Information processing with recurrent dynamical systems:

theory, characterisation and experiment.

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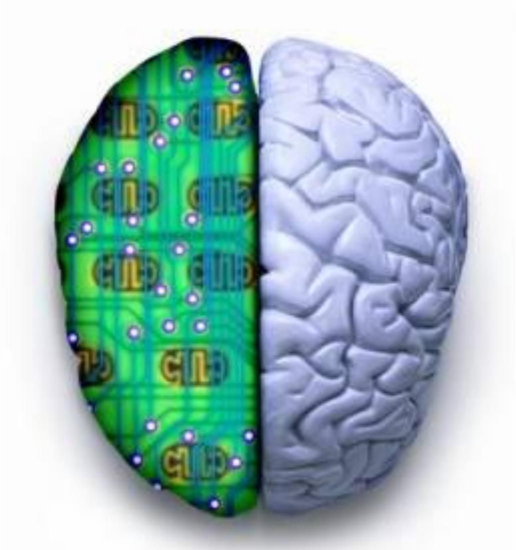
My Background

- Theoretical Physicist
  - Quantum Gravity
  - Quantum Information Theory
- Experimental Physicist
  - Quantum and Non Linear Optics
- And recently:
  - Machine Learning and Reservoir Computing

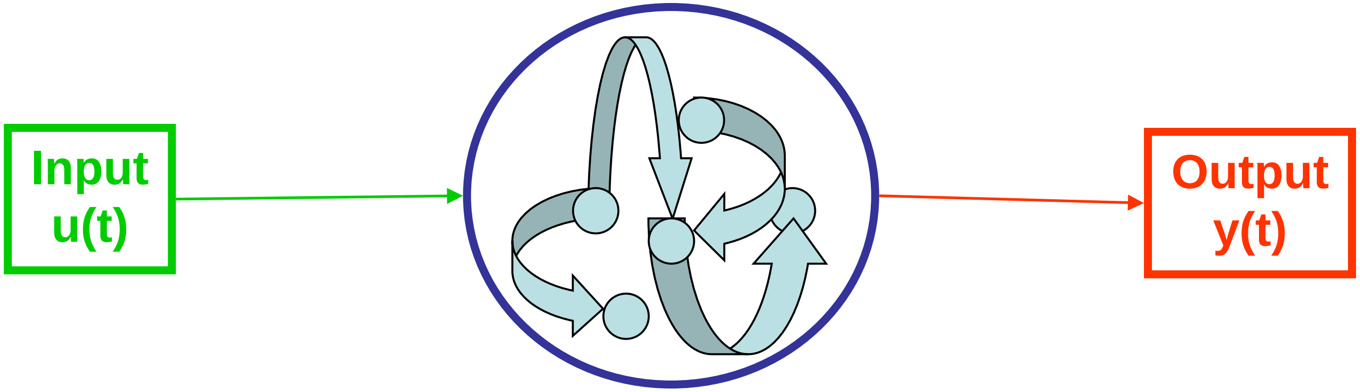
COMPUTERS VS BRAINS



- Low number of parallel units
- energy & space consumption
- Best at "hard" tasks
- Top-down architecture
- Designed
- Everything has a clear purpose

- High parallelism
- = energy and space consumption*10^5
- Good at "difficult to code" tasks
- Bottom-up architecture
- Evolved
- Doesn't even *have* to make sense

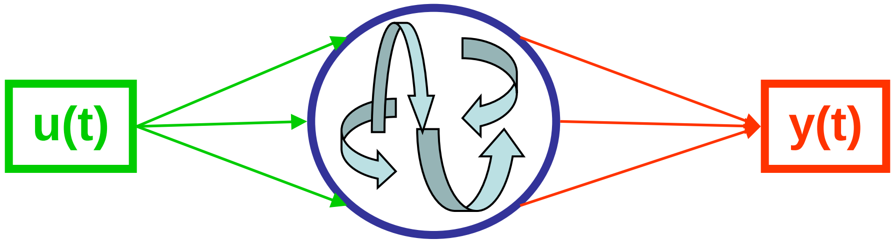
Training Recurrent Dynamical Systems



Recurrent Dynamical System

- Hard to train individual parameters using gradient descent.

Reservoir Computing
(Liquid State Machine)
(Echo State Network)



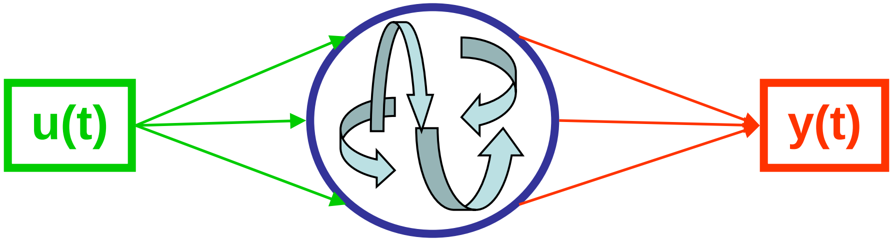
$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

$$y(t) = \sum_i W_i x_i(t)$$

Linear Readout
Only these coefficients are trained.
→Easy optimisation

State of the art performance for:
- time series prediction
- speech recognition

Echo State Network



$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

$$y(t) = \sum_i W_i x_i(t)$$

$N(0,\beta^2)$ Input Scaling

$N(0,\alpha^2)$ Feedback Strength

RANDOM COEFFICIENTS

!!ALMOST ALL RESERVOIRS ARE GOOD!!

→EXPERIMENT: Almost all experimental implementations will work.

→THEORY: Understand property of average reservoir

Plan

- Experimental reservoir computing
  - Opto-electronic reservoir computer with a single non linear node
- Characterise information processing of recurrent dynamical systems
  - Linear and Non Linear Memory Capacities
- Reservoir computing and statistical mechanics
  - Mean Field Theory of Echo State Networks

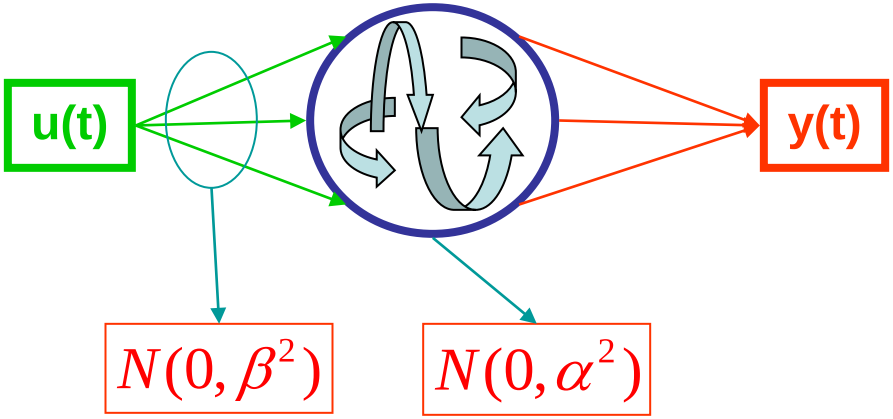
Experimental Reservoir Computer

- Experimental Demonstrations:
  - Water Bucket Fernando 2003
  - VLSI chip Schürmann 2005
  - Delay Line with a Single Non Linear Node
    - Electronic: Appeltant 2011 (in press in Nat. Comm.)
    - Opto-Electronic: Paquot 2011 (submitted)

- Why Experiments?
  - Gain new insights (e.g. noise resistance)
  - Possibility of applications:
    - Ultra fast / ultra low energy computation

- Why Optics?
  - Our labs expertise
  - Optics can in principle be super fast !!!

Echo State Network

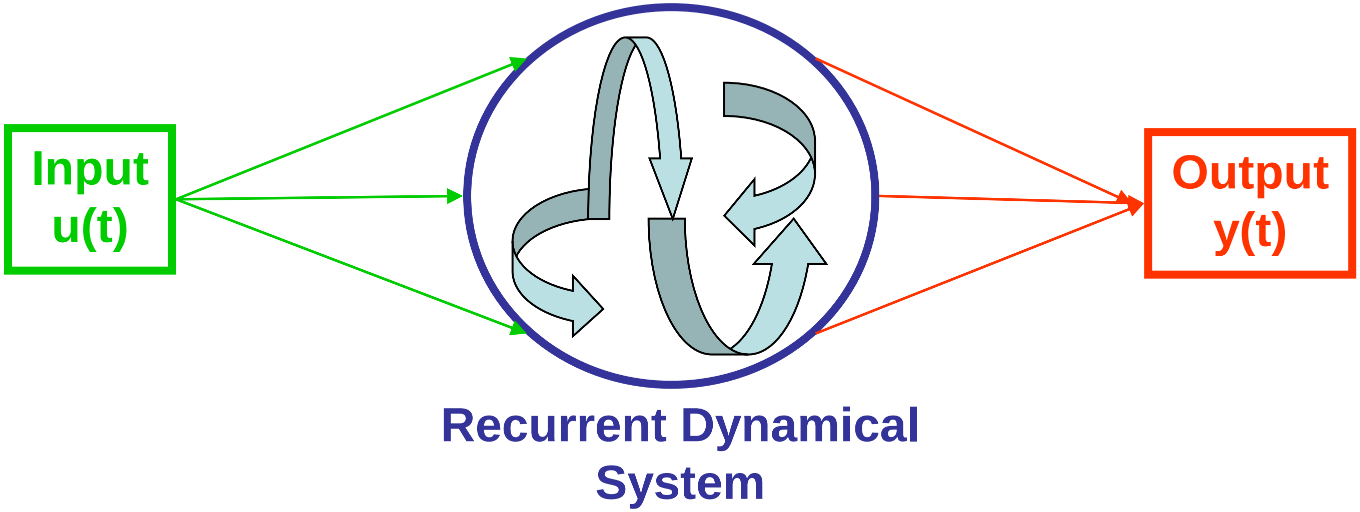


RANDOM COEFFICIENTS

!! Almost all experimental implementations will work !!

