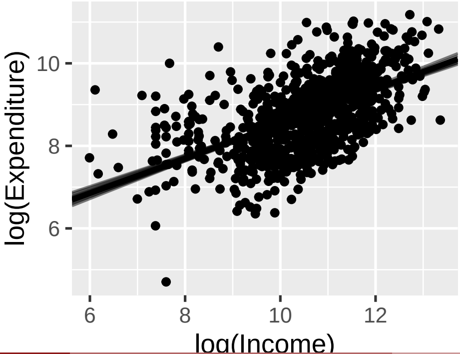
$$\tilde{\beta_0}^{(j)} + \tilde{\beta_1}^{(j)} x, \;\; j = 1, \cdots, J. \tag{18}$$

```
post <- as.mcmc(posterior_new)
post_means <- apply(post, 2, mean)
post <- as.data.frame(post)
```

post is the matrix of all the posterior draws

Simulate fits from the regression model cont'd

```
ggplot(CEData, aes(log_TotalIncome, log_TotalExp)) +
  geom_point(size=1) +
  geom_abline(data=post[1:10, ],  10 possible regression lines
              aes(intercept=beta0, slope=beta1), alpha = 0.5) +
  geom_abline(intercept = post_means[1],
              slope = post_means[2], size = 1) +
  ylab("log(Expenditure)") + xlab("log(Income)") +
  theme_grey(base_size = 10, base_family = "")
```



Learning about the expected response

- What if one wants to learn about the expected log expenditure of a CU for a specific log income value?
- One could obtain a simulated sample from the posterior of $\beta_0 + \beta_1 x$ by computing $E(Y) = \beta_0 + \beta_1 x$ on each of the simulated pairs from the posterior of (β_0, β_1).

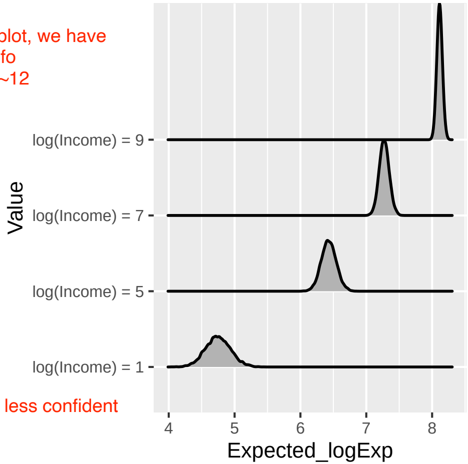
```
post <- as.data.frame(post)
one_expected <- function(x){
  lp <- post[ , "beta0"] +  x * post[ , "beta1"]
  data.frame(Value = paste("log(Income) =", x),
             Expected_logExp = lp)
}

df <- map_df(c(1, 5, 7, 9), one_expected)
```

Learning about the expected response cont'd

```
require(ggridges)
ggplot(df, aes(x = Expected_logExp, y = Value)) +
  geom_density_ridges() +
  theme_grey(base_size = 8, base_family = "")
```

Recall the scatter plot, we have more info near x=9~12

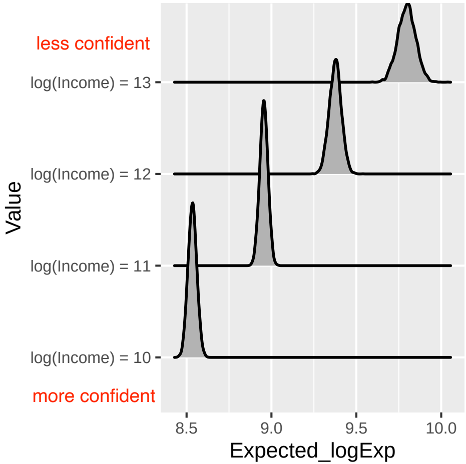


more confident

less confident

Learning about the expected response cont'd

```r
df <- map_df(c(10, 11, 12, 13), one_expected)
ggplot(df, aes(x = Expected_logExp, y = Value)) +
  geom_density_ridges() +
  theme_grey(base_size = 8, base_family = "")
```



Learning about the expected response cont'd

```
df %>% group_by(Value) %>%
  summarize(P05 = quantile(Expected_logExp, 0.05),
            P50 = median(Expected_logExp),
            P95 = quantile(Expected_logExp, 0.95))
```

```
## # A tibble: 4 x 4
##   Value                P05   P50   P95
##   <chr>              <dbl> <dbl> <dbl>
## 1 log(Income) = 10    8.49  8.53  8.58
## 2 log(Income) = 11    8.91  8.96  8.99
## 3 log(Income) = 12    9.32  9.38  9.43
## 4 log(Income) = 13    9.71  9.80  9.88
```

e.g. for a CU of log(Income) of \$5, the posterior median of the expected log(Expenditure) is \$6.43, and the probability that the expected log(Expenditure) falls between \$6.25 and \$6.62 is 90%.

