Euler Angles

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Abstract

Euler Angles, even rotations on their own, have many conventions. Euler angles have different rotation orders, rotations have different "handedness".

According to Shoemake 1994, there are $3 \times 2 \times 2 \times 2 = 24$ possible conventions with given rotation "handedness".

This code implements all 24.

The rotation matrix to euler method takes later Mike 2014 improvement.

The implementation for quaternion to euler, however is derived independently, and provides all solutions in $SU(2)$ (keeping the sign of quaternion) or $SO(3)$.

The direct implementation takes less calculations and is potentially more accurate.

After implementation, the formula is found to be equivalent to that of the Evandro, Stéphane 2022 paper on similar matter.

This article aims to provide a detailed explanation of background and implementation.

1 Conventions

There are occasions where many different terms can be commonly confused and used interchangably while they shouldn't. This section aims to clarify those terms and establish a grounding for clear delivery.

1.1 Matrix Storage

When a matrix is stored in linear memory, the elements are normally tiled either row first or column first. This does NOT affect what matrix it represents, but is meerly a choice of storage.

Similarly, one could choose to store a matrix in zig-zag order. This is often the case for what Vulkan calls `VK_IMAGE_LAYOUT_SHADER_READ_ONLY_OPTIMAL` for images since it optimizes memory contigency for nearby reads.

The following is a diagram of 4×4 matrices stored in row major and column major in memory.

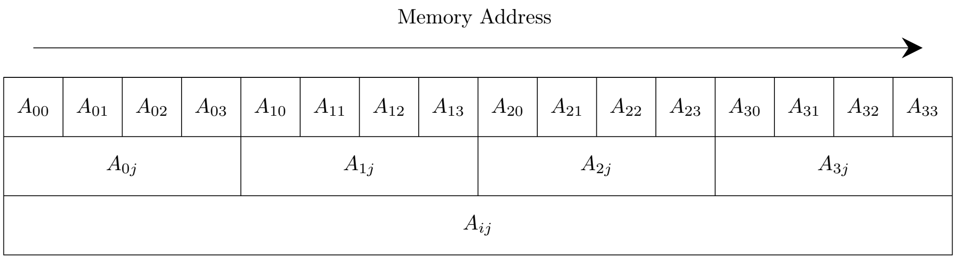

Figure 1: Row Major Storage

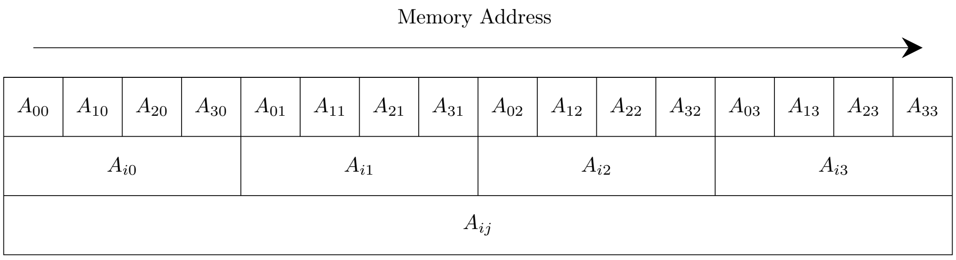

Figure 2: Column Major Storage

1.2 Vector Transforms

In geometric libraries, we represent positions as vectors, while rotations and other linear transformations as matrices. We can use column vectors, and multiply matrices at the front. Or we can use row vectors and multiply matrices at the right.

The relationship between the two is simple:

$$\begin{aligned} v' &= Mv \\ v'^T &= (Mv)^T \\ &= v^T M^T \end{aligned} \tag{1}$$

This is different from how one wants to store the matrix. One can make a matrix compose of rows while using column vectors. The mismatch only matters if one wish to use SIMD, and using same row / column composition will result in simple linear combinations while mismatch will mean memory contigency for dot products.

1.3 Coordinate Handedness

Although most of scientific work is done in right hand coordinates, there is a major share of the graphics industry that uses left hand coordinate systems so that when x axis goes to the right of screen, y axis goes up the screen, z can point into the screen, which is the direction of view.

Different handedness of coordinate system means different sign of the Levi-Civita Symbol, and a transposed rotation matrix.

But this does not change translation matrices to be in last row for column vector transforms. Only rotations are affected.

1.4 Rotations

Rotations are not vectors. Angular speed is a pseudo-vector. Pseudo-vectors behaves differently in spatial inversions, and is actually shorthand for tensors with Levi-Civita Symbol.

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \times \mathbf{B} \\ P(\mathbf{C}) &= P(\mathbf{A}) \times P(\mathbf{B}) \\ &= (-\mathbf{A}) \times (-\mathbf{B}) \\ &= \mathbf{C} \end{aligned} \tag{2}$$

Normally we would assign angular speed a direction according to the "handedness" of the coordinate system.

