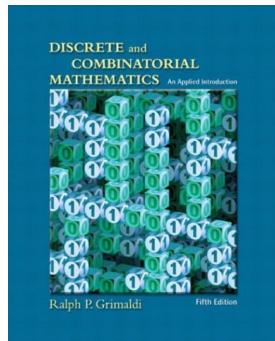
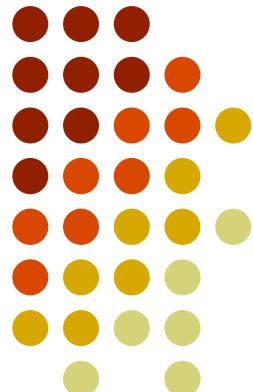


Discrete Mathematics

-- *Chapter 2: Fundamentals of Logic*



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Outline

- Basic Connectives and Truth Tables
- Logical Equivalence: The Law of Logic
- Logical Implication: Rule of Inference
- The Use of Quantifiers
- Quantifiers, Definitions, and the Proofs of Theorems



The Way to Proof

- 證明”A君是萬能”這句話是錯的
 - (1) A君可以搬動任何石頭 $\rightarrow q$
 - (2) A君可以製造出他無法搬動的石頭
 - (3) (1), (2) 相互矛盾, 原命題為假 $\neg q$

$$\begin{array}{c} p \\ \text{(1)} p \rightarrow q \\ \text{(2)} p \rightarrow r \\ \hline r \rightarrow \neg q \end{array}$$

A logical sequence of statements.

$$\begin{array}{c} p \\ p \rightarrow q \\ q \\ p \rightarrow r \\ r \\ r \rightarrow \neg q \\ \neg q \end{array}$$



2.1 Basic Connectives and Truth Tables

- Statement 敘述 (Proposition 命題): are declarative sentences that **are either true or false, but not both.**
- Primitive Statement (原始命題)
 - Examples
 - p : ‘Discrete Mathematics’ is a required course for sophomores.
 - q : Margaret Mitchell wrote ‘*Gone with the Wind*’.
 - r : $2+3=5$.
 - “What a beautiful evening!” (not a statement)
 - ”Get up and do your exercises.” (not a statement)
 - No way to make them simpler

“*The number x is an integer.*” is a statement ?



2.1 Basic Connectives and Truth Tables

- New statements can be obtained from primitive statements in two ways
 - Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read “Not p ”. 非 p (*Negation statements*)
 - Combine two or more statements into a *compound* statement, using *logical connectives*. (*Compound statements* 複合敘述)



2.1 Basic Connectives and Truth Tables

- Compound Statement (Logical Connectives)

- Conjunction: $p \wedge q$ (read “p and q”)
- Disjunction : $p \vee q$ (read “p or q”)
- Exclusive or : $p \underline{\vee} q$. ($p \oplus q$)
- Implication: $p \rightarrow q$,
“p implies q”.
 p : hypothesis
 q : conclusion
- Biconditional: $p \leftrightarrow q$
 - “p if and only if q” (若且為若)
 - “p iff q”
 - “p is necessary and sufficient for q”.

$p \rightarrow q$ is also called,

- If p , then q
- p is sufficient for q
- p is a sufficient condition for q
- q is necessary for p
- q is a necessary condition for p
- p only if q
- q whenever p



2.1 Basic Connectives and Truth Tables

- The truth and falsity of the compound statements based on the truth values of their components (primitive statements).

We do not want a true statement to lead us into believing something that is false.

| p | $\neg p$ |
|-----|----------|
| 0 | 1 |
| 1 | 0 |

| p | q | $p \wedge q$ | $p \vee q$ | $p \underline{\vee} q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|-----|-----|--------------|------------|------------------------|-------------------|-----------------------|
| 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 |

Truth Tables



Writing Down Truth Tables

- “True, False” is preferred (less ambiguous) than “1, 0”
- List the elementary truth values in a **consistent** way (from F...F to T...T or from T...T to F...F).
- They get long: with n variables, the length is 2^n .
- In the end we want to be able to reason about logic without having to write down such tables.



2.1 Basic Connectives and Truth Tables

- $p \rightarrow q$ is equivalent with $(\neg p \vee q)$
- It's not relative about the causal relationship
 - If 'Discrete Mathematics' is a required course for sophomores, then Margaret Mitchell wrote 'Gone with the Wind'. **is true**
 - If "2+3=5", then "4+2=6" **is true**
 - If "2+3=6", then "2+4=7" **is true**
- "Margaret Mitchell wrote 'Gone with the Wind' (q), and $2+3 \neq 5$ ($\neg r$), the 'Discrete Mathematics' is a required course for sophomores (p).
 - $q \wedge (\neg r \rightarrow p)$

Table 2.3

| 1 | 2 | 3 | | | |
|----------|----------|----------|----------|------------------------|-----------------------------------|
| p | q | r | $\neg r$ | $\neg r \rightarrow p$ | $q \wedge (\neg r \rightarrow p)$ |
| 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |



Implications examples

p: If it is sunny today, then we will go to the beach.

q: If today is Friday, then $2+3=5$.

r: If today is Friday, then $2+3=6$.

r is true every day except Friday, even though $2+3=6$ is false.