Prediction of future responses

- To predict future log(Expenditure) for a CU given its log(Income):

$$Y_i \mid x_i, \beta_0, \beta_1, \sigma \overset{ind}{\sim} \text{Normal}(\beta_0 + \beta_1 x_i, \sigma). \tag{19}$$

- variability in β_0 and β_1 in the expected value $\beta_0 + \beta_1 x$
- variability in the sampling model in Equation (19)

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- variability in β_0 and β_1 in the expected value $\beta_0 + \beta_1 x$
- variability in the sampling model in Equation (19)

- For a large number of S:

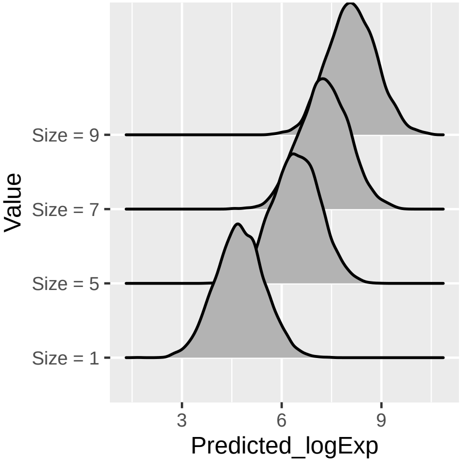
$$\begin{aligned}
\text{simulate } E[y]^{(1)} = \beta_0^{(1)} + \beta_1^{(1)} x \quad &\rightarrow \quad \text{sample } \tilde{y}^{(1)} \sim \mathrm{Normal}(E[y]^{(1)}, \sigma^{(1)}) \\
\text{simulate } E[y]^{(2)} = \beta_0^{(2)} + \beta_1^{(2)} x \quad &\rightarrow \quad \text{sample } \tilde{y}^{(2)} \sim \mathrm{Normal}(E[y]^{(2)}, \sigma^{(2)}) \\
&\vdots \\
\text{simulate } E[y]^{(S)} = \beta_0^{(S)} + \beta_1^{(S)} x \quad &\rightarrow \quad \text{sample } \tilde{y}^{(S)} \sim \mathrm{Normal}(E[y]^{(S)}, \sigma^{(S)})
\end{aligned}$$

Prediction of future responses cont'd

```
one_predicted <- function(x){
  lp <- post[ , "beta0"] +  x * post[ , "beta1"]
  y <- rnorm(5000, lp, post[, "sigma"])
  data.frame(Value = paste("Size =", x),
             Predicted_logExp = y)
}
df <- map_df(c(1, 5, 7, 9), one_predicted)
```

Prediction of future responses cont'd

```
require(ggridges)
ggplot(df, aes(x = Predicted_logExp, y = Value)) +
  geom_density_ridges() +
  theme_grey(base_size = 9, base_family = "")
```



Prediction of future responses cont'd

```
df %>% group_by(Value) %>%
  summarize(P05 = quantile(Predicted_logExp, 0.05),
            P50 = median(Predicted_logExp),
            P95 = quantile(Predicted_logExp, 0.95))
```

```
## # A tibble: 4 x 4
##   Value      P05   P50   P95
##   <chr>    <dbl> <dbl> <dbl>
## 1 Size = 1  3.54  4.73  5.97
## 2 Size = 5  5.25  6.43  7.62
## 3 Size = 7  6.05  7.28  8.48
## 4 Size = 9  6.91  8.10  9.34
```

- Recall that for a CU of log(Income) of \$5, the probability that the expected log(Expenditure) falls between \$6.25 and \$6.62 is 90%.
- A 90% prediction interval for the log(Expenditure) is (\$5.24, \$7.61), which is wider than the posterior interval estimate for the expected log(Expenditure).
- The predictive distribution incorporates the sizable uncertainty in the log(Expenditure) given the log(Income) represented by the sampling standard deviation σ.

Outline

Subjective prior: standardization

- To put different variables on similar scales.

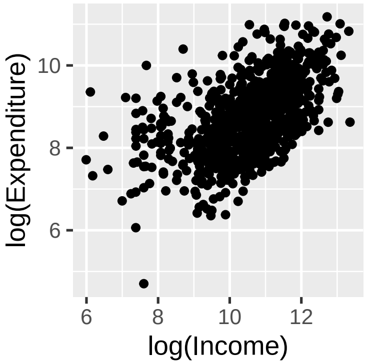
$$y_i^* = \frac{y_i - \bar{y}}{s_y}, x_i^* = \frac{x_i - \bar{x}}{s_x} \tag{20}$$

- Use the `scale` command to standardize.

```
CEData$log_TotalExpSTD <- scale(CEData$log_TotalExp)
CEData$log_TotalIncomeSTD <- scale(CEData$log_TotalIncome)
```

Subjective prior: standardization cont'd

```
g2 = ggplot(CEData, aes(x = log_TotalIncomeSTD, y = log_TotalExpSTD)) +
  geom_point(size=1) +
  xlab("log(Income) STD") + ylab("log(Expenditure) STD") +
  theme_grey(base_size = 10, base_family = "")
grid.arrange(g1, g2, ncol=2)
```



Subjective prior: SLR model after standardization

- A standardized value represents the number of standard deviations that the value falls above or below the mean.
- What does $x_i^* = -2$ mean? What does $y_i^* = 1$ mean?