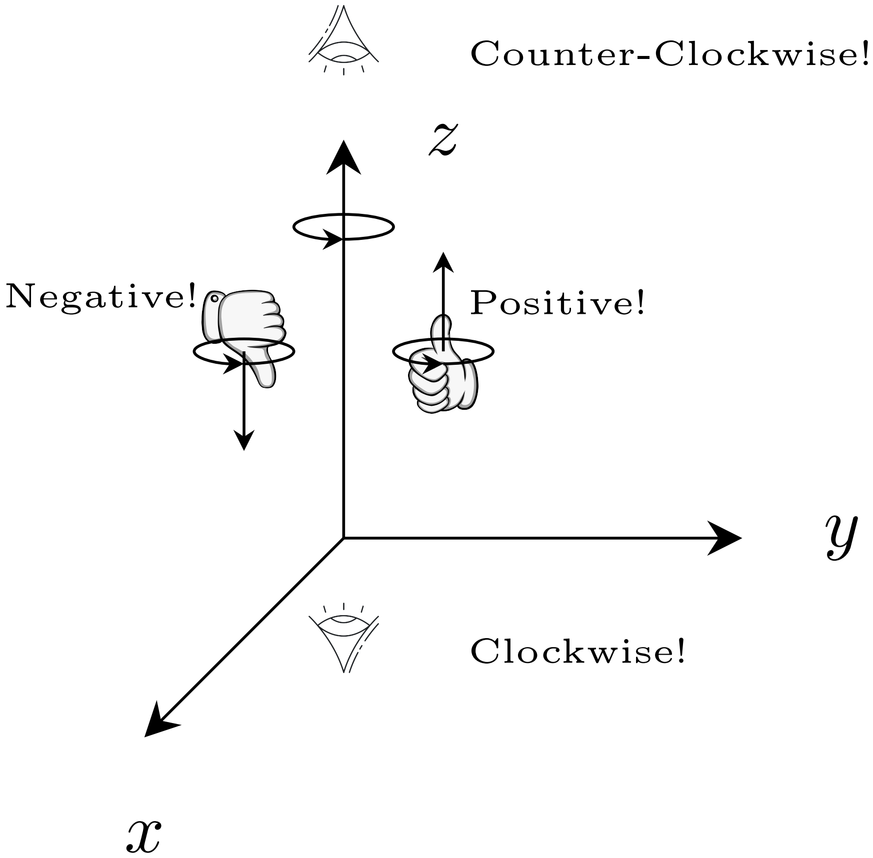

Figure 3: Direction of rotation

1.5 Euler Order

According to Shoemake 1994 article, there are only 2 fundamentally different euler angle conventions: ones with repeating axis (Proper Euler Angles originally used by Euler), and ones without repeating axis (Tait Bryant Angles).

Proper Euler Angles have cleaner analytic formulas, whereas small Tait Bryant Angles can be treated as small time delta of angular speed.

All conventions are either one combined with a permutation of axis order. The resulting order will have two types of parity. This gives $3 \times 2 \times 2$ possibilities. Then one can choose the spinning axis, which reverses the first and last rotation. Then the total is $3 \times 2 \times 2 \times 2 = 24$ possibilities.

This library assigns the names of `roll->pitch->yaw` and `bank->attitude->heading` to Tait Bryant Angles and `spin->nutation->precession` to Proper Euler Angles in rotation order in static frame. In the code, that is `rot0->rot1->rot2` in static frame or `rot2->rot1->rot0` in rotating frame.

This library encodes this metadata like this:

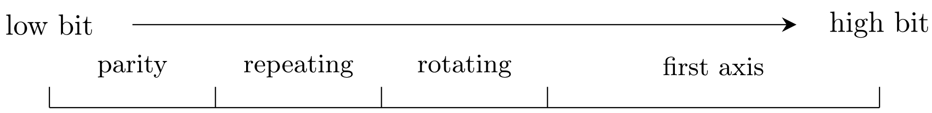

Figure 4: enum euler_angle_type encoding

This article will use ijk notation when possible. ij corresponds to θ_0, θ_1 axis, k means θ_2 axis for Tait-Bryant Angles. k means the not-rotated axis in Proper Euler Angles.

1.6 Common Conventions

Here is a non-comprehensive table of conventions for game engines and maths libraries.

Library	Storage	Indexing	Vectors	Handed	Screen In	Rotation	Euler
this library	row	(row,col)	column	right	$-z$	right hand	all
GLM / GLSL	column	[col][row]	column	right	$-z$	right hand	Tait-Bryant
DirectXMath / HLSL	row	[row][col]	row	left	$+z$	left hand	YXZs
Unity	column	[col][row]	column	left	$+z$	left hand	ZXYs
Unreal	row	[row][col]	row	left	$-y$	mixed	XYZs, XY right hand, Z left hand
Cocos	column	N/A	column	right	$-z$	right hand	ZYXs
Ogre	row	[row][col]	column	right	$-z$	right hand	Tait-Bryant
CryEngine	row	[row][col]	column	right	$+y$	right	ZYXs
CLHEP	row	[row][col]	column	right	N/A	left hand	ZYZs
Eigen	templated	(row,col)	N/A	N/A	N/A	N/A	N/A

Table 1: Common Conventions

This library's implementation of euler angles is NOT dependent on matrices being row major, but relies on vectors being column vectors (matrices multiply on the left).

