Bootstrap Methodology in Claim Reserving

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Abstract

In this paper, we use the bootstrap technique to obtain prediction errors for different claim reserving methods, namely methods based on the chain ladder technique and on generalised linear models. We discuss several forms of performing the bootstrap and illustrate the different solutions using the data set from Taylor and Ashe (1983) which has already been used by several authors.

Keywords

Claim reserving, bootstrap, generalised linear models.

1. Introduction

The prediction of an adequate amount to face the responsibilities assumed by an insurance company is a major subject in actuarial science. Despite its well-known limitations, the chain ladder technique (see for instance Taylor (2000) for a presentation of this technique) is the most widely applied claim reserving method. Moreover, in recent years, considerable attention has been given to discuss possible relationships between the chain ladder and various stochastic models (Mack (1993), (1994), Mack and Venter (2000), Verrall (1991), (2000), Renshaw and Verrall (1994), England and Verrall (1999), etc.).

The bootstrap technique has proved to be a very usefull tool in many fields and can be particularly interesting to assess the variability of the claim reserving predictions and to construct upper limits at an adequate confidence level. Some applications of the bootstrap technique to claim reserving can be found in Lowe (1994), England and Verrall (1999) and in Taylor (2000). However, the definition of the propper residuals to base the bootstrap methodology in the claim reserving process is still an open subject as well as the particular technique to use, when bootstrapping, to obtain upper limits for the predictions.

The main purpose of this paper is to discuss those different methods when combined with different stochastic models and to identify the most important differences in a benchmark example, the data set provided in Taylor and Ashe (1983) which has been used by many authors.

The problem of claim reserving can be summarised in the following way: Given the available information about the past, how can we obtain an estimate of the future payments (or eventually the number of claims to be reported) due to claims occurred in those years? Furthermore we need to determine a prudential margin which is to say we want to estimate an upper limit for the reserve with an adequate level of confidence.

Figure 1 – Pattern of the available data

Origin	Development year						
year	1	2	…	j	…	$n-1$	n
1	C_{11}	C_{12}	...	C_{1j}	...	$C_{1,n-1}$	C_{1n}
2	C_{21}	C_{22}	...	C_{2j}	...	$C_{2,n-1}$	
…	...	...	...	...			
i	C_{i1}	C_{i2}	...	$C_{i,n+1-i}$			
…							
$n-1$	$C_{n-1,1}$	$C_{n-1,2}$					
n	$C_{n,1}$						

Let C_{ij} represent either the incremental claim amounts or the number of claims arising from accident year *i* and development year *j* and let us assume that we are in year *n* and that we know all the past information, i.e. C_{ij} ($i = 1,2,...,n$ and $j = 1,2,...,n+1-i$). The available data presents a characteristic pattern, which can be seen in figure 1. From now on, and without loss of generality, we consider that the C_{ij} are the incremental claim amounts.

More than to predict the individual values, C_{ij} ($i = 2,3,...,n$ and $j = n+2-i, n+3-i,..., n$), we are interested in the prediction of the rows total, $C_{i\bullet}$ ($i = 2,3,...,n$), i.e. the amounts needed to face the claims occurred in year *i* and especially in the aggregate prediction, *C*, which represent the expected total liability. Keep in mind that we want to obtain upper limits to the forecasts and to associate a confidence level to those limits.

In section 2 we present a brief review of generalised linear models (GLM) and their application to claim reserving while in section 3 we discuss some aspects linked to the bootstrap methodology. Section 4 is devoted to the application of the different methods to the data set provided in Taylor and Ashe (1983) and to draw some conclusions.

2. Generalised Linear Models (GLM) and claim reserving methods

Following Renshaw and Verrall (1994) we can formulate most of the stochastic models for claim reserving by means of a particular family of generalised linear models (see McCullagh and Nelder (1989) for an introduction to GLM). The structure of those GLM will be given by

(1) $Y_{ij} \sim f(y;\mu_{ij},\phi)$ with independent Y_{ij}, $\mu_{ij} = E(Y_{ij})$ and where $f(.)$, the density (probability) function of Y_{ij} belongs to the exponential family. ϕ is a scale parameter.

(2) $\eta_{ij} = g(\mu_{ij})$, g(.) is called the link function.

(3) $\eta_{ij} = c + \alpha_i + \beta_j$ with $\alpha_1 = \beta_1 = 0$ to avoid over-parametrization.

