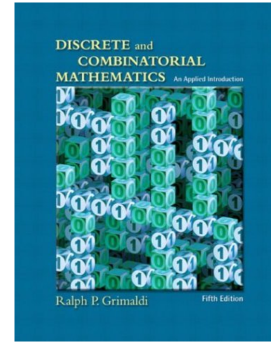
Discrete Mathematics
-- *Chapter 10: Recurrence Relations*



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First glance at "recurrence"

$$F_{n+2} = F_{n+1} + F_n$$

$$a_{n+1} = 3a_n$$

$$a_n = A*3^n$$

Outline

- The first-order linear recurrence relation
- The second-order linear homogeneous recurrence relation with constant coefficients
- The nonhomogeneous recurrence relation
- The method of generating functions

Key1: solve simple recurrence relation
Key2: model recurrence relation

The First-Order Linear Recurrence Relation

- The equation $a_{n+1} = 3a_n$ is a recurrence relation with constant coefficients. Since a_{n+1} *only depends on its immediate predecessor*, the relation is said to be *first order*.
- The expression $a_0 = A$, where A is a constant, is referred to as an initial condition.

- The **unique** solution of the recurrence relation $a_{n+1} = da_n$, where $n \geq 0$, d is a constant, and $a_0 = A$, is given by $a_n = Ad^n$.

The First-Order Linear Recurrence Relation

- **Ex 10.1** : Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$.
  - $a_n = a_0(7^n)$, $a_2 = 98 = a_0(7^2) \Rightarrow a_0 = 2$, $a_n = 2(7^n)$.

- **Ex 10.2** : A bank pay 6% annual interest on savings, compounding the interest monthly. If we deposit \$1000, how much will this deposit be worth a year later?
  - $p_{n+1} = (1.005)p_n$, $p_0 = 1000 \Rightarrow p_n = p_0(1.005)^n$
  - $p_{12} = 1000(1.005)^{12} = \$1061.68$

The First-Order Linear Recurrence Relation

- Refer to examples 1.37, 3.11, 4.12, and 9.12.
- **Ex 10.3** : Let a_n count the number of compositions of n, we find that

$$a_{n+1} = 2a_n,\ n \geq 1,\ a_1 = 1 \Rightarrow a_n = 2^{n-1}$$

		(1′)	4
		(2′)	1 + 3
(1)	3	(3′)	2 + 2
(2)	1 + 2	(4′)	1 + 1 + 2
(3)	2 + 1		
(4)	1 + 1 + 1	(1″)	3 + 1
		(2″)	1 + 2 + 1
		(3″)	2 + 1 + 1
		(4″)	1 + 1 + 1 + 1

Figure 10.1

The First-Order Linear Recurrence Relation

- The recurrence relation $a_{n+1} - da_n = 0$ is called <u>linear</u> because each term appears to the first power.
- Sometimes a nonlinear recurrence (e.g., $a_{n+1} - 3a_n a_{n-1} = 0$) relation can be transformed to a linear one by a suitable algebraic substitution.
- <u>**Ex 10.4**</u> : Find a_{12} if $a_{n+1}^2 = 5a_n^2$ where $a_n > 0$ for $n \geq 0$ and $a_0 = 2$.
  - Let $b_n = a_n^2$. Then $b_{n+1} = 5b_n$ (linear) for $n \geq 0$ and $b_0 = 4 \Rightarrow b_n = 4 \cdot 5^n$

Homogeneous and Nonhomogeneous

- The general first-order linear recurrence relation with constant coefficients has the form
  $a_{n+1} + ca_n = f(n)$.
  - $f(n) = 0$, the relation is called homogeneous.
  - Otherwise, it is called nonhomogeneous.
- **Ex 10.5** : Let a_n denote the number of comparisons needed to sort n numbers in bubble sort, we find the recurrence relation
  - $a_n = a_{n-1} + (n - 1)$, $n \geq 2$, $a_1 = 0$

i = 1									
x_1	7		7		7		7	j = 2	2
x_2	9		9		9	j = 3	2		7
x_3	2		2	j = 4	2		9		9
x_4	5	j = 5	5		5		5		5
x_5	8		8		8		8		8

Four comparisons and two interchanges.

i = 2							
x_1	2		2		2		2
x_2	7		7		7	j = 3	5
x_3	9		9	j = 4	5		7
x_4	5	j = 5	5		9		9
x_5	8		8		8		8

Three comparisons and two interchanges.

i = 3					
x_1	2		2		2
x_2	5		5		5
x_3	7		7	j = 4	7
x_4	9	j = 5	8		8
x_5	8		9		9

Two comparisons and one interchange.

i = 4		
x_1	2	
x_2	5	
x_3	7	
x_4	8	j = 5
x_5	9	

One comparison but no interchanges.