2 Euler to Matrix/Quaternion

As Shoemake's 1994 paper pointed out, there are only two matrices that matters:

$$R_{ZYX} = \begin{pmatrix} \cos\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_0\cos\theta_2 - \cos\theta_0\sin\theta_2 & \sin\theta_1\cos\theta_0\cos\theta_2 + \sin\theta_0\sin\theta_2 \\ \cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_0\sin\theta_2 + \cos\theta_0\cos\theta_2 & \sin\theta_1\cos\theta_0\sin\theta_2 - \sin\theta_0\cos\theta_2 \\ -\sin\theta_1 & \cos\theta_1\sin\theta_0 & \cos\theta_1\cos\theta_0 \end{pmatrix} \tag{3}$$

$$R_{XYX} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1\sin\theta_0 & \sin\theta_1\cos\theta_0 \\ \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_0\sin\theta_2 + \cos\theta_0\cos\theta_2 & -\cos\theta_1\cos\theta_0\sin\theta_2 - \sin\theta_0\cos\theta_2 \\ -\sin\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_0\cos\theta_2 + \cos\theta_0\sin\theta_2 & \cos\theta_1\cos\theta_0\cos\theta_2 - \sin\theta_0\sin\theta_2 \end{pmatrix} \tag{4}$$

Similarly Quaternions have the formula:

$$q_{ZYX}(wxyz) = \begin{pmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\sin\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2} - p\sin\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\cos\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2} - p\sin\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2} \end{pmatrix} \tag{5}$$

$$q_{XYX}(wxyz) = \begin{pmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos(\frac{\theta_2}{2} + \frac{\theta_0}{2}) \\ \cos\frac{\theta_1}{2}\sin(\frac{\theta_2}{2} + \frac{\theta_0}{2}) \\ \sin\frac{\theta_1}{2}\cos(\frac{\theta_2}{2} - \frac{\theta_0}{2}) \\ p\sin\frac{\theta_1}{2}\sin(\frac{\theta_2}{2} - \frac{\theta_0}{2}) \end{pmatrix} \tag{6}$$

This is quite straight-forward.

Unit quaternion is a representations of $SU(2)$, whereas rotation matrix (orthonormal 3×3 matrix with $+1$ determinant) is a representation of $SO(3)$.

$SU(2)$ double covers $SO(3)$, with q and $-q$ mapping to the same rotation matrix with $qvq^{-1} = M(q)v$.

The formulas have 2π period in $SO(3)$ (rotation matrix) whereas the period is 4π in $SU(2)$ (quaternion).

This is very detailedly explained in differences of spin $\frac{1}{2}$ spinors and spin 1 vectors.

3 Matrix/Quaternion to Euler

3.1 Matrix ($SO(3)$) to Euler

When given a 3×3 rotation matrix, there are actually normally two solutions for $\theta_0, \theta_1, \theta_2, \in [-\pi, +\pi]$. This can be shown by looking at the rotation matrix. Proper Euler Angles only limits $\cos\theta_1$ whereas Tait Bryant Angles only limits $\sin\theta_1$. This means one positive and one negative value possible for the other trig function.

The normal convention for this is to make the other trig function of θ_1 positive. This gives a solution of Tait Bryant Angles for $\theta_1 \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$, and a solution of Proper Euler Angles for $\theta_1 \in [0, \pi]$.

If the solution for this convention is known, the other solution is $\theta_0' = \pi - \theta_0$, $\theta_2' = \pi - \theta_2$, $\theta_1' = -\theta_1(repeating)$ $\theta_1' = \pi - \theta_1(non - repeating)$.

Shoemake's 1994 implementation of the general solution to solving euler angles from rotation matrices is flawed by inverting the angles in the last step. This breaks the solution choice for Proper Euler Angles if parity is different, giving $\theta_1 \in [-\pi, 0]$.

This does not involve the so-called "Gimbal Lock" when the other trig function is 0.

Mike Day's 2014 improvement on "Gimbal Lock" location can be summarized as such:

The euler angle matrix elements can be categorized into three categories. 1 Type I element that is the single one with $\sin\theta_1$ or $\cos\theta_1$. 4 Type II elements that are one term with the other trig function of θ_1. 4 Type III elementa that are the most complicated, with two terms mixing everything.

Normally we use Type I element to calculate θ_1, and then with the previously said assumption, use Type II to calculate θ_0, θ_2. Type III elements are discarded and only accessed on the special occasion of "Gimbal Lock".

When at "Gimbal Lock" Type I element goes to 1, Type II elements go to 0. Type III elements give the relation of $\theta_0 \pm \theta_2$, depending on the order of axis. Thus Type III elements stores all the information of the matrix when at "Gimbal Lock".