It is common in claim reserving to consider three possible distributions for the variable C_{ij}: Lognormal, Gamma or Poisson. For models based on Gamma or Poisson distributions, the relations (1)-(3) define a GLM with $Y_{ij} = C_{ij}$ denoting the incremental claim amounts. The link function is $\eta_{ij} = \ln(\mu_{ij})$.

When we consider that the claim amounts follow a lognormal distribution, see Kremer (1982), Verrall (1991) or Renshaw (1994) among others, we observe that $Y_{ij} = \ln(C_{ij})$ has a normal distribution and consequently the relations (1)-(3) still continue to define a GLM for the logs of the incremental claim amounts. Now the link function is given by $\eta_{ij} = \mu_{ij}$ and the scale parameter is the variance of the normal distribution, i.e. $\phi = \sigma^2$.

The linear structure given by (3) implies that the estimates for some of the parameters depend on one observation only, i.e. there is a perfect fit for these observations. If the

available data follows the pattern shown in figure 1 it is straightforward to see that $\hat{\mu}_{1,n} = y_{1,n}$ and that $\hat{\mu}_{n,1} = y_{n,1}$.

When we define a GLM, we can omit the distribution of Y_{ij} and specify only the variance function and estimate the parameters by maximum quasi-likelihood (McCullagh and Nelder (1989)) instead of maximum likelihood. The estimators remain consistent. In this case, we replace the distributional assumption by $\text{var}(Y_{ij}) = \phi V(\mu_{ij})$, where $V(.)$ is the variance function. As we know, for the normal distribution $V(\mu_{ij}) = 1$, for the Poisson (eventually "over-dispersed" when $\phi > 1$) $V(\mu_{ij}) = \mu_{ij}$ and for the Gamma $V(\mu_{ij}) = \mu_{ij}^{\ 2}$.

It is well known that a GLM with the linear structure given by (3) and $V(\mu_{ij}) = \mu_{ij}$, i.e. a quasi over-dispersed Poisson distribution gives the same predictions as those obtained by the chain ladder technique (see Renshaw and Verrall (1994)). However, if we use a quasi over-dispersed Poisson, it is necessary to impose the constraint that the sum of incremental claims in each column is greater than 0. Note that the same constraints apply to quasi gamma models and that we need to impose a stronger constraints for the lognormal, gamma or Poisson models (each incremental value should be greater than 0).

As we said, in claim reserving, the figures of interest will be the aggregate value $Y_{\bullet} = \sum_{i=2}^{n} \sum_{j=n+2-i}^{n} Y_{ij}$ and the rows total $Y_{i\bullet} = \sum_{j=n+2-i}^{n} Y_{ij}$. The predicted values will be given by $\hat{\mu}_{\bullet} = \sum_{i=2}^{n} \sum_{j=n+2-i}^{n} \hat{\mu}_{ij}$ and $\hat{\mu}_{i\bullet} = \sum_{j=n+2-i}^{n} \hat{\mu}_{ij}$ respectively. To obtain those forecasts the procedure will be:

- Define the model
- Estimate the parameters c, α_i, β_j for $i, j = 1,2,...,n$ and ϕ.
- Obtain the fitted values $\hat{\mu}_{ij}$ ($i = 1,2,...,n$ and $j = 1,2,...,n+1-i$)
- Check the model (eventually)
- Obtain the "individual" forecasts $\hat{\mu}_{ij} = \hat{c} + \hat{\alpha}_i + \hat{\beta}_j$ ($i = 2,...,n$ and $j = n+2-i,...,n$)
- Obtain the forecasts for the rows reserve $\hat{\mu}_{i\bullet} = \sum_{j=n+2-i}^{n} \hat{\mu}_{ij}$ ($i = 2,...,n$)
- Obtain the forecast for the total reserve $\hat{\mu}_{\bullet} = \sum_{i=2}^{n} \hat{\mu}_{i\bullet}$

Obtaining estimates for the standard error of prediction is a more difficult task. Renshaw (1994), using first degree Taylor expansions, deduced some approximations to the standard errors. These values are given by:

- Standard error for the "individual" predictions:

$$(4) \qquad \sqrt{E(Y_{ij} - \hat{\mu}_{ij})^2} \cong \sqrt{\text{var}(Y_{ij}) + \text{var}(\hat{\mu}_{ij})} \cong \sqrt{\phi V(\mu_{ij}) + \mu_{ij}^{\ 2} \text{var}(\hat{\eta}_{ij})},$$

where $V(.)$ is the variance function and $\text{var}(\hat{\eta}_{ij})$ is obtained as a function of the covariance matrix of the estimators and is usually available from most statistical software. The term $\mu_{ij}^{\ 2}$ is a consequence of the link function chosen.

