process ($N(t)$, $t \geq 0$) is a Poisson process if and only if the conditional distribution of the n arrival times given $N(t) = n$ is the same as the distribution of the order statistics corresponding to n independent random variables indentically and uniformly distributed on the interval $(0, t)$.

A nonhomogeneous Poisson process $N(t)$ with nondecreasing mean function $m(t)$ can be transformed into a homogeneous Poisson process $M(u) = N(m^{-1}(u))$ with intensity $\lambda = 1$, where $m^{-1}(u) = \inf_t \{t : m(t) \geq u\}$ (Parzen 1965, p. 126). Therefore, we have the following characterization. A renewal point process $N(t)$ with independent continuously distributed interarrival times is a nonhomogeneous Poisson process if and only if the conditional distribution of an interarrival time given $N(t) = n$ is the same as that of the minimum order statistic corresponding to n independent random variables identically distributed according to the distribution $G(u) = m(u)I(0, t)(u)/m(t)$. As in the homogeneous case, these characterizations can be equivalently stated in terms of the conditional distribution of the n arrival times.

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An Alternate Proof of Samuelson's Inequality and Its Extensions

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Paul Samuelson and other authors have shown that no element in a population of n items can lie farther than $(n - 1)^{1/2}$ standard deviations from the mean. Other researchers extended this by giving upper and lower bounds for the kth largest item in terms of the population mean and standard deviation. We provide an alternate proof of these results. The technique used in the proof gives insight into the conditions under which the bounds are attained and can be used to improve bounds when additional information is available.

KEY WORDS: Bounds on order statistics; Outliers.

1. BACKGROUND

Samuelson (1968) proved that in a population of n items, none can be farther than $(n - 1)^{1/2}$ standard deviations from the mean. Wolkowicz and Styan (1979) extended this to

$$\bar{x} - s\sqrt{(n - k)/k} \leq x_k \leq \bar{x} + s\sqrt{(k - 1)/(n - k + 1)}, \quad (1.1)$$

where $x_1 \leq x_2 \leq \cdots \leq x_n$ are any real numbers, $\bar{x} = (1/n)\Sigma x_i$, $s^2 = (1/n)\Sigma(x_i - \bar{x})^2$, and $k = 1, 2, \ldots, n$. (Wolkowicz and Styan used the ordering $x_n \leq x_{n-1} \leq \cdots \leq x_1$; we have chosen to use $x_1 \leq x_2 \leq \cdots \leq x_n$ so that the x_i's may simply be thought of as order statistics.) Equality holds on the right side of (1.1) if and only if

$$x_1 = x_2 = \cdots = x_{k-1}, \qquad x_k = x_{k+1} = \cdots = x_n; \quad (1.2)$$

equality holds on the left side if and only if

$$x_1 = x_2 = \cdots = x_k, \qquad x_{k+1} = x_{k+2} = \cdots = x_n. \quad (1.3)$$

The most complete list of references on (1.1) can be found in Kabe (1980) and Wolkowicz and Styan (1980).

The proof that follows depends on the simple identity (Brownlee 1965, p. 271)

$$\sum_1^n (x_i - c)^2 = \sum_1^n (x_i - \bar{x})^2 + n(\bar{x} - c)^2 \quad (1.4)$$

and the formula

$$s^2 = \sum_{i<j}^n (x_i - x_j)^2/n^2. \quad (1.5)$$

Equation (1.5), which holds for arbitrary x_i's (not just for ordered data), is not always mentioned in modern textbooks but is well suited to questions involving order statistics. Interested readers should note that (1.5) follows immediately from the more general Lagrange identity (Beckenbach and Bellman 1971, p. 3)

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$$\left(\sum_1^n x_i^2\right) \cdot \left(\sum_1^n y_i^2\right) - \left(\sum_1^n x_i y_i\right)^2 = \sum_{i<j}^n (x_i y_j - x_j y_i)^2$$

by setting all of the y_i's equal to 1.