The naive approach behaves badly at "Gimbal Lock" because Type I element $\approx 1 - \frac{1}{2}\delta^2$. Thus the error is approximately $\sqrt{2\texttt{epsilon}}$. That is 3-4 digits for IEEE float. Shoemake's original paper already takles this by using Type II elements to approximate "Gimbal Lock" θ_1 via arctan2, giving θ_1 with `epsilon` order error. But the behavior is still erratic by requiring special treatment for "Gimbal Lock".

Mike's improvement is that Type III elements is used along with prior calculated θ_0 to give θ_2. This automatically satisfies the relationship between θ_0 and θ_2 at "Gimbal Lock" while being as continuous as possible for all angles.

If the two Type II elements involving θ_2 are used to give an estimate of the other trig function of θ_1, then all the elements of the 3×3 matrix are used to calculate the result. This gives a really good result.

This is what the implementation uses to calculate euler angles from matrices. Along with adaptations to fix the flaw in Shoemake's method and using all elements to calculate.

For Proper Euler Angles:

$$\begin{aligned}
\sin\theta_1 &= \pm\sqrt{m_{ji}^2 + m_{ki}^2} \\
\cos\theta_1 &= m_{ii} \\
\sin\theta_0 &= \frac{m_{ij}}{\sin\theta_1} \\
\cos\theta_0 &= p\frac{m_{ik}}{\sin\theta_1} \\
\sin\theta_2 &= p(m_{kj} \times \cos\theta_0 - pm_{kk} \times \sin\theta_0) \\
\cos\theta_2 &= m_{jj} \times \cos\theta_0 - pm_{jk} \times \sin\theta_0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\sin\theta_1 >= 0 \\
&\quad\rightarrow \\
&\theta_1 = \arctan2(\sqrt{m_{ji}^2 + m_{ki}^2}, m_{ii}) \\
&\theta_0 = \arctan2(m_{ij}, pm_{ik}) \\
&\theta_2 = p\arctan2(m_{kj} \times \cos\theta_0 - pm_{kk} \times \sin\theta_0, m_{jj} \times \cos\theta_0 - pm_{jk} \times \sin\theta_0)
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\sin\theta_1 <= 0 \\
&\quad\rightarrow \\
&\theta_1 = \arctan2(-\sqrt{m_{ji}^2 + m_{ki}^2}, m_{ii}) \\
&\theta_0 = \arctan2(-m_{ij}, -pm_{ik}) \\
&\theta_2 = p\arctan2(-m_{kj} \times \cos\theta_0 + pm_{kk} \times \sin\theta_0, -m_{jj} \times \cos\theta_0 + pm_{jk} \times \sin\theta_0)
\end{aligned} \tag{9}$$

For Tait Bryant Angles:

$$\begin{aligned}
\sin\theta_1 &= -pm_{ki} \\
\cos\theta_1 &= \pm\sqrt{m_{ii}^2 + m_{ji}^2} \\
\sin\theta_0 &= p\frac{m_{kj}}{\cos\theta_1} \\
\cos\theta_0 &= \frac{m_{kk}}{\cos\theta_1} \\
\sin\theta_2 &= p(-m_{ij} \times \cos\theta_0 + pm_{ik} \times \sin\theta_0) \\
\cos\theta_2 &= m_{jj} \times \cos\theta_0 - pm_{jk} \times \sin\theta_0
\end{aligned} \tag{10}$$

$$
\begin{aligned}
&\cos\theta_1 >= 0\\
&\qquad\rightarrow\\
&\qquad\theta_1 = p\,\text{arctan2}(-m_{ki}, \sqrt{m_{ii}^2 + m_{ji}^2})\\
&\qquad\theta_0 = p\,\text{arctan2}(m_{kj}, m_{kk})\\
&\qquad\theta_2 = p\,\text{arctan2}(-m_{ij} \times \cos\theta_0 + pm_{ik} \times \sin\theta_0, m_{jj} \times \cos\theta_0 - pm_{jk} \times \sin\theta_0)
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\cos\theta_1 <= 0\\
&\qquad\rightarrow\\
&\qquad\theta_1 = p\,\text{arctan2}(-m_{ki}, -\sqrt{m_{ii}^2 + m_{ji}^2})\\
&\qquad\theta_0 = p\,\text{arctan2}(-m_{kj}, -m_{kk})\\
&\qquad\theta_2 = p\,\text{arctan2}(m_{ij} \times \cos\theta_0 - pm_{ik} \times \sin\theta_0, -m_{jj} \times \cos\theta_0 + pm_{jk} \times \sin\theta_0)
\end{aligned}
\tag{12}
$$

Another numerically stable method that doesn't require specially treating "Gimbal Lock" is CERN CLHEP's method. But when tried out seemd to be less accurate than this one.