- Standard error for the row totals

$$\begin{aligned}\sqrt{E(Y_{i\bullet}-\hat{\mu}_{i\bullet})^2} &\cong \sqrt{\text{var}(Y_{i\bullet})+\text{var}(\hat{\mu}_{i\bullet})} \\ &\cong \sqrt{\sum_j \phi V(\mu_{ij}) + \sum_j \mu_{ij}^{\ 2}\,\text{var}(\hat{\eta}_{ij}) + 2\sum_{\substack{j_1,j_2 \\ j_2>j_1}} \mu_{ij_1}\,\mu_{ij_2}\,\text{cov}(\hat{\eta}_{ij_1},\hat{\eta}_{ij_2})}\end{aligned} \tag{5}$$

where the summations are made for the "individual" forecasts in each row.

- Standard error for the grand total

$$\sqrt{E(Y_{\bullet}-\hat{\mu}_{\bullet})^2} \cong \sqrt{\sum_{i,j} \phi V(\mu_{ij}) + \sum_{i,j} \mu_{ij}^{\ 2}\,\text{var}(\hat{\eta}_{ij}) + \sum_{\substack{i_1,j_1 \\ i_2,j_2 \\ i_1,j_1 \neq i_2,j_2}} \mu_{i_1 j_1}\,\mu_{i_2 j_2}\,\text{cov}(\hat{\eta}_{i_1 j_1},\hat{\eta}_{i_2 j_2})} \tag{6}$$

where the summations are made for all the "individual" forecasts.

Those estimates are difficult to calculate and are only approximate values, even in the hypothesis that the model is correctly specified. This is the main reason to take advantage of the bootstrap technique.

3. The Bootstrap Technique

The bootstrap technique is a particular resampling method used to estimate, in a consistent way, the variability of a parameter. This resampling method replaces theoretical deductions in statistical analysis by repeatedly resampling the "original" data and making inferences from the resamples.

Presentation of the bootstrap technique could be easily found in the literature (see for instance Efron and Tibshirani (1993), Shao and Tu (1995) or Davison and Hinkley (1997))

The bootstrap technique must be adapted to each situation. For the linear model ("classical" or generalised) it is common to adopt one of two possible ways:

- Paired bootstrap – The resampling is done directly from the observations (values of y and the corresponding lines of the X matrix in the regression model);
- Residuals bootstrap – The resampling is applied to the residuals of the model.

Despite the fact that the paired bootstrap is more robust than the residual bootstrap, only the later could be implemented in the context of the claim reserving, given the dependence between some observations and the parameter estimates.

To implement a bootstrap analysis we need to choose a model, to define an adequate residual and to use a bootstrap prediction procedure.

To define the most adequate residuals for the bootstrap it is important to remember two points:

- The resampling is based on the hypothesis that the residuals are independent and identically distributed;

- It's indifferent to resample the residuals or the residuals multiplied by a constant, as long as we take that fact into account in the generation of the pseudo data

Within the framework of a GLM we could use different types of residual (Pearson, deviance, Anscombe…). In this paper, our starting point will be the Pearson residuals defined by

$$(7) \qquad r_{ij}^{(P)} = \frac{y_{ij} - \hat{\mu}_{ij}}{\sqrt{\hat{\text{var}}(Y_{ij})}} = \frac{y_{ij} - \hat{\mu}_{ij}}{\sqrt{\hat{\phi}\hat{V}(\mu_{ij})}}.$$

Since ϕ is constant for the data set, we can take advantage of the second point and use

$$(8) \qquad r_{ij}^{(P^*)} = \frac{y_{ij} - \hat{\mu}_{ij}}{\sqrt{\hat{V}(\mu_{ij})}}$$

instead of $r_{ij}^{(P)}$ in the bootstrap procedure, that is to ignore, at this stage, the scale parameter. When using a normal model it is trivial to see that these residuals are equivalent to the classical residuals, $y_{ij} - \hat{\mu}_{ij}$, since $V(\mu_{ij}) = 1$.