Subjective prior: SLR model after standardization

- A standardized value represents the number of standard deviations that the value falls above or below the mean.

x_i (the original value) is 2 std. dev. below the mean of x

- What does $x_i^* = -2$ mean? What does $y_i^* = 1$ mean?

y_i is 1 std. dev above the mean of y

- The SLR model after standardization:

$$Y_i^* \mid \mu_i^*, \sigma \overset{ind}{\sim} \text{Normal}(\mu_i^*, \sigma), \tag{21}$$

$$\mu_i^* = \beta_0 + \beta_1 x_i^*. \tag{22}$$

1. Y_i^* is standardized log(Expenditure), and x_i^* is standardized log(Income).
2. The intercept β_0: the expected standardized log(Expenditure) μ_i^* for a CU i that has the average log(Income) (i.e. $x_i^* = 0$).
3. The slope β_1: the change in the expected standardized log(Expenditure) μ_i^* when the standardized log(Income) x_i^* of CU i increases by 1 unit, or when the log(Income) variable increases by one standard deviation.
4. The slope β_1 equals to the correlation between x_i^* and y_i^*: positive β_1 vs negative β_1, absolute value indicates the strength.

Subjective prior: a subjective prior

- Assuming independence:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{23}$$

Subjective prior: a subjective prior

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$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{23}$$

- Prior on the intercept β_0:
  1. If one believes a CU of average log(Income) will also have an average log(Expenditure).
  $$\beta_0 \sim \mathrm{Normal}(0, s_0). \tag{24}$$
  2. The standard deviation s_0 reflects how confident the person believes in the guess of $\beta_0 = 0$. e.g. $\mathrm{Normal}(0, 1)$.

Subjective prior: a subjective prior

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$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{23}$$

- Prior on the intercept β_0:
  1. If one believes a CU of average log(Income) will also have an average log(Expenditure).
  $$\beta_0 \sim \mathrm{Normal}(0, s_0). \tag{24}$$
  2. The standard deviation s_0 reflects how confident the person believes in the guess of $\beta_0 = 0$. e.g. $\mathrm{Normal}(0, 1)$.

- Prior on the slope β_1:
  1. The slope β_1 represents the correlation between the predictor and the response.
  $$\beta_1 \sim \mathrm{Normal}(\mu_1, s_1). \tag{25}$$
  2. μ_1 represents one's best guess of the correlation, and s_1 represents the sureness of this guess. e.g. $\mathrm{Normal}(0.7, 0.15)$.

Subjective prior: a subjective prior cont'd

- Assuming independence:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{26}$$

Subjective prior: a subjective prior cont'd

- Assuming independence:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{26}$$

- Prior on the standard deviation σ: weakly informative

$$1/\sigma^2 \sim \mathrm{Gamma}(1, 1). \tag{27}$$

Subjective prior: a subjective prior cont'd

- Assuming independence:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{26}$$

- Prior on the standard deviation σ: weakly informative

$$1/\sigma^2 \sim \text{Gamma}(1, 1). \tag{27}$$

- The informative/subjective prior for $(\beta_0, \beta_1, \sigma)$ is defined as

$$\begin{aligned}
\pi(\beta_0, \beta_1, \sigma) &= \pi(\beta_0)\pi(\beta_1)\pi(\sigma), & (28)\\
\beta_0 &\sim \text{Normal}(0, 1), & (29)\\
\beta_1 &\sim \text{Normal}(0.7, 0.15), & (30)\\
1/\sigma^2 &\sim \text{Gamma}(1, 1). & (31)
\end{aligned}$$

Subjective prior: JAGS script for the standardized SLR model

```
modelString <-"
model {
## sampling
for (i in 1:N){
y[i] ~ dnorm(beta0 + beta1*x[i], invsigma2)
}

## priors
beta0 ~ dnorm(mu0, g0)
beta1 ~ dnorm(mu1, g1)
invsigma2 ~ dgamma(a, b)
sigma <- sqrt(pow(invsigma2, -1))
}
"
```

Subjective prior: JAGS script for the standardized SLR model cont'd

- Pass the data and hyperparameter values to JAGS:

```
y <- as.vector(CEData$log_TotalExpSTD)
x <- as.vector(CEData$log_TotalIncomeSTD)
N <- length(y)
the_data <- list("y" = y, "x" = x, "N" = N,
                 "mu0" = 0, "g0" = 1,
                 "mu1" = 0.7, "g1" = 44.4,
                 "a" = 1, "b" = 1)

initsfunction <- function(chain){
  .RNG.seed <- c(1,2)[chain]
  .RNG.name <- c("base::Super-Duper",
                 "base::Wichmann-Hill")[chain]
  return(list(.RNG.seed=.RNG.seed,
              .RNG.name=.RNG.name))
}
```

JAGS script for the SLR model cont'd

- Run the JAGS code for this model:

```
posterior_sub <- run.jags(modelString,
                      n.chains = 1,
                      data = the_data,
                      monitor = c("beta0", "beta1", "sigma"),
                      adapt = 1000,
                      burnin = 5000,
                      sample = 5000,
                      thin = 1,
                      inits = initsfunction)
```

Subjective prior: JAGS output for the SLR model

- Obtain posterior summaries of all parameters:

