UNIT 1 PROPOSITIONAL CALCULUS

Structure

1.0 INTRODUCTION

According to the theory of evolution, human beings have evolved from the lower species over many millennia. The chief asset that made humans "superior" to their ancestors was the ability to reason. How well this ability has been used for scientific and technological development is common knowledge. But no systematic study of logical reasoning seems to have been done for a long time. The first such study that has been found is by Greek philosopher Aristotle (384-322 BC). In a modified form, this type of logic seems to have been taught through the Middle Ages.

Then came a major development in the study of logic, its formalisation in terms of mathematics.It was mainly Leibniz (1646-1716) and George Boole (1815-1864) who seriously studied and development this theory, called **symbolic logic**. It is the basics of this theory that we aim to introduce you to in this unit and the next one.

In the introduction to the block you have read about what symbolic logic is. Using it we can formalise our arguments and logical reasoning in a manner that can easily show if the reasoning is valid, or is a fallacy. How we symbolise the reasoning is what is presented in this unit.

More precisely, in Section 1.2 (i.e., Sec. 1.2, in brief) we talk about what kind of sentences are acceptable in mathematical logic. We call such sentences statements or propositions. You will also see that a statement can either be true or false. Accordingly, as you will see, we will give the statement a truth value T or F.

In Sec. 1.3 we begin our study of the logical relationship between propositions. This is called prepositional calculus. In this we look at some ways of connecting simple propositions to obtain more complex ones. To do so, we use logical connectives like "and" and "or". We also introduce you to other connectives like "not", "implies" and "implies and is implied by". At the same time we construct tables that allow us to find the truth values of the compound statement that we get.

In Sec. 1.4 we consider the conditions under which two statements are "the same". In such a situation we can safely replace one by the other.

And finally, in Sec 1.5, we talk about some common terminology and notation which is useful for quantifying the objects we are dealing with in a statement.

It is important for you to study this unit carefully, because the other units in this block are based on it. Please be sure to do the exercises as you come to them. Only then will you be able to achieve the following objectives.

1.1 OBJECTIVES

After reading this unit, you should be able to:

- distinguish between propositions and non-propositions;
- construct the truth table of any compound proposition;
- identify and use logically equivalent statements;
- identify and use logical quantifiers.

Let us now begin our discussion on mathematical logic.

1.2 PROPOSITIONS

Consider the sentence 'In 2003, the President of India was a woman'. When you read this declarative sentence, you can immediately decide whether it is true or false. And so can anyone else. Also, it wouldn't happen that some people say that the statement is true and some others say that it is false. Everybody would have the same answer. So this sentence is either **universally true** or **universally false**.

Similarly, 'An elephant weighs more than a human being.' Is a declarative sentence which is either true or false, but not both. In mathematical logic we call such sentences **statements or propositions**.

On the other hand, consider the declarative sentence 'Women are more intelligent than men'. Some people may think it is true while others may disagree. So, it is neither universally true nor universally false. Such a sentence is not acceptable as a statement or proposition in mathematical logic.

Note that a **proposition should be either uniformly true or uniformly false**. For example, 'An egg has protein in it.', and 'The Prime Minister of India has to be a man.' are both propositions, the first one true and the second one false.

Would you say that the following are propositions?

'Watch the film.
'How wonderful!'
'What did you say?'

Actually, none of them are declarative sentences. (The first one is an order, the second an exclamation and the third is a question.) And therefore, none of them are propositions.

Now for some mathematical propositions! You must have studied and created many of them while doing mathematics. Some examples are

Two plus two equals four.
Two plus two equals five.
$x + y > 0$ for $x > 0$ and $y > 0$.
A set with n elements has 2^n subsets.

Of these statements, three are true and one false (which one?).
Now consider the algebraic sentence '$x + y > 0$'. Is this a proposition? Are we in a position to determine whether it is true or false? Not unless we know the values that x and y can take. For example, it is false for
$x = 1, y = -2$ and true if $x = 1, y = 0$. Therefore,
'$x + y > 0$' is not a proposition, while
'$x + y > 0$ for $x > 0, y > 0$' is a proposition.

Why don't you try this short exercise now?

E1) Which of the following sentences are statements? What are the reasons for your answer?
i) The sun rises in the West.
ii) How far is Delhi from here?
iii) Smoking is injurious to health.
iv) There is no rain without clouds.
v) What is a beautiful day!
vi) She is an engineering graduates.
vii) $2^n + n$ is an even number for infinitely many n.
viii) $x + y = y + x$ for all $x, y \in \mathbf{R}$.
ix) Mathematics is fun.
x) $2^n = n^2$.

Usually, when dealing with propositions, we shall denote them by lower case letters like p, q, etc. So, for example, we may denote

'Ice is always cold.' by p, or
'$\cos^2 \theta + \sin^2 \theta =1$ for $\theta \in [0, 2\pi]$' by q.
We shall sometimes show this by saying
p: Ice is always cold., or
q: $\cos^2 \theta + \sin^2 \theta = 1$ for $\theta \in [0, 2\pi]$.

Now, given a proposition, we know that it is either true or false, but not both. If it is **true**, we will allot it the **truth value T.** If it is **false**, its **truth value will be F**. So, for example, the truth value of

'Ice melts at 30^o C.' is F, while that of '$x^2 \geq 0$ for $x \in R$' is T.

Sometimes, as in the context of logic circuits (See unit 3), we will use 1 instead of T and 0 instead of F.

Here are some exercises for you now.

E2) Give the truth values of the propositions in E1.

E3) Give two propositions each, the truth values of which are T and F, respectively. Also give two examples of sentences that are not propositions.

Let us now look at ways of connecting simple propositions to obtain compound statements.

1.3 LOGICAL CONNECTIVES

When you're talking to someone, do you use very simple sentences only? Don't you use more complicated ones which are joined by words like 'and', 'or', etc? In the same way, most statements in mathematical logic are combinations of simpler statements joined by words and phrases like 'and'. 'or', 'if … then'. 'if and only if', etc. These words and phrases are called **logical connectives**. There are 6 such connectives, which we shall discuss one by one.

1.3.1 Disjunction

Consider the sentence 'Alice or the mouse went to the market.'. This can be written as 'Alice went to the market or the mouse went to the market.' So, this statement is actually made up of two simple statements connected by 'or'. We have a term for such a compound statement.

Definition: The **disjunction** of two propositions p and q is the compound statement

p or q, denoted by **$p \vee q$**.
For example, 'Zarina has written a book or Singh has written a book.' Is the disjunction of p and q, where
p : Zarina has written a book, and
q : Singh has written a book.

Similarly, if p denotes '$2 > 0$' and q denotes '$2 < 5$', then $p \vee q$ denotes the statement '2 is greater than 0 or 2 is less than 5.'.

Let us now look at how the truth value of $p \vee q$ depends upon the truth values of p and q. For doing so, let us look at the example of Zarina and Singh, given above. If even one of them has written a book, then the compound statement $p \vee q$ is true. Also, if both have written books, the compound statement $p \vee q$ is again true. Thus, if the truth value of even one out of p and q is T, then that of '$p \vee q$' is T. Otherwise, the truth value of $p \vee q$ is F. This holds for any pair of propositions p and q. To see the relation between the truth values of p, q and $p \vee q$ easily, we put this in the form of a table (Table 1), which we call a **truth table.**

Table 1: Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

How do we form this table? We consider the truth values that p can take – T or F. Now, when p is true, q can be true or false. Similarly, when p is false q can be true or false. In this way there are 4 possibilities for the compound proposition $p \vee q$. Given any of these possibilities, we can find the truth value of $p \vee q$. For instance, consider the third possibility, i.e., p is false and q is true. Then, by definition, $p \vee q$ is true. In the same way, you can check that the other rows are consistent.

Let us consider an example.

Example 1: Obtain the truth value of the disjunction of 'The earth is flat'.
and '$3 + 5 = 2$'.

Solution: Let p denote 'The earth is flat,' and q denote '$3 + 5 = 2$'. Then we know that the truth values of both p and q are F. Therefore, the truth value of $p \vee q$ is F.

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Try an exercise now.

E4) Write down the disjunction of the following propositions, and give its truth value.
   i) $2 + 3 = 7$,
   ii) Radha is an engineer.

We also use the term 'inclusive or ' for the connective we have just discussed. This is because $p \vee q$ is true even when both p and q are true. But, what happens when we want to ensure that only one of them should be true? Then we have the following connective.

Definition: The **exclusive disjunction** of two propositions p and q is the statement **'Either p is true or q is true, but both are not true.'.** Either p is true or q is true, but both are not true.'. We denote this by **$p \oplus q$.**

So, for example, if p is '$2 + 3 = 5$' and q the statement given in E4(ii), then $p \oplus q$ is the statement 'Either $2 + 3 = 5$ or Radha is an engineer'. This will be true only if Radha is not an engineer.

In general, how is the truth value of $p \oplus q$ related to the truth values of p and q? This is what the following exercise is about.

E5) Write down the truth table for $\oplus$. Remember that $p \oplus q$ is not true if both p and q are true.

Now let us look at the logical analogue of the coordinating conjunction 'and'.

1.3.2 Conjunction

As in ordinary language, we use 'and' to combine simple propositions to make compound ones. For instance, ' $1 + 4 \neq 5$ and Prof. Rao teaches Chemistry.' is formed by joining '$1 + 4 \neq 5$' and 'Prof. Rao teaches Chemistry' by 'and'. Let us define the formal terminology for such a compound statement.

Definition: We call the compound statement **'p and q'** the **conjunction** of the statements p and q. We denote this by $\mathbf{p \wedge q}$.

For instance, '$3 + 1 \neq 7 \wedge 2 > 0$' is the conjunction of '$3 + 1 \neq 7$' and '$2 > 0$'. Similarly, '$2 + 1 = 3 \wedge 3 = 5$' is the conjunction of '$2 + 1 = 3$' and '$3 = 5$'.

Now, when would $p \wedge q$ be true? Do you agree that this could happen only when both p and q are true, and not otherwise? For instance, '$2 + 1 = 3 \wedge 3 = 5$' is not true because '$3 = 5$' is false.
So, the truth table for conjunction would be as in Table 2.

Table 2: Truth table for conjunction

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see how we can use the truth table above, consider an example.

Example 2: Obtain the truth value of the conjunction of '$2 \div 5 = 1$' and 'Padma is in Bangalore.'.

Solution: Let p : $2 \div 5 = 1$, and
q: Padma is in Bangalore.

Then the truth value of p is F. Therefore, from Table 3 you will find that the truth value of $p \wedge q$ is F.

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Why don't you try an exercise now?

E6) Give the set of those real numbers x for which the truth value of $p \wedge q$ is T, where p : $x > -2$, and q : $x + 3 \neq 7$

If you look at Tables 1 and 2, do you see a relationship between the truth values in their last columns? You would be able to formalize this relationship after studying the next connective.

1.3.3 Negation

You must have come across young children who, when asked to do something, go ahead and do exactly the opposite. Or, when asked if they would like to eat, say rice and curry, will say 'No', the 'negation' of yes! Now, if p denotes the statement 'I will eat rice.', how can we denote 'I will not eat rice.'? Let us define the connective that will help us do so.

Definition: The **negation** of a proposition p is **'not p'**, denoted by ~p.

For example, if p is 'Dolly is at the study center.', then ~ p is 'Dolly is not at the study center'. Similarly, if p is 'No person can live without oxygen.', ~ p is 'At least one person can live without oxygen.'.

Now, regarding the truth value of ~ p, you would agree that it would be T if that of p is F, and vice versa. Keeping this in mind you can try the following exercises.

E7) Write down ~ p, where p is
  i) $0 - 5 \neq 5$
  ii) $n > 2$ for every $n \in \mathbf{N}$.
  iii) Most Indian children study till class 5.

E8) Write down the truth table of negation.

Let us now discuss the conditional connectives, representing 'If …, then …' and 'if and only if'.

1.3.4 Conditional Connectives

Consider the proposition 'If Ayesha gets 75% or more in the examination, then she will get an A grade for the course.'. We can write this statement as 'If p, and q', where

p: Ayesha gets 75% or more in the examination, and
q: Ayesha will get an A grade for the course.

This compound statement is an example of the implication of q by p.

Definition: Given any two propositions p and q, we denote the statement **'If p, then q' by p → q.** We also read this as 'p **implies** q'. or 'p is sufficient for q', or 'p **only if** q'. We also call p the **hypothesis** and q the conclusion. Further, a statement of the form p → q is called a **conditional statement** or a **conditional proposition**.

So, for example, in the conditional proposition 'If m is in **Z**, then m belongs to **Q**.' the hypothesis is '$m \in \mathbf{Z}$' and the conclusion is '$m \in \mathbf{Q}$'.

Mathematically, we can write this statement as

$$m \in \mathbf{Z} \rightarrow m \in \mathbf{Q}.$$

Let us analyse the statement p → q for its truth value. Do you agree with the truth table we've given below (Table 3)? You may like to check it out while keeping an example from your surroundings in mind.

Table 3: Truth table for implication

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

You may wonder about the third row in Table 3. But, consider the example '$3 < 0 \to 5 > 0$'. Here the conclusion is true regardless of what the hypothesis is. And therefore, the conditional statement remains true. In such a situation we say that the **conclusion is vacuously true.**

Why don't you try this exercise now?

E9) Write down the proposition corresponding to $p \to q$, and determine the values of x for which it is false, where
$p : x + y = xy$ where $x, y \in \mathbf{R}$
$q : x \nless 0$ for every $x \in \mathbf{Z}$.

Now, consider the implication 'If Jahanara goes to Baroda, then the she doesn't participate in the conference at Delhi.'. What would its converse be? To find it, the following definition may be useful.

Definition: The **converse** of $p \to q$ is $q \to p$. In this case we also say 'p is necessary for q', or 'p **if** q'.

So, in the example above, the converse of the statement would be 'If Jahanara doesn't participate in the conference at Delhi, then she goes to Baroda.'. This means that Jahanara's non-participation in the conference at Delhi is necessary for her going to Baroda.

Now, what happens when we combine an implication and its converse?

To show '$p \to q$ and $q \to p$', we introduce a shorter notation.

Definition: Let p and q be two propositions. The compound statement $(p \to q) \wedge (q \to p)$ is the **biconditional** of p and q. We denote it by $\mathbf{p \leftrightarrow q}$**,** and read it as 'p **if and only** q'.

The two connectives $\to$ and $\leftrightarrow$ are called **conditional connectives.**

We usually shorten 'if and only 'if' to **iff**.

We also say that 'p **implies and is implied** by q'. or 'p is **necessary and sufficient for** q'.

For example, 'Sudha will gain weight if and only if she eats regularly.' Means that 'Sudha will gain weight if she eats regularly **and** Sudha will eat regularly if she gains weight.'

One point that may come to your mind here is whether there's any difference in the two statements $p \leftrightarrow q$ and $q \leftrightarrow p$. When you study Sec. 1.4 you will realize why they are inter-changeable.

Let us now consider the truth table of the biconditional, i.e., of the **two-way** implication.

To obtain its truth values, we need to use Tables 2 and 3, as you will see in Table 4. This is because, to find the value of $(p \to q) \wedge (q \to p)$ we need to know the values of each of the simpler statements involved.

Table 4: Truth table for two-way implication.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

As you can see from the last column of the table (and from your own experience), $p \leftrightarrow q$ is true only when both p and q are true or both p and q are false. In other words, $p \leftrightarrow q$ is true only when p and q have the same truth values. Thus, for example,'Parimala is in America iff 2 + 3 = 5' is true only if 'Parimala is in America,' is true.

Here are some related exercises.

E10) For each of the following compound statements, first identify the simple propositions p, q, r, etc., that are combined to make it. Then write it in symbols, using the connectives, and give its truth value.

i) If triangle ABC is equilateral, then it is isosceles.
ii) a and b are integers if and only if ab is a rational number.
iii) If Raza has five glasses of water and Sudha has four cups of tea, then Shyam will not pass the math examination.
iv) Mariam is in Class 1 or in Class 2.

E11) Write down two propositions p and q for which $q \rightarrow p$ is true but $p \leftrightarrow q$ is false.

Now, how would you determine the truth value of a proposition which has more than one connective in it? For instance, does $\sim p \vee q$ mean $(\sim p) \vee q$ or $\sim (p \vee q)$? We discuss some rules for this below.

1.3.5 Precedence Rule

While dealing with operations on numbers, you would have realized the need for applying the BODMAS rule. According to this rule, when calculating the value of an arithmetic expression, we first calculate the value of the Bracketed portion, then apply **Of, Division, Multiplication, Addition and Subtraction, in this order.** While calculating the truth value of compound propositions involving more than one connective, we have a similar **convention** which tells us which connective to apply first.

Why do we need such a convention? Suppose we didn't have an order of preference, and want to find the truth of, say $\sim p \vee q$. Some of us may consider **the value of $(\sim p) \vee q$, and some may consider $\sim (p \vee q)$. The truth values can be different in these cases. For instance, if p and q are both true, then $(\sim p) \vee q$ is true, but $\sim (p \vee q)$ is false. So, for the purpose of unambiguity, we agree to such an order or rule. Let us see what it is.**

The rule of precedence: The order of preference in which the connectives are applied in a formula of propositions that has no brackets is

i) $\sim$
ii) $\wedge$
iii) $\vee$ and $\oplus$
iv) $\rightarrow$ and $\leftrightarrow$

Note that the 'inclusive or' and 'exclusive or' are both third in the order of preference. However, if both these appear in a statement, we first apply the left most one. So, for instance, in $p \vee q \oplus \sim p$, we first apply $\vee$ and then $\oplus$. The same applies to the 'implication' and the 'biconditional', which are both fourth in the order of preference.

To clearly understand how this rule works, let us consider an example.

Example 3: Write down the truth table of $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$

Solution: We want to find the required truth value when we are given the truth values of p, q and r. According to the rule of precedence given above, we need to first find the truth value of $\sim r$, then that of $(q \wedge \sim r)$, then that of $(r \oplus q)$, and then that of $p \rightarrow (q \wedge \sim r)$, and finally the truth value of $[p \rightarrow (q \wedge \sim r)] \leftrightarrow r \oplus q$.

So, for instance, suppose p and q are true, and r is false. Then $\sim r$ will have value T, $q \wedge \sim r$ will be T, $r \oplus q$ will be T, $p \rightarrow (q \wedge \sim r)$ will be T, and hence, $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$ will be T.

You can check that the rest of the values are as given in Table 5. Note that we have 8 possibilities ($=2^3$) because there are 3 simple propositions involved here.

Table 5: Truth table for $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$

p	q	r	$\sim r$	$q \wedge \sim r$	$r \oplus q$	$p \rightarrow q \wedge \sim r$	$p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$
T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F

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You may now like to try some exercises on the same lines.

E12) In Example 3, how will the truth values of the compound statement change if you first apply $\leftrightarrow$ and then $\rightarrow$?

E13) In Example 3, if we replace $\oplus$ by $\wedge$, what is the new truth table?