GOAL: Build a SIMPLE system with GOOD performance

Opto-Electronic Reservoir based on a single non linear node and a delay line.



- Need to describe each part of the reservoir

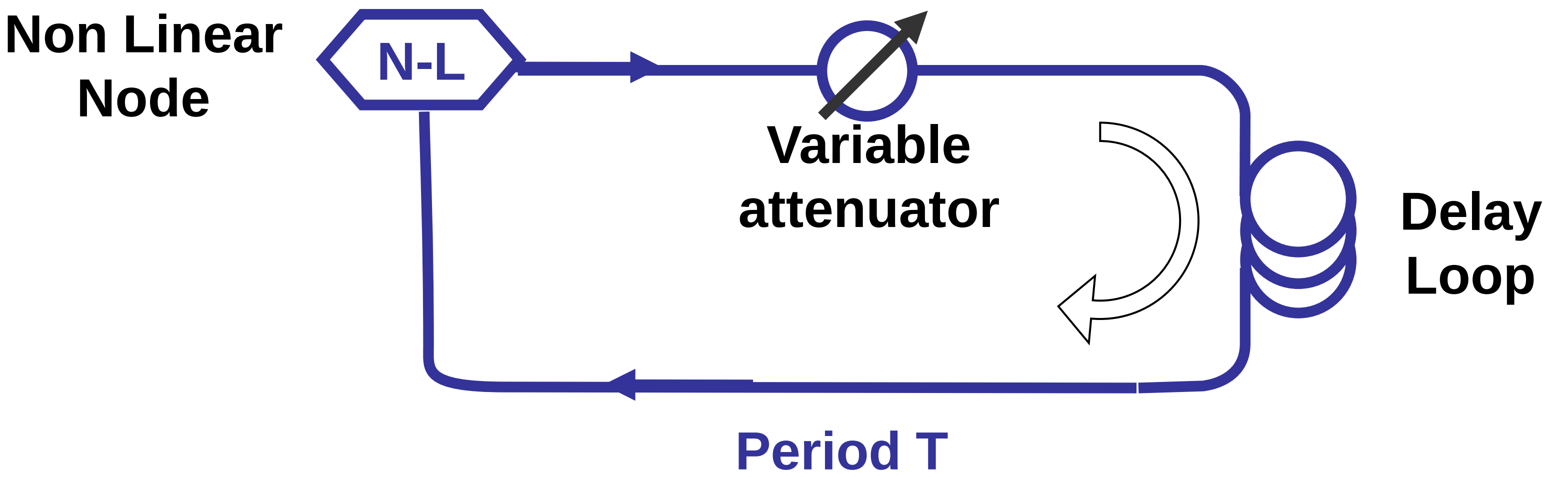
The Reservoir:
Single Non Linear Node
with Delayed Feedback

Delay systems:

✓Many internal variables

✓Rich dynamics with transition to chaos

✓Few components→Easy to build

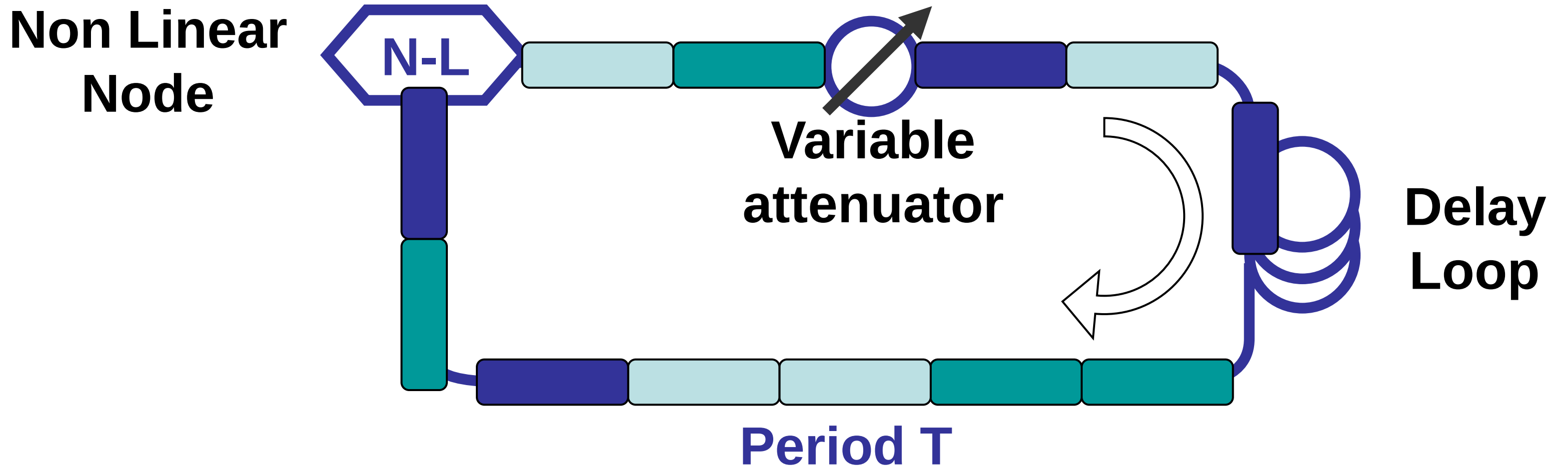


$$x(t+T)=F[\alpha x(t)]$$

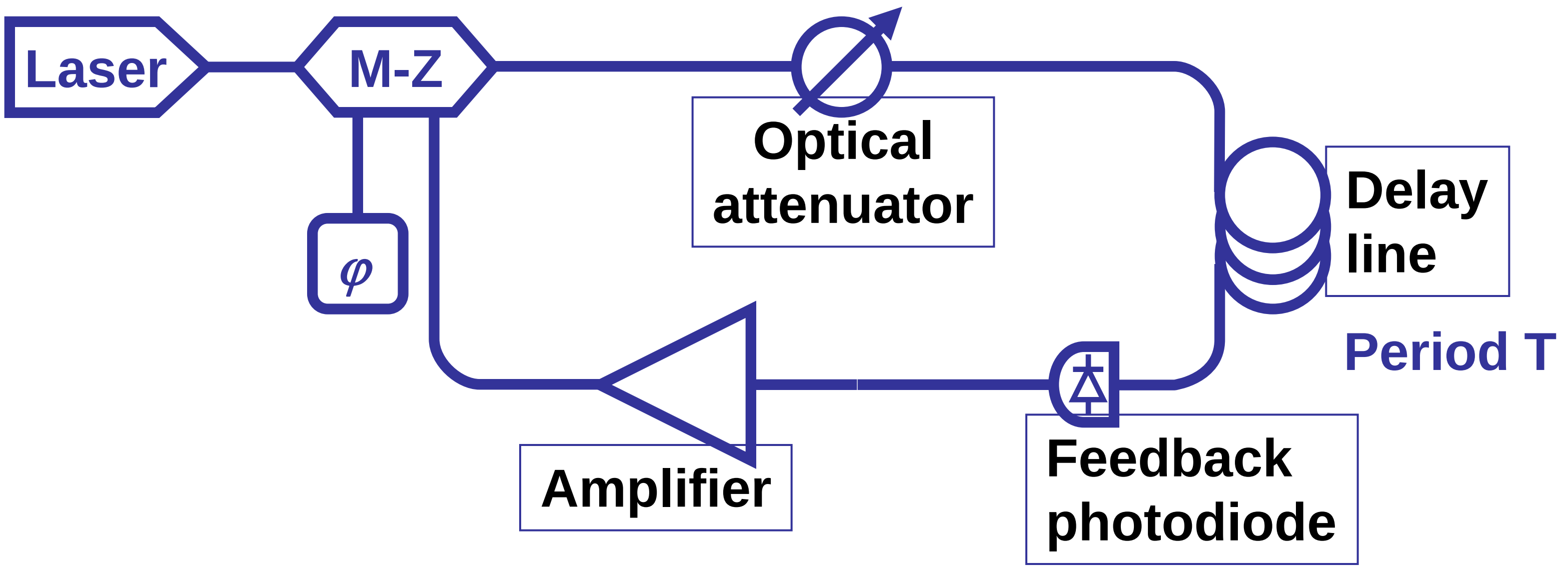
The Reservoir: Single Non Linear Node with Delayed Feedback

Dynamical variables=internal states inside delay loop

$x(t+T)=F[\alpha x(t)]$ → $x_i(n)=F[\alpha x_i(n-1)]$



Experimental Realisation



$$x(t) = \frac{I(t) - \langle I \rangle}{\langle I \rangle}$$

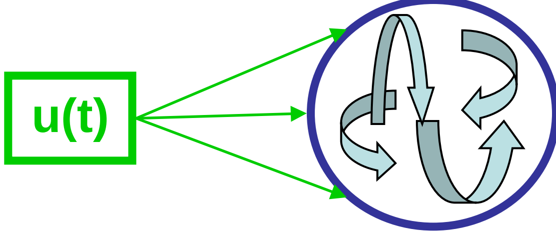
$$x(t+T) = \sin\left(\alpha\, \mathrm{x}(\mathrm{t}) + \varphi\right)$$

