

# Notes on Lie Algebra

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## 1. Eigenspace decomposition

$\mathfrak{sl}_2(\mathbb{C})$  is the set of  $2 \times 2$  matrices with zero trace. It is spanned by three basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Here we have three relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \quad (1)$$

**Definition 1.1.** A Lie algebra  $\mathfrak{g}$  is a vector space  $V$  over  $\mathbb{R}$  or  $\mathbb{C}$  with the Lie bracket  $[\cdot, \cdot] : V \times V \rightarrow V$ , which satisfies the following properties:

- $[\cdot, \cdot]$  is bilinear.
- $[X, Y] = -[Y, X]$  for all  $X, Y \in V$ .
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$  for all  $X, Y, Z \in V$ .

Let  $V$  be an irreducible finite dimensional representation of  $\mathfrak{sl}_2(\mathbb{C})$ . The action of  $H$  on  $V$  is diagonalizable. Equivalently, that there is a basis of  $V$  consisting of eigenvectors:

$$V = \bigoplus V_\alpha \quad (2)$$

where the  $\alpha$  is the eigenvalue for each eigenspace, namely  $V_\alpha$ . For any  $v \in V$ ,

$$H(v) = \alpha \cdot v. \quad (3)$$

We have seen that  $H$  act on a vector in the eigenspaces by scaling the vector with a complex number. To see the actions of  $X, Y$  on  $v \in V_\alpha$ , we need to compute how  $H$  acts on  $X(v), Y(v)$ :

$$\begin{aligned} H(X(v)) &= X(H(v)) + H(X(v)) - X(H(V)) \\ &= X(H(v)) + [H, X](v) \\ &= X(\alpha \cdot v) + 2X(v) \\ &= (\alpha + 2) \cdot X(v) \end{aligned} \quad (4)$$

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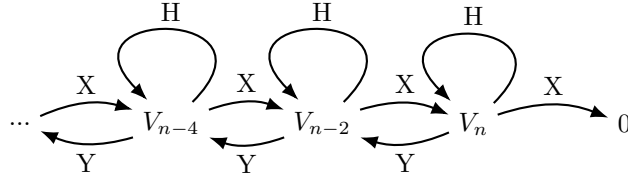
Similar for  $Y$ :

$$H(Y(v)) = (\alpha - 2) \cdot Y(v) \quad (5)$$

So  $X(v)$  and  $Y(v)$  are both eigenvectors for  $H$  with eigenvalues  $\alpha + 2$ ,  $\alpha - 2$ , respectively. In another words,  $X(V_\alpha) \subset V_{\alpha+2} \subset V$ .  $Y$  is similar. Recall that we started from the irreducible representation, since  $V$  is invariant under the actions of  $X, Y$ , we conclude that the eigenvalues are must be congruent to one another mod 2, i.e., for a fixed  $\alpha_0$ ,

$$V = \bigoplus_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}. \quad (6)$$

Since  $V$  is finite dimensional, the eigenvalue cannot be arbitrarily large. Choose  $n$  to be the maximum eigenvalue, we have  $V_n \subseteq \ker x$ ,  $V_{n+2} = 0$ . By far, we have the following picture:



Now we consider the repeated action of  $Y$  on  $v \in V_n$ .

**Lemma 1.2.**  $X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v)$ .

*Proof.* The two base cases are:

$$X(Y(v)) = [X, Y](v) + Y(X(v)) = H(v) + Y(0) = n \cdot v$$

$$X(Y^2(v)) = (n - 2) \cdot Y(v) + n \cdot Y(v).$$

By induction:

$$\begin{aligned} X(Y^m(v)) &= (n + (n - 2) + (n - 4) + \dots + (n - 2m + 2)) \cdot Y^{m-1}(v), \\ &= m(n - m + 1) \cdot Y^{m-1}(v) \end{aligned}$$

□

Similar to the previous case, we can choose  $m$  to be the minimal integer such that  $Y$  annihilates  $v$ . Explicitly,

$$0 = X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v).$$

So we have the relation:  $n = m - 1$ . This calculation tells us several important facts.

**Lemma 1.3.** *Eigenvalues for the representation  $V$  are integers symmetric about zero.*

**Proof.**  $m$  is a positive integer, so  $n$  is a non-negative integer.  $\square$

**Theorem 1.4.** *The vectors  $\{v, Y(v), Y^2(v) \dots\}$  span  $V$ .*

**Proof.** Let  $W = \text{span}\{v, Y(v), Y^2(v) \dots\} \subset V$ . Since  $V$  is irreducible, it suffices to prove that all the actions of  $\mathfrak{sl}_2(\mathbb{C})$  preserves  $W$ .  $Y$  preserves  $Y^m$  trivially, since  $Y(Y^m(v)) = Y^{m+1}(v)$ . From equation 5, we have seen that the action of  $Y$  decreases the eigenvalue by two, which means  $H(Y^m(v)) = (n - 2m) \cdot Y(v)$ . By Lemma 1.2,  $X(Y^m(v)) \subset W$ .  $W$  is non-trivial, so  $W = V$ .  $\square$