However, these residuals need to be corrected since the available data combined with the linear structure adopted in the model leads to some residuals of value 0 (as we have already mentioned, in the typical case, $y_{1,n} = \hat{\mu}_{1,n}$ and $y_{n,1} = \hat{\mu}_{n,1}$). These residuals should not be considered as observations of the underlying random variable and consequently should not be considered in the bootstrap procedure.

As in the classical linear model (see Efron and Tibshirani (1993)), it is more adequate to work with the standardised Pearson residuals and not the Pearson residuals since only the former could be considered as identically distributed. As it is well known the standardised Pearson residual should be given by

$$(9) \qquad r_{ij}^{(P^{**})} = \frac{r_{ij}^{P}}{\sqrt{1 - h_{ij}}},$$

where the factor h_{ij} is the corresponding element of the diagonal of the "hat" matrix. For the "classical" linear model, this matrix is given by

$$H = X(X^T X)^{-1} X^T$$

and for a GLM it can be generalised using

$$H = X(X^T W X)^{-1} X^T W$$

where W is a diagonal matrix with generic element given by

$$w_{ii} = \left(V(\mu_i) \left(\frac{\partial \eta_i}{\partial \mu_i} \right)^2 \right)^{-1}$$

(see McCullagh and Nelder (1989)).

Considering the structure of our models (log link functions and quasi distributions), we have

$$w_{ii} = \mu_i^{2-\kappa},$$

with $\kappa = 1$ for the quasi over dispersed Poisson and $\kappa = 2$ for the quasi gamma model.

Note that similar procedure could be defined if we use another kind of residuals, namely the deviance residuals.

Let us now briefly discuss the bootstrap prediction procedure. To obtain an upper confidence limit for the forecasts of the aggregate values we can use two approaches:

The first one takes advantage of the Central Limit Theorem and consists on approximating the distribution of the reserve by means of a normal distribution with expected value given by the initial forecast (with the original data) and standard deviation given by the **standard error of prediction**. The main difference between the bootstrap estimation of these standard errors and the theoretical approximation obtained in the preceding section is that we estimate the variance of the estimator by means of a bootstrap estimate instead of using the (approximate) theoretical expression. For a detailed presentation of this method (in a general environment) see Efron and Tibshirani (1993). England and Verrall (1999) use this approach in claim reserving and suggest a bias correction for the bootstrap estimate to allow the comparison between the bootstrap standard error of prediction and the theoretical approximation presented in section 2. The bootstrap standard error of prediction will be given by

$$(10) \qquad SEP_b(\mu) = \sqrt{\hat{\phi}\hat{\mu} + \frac{N}{N-p}\left(SE_b(\hat{\mu})\right)^2}$$

where μ stands for the row totals, $\mu_{i\bullet}$ ($i = 2,3,...,n$), or the aggregate total, $\mu_{\bullet\bullet}$. $\hat{\phi}$ and $\hat{\mu}$ are quasi-maximum likelihood estimates of the corresponding parameters, N is the number of observations, p the number of parameters (usually $N = n(n-1)$ and $p = 2n-1$) while $SE_b(\hat{\mu})$ is the bootstrap estimate of the standard error of the estimator $\hat{\mu}$, i.e.

$$SE_b(\hat{\mu}) = \sqrt{\frac{1}{B}\sum_{k=1}^{B}\left(\mu_k^* - \hat{\mu}\right)^2}$$

where B is the number of bootstrap replicates and μ_k^* is the bootstrap estimate of μ in the k-th replicate ($k = 1,2,...,B$).

The second approach (see Davison and Hinkley (1997)) is more computer intensive since it require two resampling procedures in the same bootstrap "iteration" but the results should be more robust against deviations from the hypothesis of the model. The idea is to define an adequate **prediction error** as a function of the bootstrap estimate and a bootstrap simulation of the future reality and to record the value of this prediction error for each bootstrap "iteration". We use the desired percentile of this prediction error and combine it with the initial prediction to obtain the upper limit of the prediction interval.

Figure 2 presents the different stages of the bootstrap procedures.