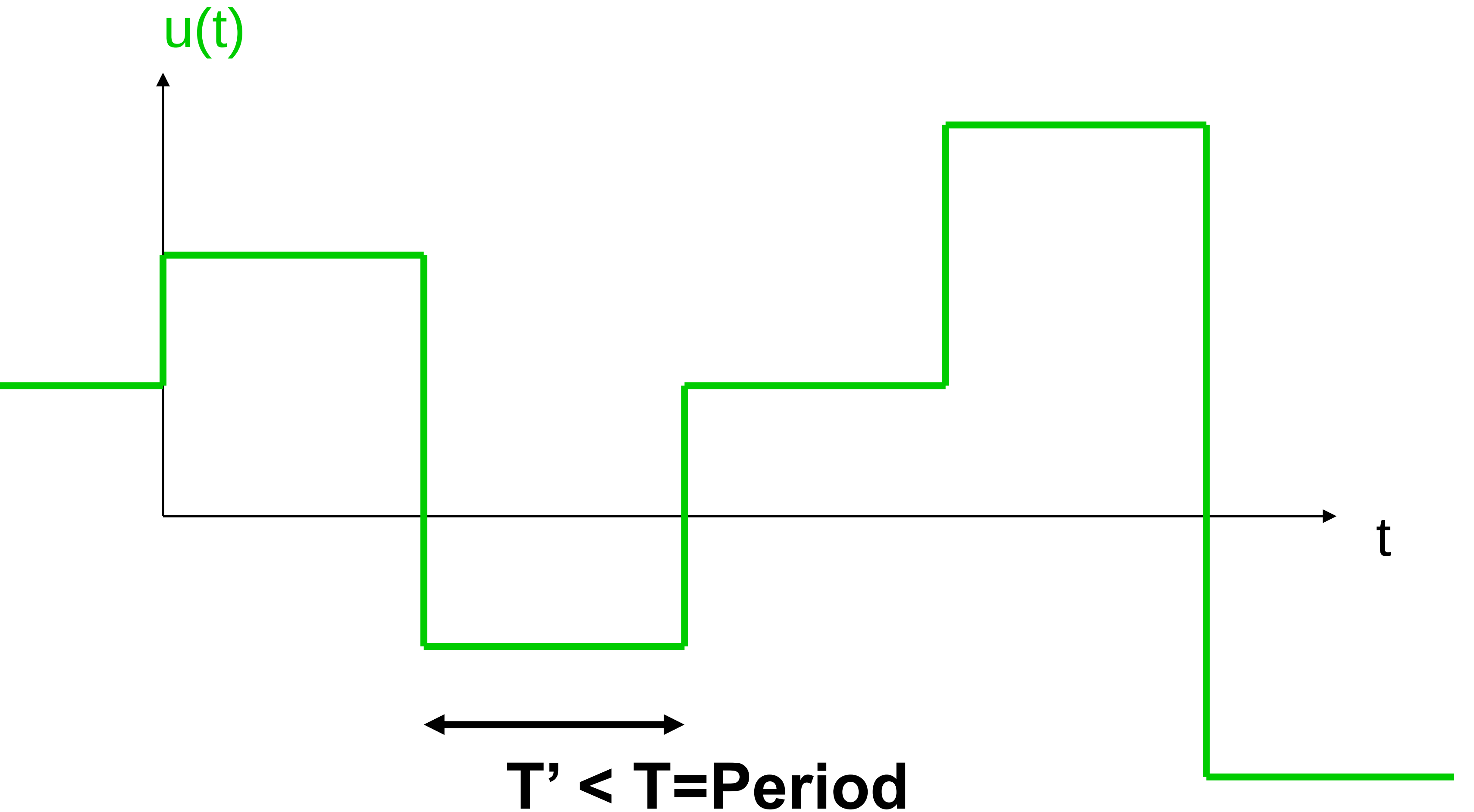
$$x_i(n) = \sin[\alpha x_i(n-1)]$$

Input

- Break symmetry
- Add richness to the reservoir dynamics

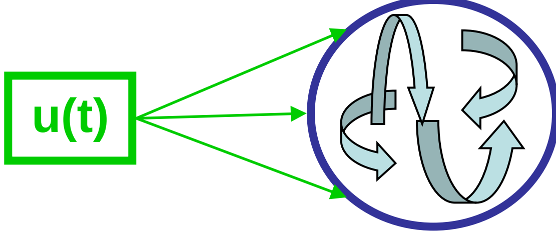
1) De-synchronyze





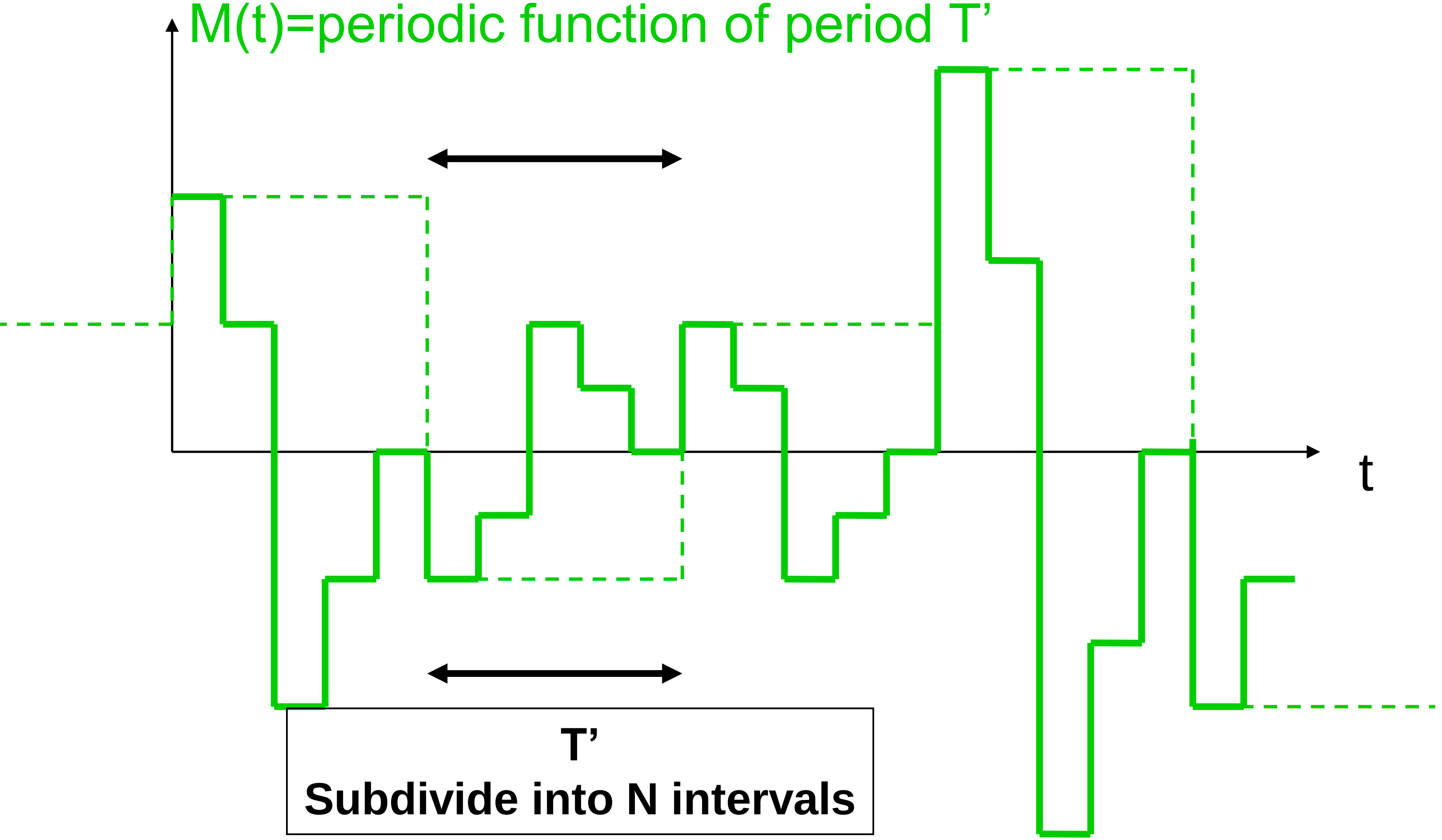
$$x_i(n)=\sin[\alpha x_{i-1}(n-1)+u(n)]$$

