THE STANDARD ERROR OF CHAIN LADDER RESERVE ESTIMATES: RECURSIVE CALCULATION AND INCLUSION OF A TAIL FACTOR

BY

Thomas MACK

Munich Re, Munich

ABSTRACT

In Mack (1993), a formula for the standard error or chain ladder reserve estimates has been derived. In the present communication, a very intuitive and easily programmable recursive way of calculating the formula is given. Moreover, this recursive way shows how a tail factor can be implemented in the calculation of the standard error.

KEYWORDS

Chain Ladder, Standard Error, Recursive Calculation, Tail Factor

INTRODUCTION

Let C_{ik} denote the cumulative loss amount of accident year $i = 1, \ldots, n$ at the end of development year (age) $k = 1, \ldots, n$. The amounts C_{ik} have been observed for $k \leq n + 1 - i$ whereas the other amounts have to be predicted. The chain ladder algorithm consists of the stepwise prediction rule

$$\hat{C}_{i,k+1} = \hat{C}_{ik}\hat{f}_k$$

starting with $\hat{C}_{i,n+1-i} = C_{i,n+1-i}$. Here, the age-to-age factor $\hat{f}_k$ is defined by

$$\hat{f}_k = \sum_{i=1}^{n-k} w_{ik} C_{ik}^{\alpha} F_{ik} \Big/ \sum_{i=1}^{n-k} w_{ik} C_{ik}^{\alpha}, \qquad \alpha \in \{0;\ 1;\ 2\},$$

where

$$F_{ik} = C_{i,k+1}/C_{ik}, 1 \leq i \leq n,\ 1 \leq k \leq n-1,$$

are the individual development factors and where

$$w_{ik} \in [0;\ 1]$$

are arbitrary weights which can be used by the actuary to downweight any outlying F_{ik}. Normally, $w_{ik} = 1$ for all i, k. Then, $\alpha = 1$ gives the historical chain ladder age-to-age factors, $\alpha = 0$ gives the straight average of the observed individual development factors and $\alpha = 2$ is the result of an ordinary regression of $C_{i,k+1}$ against C_{ik} with intercept 0. Note that in case $C_{ik} = 0$, the corresponding two summands should be omitted when calculating $\hat{f}_k$.

The above stepwise rule finally leads to the prediction

$$\hat{C}_{in} = C_{i,n+1-i}\hat{f}_{n+1-i} \cdot \ \ldots \ \cdot \hat{f}_{n-1}$$

of C_{in} but – because of limited data – the loss development of accident year i does not need to be finished at age n. Therefore, the actuary often uses a tail factor $\hat{f}_{ult} > 1$ in order to estimate the ultimate loss amount $C_{i,ult}$ by

$$\hat{C}_{i,ult} = \hat{C}_{in}\hat{f}_{ult} \ .$$

A possible way to arrive at an estimate for the tail factor is a linear extrapolation of $\ln(\hat{f}_k - 1)$ by a straight line $a \cdot k + b$, $a < 0$, together with

$$\hat{f}_{ult} = \prod_{k=n}^{\infty} \hat{f}_k \ .$$

However, the tail factor used must be plausible and, therefore, the final tail factor is the result of the personal assessment of the future development by the actuary.

In Mack (1993), a formula for the standard error of the predictor $\hat{C}_{in}$ was derived for $\alpha = 1$ and all $w_{ik} = 1$. In the next section, this formula is generalized for the cases $\alpha = 0$ or $\alpha = 2$ and $w_{ik} < 1$. Furthermore, a recursive way of calculating the standard error is given. In the last section it is shown how a tail factor can be implemented in the calculation of the standard error.

Recursive Calculation of the Standard Error

In order to calculate the standard error of the prediction $\hat{C}_{in}$ as compared to the true loss amount C_{in}, Mack (1993) introduced an underlying stochastic model (for $\alpha = 1$ and $w_{ik} = 1$) which is given here in its more general form without the restriction on α and w_{ik}:

(CL1) $E(F_{ik}|C_{i1}, \ldots, C_{ik}) = f_k$, $\quad 1 \le i \le n,\ 1 \le k \le n-1$,

(CL2) $\mathrm{Var}(F_{ik}|C_{i1}, \ldots, C_{ik}) = \frac{\sigma_k^2}{w_{ik} C_{ik}^{\alpha}}$, $\quad 1 \le i \le n,\ 1 \le k \le n-1$,

(CL3) The accident years $(C_{i1}, \ldots, C_{in})$, $1 \le i \le n$, are independent.

Within this model, the following statements hold (see Mack (1993)):

$E(C_{i,k+1}|C_{i1}, \ldots, C_{ik}) = C_{ik}f_k$,

$E(C_{in}|C_{i1}, \ldots, C_{i,n+1-i}) = C_{i,n+1-i}f_{n+1-i} \cdot \ldots \cdot f_{n-1}$,

$\hat{f}_k$ is the minimum variance unbiased linear estimator of f_k (for w_{ik} and α given),

$\hat{f}_{n+1-i} \cdot \ldots \cdot \hat{f}_{n-1}$ is an unbiased estimator of $f_{n+1-i} \cdot \ldots \cdot f_{n-1}$.