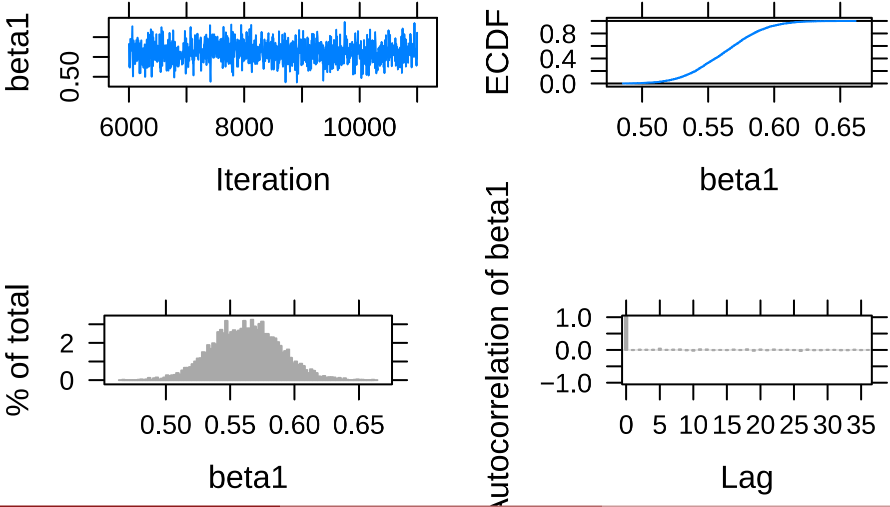
```r
summary(posterior_sub)
```

```
##          Lower95        Median   Upper95          Mean         SD Mode
## beta0 -0.0484962 -0.0000908726 0.0544673 -5.334241e-05 0.02637737   NA
## beta1  0.5131890  0.5623200000 0.6144910  5.622705e-01 0.02616385   NA
## sigma  0.7958380  0.8314665000 0.8684720  8.318869e-01 0.01866642   NA
##              MCerr MC%ofSD SSeff       AC.10 psrf
## beta0 0.0003561888     1.4  5484 -0.02064139   NA
## beta1 0.0003700127     1.4  5000 -0.02451218   NA
## sigma 0.0002502895     1.3  5562  0.02083092   NA
```

Subjective prior: JAGS output for the SLR model con'td

```
plot(posterior_sub, vars = "beta1")
```

```
## Generating plots...
```



Conditional means prior: a conditional means prior

- We have seen two methods for constructing a prior on the regression coefficient parameters:
  1. Weakly informative/vague prior on a model on the original data
  2. Subjective/informative prior on a model on standardized data

Conditional means prior: a conditional means prior

- We have seen two methods for constructing a prior on the regression coefficient parameters:
  1. Weakly informative/vague prior on a model on the original data
  2. Subjective/informative prior on a model on standardized data
- A third approach:
  1. On a model on the original data
  2. Stating prior beliefs about the expected response value conditional on specific values of the predictor variable

Conditional means prior: a conditional means prior

- We have seen two methods for constructing a prior on the regression coefficient parameters:
  1. Weakly informative/vague prior on a model on the original data
  2. Subjective/informative prior on a model on standardized data
- A third approach:
  1. On a model on the original data
  2. Stating prior beliefs about the expected response value conditional on specific values of the predictor variable
- Assuming independence:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0, \beta_1)\pi(\sigma). \tag{32}$$

- The linear relationship:

$$\mu_i = \beta_0 + \beta_1 x_i. \tag{33}$$

Conditional means prior: a conditional means prior cont'd

- The linear relationship:
$$\mu_i = \beta_0 + \beta_1 x_i. \tag{34}$$
- Easier to formulate prior opinion about the mean values, μ_i
- For predictor value x_1, on can construct a Normal prior for the mean value μ_1:
$$\mu_1 \sim \mathrm{Normal}(m_1, s_1) \tag{35}$$
e.g. if $x_1 = 10$, the mean $\mu_1 = \beta_0 + \beta_1(10) \sim \mathrm{Normal}(8, 2)$

Conditional means prior: a conditional means prior cont'd

- The linear relationship:

E(y_i)=mu_i

$$\mu_i = \beta_0 + \beta_1 x_i. \tag{34}$$

- Easier to formulate prior opinion about the mean values, μ_i
- For predictor value x_1, on can construct a Normal prior for the mean value μ_1:

$$\mu_1 \sim \mathrm{Normal}(m_1, s_1) \tag{35}$$

e.g. if $x_1 = 10$, the mean $\mu_1 = \beta_0 + \beta_1(10) \sim \mathrm{Normal}(8, 2)$

- Similarly, for predictor value x_2, on can construct a Normal prior for the mean value μ_2:

$$\mu_2 \sim \mathrm{Normal}(m_2, s_2) \tag{36}$$

e.g. if $x_2 = 1\cancel{5}$ 2, the mean $\mu_2 = \beta_0 + \beta_1(12) \sim \mathrm{Normal}(11, 2)$

Conditional means prior: a conditional means prior cont'd

- Assuming independence:

$$\pi(\mu_1, \mu_2) = \pi(\mu_1)\pi(\mu_2) \tag{37}$$

- One can then solve β_0 and β_1 in $\mu_i = \beta_0 + \beta_1 x_i$ given μ_1, μ_2, x_1, x_2:

$$\beta_1 = \frac{\mu_2 - \mu_1}{x_2 - x_1}, \tag{38}$$

$$\beta_0 = \mu_1 - x_1\left(\frac{\mu_2 - \mu_1}{x_2 - x_1}\right). \tag{39}$$

Conditional means prior: a conditional means prior cont'd

- Assuming independence:

$$\pi(\mu_1, \mu_2) = \pi(\mu_1)\pi(\mu_2) \tag{37}$$

- One can then solve β_0 and β_1 in $\mu_i = \beta_0 + \beta_1 x_i$ given μ_1, μ_2, x_1, x_2:

$$\beta_1 = \frac{\mu_2 - \mu_1}{x_2 - x_1}, \tag{38}$$

$$\beta_0 = \mu_1 - x_1 \left(\frac{\mu_2 - \mu_1}{x_2 - x_1} \right). \tag{39}$$