Figure 10.3

The First-Order Linear Recurrence Relation

- **Ex 10.6** : In Example 9.6 we sought the generating function for the sequence 0, 2, 6, 12, 20, 30, 42,…, due to the observation $a_n = n^2 + n$. If we fail to see this, alternatively

$$\begin{aligned} a_1 - a_0 &= 2 \\ a_2 - a_1 &= 4 \\ a_3 - a_2 &= 6 \\ \vdots \quad & \quad \vdots \\ a_n - a_{n-1} &= 2n. \end{aligned}$$

$$\begin{aligned} a_n - a_0 &= 2 + 4 + 6 + \cdots + 2n = 2(1 + 2 + 3 + \cdots + n) \\ &= 2[n(n+1)/2] = n^2 + n. \end{aligned}$$

The First-Order Linear Recurrence Relation

- **Ex 10.7** : Solve the relation $a_n = n \cdot a_{n-1}$, $n \geq 1$, $a_0 = 1$.

	1	2
2	1	

x 2

	1		2	3
	1	3	2	
3	1		2	
3	2		1	
	2	3	1	
	2		1	3

x 3

10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

- Linear recurrence relation of order k:
  - $C_0a_n+C_1a_{n-1}+C_2a_{n-2}+\ldots+C_ka_{n-k}=f(n)$, $n \geq 0$.
- Homogeneous relation of order 2:
  - $C_0a_n+C_1a_{n-1}+C_2a_{n-2}=0$, $n \geq 2$.
- Substituting $a_n = cr^n$ into the equation, we have
  - $C_0cr^n + C_1cr^{n-1} + C_2cr^{n-2} = 0$, $n \geq 2$.
  - Characteristic equation: $C_0r^2 + C_1r + C_2 = 0$, $n \geq 2$.
- The roots r_1, r_2 of this equation are called characteristic roots.
- Three cases for the roots:

(A) distinct real roots

(B) complex conjugate roots

(C) equivalent real roots

Case (A): Distinct Real Roots

- **Ex 10.9** : Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$, $n \geq 2$, and $a_0 = -1$ and $a_1 = 8$.
  - **Solution**

Let $a_n = cr^n$

$r^2 + r - 6 = 0 \Rightarrow r = 2, -3$

$a_n = c_1(2)^n + c_2(-3)^n$

$-1 = a_0 = c_1 + c_2$

$8 = a_1 = 2c_1 - 3c_2 \Rightarrow c_1 = 1, c_2 = -2$

$\Rightarrow a_n = (2)^n - 2(-3)^n$

$a_n=2^n$ and $a_n=(-3)^n$ are both solutions

Linearly independent solutions

$2^n + 2^{n-1} - 6*2^{n-2} = 2^{n-2}(2^2+2-6) = 0$

Distinct Real Roots

- **Ex 10.10** : Solve Fibonacci relation, $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = 0$, $F_1 = 1$.
  - **Solution**

    Let $F_n = cr^n$,

    $r^2 - r - 1 = 0 \Rightarrow \qquad r = (1 \pm \sqrt{5})/2$

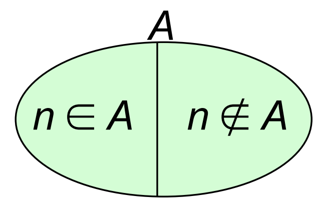
$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \qquad n \geq 0.$$

a3=5→Φ,{1}, {2}, {3}, {1,3}
a4=8→Φ, {1}, {2}, {3},{4}, {1,3}, {2,4}, {1,4}
a5=13→ Φ, {1},{2},{3},{4}, {1,3}, {2,4}, {1,4}, {5}, {1,5}, {2,5}, {3,5}, {1,3,5}

Distinct Real Roots

- **Ex 10.11** : For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, and let a_n denote the number of subsets of S that contains no consecutive integers. Find and solve a recurrence relation for a_n.
  - **Solution**
  - $a_0 = 1$ and $a_1 = 2$ and $a_2 = 3$ and $a_3 = 5$ (Fibonacci?)
  - If $A \subseteq S$ and A is to be counted in a_n, there are two cases
    - (1) $n \in A$, then $A - \{n\}$ would be counted in a_{n-2}
    - (2) $n \notin A$, then A would be counted in a_{n-1}
  - $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$



$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right], \qquad n \geq 0.$$

Exhaustive and mutually disjoint

Distinct Real Roots

- **Ex 10.12** : Suppose we have a $2 \times n$ chessboard. We wish to cover such a chessboard using 2×1 vertical dominoes or 1×2 horizontal dominoes.