只考慮邏輯間關係，不考慮因果關係

*The Mathematical concept of an **implication** is independent of a **cause-and-effect relationship** between hypothesis and conclusion.*



2.1 Basic Connectives and Truth Tables

- **Ex 2.1:** Let s , t , and u denote the primitive statements.
 - s : Phyllis goes out for a walk.
 - t : The moon is out.
 - u : It is snowing.
- English sentences for compound statements.
 - $(t \wedge \neg u) \rightarrow s$:
 - If the moon is out **and** it is not snowing, **then** Phyllis goes out for a walk.
 - $t \rightarrow (\neg u \rightarrow s)$:
 - If the moon is out, **then if** it is not snowing Phyllis goes out for a walk.
 - $\neg(s \leftrightarrow (u \vee t))$:
 - **It is not the case** that Phyllis goes out for a walk **if and only if** it is snowing **or** the moon is out.



2.1 Basic Connectives and Truth Tables

- Let s , t , and u denote the primitive statements.
 - s : Phyllis goes out for a walk.
 - t : The moon is out.
 - u : It is snowing.
- Reversely, examine the logical form for given English sentences.
 - “Phyllis will go out walking if and only if the moon is out.”
 - $s \leftrightarrow t$
 - If it is snowing and the moon is not out, then Phyllis will not go out for a walk.”
 - $(u \wedge \neg t) \rightarrow \neg s$
 - It is snowing but Phyllis will still go out for a walk.
 - $u \wedge s$ 都發生



2.1 Basic Connectives and Truth Tables

- Ex 2.3:
 - Decision (selection) structure
 - In computer science, the **if-then** and **if-then-else** decision structure arise in high-level programming languages such as Java and C++.
 - E.g., “if p then q else r ,” q is executed when p is true and r is executed when p is false.



2.1 Basic Connectives and Truth Tables

恒真式

- **Tautology, T_0 :** If a compound statement is true for all truth value assignments for its component statements.
 - Example: $p \vee \neg p$
- **Contradiction, F_0 :** If a compound statement is false for all truth value assignments for its component statements.
 - Example: $p \wedge \neg p$
- Examples
 - “ $2 = 3 - 1$ ” is not a tautology, but “ $2=1$ or $2 \neq 1$ ” is;
 - “ $1+1=3$ ” is not a contradiction, but “ $1=1$ and $1 \neq 1$ ” is.



2.1 Basic Connectives and Truth Tables

論述

- An **argument** starts with a list of given statements called premises (*hypothesis*) and a statement called the conclusion of the argument.
 - $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$



2.2 Logical Equivalence: The Laws of Logic

- **Logically equivalent**, $s_1 \Leftrightarrow s_2$: When the statement s_1 is true (false) if and only if s_2 is true (false).

$$\neg p \vee q \Leftrightarrow p \rightarrow q$$

$$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow p \Leftrightarrow q$$

the same truth tables

| p | q | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \wedge (q \rightarrow p)$ | $p \Leftrightarrow q$ |
|-----|-----|----------|-----------------|-------------------|-------------------|--|-----------------------|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |



2.2 Logical Equivalence: The Laws of Logic

- Negation, \neg (exclusive)

Table 2.8

| p | q | $p \vee q$ | $p \vee q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $(p \vee q) \wedge \neg(p \wedge q)$ |
|-----|-----|------------|------------|--------------|--------------------|--------------------------------------|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 |



2.2 Logical Equivalence: The Laws of Logic

- Logically equivalent examples
 - $p \Leftrightarrow (p \vee p)$
 - “ $1+1=2$ ” \Leftrightarrow “ $1+1=2$ or $1+1=2$ ”
 - $\neg \neg p \Leftrightarrow p$ Double negation
 - “He did not not do it” \Leftrightarrow “He did it”
 - $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ DeMorgan
 - “If p then not p ” \Leftrightarrow “not p ”

$$p \rightarrow \neg p \Leftrightarrow \neg p \vee \neg p \Leftrightarrow \neg p$$



The Laws of Logic (1/2)

1) Law of Double Negation : $\neg \neg p \Leftrightarrow p$

2) DeMorgan's Laws :
$$\begin{cases} \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \\ \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \end{cases}$$

3) Commutative Laws :
$$\begin{cases} p \vee q \Leftrightarrow q \vee p \\ p \wedge q \Leftrightarrow q \wedge p \end{cases}$$

4) Associative Laws :
$$\begin{cases} p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \\ p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \end{cases}$$

5) Distributive Laws :
$$\begin{cases} p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \end{cases}$$



The Laws of Logic (2/2)

6) Idempotent Laws: $\begin{cases} p \vee p \Leftrightarrow p \\ p \wedge p \Leftrightarrow p \end{cases}$

7) Identity Laws: $\begin{cases} p \vee F_0 \Leftrightarrow p \\ p \wedge T_0 \Leftrightarrow p \end{cases}$

8) Inverse Laws: $\begin{cases} p \vee \neg p \Leftrightarrow T_0 \\ p \wedge \neg p \Leftrightarrow F_0 \end{cases}$

9) Domination Laws: $\begin{cases} p \vee T_0 \Leftrightarrow T_0 \\ p \wedge F_0 \Leftrightarrow F_0 \end{cases}$

10) Absorption Laws: $\begin{cases} p \vee (p \wedge q) \Leftrightarrow p \\ p \wedge (p \vee q) \Leftrightarrow p \end{cases}$



2.2 Logical Equivalence: The Laws of Logic

- DeMorgan's Laws: $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
 $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

| p | q | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|-----|-----|--------------|--------------------|----------|----------|----------------------|------------|------------------|------------------------|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |



The Principle of Duality

- **Definition 2.3:** Let s be a statement. Dual of s , denoted s^d , is the statement obtained from s by replacing each occurrence of \wedge and \vee by \vee and \wedge , respectively, and each occurrence of T_0 and F_0 by F_0 and T_0 , respectively.