E14) From the truth table of $p \wedge q \vee \sim r$ and $(p \wedge q) \vee (\sim r)$ and see where they differ.

E15) How would you bracket the following formulae to correctly interpret them? [For instance, $p \vee \sim q \wedge r$ would be bracketed as $p \vee ((\sim q) \wedge r)$.]

i) $p \vee q$,
ii) $\sim q \rightarrow \sim p$,
iii) $p \rightarrow q \leftrightarrow \sim p \vee q$,
iv) $p \oplus q \wedge r \rightarrow \sim p \vee q \leftrightarrow p \wedge r$.

So far we have considered different ways of making new statements from old ones. But, are all these new ones distinct? Or are some of them the same? And "same" in what way? This is what we shall now consider.

1.4 LOGICAL EQUIVALENCE

'Then you should say what you mean', the March Have went on. 'I do,' Alice hastily replied, 'at least … at least I mean what I say – that's the same thing you know.' 'Not the same thing a bit!' said the Hatter. 'Why you might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'

-from 'Alice in Wonderland'
by Lewis Carroll

In Mathematics, as in ordinary language, there can be several ways of saying the same thing. In this section we shall discuss what this means in the context of logical statements.

Consider the statements 'If Lala is rich, then he must own a car.'. and 'if Lala doesn't own a car, then he is not rich.'. Do these statements mean the same thing? If we write the first one as $p \to q$, then the second one will be $(\sim q) \to (\sim p)$. How do the truth values of both these statements compare?

We find out in the following table.

Table 6

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Consider the last two columns of Table 6. You will find that '$p \to q$' and '$q \to \sim p$' have the same truth value for every choice of truth values of p and q. When this happens, we call them equivalent statements.

Definition: We call two propositions r and s **logically equivalent** provided they have the same truth value for every choice of truth values of simple propositions involved in them. We denote this fact by $r \equiv s$.

So, from Table 6 we find that $(p \to q) \equiv (\sim q \to \sim p)$.

You can also check that $(p \leftrightarrow q) \equiv (q \leftrightarrow p)$ for any pair of propositions p and q.

As another example, consider the following equivalence that is often used in mathematics. You could also apply it to obtain statements equivalent to 'Neither a borrower, nor a lender be.'!

Example 4: For any two propositions p and q, show that $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

Solution: Consider the following truth table.

Table 7

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

You can see that the last two columns of Table 7 are identical. Thus, the truth values of $\sim (p \vee q)$ and $\sim p \wedge \sim q$ agree for every choice of truth values of p and q. Therefore, $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

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Fig. 1: Augustus De Morgan (1806-1871) was born in Madurai

The equivalence you have just seen is one of **De Morgan's laws.** You might have already come across these laws in your previous studies of basic Mathematics.

The other law due to De Morgan is similar : $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

In fact, there are several such laws about equivalent propositions. Some of them are the following, where, as usual, p, q and r denote propositions.

a) **Double negation law** : $\sim(\sim p) \equiv p$

b) **Idempotent laws:** $p \wedge p \equiv p$,
$p \vee p \equiv p$

c) **Commutativity:** $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

d) **Associativity:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

e) **Distributivity:** $\vee (q \wedge r) \equiv \quad (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv \ (p \wedge q) \vee (p \wedge r)$

We ask you to prove these laws now.

E16) Show that the laws given in (a)-(e) above hold true.

E17) Prove that the relation of 'logical equivalence' is an equivalence relation.

E18) Check whether $(\sim p \vee q)$ and $(p \rightarrow q)$ are logically equivalent.

The laws given above and the equivalence you have checked in E18 are commonly used, and therefore, **useful to remember.** You will also be applying them in Unit 3 of this Block in the context of switching circuits.

Let us now consider some prepositional formulae which are always true or always false. Take, for instance, the statement 'If Bano is sleeping and Pappu likes ice-cream, then Beno is sleeping'. You can draw up the truth table of this compound proposition and see that it is always true. This leads us to the following definition.

Definition: A compound proposition that is true for all possible truth values of the simple propositions involved in it is called a **tautology.** Similarly, a proposition that is false for all possible truth values of the simple propositions that constitute it is called a **contradiction.**

Let us look at some example of such propositions.

Example 5: Verify that $p \wedge q \wedge \sim p$ is a contradiction and $p \rightarrow q \leftrightarrow \sim p \vee q$ is a tautology.

Solution: Let us simultaneously draw up the truth tables of these two propositions below.

Table 8

p	q	$\sim p$	$p \wedge q$	$p \wedge q \wedge \sim p$	$p \rightarrow q$	$\sim p \vee q$	$p \rightarrow q \leftrightarrow \sim p \vee q$
T	T	F	T	F	T	T	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	T	T
F	F	T	F	F	T	T	T

Looking at the fifth column of the table, you can see that $p \wedge q \wedge \sim p$ is a contradiction. This should not be surprising since $p \wedge q \wedge \sim p \equiv (p \wedge \sim p) \wedge q$ (check this by using the various laws given above).

And what does the last column of the table show? Precisely that $p \rightarrow q \quad \leftrightarrow \sim p \vee q$ is a tautology.

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Why don't you try an exercise now?

E19) Let *T* denote a tautology (i.e., a statement whose truth value is always T) and *F* a contradiction. Then, for any statement p, show that

i) $p \vee T \equiv T$
ii) $p \wedge T \equiv p$
iii) $p \vee F \equiv p$
iv) $p \wedge F \equiv F$

Another way of proving that a proposition is a tautology is to use the properties of logical equivalence. Let us look at the following example.

Example 6: Show that $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Complementation law: $q \wedge \sim q$ is a contradiction.

Solution: $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv [(\sim p \vee q) \wedge \sim q] \rightarrow \sim p$, using E18, and symmetricity of $\equiv$.

$\equiv [(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \rightarrow \sim p$, by De Morgan's laws.

$\equiv [(\sim p \wedge \sim q) \vee F] \rightarrow \sim p$, since $q \wedge \sim q$ is always false.

$\equiv (\sim p \wedge \sim q) \rightarrow \sim p$, using E18.

Which is tautology.

And therefore the proposition we started with is a tautology.

***

The laws of logical equivalence can also be used to prove some other logical equivalences, without using truth tables. Let us consider an example.

Example 7: Show that $(p \rightarrow \sim q) \wedge (p \rightarrow \sim r) \equiv \sim [p \wedge (q \vee r)]$.

Solution: We shall start with the statement on the left hand side of the equivalence that we have to prove. Then, we shall apply the laws we have listed above, or the equivalence in E 18, to obtain logically equivalent statements. We shall continue this process till we obtain the statement on the right hand side of the equivalence given above. Now

$(p \rightarrow \sim q) \wedge (p \rightarrow \sim r)$

$\equiv (\sim p \vee q) \wedge (\sim p \vee \sim r)$, by E18

$\equiv \sim p \vee (\sim q \wedge \sim r)$, by distributivity

$\equiv \sim p \vee [\sim (q \vee r)]$, by De Morgan's laws

$\equiv \sim [p \wedge (q \vee r)]$, by De Morgan's laws

So we have proved the equivalence that we wanted to.

***

You may now like to try the following exercises on the same lines.

E20) Use the laws given in this section to show that $\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p$.

E21) Write down the statement 'If it is raining and if rain implies that no one can go to see a film, then no one can go to see a film.' As a compound proposition. Show that this proposition is a tautology, by using the properties of logical equivalence.

E22) Give an example, with justification, of a compound proposition that is neither a tautology nor a contradiction.

Let us now consider proposition-valued functions.

1.5 LOGICAL QUANTIFIERS

In Sec. 1.2, you read that a sentence like ‘She has gone to Patna.’ Is not a proposition, unless who ‘she’ is clearly specified.
Similarly, ‘$x > 5$’ is not a proposition unless we know the values of x that we are considering. Such sentences are examples of ‘propositional functions’.

Definition: A propositional function, or a **predicate,** in a variable x is a sentence p(x) involving x that becomes a proposition when we give x a definite value from the set of values it can take. We usually denote such functions by p(x), q(x), etc. The set of values x can take is called the universe of discourse.

So, if p(x) is ‘$x > 5$’, then p(x) is not a proposition. But when we give x particular values, say $x = 6$ or $x = 0$, then we get propositions. Here, p(6) is a true proposition and p(0) is a false proposition.

Similarly, if q(x) is ‘x has gone to Patna.’, then replacing x by ‘Taj Mahal’ gives us a false proposition.

Note that a predicate is usually not a proposition. But, of course, every proposition is a prepositional function in the same way that every real number is a real-valued function, namely, the constant function.

Now, can all sentences be written in symbolic from by using only the logical connectives? What about sentences like ‘x is prime and x + 1 is prime for some x.’? How would you symbolize the phrase ‘for some x’, which we can rephrase as ‘there exists an x’? You must have come across this term often while studying mathematics. **We use the symbol ‘∃’ to denote this quantifier, ‘there exists’.** The way we use it is, for instance, to rewrite ‘There is at least one child in the class.’ as‘(∃ x in U)p(x)’, where p(x) is the sentence ‘x is in the class.’ and U is the set of all children.

∃ is called the **existential quantifier.**

Now suppose we take the negative of the proposition we have just stated. Wouldn’t it be ‘There is no child in the class.’? We could symbolize this as ‘for all x in U, q(x)’ where x ranges over all children and q(x) denotes the sentence ‘x is not in the class.’, i.e., $q(x) \equiv \sim p(x)$.

We have a **mathematical symbol for the quantifier ‘for all’, which is ‘∀’.** So the proposition above can be written as

∀ is called the **universal quantifier**.

‘$(\forall x \in U)q(x)$’, or ‘$q(x), \forall x \in U$’.

An example of the use of the existential quantifier is the true statement.

$(\exists x \in \mathbf{R})(x + 1 > 0)$, which is read as ‘There exists an x in **R** for which $x + 1 > 0$.’.

Another example is the false statement
$(\exists x \in \mathbf{N})(x - \frac{1}{2} = 0)$, which is read as ‘There exists an x in **N** for which $x - \frac{1}{2} = 0$.’.

An example of the use of the universal quantifier is $(\forall x \notin \mathbf{N})(x^2 > x)$, which is read as ‘for every x not in **N**, $x^2 > x$.’. Of course, this is a false statement, because there is at least one $x \notin \mathbf{N}$, $x \in \mathbf{R}$, for which it is false.

We often use both quantifiers together, as in the statement called **Bertrand’s postulate:**

$$(\forall n \in \mathbf{N} \setminus \{1\})(\exists x \in \mathbf{N})(x \text{ is a prime number and } n < x < 2n).$$

In words, this is 'for every integer $n > 1$ there is a prime number lying strictly between n and 2n.'

As you have already read in the example of a child in the class,
$(\forall x \in U)p(x)$ is logically equivalent to $\sim (\exists x \in U)(\sim p(x))$. Therefore,
$\sim(\forall x \in U)p(x) \equiv \sim\sim (\exists x \in U)(\sim p(x)) \equiv (\exists x \in U)(\sim p(x))$.

This is one of the rules for negation that relate $\forall$ and $\exists$. The two rules are

$\sim (\forall \mathbf{x} \in \mathbf{U})\mathbf{p(x)} \equiv (\exists x \in \mathbf{U})\ \mathbf{(\sim p(x))}$, and
$\sim (\exists \mathbf{x} \in \mathbf{U})\mathbf{p(x)} \equiv (\forall x \in \mathbf{U})\ \mathbf{(\sim p(x))}$

Where U is the set of values that x can take.

Now, consider the proposition

'There is a criminal who has committed every crime.'

We could write this in symbols as

$(\exists c \in A)(\forall x \in B)$ (c has committed x)

Where, of course, A is the set of criminals and B is the set of crimes (determined by law).

What would its negation be? It would be

$\sim (\exists c \in A)(\forall x \in B)$ (c has committed x)

Where, of course, A is the set of criminals and B is the set of crimes (determined by law).

What would its negation be? It would be

$\sim (\exists c \in A)(\forall x \in B)$ (c has committed x)
$\equiv (\forall c \in A)[\sim (\forall x \in B)$ (c has committed x)
$\equiv (\forall c \in A)(\exists x \in B)$ (c has not committed x).

We can interpret this as 'For every criminal, there is a crime that this person has not committed.'.

A predicate can be a function in two **or more** variables.

These are only some examples in which the quantifiers occur singly, or together. Sometimes you may come across situations (as in E23) where you would use $\exists$ or $\forall$ twice or more in a statement. It is in situations like this or worse [say, $(\forall x_i \in U_1)(\exists x_2 \in U_2)(\exists x_3 \in U_2)(\exists x_3 \in U_3)(\forall x_4 \in U_4) \ldots (\exists x_n \in U_n)p$]
where our rule for negation comes in useful. In fact, applying it, in a trice we can say that the negation of this seemingly complicated example is

$(\exists x_1 \in U_1)(\forall x_2 \in U_2)(\forall x_3 \in U_3)(\exists x_4 \in U_4) \ldots (\forall x_n \in U_n)(\sim p)$.

Why don't you try some exercise now?

E23) How would you present the following propositions and their negations using logical quantifiers? Also interpret the negations in words.
- i) The politician can fool all the people all the time.
- ii) Every real number is the square of some real number.
- iii) There is lawyer who never tell lies.

E24) Write down suitable mathematical statements that can be represented by the following symbolic propostions. Also write down their negations. What is the truth value of your propositions?
- i) $(\forall x)(\exists y)p$
- ii) $(\exists x)(\exists y)(\forall z)p$.

And finally, let us look at a very useful quantifier, which is very closely linked to ∃. You would need it for writing, for example, 'There is one and only one key that fits the desk's lock.' In symbols. The symbol is ∃! X which stands for **'there is one and only one x'** (which is the same as **'there is a unique** x' or **'there is exactly one** x').

So, the statement above would be $(\exists!\ X \in A)$ (x fits the desk's lock), where A is the set of keys.

For other examples, try and recall the statements of uniqueness in the mathematics that you've studied so far. What about 'There is a unique circle that passes through three non-collinear points in a plane.'? How would you represent this in symbols? If x denotes a circle, and y denotes a set of 3 non-collinear points in a plane, then the proposition is

$$(\forall\ y \in P)\ (\exists!\ X \in C)\ (x \text{ passes through } y).$$

Here C denotes the set of circles, and P the set of sets of 3 non-collinear points. And now, some short exercises for you!

E25) Which of the following propositions are true (where x, y are in R)?

i) $(x \geq 0) \rightarrow (\exists\ y)\ (y^2 = x)$

ii) $(\forall\ x)\ (\exists!\ y)\ (y^2 = x^3)$

iii) $(\exists x)\ (\exists!\ y)\ (xy = 0)$

Before ending the unit, let us take quick look at what e have covered in it.

1.6 SUMMARY

In this unit, we have considered the following points.

1. What a mathematically acceptable statement (or proposition) is.
2. The definition and use of logical connectives:

   Give propositions p and q,

   i) their disjunction is 'p and q', denoted by $p \vee q$;
   ii) their exclusive disjunction is 'either p or q', denoted by $p \oplus q$;
   iii) their conjunction is 'p and q', denoted by $p \wedge q$;
   iv) the negation of p is 'not p', denoted by $\sim p$;
   v) 'if p, then q' is denoted by $p \rightarrow q$;
   vi) 'p if and only if q' is denoted by $p \leftrightarrow q$;
3. The truth tables corresponding to the 6 logical connectives.
4. Rule of precedence : In any compound statement involving more than one connective, we first apply '∼', then '∧', then '∨' and '⊕', and last of all '→' and '↔'.
5. The meaning and use of logical equivalence, denoted by '≡'.
6. The following laws about equivalent propositions:

   i) **De Morgan's laws:** $\sim (p \wedge q) \equiv \sim p \vee \sim q$
   $\sim (p \vee q) \equiv \sim p \wedge \sim q$
   ii) **Double negation law:** $\sim (\sim p) \equiv p$
   iii) **Idempotent laws:** $p \wedge p \equiv p$,
   $p \vee p \equiv p$
   iv) **Commutativity:** $p \vee q \equiv q \vee p$
   $p \wedge q \equiv q \wedge p$
   v) **Associativity:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   vi) **Distributivity:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

vii) $(\sim p \vee q) \equiv p \to q$ (ref. E18).

7. Logical quantifiers: 'For every' denoted by '$\forall$', 'there exist' denoted by '$\exists$', and 'there is one and only one' denoted by '$\exists!$'.
8. The rule of negation related to the quantifiers:
   $\sim (\forall\ x \in U)p(x) \equiv (\exists\ x \in U)(\sim p(x))$
   $\sim (\exists\ x \in U)\ p(x) \equiv (\forall\ x \in U)(\sim p(x))$

Now we have come to the end of this unit. You should have tried all the exercises as you came to them. You may like to check your solutions with the ones we have given below.

1.7 SOLUTIONS/ ANSWERS

E1) (i), (iii), (iv), (vii), (viii) are statements because each of them is universally true or universally false.
(ii) is a question.
(v) is an exclamation.
The truth or falsity of (vi) depends upon who 'she' is.
(ix) is a subjective sentence.
(x) will only be a statement if the value(s) n takes is/are given.
Therefore, (ii), (v), (vi), (ix) and (x) are not statements.

E2) The truth value of (i) is F, and of all the others is T.

E3) The disjunction is
'2+3 = 7 or Radha is an engineer.'.
Since '2+3 = 7' is always false, the truth value of this disjunction depends on the truth value of 'Radha is an engineer.'. If this is T, them we use the third row of Table 1 to get the required truth value as T. If Radha is not an engineer, then we get the required truth value as F.

Table 9: Truth table for 'exclusive or'

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

E4) p will be a true proposition for $x \in\ ]-2, \infty[$ and
$x \neq 4$, i.e., for $x \in]-2, 4[\ U\ ]4, \infty[$.

E5) i) $0 - 5 = 5$
ii) 'n is not greater than 2 for every $n \in \mathbf{N}$.', or 'There is at least one n $n \in \mathbf{N}$ for which $n \leq 2$.'
iii) There are some Indian children who do not study till Class 5.

E6)

Table 10: Truth table for negation

p	$\sim p$
T	F
F	T

E7) $p \to q$ is the statement 'If $x + y = xy$ for $x, y \in R$, then $x \not\subset 0$ for every $\in \mathbf{Z}$'.

In this case, q is false. Therefore, the conditional statement will be true if p is false also, and it will be false for those values of x and y that make p true.

So, $p \to q$ is false for all those real numbers x of the form $\frac{y}{y-1}$, where $y \in \mathbf{R} \setminus \{1\}$. This is because if $x = \frac{y}{y-1}$ for some $y \in \mathbf{R} \setminus \{1\}$, then $x + y = xy$, i.e., p will be true.

E8) i) $p \to q$, where p : ΔABC is isosceles. If q is true, then $p \to q$ is true. If q is false, then $p \to q$ is true only when p is false. So, if ΔABC is an isosceles triangle, the given statement is always true. Also, if ΔABC is not isosceles, then it can't be equilateral either. So the given statement is again true.

ii) p : a is an integer.
q : b is an integer.
r : ab is a rational number
The given statement is $(p \wedge q) \leftrightarrow r$.
Now, if p is true and q is true, then r is still true.