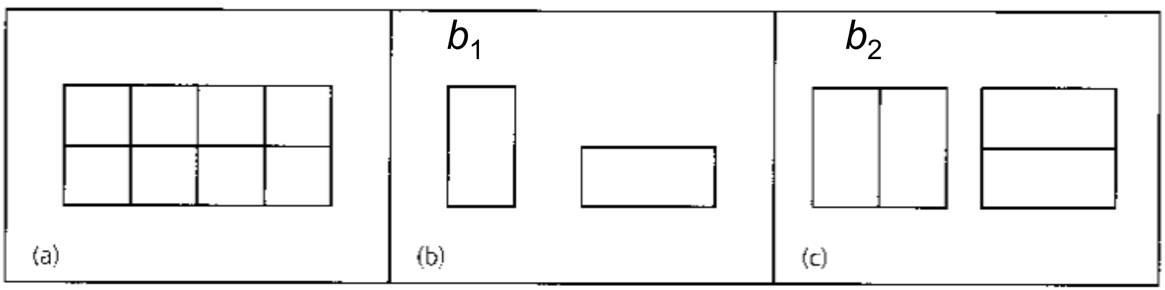
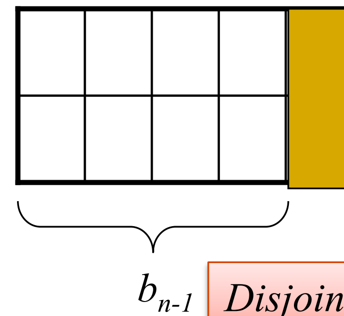


Figure 10.5

Distinct Real Roots

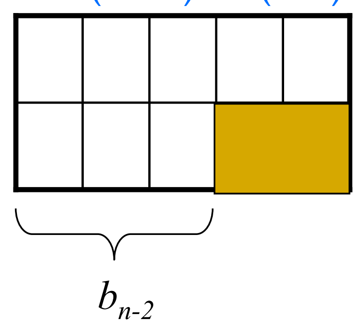
- Let b_n count the number of ways we can cover a $2\times n$ chessboard by using 2×1 vertical dominoes or 1×2 horizontal dominoes.
- $b_1=1$ and $b_2=2$
- For $n \geq 3$, consider the last (n-th) column of the chessboard
  1) By one 2×1 vertical domino: Now the remaining $2 \times(n -1)$ subboard can be covered in b_{n-1} ways.
  2) By two 1×2 horizontal dominoes, place one above the other: Now the remaining $2 \times(n -2)$ subboard can be covered in b_{n-2} ways.
- $b_n= b_{n-1} + b_{n-2}$, $n \geq 3$, $b_1=1$ and $b_2=2$
  $\Rightarrow b_n = F_{n+1}$

2n = 2×(n - 1) + 2×1



Disjoint?

2n = 2×(n - 2) + 2(1×2)



Distinct Real Roots

- **Ex 10.14** : Suppose the symbols of legal arithmetic expressions include 0, 1,…, 9, +, *, /.
- Let a_n be the number of legal arithmetic expressions that are made up of n symbols. Solve a_n. a_1=10 and a_2=100
  0, 1, 2, ⋯, 9　00, 01, 02, ⋯, 99

Solution:

- For n≥3, consider the two cases of length n -1:
  1) If x is an expression of n -1 symbols, add a digit to the right of x. ⇒ $10a_{n-1}$ 補上 0, 1, 2, ⋯, 9
  2) If x is an expression of n - 2 symbols, we adjoin to the right of x one of the 29 two-symbol expressions: +0, …, +9, *0, …,*9, /1, /2,…, /9. ⇒ $29a_{n-2}$ 不能補2個數字，會與n -1相同
- $a_n = 10a_{n-1} + 29a_{n-2}$, $n \geq 3$