- Currently, we have $x_1 = 10, x_2 = 12$, and

$$\mu_1 = \beta_0 + \beta_1(10) \sim \mathrm{Normal}(8, 2), \tag{40}$$

$$\mu_2 = \beta_0 + \beta_1(12) \sim \mathrm{Normal}(11, 2). \tag{41}$$

Conditional means prior: JAGS script

```
modelString <-"
model {
## sampling
for (i in 1:N){
y[i] ~ dnorm(beta0 + beta1*x[i], invsigma2)
}

## priors
beta1 <- (mu2 - mu1)/(x2 - x1)
beta0 <- mu1 - x1*(mu2 - mu1)/(x2 - x1)
mu1 ~ dnorm(m1, g1)
mu2 ~ dnorm(m2, g2)
invsigma2 ~ dgamma(a, b)
sigma <- sqrt(pow(invsigma2, -1))
}
"
```

Outline

The CE sample

The CE sample comes from the 2017 Q1 CE PUMD. \ 4 variables, 994 observations.

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)
Rural	Binary; the urban/rural status of CU: 0 = Urban, 1 = Rural
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race

How can we include additional information about the urban/rural status and race category to predict a CU's log(Expenditure)?

A multiple linear regression model

- Similar to SLR, MLR assumes an observation specific mean μ_i for Y_i:
$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma), \quad i = 1, \cdots, n. \tag{42}$$

A multiple linear regression model

- Similar to SLR, MLR assumes an observation specific mean μ_i for Y_i:

$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \text{Normal}(\mu_i, \sigma), \quad i = 1, \cdots, n. \tag{42}$$

- In addition, MLR assumes the mean of Y_i is a linear function of all predictors:

$$\mu_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_r x_{i,r}. \tag{43}$$

  - $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,r})$ is a vector of known predictors for observation i
  - $\beta = (\beta_0, \beta_1, \cdots, \beta_r)$ is a vector of unknown regression coefficient parameters (shared among all observations)

Regression coefficient interpretation

$$\mu_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_r x_{i,r}. \tag{44}$$

- When all r predictors are continuous,
  - What does β_0 mean?
  - What does each $\beta_j, j = 1, \cdots, r$ mean?

Regression coefficient interpretation

$$\mu_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_r x_{i,r}. \tag{44}$$

- When all r predictors are continuous,
  - What does β_0 mean?
  - What does each $\beta_j, j = 1, \cdots, r$ mean?
- In the CE example, the predictors are not all continuous.

Variable	Description
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)
Rural	Binary; the urban/rural status of CU: 0 = Urban, 1 = Rural
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race

Adding a binary predictor

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
Rural	Binary; the urban/rural status of CU: 0 = Urban, 1 = Rural

- While it is possible to consider Rural as a continuous variable: change by one unit from urban to rural...
- It is much more common to consider it as a binary categorical variable to classify two groups:
  - The urban group
  - The rural group
- Such classification puts an emphasis on the difference of the expected outcomes between the two groups.

With only one binary predictor

- For simplicity, consider a simplified regression model with a single predictor: the binary indicator for rural area x_i.

$$\mu_i = \beta_0 + \beta_1 x_i = \begin{cases} \beta_0, & \text{the urban group;} \\ \beta_0 + \beta_1, & \text{the rural group.} \end{cases} \tag{45}$$

- The expected outcome μ_i for CUs in the urban group: β_0.
- The expected outcome μ_i for CUs in the rural group: $\beta_0 + \beta_1$.
- β_1 represents the change in the expected outcome μ_i from the urban group to the rural group.

Adding a multi-category categorical predictor

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race

- It is common to consider it as a categorical variable to classify multiple groups:
  - How many groups? What are the groups? 6 groups
- Such classification puts an emphasis on the difference of the expected outcomes between one group to the reference group.

With only one categorical predictor

- For simplicity, consider a simplified regression model with a single predictor: the race category of the reference person x_i.

$$
\begin{aligned}
\mu_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \beta_3 x_{i,3} + \beta_4 x_{i,4} + \beta_5 x_{i,5} \\
&= \begin{cases} \beta_0, & \text{White;} \\ \beta_0 + \beta_1, & \text{Black;} \\ \beta_0 + \beta_2, & \text{Native American;} \\ \beta_0 + \beta_3, & \text{Asian;} \\ \beta_0 + \beta_4, & \text{Pacific Islander;} \\ \beta_0 + \beta_5, & \text{Multi-race.} \end{cases}
\end{aligned} \tag{46}
$$

- What is the expected outcome μ_i for CUs in the White group? beta_0
- What is the expected outcome μ_i for CUs in the Asian group? beta_0+beta_3
- What does β_5 represent?