So, $(p \wedge q) \leftrightarrow r$ will be true if $p \wedge q$ is true, or when $p \wedge q$ is false and r is false.
In all the other cases $(p \wedge q) \leftrightarrow r$ will be false.

iii) p : Raza has 5 glasses of water.
q : Sudha has 4 cups of tea.
r : Shyam will pass the math exam.

The given statement is $(p \wedge q) \to \sim r$.
This is true when $\sim r$ is true, or when r is true and $p \wedge q$ is false.
In all the other cases it is false.

iv) p : Mariam is in Class 1.
q : Mariam is in Class 2.

The given statement is $p \oplus q$.
This is true only when p is true or when q is true.

E9) There are infinitely many such examples. You need to give one in which p is true but q is false.

E10) Obtain the truth table. The last column will now have entries TTFTTTTT.

E11) According to the rule of precedence, given the truth values of p, q, r you should first find those of $\sim r$, then of $q \wedge \sim r$, and $r \wedge q$, and $p \to q \wedge \sim r$, and finally of $(p \to q \wedge \sim r) \leftrightarrow r \wedge q$.

Referring to Table 5, the values in the sixth and eighth columns will be replaced by

$r \wedge q$
T
F
F
F
T
F
F
F

$p \to q \wedge \sim r \leftrightarrow r \wedge q$
F
F
T
T
T
F
F
F

E12) They should both be the same, viz.,

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee (\sim r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

E13) i) $(\sim p) \vee q$

ii) $(\sim q) \rightarrow (\sim p)$

iii) $(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$

iv) $[(p \oplus (q \wedge r) \rightarrow [(\sim p) \vee q]] \leftrightarrow (p \wedge r)$

E14) a)

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

The first and third columns prove the double negation law.

b)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The third and fourth columns prove the commutativity of ∨.

E15) For any three propositions p, q, r:

i) $p \equiv p$ is trivially true.

ii) if $p \equiv q$, then $q \equiv p$ (if p has the same truth value as q for all choices of truth values of p and q, then clearly q has the same truth values as p in all the cases.

iii) if $p \equiv q$ and $q \equiv r$, then $p \equiv r$ (reason as in (ii) above).

Thus, ≡ is reflexive, symmetric and transitive.

E16)

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last two columns show that $[(\sim p) \vee q] \equiv (p \rightarrow q)$.

E17) i)

p	T	$p \vee \mathcal{T}$
T	T	T
F	T	T

The second and third columns of this table show that $p \vee \mathcal{T} = T$.

ii)

p	$\mathcal{F}$	p ∧ $\mathcal{F}$
T	F	F
F	F	F

The second and third columns of this table show that $p \wedge \mathcal{F} = F$.
You can similarly check (ii) and (iii).

E18) $\sim(\sim p \wedge q) \wedge (p \vee q)$
$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \wedge q)$, by De Morgan's laws.
$\equiv (p \vee \sim q) \wedge (p \vee q)$, by the double negation law.
$\equiv p \vee (\sim q \wedge q)$, by distributivity
$\equiv p \vee \mathcal{F}$, where $\mathcal{F}$ denotes a contradiction
$\equiv p$, using E 19.

E19) p: It is raining.
q: Nobody can go to see a film.
Then the given proposition is
$[p \wedge (p \rightarrow q)] \rightarrow q$
$\equiv p \wedge (\sim p \vee q) \rightarrow q$, since $(p \rightarrow q) \equiv (\sim p \vee q)$
$\equiv (p \wedge \sim p) \vee (p \wedge q) \rightarrow q$, by De Morgan's law
$\equiv \mathcal{F} \vee (p \wedge q) \rightarrow q$, since $p \wedge \sim p$ is a contradiction
$\equiv (\mathcal{F} \vee p) \wedge (F \vee q) \rightarrow q$, by De Morgan's law
$\equiv p \wedge q \rightarrow q$, since $\mathcal{F} \vee p \equiv p$.
which is a tautology.

E20) There are infinitely many examples. One such is:
'If Venkat is on leave, then Shabnam will work on the computer'.This is of the form $p \rightarrow q$. Its truth values will be T or F, depending on those of p and q.

E21) i) $(\forall\, t \in [0, \infty[)\ (\forall\, x \in H)p(x,t)$ is the given statement where p(x, t) is the predicate 'The politician can fool x at time t second.', and H is the set of human beings.
Its negation is $(\exists\, t \in [0, \infty[)\ (\exists\, x \in \mathbf{H})\ (\sim p(x, t))$, i.e., there is somebody who is not fooled by the politician at least for one moment.

ii) The given statement is
$(\forall\, x \in \mathbf{R})\ (\exists\, y \in \mathbf{R})\ (x = y^2)$. Its negation is
$(\exists\, x \in \mathbf{R})\ (\forall\, y \in \mathbf{R})\ (x \neq y^2)$, i.e.,
there is a real number which is not the square of any real number.

iii) The given statement is
$(\exists\, x \in L)\ (\forall\, t \in [0, \infty[)p(x, t)$, where L is the set of lawyers and p(x, t) : x does not lie at time t. The negation is
$(\forall\, x \in L)\ (\exists\, t \in [0, \infty[)\ (\sim p)$, i.e., every lawyer tells a lie at some time.

E22) i) For example,

$(\forall\, \mathbf{x} \in \mathbf{N})\ (\exists\, y \in \mathbf{Z})\ (\frac{x}{y} \in \mathbf{Q})$ is a true statement. Its negation is

$\exists\, \mathbf{x} \in \mathbf{N})\ (\forall\, y \in \mathbf{Z})\ (\frac{x}{y} \notin \mathbf{Q})$

You can try (ii) similarly.

E23) (i), (iii) are true.
(ii) is false (e.g., for x = -1 there is no y such that $y^2 = x^3$).

(iv) is equivalent to $(\forall\ x \in \mathbf{R})\ [\sim (\exists!\ y \in \mathbf{R})\ (x + y = 0)]$, i.e., for every x there is no unique y such that $x + y = 0$. This is clearly false, because for every x **there is** a unique $y(= - x)$ such that $x + y = 0$.

UNIT 2 METHODS OF PROOF

Structure

2.0 INTRODUCTION

In the previous unit you studied about statements and their truth values. In this unit, we shall discuss ways in which statements can be linked to form a logically valid argument. Throughout your mathematical studies you would have come across the terms 'theorem' and 'proof'. In sec. 2.2, we shall talk about what a theorem is and what constitutes a mathematically acceptable proof.

In Sec 2.3, we shall discuss some ideas formalised by the English mathematician Boole and the German logician Frege (1848-1925). These are the different methods used for proving or disproving a statement. As you go through the different types of **valid arguments,** please try and find connections with what we discussed in Block 1.

Fig. 1: George Boole (1815-1864)

The principle of mathematical induction has a very special place in mathematics because of its simplicity and vast applicability. You will revisit this tool for proving statements in sec. 2.4.

Please go through this unit carefully. You need to be able to convince your learners that its contents are part of the foundation on which all mathematical knowledge is built.

2.1 OBJECTIVES

After reading this unit, you should be able to develop in your learners the ability to:

- explain the terms 'theorem', 'proof' and 'disproof';
- describe the direct method and some indirect methods of proof;
- state and apply both forms of the principle of induction

2.2 WHAT IS A PROOF?

Suppose I tell somebody, "I am stronger than you." The person is quite likely to turn around, look menacingly at me, and say, "Prove it!" What she or he really wants is to be convinced of my statement by some evidence. (In this case it would probably be a big physical push!)

Convincing evidence is also what the world asks for before accepting a scientist's predictions, or a historian's claims.

In the same way, if you want a mathematical statement to be accepted as true, you would need to provide **mathematically acceptable** evidence to support it. This means that you would need to show that the statement is **universally** true. And this would be done in the form of a logically valid argument.

Definition: An **argument** (in mathematics or logic) is a finite sequence of statements $p_1, \cdots, p_n$, p such that $(p_1 \wedge p_2 \wedge \cdots \wedge p) \rightarrow p$.

Each statement in the sequence, $p_1, p_2, \ldots, p_n$ is called a **premise** (or an **assumption,** or a **hypothesis**). The final statement p is called the **conclusion.**

Let's consider an example of an argument that shows that a given statement is true.

Example 1: Give an argument to show that the mathematical statement
'For any two sets A and B, $A \cap B \subseteq A$' is true.

Solution: One argument could be the following.
Let x be an arbitrary element of $A \cap B$.
Then $x \in A$ and $x \in B$, by definition of '$\cap$'.
Therefore, $x \in A$.
This is true for every x in $A \cap B$.
Therefore, $A \cap B \subseteq A$, by definition of `$\subseteq$'.

***

The argument in Example 1 has a peculiar nature. The truth of each of the 4 premises and of its conclusion follows from the truth of the earlier premises in it. Of course, we start by assuming that the first statement is true. Then, assuming the definition of `intersection', the second statement is true. The third one is true, whenever the second one is true because of the properties of logical implication. The fourth statement is true whenever the first three are true, because of the definition and properties of the term `for all'. And finally, the last statement is true whenever all the earlier ones are. In this way we have shown that the given statement is true. In other words, we have proved the given statement, as the following definition show.

Definitions: We say that a proposition p **follows logically from** propositions $p_1, p_2, \cdots, p_n$ if p must be true whenever $p_1, p_2, \cdots, p_n$ are true, i.e.,
$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow p$.
[Here, **note** the use of the implication arrow `$\Rightarrow$'. For any two propositions r and s, `$r \Rightarrow s$' **denotes 's is true whenever r is true.'** Note that, using the contrapositive, this also denotes `r is false whenever s is false'. Thus `$r \rightarrow s$' and `$r \Rightarrow s$' are different except when both r and s are true or both are false.]

A **proof** of a proposition p is a mathematical argument consisting of a sequence of statements $p_1, p_2, \cdots, p_n$ from which p logically follows. So, p is the conclusion of this argument.

The statement that is proved to be true is called a **theorem.**

Sometimes, as you will see in Sec.2.3.3, instead of showing that a statement p is true, we try to prove that it is false, i.e., that ~ p is true. Such a proof is called a **disproof** of p. In the next section you will read about some ways of disproving a statement.

Sometimes it happens that we feel a certain statement is true, but we don't succeed in proving it. It may also happen that we can't disprove it. Such statements are called **conjectures.** If and when a conjecture is proved, it would be called a theorem. If it is disproved, then its negative will be a theorem!

In this context, there's a very famous conjecture which was made by a mathematician **Goldbach** in 1742. He stated that :

For every n ∈ N. If n is even and n > 2, then n is the sum of two primes.
To this day, no one has been able to prove it or disprove it. To disprove it several people have hunting for an example for which the statement is not true, i.e., an even number n>2 such that n cannot be written as the sum of two prime numbers.

Now, as you have seen, a mathematical proof of a statement consists of one or more premises. These premises could be of four types:

i) a proposition that has been proved earlier (e.g., to prove that the complex roots of a polynomial in **R**[x] occur in pairs, we use the division algorithm); or
ii) a proposition that follows logically from the earlier propositions given in the proof (as you have seen in Example 1); or
iii) a mathematical fact that has never been proved, but is universally accepted as true (e.g., two points determine a line). Such a fact is called an **axiom** (or a **postulate**);
iv) the definition of a mathematical term (e.g., assuming the definition of '$\subseteq$' in the proof of $A \cap B \subseteq A$).

You will come across more examples of each type while doing the following exercises, and while going through proofs in this course and other course.

E1) Write down an example of a theorem, and its proof (of at least 4 steps), taken from school-level algebra. At each step, indicate which of the four types of premise it is.
E2) Is every statement a theorem? Why?

So far we have spoken about valid, or acceptable, arguments. Now let us see an example of a sequence of statements that will not form a valid argument. Consider the following sequence.

> If Maya sees the movie, she won't finish her homework.
> Maya won't finish her homework.
> Therefore, Maya sees the movie.

Looking at the argument, can you say whether it is valid or not? Intuitively you may feel that the argument isn't valid. But, is there a formal logical tool that you can apply check if your intuition is correct? What about truth tables? Let's see.

The given argument is of the form
$[(p \to q) \wedge q] \Rightarrow p$, where
p: Maya sees the movie, and
q: Maya won't finish here homework.

Let us look at the truth table related to this argument (see Table 1).

Table 1.

p	q	$p \to q$	$(p \to q) \wedge q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	F

This last column gives the truth values of the premises. The first column given corresponding truth values of the conclusion. **Now, the argument will only be valid if whenever both the premises are true, the conclusion is true.** This happens in the first row, but not in the third row. **Therefore, the argument is** not valid.

Why don't you check an argument for validity now?

E3) Check whether the following argument is valid
$(p \to q \vee \sim r) \wedge (q \to p) \Rightarrow (p \to r)$

You have seen that a proof is a logical argument that verifies the truth of a theorem. There are several ways of proving a theorem, as you will see in the next section. All of them are based on one or more **rules of inference**, which are different forms of arguments. We shall now present four of the most commonly used rules.

'Modus ponens' is a Latin term which means 'method of affirmation'.

i) Law of detachment (or modus ponens)
Consider the following argument:

If Kali can draw, she will get a job.
Kali can draw.
Therefore, she will get a job.

To study the form of the argument, let us take p to be the proposition 'Kali can draw'. And q to be the proposition 'Kali will get a job.' Then the premises are $(p \to q)$ and p. The conclusion is q.

So, the form of the argument is

$$\begin{array}{l} p \to q \\ \underline{\quad p \quad} \\ \therefore q \end{array} \text{, i.e., } [(p \to q) \wedge p] \Rightarrow q.$$

$\therefore$ denotes 'therefore'.

Is this argument valid? To find out, let's construct its truth table (see Table 2).

Table 2: Truth table for $[(p \to q) \wedge p] \Rightarrow q$

p	q	$p \to q$	$(p \to q) \wedge q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

In the table, look at the second column (the conclusion) and the fourth column (the premises). Whenever the premises are true, i.e., in Row 1, the conclusion is true. Therefore, the argument is valid.

This form of valid argument is called the law of detachment because the conclusion q is detached from a premise (namely, $p \to q$). It is also called the **law of direct inference**.

'Modus tollens' means 'method of denial'.

ii) Law of contraposition (or modus tollens)
To understand this law, consider the following argument:

If Kali can draw, then she will get a job.
Kali will not get a job.
Therefore, Kali can't draw.

Taking p and q as in (i) above, you can see that the premises are $p \to q$ and $\sim q$. The conclusion is $\sim p$

So the argument is
$p \to q$
$\sim q$, i.e., $[(p \to q) \wedge \sim] \Rightarrow \sim p$.

If you check, you'll find that this is a valid form of argument. There are two more rules of inference that most commonly form the basis of several proofs. The following exercise is about them.

E4) You will find three arguments below. Convert each of them into the language of symbols, and check if they are valid.

i) Either the eraser is white or oxygen is a metal.
The eraser is black.
Therefore, oxygen is a metal.

ii) If madhu is a 'sarpanch', she will head the 'panchayat'.
If Madhu heads the 'panchayat', she will decide on property disputes.
Therefore, if Madhu is a 'sarpanch', she will decide on property disputes.

iii) Either Munna will cook or Munni will practise Karate.
If Munni practices Karate, then Munna studies.
Munna does not study.
Therefore, Munni will practise Karate.

E5) Write down one example each of modus ponens and modus tollens.

As you must have discovered, the arguments in E4(i) and (ii) are valid. The first one is an example of a **disjunctive syllogism**. The second one is an example of a **hypothetical syllogism.**

Thus, a disjunctive syllogism is of the form

$$\begin{array}{l} p \vee q \\ \sim \dfrac{p}{q} \end{array} \quad \text{i.e., } [(p \vee q) \wedge {\sim}p] \Rightarrow q.$$

And, a hypothetical syllogism is of the form

$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array} , \quad \text{i.e., } [(p \to q) \wedge (q \to r)] \Rightarrow (p \to r).$$

Let us now see how different forms of arguments can be put together to prove or disprove a statement.

2.3 DIFFERENT METHODS OF PROOF

In this section we shall consider three different strategies for proving a statement. We will also discuss a method that is used only for disproving a statement.

Let us start with a proof strategy based on the first rule of inference that we discussed in the previous section.

2.3.1 Direct Proof

This form of proof is based entirely on modus ponens. Let us formally spell out the strategy.

Definition: A direct proof of $p \Rightarrow q$ is a logically valid argument that begins with the assumptions that p is true and, in one or more applications of the law of detachment, concludes that q must be true.

So, to construct a direct proof of $p \Rightarrow q$, we start by assuming that p is true. Then, in one or more steps of the form $p \Rightarrow q_1$, $q_1 \Rightarrow q_2, \ldots\ldots, q_n \Rightarrow q$, we conclude that q is true. Consider the following examples.

Example 2: Give a direct proof of the statement 'The product of two odd integers is odd'.

Solution: Let us clearly analyse what our hypotheses are, and what we have to prove. We start by considering any two odd integers x and y. So our hypothesis is p: x and y are odd.

The conclusion we want to reach is

q : xy is odd.

Let us first prove that $p \Rightarrow q$.
Since x is odd, $x = 2m + 1$ for some integer m.
Similarly, $y = 2n + 1$ for some integer n.
Then $xy = (2m + 1) (2n + 1) = 2(2mn + m + n) +1$
Therefore, xy is odd.
So we have shown that $p \Rightarrow q$.

Now we can apply modus ponens to $p \wedge (p \Rightarrow q)$ to get the required conclusion.

Note that the essence of this direct proof lies in showing $p \Rightarrow q$.

***

Example 3: Give a direct proof of the theorem 'The square of an even integer is an even integer.'

Solution: First of all, let us write the given statement symbolically, as

$(\forall\ x \in \mathbf{Z})(p(x) \Rightarrow q(x))$

where p(x) : x is even, and

q(x) : x^2 is even, i.e., q(x) is the same as p (x^2).

The direct proof, then goes as follows.

Let x be an even number (i.e., we assume p(x) is true).
Then x = 2n, for some integer n (we apply the definition of an even number).
Then $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$.
x^2 is even (i.e., q (x) is true).

***

Why don't you try an exercise now?

E6) Give a direct proof of the statement 'If x is a real number such that $x^2 = 9$, then either x=3 or x = -3. '.

Let us now consider another proof strategy.

2.3.2 Indirect Proofs

In this sub-section we shall consider two roundabout methods for proving $p \Rightarrow q$.

Proof by contrapositive: In the first method, we use the fact that the proposition $p \Rightarrow q$ is logically equivalent to its contrapositive $(\sim q \Rightarrow \sim p)$,
i.e.,

$$(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p).$$

For instance, 'If Ammu does not agree with communalists, then she is not orthodox.' is the same as 'If Ammu is orthodox, then she agrees with communalists.'.

