Notes on Lie Algebra

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1. Eigenspace decomposition

$\mathfrak{sl}_2(\mathbb{C})$ is the set of 2×2 matrices with zero trace. It is spanned by three basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Here we have three relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{1}$$

Definition 1.1. A Lie algebra $\mathfrak{g}$ is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with the Lie bracket $[,] : V \times V \to V$, which satisfies the following properties:

- $[,]$ is bilinear.
- $[X, Y] = -[Y, X]$ for all $X, Y \in V$.
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for all $X, Y, Z \in V$.

Let V be an irreducible finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. The action of H on V is diagonalizable. Equivalently, that there is a basis of V consisting of eigenvectors:

$$V = \bigoplus V_\alpha \tag{2}$$

where the α is the eigenvalue for each eigenspace, namely V_α. For any $v \in V$,

$$H(v) = \alpha \cdot v. \tag{3}$$

We have seen that H act on a vector in the eigenspaces by scaling the vector with a complex number. To see the actions of X, Y on $v \in V_\alpha$, we need to compute how H acts on $X(v), Y(v)$:

$$\begin{aligned} H(X(v)) &= X(H(v)) + H(X(v)) - X(H(V)) \\ &= X(H(v)) + [H, X](v) \\ &= X(\alpha \cdot v) + 2X(v) \\ &= (\alpha + 2) \cdot X(v) \end{aligned} \tag{4}$$

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Similar for Y:

$$H(Y(v)) = (\alpha - 2) \cdot Y(v) \tag{5}$$

So X(v) and Y(v) are both eigenvectors for H with eigenvalues $\alpha + 2$, $\alpha - 2$, respectively. In another words, $X(V_\alpha) \subset V_{\alpha+2} \subset V$. Y is similar. Recall that we started from the irreducible representation, since V is invariant under the actions of X, Y, we conclude that the eigenvalues are must be congruent to one another mod 2, i.e., for a fixed α_0,

$$V = \bigoplus_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}. \tag{6}$$

Since V is finite dimensional, the eigenvalue cannot be arbitrarily large. Choose n to be the maximum eigenvalue, we have $V_n \subseteq \ker x$, $V_{n+2} = 0$. By far, we have the following picture:

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "X"] & V_{n-4} \arrow[l, bend left, "Y"] \arrow[loop above, "H"] \arrow[r, bend left, "X"] & V_{n-2} \arrow[l, bend left, "Y"] \arrow[loop above, "H"] \arrow[r, bend left, "X"] & V_n \arrow[l, bend left, "Y"] \arrow[loop above, "H"] \arrow[r, bend left, "X"] & 0
\end{tikzcd}$$

Now we consider the repeated action of Y on $v \in V_n$.

Lemma 1.2. $X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v)$.

Proof. The two base cases are:

$$\begin{aligned}
X(Y(v)) &= [X, Y](v) + Y(X(v)) = H(v) + Y(0) = n \cdot v \\
X(Y^2(v)) &= (n-2) \cdot Y(v) + n \cdot Y(v).
\end{aligned}$$

By induction:

$$\begin{aligned}
X(Y^m(v)) &= (n + (n-2) + (n-4) + ... + (n - 2m + 2)) \cdot Y^{m-1}(v), \\
&= m(n - m + 1) \cdot Y^{m-1}(v)
\end{aligned}$$

□

Similar to the previous case, we can choose m to be the minimal integer such that Y annihilates v. Explicitly,

$$0 = X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v).$$

So we have the relation: $n = m - 1$. This calculation tells us several important facts.

Lemma 1.3. *Eigenvalues for the representation V are integers symmetric about zero.*

Proof. m is a positive integer, so n is a non-negative integer. $\square$

Theorem 1.4. *The vectors* $\{v, Y(v), Y^2(v)...\}$ *span* V.

Proof. Let $W = span\{v, Y(v), Y^2(v)...\} \subset V$. Since V is irreducible, it is suffices to prove that all the actions of $\mathfrak{sl}_2(\mathbb{C})$ preserves W. Y preserves Y^m trivially, since $Y(Y^m(v)) = Y^{m+1}(v)$. From equation 5, we have seen that the action of Y decreases the eigenvalue by two, which means $H(Y^m(v)) = (n - 2m) \cdot Y(v)$. By Lemma 1.2, $X(Y^m(v)) \subset W$. W is non-trivial, so $W = V$. $\square$