Idea: use a_{n-1} (or more) to represent a_n

Distinct Real Roots

- **Ex 10.15** (9.13): Palindromes are the compositions of numbers that read the same left to right as right to left.
- Let p_n count the number of palindromes of n.
- $p_n = 2p_{n-2}$, $n \geq 3$, $p_1=1$, $p_2=2$

p_3		p_5		p_4		p_6	
(1)	3	(1′)	5	(1)	4	(1′)	6
(2)	1 + 1 + 1	(2′)	2 + 1 + 2	(2)	1 + 2 + 1	(2′)	2 + 2 + 2
		(1″)	1 + 3 + 1	(3)	2 + 2	(3′)	3 + 3
		(2″)	1 + 1 + 1 + 1 + 1	(4)	1 + 1 + 1 + 1	(4′)	2 + 1 + 1 + 2
						(1″)	1 + 4 + 1
						(2″)	1 + 1 + 2 + 1 + 1
						(3″)	1 + 2 + 2 + 1
						(4″)	1 + 1 + 1 + 1 + 1 + 1

(‘) Add 1 to the fist and last summands

(“) Append “1+” to the start and “+1” to the end

Figure 10.6

Distinct Real Roots

$$p_n = 2p_{n-2}, \qquad n \geq 3, \qquad p_1 = 1, \qquad p_2 = 2.$$

Substituting $p_n = cr^n$, for $c, r \neq 0, n \geq 1$, into this recurrence relation, the resulting characteristic equation is $r^2 - 2 = 0$. The characteristic roots are $r = \pm\sqrt{2}$, so $p_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$. From

$$1 = p_1 = c_1(\sqrt{2}) + c_2(-\sqrt{2})$$
$$2 = p_2 = c_1(\sqrt{2})^2 + c_2(-\sqrt{2})^2$$

we find that $c_1 = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$, $c_2 = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)$, so

$$p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n, \qquad n \geq 1.$$

we consider n even, say $n = 2k$

$$p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^{2k} + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^{2k}$$
$$= \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)2^k + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)2^k = 2^k = 2^{n/2}$$

Distinct Real Roots

- **Ex 10.16** : Find the number of recurrence relation for the number of binary sequences of length n that have no consecutive 0's.
  - Let a_n be the number of such sequences of length n.
    - Let $a_n^{(0)}$ count those end in 0, and $a_n^{(1)}$ count those end in 1
      $\Rightarrow a_n = a_n^{(1)} + a_n^{(0)}$

      結尾為1，能為1、0
      結尾為0，只能為1
  - Consider x of length n -1
    - If x ends in 1, we can append a 0 or a 1 to it ($2a_{n-1}^{(1)}$).
    - If x ends in 0, we can append a 1 to it ($a_{n-1}^{(0)}$).
    - $a_n = 2a_{n-1}^{(1)} + a_{n-1}^{(0)} = a_{n-1}^{(1)} + \underline{a_{n-1}^{(1)} + a_{n-1}^{(0)}} \rightarrow a_{n-1}$
  - If y is counted in a_{n-2} $\Leftrightarrow$ sequence y1 is counted in $a_{n-1}^{(1)}$
    - So, $a_{n-2} = a_{n-1}^{(1)}$.
  - $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, $a_1=2$, $a_2=3$

- Try Ex10.17

Second- or Higher-Order Recurrence Relation

- **Ex 10.18** : Solve $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \geq 0$, $a_0=0$, $a_1=1$, $a_2=2$
  - Let $a_n = cr^n$
  - Characteristic equation: $2r^3-r^2-2r+1=0 \Rightarrow r = 1, 1/2, -1$
  - $a_n = c_1(1)^n + c_2(-1)^n + c_3(1/2)^n$
  - From $a_0=0$, $a_1=1$, $a_2=2$, derive $c_1=5/2$, $c_2=1/6$, $c_3=-8/3$
  - $a_n = (5/2)+(1/6)(-1)^n+(-8/3)(1/2)^n$

Second- or Higher-Order Recurrence Relation

- **Ex 10.19** : We want to tile a $2 \times n$ chessboard using two types of tiles shown in Figure 10.8.