Because of this equivalence, to prove p $\Rightarrow$ q, we can, instead, prove ~ q $\Rightarrow$ ~ p. This means that we can assume that ~ q is true, and then try to prove that ~ p is true. In other words, **what we do to prove p $\Rightarrow$ q in this method is to assume that q is false and then show that p is false.** Let us consider an example.

Example 4: Prove that 'If x, y $\in$ **Z** such that xy is odd, then both x and y are odd.', by proving its contrapositive.

Solution: Let us name the statements involved as below.
p : xy is odd
q : both x and y are odd.
So,
~ p : xy is even, and
~ q : x is even or y is even, or both are even.

We want to prove p $\Rightarrow$ q, by proving that ~ q $\Rightarrow$ ~ p. So we start by assuming that ~ q is true, i.e., we suppose that x is even.

The x = 2n for some n $\in$ **N**.
Therefore, xy = 2ny.
Therefore xy is even, by definition.
That is, ~ p is true.

So, we have shown that ~ q $\Rightarrow$ ~p. Therefore, p $\Rightarrow$ q.

Why don't you ask your students to try some related exercises now?

E7) Write down the contrapositive of the statement 'If f is a 1-1 function from a finite set X into itself, then f must be subjective.'.
E8) Prove the statement 'If x is an integer and x^2 is even, then x is also even.' By proving its contrapositive.

And now let us consider the other way of proving a statement indirectly.

Proof by contradiction: In this method, to prove q is true, we start by assuming that q is false (i.e., ~ q is true). Then, by a logical argument we arrive at a situation where a statement is true as well as false, i.e., we reach a contradiction r $\wedge$ ~ r for some statement that is always false. This can only happen when ~ q is false also. Therefore, q must be true.

This method is called **proof by contradiction**. It is also called reductio ad absurdum (a Latin phrase) because it relies on reducing a given assumption to an absurdity.

Let us consider an example of the use of this method.

Example 5: Show that $\sqrt{5}$ is irrational.

Solution: Let us try and prove the given statement by contradiction. For this, we begin by assuming that $\sqrt{5}$ is rational. This means that there exist positive integers a and b such that $\sqrt{5} = \frac{a}{b}$, where a and b have no common factors.

This implies a = $\sqrt{5}$ b $\Rightarrow a^2 = 5b^2 \Rightarrow 5|a^2 \Rightarrow 5|a$.

Therefore, by definition, a = 5c for some c $\in$ **Z**.
Therefore, $a^2 = 25c^2$.

But $a^2 = 5b^2$ also.

So $25c^2 = 5b^2 \Rightarrow 5c^2 = b^2 \Rightarrow 5|b^2 \Rightarrow 5|b$.

But now we find that 5 divides both a and b, which contradicts our earlier assumption that a and b have no common factor.
Therefore, we conclude that our assumption that $\sqrt{5}$ is rational is false, i.e, $\sqrt{5}$ is irrational.

***

We can also use the method of contradiction to prove an implication $r \Rightarrow s$. Here we can use the equivalence $\sim (r \rightarrow s) \equiv r \wedge \sim s$. So, to prove $r \Rightarrow s$, we can begin by assuming that $r \Rightarrow s$ is false, i.e., r is true and s is false. Then we can present a valid argument to arrive at a contradiction.

Consider the following example from plane geometry.

Example 6: Prove the following:
If two distinct lines L_1 and L_2 intersect, then their intersection consists of exactly one point.

Solution: To prove the given implication by contradiction, let us begin by assuming that the two distinct lines L_1 and L_2 intersect in more than one point. Let us call two of these distinct points A and B. Then, both L_1 and L_2 contain A and B. This contradicts the axiom from geometry that says 'Given two distinct points, there is exactly one line containing them.'.

Therefore, if L_1 and L_2 intersect, then they must intersect in only one point.

***

The contradiction rule is also used for solving many logical puzzles by discarding all solutions that educe to contradictions. Consider the following example.

Example 7: There is a village that consists of two types of people – those who always tell the truth, and those who always lie. Suppose that you visit the village and two villagers A and B come up to you. Further, suppose
A says, "B always tells the truth," and
B says, "A and I are of opposite types".
What types are A and B ?

Solution: Let us start by assuming A is a truth-teller.
$\therefore$ What A says is true.
$\therefore$ B is a truth-teller.
$\therefore$ What B says is true.
$\therefore$ A and B are of opposite types.

This is a contradiction, because our premises say that A and B are both truth-tellers.
$\therefore$ The assumption we started with is false.
$\therefore$ A always tells lies.
$\therefore$ What A has told you is lie.
$\therefore$ B always tells lies.
$\therefore$ A and B are of the same type, i.e., both of them always lie.

***

Here are a few exercises for you now. While doing them you would realize that there are situations in which all the three methods of proof we have discussed so far can be used.

E9) Use the method of proof by contradiction to show that
ii) For $x \in \mathbf{R}$, if $x^3 + 4x = 0$, then $x = 0$

E10) Prove E 9(ii) directly as well as by the method of contrapositive.

E11) Suppose you are visiting the village described in Example 7 above. Another two villagers C and D approach you. C tells you, "Both of us always tell the truth," and D says, "C always lies." What types are C and D?

There can be several ways of proving a statement.

Let us now consider the problem of showing that a statement is false.

2.3.3 Counterexamples

Suppose I make the statement 'All human beings are 5 feet tall.'. You are quite likely to show me an example of a human being standing nearby for whom the statement is not true. And, as you know, the moment we have even one example for which the statement $(\forall x)p(x)$ is false [i.e.,$(\exists x)$ $(\sim p(x))$ is true], then the statement is false.

An example that shows that a statement is false is a **counterexample** to such a statement. The name itself suggests that it is an example to counter a given statement.

A common situation in which we look for counterexamples is to disprove statements of the form $p \rightarrow q$ needs to be an example where $p \wedge \sim q$. Therefore, a counterexample to $p \rightarrow q$ needs to be an example where $p \wedge \sim q$ is true, i.e., p is true and $\sim q$ is true, i.e., the hypothesis p holds but the conclusion q does not hold.

For instance, to disprove the statement 'If n is an odd integer, then n is prime.', we need to look for an odd integer which is not a prime number. 15 is one such integer. So, n = 15 is a counterexample to the given statement.

Notice that a **counterexample to a statement p proves that** p is false, i.e., ~ **p is true**.

Let us consider another example.

Example 8: Disprove the following statement:

$(\forall\, a \in \mathbf{R})\ (\forall\, b \in \mathbf{R})\ [(a^2 = b^2) \Rightarrow (a = b)]$.

Solution: A good way of disproving it is to look for a counterexample, that is, a pair of real numbers a and b for which $a^2 = b^2$ but $a \neq b$. Can you think of such a pair? What about a = 1 and b = -1? They serve the purpose.

In fact, there are infinitely many counterexamples. (Why?)

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Now, an exercise!

E12) Disprove the following statements by providing a suitable counterexample.
i) $\forall\, x \in \mathbf{Z}, x \in \mathbf{Q} \setminus \mathbf{N}$.
ii) $(x+y)^n = x^n + y^n\ \forall\, n \in \mathbf{N}, x, y \in \mathbf{Z}$.
iii) $f : \mathbf{N} \rightarrow \mathbf{N}$ is 1-1 iff f is onto.
(**Hint:** To disprove $p \Leftrightarrow q$ it is enough to prove that $p \Leftrightarrow q$ is false or $q \Rightarrow p$ is false.)

There are some other strategies of proof, like a constructive proof, which you must have come across in other mathematics courses. We shall not discuss this method here.

Other proof-related adjectives that you will come across are **vacuous** and **trivial**.

A **vacuous proof** make use of the fact that if p is false, the p → q is true, regardless of the truth value of q. So, to vacuously prove p → q, all we need to do is to show that p is false. For instance, suppose we want to prove that 'If n > n + 1 for n ∈ **Z**, then n^2 = 0.'.

Since 'n . n + 1' is false for every n ∈ **Z**, the given statement is vacuously true, or true by default.

Similarly, a **trivial proof** of p → q is one based on the fact that if q is true, then p → q is true, regardless of the truth value of p. So, for example, 'If n > n + 1 for n ∈ **Z**, then n + 1 > n' is trivially true since n + 1 > n ∀ n ∈ **Z**. The truth value of the hypothesis (which is false in this example) does not come into the picture at all.

Here's a chance for you to think up such proofs now!

E13) Give one example each of a vacuous proof and trivial proof.

And now let us study a very important technique of proof for statements that are of the form p(n), n ∈ **N**.

2.4 PRINCIPLE OF INDUCTION

In a discussion with some students the other day, of them told me very cynically that all Indian politicians are corrupt. I asked him how he had reached such a conclusion. As an argument he gave me instances of several politicians, all of whom were known to be corrupt. What he had done was to formulate his general opinion of politicians on the basis of several particular instances. This is an example of **inductive logic**, a process of reasoning by which general rules are discovered by the observation of several individual cases. Inductive reasoning is used in all the sciences, including mathematics. But in mathematics we use a more precise form.

Precision is required in mathematical induction because, as you know, a statement of the form (∀ n ∈**N**)p(n) is true only if it can be shown to be true for each n in **N**. (In the example above, even if the student is given an example of one clean politician, he is not likely to change his general opinion.)

How can we make sure that our statement p(n) is true for each n that we are interested in? To answer this, let us consider an example.

Suppose we want to prove that 1+2+3+……+n = $\frac{n(n+1)}{2}$ for each n ∈ **N**. Let us call p(n) the predicate '1+2+…+n = $\frac{n(n+1)}{2}$. Now, we can verify that it is true for a few values, say, n = 1, n = 5, n = 10, n = 100, and so on. But we still can't be sure that it will be true for some value of n that we haven't tried.

But now, suppose we can show that if p(n) is true for some n,n = k say, then it will be true for n = k + 1. Then we are in a very good position because we already know that p(1) is true. And, since p(1) is true, so is p(1+1), i.e., p(2), and so on. In this way we can show that p(n) is true for every n ε **N**. So, our proof boils down to two steps, namely,

i) Checking that p(1) is true;

ii) Proving that whenever p(k) is true, then p(k+1) is true, where $k \in \mathbf{N}$.

This is the principle that we will now state formally, in a more general form.

Principle of Mathematical Induction (PMI): Let p(n) be a predicate involving a natural number n. Suppose the following two conditions hold:

i) p(m) is true for some $m \varepsilon \mathbf{N}$;
ii) If p(k) is true, then p(k+1) is true, where $k(\geq m)$ is any natural number.

Then p(n) is true for every $n \geq m$.

Looking at the two conditions in the principle, can you make out why it works? (As a hint, put m = 1 in our example above.)

Well, (i) tells us that p(m) is true. Then putting k = m in (ii), we find that p(m + 1) is true. Again, since p(m+1) is true, p(m+2) is true, and so on.

Going back to the example above, let us complete the second step. We know that p(k) is true, i.e.,$1 + 2+\ldots+k = \frac{k(k+1)}{2}$. We want to check if p(k + 1) is true. So let us find

$$\begin{aligned} 1 + 2+\ldots + (k + 1) &= (1+2+\ldots+k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k + 1), \text{ since p(k) is true} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

So, p(k + 1) is true.

And so, by the principle of mathematical induction, we know that p(n) is true for every $n \in \mathbf{N}$.

What does this principle really say? It says that if you can walk a few steps, say m steps, and if at each stage you can walk one more step, then you can walk and distance. It sounds very simple, but you may be surprised to know that the technique in this principle was first used by Europeans only as late as the 16th century by the Venetian F. Maurocylus (1494-1573). He used it to show that $1+3+ \ldots +(2n - 1) = n^2$. Pierre de Fermat (1601 – 1665) improved on the technique and proved that this principle is equivalent to the following often-used principle of mathematics.

The Well-ordering Principle: Any non-empty subset of N contains a smallest element.

You may be able to see the relationship between the two principles if we reword the PMI in the following form.

Principle of Mathematical Induction (Equivalent form): Let $\mathbf{S} \subseteq \mathbf{N}$ be such that
  i) $m \in \mathbf{S}$
  ii) For each $k \in \mathbf{N}$, $k \geq m$, the following implication is true: $K \in \mathbf{S} \Rightarrow k \;\; 1 \in S$.
    Then $\mathbf{S} = \{m, m + 1, m + 2, \ldots\}$.

The term 'mathematical induction' was first used by De Morgan.

Can you see the equivalence of the two forms of the PMI? If you take $S = \{ n \in \mathbf{N} \mid p(n) \text{ is true } \}$ then you can see that the way we have written the principle above is a mere rewrite of the earlier form.

Now, let us consider an example of proof using PMI.

Example 9: Use mathematical induction to prove that

Note that p(n) is a predicate, not a statement, unless we know the value of n.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}\ (n + 1)\ (2n + 1)\ \forall\ n \in \mathbf{N}.$$

Solution: We call p(n) the predicate

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}\ (n + 1)\ (2n + 1).$$

Since we want to prove it for every $n \in \mathbf{N}$, we take m = 1.

Step 1: p(1) is $1^2 = \frac{1}{6}\ (1 + 1)\ (2 + 1)$, which is true

Step 2: Suppose for an arbitrary $k \in \mathbf{N}$, p(k) is true, i.e.,

$$1^2 + 2^2 + \ldots + k^2 = \frac{k}{6}(k + 1)\ (2k + 1) \text{ is true.}$$

Step 3: To check if the assumption in step 2 implies that p(k + 1) is true. Let's see.

$$P(k + 1) \text{ is } 1^2 + 2^2 + \ldots + k^2 + (k + 1)^2 = \frac{k+1}{6}\ (k + 2)\ (2k + 3)$$

$$\Leftrightarrow (1^2 + 2^2 + \ldots + k^2) + (k + 1)^2 \quad \frac{k+1}{6}(k + 2)\ (2k + 3)$$

$$\Leftrightarrow \frac{k}{6}(k + 1)\ (2k + 1) + (k + 1)^2 = \frac{k+1}{6}\ (k + 2)\ (2k + 3),$$

since p(k) is true.

$$\Leftrightarrow \frac{k+1}{6}[k(2k + 1) + 6(k + 1)] = \frac{k+1}{6}\ (k + 2)\ (2k + 3)$$

$$\Leftrightarrow 2k^2 + 7k + 6 = (k + 2)\ (2k + 3), \text{ dividing throughout by } \frac{k+1}{6},$$

which is true.

So, p(k) is true implies that p(k + 1) is true.

So, both the conditions of the principle of mathematical induction hold. Therefore, its conclusion must hold, i.e., p(n) is true for every n ε **N**.

***

Have you gone through Example 9 carefully? If so, you would have noticed that the proof consists of three steps:

Step 1: (called the **basis of induction**): Checking if p(m) is true for some $m \in \mathbf{N}$.

Step 2: (called the **induction hypothesis**): Assuming that p(k) is true for an arbitrary $k \in \mathbf{N}$, $k \geq m$.

Step 3: (called the **induction step**): Showing that p(k + 1) is true, by a direct or an indirect proof.

Now let us consider an example in which $m \neq 1$.

Example 10: Show that $2^n > n^3$ for ≥ 10.

Solution: We write p(n) for the predicate '$2^n > n^3$'.

Step 1: For n = 10, $2^{10} = 1024$, which is greater that 10^3. Therefore, p(10) is true.

Step 2: We assume that p(k) is true for an arbitrary $k \geq 10$. Thus, $2^k > k^3$.

Step 3: Now, we want to prove that $2^{k+1} > (k + 1)^3$.

$2^{k+1} = 2.2^k > 2.k^3$, by our assumption

$> (1 + \frac{1}{10})^3 .k^3$, since $2 > (1 + \frac{1}{10})^3$

$\geq (1 + \frac{1}{k})^3 \ .k^3$, since ≥ 10

$= (k + 1)^3$.

Thus, p(k + 1) is true if p(k) is true for $k \geq 10$.

Therefore by the principle of mathematical induction, p(n) is true $\forall$ $n \geq 10$.

***

Why don't you try to apply the principle now?

E14) Use mathematical induction to prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n} \ \forall \ n \in \mathbf{N}.$$

E15) Show that for any integer n > 1, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

(**Hint**: The basis of induction is p(2).).

Before going further a **note of warning!** To prove that p(n) is true $\forall$ $n \geq m$, both the basis of induction **as well as** the induction step must hold. If even one of these conditions does not hold, we cannot arrive at the conclusion that p(n) is true $\forall$ $n \geq m$.

For example, suppose p(n) is $(x + y)^n \leq x^n + y^n \ \forall \ x,y \in \mathbf{R}$. Then p(1) is true. But Steps 2 and 3 do not hold. Therefore, p(n) is not true for every $n \in \mathbf{N}$. (Can you find a value of n for which p(n) is false?)

As another example, take p(n) to be the statement ' $1 + 2 + \ldots + n < n$'. Then, if p(k) is true, so is p(k + 1) (prove it!). But the basis step does not hold for any $m \in \mathbf{N}$. And, as you can see, p(n) is false.

Now let us look at a situation in which we may expect the principle of induction to work, but it doesn't. Consider the sequence of numbers 1,1,2,3,5,8,…. These are the **Fibonacci numbers**, named after the Italian mathematician Fibonacci. Each term in the sequence, from the third on, is obtained by Fibonacci. Each term in the sequence, from the third term on, is obtained by adding the previous 2 terms. So, if a_n is the nth term, then $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2} \ \forall \ n \geq 3$.

Suppose we want to show that $a_n < 2^n \ \forall \ n \in \mathbf{N}$ using the PMI. Then, if p(n) is the predicate $a_n < 2^n$, we know that p(1) is true.

Now suppose we know that p(k) is true for an arbitrary $k \in \mathbf{N}$, i.e., $a_k < 2^k$. We want to show that $a_{k+1} < 2^{k+1}$, i.e., $a_k + a_{k-1} < 2^{k+1}$. But we don't know anything about a_{k-1} . So how can we apply the principle of induction in the form that we have stated it? In such a situation, a stronger, more powerful, version of the principle of induction comes in handy. Let's see what this is.

Principle of Strong Mathematical Induction: Let p(n) be a predicate that involves a natural number n. Suppose we can show that

i) p(m) is true for some $m \in \mathbf{N}$, and
ii) Whenever p(m), p(m +1), …, p(k) are true, then p(k + 1) is true, where $k \geq m$.

Then we can conclude that p(n) is true for all natural numbers $n \geq m$.

Why do we call this principle stronger than the earlier one? This is because, in the induction step we are making more assumptions, i.e., that p(n) is true for every n lying between m and k, not just that p(k) is true.

In using the strong form we often need to check Step 1 for more than one value of n.

Let us now go back to the fibonacci sequence. To use the strong form of the PMI, we take m = 1. We have seen that p(1) is true. We also need to see if p(2) is true. This is because we have to use the relation $a_n = a_{n-1} + a_{n-2,}$ which is valid for $n \geq 3$.

Now that we know that both p(1) and p(2) are true, let us go the next step. In step 2, for an arbitrary $k \geq 2$, we assume that p(n) is true for every n such that $1 \leq n \leq k$, i.e., $a_n < 2^n$ for $1 \leq n \leq k$.