Figure 2 - Bootstrap procedures

Stage 1 – The preliminaries

- Estimation of the model parameters c, α_i, β_j ($i, j = 1,2,...,n$) and ϕ;
- Calculation of the fitted values, $\hat{\mu}_{ij}$ ($i = 1,2,...,n$ and $j = 1,2,...,n+1-i$);
- Calculation of the residuals $r_{ij} = h(y_{ij}, \hat{\mu}_{ij})$;
- Forecasts with the original data $\hat{\mu}_{ij}$, $\hat{\mu}_{i\bullet}$ and $\hat{\mu}_{\bullet}$ ($i = 2,...,n$ and $j = n+2-i,...,n$)

Stage 2 – bootstrap loop (to be repeated *B* times)

Sub stage 2.1 – Bootstrap estimates

- Resampling of the residuals obtained in stage 1 (original data) using replacement $\rightarrow r_{ij}^*$
- Create the pseudo data y_{ij}^*, solving $r_{ij}^* = h(y_{ij}^*, \hat{\mu}_{ij})$.
- Estimate the model with the pseudo data and obtain the bootstrap forecast $\hat{\mu}_{ij}^*$, $\hat{\mu}_{i\bullet}^*$ and $\hat{\mu}_{\bullet}^*$.
- Keep the bootstrap forecasts $\hat{\mu}_{i\bullet}^{(b)} = \hat{\mu}_{i\bullet}^*$ and $\hat{\mu}_{\bullet}^{(b)} = \hat{\mu}_{\bullet}^*$, *b* being the index of the cycle.

Sub stage 2.2 – Pseudo reality (only for procedure 2)

- Resampling again the residuals obtained in stage 1 and select (with replacement) as much values as there are "individual" forecasts to be done $\rightarrow r_{ij}^{**}$, ($i = 2,...,n$ and $j = n+2-i,...,n$).
- Create the pseudo reality, y_{ij}^{**}, solving $r_{ij}^{**} = h(y_{ij}^{**}, \hat{\mu}_{ij})$ ($i = 2,...,n$ and $j = n+2-i,...,n$). $\hat{\mu}_{ij}$ are the predictions obtained in stage 1.
- Obtain the prediction errors $r_{i\bullet}^{(b)} = h(y_{i\bullet}^{**}, \hat{\mu}_{i\bullet}^*)$ and $r_{\bullet}^{(b)} = h(y_{\bullet}^{**}, \hat{\mu}_{\bullet}^*)$ and keep them.
- Return to the beginning of stage 2 until the *B* repetitions are completed.

Stage 3 – Bootstrap data analysis

Stage 3.1 – (Essentially for procedure 1)

- Obtain the bootstrap estimate for $\text{var}(\hat{\mu}_{i\bullet})$ and $\text{var}(\hat{\mu}_{\bullet})$ by mean of the empirical variance of the corresponding *B* bootstrap estimates. England and Verrall (1999) suggest that we could correct the bias of such estimates by multiplying them by a factor equal to $n/(n-p)$, n being the number of observations in the data triangle and *p* the number of parameters in the linear structure.
- Apply the theoretical expressions of the standard error of prediction and use those estimates.

Stage 3.2 (only for procedure 2)

- Use the percentile *k*% of the bootstrap observations of prediction error, for instance $r_{\bullet,k}^*$ for the grand total, and obtain the corresponding percentile of the provisions by solving $r_{\bullet,k}^* = h(y_{\bullet,k}^*, \hat{\mu}_{\bullet})$. $\hat{\mu}_{\bullet}$ is the prediction with the original data (stage 1) .

4. An application

Let us consider the data from Taylor and Ashe (1983) which are presented in table 1 in incremental form. As already said, this data set has been used by several authors and acts as a sort of benchmark for claims reserving methods. England and Verrall (1999) have summarised the main results obtained by those authors and they also use bootstrap technique to evaluate the predictions standard errors related to the chain

ladder approach. For that purpose, they consider a model with a quasi over dispersed Poisson data to define the residuals, taking advantage from the fact that this particular GLM generates the same estimates as the chain ladder technique. They use Pearson residuals without correction (given in relation (8)) and they follow the first procedure for the bootstrap, based on the estimation of a standard error of prediction. Despite they use a GLM for the residual definition they obtain the predictions by means of the chain ladder. This difference is relevant since, in that particular example, the two methods do not agree for some pseudo data sets generated by the bootstrap. In fact, for some pseudo data sets we could have negative values in the north-east corner which do not allow the use of this GLM.