a_2: 2×2 chessboard

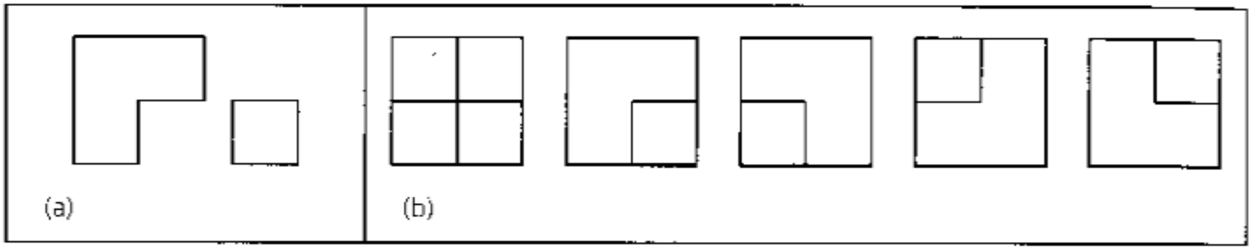


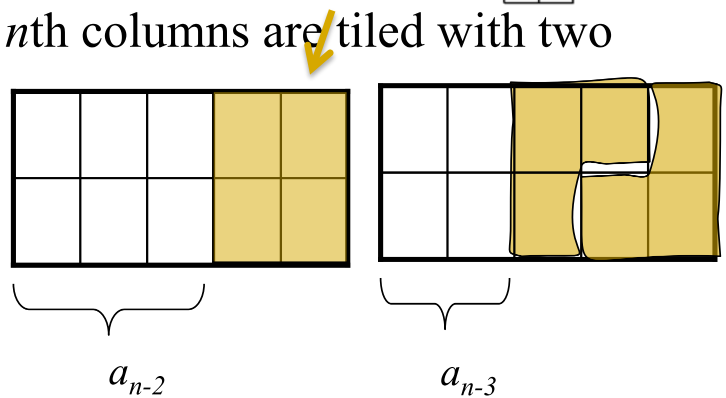
Figure 10.8

a_3: 2×3 chessboard

Second- or Higher-Order Recurrence Relation

- Let a_n count the number of such tilings.
- $a_1=1$ and $a_2=5$ and $a_3=$11 (try it)
- For $n \geq 4$, consider the last column of the chessboard
  1) the nth column is covered by two 1×1 tiles $\Rightarrow a_{n-1}$
  2) the $(n-1)$st and the nth column are tiled with one 1×1 tile and a larger tile $\Rightarrow 4a_{n-2}$

  3) the $(n-2)$nd, $(n-1)$st and the nth columns are tiled with two large tiles $\Rightarrow 2a_{n-3}$
- $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$, $n \geq 4$



Case (B): Complex Roots

- DeMoivre's Theorem:
  $(\cos\theta + i\ \sin\theta)^n = \cos n\theta\ + i\ \sin n\theta$

- If $z = x + iy \in \mathbf{C} \Rightarrow z = r(\cos\theta + i\sin\theta), r = \sqrt{x^2 + y^2}, \dfrac{y}{x} = \tan\theta$, for x $\neq 0$

  If $x = 0, \begin{cases} y > 0, z = yi = yi\sin(\pi/2) = y(\cos(\pi/2) + i\sin(\pi/2)) \\ y < 0, z = |y|\sin(3\pi/2) = |y|(\cos(3\pi/2) + i\sin(3\pi/2)) \end{cases}$

  By DeMoivre's Theorem, $z^n = r^n(\cos n\theta + i\sin n\theta)$

Complex Roots

- **Ex 10.20** : Determine $(1+\sqrt{3}i)^{10}$
  - Solution

Complex number $1+\sqrt{3}i$ is represented as the point $(1,\sqrt{3})$ in the xy-plane

$$r=\sqrt{1^2+(\sqrt{3})^2}=2, \theta=\pi/3$$

$$1+\sqrt{3}i=2(\cos(\pi/3)+i\sin(\pi/3))$$

$$(1+\sqrt{3}i)^{10}=2^{10}(\cos(10\pi/3)+i\sin(10\pi/3))=2^{10}(\cos(4\pi/3)+i\sin(4\pi/3))$$

$$=2^{10}((-1/2)-(\sqrt{3}/2)i)=(-2^9)(1+\sqrt{3}i)$$

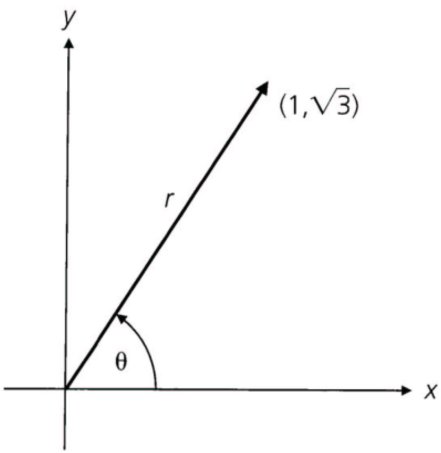


Figure 10.9

Complex Roots

- Ex 10.21 : Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ where $n \geq 2$, $a_0 = 1$, $a_1 = 2$.
  - Let $a_n = cr^n$
  - $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$
  - $1 + i = \sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$
  - $1 - i = \sqrt{2}(\cos(\pi/4) - i \sin(\pi/4))$