That method uses the following relations:

For Proper Euler Angles:

$$
\begin{aligned}
m_{jj} + m_{kk} &= +\cos(\theta_2 + \theta_0)(1 + \cos\theta_1)\\
m_{jj} - m_{kk} &= +\cos(\theta_2 - \theta_0)(1 - \cos\theta_1)\\
m_{kj} - m_{jk} &= +p\sin(\theta_2 + \theta_0)(1 + \cos\theta_1)\\
m_{kj} + m_{jk} &= +p\sin(\theta_2 - \theta_0)(1 - \cos\theta_1)\\
m_{ki}m_{ik} - m_{ji}m_{ij} &= -\cos(\theta_2 - \theta_0)\sin^2\theta_1\\
m_{ki}m_{ik} + m_{ji}m_{ij} &= -\cos(\theta_2 + \theta_0)\sin^2\theta_1\\
m_{ji}m_{ik} - m_{ki}m_{ij} &= +p\sin(\theta_2 - \theta_0)\sin^2\theta_1\\
m_{ji}m_{ik} + m_{ki}m_{ij} &= +p\sin(\theta_2 + \theta_0)\sin^2\theta_1
\end{aligned}
\tag{13}
$$

For Tait Bryant Angles:

$$
\begin{aligned}
m_{ij} - m_{jk} &= -p\sin(\theta_2 - \theta_0)(1 + p\sin\theta_1)\\
m_{ij} + m_{jk} &= -p\sin(\theta_2 + \theta_0)(1 - p\sin\theta_1)\\
m_{jj} - m_{ik} &= +\cos(\theta_2 + \theta_0)(1 - p\sin\theta_1)\\
m_{jj} + m_{ik} &= +\cos(\theta_2 - \theta_0)(1 + p\sin\theta_1)\\
m_{ii}m_{kk} - m_{kj}m_{ji} &= +\cos(\theta_2 + \theta_0)\cos^2\theta_1\\
m_{ii}m_{kk} + m_{kj}m_{ji} &= +\cos(\theta_2 - \theta_0)\cos^2\theta_1\\
m_{ji}m_{kk} - m_{kj}m_{ii} &= +p\sin(\theta_2 - \theta_0)\cos^2\theta_1\\
m_{ji}m_{kk} + m_{kj}m_{ii} &= +p\sin(\theta_2 + \theta_0)\cos^2\theta_1
\end{aligned}
\tag{14}
$$

3.2 Quaternion ($SU(2)$) to Euler

There are a total of 4 solution of θ_1 in the θ_1 period $[-2\pi, +2\pi]$. θ_0, θ_2 range is elaborated later.

For Proper Euler Angles:

$$
\begin{aligned}
q_0 &= \cos\frac{\theta_1}{2}\cos(\frac{\theta_2}{2}+\frac{\theta_0}{2}) \\
q_i &= \cos\frac{\theta_1}{2}\sin(\frac{\theta_2}{2}+\frac{\theta_0}{2}) \\
q_j &= \sin\frac{\theta_1}{2}\cos(\frac{\theta_2}{2}-\frac{\theta_0}{2}) \\
q_k &= p\sin\frac{\theta_1}{2}\sin(\frac{\theta_2}{2}-\frac{\theta_0}{2})
\end{aligned}
$$

$$
\begin{aligned}
\cos^2\frac{\theta_1}{2} &= q_0^2+q_i^2 \\
\sin^2\frac{\theta_1}{2} &= q_j^2+q_k^2 \\
\cos(\frac{\theta_2}{2}+\frac{\theta_0}{2}) &= \frac{q_0}{\cos\frac{\theta_1}{2}} \\
\sin(\frac{\theta_2}{2}+\frac{\theta_0}{2}) &= \frac{q_i}{\cos\frac{\theta_1}{2}} \\
\cos(\frac{\theta_2}{2}-\frac{\theta_0}{2}) &= \frac{q_j}{\sin\frac{\theta_1}{2}} \\
\sin(\frac{\theta_2}{2}-\frac{\theta_0}{2}) &= p\frac{q_k}{\sin\frac{\theta_1}{2}}
\end{aligned} \tag{15}
$$

Solution 0:

$$
\begin{aligned}
\theta_1\in[0,\pi] &\rightarrow \frac{\theta_1}{2}\in[0,\frac{\pi}{2}] \\
&\rightarrow \cos\frac{\theta_1}{2}\geq 0,\ \sin\frac{\theta_1}{2}\geq 0 \\
\theta_1 &= 2\,\mathrm{arctan2}(\sqrt{q_j^2+q_k^2},\sqrt{q_0^2+q_i^2}) \\
\frac{\theta_0}{2}+\frac{\theta_2}{2} &= \mathrm{arctan2}(q_i,q_0) \\
&= \alpha \\
\frac{\theta_0}{2}-\frac{\theta_2}{2} &= \mathrm{arctan2}(pq_k,q_j) \\
&= \beta
\end{aligned} \tag{16}
$$