Table 1 – Available data

	1	2	3	4	5	6	7	8	9	10
1	357 848	766 940	610 542	482 940	527 326	574 398	146 342	139 950	227 229	67 948
2	352 118	884 021	933 894	1 183 289	445 745	320 996	527 804	266 172	425 046	
3	290 507	1 001 799	926 219	1 016 654	750 816	146 923	495 992	280 405		
4	310 608	1 108 250	776 189	1 562 400	272 482	352 053	206 286			
5	443 160	693 190	991 983	769 488	504 851	470 639				
6	396 132	937 085	847 498	805 037	705 960					
7	440 832	847 631	1 131 398	1 063 269						
8	359 480	1 061 648	1 443 370							
9	376 686	986 608								
10	344 014									

We extend the work of England and Verrall in three directions: First, we discuss the consequences of the introduction of the residual corrections. Second, we examine the differences in the results obtained by the two bootstrap procedures. Finally we consider the bootstrap predictions obtained by the gamma model. For this data set the gamma model presents a clear advantage: all the pseudo data generated by this model allow their estimation by the same model.

To discuss the first point we consider the same methodology as England and Verall, which will be called by mixed model (quasi over dispersed Poisson for the residual definition and chain ladder for the predictions) and the first bootstrap procedure. We use the Pearson residuals without corrections, with the zeros corrected - we eliminate the residuals corresponding to the last column of row 1 and to the first column of row 10 – and the residuals with the zeros corrected and the factor h (see relation (9)).

Table 2 presents the standard errors of prediction for the three situations considered as well as the upper limits for a confidence level of 95%. Remember that the estimated reserve is the same.

As expected, the standard errors of prediction grow up when we introduce the corrections and consequently the same happens to the upper limits. For the standard errors, the changes are between 16.9% and 24.4% when we compare the estimates with the zeros corrected and the h factor against the estimates obtained without corrections. The values for the situation where only the zeros are discarded are located

between the two extremes. We can also see that the impact of the introduction of the *h* factor is greater than the discard of the two zero values. When we look to the upper limits the differences act in the same way but the figures are necessarily smaller (between 4% and 16%).

Table 2 – Quasi over-dispersed Poisson model (chain ladder)

Year	Estimated Reserve	Without correction		0's corrected		0's and h corrected	
		SEP	upper 95%	SEP	upper 95%	SEP	upper 95%
2	94 634	108 949	273 840	112 665	279 952	135 339	317 246
3	469 511	216 284	825 266	216 351	825 376	259 378	896 149
4	709 638	258 377	1 134 631	261 856	1 140 353	313 130	1 224 691
5	984 889	304 002	1 484 928	307 373	1 490 473	365 791	1 586 561
6	1 419 459	376 754	2 039 163	377 001	2 039 570	449 639	2 159 048
7	2 177 641	488 362	2 980 925	497 648	2 996 199	607 530	3 176 938
8	3 920 301	792 406	5 223 693	797 738	5 232 463	943 367	5 472 000
9	4 278 972	1 081 289	6 057 533	1 083 187	6 060 655	1 264 126	6 358 273
10	4 625 811	2 034 469	7 972 214	2 091 412	8 065 876	2 516 470	8 765 034
Total	**18 680 856**	**2 993 352**	**23 604 480**	**3 109 410**	**23 795 378**	**3 590 809**	**24 587 209**

Table 3 - SEP against PPE bootstrap approaches

Over-dispersed Poisson model (chain ladder)

Year	Estimated Reserve	0's corrected		0's and h corrected	
		SEP- 95%	PPE - 95%	SEP- 95%	PPE - 95%
2	94 634	279 952		317 246	
3	469 511	825 376	785 911	896 149	886 168
4	709 638	1 140 353	1 090 954	1 224 691	1 175 163
5	984 889	1 490 473	1 474 210	1 586 561	1 520 295
6	1 419 459	2 039 570	1 990 773	2 159 048	2 106 503
7	2 177 641	2 996 199	2 867 049	3 176 938	3 085 471
8	3 920 301	5 232 463	4 994 792	5 472 000	5 286 592
9	4 278 972	6 060 655	5 900 404	6 358 273	6 215 378
10	4 625 811	8 065 876	8 182 354	8 765 034	9 370 058
Total	**18 680 856**	**23 795 378**	**22 516 534**	**24 587 209**	**23 678 710**

The second point is to compare the two bootstrap approaches. For that purpose we consider again the mixed model and we obtain the upper limits using the two bootstrap procedures: SEP, that is the approach based in the standard error of prediction, against PPE, that is the other procedure to obtain the adequate percentile of

the prediction error. Table 3 presents the results for the upper 95% limit using two residual corrections. Since some generated pseudo data set present negative values in the north-east corner, it is not possible to obtain the upper limit for year 2 when using PPE bootstrap procedure.