Finally, in Step 3, we must show that p(K + 1) is true, i.e., $a_{k+1} < 2^{k+1}$. Now

$$a_{k+1} = a_k$$
$$< 2^k + 2^{k-1}, \text{ by our assumption in Step 2.}$$
$$= 2^{k-1}(2+1)$$
$$< 2^{k-1}.2^2$$
$$= 2^{k+1}$$

P(k + 1) is true.

P(n) is true $\forall$ n $\in$ **N**.

Though the "strong" form of the PMI appears to be different from the "weak" from, the **two are actually equivalent**. This is because each can be obtained from the other. So, we can use either form of mathematical induction. In a given problem we use the form that is more suitable. For instance, in the following example, as in the case of the one above, you would agree that it is better to use the strong form of the PMI.

Example 11: Use induction to prove that any integer $n \geq 2$ is either a prime or a product of primes.

Solution: Here p(n) is the predicate 'n is a prime or n is a product of primes.'.

Step 1: (**basis of induction**) : since 2 is a prime, p(2) is true.

Step 2: (**induction hypothesis**): Assume that p(n) is true for any integer n such that $2 \leq n \leq k$, i.e., p(3), p(4),…, p(k) are true.

Step 3: (**induction step**): Now consider p(k +1). If k + 1 is a prime, then p(k + 1) is true. If k + 1 is not a prime, then k + 1 = rs, where $2 \leq r \leq k$ and $2 \leq s \leq k$. But, by our induction hypothesis, p(r) is true and p(s) is true. Therefore, r and s are either primes or products of primes. And therefore, k + 1 is a product of primes. So, p(k + 1) is true.

Therefore, p(n) is true $\forall\ n \geq 2$.

***

Why don't you try some exercises now?

E16) If $a_1, a_2, \ldots$ are the terms in the Fibonacci sequence, use the weak as well as the strong forms of the principle of mathematical induction to show that $a_n > \frac{3}{2}\ \forall n \geq 3$. Which form did you find more convenient?

E17) Consider the following "proof" by induction of the statement. 'Any n marbles are of the same size.', and say why it is wrong.

Basis of induction : For n = 1, the statement is clearly true.

Induction hypothesis: Assume that the statement is true for n = k.

Induction step: Now consider any k + 1 marbles 1,2, …, k + 1. By the induction hypothesis the k marbles 2,3,…,k + 1 are of the same size. Therefore, all the k + 1 marbles are of the same size.Therefore, the given statement is true for every n.

E18) Prove that the following result is equivalent to the principle of mathematical induction (strong form):
Let $S \subseteq \mathbf{N}$ such that
i) $m \in \mathbf{S}$
ii) If m, m + 1, m + 2, …, k are in S, then $k + 1 \in S$.
Then $S = \{n \in \mathbf{N} \mid n \geq m\}$.

E19) To prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1 \;\forall\; n \in \mathbf{N}$, which form of the principle of mathematical induction would you use, and why? Also, prove the inequality.

With this we come the end of our discussion on various techniques of proving of disproving mathematical statements. Let us take a brief look at what you have read in this unit.

2.5 SUMMARY

In this unit, you have studied the following points.

1. What constitutes a proof of a mathematical statement, including 4 commonly used rules of inference, namely,
    i) law of detachment (or modus ponen) : $[(p \rightarrow q) \wedge p] \Rightarrow q$
    ii) law of contraposition (or modus tollens) : $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$
    iii) disjunctive syllogism : $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$
2. The description and examples of a direct proof, which is based on modus ponens.
3. Two types of indirect proofs : proof by contrapositive and proof by contradiction.
4. The use of counterexamples for disproving a statement.
5. The "strong" and "weak" forms of the principle of mathematical induction, and their equivalence with the well-ordering principle.

2.6 SOLUTIONS/ ANSWERS

E1) For example,

Theorem: $(x + y)^2 = x^2 + 2xy + y^2$ for $x, y \in \mathbf{R}$.

Proof: for $x, y \in \mathbf{R}$, $(x + y)^2 = (x + y)(x + y)$ (by definition of 'square')
$(x + y)(x + y) = x(x + y) + y(x + y)$ (by distributivity, and by definition of addition and multiplication of algebraic terms).

Therefore, $(x + y)^2 = x^2 + 2xy + y^2$ (using an earlier proved statement that a =b and b = c implies that a = c).

E2) No, not unless it has been proved to be true

E3) premises ↓ conclusion ↓

p	q	r	$\sim r$	$q \vee \sim r$	$p \to q \vee \sim r$	$q \to p$	$p \to r$
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	F
T	F	T	F	F	F	T	T
T	F	F	T	T	T	T	F
F	T	T	F	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	F	F	T	T	T
F	F	F	T	T	T	T	T

The premises are true in Rows 1, 2, 4, 7, 8. So, the argument will be valid if the conclusion is also true in these rows. But this does not happen in Row 2, for instance. Therefore, the argument is invalid.

E4) i) Let p : The eraser is white,
q : Oxygen is a metal.
Then the argument is

$$\begin{array}{l} p \vee q \\ \sim \underline{p} \\ \quad q \end{array}$$

Its truth table is given below.

conclusion ↓ premises ↓

p	q	$\sim p$	$p \vee q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

All the premises are true only in the third row. Since the conclusion in this row is also true, the argument is valid.

ii) The argument is $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$
where p: Madhu is a 'sarpanch',
q : Madhu heads the 'Panchayat'.
r : Madhu decides on property disputes.
This is valid because, whenever both the premises are true, so is the conclusion (see the following table.)

premises ↓ ↓ conclusion ↓

P	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

iii) The argument is $[(p \vee q) \wedge (q \rightarrow r) \wedge \sim r] \Rightarrow q$
Where p: Munna will cook.
q: Munni will practise Karate.
r: Munna studies.

This is **not valid**, as you can see from Row 4 of the following truth table.

conclusion ↓ premises ↓

↓

p	Q	r	~ r	p ∨ q	Q → r
T	T	T	F	T	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	T	T	F
F	F	T	F	F	T
F	F	F	T	F	T

E5) We need to prove $p \Rightarrow q$, where
p: $x \in \mathbf{R}$ such that $x^2 = 9$, and
q: $x = 3$ or $x = -3$.
Now, $x^2 = 9 \Rightarrow \sqrt{x^2} = \pm\sqrt{9} \Rightarrow x = \pm 3$.

Therefore, p is true and $(p \Rightarrow q)$ is true, allows us to conclude that q is True.

E6) If f is not surjective, then f is not a 1-1 function from X into itself.

E7) We want to prove $\sim q \Rightarrow \sim p$, where
p: $x \in \mathbf{Z}$ such that x^2 is even,
q: x is even.
Now, we start by assuming that q is false, i.e., x is odd.
Then $x = 2m + 1$ for some $m \in Z$.
Therefore, $x^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$
Therefore, x^2 is odd, i.e., p is false.
Thus, $\sim q \Rightarrow \sim p$, and hence, $p \Rightarrow q$.

E8) i) This is on the lines of Example 5.
ii) Let us assume that $x^3 + 4x = 0$ and $x \neq 0$. Then $x(x^2 + 4) = 0$ and $x \neq 0$. Therefore, $x^2 + 4 = 0$, i.e., $x^2 = -4$. But $x \in \mathbf{R}$ and $x^2 = -4$ is a contradiction. Therefore, our assumption is false. Therefore, the given statement is true.

E9) Direct proof: $x^3 + 4x = 0 \Rightarrow x(x^2 + 4) = 0$
$\Rightarrow x = 0$ or $x^2 + 4 = 0$
$\Rightarrow x = 0$, since $x^2 \neq -4 \ \forall \ x \in R$.
Proof by contrapositive: Suppose $x \neq 0$. Then $x(x^2 + 4) \neq 0$ for any $x \in \mathbf{R}$.
$x^3 + 4x \neq 0$ for every $x \in \mathbf{R}$.
So we have proved that 'For $x \in \mathbf{R}$, $x \neq 0 \Rightarrow x^3 + 4x \neq 0$.'.
That is, 'For $x \in \mathbf{R}$, $x^3\ \ 4x = 0 \Rightarrow x = 0$.'.

E10) Suppose C tells the truth. Therefore, D always tells the truth. Therefore, C always lies, which is a contradiction. Therefore, C can't be a truth-teller, i.e., C is a liar. Therefore, D is a truth-teller.

E11) i) What about x = 1?

ii) Take n= 2, x = 1 and y = -1, for instance.

iii) Here we can find an example f such that f is 1-1 but not onto, or such that f is onto but not 1-1.
Consider f: $\mathbf{N} \to \mathbf{N}$: (f(x) = x + 10. Show that this is 1-1, but not surjective.

E12) i) **Theorem:** The area of every equilateral triangle of side a and perimeter 2a is divisible by 3.

Proof: Since there is no equilateral triangle that satisfies the hypothesis, the proposition is vacuously true.

ii) **Theorem:** If a natural number c is divisible by 5, then the perimeter of the equilateral triangle of side c is 3c.

Proof: Since the conclusion is always true, the proposition is trivially true.

E13) Let p(n) be the given predicate.

Step 1: p(1) : $1 \leq 2 - 1$, which is true.

Step 2: Assume that p(k) is true for some $k \geq 1$, i.e., assume that $1 + \frac{1}{4} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.

Step 3: To show that p(k + 1) is true, consider

$$1 + \frac{1}{4} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} = \left(1 + \frac{1}{4} + \ldots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2}$$
$$\leq \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2}, \text{ by step 2.}$$

Now, $\left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

iff $\frac{1}{(k+1)^2} \leq \frac{1}{k} - \frac{1}{(k+1)}$

iff $k \leq k + 1$, which is true.

Therefore, $\left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

Therefore, p(k + 1) is true.
Thus, by the PMI, p(n) is true $\forall$ n $\in \mathbf{N}$.

E14) p(2) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$, which is true.

Now, assume that p(k) is true for some $k \geq 2$. Then

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k}+\frac{1}{\sqrt{k+1}},\ \text{since p(k) is true.}$$

$$= \frac{\sqrt{k(k+1)}}{\sqrt{k+1}}+1$$

$$> \sqrt{k+1},\ \text{since } \sqrt{k+1} > \sqrt{k}\ .$$

Hence p(k + 1) is true.
P(n) is true ∀ n ≥ 2.

E15) We shall apply the strong form of the PMI here.

Let p(n) : $a_n > \frac{3}{2}$.

Step 1: p(3) and p(4) are true.

Step 2: Assume now that for k ∈ **N**, ≥ 3, p(n) is true for every n such that 3 ≤ n ≤ k.

Step 3: We want to show that p(k + 1) is true. Now

$$a_{k+1} = a_k + a_{k-1} > \frac{3}{2}+\frac{3}{2},\ \text{by step 2}$$

$$> \frac{3}{2}.$$

p(k + 1) is true.
Thus, p(n) is true ∀ n ≥ 3.

In this case, you will be able to use the weak form conveniently too since $a_k > \frac{3}{2}$ is enough for showing that p(k + 1) is true.

Thus, **in this case the weak form is more appropriate** since fewer assumptions give you the same result.

E16) The problem is at the induction step. The first marble may be a different size from the other k marbles. So, we have not shown that p(k+1) is true whenever p(k) is true.

E17) With reference to the statement of the strong form of the PMI, let
S = { n ∈ **N** | p(n) is true }.

Then you can show how the form in this problem is the same as the statement of the strong form of the PMI.

E18) Let p(n) : $\sum_{i=i}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}-1.$

The weak form suffices here, since the assumption that p(k) is true is enough to prove that p(k + 1) is true. We don't need to assume that p(1), p(2),…,p(k – 1) are also true to show that p(k + 1) is true. Let's prove that p(n) is true ∀ n ∈ **N**.

Now, p(1) : 1 ≤ 2 – 1, which is true.
Next, assume that p(k) is true for some k ∈ **N**.

Then $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \leq (2\sqrt{k}-1)+\frac{1}{\sqrt{k+1}}$, since p(k) is true.

Now $2\sqrt{k}-1+\frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}-1$

$\Leftrightarrow 2(\ \sqrt{k+1}-\sqrt{k}) \geq \frac{1}{\sqrt{k+1}}$

$\Leftrightarrow$ 2(k + 1 - $\sqrt{k(k+1)} \geq 1$

$\Leftrightarrow 1 \geq 0$, which is true.

p(k + 1) is true.

p(n) is true $\forall$ n $\in$ **N**.

UNIT 3 BOOLEAN ALGEBRA AND CIRCUITS

Structure

3.0 INTRODUCTION

Fig. 1: Claude Shannon

This unit is very closely linked with Unit 1. It was C.E.Shannon, the founder of information theory, who observed an analogy between the functioning of switching circuits and certain operations of logical connectives. In 1938 he gave a technique based on this analogy to **express and manipulate** simple switching circuits algebraically. Later, the discovery of some new solid state devices (called **electronic switches** or **logic gates**) helped to modify these algebraic techniques and, thereby, paved a way to solve numerous problems related to digital systems algebraically.

In this unit, we shall discuss the symbolic logic techniques which are required for the algebraic understanding of circuits and computer logic. In Sec. 3.2, we shall introduce you to **Boolean algebras** with the help of certain examples based on objects you are already familiar with. You will see that such algebras are apt for describing operations of logical circuits used in computers.

In Sec. 3.3, we have discussed the linkages between **Boolean expressions** and logic circuits.

In Sec. 3.4, you will read about how to express the overall functioning of a circuit mathematically in terms of certain suitably defined functions called **Boolean functions**. In this section we shall also consider a simple **circuit design problem** to illustrate the applications of the relationship between Boolean functions and circuits.

Let us now consider the objectives of this unit.

3.1 OBJECTIVES

After reading this unit, you should be able to:

- define and give examples of Boolean algebras, expressions and functions;
- give algebraic representations of the functioning of logic gates;
- obtain and simplify the Boolean expression representing a circuit;
- construct a circuit for a Boolean expression;
- design and simplify some simple circuits using Boolean algebra techniques.

3.2 BOOLEAN ALGEBRAS

Let us start with some questions: Is it possible to design an electric/electronic circuit without actually using switches(or logic gates) and wires? Can a circuit be redesigned, to get a simpler circuit with the help of pen and paper only?

The answer to both these questions is `Yes'. What allows us to give this reply is the concept of **Boolean algebras**. Before we start a formal discussion on these types of algebras, let us take another look at the objects treated in Unit 1.

As before, let the letters p, q, r,. . . denote statements (or propositions). We write $\mathcal{S}$ for the set of all propositions. As you may recall, a tautology $\mathcal{T}$(or a contradiction $\mathcal{F}$) is any proposition which is always true (or always false, respectively). By abuse of notation, we shall **let $\mathcal{T}$denote the set of all tautologies and $\mathcal{F}$ denote the set of all contradictions.** Thus,

$$\mathcal{T} \leq \mathsf{S},\ \mathcal{F} \leq \mathsf{S}.$$

You already know from Unit 1 that, given two propositions p and q, both p ^ q and p v q are again propositions. And so, by the definition of a binary operation, you can see that both $\wedge$ (**conjunction**) and $\vee$ (**disjunction**) are binary operations on the set S, where we are writing $\wedge$ (p, q) as $p \wedge q$ and v (p, q) as $p \vee q$ $\forall$ p, q $\in$ S.

Again, since ~ p is also a proposition, the operation ~ **(negation)** defines a unary function $\sim: \mathsf{S} \rightarrow \mathsf{S}$. Thus, the set of propositions S, with these operations, acquires an algebraic structure.

As is clear from Sec.1.3, under these three operations, the elements of S satisfy **associative laws, commutative laws, distributive laws** and **complementation laws.**

Also, by E19 of Unit 1, you know that $p \vee \mathcal{F} = p$ and $p \wedge \mathcal{T} = p$, for any proposition p. These are called the **identity laws**. The set S with the three operations and properties listed above is a particular case of an algebraic structure which we shall now define.

In the NCERT textbook, '+' and '.' are used instead of '$\vee$' and '$\wedge$', respectively.

Definition: A Boolean algebra B is an algebraic structure which consists of a set X ($\neq \varnothing$) having two binary operations (denoted by $\vee$ and $\wedge$), one unary operation (denoted by ') and two specially defined elements **O** and **I** (say), which satisfy the following five laws for all x, y, z $\in$ **X**.

B1. Associative Laws: $x \vee (y \vee z) = (x \vee y) \vee z$,
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$

B2. Commutative Laws: $x \vee y = y \vee x$,
$x \wedge y = y \wedge x$

B3. Distributive Laws: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

B4. Identity Laws: $x \vee \mathbf{O} = x$,
$x \wedge \mathbf{I} = x$

B5. Complementation Laws: $x \wedge x' = \mathbf{O}$,
$x \vee x' = \mathbf{I}$.

We write this algebraic structure as **B** = (**X**, $\vee$, $\wedge$, ' , **O, I**), or simply **B**, if the context makes the meaning of the other terms clear. The two operations $\vee$ and $\wedge$ are called the **join operation** and **meet operation**, respectively. The unary operation ' is called the **complementation**.

From our discussion preceding the definition above, you would agree that the set S of propositions is a Boolean algebra, where $\mathcal{T}$and $\mathcal{F}$ will do the job of **I** and **O**, respectively. Thus, (S, $\wedge$, $\vee$, ~, $\mathcal{F}$, $\mathcal{T}$) is an example of a Boolean algebra.

We give another example of a Boolean algebra below.

Example 1: Let X be a non-empty set, and $\mathcal{P}$(**X**) denote its power set, i.e., $\mathcal{P}$(**X**) is the set consisting of all the subsets of the set X. Show that $\mathcal{P}$(**X**) is a Boolean algebra.

Solution: We take the usual set-theoretic operations of intersection ($\cap$), union ($\cup$), and complementation (^{c}) in $\mathcal{P}$(X) as the three required operations. Let ø and X play the roles of **O** and **I**, respectively. Then you can verify that all the conditions for ($\mathcal{P}$ (**X**), $\cup$, $\cap$, ^{c} , **Φ**, **X**) to be a Boolean algebra hold good.

For instance, the identity laws (B4) follow from two set-theoretic facts, namely, `the intersection of any subset with the whole set is the set itself' and `the union of any set with the empty set is the set itself'. On the other hand, the complementation laws (B5) follow from another set of facts from set theory, namely, `the intersection of any subset with its complement is the empty set' and `the union of any set with its complement is the whole set'.

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Yet another example of a Boolean algebra is based on **switching circuits**. For this, we first need to elaborate on the functioning of ordinary switches in a mathematical way. In fact, we will present the basic idea which helped the American, C.E.Shannon, to detect the connection between the functioning of switches and Boole's symbolic logic.

You may be aware of the functioning of a simple on-off switch which is commonly used as an essential component in the electric (or electronic) networking systems. A switch is a device which allows the current to flow only when it is placed in the **ON** position, i.e., when the gap is **closed** by a conducting rod. Thus, the **ON** position of a switch is one state of a switch, called a **closed state**. The other state of a switch is the open state, when it is placed in the **OFF** position. So, a switch has two stable states.