Solution 1:

$$
\begin{aligned}
\theta_1\in[-\pi,0] &\rightarrow \frac{\theta_1}{2}\in[-\frac{\pi}{2},0] \\
&\rightarrow \cos\frac{\theta_1}{2}\geq 0,\ \sin\frac{\theta_1}{2}\leq 0 \\
\theta_1 &= 2\,\mathrm{arctan2}(-\sqrt{q_j^2+q_k^2},\sqrt{q_0^2+q_i^2}) \\
\frac{\theta_0}{2}+\frac{\theta_2}{2} &= \mathrm{arctan2}(q_i,q_0) \\
&= \alpha \\
\frac{\theta_0}{2}-\frac{\theta_2}{2} &= \mathrm{arctan2}(-pq_k,-q_j) \\
&= \beta
\end{aligned} \tag{17}
$$

Solution 2:

$$
\begin{aligned}
\theta_1 \in [\pi, 2\pi] &\rightarrow \frac{\theta_1}{2} \in [\frac{\pi}{2}, \pi] \\
&\rightarrow \cos\frac{\theta_1}{2} \leq 0,\ \sin\frac{\theta_1}{2} \geq 0 \\
\theta_1 &= 2\arctan2(\sqrt{q_j^2 + q_k^2}, -\sqrt{q_0^2 + q_i^2}) \\
\frac{\theta_0}{2} + \frac{\theta_2}{2} &= \arctan2(-q_i, -q_0) \\
&= \alpha \\
\frac{\theta_0}{2} - \frac{\theta_2}{2} &= \arctan2(pq_k, q_j) \\
&= \beta
\end{aligned}
\tag{18}
$$

Solution 3:

$$
\begin{aligned}
\theta_1 \in [-2\pi, -\pi] &\rightarrow \frac{\theta_1}{2} \in [-\pi, -\frac{\pi}{2}] \\
&\rightarrow \cos\frac{\theta_1}{2} \leq 0,\ \sin\frac{\theta_1}{2} \leq 0 \\
\theta_1 &= 2\arctan2(-\sqrt{q_j^2 + q_k^2}, -\sqrt{q_0^2 + q_i^2}) \\
\frac{\theta_0}{2} + \frac{\theta_2}{2} &= \arctan2(-q_i, -q_0) \\
&= \alpha \\
\frac{\theta_0}{2} - \frac{\theta_2}{2} &= \arctan2(-pq_k, -q_j) \\
&= \beta
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\theta_0 &= \alpha + \beta \\
\theta_2 &= \alpha - \beta
\end{aligned}
\tag{20}
$$

For Tait Bryant Angles:

$$
\begin{aligned}
q_0 &= \cos\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\sin\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\
q_i &= \cos\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2} - p\sin\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\
q_j &= \sin\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\cos\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2} \\
q_k &= \cos\frac{\theta_1}{2}\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2} - p\sin\frac{\theta_1}{2}\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2}
\end{aligned}
$$

$$
\begin{aligned}
q_iq_k - pq_0q_j &= -p\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2} \\
q_iq_k + pq_0q_j &= 2\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2} + p\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right)\left(\cos^2\frac{\theta_2}{2} - \sin^2\frac{\theta_2}{2}\right) \\
q_0 + q_j &= \left(\cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2}\right)\left(\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2}\right) \\
q_0 - q_j &= \left(\cos\frac{\theta_1}{2} - \sin\frac{\theta_1}{2}\right)\left(\cos\frac{\theta_0}{2}\cos\frac{\theta_2}{2} - p\sin\frac{\theta_0}{2}\sin\frac{\theta_2}{2}\right) \\
q_i + pq_k &= \left(\cos\frac{\theta_1}{2} - \sin\frac{\theta_1}{2}\right)\left(\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2} + p\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2}\right) \\
q_i - pq_k &= \left(\cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2}\right)\left(\sin\frac{\theta_0}{2}\cos\frac{\theta_2}{2} - p\cos\frac{\theta_0}{2}\sin\frac{\theta_2}{2}\right)
\end{aligned}
\tag{21}
$$

Taking parameters:

$$
\begin{aligned}
f_{00} &= q_0 + q_j \\
f_{01} &= q_0 - q_j \\
f_{10} &= q_i - pq_k \\
f_{11} &= q_i + pq_k \\
A &= \cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2} \\
B &= \cos\frac{\theta_1}{2} - \sin\frac{\theta_1}{2}
\end{aligned} \tag{22}
$$

We have:

$$
\begin{aligned}
A^2 &= f_{10}^2 + f_{00}^2 \\
B^2 &= f_{11}^2 + f_{01}^2 \\
\sin\left(\frac{\theta_0}{2} + p\frac{\theta_2}{2}\right) &= \frac{f_{11}}{B} \\
\cos\left(\frac{\theta_0}{2} + p\frac{\theta_2}{2}\right) &= \frac{f_{01}}{B} \\
\sin\left(\frac{\theta_0}{2} - p\frac{\theta_2}{2}\right) &= \frac{f_{10}}{A} \\
\cos\left(\frac{\theta_0}{2} - p\frac{\theta_2}{2}\right) &= \frac{f_{00}}{A} \\
\sin\theta_1 &= 2(q_0q_j - q_iq_k) \\
\cos\theta_1 &= AB
\end{aligned} \tag{23}
$$