- E.g. $s : (p \wedge \neg q) \vee (r \wedge T_0)$

$$s^d : (p \vee \neg q) \wedge (r \vee F_0) \quad \text{Keep negation!}$$

- **The Principle of Duality:** Let s and t be statements that contain no logical connectives other than \wedge and \vee .

If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Ex: $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
 $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$



Substitution Rules

- **Rule 1:** Suppose that the compound statement P is a **tautology**.
 - If p is a primitive statement that appears in P and we replace each occurrence of p by the same statement q , then the resulting compound statement P_1 is also a tautology.
- **Rule 2:** Let P be a compound statement where p is an arbitrary statement that appears in P , and let q be a statement such that $q \Leftrightarrow p$.
 - Suppose that in P we replace one or more occurrences of p by q . Then this replacement yields the compound statement P_1 . Under these circumstances $P_1 \Leftrightarrow P$.



Simplification of Compound Statements

- **Ex 2.10:**

- From the first of DeMorgan's Laws

$P: \neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ is a tautology

- from the first substitution rule

$P_1: \neg[(r \wedge s) \vee q] \Leftrightarrow [\neg(r \wedge s) \wedge \neg q]$ is also a tautology

- replace each occurrence of q by $t \rightarrow u$

$P_2: \neg[(r \wedge s) \vee (t \rightarrow u)] \Leftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)]$



Simplification of Compound Statements

- **Ex 2.11:**

- $P: (p \rightarrow q) \rightarrow r$, and because $(p \rightarrow q) \Leftrightarrow \neg p \vee q$
- from the second substitution rule

if $P_1: (\neg p \vee q) \rightarrow r$, then $P_1 \Leftrightarrow P$

- Check 2.11 (b) for another example using the second rule
 - $p \rightarrow (p \vee q), \neg \neg p \Leftrightarrow p$
 - $P_1: p \rightarrow (\neg \neg p \vee q), P_1 \Leftrightarrow P$
 - $P_2: \neg \neg p \rightarrow (\neg \neg p \vee q), P_2 \Leftrightarrow P$



Simplification of Compound Statements

- **Ex 2.12:**

- Negate and simplify the compound statement $(p \vee q) \rightarrow r$
 1. $(p \vee q) \rightarrow r \Leftrightarrow \neg(p \vee q) \vee r$ (First substitution rule)
 2. $\neg[(p \vee q) \rightarrow r] \Leftrightarrow \neg[\neg(p \vee q) \vee r]$ (Negating)
 3. $\neg[\neg(p \vee q) \vee r] \Leftrightarrow \neg\neg(p \vee q) \wedge \neg r$ (DeMorgan's Laws)
 4. $\neg\neg(p \vee q) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r$ (Law of double Negation)
 5. $\neg[(p \vee q) \rightarrow r] \Leftrightarrow (p \vee q) \wedge \neg r$



Simplification of Compound Statements

- **Ex 2.13:**

- p : Joan goes to Lake George
- q : Mary pays for Joan's shopping spree.
- $p \rightarrow q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.
- The negation of $p \rightarrow q$:
 - One way: $\neg(p \rightarrow q)$. *It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree.*
 - Another way: $p \wedge \neg q$. Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

$$\begin{aligned}\neg(p \rightarrow q) \\ \Leftrightarrow \neg(\neg p \vee q) \\ \Leftrightarrow \neg\neg p \wedge \neg q \\ \Leftrightarrow p \wedge \neg q\end{aligned}$$



Relevant Statements to Implication Statement

- Ex 2.15:

contrapositive

converse

inverse

| p | q | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ | $q \rightarrow p$ | $\neg p \rightarrow \neg q$ |
|-----|-----|-------------------|-----------------------------|-------------------|-----------------------------|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

- $p \rightarrow q \Leftrightarrow (\neg q \rightarrow \neg p)$
- $q \rightarrow p \Leftrightarrow (\neg p \rightarrow \neg q)$
- **Contrapositive** of $p \rightarrow q : \underline{\neg q \rightarrow \neg p}$
- **Converse** of $p \rightarrow q : \underline{q \rightarrow p}$
- **Inverse** of $p \rightarrow q : \underline{\neg p \rightarrow \neg q}$