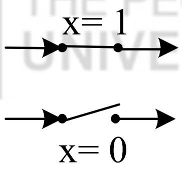


Fig. 2: OFF-ON position

There is another way to talk about the functioning of a switch. We can denote a switch by x, and use the values 0 and 1 to depict its two states, i.e., to convey that x is open we write x = 0, and to convey that x is closed we write x = 1 (see Fig.2).

These values which denote the state of a switch x are called the **state-values** (**s.v.,** in short) of that switch.

We shall also write x′ for a switch which is always in a state opposite to x. So that,

x **is open** → x' **is closed** and x **is closed** → x' **is open**.

The switch x' is called the invert of the switch x. For example, the switch a' shown in Fig.3 is an invert of the switch a.

Table 1 alongside gives the state value of x' for a given state value of the switch x. These values are derived from the definition of x' and our preceding discussion.

Table 1: s.v. of x'

x	x'
0	1
1	0

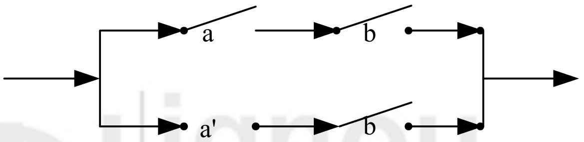
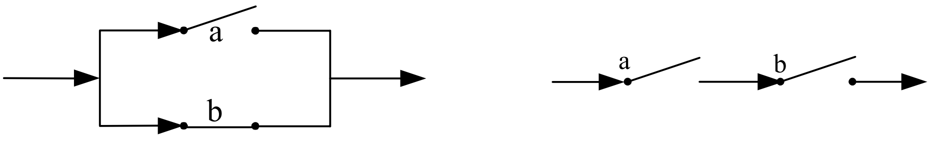


Fig. 3: a` is the invert of a.

Note that the variable x that denotes a switch can only take on 2 values, 0 and 1. Such a variable (which can only take on two values) is called a **Boolean variable**. Thus, if x is a Boolean variable, so is x'. Now, in order to design a circuit consisting of several switches, there are two ways in which two switches can be connected: **parallel connections** and **series connections** (see Fig.4).

Do you see a connection between Table 1 above and Table 10, Unit 1 ?



(i) Parallel Connection

(ii) Series Connection

Fig. 4: Two ways of connecting switches

From Fig.4(i) above, you can see that in case of a parallel connection of switches a and b (say), current will flow from the left to the right extreme if **at** least one of the two switches is closed. Note that 'parallel' **does not** mean that both the switches are in the same state.

On the other hand, current can flow in a series connection of switches only when **both** the switches a and b are closed (see Fig.4 (ii)).

Given two switches a and b, we write a **par** b and a **ser** b for these two types of connections, respectively.

In view of these definitions and the preceding discussion, you can see that the state values of the connections a **par** b and a **ser** b, for different pairs of state values of switches a and b, are as given in the tables below.

Table 2: State values of a par b and a ser b.

s.v. of a	**s.v.** of b	**s.v.** of a **par** b
0	0	0
0	1	1
1	0	1
1	1	1

s.v. of a	**s.v.** of b	**s.v.** of a **ser** b
0	0	0
0	1	0
1	0	0
1	1	1

We have now developed a sufficient background to give you the example of a Boolean algebra which is based on switching circuits.

Example 2: The set S = {0, 1} is a Boolean algebra.

Solution: Take **ser** and **par** in place of $\wedge$and $\vee$, respectively, and inversion(') instead of ~ as the three required operations in the definition of a Boolean algebra,. Also take 0 for the element **O** and 1 for the element **I** in this definition.
Now, using Tables 1 and 2, you can check that the five laws B1-B5 hold good. Thus, (**S, par, ser,**', 0, 1) is a Boolean algebra.

***

A Boolean algebra whose underlying set has only two elements is very important in the study of circuits. We call such an algebra a **two-element Boolean algebra**, and denote it by $\mathcal{B}$. From this Boolean algebra we can build many more, as in the following example.

Example 3: Let $\mathcal{B}^n = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} = \{(e_1 , e_2 , . . . , e_n) \mid \text{each } e_i = 0 \text{ or } 1\}$, for n $\geq$1, be the Cartesian product of n copies of $\mathcal{B}$. For $i_k , j_k \in \{0, 1\}$ $(1 \leq k \leq n)$, define

$$(i_1 , i_2 , . . . , i_n) \wedge (j_1 , j_2 , . . . , j_n) = (i_1 \wedge j_1 , i_2 \wedge j_2 , . . . , i_n \wedge j_n) ,$$
$$(i_1 , i_2 , . . . , i_n) \vee (j_1 , j_2 , . . . , j_n) = (i_1 \vee j_1 , i_2 \vee j_2 , . . . , i_n \vee j_n) , \text{ and}$$
$$(i_1 , i_2 , . . . , i_n)' = (i'_1 , i'_2 , . . . , i'_n) .$$

Then B^n is a Boolean algebra, for all n ≥ 1.

Solution: Firstly, observe that the case n = 1 is the Boolean algebra $\mathcal{B}$.
Now, let us write 0 = (0, 0, . . . , 0) and **I** = (1, 1, . . . , 1), for the two elements of $\mathcal{B}^n$ consisting of n-tuples of 0's and 1's, respectively. Using the fact that $\mathcal{B}$ is a Boolean algebra, you can check that $\mathcal{B}^n$, with operations as defined above, is a Boolean algebra for every n $\geq$1.

***

The Boolean algebras $\mathcal{B}^n$, n ≥ 1, (called **switching algebras**) are very useful for the study of the hardware and software of digital computers.

We shall now state, without proof, some other properties of Boolean algebras, which can be deduced from the five laws (B1-B5).

Theorem 1: Let $\mathcal{B}$ = (**S**, $\vee$, $\wedge$, ', **O, I**) be a Boolean algebra. Then the following laws hold $\forall x, y \in S$.

a) **Idempotent laws** : $x \vee x = x$, $x \wedge x = x$.
b) **Absorption laws :** $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$.
c) **Involution law :** $(x')' = x$.
d) **De Morgan's laws :** $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$.

In fact, you have already come across some of these properties for the Boolean algebras of propositions in Unit 1. In the following exercise we ask you to verify them.

E1) a) Verify the identity laws and absorption laws for the Boolean algebra (**S**, ^, v, ~, $\mathcal{T}$, $\mathcal{F}$) of propositions.
b) Verify the absorption laws for the Boolean algebra ($\mathcal{P}$(**X**),$\cup$,$\cap$, ^{c} ,Φ, **X**).

In Theorem 1, you may have noticed that for each statement involving $\vee$ and $\wedge$, there is an analogous statement with $\wedge$ (instead of $\vee$) and $\vee$ (instead of $\wedge$). This is not a coincidence, as the following definition and result shows.

Definition : If p is a proposition involving ~, $\wedge$ and $\vee$, the **dual** of p, denoted by p^d, is the proposition obtained by replacing each occurrence of $\wedge$ (and/or $\vee$) in p by $\vee$ (and/or $\wedge$, respectively) in p^d .
For example, $x \vee (x \wedge y) = x$ is the **dual** of $x \wedge (x \vee y) = x$.

Now, the following principle tells us that if a statement is proved true, then we have simultaneously proved that its dual is true.

Theorem 2 (The principle of duality): If s is a theorem about a Boolean algebra, then so is its dual s^d .
It is because of this principle that the statements in Theorem 1 look so similar.

Let us now see **how to apply Boolean algebra methods to circuit design.**

While expressing circuits mathematically, we identify each circuit in terms of some Boolean variables. Each of these variables represents either a simple switch or an input to some electronic switch.

Definition: Let $\mathcal{B}$ = (S, $\vee$, $\wedge$, ', **O, I**) be a Boolean algebra. **A Boolean expression** in variables $x_1, x_2, \ldots, x_k$ (say), each taking their values in the set **S** is defined recursively as follows:

i) Each of the variables x_1 , $x_2, \ldots, x_k$, as well as the elements **O** and **I** of the Boolean algebra $\mathcal{B}$ are Boolean expressions.

ii) If $\mathbf{X_1}$ and $\mathbf{X_2}$ are previously defined Boolean expressions, then $\mathbf{X_1} \wedge \mathbf{X_2}$, $\mathbf{X_1} \vee \mathbf{X_2}$ and $\mathbf{X'_1}$ are also Boolean expressions.

For instance, $x_1 \wedge x'_3$ is a Boolean expression because so are x_1 and x'_3 , Similarly, because $x_1 \wedge x_2$ is a Boolean expression, so is $(x_1 \wedge x_2) \wedge (x_1 \wedge x'_3)$.

If **X** is a Boolean expression in n variables $x_1, x_2, \ldots, x_n$ (say), we write this as
$\mathbf{X = X(x_1, \ldots, x_n)}$ **.**
In the context of simplifying circuits, we need to reduce Boolean expressions to

simpler ones. `Simple' means that the expression has fewer connectives, and all the literals involved are distinct. We illustrate this technique now.

Example 4: Reduce the following Boolean expressions to a simpler form.

(a) $X(x_1, x_2) = (x_1 \wedge x_2) \wedge (x_1 \wedge x'_2)$;
(b) $X(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3)$.

Solution: (a) Here we can write

$(x_1 \wedge x_2) \wedge (x_1 \wedge x'_2) = ((x_1 \wedge x_2) \wedge x_1) \wedge x'_2$ (Associative law)
$= (x_1 \wedge x_2) \wedge x'_2$ (Absorption law)
$= x_1 \wedge (x_2 \wedge x'_2)$ (Associative law)
$= x_1 \wedge \mathbf{O}$ (Complementation law)
$= \mathbf{O}.$ (Identity law)

Thus, in its simplified form, the expression given in (a) above is **O**, i.e., a **null expression.**

(b) We can write
$(x_1 \wedge x_2) \vee (x_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_3)$
$= [x_1 \wedge \{x_2 \vee (x'_2 \wedge x_3\}] \wedge (x_1 \wedge x_3)$ (Distributive law)
$= [x_1 \wedge \{(x_2 \vee x'_2) \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3)$ (Distributive law)
$= [x_1 \wedge \{\mathbf{I} \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3)$ (Complementation law)
$= [x_1 \wedge (x_2 \vee x_3)] \wedge (x_1 \wedge x_3)$ (Identity law)
$= [(x_1 \wedge x_2) \vee (x_1 \wedge x_3)] \wedge (x_1 \wedge x_3)$ (Distributive law)
$= [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee [(x_1 \wedge x_3) \wedge (x_1 \wedge x_3)]$ (Distributive law)
$= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3)$ (Idemp.,& assoc. laws)
$= x_1 \wedge [(x_2 \wedge x_3) \vee x_3]$ (Distributive law)
$= x_1 \wedge x_3$ (Absorption law)

Thus, the simplified form of the expression given in (b) is $(x_1 \wedge x_3)$.

***

Now you should find it easy to solve the following exercise.

E2) Simplify the Boolean expression
$\mathbf{X}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3) \vee (x_2 \wedge x_3)$.

With this we conclude this section. In the next section we shall give an important application of the concepts discussed here.

3.3 LOGIC CIRCUITS

If you look around, you would notice many electric or electronic appliances of daily use. Some of them need a simple switching circuit to control the auto-stop (such as in a stereo system). Some would use an auto-power off system used in transformers to control voltage fluctuations. Each circuit is usually a combination of on-off switches, wired together in some specific configuration. Nowadays certain types of **electronic blocks** (i.e., solid state devices such as transistors, resistors and capacitors) are more in use. We call these electronic blocks **logic gates,** or simply, **gates.** In Fig. 5 we have shown a box which consists of some electronic switches (or logic gates), wired together in a specific manner. Each line which is entering the box from the left represents an independent power source (called **input**), where all of them need not supply voltage to the box at a given moment. A single line coming out of the box gives the **final output** of the circuit box. The output depends on the type of input.

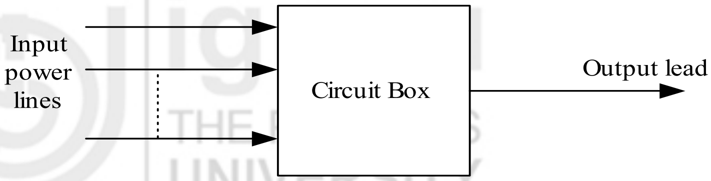


Fig. 5: A Logic circuit

This sort of arrangement of **input power lines, a circuit box** and **output lead** is basic to all electronic circuits. Throughout the unit, any such interconnected assemblage of logic gates is referred to as a **logic circuit.**

As you may know, computer hardwares are designed to handle only two levels of voltage, both as inputs as well as outputs. These two levels, denoted by 0 and 1, are called **bits** (an acronym for **binary digits**). When the bits are applied to the logic gates by means of **one** or **two** wires (input leads), the output is again in the form of voltages 0 and 1. Roughly speaking, **you may think of a gate to be on or off according to whether the output voltage is at level 1 or 0, respectively.**

Table 3:Outputs of AND-gate

x_1	x_2	$x_1 \wedge x_2$
0	0	0
1	1	0
0	1	0
1	1	1

Three basic types of logic gates are an **AND-gate,** an **OR-gate** and a **NOT-gate.** We shall now define them one by one.

Definition : Let the Boolean variables x_1 and x_2 represent any two bits. An **AND-gate** receives inputs x_1 and x_2 and produces the output, denoted by $x_1 \wedge x_2$, as given in Table 3 alongside.

The standard pictorial representation of an **AND**-gate is shown in Fig.6 below.

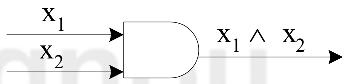


Fig. 6: Diagrammatic representation of an AND -gate

From the first three rows of Table 3, you can see that whenever the voltage in any one of the input wires of the **AND**-gate is at level 0, then the output voltage of the gate is also at level 0. You have already encountered such a situation in Unit 1. In the following exercise we ask you to draw an analogy between the two situations.

Table 4: Output of an OR-gate.

x_1	x_2	$x_1 \vee x_2$
0	0	0
0	1	1
1	0	1
1	1	1

E3) Compare Table 3 with Table 2 of Unit 1. How would you relate $x_1 \wedge x_2$ with $p \wedge q$, where p and q denote propositions?

Let us now consider another elementary logic gate.

Definition : An **OR-gate** receives inputs x_1 and x_2 and produces the output, denoted by $x_1 \vee x_2$, as given in Table 4. The standard pictorial representation used for the **OR**-gate is as shown in Fig.7.

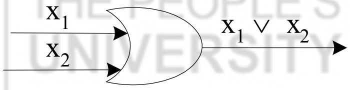


Fig. 7: Diagrammatic representation of an OR-gate

From Table 4 you can see that the situation is the other way around from that in Table 3, i.e., the output voltage of an **OR**-gate is at level 1 whenever the level of voltage in even one of the input wires is 1. What is the analogous situation in the context of propositions? The following exercise is about this.

E4) Compare Table 4 with Table 1 of Unit 1. How would you relate $x_1 \vee x_2$ with $p \vee q$, where p and q are propositions?

And now we will discuss an electronic realization of the invert of a simple switch about which you read in Sec. 3.2.

Definition : A NOT-gate receives bit x as input, and produces an output denoted by x', as given in Table 5. The standard pictorial representation of a **NOT-**gate is shown in Fig. 8 below.

Table 5: Output of a NOT-gate

x	x'
0	1
1	0

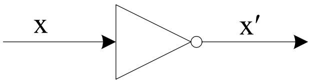


Fig. 8: Diagrammatic representation of NOT-gate

If you have solved E5 and E6, you would have noticed that Tables 3 and 4 are the same as the truth tables for the logic connectives $\wedge$ (conjunction) and $\vee$(disjunction). Also Table 3 of Unit 1, after replacing T by 1 and F by 0, gives Table 5. This is why the output tables for the three elementary gates are called **logic tables.** You may find it useful to remember these logic tables because they are needed very often for computing the logic tables of logic circuits.

Another important fact that these logic tables will help you prove is given in the following exercise.

E5) Let $\mathcal{B} = \{0, 1\}$ consist of the bits 0 and 1. Show that $\mathcal{B}$ is a Boolean algebra, i.e., that the bits 0 and 1 form a two-element Boolean algebra.

As said before, a logic circuit can be designed using elementary gates, where the output from an **AND-**gate, or an **OR**-gate, or a **NOT**-gate is used as an input to other such gates in the circuitry. The different levels of voltage in these circuits, starting from the input lines, move only in the direction of the arrows as shown in all the figures given below. For instance, one combination of the three elementary gates is shown in Fig.9.

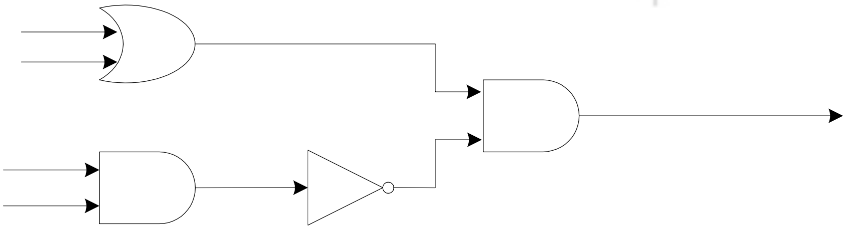


Fig. 9: A logic circuit of elementary gates.

Now let us try to see the connection between logic circuits and Boolean expressions. We first consider the elementary gates. For a given pair of inputs x_1 and x_2 , the output in the case of each of these gates is an expression of the form $x_1 \wedge x_2$ or $x_1 \vee x_2$ or x' .

Next, let us look at larger circuits. Is it possible to find an expression associated with a logic circuit, using the symbols $\wedge$, $\vee$ and ' ? Yes, it is. We will illustrate the technique of finding a Boolean expression for a given logic circuit with the help of some examples. But first, note that the output of a gate in a circuit may serve as an input to some other gate in the circuit, as in Fig. 9. So, to get an expression for a logic circuit the process always moves in the direction of the arrows in the circuitry. With this in mind, let us consider some circuits.

Example 5: Find the Boolean expression for the logic circuit given in Fig.9 above.

Solution: In Fig.9, there are four input terminals. Let us call them x_1 , x_2 , x_3 and x_4 . So, x_1 and x_2 are inputs to an **OR**-gate, which gives $x_1 \vee x_2$ as an output expression (see Fig. 9(a)).

Similarly, the other two inputs x_3 and x_4 , are inputs to an **AND-**gate. They will give $x_3 \wedge x_4$ as an output expression. This is, in turn, an input for a **NOT**-gate in the circuit. So, this yields $(x_3 \wedge x_4)'$ as the output expression. Now, both the expressions $x_1 \vee x_2$ and $(x_3 \wedge x_4)'$ are inputs to the extreme right **AND**-gate in the circuit. So, they give $(x_1 \vee x_2) \wedge (x_3 \wedge x_4)'$ as the final output expression, which represents the logic circuit.