2) PeriodicMask

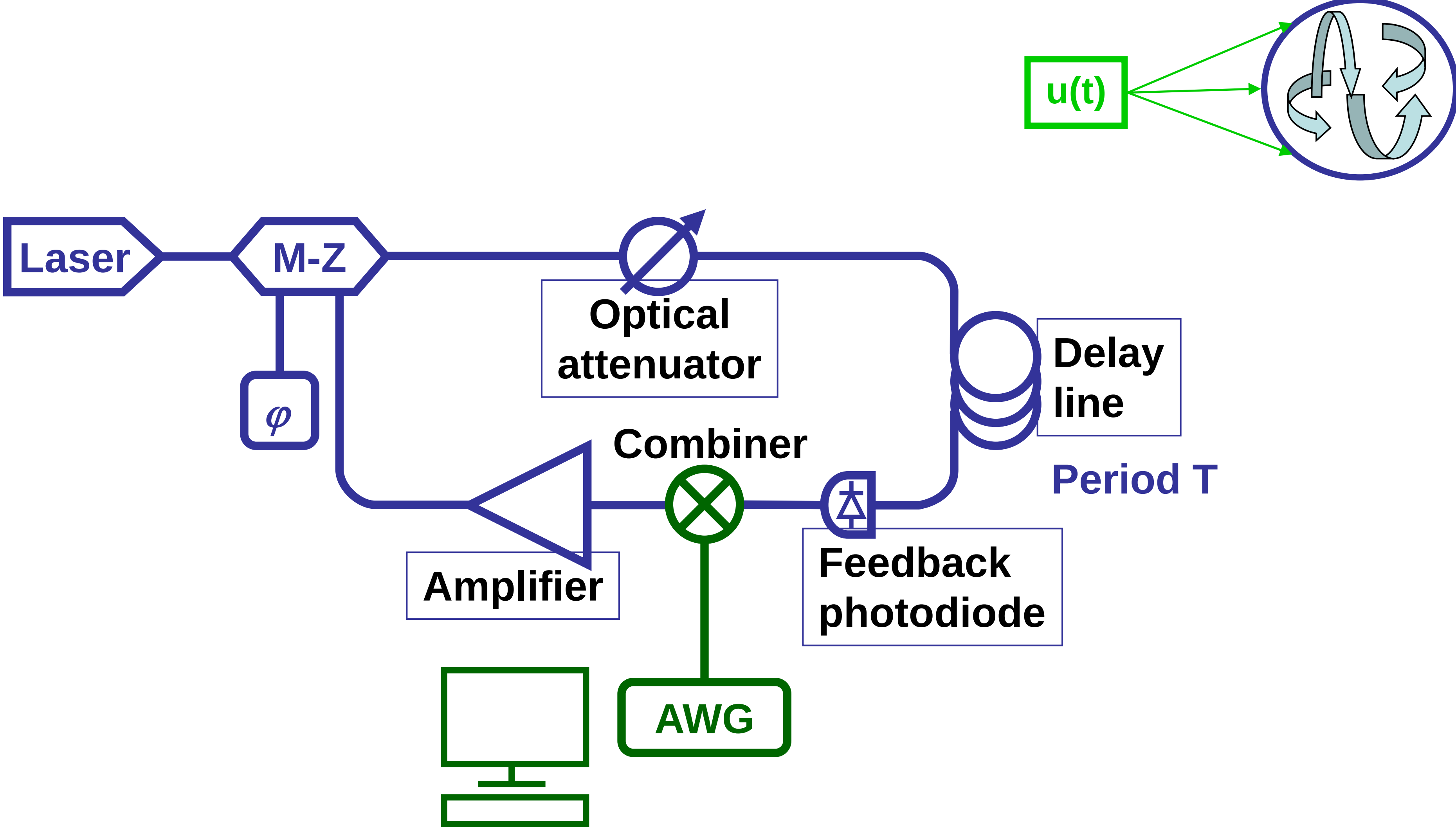


$J(t)=u(t)*M(t)$

$M(t)$=periodic function of period T'