Simplification of Compound Statements

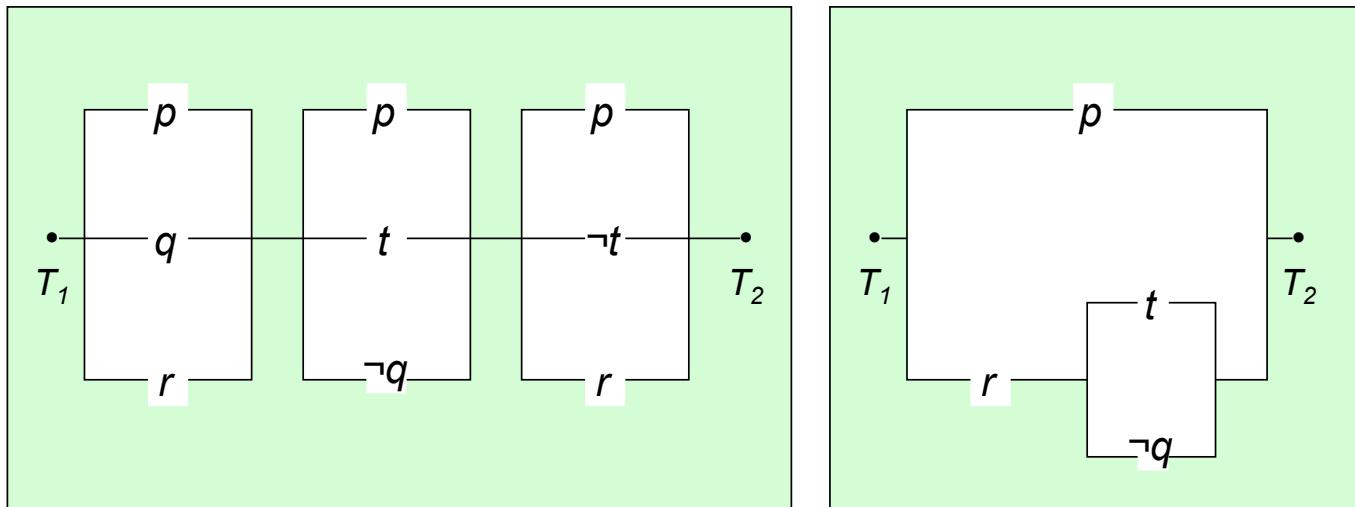
- **Ex 2.16:** How to simply express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$?

$$\begin{aligned} & (p \vee q) \wedge \neg(\neg p \wedge q) && \text{Reasons} \\ \Leftrightarrow & (p \vee q) \wedge (\neg \neg p \vee \neg q) && (\text{DeMorgan's Law}) \\ \Leftrightarrow & (p \vee q) \wedge (p \vee \neg q) && (\text{Law of Double Negation}) \\ \Leftrightarrow & p \vee (q \wedge \neg q) && (\text{Distributive Law of } \vee \text{ over } \wedge) \\ \Leftrightarrow & p \vee F_0 && (\text{Inverse Law}) \\ \Leftrightarrow & p && (\text{Identity Law}) \end{aligned}$$



Simplification of Compound Statements

- **Ex 2.18:** A switching network is made up of wires and switches connecting two terminals T_1 and T_2

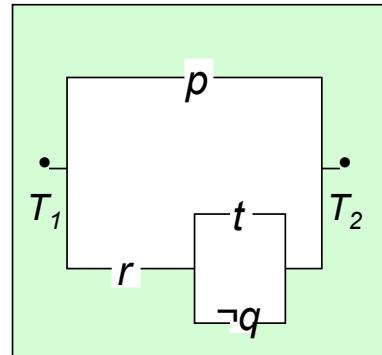


$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$$



Simplification of Compound Statements

$$\begin{aligned}& (p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \\ \Leftrightarrow & p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)] \\ \Leftrightarrow & p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)] \\ \Leftrightarrow & p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)] \\ \Leftrightarrow & p \vee [((q \wedge \neg t) \wedge \neg(\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)] \\ \Leftrightarrow & p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)] \\ \Leftrightarrow & p \vee [r \wedge (t \vee \neg q)]\end{aligned}$$





Ex 2.2-15

- Define “Nand” as $(p \uparrow q) \Leftrightarrow \neg(p \wedge q)$, represent the following using only this connective.
 - $\neg p$
 - $\Leftrightarrow \neg(p \wedge p) \Leftrightarrow (p \uparrow p)$
 - $p \vee q$
 - $\Leftrightarrow \neg(\neg p \wedge \neg q) \Leftrightarrow \neg p \uparrow \neg q \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$
 - $p \wedge q \Leftrightarrow \neg\neg(p \wedge q) \Leftrightarrow \neg(p \uparrow q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$
 - $p \rightarrow q$
 - $\Leftrightarrow \neg p \vee q \Leftrightarrow (\neg p \uparrow \neg p) \uparrow (q \uparrow q) \Leftrightarrow \neg\neg p \uparrow (q \uparrow q) \Leftrightarrow p \uparrow (q \uparrow q)$
 - $p \Leftrightarrow q \Leftrightarrow (t \uparrow u) \uparrow (t \uparrow u)$
 - $t : p \uparrow (q \uparrow q)$
 - $u : q \uparrow (p \uparrow p)$

$$\text{NOR: } (p \downarrow q) \Leftrightarrow \neg(p \vee q)$$

$$\circ \quad \neg p$$

$$\triangleright \Leftrightarrow \neg(p \vee p) \Leftrightarrow (p \downarrow p)$$

$$\circ \quad p \vee q$$

$$\triangleright \Leftrightarrow \neg \neg(p \vee q) \Leftrightarrow \neg(p \downarrow q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$$

$$\circ \quad p \wedge q$$

$$\triangleright \Leftrightarrow \neg(\neg p \vee \neg q) \Leftrightarrow (\neg p \downarrow \neg q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$$

$$\circ \quad p \rightarrow q$$

$$\triangleright \Leftrightarrow \neg p \vee q \Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q)$$

$$\Leftrightarrow [(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$$

$$\circ \quad p \leftrightarrow q$$

$$\triangleright \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\Leftrightarrow [(p \rightarrow q) \downarrow (p \rightarrow q)] \downarrow [(q \rightarrow p) \downarrow (q \rightarrow p)]$$

$$\Leftrightarrow (r \downarrow r) \downarrow (s \downarrow s)$$

$$r : [(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$$

$$s : [(q \downarrow q) \downarrow p] \downarrow [(q \downarrow q) \downarrow p]$$



2.3 Logic Implication: Rules of Inference

- Argument: $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$
Premiss: p_1, p_2, \dots, p_n
Conclusion: q
- Ex 2.19:
 - p : Roger studies.
 - q : Roger plays racketball.
 - r : Roger passes discrete mathematics.