$$\begin{aligned}
a_n &= c_1(1+i)^n + c_2(1-i)^n \\
&= c_1[\sqrt{2}(\cos(\pi/4) + i\ \sin(\pi/4))]^n + c_2[\sqrt{2}(\cos(-\pi/4) + i\ \sin(-\pi/4))]^n \\
&= c_1(\sqrt{2})^n(\cos(n\pi/4) + i\ \sin(n\pi/4)) + c_2(\sqrt{2})^n(\cos(-n\pi/4) + i\ \sin(-n\pi/4)) \\
&= c_1(\sqrt{2})^n(\cos(n\pi/4) + i\ \sin(n\pi/4)) + c_2(\sqrt{2})^n(\cos(n\pi/4) - i\ \sin(n\pi/4)) \\
&= (\sqrt{2})^n[k_1 \cos(n\pi/4) + k_2 \sin(n\pi/4)],
\end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = (c_1 - c_2)i$.

$$1 = a_0 = [k_1 \cos 0 + k_2 \sin 0] = k_1$$

$$2 = a_1 = \sqrt{2}[1 \cdot \cos(\pi/4) + k_2 \sin(\pi/4)],\ \text{or } 2 = 1 + k_2,\ \text{and } k_2 = 1.$$

The solution for the given initial conditions is then given by

$$a_n = (\sqrt{2})^n[\cos(n\pi/4) + \sin(n\pi/4)], \qquad n \geq 0.$$

Complex Roots

- **Ex 10.22** : Let a_n denote the value of the $n \times n$ determinant D_n
  - $a_1 = b$, $a_2 = 0$ and $a_3 = -b^3$
  - $D_n = bD_{n-1} - b^2 D_{n-2}$
  - $a_n = ba_{n-1} - b^2 a_{n-2}$

$$\begin{vmatrix} b & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix}$$

$$a_1 = |b| = b \quad \text{and} \quad a_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = 0 \quad \left(\text{and} \quad a_3 = \begin{vmatrix} b & b & 0 \\ b & b & b \\ 0 & b & b \end{vmatrix} = -b^3\right).$$

$$= b \underbrace{\begin{vmatrix} b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b \end{vmatrix}}_{\text{(This is } D_{n-1}.)} - b \begin{vmatrix} b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b \end{vmatrix}$$

If we let $a_n = cr^n$ for $c, r \neq 0$ and $n \geq 1$, the characteristic equation produces the roots $b[(1/2) \pm i\sqrt{3}/2]$.

Hence

$$\begin{aligned} a_n &= c_1[b((1/2) + i\sqrt{3}/2)]^n + c_2[b((1/2) - i\sqrt{3}/2)]^n \\ &= b^n[c_1(\cos(\pi/3) + i\ \sin(\pi/3))^n + c_2(\cos(\pi/3) - i\ \sin(\pi/3))^n] \\ &= b^n[k_1 \cos(n\pi/3) + k_2 \sin(n\pi/3)]. \end{aligned}$$

$b = a_1 = b[k_1 \cos(\pi/3) + k_2 \sin(\pi/3)]$, so $1 = k_1(1/2) + k_2(\sqrt{3}/2)$, or $k_1 + \sqrt{3}\,k_2 = 2$.

$0 = a_2 = b^2[k_1 \cos(2\pi/3) + k_2 \sin(2\pi/3)]$, so $0 = (k_1)(-1/2) + k_2(\sqrt{3}/2)$, or

$$k_1 = \sqrt{3}\,k_2.$$

Hence $k_1 = 1$, $k_2 = 1/\sqrt{3}$ and the value of D_n is

$$b^n[\cos(n\pi/3) + (1/\sqrt{3}) \sin(n\pi/3)].$$

Case (C): Repeated Real Roots

- **Ex 10.23** : Solve the recurrence relation
  $a_{n+2} = 4a_{n+1} - 4a_n$ where $n \geq 0$, $a_0=1$, $a_1=3$
  - **Solution**