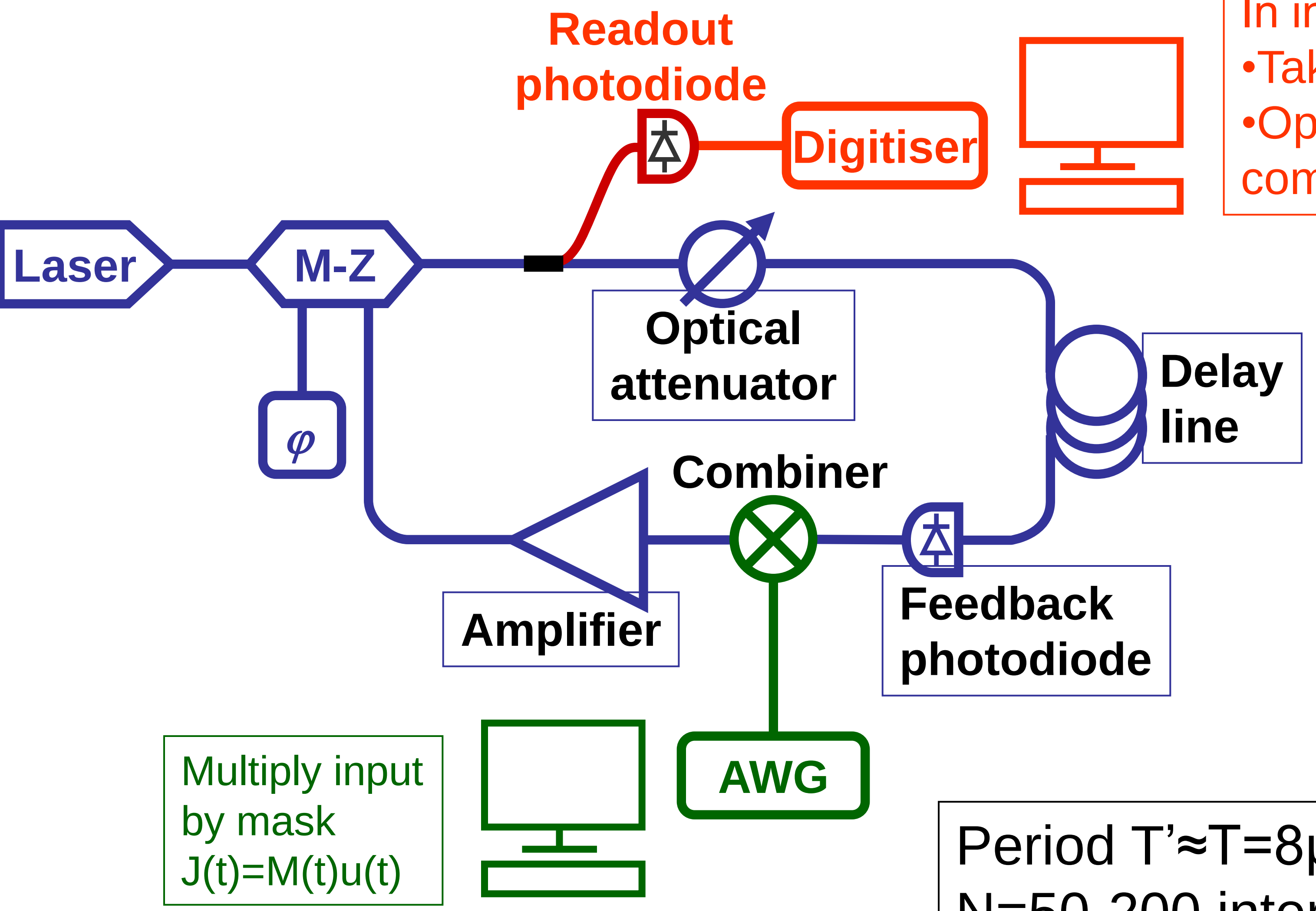


$$x_i(n)=\sin[\alpha x_{i-1}(n-1)+m_i u(n)]$$



$$x_i(n)=\sin[\alpha x_{i-1}(n-1)+m_i u(n)+\varphi]$$

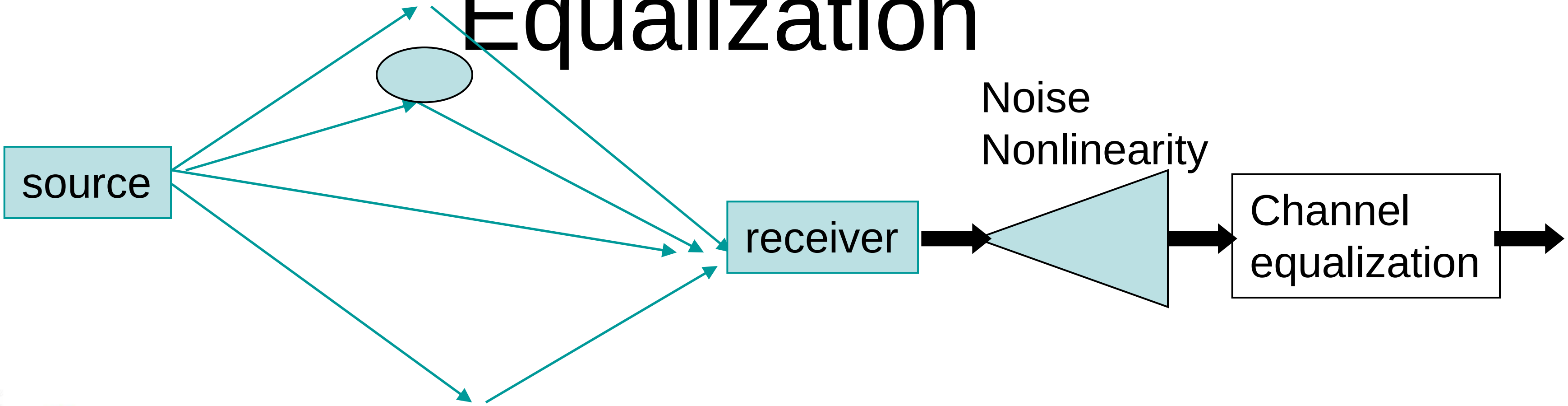
Opto-Electronic Reservoir

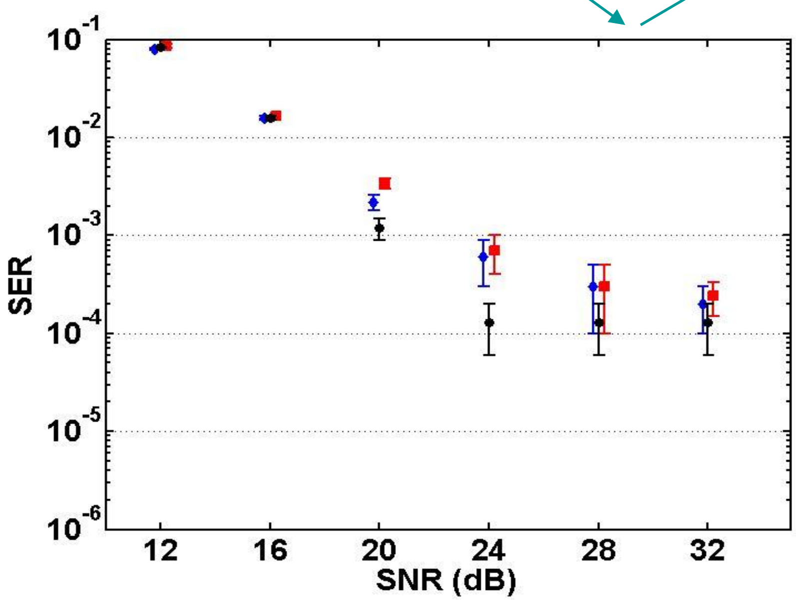


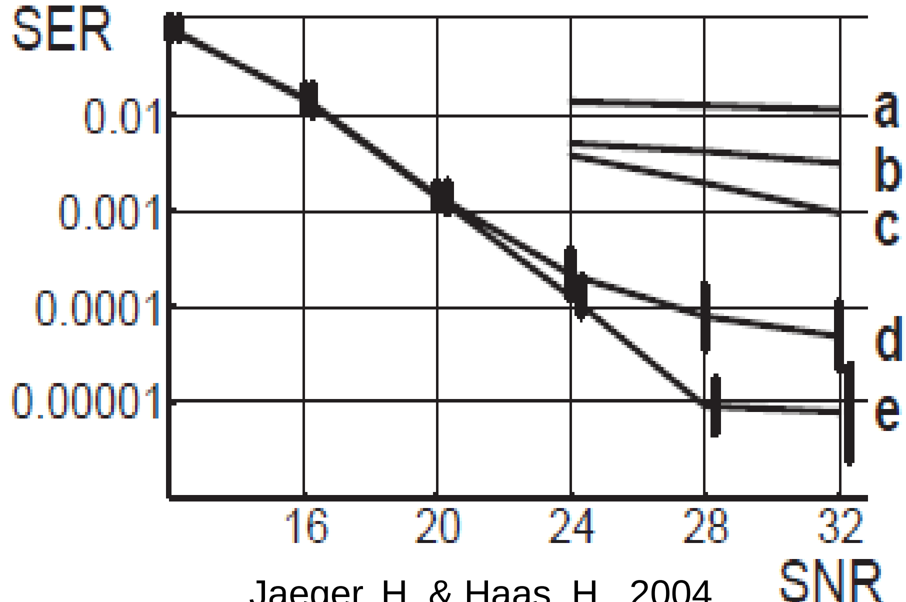
In interval T'
- Take N samples
- Optimize linear combination

Period T'≈T=8μs
N=50-200 internal nodes

Task 1: Non Linear Channel Equalization



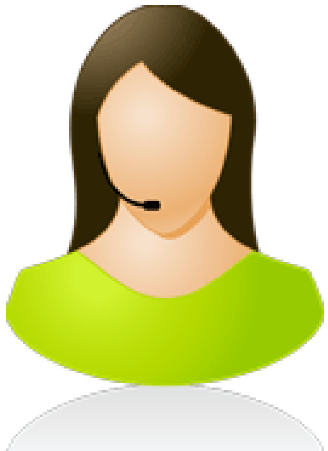




Jaeger, H. & Haas, H., 2004. *Science*

Our results (**two simulations** and **experiment**) comparable with average results (d) of Jaeger (same reservoir size).

Results:
2) Spoken Digit Recognition



•Five speakers pronounce several times digits from 0 to 9

•Goal: recognize the digits from a pre-processed version (cochlear model) of the audio files

- Word Error Rate of 0.4% – that's 2 errors on the whole 500 digits set

Method	WER
Opto-Electronic (Paquot 11)	0.4%
Electronic (Appeltant 11)	0.2%
First reservoir (Verstraeten 05)	4.3%
Verstraeten 06	0.2%
SPHINX 4	0.55%

Experiment: Summary

- First analog RC with performance comparable to digital RC
  - Very simple architecture: Non Linear node with delayed feedback

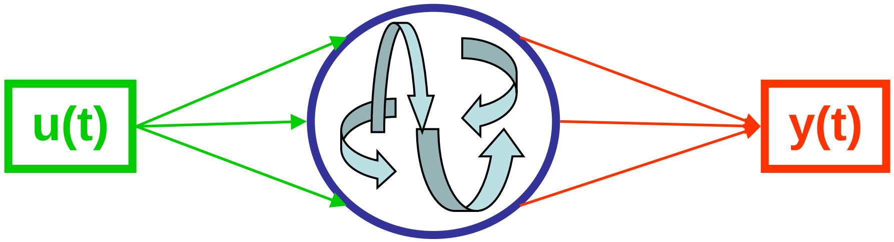
- Perspectives:
  - Remove electronic pre- and post-processing
  - Low energy / High speed

Recipe to build experimental reservoirs

- Non Linear Dynamical System
  - 20-100 internal variables
  - Driven by external input u(t)
- Adjustable parameters
  - Feedback strength → threshold of instability
  - Input gain → non linearity of reservoir
- Readout
  - Measure internal variables
  - Adjust weights of linear combination

- Controled feedback loops→autonomous behaviour

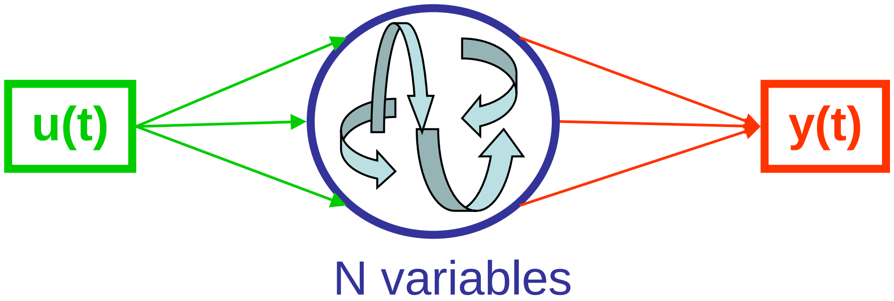
Linear and Non Linear Memory Capacity



- Characterise the information processing by recurrent dynamical systems?
  - Are some systems better than others?
  - Can we associate performance on tasks to properties of the system?

Linear Memory Capacity
(Jaeger 2002)

IID:Independent Identically Distributed



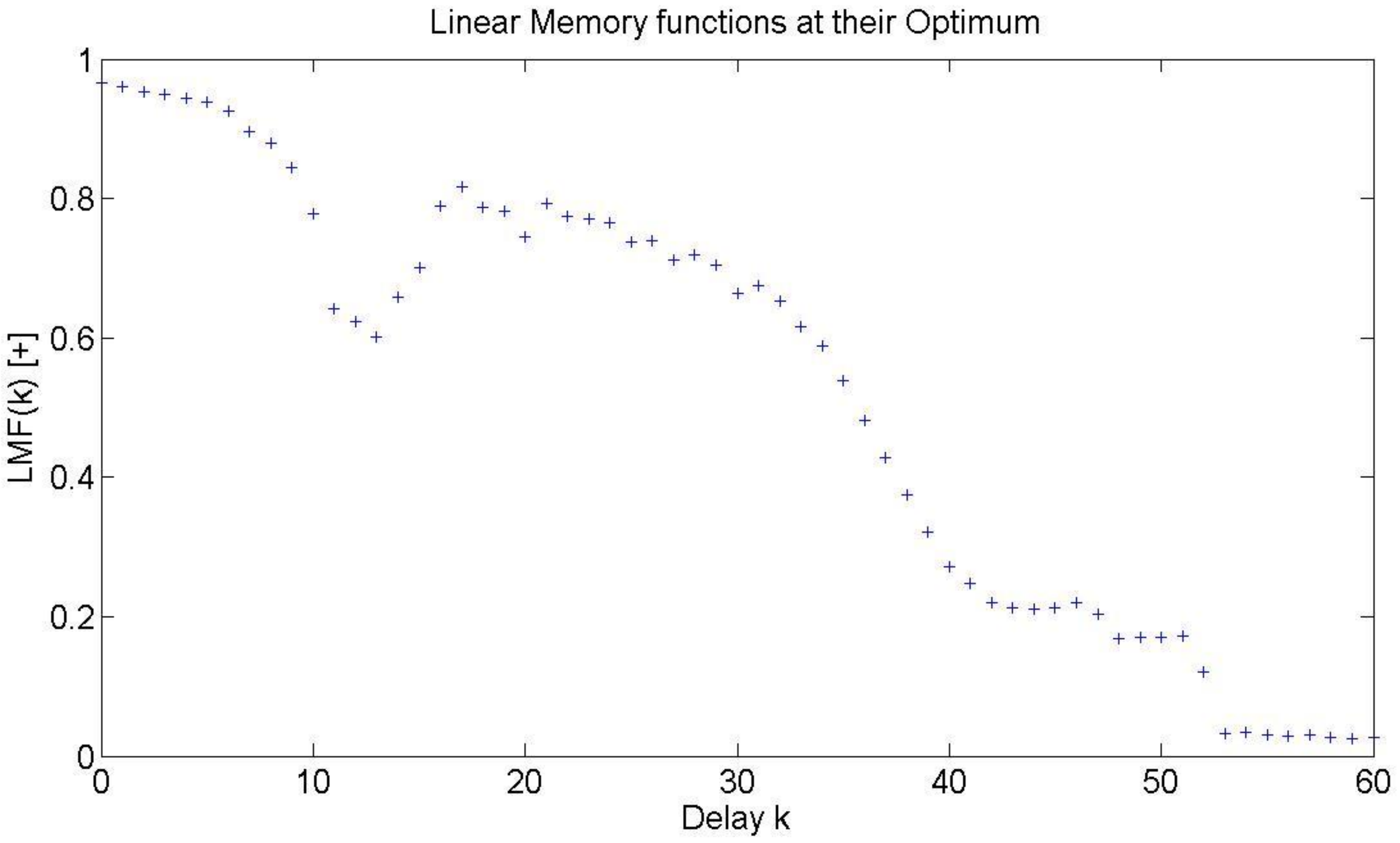
Goal: Reproduce Input k steps in the past:
y(t)≈u(t-k)

$C_L(k)$ = ability to reproduce input k steps in the past

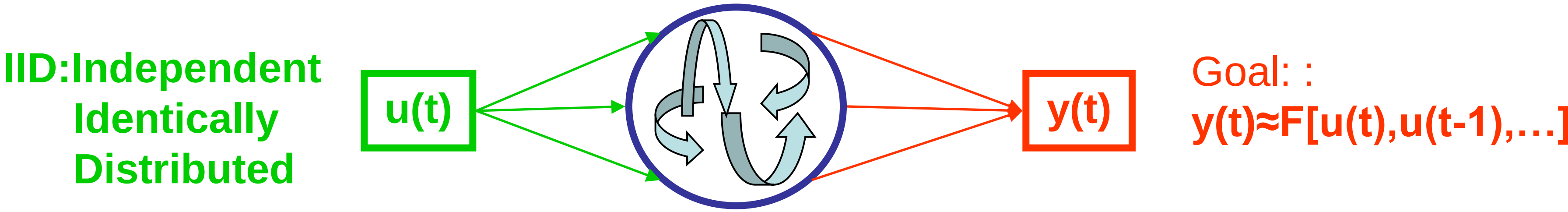
$0 \le C_L(k) \le 1$

$\sum_k C_L(k) \le N$ =size of reservoir (Equality for linear reservoirs and linearly independent internal states)

Linear Memory Function



Non Linear Memory Capacity



$C(F)$ = ability to reproduce F[u(t-1),u(t-2),...]