Solution 0:

$$
\begin{aligned}
\theta_1 \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \rightarrow \frac{\theta_1}{2} &\in [-\frac{\pi}{4}, +\frac{\pi}{4}] \\
&\rightarrow \cos\frac{\theta_1}{2} \geq |\sin\frac{\theta_1}{2}| \\
&\rightarrow A \geq 0, B \geq 0 \\
\theta_1 &= \operatorname{arctan2}(2(q_0q_j - q_iq_k), \sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \operatorname{arctan2}(f_{11}, f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \operatorname{arctan2}(f_{10}, f_{00}) \\
&= \beta
\end{aligned} \tag{24}
$$

Solution 1:

$$
\begin{aligned}
\theta_1 &\in [-\pi, -\frac{\pi}{2}) \cup (+\frac{\pi}{2}, +\pi] \\
\sin\theta_1 &= 2(q_0q_j - q_iq_k) >= 0 \\
&\rightarrow \theta_1 \in (+\frac{\pi}{2}, +\pi] \\
&\rightarrow A > 0, B < 0 \\
\theta_1 &= \operatorname{arctan2}(2(q_0q_j - q_iq_k), -\sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \operatorname{arctan2}(-f_{11}, -f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \operatorname{arctan2}(f_{10}, f_{00}) \\
&= \beta
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\sin\theta_1 &= 2(q_0q_j - q_iq_k) <= 0 \\
&\rightarrow \theta_1 \in [-\pi, -\frac{\pi}{2}) \\
&\rightarrow A < 0, B > 0 \\
\theta_1 &= \text{arctan2}(2(q_0q_j - q_iq_k), -\sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)})+ \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \text{arctan2}(f_{11}, f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \text{arctan2}(-f_{10}, -f_{00}) \\
&= \beta
\end{aligned}
\tag{26}
$$

Solution 2:

$$
\begin{aligned}
\theta_1 &\in [-\frac{3\pi}{2}, -\pi) \cup (+\pi, +\frac{3\pi}{2}] \\
\sin\theta_1 &= 2(q_0q_j - q_iq_k) > 0 \\
&\rightarrow \theta_1 \in [-\frac{3\pi}{2}, -\pi) \\
&\rightarrow A < 0, B \geq 0 \\
\theta_1 &= \text{arctan2}(2(q_0q_j - q_iq_k), -\sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) - 2\pi \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \text{arctan2}(f_{11}, f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \text{arctan2}(-f_{10}, -f_{00}) \\
&= \beta
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\sin\theta_1 &= 2(q_0q_j - q_iq_k) < 0 \\
&\rightarrow \theta_1 \in (+\pi, +\frac{3\pi}{2}] \\
&\rightarrow A \geq 0, B < 0 \\
\theta_1 &= \text{arctan2}(2(q_0q_j - q_iq_k), -\sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) + 2\pi \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \text{arctan2}(-f_{11}, -f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \text{arctan2}(f_{10}, f_{00}) \\
&= \beta
\end{aligned}
\tag{28}
$$

Solution 3:

$$
\begin{aligned}
\theta_1 &\in [-2\pi, -\frac{3\pi}{2}) \cup (+\frac{3\pi}{2}, +2\pi] \\
\sin\theta_1 &= 2(q_0q_j - q_iq_k) \geq 0 \\
&\rightarrow \theta_1 \in [-2\pi, -\frac{3\pi}{2}) \\
&\rightarrow A < 0, B < 0 \\
\theta_1 &= \text{arctan2}(2(q_0q_j - q_iq_k), \sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) - 2\pi \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \text{arctan2}(-f_{11}, -f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \text{arctan2}(-f_{10}, -f_{00}) \\
&= \beta
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\sin\theta_1 &= 2(q_0q_j - q_iq_k) \leq 0 \\
&\rightarrow \theta_1 \in (+\frac{3\pi}{2}, +2\pi] \\
&\rightarrow A < 0, B < 0 \\
\theta_1 &= \operatorname{arctan2}(2(q_0q_j - q_iq_k), \sqrt{(f_{11}^2 + f_{01}^2)\,(f_{10}^2 + f_{00}^2)}) + 2\pi \\
\frac{\theta_0}{2} + p\frac{\theta_2}{2} &= \operatorname{arctan2}(-f_{11}, -f_{01}) \\
&= \alpha \\
\frac{\theta_0}{2} - p\frac{\theta_2}{2} &= \operatorname{arctan2}(-f_{10}, -f_{00}) \\
&= \beta
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\theta_0 &= \alpha + \beta \\
\theta_2 &= p(\alpha - \beta)
\end{aligned}
\tag{31}
$$

The solutions do not need to work on "Gimbal Lock" either, since they work in relations of $\frac{\theta_0}{2} \pm \frac{\theta_2}{2}$.

This direct quaternion to euler approach keeps the sign of quaternion, as it works in $SU(2)$.