2. PROOF

Letting $A_j = \Sigma_{i=1}^{j-1}(x_i - x_j)^2$ for $j = 2, 3, \ldots, n$ we can rewrite (1.5) as

$$n^2 s^2 = \sum_2^n A_j. \tag{2.1}$$

From (2.1) or (1.5) it is easy to see that the A_j's (and hence s^2) must increase as the number of distinct x_i's increases.

For a given k, consider the special case

$$x_1 \leq x_2 \leq \cdots \leq x_{k-1}, \qquad x_k = x_{k+1} = \cdots = x_n. \tag{2.2}$$

Then, because $\Sigma_k^n(x_i - x_k)^2 = 0$, we have $A_j = A_k = \Sigma_1^n(x_i - x_k)^2$ for $j = k, k + 1, \ldots, n$. Using this fact along with (1.4) in (2.1) gives $n^2 s^2 \geq (n - k + 1)A_k = (n - k + 1)\Sigma_1^n(x_i - x_k)^2 = (n - k + 1)(n)[s^2 + (x_k - \bar{x})^2]$, which, after some algebra, implies

$$|(x_k - \bar{x})| \leq s\sqrt{(k-1)/(n-k+1)}$$

and, consequently,

$$x_k \leq \bar{x} + s\sqrt{(k-1)/(n-k+1)}. \tag{2.3}$$

Following a similar argument for the special case

$$x_1 = x_2 = \cdots = x_k, \qquad x_{k+1} \leq x_{k+2} \leq \cdots \leq x_n, \tag{2.4}$$

since each $A_j \geq k(x_j - x_k)^2$ for $j = k + 1, \ldots, n$ and $\Sigma_1^k(x_i - x_k)^2 = 0$, we have $n^2 s^2 \geq k\Sigma_1^n(x_i - x_k)^2$. Using (1.4) with $c = x_k$ in the last inequality yields $n^2 s^2 \geq k[\Sigma_1^n(x_i - \bar{x})^2 + n(\bar{x} - x_k)^2]$, from which it follows that

$$|(\bar{x} - x_k)| \leq s\sqrt{(n-k)/k}$$

and, in particular,

$$\bar{x} - s\sqrt{(n-k)/k} \leq x_k. \tag{2.5}$$

For the general case $x_1 \leq x_2 \leq \cdots \leq x_n$, consider the two related sequences in (2.2) and (2.4). Since going from (2.2) to the general case can only increase $\bar{x}$ and s, we see that (2.3) continues to hold. Likewise, going from (2.4) to the general case decreases $\bar{x}$ while increasing s, so (2.5) continues to hold. Notice that the inequalities leading to (2.3) become equalities if and only if (1.2) holds and that (1.3) similarly leads to equality in (2.5).

3. CONCLUSIONS

The proof relies on elementary statistical identities and arguments and lends insight to the nature of the equality conditions (1.2) and (1.3). Furthermore, the decomposition in (2.1) provides a vehicle for improving the inequalities. As an example, consider the largest item, x_n. Letting s_{n-1}^2 denote the variance of $x_1, x_2, \ldots, x_{n-1}$ and adding the term $(x_n - x_n)^2 = 0$ to A_n, (2.1) becomes

$$n^2 s^2 = \sum_2^{n-1} A_j + A_n = (n-1)^2 s_{n-1}^2 + \sum_1^n (x_i - x_n)^2.$$

Applying (1.4) and solving for x_n,

$$x_n = \bar{x} + s\sqrt{n-1} \cdot \{1 - (n-1)s_{n-1}^2/ns^2\}. \tag{3.1}$$

It is clear from (1.5) that $(n - 1)s_{n-1}^2 \leq ns^2$, so the term in braces in (3.1) never exceeds 1, thus proving Samuelson's original result. The identity (3.1) can be used to improve the bound on x_n in the presence of additional information about the x_i's (e.g., symmetry or distributional assumptions). Along these lines, note that an identity equivalent to (3.1) has long been known in the theory of outliers (Barnett and Lewis 1984, p. 167).

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