$0 \le C(F) \le 1$

$$\sum_{F \in \text{Complete Set}} C(F) \le N = \text{size of Reservoir (Equality for reservoirs with FADING MEMORY}$$

and linearly independent internal states)

Hilbert Space of Fading Memory Functions

$u(t)$ i.i.d.

One Input

$P(u)$ = probability distribution over u

F(u) = real valued function of u

Hilbert space: $\langle \mathrm{F,G} \rangle = \int du P(u) F(u) G(u)$

Many Inputs

$u^{-\infty} = u(t), u(t-1), u(t-2), \ldots$

$F[u^{-\infty}] = F[u(t), u(t-1), u(t-2), \ldots]$ = function all of previous inputs

H_{FM} =Hilbert space of Fading Memory Functions:

$$\langle \mathrm{F,G} \rangle = \int du(t) du(t-1) du(t-2) \ldots F[u(t), u(t-1), u(t-2), \ldots] G[u(t), u(t-1), u(t-2), \ldots]$$

$$= \int du^{-\infty} P(u^{-\infty}) F[u^{-\infty}] G[u^{-\infty}]$$

Reservoir with Fading Memory

Dynamical System has Fading Memory: $x_i(t) = x_i[u^{-\infty}] \in H_{FM}$

choose orthonormal basis:

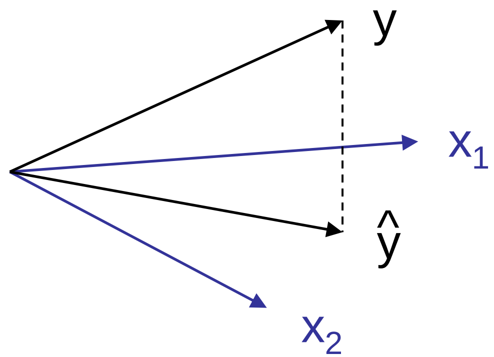
$$\tilde{\mathrm{x}}_{\mathrm{i}}[u^{-\infty}] = \sum_j c_{ij} x_j [u^{-\infty}]$$

$$\left\langle \tilde{\mathrm{x}}_{\mathrm{i}}, \tilde{\mathrm{x}}_{\mathrm{j}} \right\rangle = \delta_{ij}$$

Target function= $\mathrm{y}[u^{-\infty}]$

$\hat{y}[u^{-\infty}] = \sum_i W_i x_i [u^{-\infty}] =$ best linear approximation of $\mathrm{y}[u^{-\infty}]$

$\hat{y}[u^{-\infty}] = \sum_i \left\langle \tilde{\mathrm{x}}_{\mathrm{i}}, y \right\rangle x_i [u^{-\infty}] =$ projection of $y(t)$ onto space spanned by $x_i[u^{-\infty}]$



Non Linear Memory Capacity =capacity to reconstruct y[u$^{-\infty}$]

Target function= y[$u^{-\infty}$]

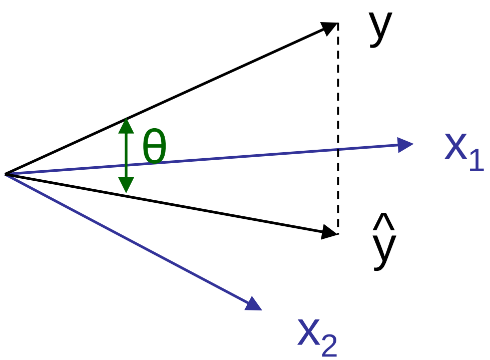
$$\hat{y}[u^{-\infty}] = \sum_i W_i x_i[u^{-\infty}] = \text{best linear approximation of y}[u^{-\infty}]$$

$$\hat{y}[u^{-\infty}] = \sum_i \langle \tilde{\mathrm{x}}_\mathrm{i}, y\rangle x_i[u^{-\infty}] = \text{ projection of } y(t) \text{ onto space spanned by } x_i[u^{-\infty}]$$

$$C[y] = \frac{\sum_i \langle \tilde{\mathrm{x}}_\mathrm{i}, y\rangle^2}{\langle y, y\rangle^2} = \frac{\text{norm square of best linear approximation of y}}{\text{norm square of y}}$$

$$0 \le C[y] \le 1$$

$\boxed{C[y]=\cos^2(\theta)}$



Completeness

$$C[y] = \frac{\sum_i \left\langle \tilde{\mathrm{x}}_{\mathrm{i}}, y \right\rangle^2}{\left\langle y, y \right\rangle^2}$$

$\{y_k\}$ = complete orthonormal basis of H_{FM}

$$\left\langle y_k, y_{k'} \right\rangle = \delta_{kk'}$$

$$\sum_{k \in Basis} C[y_k] = \sum_{k \in Basis} \sum_i \left\langle \tilde{\mathrm{x}}_{\mathrm{i}}, y_k \right\rangle = \sum_i \left(\sum_{k \in Basis} \left\langle \tilde{\mathrm{x}}_{\mathrm{i}}, y_k \right\rangle \right) = \sum_i 1 = N$$

In practice

$$u(t) \in [-1, +1]$$

$$\{y_k\} = \{\text{Legendre Polynomials and products of Legendre Polynomials}\}$$