Additionally, this library allows specifying θ_1 outside $[-2\pi, +\pi]$ range. The convention for solution index is shown in figure 5:

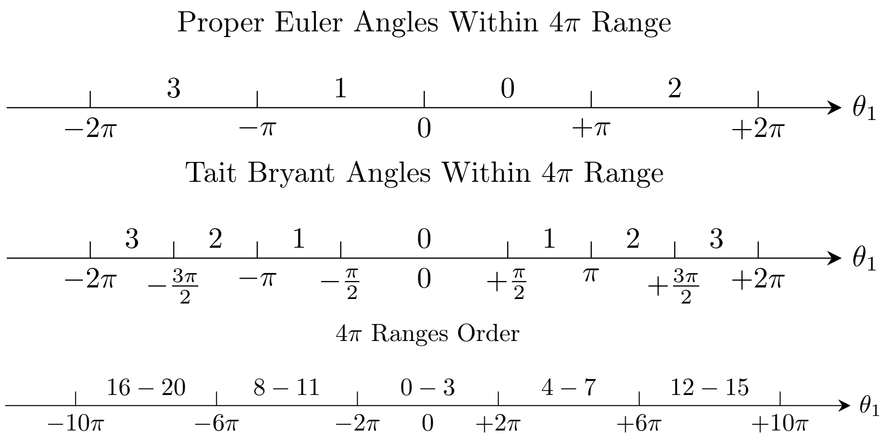

Figure 5: θ_1 Solution Index for $SU(2)$

The solutions of θ_0, θ_2 satisfies $\frac{\theta_0}{2} \pm \frac{\theta_2}{2} \in [-\pi, +\pi]$. This is a larger range than $\theta_0, \theta_2 \in [-\pi, +\pi]$, but smaller range than $\theta_0, \theta_2 \in [-2\pi, +2\pi]$. As shown in figure 6.

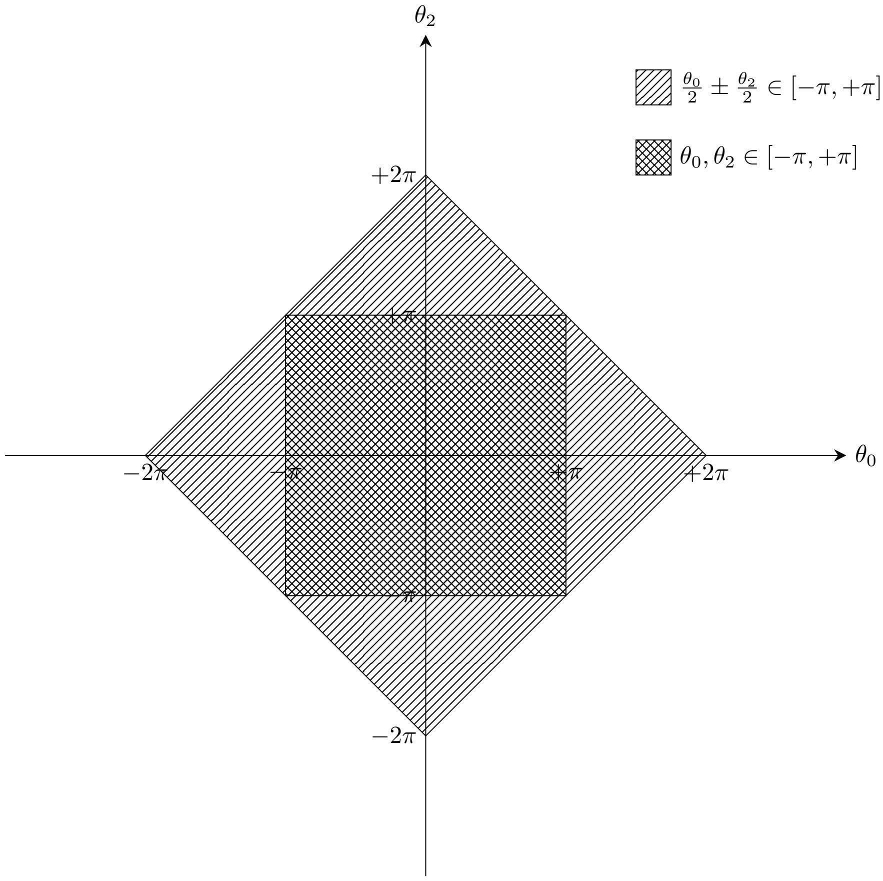

Figure 6: Euler Angles θ_0, θ_2 from $SU(2)$ to $SO(3)$

If one "corrects" one of θ_0, θ_2 if it lies outside the range $\theta_0, \theta_2 \in [-\pi, +\pi]$, one will get the result back to single covering $SO(3)$, and inverting the quaternion in the process.

There is NO guarantee of having a SU(2) solution in $\theta_0, \theta_1, \theta_2 \in [-\pi, +\pi]$ or $\theta_0, \theta_1, \theta_2 \in [0, 2\pi]$.