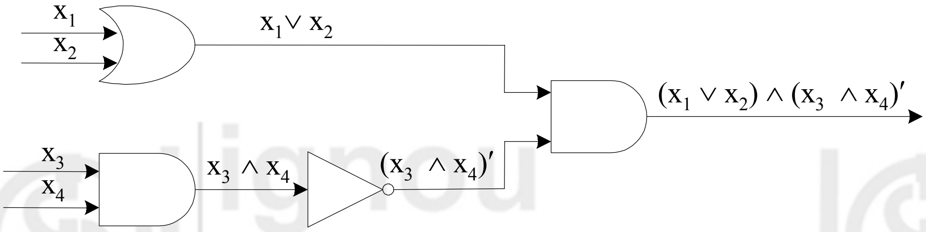


Fig. 9 (a)

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You have just seen how to find a Boolean expression for a logic circuit. For more practice, let us find it for another logic circuit.

Example 6: Find the Boolean expression C for the logic circuit given in Fig. 10.

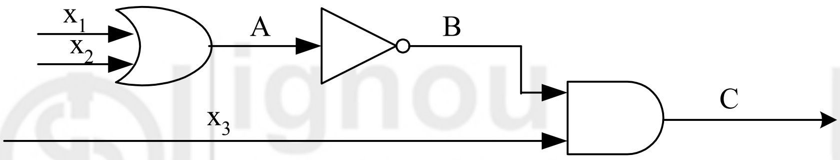

Fig. 10

Solution: Here the first output is from an **OR**-gate, i.e., A is $x_1 \vee x_2$. This, in turn, serves as the input to a **NOT-**gate attached to it from the right. The resulting bit B is $(x_1 \vee x_2)'$. This, and x_3 , serve as inputs to the extreme right **AND**-gate in the circuit given above. This yields an output expression $(x_1 \vee x_2)' \wedge x_3$, which is C, the required expression for the circuit given in Fig.10.

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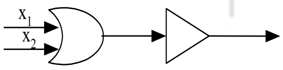
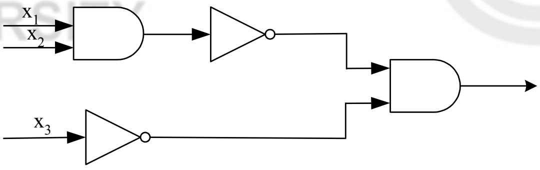
Why don't you try to find the Boolean expressions for some more logic circuits now?

E6) Find the Boolean expression for the output of the logic circuits given below.



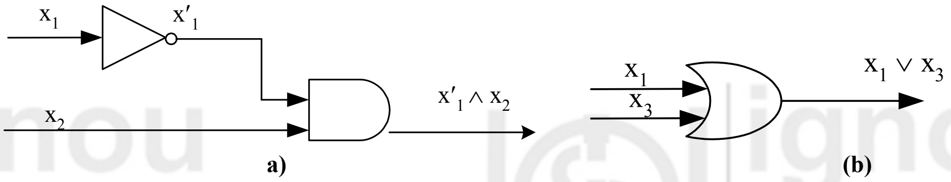
(a)



(b)

So far, you have seen how to obtain a Boolean expression that represents a given circuit. Can you do the converse? That is, can you construct a logic circuit corresponding to a given Boolean expression? In fact, this is done when a circuit designing problem has to be solved. The procedure is quite simple. We illustrate it with the help of some examples.

Example 7: Construct the logic circuit represented by the Boolean expression $(x'_1 \wedge x_2) \vee (x_1 \vee x_3)$, where x_i $(1 \leq i \leq 3)$ are assumed to be inputs to that circuitry.
Solution: Let us first see what the portion $(x'_1 \wedge x_2)$ of the given expression contributes to the complete circuit. In this expression the literals x'_1 and x_2 are connected by the connective $\wedge$ (**AND**). Thus the circuit corresponding to it is as shown in Fig.11(a) below, by the definitions of **NOT**-gate and **AND**-gate.



a) **(b)**

Fig. 11: Logic circuits for the expressions $x'_1 \wedge x_2$ and $x_1 \vee x_3$.

Similarly, the gate corresponding to the expression $x_1 \vee x_3$ is as shown in Fig.11(b) above. Finally, note that the given expression has two parts, namely, $x'_1 \wedge x_2$ and $x_1 \vee x_3$, which are connected by the connective $\vee$ (**OR**). So, the two logic circuits given in Fig.11 above, when connected by an **OR**-gate, will give us the circuit shown in Fig. 12 below.

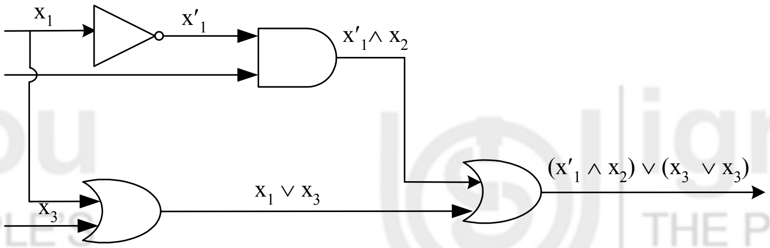


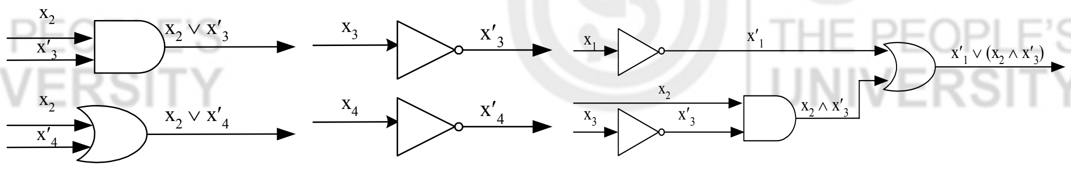
Fig.12: Circuitry for the expression $(x'_1 \wedge x_2) \vee (x_1 \vee x_3)$

This is the required logic circuit which is represented by the given expression.

***

Example 8: Given the expression $(x'_1 \vee (x_2 \wedge x'_3)) \wedge (x_2 \vee x'_4)$, find the corresponding circuit, where x_i $(1 \leq i \leq 4)$ are assumed to be inputs to the circuitry.

Solution: We first consider the circuits representing the expressions $x_2 \wedge x'_3$ and $x_2 \vee x'_4$. They are as shown in Fig.13(a).



(a) **(b)** **(c)**

Fig. 13: Construction of a logic circuitry.

Also you know that the literals x'_3 and x'_4 are outputs of the **NOT**-gate. So, these can be represented by logic gates as shown in Fig.13(b). Then the circuit for the part $x'_1 \vee (x_2 \wedge x'_3)$ of the given expression is as shown in Fig.13(c). You already know how to construct a logic circuit for the expression $x_2 \vee x'_4$.

Finally, the two expressions $(x'_1 \vee (x_2 \wedge x'_3))$ and $(x_2 \vee x'_4)$ being connected by the connective $\wedge$ (**AND**), give the required circuit for the given expression as shown in Fig.14.

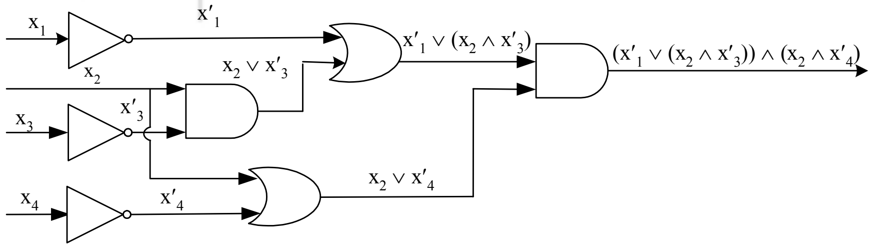


Fig. 14: Circuitry for the expression $(x'_1 \vee (x_2 \wedge x'_3)) \wedge (x_2 \vee x'_4)$.

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Why don't you try to solve some exercises now?

E7) Find the logic circuit corresponding to the expression $x'_1 \wedge (x_2 \vee x'_3)$.

E8) Construct the logic circuit and obtain the logic table for the expression $x_1 \vee (x'_2 \wedge x_3)$.

So far we have established a one-to-one correspondence between logic circuits and Boolean expressions. You may wonder about the utility of this. The mathematical view of a circuit can help us understand the **overall functioning** of the circuit. To understand how, consider the circuit given in Fig.10 earlier.

You may think of the inputs bits x_1 , x_2 , and x_3 as three variables, each one of which is known to have two values only, namely, 0 or 1, depending upon the level of voltage these inputs have at any moment of time. Then the idea is to evaluate the expression $(x_1 \vee x_2)' \wedge x_3$, which corresponds to this circuit, for different values of the 3-tuple (x_1 , x_2 , x_3).

How does this evaluation help us to understand the functioning of the circuit? To see this, consider a situation in which the settings of x_1 , x_2 and x_3 at a certain stage of the process are $x_1 = x_3 = 0$ and $x_2 = 1$. Then we know that $x_1 \vee x_2 = 0 \vee 1 = 1$ (see the second row of Table 3 given earlier). Further, using the logic table of a **NOT**-gate, we get $(x_1 \vee x_2)' = 1' = 0$. Finally, from Table 3, we get $(x_1 \vee x_2)' \wedge x_3 = 0 \wedge 1 = 0$. Thus, the expression $(x_1 \vee x_2)' \wedge x_3$ has value 0 for the set of values (0, 1, 0) of input bits (x_1 , x_2 , x_3). **Thus, if x 1 and x 3 are closed, while x_2 is open, the circuit remains closed.**

Using similar arguments, you can very easily calculate the other values of the expression $(x_1 \vee x_2)' \wedge x_3$ in the set

$$\{0,1\}^3 = \{(x_1 , x_2 , x_3) \mid x_i = 0 \text{ or } 1, 1 \leq i \leq 3\}$$

of values of input bits. We have recorded them in Table 6.

Observe that the row entries in the first three columns of Table 6 represent the different values which the input bits (x_1 , x_2 , x_3) may take. Each entry in the last column of the table gives the output of the circuit represented by the expression $(x_1 \vee x_2)' \wedge x_3$ for the corresponding set of values of (x_1 , x_2 , x_3). For example, if (x_1 , x_2 , x_3) is (0,1,0), then the level of voltage in the output lead is at a level 0 (see the third row of Table 6).

You should verify that the values in the other rows are correct.

Table 6: Logic table for the expression $(x_1 \vee x_2)' \wedge x_3$.

Table 6 is the **logic table** for the circuit given in Fig. 10.

x_1	x_2	x_3	$x_1 \vee x_2$	$(x_1 \vee x_2)$	$(x_1 \vee x_2)' \wedge x_3$
0	0	0	0	1	0
0	0	1	0	1	1
0	1	0	1	0	0
1	0	0	1	0	0
0	1	1	1	0	0
1	1	0	1	0	0
1	0	1	1	0	0
1	1	1	1	0	0

Why don't you try an exercise now?

E9) Compute the logic table for the circuit given in E6(b) above.

You have seen how the logic table of an expression representing a circuit provides a functional relationship between the state (or level) of voltage in the input terminals and that in the output lead of that logic circuitry. This leads us the concept of Boolean functions, which we will now discuss.

3.4 BOOLEAN FUNCTIONS

In the last section you studied that an output expression is not merely a device for representing an interconnection of gates. It also defines output values as a function of input bits. This provides information about the overall functioning of the corresponding logic circuit. So, this function gives us a relation between **the inputs to the circuit** and its **final output** .

This is what helps us to understand control over the functioning of logic circuits from a mathematical point of view. To explain what this means, let us reformulate the logic tables in terms of functions of the input bits.

Let us first consider the Boolean expression

$$X(x_1 , x_2) = x_1 \wedge x'_2,$$

where x_1 and x_2 take values in $\mathcal{B} = \{0, 1\}$. You know that all the values of this expression, for different pairs of values of the variables x_1 and x_2 , can be calculated by using properties of the Boolean algebra $\mathcal{B}$. For example,

$$0 \wedge 1' = 0 \wedge 0 = 0 \Rightarrow X(0, 1) = 0.$$

Similarly, you can calculate the other values of $X(x_1, x_2) = x_1 \wedge x'_2$ over $\mathcal{B}$.

In this way we have obtained a function $f : \mathcal{B}_2 \to B$, defined as follows:

$$f(e_1 , e_2) = X(e_1 , e_2) = e_1 \wedge e'_2 , \text{ where } e_1 , e_2 \,\varepsilon\, \{0, 1\}.$$

So f is obtained by replacing x_i with e_i in the expression $X(x_1 , x_2)$. For example, when $e_1 = 1$, $e_2 = 0$, we get $f(1, 0) = 1 \wedge 0' = 1$.

More generally, each Boolean expression $X(x_1 , x_2 , . . . , x_k)$ in k variables, where each variable can take values from the two-element Boolean algebra $\mathcal{B}$, defines a function $f : B^k \to B : f(e_1 , \dots , e_k) = X(e_1 , \dots , e_k)$.

Any such function is called a **Boolean function.**
Thus, each Boolean expression over $\mathcal{B} = \{0, 1\}$ gives rise to a Boolean function.
In particular, corresponding to each circuit, we get a Boolean function.
Therefore, the logic table of a circuit is just another way of representing the Boolean function corresponding to it.

For example, the logic table of an **AND**-gate can be obtained using the function $\wedge$: $\mathcal{B}^2 \to B : \wedge (e_1 , e_2) = e_1 \wedge e_2$.

To make matters more clear, let us work out an example.

Example 9: Let $f : \mathcal{B}^2 \to \mathcal{B}$ denote the function which is defined by the Boolean expression $X(x_1 , x_2) = x'_1 \wedge x'_2$. Write the values of f in tabular form.

Solution: f is defined by $f(e_1 , e_2) = e'_1 \wedge e'_2$ for $e_1 , e_2 \in \{0, 1\}$. Using Tables 3, 4 and 5, we have

$f(0, 0) = 0' \wedge 0' = 1 \wedge 1 = 1,$ $f(0, 1) = 0' \wedge 1' = 1 \wedge 0 = 0,$
$f(1, 0) = 1' \wedge 0' = 0 \wedge 1 = 0,$ $f(1, 1) = 1' \wedge 1' = 0 \wedge 0 = 0.$

We write this information in Table 7.

Table 7: Boolean function for the expression x'_1 ^ x'_2 .

e_1	e_2	e'_1	e'_2	$f(e_1, e_{}) = e'_1 \wedge e'_2$
0	0	1	1	$1\wedge1=1$
0	1	1	0	$1\wedge0=0$
1	0	0	1	$0\wedge1=0$
1	1	0	0	$0\wedge0=0$

***

Why don't you try an exercise now?

E10) Find all the values of the Boolean function $f : \mathcal{B}_2 \to \mathcal{B}$ defined by the Boolean expression $(x_1 \wedge x_2) \vee (x_1 \wedge x'_3)$.

Let us now consider the Boolean function $g : \mathcal{B}_2 \to \mathcal{B}$, defined by the expression $X(x_1 , x_2) = (x_1 \vee x_2)'$.
Then $g(e_1 , e_2) = (e_1 \vee e_2)' , e_1 , e_2 \varepsilon \mathcal{B}$.

So, the different values that g will take are

$g(0, 0) = (0 \vee 0)' = 0' = 1,$ $g(0, 1) = (0 \vee 1)'= 1' = 0,$
$g(1, 0) = (1 \vee 0)' = 1' = 0,$ $g(1, 1) = (1 \vee 1)' = 1' = 0.$

In tabular form, the values of g can be presented as in Table 8.

Table 8: Boolean function of the expression $(x_1 \text{ v } x_2)'$.

e_1	e_2	$e_1 \vee e_2$	$g(e_1, e_2) = (e_1 \text{ v } e_2)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

By comparing Tables 7 and 8, you can see that $f(e_1 , e_2) = g(e_1 , e_2)$ for all

$(e_1, e_2) \in \mathcal{B}^2$. So f and g are the same function.

What you have just seen is that **two (seemingly) different Boolean expressions can have the same Boolean function specifying them.** Note that if we replace the input bits by propositions in the two expressions involved, then we get logically equivalent statements. This may give you some idea of how the two Boolean expressions are related. We give a formal definition below.

Definition : Let $X = X(x_1, x_2, \ldots, x_k)$ and $Y = Y(x_1, x_2, \ldots, x_k)$ be two Boolean expressions in the k variables $x_1, \ldots, x_k$. We say **X is equivalent to Y** over the Boolean algebra $\mathcal{B}$, and write **$X \equiv Y$,** if both the expressions X and Y define the same Boolean function over $\mathcal{B}$, i.e.,

$$X(e_1, e_2, \ldots, e_k) = Y(e_1, e_2, \ldots, e_k), \text{ for all } e_i \in \{0, 1\}.$$

So, the expressions to which f and g (given by Tables 7 and 8) correspond are equivalent.

Why don't you try an exercise now?

E11) Show that the Boolean expressions
$X = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and $Y = x_1 \wedge (x_2 \vee x'_3)$
are equivalent over the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$.

So far you have seen that given a circuit, we can define a Boolean function corresponding to it. You also know that given a Boolean expression over $\mathcal{B}$, there is a circuit corresponding to it. Now, you may ask:

Given a Boolean function **$f : \mathcal{B}^n \to \mathcal{B}$,** is it always possible to get a Boolean expression which will specify f over B? The answer is `yes', i.e., for every function $f : \mathcal{B}^n \to \mathcal{B}$ ($n \geq 2$) there is a Boolean expression (in n variables) whose Boolean function is f itself.

To help you understand the underlying procedure, consider the following examples.

Example 10: Let $f : \mathcal{B}^2 \to \mathcal{B}$ be a function which is defined by
$f(0, 0) = 1, f(1, 0) = 0, f(0, 1) = 1, f(1, 1) = 1.$

Find the Boolean expression specifying the function f.

Solution: f can be represented by the following table.

Input		Output
x_1	x_2	$f(x_1, x_2)$
0	0	1
1	0	0
0	1	1
1	1	1

In Boolean algebra terminology this is known as the 'disjunctive normal form' (DNF) of the expression.

We find the Boolean expression according to the following algorithm:

Step 1: Identify all rows of the table where the output is 1: these are the 1st, 3rd and 4th rows.

Step 2: Combine the variables in each of the rows identified in Step 1 with `and'. Simultaneously, apply `not' to the variables with value zero in these rows. So, for the
1st row: $x'_1 \wedge x'_2$,

3rd row: $x'_1 \wedge x_2$,
4th row: $x_1 \wedge x_2$.

Step 3: Combine the Boolean expressions obtained in Step 2 with `or' to get the compound expression representing f:
So, $f(x_1, x_2) = (x'_1 \wedge x'_2) \vee (x'_1 \wedge x_2) \vee (x_1 \wedge x_2)$.

***

You can complete Example 10, by doing the following exercise.

E12) In the previous example, show that $X(e_1, e_2) = f(e_1, e_2) \ \forall e_1, e_2 \in \mathcal{B}$.