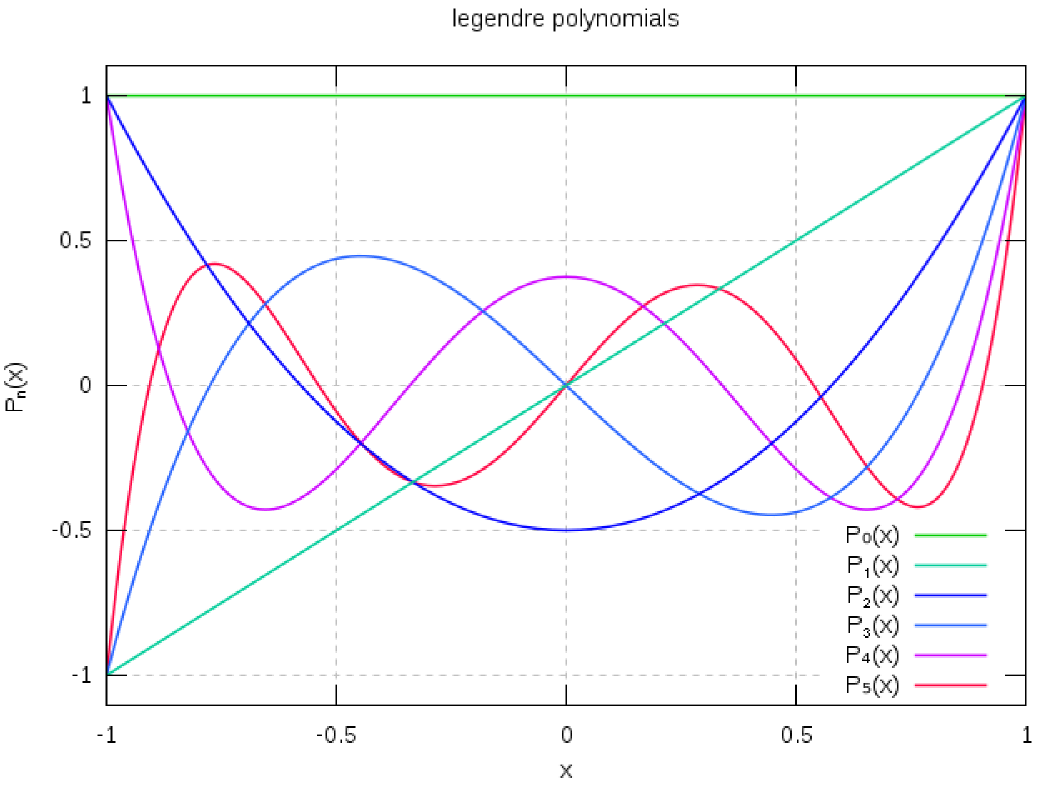
$$P_0 = 1$$

$$P_{1;k} = u(t-k)$$

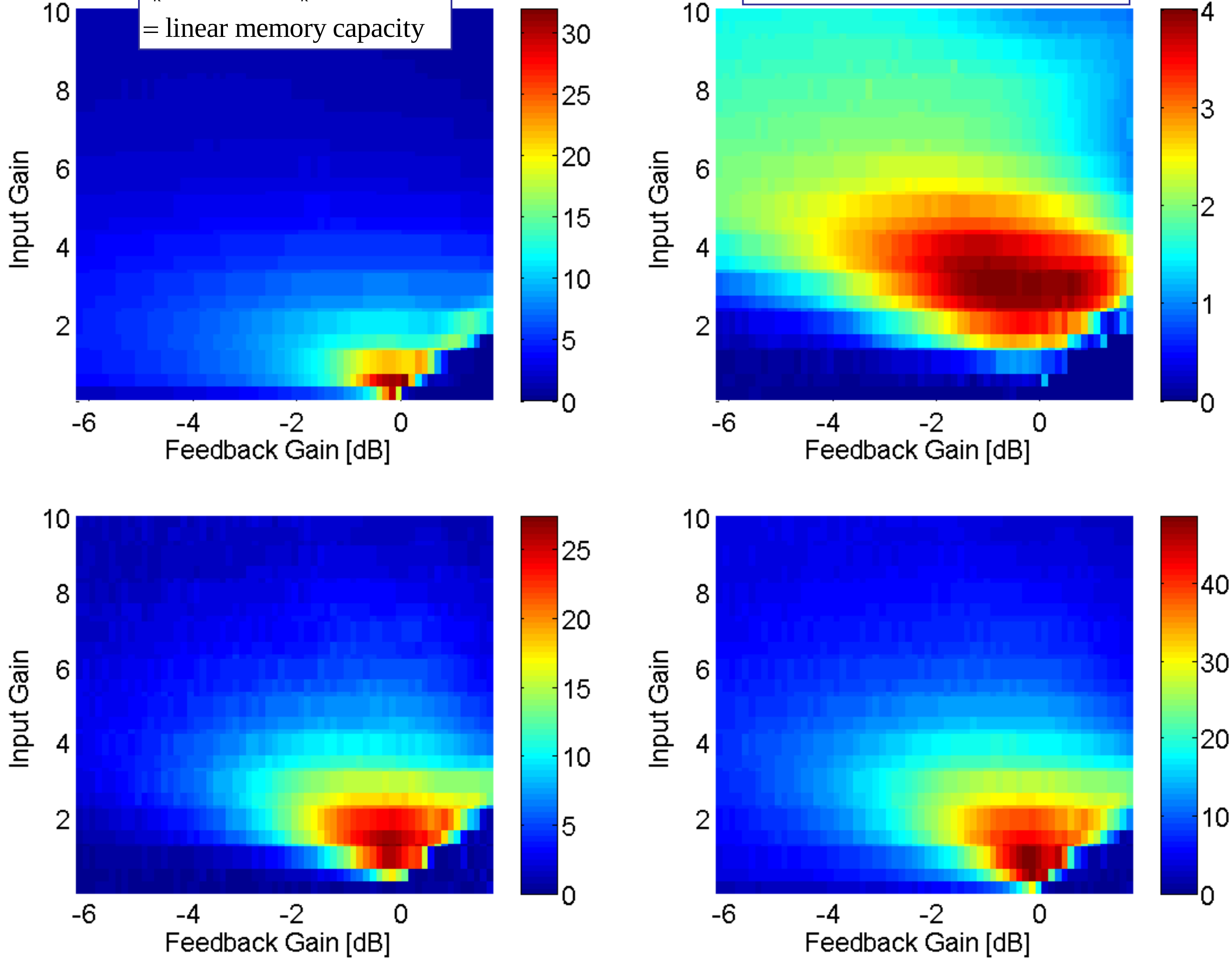
$$P_{2;k} = \frac{1}{2}\left(3u^2(t-k) - 1\right)$$

$$P_{11;kk'} = u(t-k)u(t-k')$$

$$etc...$$



In practice: Experimental Data (N=50)



$$\sum_k C\left[P_{1;k}\right] = \sum_k C\left[u(t-k)\right]$$
= linear memory capacity

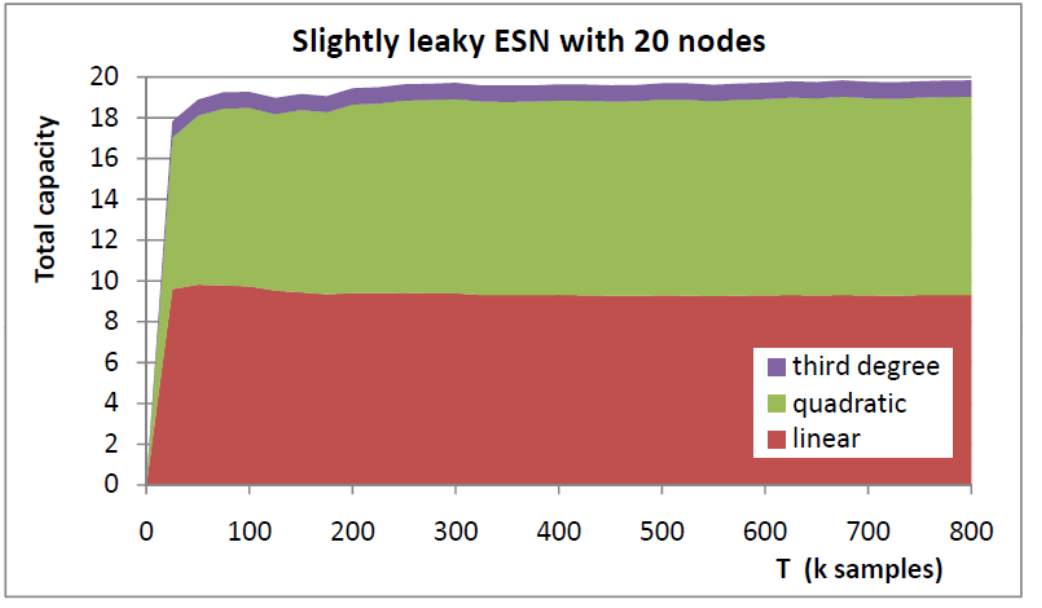
$$\sum_k C\left[P_{2;k}\right] = \sum_k C\left[\frac{3u^2(t-k)-1}{2}\right]$$

$$\sum_{kk'} C\left[P_{11;kk'}\right] = \sum_{kk'} C\left[u(t-k)u(t-k')\right]$$

Sum of capacities of order 1 and 2

Linear & Non Linear Memory Capacity

- New tool to characterise information processing by dynamical systems
  - !!!Convergence!!!



- Perspective: relate non linear memory capacity to performance on tasks

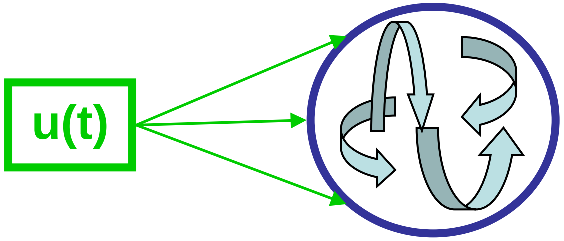
- Questions??

Mean Field Theory

- Why a Theory of Reservoir Computing?
  - Fundamental Curiosity: what enables computation?
  - Help in designing better reservoirs

- Aim: predict *ab initio* the memory capacities, the performance on tasks

Mean Field Theory

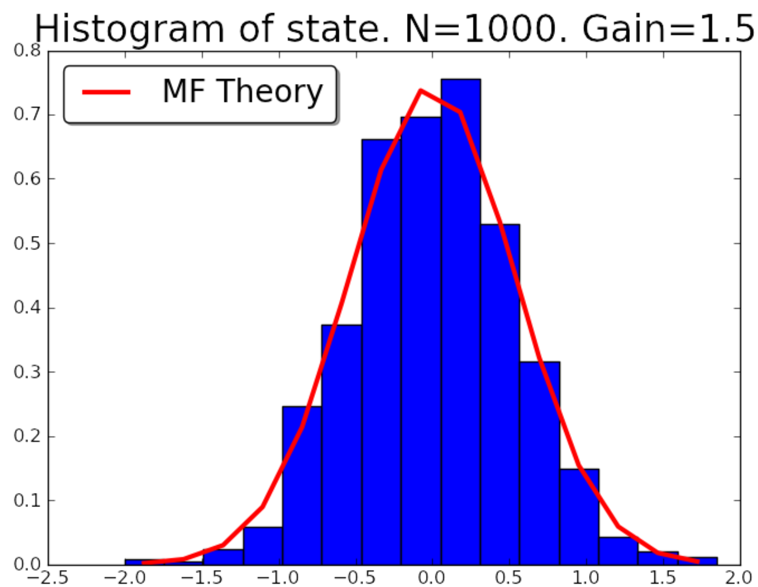


$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

Key Insight:

Neglect correlations between $x_i(t)$

Then $\sum_j \alpha_{ij} x_j(t)$ = sum of independent variables = GAUSSIAN

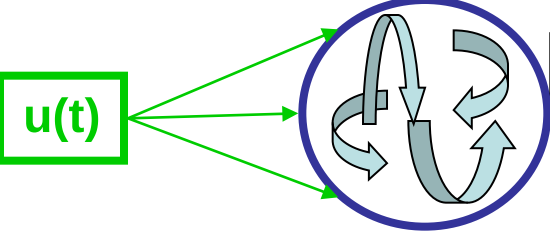


Mean Field Theory: No Source

$$\boxed{x_i\left(t+1\right) = \tanh\left(\sum_j \alpha_{ij} x_j(t)\right)}$$

$$a_i(t) = \sum_j \alpha_{ij} x_j(t) \quad \sim \quad N\left(0, \Sigma^2(t)\right)$$

$$x_i\left(t+1\right) = \tanh\left(a_i(t)\right) \qquad \rightarrow \text{ compute } \quad \mathrm{var}(x_i(t)) = \sigma^2(t)$$



Mean Field Theory

$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

$$a_i(t) = \sum_j \alpha_{ij} x_j(t) \quad \square \quad N\left(0, \Sigma^2(t)\right)$$

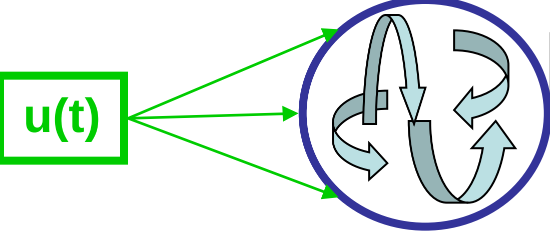
$$x_i(t+1) = \tanh\left(a_i(t)\right) \quad \to \text{ compute } \quad \operatorname{var}(x_i(t)) = \sigma^2(t)$$