Consider all predictors

- The linear combination of the log(income), rural indicator, and race category:

$$
\begin{aligned}
\mu_i^* = \beta_0 &+ \beta_1 x_{i,income}^* + \beta_2 x_{i,rural} + \beta_3 x_{i,race_B} \\
&+ \beta_4 x_{i,race_N} + \beta_5 x_{i,race_A} + \beta_6 x_{i,race_P} + \beta_7 x_{i,race_M}.
\end{aligned} \tag{47}
$$

- The MLR is written as

$$
\begin{aligned}
Y_i^* \mid \beta_0, \beta_1, \cdots, \beta_7, \sigma, \mathbf{x}_i^* \overset{ind}{\sim} \text{Normal}(\beta_0 &+ \beta_1 x_{i,income} + \beta_2 x_{i,rural} \\
&+ \beta_3 x_{i,race_B} + \beta_4 x_{i,race_N} \\
&+ \beta_5 x_{i,race_A} + \beta_6 x_{i,race_P} \\
&+ \beta_7 x_{i,race_M}, \sigma).
\end{aligned} \tag{48}
$$

9 param in total

Note: * indicates standardized values.

Consider all predictors cont'd

$$
\begin{aligned}
\mu_i^* = \beta_0 \quad &+ \quad \beta_1 x_{i,income}^* + \beta_2 x_{i,rural} + \beta_3 x_{i,race_B} \\
&+ \quad \beta_4 x_{i,race_N} + \beta_5 x_{i,race_A} + \beta_6 x_{i,race_P} + \beta_7 x_{i,race_M}.
\end{aligned} \tag{49}
$$

- What does intercept β_0 mean?
- What does regression coefficient β_1 mean?
- What about the expected outcome of a CU with Native American reference person, living in rural area with standardized log(income) = 1? mu_i^*=beta_0+beta_1*1+beta_2+beta_4

A weakly informative prior

- With more than one predictors, the subjective prior and the conditional means prior are more difficult to specify. Why?
- Let's try giving a weakly informative prior.

A weakly informative prior

- With more than one predictors, the subjective prior and the conditional means prior are more difficult to specify. Why?
- Let's try giving a weakly informative prior.
- Assuming independence:

$$\pi(\beta_0, \beta_1, \cdots, \beta_7, \sigma) = \pi(\beta_0, \beta_1, \cdots, \beta_7)\pi(\sigma). \tag{50}$$

  - Assuming independence among β_j's:

$$\pi(\beta_0, \beta_1, \cdots, \beta_7) = \prod_{j=0}^{7} \pi(\beta_j), \tag{51}$$

$$\beta_j \sim \mathrm{Normal}(\mu_j, s_j). \tag{52}$$

    e.g. $\mathrm{Normal}(0, 1)$. for standardized data

  - Assigning a weakly informative prior for the standard deviation σ:

$$1/\sigma^2 \sim \mathrm{Gamma}(a, b). \tag{53}$$

    e.g. $\mathrm{Gamma}(1, 1)$.

JAGS script for the MLR model

- Need to standardize log(Expenditure) and log(Income).
- Also, need to create indicator variables (0 or 1) for each category of the categorical variable, except for the reference category.

```
CEData$log_TotalExpSTD <- scale(CEData$log_TotalExp)
CEData$log_TotalIncomeSTD <- scale(CEData$log_TotalIncome)

library(fastDummies)
## create indictor variable for Rural
CEData$Rural = fastDummies::dummy_cols(CEData$UrbanRural)[,names(fastDummie
 == ".data_2"]
```

- Do this for Race as well, 5 indicator variables.

JAGS script for the MLR model cont'd

```
modelString <-"
model {
## sampling
for (i in 1:N){
y[i] ~ dnorm(beta0 + beta1*x_income[i] + beta2*x_rural[i] +
beta3*x_race_B[i] + beta4*x_race_N[i] +
beta5*x_race_A[i] + beta6*x_race_P[i] +
beta7*x_race_M[i], invsigma2)
}
## priors
beta0 ~ dnorm(mu0, g0)
beta1 ~ dnorm(mu1, g1)
beta2 ~ dnorm(mu2, g2)
beta3 ~ dnorm(mu3, g3)
beta4 ~ dnorm(mu4, g4)
beta5 ~ dnorm(mu5, g5)
beta6 ~ dnorm(mu6, g6)
beta7 ~ dnorm(mu7, g7)
invsigma2 ~ dgamma(a, b)
sigma <- sqrt(pow(invsigma2, -1))
}
```

JAGS script for the MLR model cont'd

- Pass the data and hyperparameter values to JAGS:

```
y = as.vector(CEData$log_TotalExpSTD)
x_income = as.vector(CEData$log_TotalIncomeSTD)
x_rural = as.vector(CEData$Rural)
x_race_B = as.vector(CEData$Race_Black)
x_race_N = as.vector(CEData$Race_NA)
x_race_A = as.vector(CEData$Race_Asian)
x_race_P = as.vector(CEData$Race_PI)
x_race_M = as.vector(CEData$Race_M)
N = length(y)  # Compute the number of observations
```