    $a_n = c_1 2^n + c_2 2^n$? X

    Let $a_n = cr^n$

    $r^2 - 4r + 4 = 0 \Rightarrow r = 2 \Rightarrow 2^n$ and 2^n are not independent solutions, need another independent solution

    So, try $g(n)2^n$, where $g(n)$ is not a constant

    $\Rightarrow g(n+2)2^{n+2} = 4g(n+1)2^{n+1} - 4g(n)2^n$

    $\Rightarrow g(n+2) = 2g(n+1) - g(n) \Rightarrow g(n) = n, \therefore n2^n$ is a solution

    $a_n = c_1(2^n) + c_2(n2^n)$ with $a_0=1$, $a_1=3$

    $a_n = 1(2^n) + (1/2)(n2^n)$

Repeated Real Roots

- In general, if

  $C_0 a_n + C_1 a_{n\text{-}1} + C_2 a_{n\text{-}2} + \ldots + C_k a_{n\text{-}k} = 0$

  with r, a characteristic root of multiplicity m, then the part of the general solution that involves the root r has the form

  $A_0 r^n + A_1 n r^n + A_2 n^2 r^n + A_3 n^3 r^n + \ldots + A_{m\text{-}1} n^{m\text{-}1} r^n$

$=(A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \ldots + A_{m\text{-}1} n^{m\text{-}1}) r^n$

Repeated Real Roots

- **Ex 10.24** : Let p_n denote the probability that at least one case of measles is reported during the nth week after the first recorded case. School records provide evidence that $p_n = p_{n-1} - (0.25)p_{n-2}$, for $n \geq 2$. Since $p_n = 0$ and $p_1 = 1$, if the first case is recorded on Monday, March 3, 2003, when did the probability for the occurrence of a new case decrease to less than 0.01 for the first time?
  - **Solution**

    Let $p_n = cr^n$

    $r^2 - r + (1/4) = 0 \Rightarrow r = 1/2$

    $p_n = (c_1 + c_2 n)(1/2)^n \Rightarrow c_1 = 0$ and $c_2 = 2 \Rightarrow p_n = n2^{-n+1}$

    $p_n < 0.01 \Rightarrow$ the first n is 12, the week of May 19, 2003.

10.3 The Nonhomogeneous Recurrence Relation

- $a_n + C_1 a_{n-1} = f(n),\ n \geq 1,$
- $a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n),\ n \geq 2$
- Let $a_n^{(h)}$ denote the general solution of the associated homogeneous relation.
- Let $a_n^{(p)}$ denote a solution of the given nonhomogeneous relation. (particular solution)
- Then $a_n = a_n^{(h)} + a_n^{(p)}$ is the general solution of the recurrence relation.

The Nonhomogeneous Recurrence Relation

$a_n - a_{n-1} = f(n)$, we have

$$
\begin{aligned}
a_1 &= a_0 + f(1) \\
a_2 &= a_1 + f(2) = a_0 + f(1) + f(2) \\
a_3 &= a_2 + f(3) = a_0 + f(1) + f(2) + f(3) \\
&\vdots \\
a_n &= a_{n-1} + f(n) = a_0 + f(1) + \cdots + f(n) = a_0 + \sum_{i=1}^{n} f(i).
\end{aligned}
$$

- **Ex 10.25**

Solve the recurrence relation $a_n - a_{n-1} = 3n^2$, where $n \geq 1$ and $a_0 = 7$.

Here $f(n) = 3n^2$, so the unique solution is

$$a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum_{i=1}^{n} i^2 = 7 + \frac{1}{2}(n)(n+1)(2n+1).$$

The Nonhomogeneous Recurrence Relation

- **Ex 10.26** : Solve the recurrence relation $a_n - 3a_{n-1} = 5(7^n)$ for $n \geq 1$ and $a_0 = 2$.
  - **Solution**

    The solution for $a_n - 3a_{n-1} = 0$ is $a_n^{(h)} = c(3^n)$. First order linear recurrence

    Since $f(n) = 5(7^n)$, let $a_n^{(p)} = A(7^n)$

    $\Rightarrow A(7^n) - 3A(7^{n-1}) = 5(7^n) \Rightarrow A = 35/4$

    $a_n^{(p)} = (35/4)7^n = (5/4)7^{n+1}$.