- p_1 : Roger studies, then he will pass discrete mathematics.
- p_2 : If Roger don't play racketball, then he'll study.
- p_3 : Roger failed discrete mathematics.
- Determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is valid.

$$\begin{aligned}p_1 &: p \rightarrow r \\p_2 &: \neg q \rightarrow p \\p_3 &: \neg r\end{aligned}$$



Logic Implication: Rules of Inference

- Examine the truth table for the implication

$$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$$

| | | | p_1 | p_2 | p_3 | $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ |
|-----|-----|-----|-------------------|------------------------|----------|---|
| p | q | r | $p \rightarrow r$ | $\neg q \rightarrow p$ | $\neg r$ | $[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$ |
| 0 | 0 | 0 | 1 | 0 | 1 | |
| 0 | 0 | 1 | 1 | 0 | 0 | |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

- $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is a valid argument.



Logic Implication: Rules of Inference

- Definition 2.4: p logically implies q , $p \Rightarrow q$.
 - If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology.
- Rules of inference ← *Use instead of constructing the huge truth table*
 - Using these techniques will enable us to consider only the cases wherein all the premises are true.
 - Development of a step-by-step validation of how the conclusion q logically follows from the premises p_1, p_2, \dots, p_n in an implication of the form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$



Logic Implication: Rules of Inference

- Rule of Detachment (分離)
 - Modus Ponens: the method of affirming
- Example: $[p \wedge (p \rightarrow q)] \rightarrow q$

| p | q | $p \rightarrow q$ | $p \wedge (p \rightarrow q)$ | $[p \wedge (p \rightarrow q)] \rightarrow q$ |
|-----|-----|-------------------|------------------------------|--|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- Tabular form:
$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \text{therefore } \rightarrow \therefore q \end{array}$$
$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$



Logic Implication: Rules of Inference

- Law of the Syllogism (演繹推理)

- Tabular form:

$$p \rightarrow q$$

$$\frac{q \rightarrow r}{\therefore p \rightarrow r}$$

- Modus Tollens: Method of Denying

- Tabular form:

$$p \rightarrow q$$

$$\frac{\neg q}{\therefore \neg p}$$



Table 2.19

Rules of Inference

| Rule of Inference | Related Logical Implication | Name of Rule |
|--|---|--|
| 1) $\frac{p \quad p \rightarrow q}{\therefore q}$ | $[p \wedge (p \rightarrow q)] \rightarrow q$ | Rule of Detachment (Modus Ponens) |
| 2) $\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$ | $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ | Law of the Syllogism |
| 3) $\frac{p \rightarrow q \quad \neg q}{\therefore \neg p}$ | $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ | Modus Tollens |
| 4) $\frac{p \quad q}{\therefore p \wedge q}$ | | Rule of Conjunction |
| 5) $\frac{p \vee q \quad \neg p}{\therefore q}$ | $[(p \vee q) \wedge \neg p] \rightarrow q$ | Rule of Disjunctive Syllogism |
| 6) $\frac{\neg p \rightarrow F_0}{\therefore p}$ | $(\neg p \rightarrow F_0) \rightarrow p$ | Rule of Contradiction |
| 7) $\frac{p \wedge q}{\therefore p}$ | $(p \wedge q) \rightarrow p$ | Rule of Conjunctive Simplification |
| 8) $\frac{p}{\therefore p \vee q}$ | $p \rightarrow p \vee q$ | Rule of Disjunctive Amplification |
| 9) $\frac{p \wedge q \quad p \rightarrow (q \rightarrow r)}{\therefore r}$ | $[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$ | Rule of Conditional Proof |
| 10) $\frac{p \rightarrow r \quad q \rightarrow r}{\therefore (p \vee q) \rightarrow r}$ | $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$ | Rule for Proof by Cases |
| 11) $\frac{p \rightarrow q \quad r \rightarrow s \quad p \vee r}{\therefore q \vee s}$ | $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$ | Rule of the Constructive Dilemma |
| 12) $\frac{p \rightarrow q \quad r \rightarrow s \quad \neg q \vee \neg s}{\therefore \neg p \vee \neg r}$ | $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$ | Rule of the Destructive Dilemma |



Logic Implication: Rules of Inference

Ex 2.30:

| $p \rightarrow q$ $q \rightarrow (r \wedge s)$ $\neg r \vee (\neg t \vee u)$ $\frac{p \wedge t}{\therefore u}$ | Steps | Reasons |
|---|----------------------------------|---|
| | 1) $p \rightarrow q$ | Premise |
| | 2) $q \rightarrow (r \wedge s)$ | Premise |
| | 3) $p \rightarrow (r \wedge s)$ | Steps (1) and (2) and the Law of the Syllogism |
| | 4) $p \wedge t$ | Premise |
| | 5) p | Step (4) and the Rule of Conjunctive Simplification |
| | 6) $r \wedge s$ | Steps (5) and (3) and the Rule of Detachment |
| | 7) r | Step (6) and the Rule of Conjunctive Simplification |
| | 8) $\neg r \vee (\neg t \vee u)$ | Premise |
| | 9) $\neg(r \wedge t) \vee u$ | Step (8), the Associative Law of \vee , and DeMorgan's Laws |
| | 10) t | Step (4) and the Rule of Conjunctive Simplification |
| | 11) $r \wedge t$ | Steps (7) and (10) and the Rule of Conjunction |
| | 12) $\therefore u$ | Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism |



Logic Implication: Rules of Inference

- Ex 2.31:

- If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been cancelled and Alicia would have been angry.
- If the party were cancelled, then refunds would have had to be made.
- No refunds were made.
- Therefore the band could play rock music.

p : The band could play rock music.

q : The refreshments were delivered on time

r : The New Year's party was cancelled .

s : Alicia was angry.

t : Refunds had to be made.

$$(\neg p \vee \neg q) \rightarrow (r \wedge s)$$

$$r \rightarrow t$$

$$\neg t$$

$$\therefore p$$



Logic Implication: Rules of Inference

Steps

- (1) $r \rightarrow t$
- (2) $\neg t$
- (3) $\neg r$
- (4) $\neg r \vee \neg s$
- (5) $\neg(r \wedge s)$
- (6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$
- (7) $\neg(\neg p \vee \neg q)$
- (8) $p \wedge q$
- (9) $\therefore p$

Reasons

- Premise
Premise
(1) and (2), and Method of Denying ([Modus Tollens](#))
(3) and Rule of Disjunctive Amplification
(4) and DeMorgan's Laws
Premise
(6) and (5), and Method of Denying
(7), DeMorgan's Laws, and Law of Double Negation
(8) and Rule of Conjunctive Simplification

$$(\neg p \vee \neg q) \rightarrow (r \wedge s)$$

$$r \rightarrow t$$

$$\neg t$$

$$\therefore p$$



Proof by Contradiction

- Ex 2.35:

- Is the right argument valid or invalid?

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \hline \neg p \vee r \\ \therefore \neg p \end{array}$$

- Let conclusion $\neg p$ be false (the argument invalid)
 - $\Rightarrow p$ is true
 - $\Rightarrow q$ is true
 - $\Rightarrow s$ is true
 - $\Rightarrow r$ is false
 - $\Rightarrow r$ is true ($\neg p \vee r$, now p is true)
 - \Rightarrow contradiction
 - \Rightarrow argument is valid



Interesting examples

- A. 考前猜題 (question selection)
 - 習題A-1, A-2兩題中必出一題, 但不會兩題都出
 - 習題A-2和A-3兩題要嘛都出, 要嘛都不出
 - 如果不出A-1, 也不會出A-3
 - 請問哪一題會出哪一題不會出?

p: A-1 is selected, q: A-2 is selected, r: A-3 is selected

($p \vee q$) and not($p \wedge q$)
 $(q \wedge r) \vee \neg(q \vee r)$
 $\neg p \rightarrow \neg r$
Test p (success), not p (fail)

Ans: A-1 selected, A-2 not selected, A-3 not selected



Interesting examples

- B. 誰是兇手
 - 角頭老大A死了，警方請來他幫派裡兩位二哥級人物B和C來協助瞭解死因。
 - B說：如果A被謀殺，那肯定是C幹的
 - C說：如果A不是自殺，那就是被謀殺
 - 警方不知B,C供詞真偽，但可以確定的是
 - A死因只有三種：意外，自殺和謀殺
 - 如果B和C都沒有說謊，那A就死於一場意外
 - 如果B和C兩人中有一人說謊，那麼A就不是死於意外
 - 請問A的死因為何？
- Ans: 謀殺, B說謊



2.4 The Use of Quantifiers

- Definition 2.5: A declarative sentence is an **open statement** if
 - it **contains one or more variables**, and
 - it **is not a statement**, but
 - it becomes a statement when ***the variables in it are replaced by certain allowable choices***.
- E.g.,
 - $p(x)$: The number $x+2$ is an even integer.
 - $q(x, y)$: The number $y+2$, $x - y$, and $x+2y$ are even integers.
 - For some x , $p(x)$. For some x, y , $q(x, y)$.
 - For some x , $\neg p(x)$. For some x, y , $\neg q(x, y)$.
- **Existential quantifier**: $\exists x$, “for some x ”, “for at least one x ” or “there exists an x such that”
- **Universal quantifier**: $\forall x$, “for all x ”, “for any x ” “for every x ” or “for each x such that”



The Use of Quantifiers

- **Ex 2.36:**

- Given the open statements $p(x): x \geq 0$, $q(x): x^2 \geq 0$
 $r(x): x^2 - 3x - 4 = 0$, $s(x): x^2 - 3 > 0$

- The statement $\forall x [p(x) \rightarrow q(x)]$ is true
 - For every real number x , if $x \geq 0$, then $x^2 \geq 0$.
 - Every nonnegative real number has a nonnegative square.
 - The square of any nonnegative real number is a nonnegative real number.
 - All nonnegative real numbers have nonnegative squares.
- The statement $\exists x [p(x) \wedge r(x)]$ is true.
- The statement $\forall x [q(x) \rightarrow s(x)]$ is false. ← Find counterexample!