JAGS script for the MLR model cont'd

- Pass the data and hyperparameter values to JAGS:

```
the_data <- list("y" = y, "x_income" = x_income,
                 "x_rural" = x_rural, "x_race_B" = x_race_B,
                 "x_race_N" = x_race_N, "x_race_A" = x_race_A,
                 "x_race_P" = x_race_P, "x_race_M" = x_race_M,
                 "N" = N,
                 "mu0" = 0, "g0" = 1, "mu1" = 0, "g1" = 1,
                 "mu2" = 0, "g2" = 1, "mu3" = 0, "g3" = 1,
                 "mu4" = 0, "g4" = 1, "mu5" = 0, "g5" = 1,
                 "mu6" = 0, "g6" = 1, "mu7" = 0, "g7" = 1,
                 "a" = 1, "b" = 1)
```

JAGS script for the MLR model cont'd

- Pass the data and hyperparameter values to JAGS:

```
initsfunction <- function(chain){
  .RNG.seed <- c(1,2)[chain]
  .RNG.name <- c("base::Super-Duper",
                 "base::Wichmann-Hill")[chain]
  return(list(.RNG.seed=.RNG.seed,
              .RNG.name=.RNG.name))
}
```

JAGS script for the MLR model cont'd

- Run the JAGS code for this model:

```
posterior_MLR <- run.jags(modelString,
                      n.chains = 1,
                      data = the_data,
                      monitor = c("beta0", "beta1", "beta2",
                                  "beta3", "beta4", "beta5",
                                  "beta6", "beta7", "sigma"),
                      adapt = 1000,
                      burnin = 5000,
                      sample = 5000,
                      thin = 1,
                      inits = initsfunction)
```

```
## Warning: Convergence cannot be assessed with only 1 chain
```

JAGS output for the MLR model

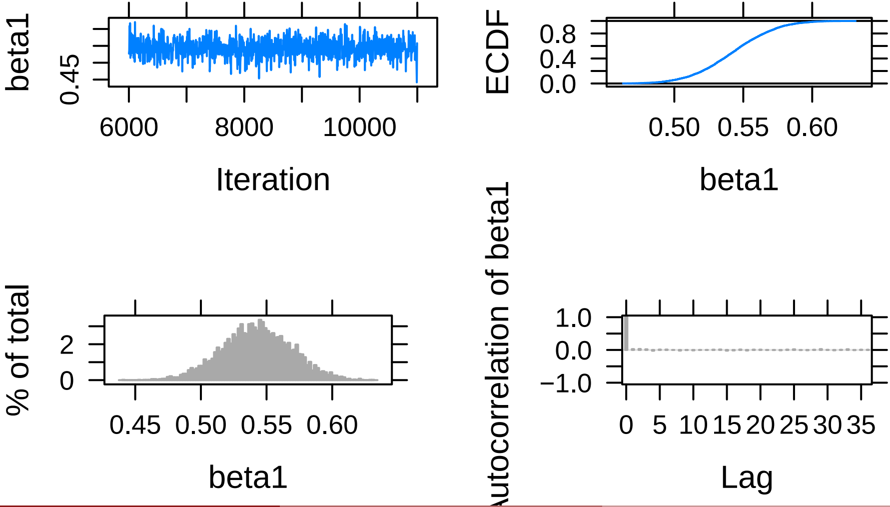
```r
summary(posterior_MLR)
```

```
##          Lower95      Median    Upper95        Mean         SD Mode
## beta0 -0.0261884  0.03072615  0.0898729  0.03111402 0.02974211   NA
## beta1  0.4889750  0.54241750  0.5926740  0.54237419 0.02648363   NA
## beta2 -0.4918380 -0.26816300 -0.0262582 -0.26872848 0.11800343   NA
## beta3 -0.3996750 -0.23509350 -0.0678321 -0.23338067 0.08548032   NA
## beta4 -0.6071250 -0.01422395  0.5520270 -0.01764766 0.29634497   NA
## beta5 -0.0922392  0.18444550  0.4434330  0.18556012 0.13734992   NA
## beta6 -0.5323690  0.08998935  0.6842530  0.09247921 0.31739975   NA
## beta7 -0.3609250  0.03685400  0.4281700  0.03163386 0.20378584   NA
## sigma  0.7892530  0.82702250  0.8624090  0.82749520 0.01860583   NA
##              MCerr MC%ofSD SSeff        AC.10 psrf
## beta0 0.0005408114     1.8  3024 -0.004175826   NA
## beta1 0.0003934179     1.5  4532 -0.012275215   NA
## beta2 0.0018827201     1.6  3928 -0.001199134   NA
## beta3 0.0014087613     1.6  3682 -0.006798767   NA
## beta4 0.0041909508     1.4  5000 -0.024524053   NA
## beta5 0.0020743101     1.5  4384 -0.007574647   NA
## beta6 0.0044887103     1.4  5000  0.007878188   NA
## beta7 0.0030045388     1.5  4600  0.011216547   NA
## sigma 0.0002631261     1.4  5000  0.011646253   NA
```

JAGS output for the MLR model cont'd

```
plot(posterior_MLR, vars = "beta1")
```