The second bootstrap procedure generates smaller values for all but the last occurrence year. The differences amongst the results obtained with the two bootstrap procedures are not very important (less than 7% in absolute terms).

Let us now compare the results for the quasi over-dispersed Poisson against the Gamma model. Table 4 shows the estimated reserve as well as the theoretical approximation to the standard error of prediction given by relations (5) and (6). Combining these two estimates and using the normal distribution we obtain the upper 95% confidence limits which are also presented. Note that the gamma model provides a better fit than the over-dispersed Poisson.

Table 4 - Over-dispersed Poisson against Gamma model

	Over dispersed Poisson			Gamma model		
Year	Est. Reserve	SEP	Upper 95%	Est. Reserve	SEP	upper 95%
2	94 634	110 258	275 992	93 316	46 505	169 810
3	469 511	216 265	825 235	446 504	165 315	718 423
4	709 638	261 114	1 139 132	611 145	182 889	911 971
5	984 889	303 822	1 484 632	992 023	262 013	1 422 996
6	1 419 459	375 374	2 036 894	1 453 085	361 748	2 048 107
7	2 177 641	495 911	2 993 342	2 186 160	541 888	3 077 486
8	3 920 301	791 169	5 221 658	3 665 065	969 223	5 259 294
9	4 278 972	1 048 624	6 003 804	4 122 398	1 210 801	6 113 988
10	4 625 811	1 984 733	7 890 405	4 516 073	1 716 813	7 339 978
Total	**18 680 856**	**2 951 829**	**23 536 181**	**18 085 769**	**2 782 816**	**22 663 092**

Two main conclusions can be drawn:

- Usually the gamma model produces smaller estimated reserves but the figures are not very different. The exception is the reserve for year 4 where the value obtained with the gamma model is 14% less than those estimated with the over-dispersed Poisson model. For the global prediction the same ratio is –3%.
- However the standard errors of prediction are quite different and consequently the estimated upper limits. These differences tend to be greater in the first years (estimation based on few predictions). The upper confidence limits present smoother differences since they combine the standard errors of prediction with the estimated reserves. The upper limit for the global prediction is 4% lesser with the gamma model than with the over-dispersed Poisson model.

Finally we compare the results obtained with the different models and the two bootstrap procedures. Table 5 presents those results when the residuals are corrected.

The main conclusion is that the aggregate predictions are not very different. The smaller value (23 460 724) is 92% of the biggest (24 857 209). Those differences are much more influenced by the choice of a model than by the selection of a bootstrap procedure.

When we look at the results for each year the differences are more significant but, again, we see that the main problem is the choice of a model. In this particular example the gamma model presents a better fit to the observed values.

Table 5 - Bootstrap results (corrected residuals)

	Poisson based			Gamma		
Year	Est. Reserve	SEP- 95%	PPE - 95%	Est. Reserve	SEP- 95%	PPE - 95%
2	94 634	317 246		93 316	178 376	224 222
3	469 511	896 149	886 168	446 504	742 285	797 805
4	709 638	1 224 691	1 175 163	611 145	943 805	996 543
5	984 889	1 586 561	1 520 295	992 023	1 492 281	1 522 673
6	1 419 459	2 159 048	2 106 503	1 453 085	2 128 274	2 117 230
7	2 177 641	3 176 938	3 085 471	2 186 161	3 206 829	3 240 837
8	3 920 301	5 472 000	5 286 592	3 665 066	5 563 783	5 649 816
9	4 278 972	6 358 273	6 215 378	4 122 398	6 519 504	7 063 204
10	4 625 811	8 765 034	9 370 058	4 516 073	7 971 008	9 911 301
Total	**18 680 856**	**24 587 209**	**23 678 710**	**18 085 772**	**23 675 062**	**23 460 724**

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