Therefore, the model CL1-3 can be called underlying the chain ladder algorithm. Furthermore,

$$\hat{\sigma}_k^2 = \frac{1}{n-k-1}\sum_{i=1}^{n-k} w_{ik}C_{ik}^{\alpha}(F_{ik} - \hat{f}_k)^2, \qquad 1 \le k \le n-2,$$

is an unbiased estimator for $\hat{\sigma}_k^2$ which can be supplemented by

$$\hat{\sigma}_{n-1}^2 = \min\left(\hat{\sigma}_{n-2}^4/\hat{\sigma}_{n-3}^2,\ \min\left(\hat{\sigma}_{n-3}^2, \hat{\sigma}_{n-2}^2\right)\right).$$

Based on this model for $\alpha = 1$ and all $w_{ik} = 1$, Mack (1993) derived the following formula for the standard error of $\hat{C}_{in}$, which at the same time is the standard error of the estimate $\hat{R}_i = \hat{C}_{in} - C_{i,n+1-i}$ for the claims reserve $R_i = C_{in} - C_{i,n+1-i}$:

$$\left(\text{s.e.}(\hat{C}_{in})\right)^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{n-k} C_{jk}}\right).$$

This formula can be rewritten as

$$(*) \qquad (\text{s.e.}(\hat{C}_{in}))^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \left((\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2\right)\Big/\hat{f}_k^2$$

where $(\text{s.e.}(F_{ik}))^2$ is an estimate of $\text{Var}(F_{ik}|C_{i1}, \ldots, C_{ik})$ and $(\text{s.e.}(\hat{f}_k))^2$ is an estimate of

$$\text{Var}(\hat{f}_k) = \sigma_k^2 \Big/ \sum_{j=1}^{n-k} w_{jk}C_{jk}^{\alpha}.$$

In this last form, formula (*) also holds for $\alpha = 0$ and $\alpha = 2$ and any $w_{ik} \in [0;\ 1]$ as can be seen by applying the proof for $\alpha = 0$ and $w_{ik} = 1$ analogously. Moreover, from this proof the following easily programmable recursion can be gathered:

$$(\text{s.e.}(\hat{C}_{i,k+1}))^2 = \hat{C}_{ik}^2\left((\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2\right) + (\text{s.e.}(\hat{C}_{ik}))^2\hat{f}_k^2$$

with starting value s.e.$(\hat{C}_{i,n+1-i}) = 0$. This recursion, which leads to formula (*), is very intuitive: $(\text{s.e.}(F_{ik}))^2$ estimates the (squared) random error $\text{Var}(F_{ik}) = E(F_{ik} - f_k)^2$, i.e. the mean squared deviation of an individual F_{ik} from its true mean f_k, and $(\text{s.e.}(\hat{f}_k))^2$ estimates the (squared) estimation error $\text{Var}(\hat{f}_k) = E(\hat{f}_k - f_k)^2$, i.e. the mean squared deviation of the estimated average $\hat{f}_k$ of the F_{ik}, $1 \le i \le n$, from the true f_k. From this interpretation it is clear that we have $\text{Var}(\hat{f}_k) < \text{Var}(F_{ik})$ if $\hat{f}_k$ is unbiased and accident year i belongs to those years over which $\hat{f}_k$ is the average.

Inclusion of a Tail Factor

The recursion can immediately be extended to include a tail factor $\hat{f}_{ult}$:

$$\left(\text{s.e.}(\hat{C}_{i,ult})\right)^2 = \hat{C}_{in}^2\left((\text{s.e.}(F_{i,ult}))^2 + (\text{s.e.}(\hat{f}_{ult}))^2\right) + (\text{s.e.}(\hat{C}_{in}))^2 \hat{f}_{ult}^2$$

and an actuary who develops an estimate for f_{ult} should also be able to develop an estimate s.e.$(\hat{f}_{ult})$ for its estimation error $\sqrt{\text{Var}(\hat{f}_{ult})}$ (How far will $\hat{f}_{ult}$ deviate from f_{ult}?) and an estimate s.e.$(F_{i,ult})$ for the corresponding random error $\sqrt{\text{Var}(F_{i,ult})}$ (How far will any individual $F_{i,ult}$ deviate from f_{ult} on average?). Note that at F_{ik}, f_k and σ_k, index $k = ult$ is the same as $k = n$ whereas at C_{ik} we have $ult = n + 1$.

As a plausibility consideration, we will usually be able to find an index $k < n$ with

$$\hat{f}_{k-1} > \hat{f}_{ult} > \hat{f}_k \; .$$

Then we can check whether it is reasonable to assume that the inequalities

$$\text{s.e.}(\hat{f}_{k-1}) > \text{s.e.}(\hat{f}_{ult}) > \text{s.e.}(\hat{f}_k)$$

and

$$\text{s.e.}(F_{i,k-1}) > \text{s.e.}(F_{i,ult}) > \text{s.e.}(F_{ik})$$

hold, too, or whether there are reasons to fix s.e.$(\hat{f}_{ult})$ and/or s.e.$(F_{i,ult})$ outside these inequalities.