Summarization

Table 2.21

| Statement | When Is It True? | When Is It False? |
|-----------------------|--|--|
| $\exists x p(x)$ | For some (at least one) a in the universe, $p(a)$ is true. | For every a in the universe, $p(a)$ is false. |
| $\forall x p(x)$ | For every replacement a from the universe, $p(a)$ is true. | There is at least one replacement a from the universe for which $p(a)$ is false. |
| $\exists x \neg p(x)$ | For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true. | For every replacement a in the universe, $p(a)$ is true. |
| $\forall x \neg p(x)$ | For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true. | There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true. |

$$\begin{aligned}\neg \exists x: p(x) \text{ v.s. } \exists x: \neg p(x) &\quad \times \\ \neg \forall x: p(x) \text{ v.s. } \forall x: \neg p(x) &\quad \checkmark\end{aligned}$$



The Use of Quantifiers

- For open statements $p(x)$, $q(x)$, the universally quantified statement $\forall x[p(x) \rightarrow q(x)]$, we define
 - Contrapositive: $\forall x[\neg q(x) \rightarrow \neg p(x)]$
 - Converse: $\forall x[q(x) \rightarrow p(x)]$
 - Inverse: $\forall x[\neg p(x) \rightarrow \neg q(x)]$



Logical Equivalence and Implication for Quantified Statements

$$\exists x [p(x) \wedge q(x)] \Rightarrow [\exists x p(x) \wedge \exists x q(x)]$$

$$\exists x [p(x) \vee q(x)] \Leftrightarrow [\exists x p(x) \vee \exists x q(x)]$$

$$\forall x [p(x) \wedge q(x)] \Leftrightarrow [\forall x p(x) \wedge \forall x q(x)]$$

$$[\forall x p(x) \vee \forall x q(x)] \Rightarrow \forall x [p(x) \vee q(x)]$$



The Use of Quantifiers

- Rules for **negating** statements with one quantifier

a) $\forall x \neg\neg p(x) \Leftrightarrow \forall x p(x)$

b) $\forall x \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x [\neg p(x) \vee \neg q(x)]$

c) $\forall x \neg[p(x) \vee q(x)] \Leftrightarrow \forall x [\neg p(x) \wedge \neg q(x)]$

$\neg[\forall x p(x)] \Leftrightarrow \exists x \neg p(x)$

$\neg[\exists x p(x)] \Leftrightarrow \forall x \neg p(x)$

$\neg[\forall x \neg p(x)] \Leftrightarrow \exists x \neg\neg p(x) \Leftrightarrow \exists x p(x)$

$\neg[\exists x \neg p(x)] \Leftrightarrow \forall x \neg\neg p(x) \Leftrightarrow \forall x p(x)$



Commuting Quantifiers

Identical quantifiers commute:

$$\exists x \exists y: P(x,y) \Leftrightarrow \exists y \exists x: P(x,y) \text{ and}$$

$$\forall x \forall y: P(x,y) \Leftrightarrow \forall y \forall x: P(x,y)$$

But non-identical ones do not, see:

$$\exists x \forall y: x=y \text{ v.s. } \forall y \exists x: x=y$$



The Use of Quantifiers

- Ex 2.49:
 - What is the negation of the following statement?
$$\forall x \exists y [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$
$$\neg [\forall x \exists y [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]$$
$$\Leftrightarrow \exists x \underline{\neg \exists y} [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$
$$\Leftrightarrow \exists x \forall y \underline{\neg} [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$
$$\Leftrightarrow \exists x \forall y \neg [\neg [p(x, y) \wedge q(x, y)] \vee r(x, y)]$$
$$\Leftrightarrow \exists x \forall y [\neg \neg [p(x, y) \wedge q(x, y)] \wedge \neg r(x, y)]$$
$$\Leftrightarrow \exists x \forall y [p(x, y) \wedge q(x, y)] \wedge \neg r(x, y)]$$



2.5 Quantifiers, Definitions, and the Proofs of Theorems

- Ex 2.52:
 - show that for all n , $n=2, 4, 6, \dots, 24, 26$, we can write n as the sum of at most three perfect squares.
 - Exhaustion method 窮舉法

| | | |
|-----------|-------------|-------------|
| $2=1+1$ | $10=9+1$ | $20=16+4$ |
| $4=4$ | $12=4+4++4$ | $22=9+9+4$ |
| $6=4+1+1$ | $14=9+4+1$ | $24=16+4+4$ |
| $8=4+4$ | $16=16$ | $26=25+1$ |
| | $18=16+1+1$ | |



Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Specification:** If an open statement becomes true for all replacements by the members in a given universe, then that open statement is true for each specific individual member in that universe.

if $\forall x p(x)$ is true, then $p(x)$ is true for each a in the universe.

- E.g.

$$\frac{m(a)}{\therefore c(a)}$$

Steps

(1) $\forall x[m(x) \rightarrow c(x)]$

(2) $m(a)$

(3) $m(a) \rightarrow c(a)$

(4) $\therefore c(a)$

Reasons

Premise

Premise

(1) and the rule of Universal Specification

(2) and (3) and the Rule of Detachment



Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Generalization:** If an open statement $p(x)$ is proved to be true when x is replaced by any arbitrarily chosen element c from our universe, then the universally quantified statement $\forall x p(x)$ is true. Furthermore, the rule extends beyond a single variable.