Recurrence for the Variances:

$$\Sigma^2(t) = g\sigma^2(t) + \beta^2 u^2(t) \qquad \text{g=N*var}(\alpha_{\text{ij}}) \qquad \beta^2 = \operatorname{var}(\beta_i)$$

$$\sigma^2(t+1) = F\left(\Sigma^2(t)\right)$$

$$F\left(\Sigma^2\right) = \int da \tanh^2(a) \frac{\exp\left[-a^2 / 2\Sigma^2\right]}{\sqrt{2\pi\Sigma}}$$



Mean Field Theory

$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

Recurrence for the Variances:

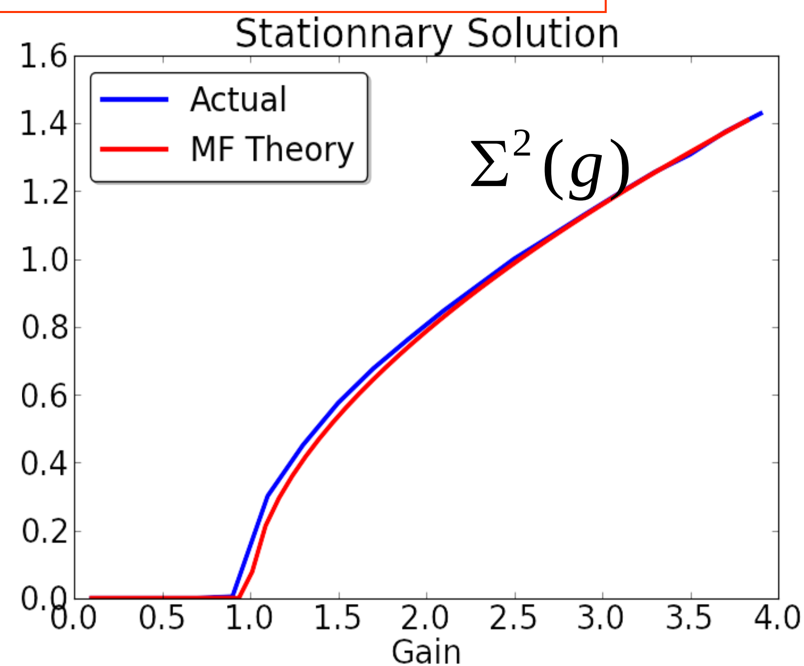
$$\Sigma^2(t) = g\sigma^2(t) + \beta^2 u^2(t) \qquad \text{g=N*var}(\alpha_{ij}) \qquad \beta^2 = \text{var}(\beta_i)$$

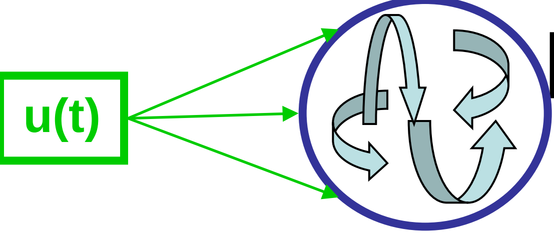
$$\sigma^2(t+1) = F\left(\Sigma^2(t)\right)$$

$$F\left(\Sigma^2\right) = \int da \tanh^2(a) \frac{\exp\left[-a^2/2\Sigma^2\right]}{\sqrt{2\pi\Sigma}}$$

Σ^2

No Source





Mean Field Theory

$$x_i(t+1) = \tanh\left(\sum_j \alpha_{ij} x_j(t) + \beta_i u(t)\right)$$

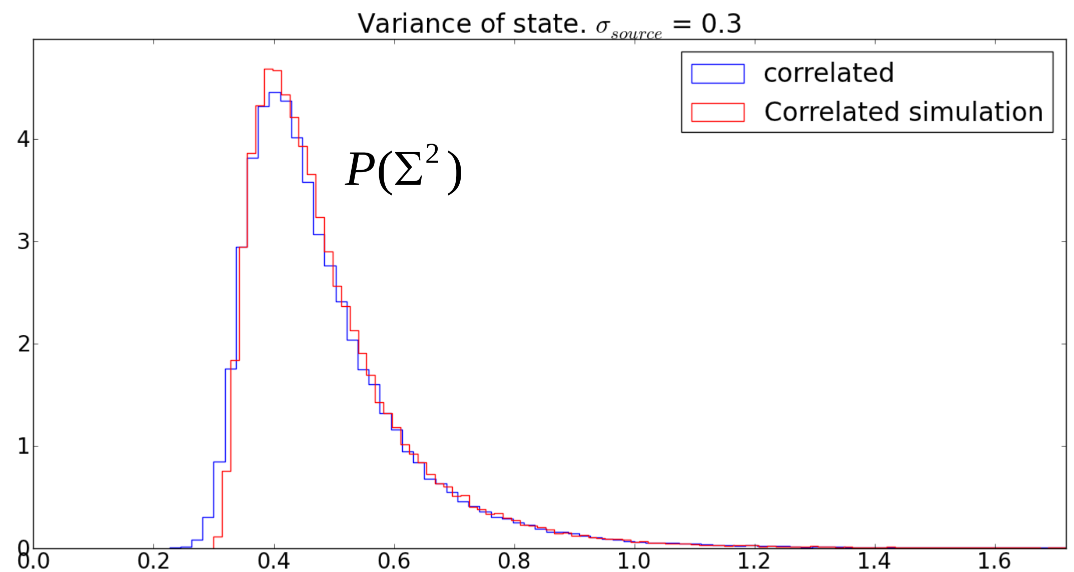
Recurrence for the Variances:

$$\Sigma^2(t) = g\sigma^2(t) + \beta^2 u^2(t) \qquad g = N^*\mathrm{var}(\alpha_{ij}) \qquad \beta^2 = \mathrm{var}(\beta_i)$$

$$\sigma^2(t+1) = F\left(\Sigma^2(t)\right)$$

$$F\left(\Sigma^2\right) = \int da \tanh^2(a) \frac{\exp\left[-a^2/2\Sigma^2\right]}{\sqrt{2\pi}\Sigma}$$

Correlated Source
→u(t) changes in time
→Probability distribution for $\sigma^2(t)$



Mean Field Theory: Computational Power

$$a_i(t) = \sum_j \alpha_{ij} x_j(t)$$

$$x_i(t+1) = \tanh\left(a_i(t) + \beta_i u(t)\right)$$

Consider two different inputs $u(t)$ and $u'(t)$

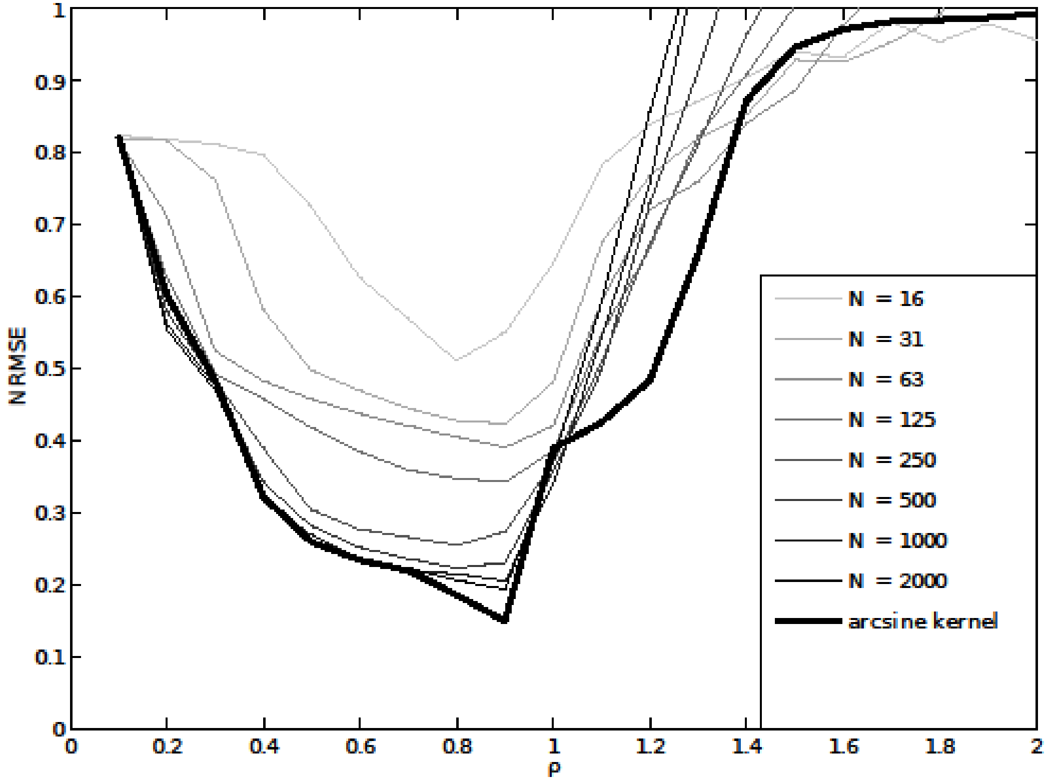
Mean field theory enables computation of the scalar product between $x_i[\mathrm{u}]$ and $x_i[u']$:

$$\sum_i \langle x_i[\mathrm{u}]\ , x_i[u']\ \rangle$$

From the scalar product, derive the performance on computational tasks

Herman and Schrauwen (2011)

Mean Field Theory: Computational Power



Performance on NARMA10 for reservoirs of different sizes versus spectral radius
Bold=mean field theory

Herman and Schrauwen (2011)

Summary

- First Experimental Reservoirs with performance comparable to digital implementation
  - Simple architecture: NL node + delay line
- Memory Capacities
  - New tool for characterising information processing in dynamical systems.
- Mean Field Theory = Initial steps towards understanding the dynamics of reservoirs