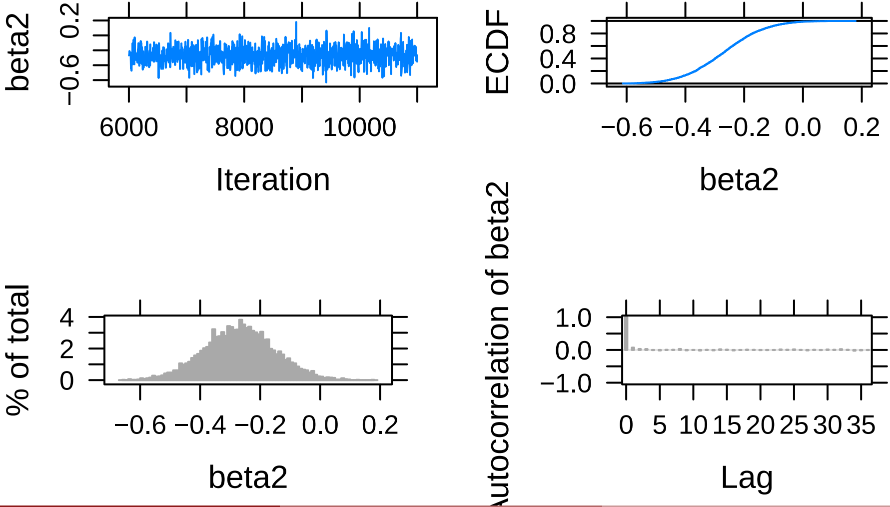
```
## Generating plots...
```



JAGS output for the MLR model cont'd

```
plot(posterior_MLR, vars = "beta2")
```

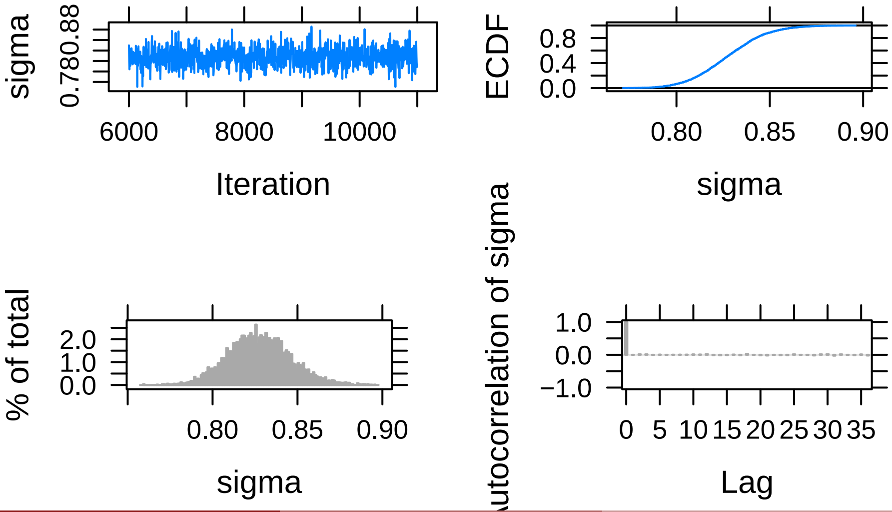
```
## Generating plots...
```



JAGS output for the MLR model cont'd

```
plot(posterior_MLR, vars = "sigma")
```

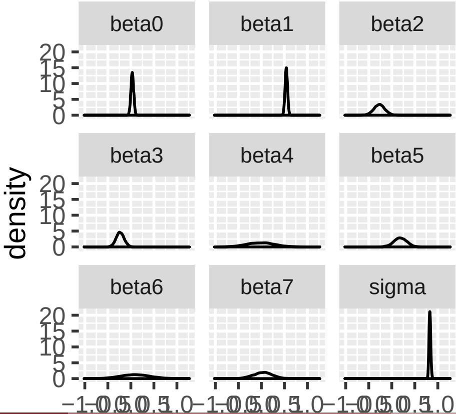
```
## Generating plots...
```



JAGS output for the MLR model cont'd

```
post <- as.mcmc(posterior_MLR)
post %>% as.data.frame %>%
  gather(parameter, value) -> post2
ggplot(post2, aes(value)) +
  geom_density() + facet_wrap(~ parameter, ncol = 3) +
  theme(strip.text.x = element_text(size=8))
```

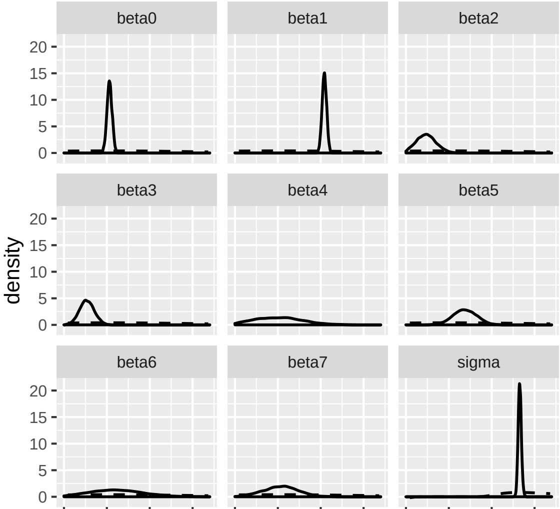
flat >> large std dev
>> uncertain



JAGS output for the MLR model cont'd

```
## Warning: Removed 548 rows containing non-finite values (stat_density).

## Warning: Removed 72 rows containing missing values (geom_path).
```



Bayesian inferences with MLR

- Learning about the expected response
- Predictor of future responses
- Posterior predictive checks
- Experiment with several priors to show the impact on the posterior inference