As an example, we take the data of Table 4 from Mack (1993). From these (using $\alpha = 1$ and all $w_{ik} = 1$, we get the results given in Table 1 for $k = 1, \ldots, 8$:

TABLE 1

PARAMETER ESTIMATES FOR THE DATA OF TABLE 4 OF MACK (1993)

k	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*ult*
$\hat{f}_k$	11.10	4.092	1.708	1.276	1.139	1.069	1.026	1.023	1.05
s.e.$(\hat{f}_k)$	2.24	0.517	0.122	0.051	0.042	0.023	0.015	0.012	0.02
s.e.(F_{3k})	7.38	1.89	0.357	0.116	0.078	0.033	0.015	0.007	0.03
$\hat{\sigma}_k$	1337	988.5	440.1	207.0	164.2	74.60	35.49	16.89	71.0

The parameter estimates $\hat{f}_k$ and $\hat{\sigma}_k$ for $1 \leq k \leq 8$ are the same as in Mack (1993). From these, the estimates s.e.$(\hat{f}_k) = \hat{\sigma}_k \Big/ \sqrt{\sum_{j=1}^{n-k} C_{jk}}$ and s.e. $(F_{ik}) = \hat{\sigma}_k / \sqrt{C_{ik}}$ for $k \leq n+1-i$ or s.e.$(F_{ik}) = \hat{\sigma}_k \Big/ \sqrt{\hat{C}_{ik}}$ for $k > n+1-i$ are calculated which give the estimation error and the random error, respectively. Note that the random error s.e.(F_{ik}) varies also over the accident years because model assumption CL2 states that for $\alpha = 1$ the variance of the individual development factor F_{ik} is the smaller the greater the previous claims amount (volume) C_{ik} is. Therefore, only the values of s.e.(F_{ik}) for accident year $i = 3$ of average volume are given. The last column of Table 1 shows a possible tail estimation by the actuary: He expects a tail factor of 1.05 with an estimation error of ± 0.02 and a random error of ± 0.03 for accident year $i = 3$. From this, the estimate $\hat{\sigma}_{ult} = \text{s.e.}(F_{3,ult})\sqrt{\hat{C}_{3,n}} = 71.0$ has been deduced and is used to calculate s.e.$(F_{i,ult})$ for the other accident years. These tail estimates fit well between the columns $k = 6$ and $k = 7$. (Note that the extrapolated estimate for σ_8 leads to a rather small s.e.$(F_{3,8})$ as compared to s.e.$(\hat{f}_8)$. This is due to the fact that $\hat{f}_8$ does not follow a loglinear decay as it was assumed for the calculation of σ_8. Therefore, an estimate $\hat{\sigma}_8 \approx 30$ would have been more reasonable.)

Table 2 shows the resulting estimates for the ultimate claims amounts. The rows $\hat{C}_{i9}$ and s.e.$(\hat{C}_{i9})$ are identical to the results given in Mack (1993). Row $\hat{C}_{i,ult}$ is 5% higher than row $\hat{C}_{i,9}$ and the last row s.e.$(\hat{C}_{i,ult})$ shows the standard errors which result from the formula given above.

TABLE 2

ESTIMATED ULTIMATE CLAIMS AMOUNTS AND THEIR STANDARD ERRORS (ALL AMOUNTS IN 1000S)

i	1	2	3	4	5	6	7	8	9
$\hat{C}_{i9}$	1950	4219	5608	7698	7216	9563	5442	3241	1660
$\hat{C}_{i,ult}$	2048	4420	5888	8073	7577	10041	5714	3403	1743
s.e.$(\hat{C}_{i9})$	0	61	140	319	596	1038	1298	1806	2182
s.e.$(\hat{C}_{i,ult})$	107	180	250	418	670	1128	1377	1902	2293

Finally, we give a recursive formula for the total reserve of all accident years together:

$$\left(\text{s.e.}\left(\sum_{i=n+1-k}^{n} \hat{C}_{i,k+1}\right)\right)^2 = \left(\text{s.e.}\left(\sum_{i=n+2-k}^{n} \hat{C}_{ik}\right)\right)^2 \cdot \hat{f}_k^2 +$$
$$+ \sum_{i=n+1-k}^{n} \hat{C}_{ik}^2 \cdot \left(\text{s.e.}(F_{ik})\right)^2 + \left(\sum_{i=n+1-k}^{n} \hat{C}_{ik}\right)^2 \cdot \left(\text{s.e.}(\hat{f}_k)\right)^2$$

starting at $k = 1$. This formula can also be gathered from the proof of the corollary to Theorem 3 in Mack (1993). In the above example, this formula yields

$$\text{s.e.}\left(\sum_{i=1}^{9} \hat{C}_{i,ult}\right) = 4054$$

as standard error of the ultimate total claims amount $\sum_{i=1}^{9} \hat{C}_{i,ult} = 48906$ (amounts in 1000s).

REFERENCE

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THOMAS MACK
Münchener Rückversicherungs-Gesellschaft
Königinstrasse 107
D-80791 München
e-mail: tmack@munichre.com