E13) By Theorem 2, we could also have obtained the expression of f in Example 10 in `conjunctive normal form' (CNF). Please do so.

An important remark: To get a Boolean expression for a Boolean function h (say), we should first see how many points v_i there are at which $h(v_i) = 0$, and how many points v_i there are at which $h(v_i) = 1$. **If the number of values for which the function h is 0 is less than the number of values at which h is 1, then we shall choose to obtain the expression in CNF, and not in DNF.** This will give us a shorter Boolean expression, and hence, a simpler circuit. For similar reasons, we will prefer DNF if the number of values at which h is 0 is more.

Why don't you apply this remark now?

E14) Find the Boolean expressions, in DNF or in CNF (keeping in mind the remark made above), for the functions defined in tabular form below.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

(a)

x_1	x_2	x_3	$g(x_1, x_2, x_3)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

(b)

Boolean functions tell us about the functioning of the corresponding circuit. Therefore, circuits represented by two equivalent expressions should essentially do the same job. We use this fact while redesigning a circuit to create a simpler one. In fact, in such a simplification process of a circuit, we write an expression for the circuit and then evaluate the same (over two-element Boolean algebra $\mathcal{B}$) to get the Boolean function. Next, we proceed to get an equivalent, simpler expression. Finally, the process terminates with the construction of the circuit for this simpler expression. Note that, **as the two expressions are equivalent, the circuit represented by the simpler expression will do exactly the same job as the circuit represented by the original expression.**

Let us illustrate this process by an example in some detail.

Example 11: Design a logic circuit capable of operating a central light bulb in a hall by three switches x_1, x_2, x_3 (say) placed at the three entrances to that hall.

Solution: Let us consider the procedure stepwise.

Step 1: To obtain the function corresponding to the unspecified circuit.

To start with, we may assume that the bulb is off when all the switches are off. Mathematically, this demands a situation where $x_1 = x_2 = x_3 = 0$ implies $f(0, 0, 0)= 0$, where f is the function which depicts the functional utility of the circuit to be designed.
Let us now see how to obtain the other values of f. Note that every change in the state of a switch should alternately put the light bulb on or off. Using this fact repeatedly, we obtain the other values of the function f.

Now, if we assign the value (1,0,0) to (x_1 , x_2 , x_3), it brings a single change in the state of the switch x_1 only. So, the light bulb must be on. This can be written mathematically in the form $f(1, 0, 0) = 1$. Here the value 1 of f stands for the on state of the light bulb.
Then, we must have $f(1, 1, 0) = 0$, because there is yet another change, now in the state of switch x_2 .

You can verify that the other values of $f(x_1 , x_2 , x_3)$ are given as in Table 9.

Table 9: Function of a circuitry for a three-point functional bulb.

x_1	x_2	X_3	$f(x_1, x_2, x_3)$
0	0	0	0
1	0	0	1
1	1	0	0
1	1	1	1
0	1	0	1
0	1	1	0
0	0	1	1
1	0	1	0

Step 2: To obtain a Boolean expression which will specify the function f. Firstly, note that the number of 1's in the last column of Table 9 are fewer than the number of 0's. So we shall obtain the expression in DNF (instead of CNF).

By following the stepwise procedure of Example 10, you can see that the required Boolean expression is given by

$$X(x_1 , x_2 , x_3) = (x_1 \wedge x'_2 \wedge x_3) \vee (x'_1 \wedge x_2 \wedge x'_3) \vee (x'_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

At this stage we can directly jump into the construction of the circuit for this expression (using methods discussed in Sec.3.3). But why not try to get a simpler circuit?

Step 3 : To simplify the expression $X(x_1 , x_2 , x_3)$ given above. Firstly, observe that

$$\begin{aligned}(x_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3) &= x_1 \wedge [(x'_2 \wedge x_3) \vee (x_2 \wedge x_3)]\\ &= x_1 \wedge [(x'_2 \vee x_2) \wedge x_3]\\ &= x_1 \wedge (1 \wedge x_3)\\ &= x_1 \wedge x_3 ,\end{aligned}$$

by using distributive, complementation and identity laws (in that order).
Similarly, you can see that

$$(x'_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_3) = (x'_2 \vee x_1) \wedge x_3 .$$

We thus have obtained a simpler (and equivalent) expression, namely,

$$X(x_1 , x_2 , x_3) = (x'_1 \wedge x_2 \wedge x'_3) \vee [(x'_2 \vee x_1) \wedge x_3] ,$$

whose Boolean function is same as the function f. (Verify this!)

Step 4: To design a circuit for the expression obtained in Step 3.

Now, the logic circuit corresponding to the simpler (and equivalent) expression obtained in Step 3 is as shown in Fig.15.

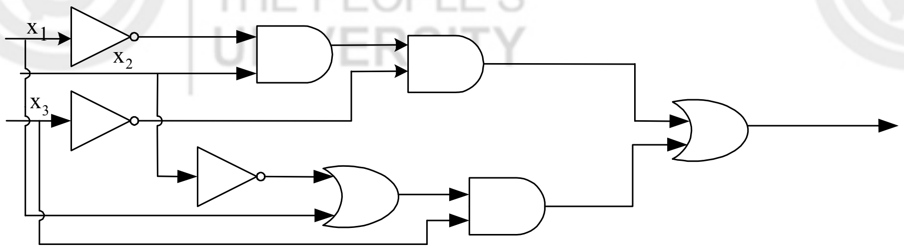

Fig. 15: A circuit for the expression $(x'_1 \wedge x_2 \wedge x'_3) \vee ((x'_2 \vee x_1) \wedge x_3)$

So, in 4 steps we have designed a 3-switch circuit for the hall.

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We can't claim that the circuit designed in the example above is the simplest circuit. How to get that is a different story and is beyond the scope of the present course.

Why don't you try an exercise now?

E15) Design a logic circuit to operate a light bulb by two switches, x_1 and x_2 (say).

We have now come to the end of our discussion on applications of logic. Let us briefly recapitulate what we have discussed here.

3.5 SUMMARY

In this unit, we have considered the following points.

1. The definition and examples of a Boolean algebra. In particular, we have discussed the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, and the switching algebras B_n, $n \geq 2$.
2. The definition and examples of a Boolean expression.
3. The three elementary logic gates, namely, **AND**-gate, **OR**-gate and **NOT-**gate; and the analogy between their functioning and operations of logical connectives.
4. The method of construction of a logic circuit corresponding to a given Boolean expression, and vice-versa.
5. How to obtain the logic table of a Boolean expression, and its utility in the understanding of the overall functioning of a circuit.
6. The method of simplifying a Boolean expression.
7. The method of construction of a Boolean function $f : \mathcal{B}^n \rightarrow B$, corresponding to a Boolean expression, and the concept of **equivalent** Boolean expressions.
8. Examples of the use of Boolean algebra techniques for constructing a logic circuit which can function in a specified manner.

3.6 SOLUTIONS/ ANSWERS

E1) a) In E19 of Unit 1, you have already verified the Identity laws. Let us proceed to show that the propositions $p \vee (p \wedge q)$ and p are logically equivalent. It suffices to show that the truth tables of both these propositions are the same. This follows from the first and last columns of the following table.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
F	F	F	F
F	T	F	F
T	F	F	T
T	T	T	T

Similarly, you can see that the propositions $p \wedge (p \vee q)$ and p are equivalent propositions. This establishes the absorption laws for the Boolean algebra $(S, \wedge, \vee\ ', \mathcal{T}, \mathcal{F})$.

b) Let A and B be two subsets of the set X. Since $A \cap B \subseteq A$, $(A \cap B) \cup A = A$. Similarly, as $A \subseteq A \cup B$, we have $(A \cup B) \cap A = A$. Thus, both the forms of the absorption laws hold good for the Boolean algebra $(\mathcal{P}(X), \cup, \cap, ^c, X,)$.

E2) We can write

$$X(x_1, x_2, x_3) = ((x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3)) \vee (x_2 \wedge x_3)$$
$$= (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \qquad \text{(by Absorption law)}$$
$$= x_2 \wedge (x_1 \vee x_3) \qquad \text{(by Distributive law)}$$

This is the simplest form of the given expression.

E3) Take the propositions p and q in place of the bits x_1 and x_2, respectively. Then, when 1 and 0 are replaced by T and F in Table 3 here, we get the truth table for the proposition $p \wedge q$ (see Table 2 of Unit 1).
This establishes the analogy between the functioning of the **AND**-gate and the conjunction operation on the set of propositions.

E4) Take the propositions p and q in place of the bits x_1 and x_2, respectively. Then, when 1 and 0 are replaced by T and F in Table 4 here, we get the truth table for the proposition $p \vee q$ (see Table 1 of Unit 1).
This establishes the analogy between the functioning of the **OR**-gate and the disjunction operation on the set of propositions.

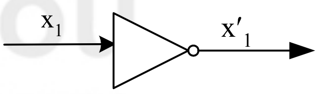
E5) Firstly, observe that the information about the outputs of the three elementary gates, for different values of inputs, can also be written as follows:

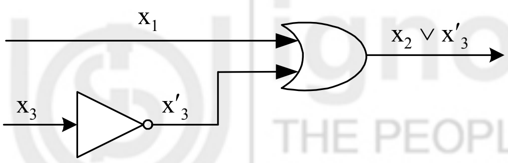
$$0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1; \qquad \text{(see Table 3)}$$
$$0 \vee 0 = 0,\ 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1; \text{ and} \qquad \text{(see Table 4)}$$
$$0' = 1,\ 1' = 0. \qquad \text{(see Table 5)}$$

Clearly, then both the operations # and # are the binary operations on $\mathcal{B}$ and ' : $\mathcal{B} \rightarrow \mathcal{B}$ is a unary operation. Also, we may take 0 for **O** and 1 for **I** in the definition of a Boolean algebra.
Now, by looking at the logic tables of the three elementary gates, you can see that all the five laws B1-B5 are satisfied. Thus, $\mathcal{B}$ is a Boolean algebra.

E6) a) Here x_1 and x_2 are inputs to an **OR**-gate, and so, we take $x_1 \vee x_2$ as input to the **NOT**-gate next in the chain which, in turn, yields $(x_1 \vee x_2)'$ as the required output expression for the circuit given in (a).
b) Here x_1 and x_2 are the inputs to an **AND**-gate. So, the expression $x_1 \wedge x_2$ serves as an input to the **NOT**-gate, being next in the chain.
This gives the expression $(x_1 \wedge x_2)'$ which serves as one input to the extreme right **AND**-gate. Also, since x'_3 is another input to this **AND**-gate (coming out of a **NOT**-gate), we get the expression $(x_1 \wedge x_2)' \wedge x'_3$ as the final output expression which represents the circuit given in (b).

E7) You know that the circuit representing expressions x_1 and $x_2 \vee x'_3$ are as shown in Fig.16 (a) and (b) below.





(a) **(b)**

Fig. 16

Thus, the expression $x'_1 \vee (x_2 \vee x'_3)$, being connected by the symbol $\wedge$, gives the circuit corresponding to it as given in Fig.17 below.

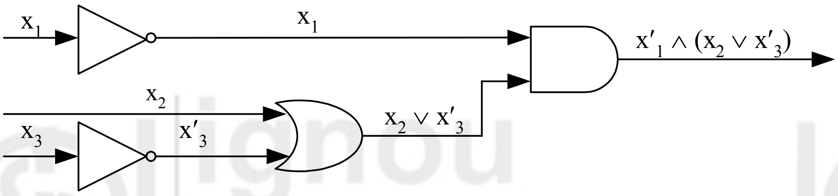


Fig. 17: A logic circuit for the expression $x'_1 \wedge (x_2 \vee x'_3)$

E8) You can easily see, by following the arguments given in E9, that the circuit represented by the expression $x_1 \vee (x'_2 \wedge x_3)$ is as given in Fig.18.

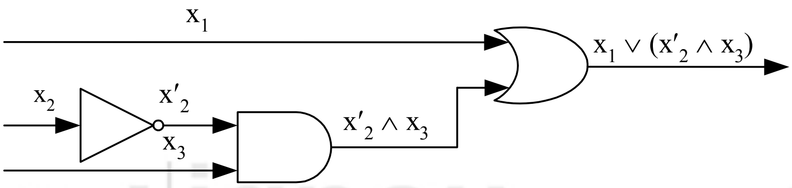


Fig. 18

The logic table of this expression is as given below.

x_1	x_2	x_3	x'_2	$x'_2 \wedge x_3$	$X_1 \vee (x'_2 \wedge x_3)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	0
1	0	0	1	0	1
0	1	1	0	0	0
1	1	0	0	0	1
1	0	1	1	1	1
1	1	1	0	0	1

E9) Since the output expression representing the circuit given in E8(b) is found to be $(x_1 \wedge x_2)' \wedge x'_3$, the logic table for this circuit is as given below.

x_1	x_2	x_3	$x_1 \wedge x_2$	$(x_1 \wedge x_2)'$	x'_3	$(x_1 \wedge x_2)' \wedge x'_3$
0	0	0	0	1	1	1
0	0	1	0	1	0	0
0	1	0	0	1	1	1
1	0	0	0	1	1	1
0	1	1	0	1	0	0
1	1	0	1	0	1	0
1	0	1	0	1	0	0
1	1	1	1	0	0	0

E10) Because the expression $(x_1 \wedge x_2) \vee (x_1 \wedge x'_3)$ involves three variables, the corresponding Boolean function, f (say) is a three variable function, i.e. f : $B_3 \to B$. It is defined by

$$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e'_3),\ e_1, e_2 \text{ and } e_3 \in B.$$

Now, you can verify that the values of f in tabular form are as given in the following table.

e_1	e_2	E_3	$e_1 \wedge e_2$	e'_3	$e_1 \wedge e'_3$	$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e'_3)$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	0	1	0	0
1	0	0	0	1	1	1
0	1	1	0	0	0	0
1	1	0	1	1	1	1
1	0	1	0	0	0	0
1	1	1	1	0	0	1

E11) To show that the Boolean expressions X and Y are equivalent over the two-element Boolean algebra $B = \{0, 1\}$, it suffices to show that the Boolean functions f and g (say) corresponding to the expressions X and Y, respectively, are the same. As you can see, the function f for the expression X is calculated in E10 above.
Similarly, you can see that the Boolean function g for the expression Y in tabular form is as given below.

x_1	x_2	x_3	x'_3	$x_2 \vee x'_3$	$G(x_1, x_2, x_3) = X_1 \wedge (x_2 \vee x'_3)$
0	0	0	1	1	0
0	0	1	0	0	0
0	1	0	1	1	0
1	0	0	1	1	1
0	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	0	0
1	1	1	0	1	1

Comparing the last columns of this table and the one given in E10 above, you can see that $f(e_1, e_2, e_3) = g(e_1, e_2, e_3)\ \forall\ e_1, e_2, e_3 \in B = \{0, 1\}$. Thus, **X** and **Y** are equivalent.

E12) Firstly, let us evaluate the given expression $X(x_1, x_2)$ over the two-element Boolean algebra $B = \{0, 1\}$ as follows:

$$\begin{aligned} X(0, 0) &= (0' \wedge 0') \vee (0' \wedge 0) \vee (0 \wedge 0) \\ &= (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) \\ &= 1 \vee 0 \vee 0 = 1 = f(0, 0); \\ X(1, 0) &= (1' \wedge 0') \vee (1' \wedge 0) \vee (1 \wedge 0) \\ &= (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \\ &= 0 \vee 0 \vee 0 = 0 = f(1, 0); \end{aligned}$$

$$\begin{aligned} X(0, 1) &= (0' \wedge 1') \vee (0' \wedge 1) \vee (0 \wedge 1) \\ &= (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ &= 0 \vee 1 \vee 0 = 1 = f(0, 1); \end{aligned}$$

$$\begin{aligned} X(1, 1) &= (1' \wedge 1') \vee (1' \wedge 1) \vee (1 \wedge 1) \\ &= (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \end{aligned}$$

$$= \quad 0 \vee 0 \vee 1 = 1 = f(1, 1).$$

It thus follows that $\mathbf{X}(e_1, e_2) = f(e_1, e_2)\ \forall\ e_1, e_2 \in \boldsymbol{B} = \{0, 1\}$.

E13) **Step 1:** Identify all rows of the table where output is 0: This is the 2nd row.
Step 2: Combine x_1 and x_2 with `or' in these rows, simultaneously applying `not' to x_i if its value is 0 in the row: So, for the 2nd row the expression we have is $x_1 \vee x_2$.
Step 3: Combine all the expressions obtained in Step 2 with `and' to get the **CNF** form representing f: In this case there is only 1 expression.
So f is represented by $x_1 \vee x_2$ in CNF.

E14) a) Observe from the given table that, among the two values 0 and 1 of the function $f(x_1, x_2, x_3)$, the value 1 occurs the least number of times. Therefore, by the remark made after **E 13**, we would prefer to obtain the Boolean expression in **DNF**. To get this we will use the stepwise procedure adopted in Example 10.
Accordingly, the required Boolean expression in DNF is given by

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2 \wedge x_3) \wedge (x_1 \wedge x'_2 \wedge x'_3) \vee (x'_1 \wedge x'_2 \wedge x'_3).$$

b) By the given table, among the two values 0 and 1 of the function the points v i at which $g(v_i) = 0$ are fewer than the points v_i at which $g(v_i) = 1$. So we would prefer to obtain the corresponding Boolean expression in CNF. Applying the stepwise procedure in the solution to E13, the required Boolean expression (in CNF) is given by

$$X(x_1, x_2, x_3) = (x'_1 \vee x_2 \vee x'_3) \wedge (x_1 \vee x'_2 \vee x'_3) \wedge (x_1 \vee x'_2 \vee x_3).$$

E15) Let g denote the function which depicts the functional utility of the circuit to be designed. We may assume that the light bulb is off when both the switches x_1 and x_2 are off, i.e., we write $g(0, 0) = 0$.
Now, by arguments used while calculating the entries of Table 9, you can easily see that all the values of the function g are as given below:

$$g(0, 0) = 0,\ g(0, 1) = 1,\ g(1, 0) = 1,\ g(1, 1) = 0.$$

Thus, proceeding as in the previous exercise, it can be seen that the Boolean expression (in DNF), which yields g as its Boolean function, is given by the expression

$$X(x_1, x_2) = (x'_1 \wedge x_2) \vee (x_1 \wedge x'_2),$$

because $g(0, 1) = 1$ and $g(1, 0) = 1$.

Finally, the logic circuit corresponding to this Boolean expression is shown in Fig. 19.

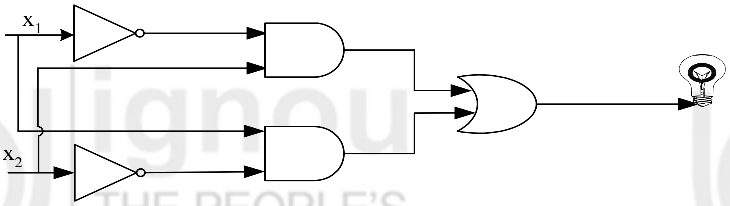


Fig. 19