- Ex 2.56:

$$\begin{array}{c} \frac{\forall x [p(x) \vee q(x)] \quad [\neg r(x) \rightarrow \neg(\neg p(x) \wedge q(x))] }{\forall x [(\neg p(x) \wedge q(x)) \rightarrow r(x)]} \\ \downarrow \\ \therefore \forall x [\neg r(x) \rightarrow p(x)] \end{array}$$
$$\frac{[\neg r(x) \rightarrow \neg(\neg p(x) \wedge q(x))] \quad \downarrow}{[\neg r(x) \rightarrow p(x) \vee \neg q(x)]}$$



Quantifiers, Definitions, and the Proofs of Theorems

- **Ex 2.56:**

Steps

1) $\forall x [p(x) \vee q(x)]$

2) $p(c) \vee q(c)$

3) $\forall x [(\neg p(x) \wedge q(x)) \rightarrow r(x)]$

4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$

5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$

6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$

7) $\boxed{\neg r(c)}$

8) $p(c) \vee \neg q(c)$

9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$

10) $p(c) \vee [q(c) \wedge \neg q(c)]$

11) $p(c)$

12) $\therefore \forall x [\neg r(x) \rightarrow p(x)]$

Reasons

Premise

Step (1) and the Rule of Universal Specification

Premise

Step (3) and the Rule of Universal Specification

Step (4) and $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg s$

Step (5), DeMorgan's Law, and the Law of Double Negation

Premise (assumed)

Steps (7) and (6) and Modus Ponens

Steps (2) and (8) and the Rule of Conjunction

Step (9) and the Distributive Law of \vee over \wedge

Step (10), $q(c) \wedge \neg q(c) \Leftrightarrow F_0$, and
 $p(c) \vee F_0 \Leftrightarrow p(c)$

Steps (7) and (11) and the Rule of Universal Generalization



Quantifiers, Definitions, and the Proofs of Theorems

- Argument
$$\begin{array}{c} \forall x [(j(x) \vee s(x)) \rightarrow \neg p(x)] \\ p(m) \\ \hline \therefore \neg s(m) \end{array}$$
- Establish the validity of this argument

| Steps | Reasons |
|---|--|
| 1) $\forall x [j(x) \vee s(x) \rightarrow \neg p(x)]$ | Premise |
| 2) $p(m)$ | Premise |
| 3) $j(m) \vee s(m) \rightarrow \neg p(m)$ | (1) and Rule of Universal Specification |
| 4) $p(m) \rightarrow \neg(j(m) \vee s(m))$ | (3), $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg q)$ and Law of Double Negation |
| 5) $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$ | (4) and DeMorgan's Law |
| 6) $\neg j(m) \wedge \neg s(m)$ | (2) and (5) and Rule of Detachment |
| 7) $\therefore \neg s(m)$ | (6) and Rule of Conjunctive Simplification |

- No junior or senior is enrolled in a physical education class.
- Mary is enrolled in a physical education class.
- Thus Mary is not a senior.



Quantifiers, Definitions, and the Proofs of Theorems

- Definition 2.8:
 - Let n be an integer. We call n even if n is divisible by 2, that is, if there exists an integer r so that $n = 2r$. If n is not even, then we call n odd and find for this case that there exists an integer s where $n = 2s + 1$.
- Theorem 2.2: For all integers k and l , if k and l are both odd, then $k + l$ is even.
 - Proof
 - 1. Since k and l are odd, we may write $k = 2a+1$ and $l = 2b+1$, for some integers a, b . (*the Rule of Universal Specification*)
 - 2. Then $k + l = (2a+1)+(2b+1) = 2(a+b+1)$, which hold for integers. (apply Commutative, Associative, and Distributive Laws)
 - 3. Since a, b are integers, $a+b+1 = c$ is an integer; with $k + l = 2c$, so $k + l$ is even.



Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.3: For all integers k and l , if k and l are both odd, then kl is also odd.
 - Proof?



Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.4: If m is an even integer, then $m + 7$ is odd.
 - Proof
 - Since m is even, we have $m = 2a$ for some integer a . Then $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a + 3) + 1$. since $a+3$ is an integer, we know that $m + 7$ is odd.
 - Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer b , and $m = 2b - 7 = 2b - 8 + 1 = 2(b - 4) + 1$, where $b - 4$ is an integer. Hence m is odd. (*contraposition method*)
 - Assume that m is even and $m + 7$ is also even. Then $m + 7$ even implies that $m + 7 = 2c$ for some integer c . Consequently, $m = 2c - 7 = 2c - 8 + 1 = 2(c - 4) + 1$ with $c - 4$ an integer, so m is odd. Now we have contradiction. So the assumption is false ($m + 7$ is even), and we have $m + 7$ odd. (*contradiction method*)

| | Assumption | Result Derived |
|----------------|------------------------|----------------|
| Contraposition | $\neg q(m)$ | $\neg p(m)$ |
| Contradiction | $p(m)$ and $\neg q(m)$ | F_0 |



Proof

推論證明

推論證明有下列三種方式：

1. 正證法：以 $P \Rightarrow Q$ 的方式完成推論證明的過程，稱之為正證法。
2. 反證法：以 $\sim Q \Rightarrow \sim P$ 的方式完成推論證明的過程，稱之為反證法。
3. 矛盾證法：以 $\sim P \vee Q$ 的方式完成推論證明的過程，稱之為矛盾證法。

Prove (p and $\neg q$) is F₀ → prove ($\neg p$ or q) is true



Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.5:
 - For all positive real numbers x and y , if the product xy exceeds 25, then $x > 5$ or $y > 5$.
 - Proof: (*Contrapositive*)
 - Suppose that $0 < x \leq 5$ and $0 < y \leq 5$ ($\neg q(x, y)$)
 - We find that $0 = 0 \cdot 0 < x \cdot y \leq 5 \cdot 5 = 25$ ($\neg p(x, y)$)



- 糖果 / 我們 / 太多 / 牙醫 / 吃 / 告訴 / 不可以
- 牙醫告訴我們不可以吃太多糖果
- 糖果吃太多不可以告訴牙醫