    The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = c(3^n) + (5/4)7^{n+1}$

    So, $a_n = (-1/4)(3^{n+3}) + (5/4)7^{n+1}$

The Nonhomogeneous Recurrence Relation

- **Ex 10.27** : Solve the recurrence relation $a_n - 3a_{n-1} = 5(3^n)$ for $n \geq 1$ and $a_0 = 2$.
  - **Solution**

    Let $a_n^{(h)} = c(3^n)$. 多乘一個n

    Since *$a_n^{(h)}$ and f(n) are not linearly independent*, let $a_n^{(p)} = Bn(3^n) \Rightarrow Bn(3^n) - 3B(n-1)(3^{n-1}) = 5(3^n). \Rightarrow B=5.$

    The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = c(3^n) + (5)n3^{n+1}$

    $a_n = (2 + 5n)(3^n)$

Solution for the Nonhomogeneous First-Order Relation

- $a_n + C_1 a_{n-1} = kr^n$.
  - If r^n is not a solution of the homogeneous relation $a_n + C_1 a_{n-1} = 0$, then $a_n^{(p)} = Ar^n$.
  - If r^n is a solution of the homogeneous relation, then $a_n^{(p)} = Bnr^n$.

Solution for the Nonhomogeneous Second-Order Relation

- $a_n + C_1 a_{n-1} + C_2 a_{n-2} = kr^n$.
  - If r^n is not a solution of the homogeneous relation, then $a_n^{(p)} = Ar^n$.
  - If $a_n^{(h)} = c_1 r^n + c_2 r_1^{\,n}$, where $r \neq r_1$, then $a_n^{(p)} = Bnr^n$.
  - If $a_n^{(h)} = (c_1 + c_2 n) r^n$, then $a_n^{(p)} = Cn^2 r^n$.

The Nonhomogeneous Recurrence Relation

- **Ex 10.28** : The Towers of Hanoi.
  - Let count the minimum number of moves it takes to transfer n disks from peg 1 to peg 3.
  - $a_{n+1} = 2a_n + 1$
    - Transfer the top n disks from peg 1 to peg 2, need a_n moves.
    - Transfer the largest disk from peg 1 to peg 3, need 1 moves.
    - Transfer the n disks on peg 2 onto the largest disk, need a_n moves.

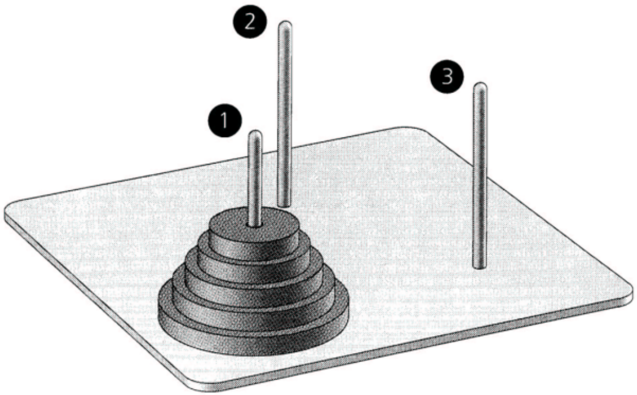


Figure 10.11

For $a_{n+1} - 2a_n = 1$, we know that $a_n^{(h)} = c(2^n)$. Since $f(n) = 1 = (1)^n$ is not a solution of $a_{n+1} - 2a_n = 0$, we set $a_n^{(p)} = A(1)^n = A$ and find from the given relation that $A = 2A + 1$, so $A = -1$ and $a_n = c(2^n) - 1$. From $a_0 = 0 = c - 1$ it then follows that $c = 1$, so $a_n = 2^n - 1$, $n \geq 0$.

The Nonhomogeneous Recurrence Relation

- **Ex 10.29** : Let a_n denote the amount still owed on the loan at the end of the nth period.

  (r is the interest rate, P is payment, S is loan)

  $a_{n+1} = a_n + ra_n - P$, $0 \leq n \leq T-1$, $a_0 = S$, $a_T = 0$

  $a_n^{(h)} = c(1+r)^n$.

  Let $a_n^{(p)} = A$, $A - (1+r)A = -P \Rightarrow A = P/r$, $a_n^{(p)} = P/r$.

  $a_n = a_n^{(h)} + a_n^{(p)} = c(1+r)^n + P/r \Rightarrow c = S - (P/r)$

  $a_n = (S - (P/r))(1+r)^n + (P/r)$

  Since $0 = a_T$, we have $P = (Sr)[1 - (1 + r)^{-T}]^{-1}$

The Nonhomogeneous Recurrence Relation

- **Ex 10.30** : Let S be a set containing 2^n real numbers. Find the maximum and minimum in S. We wish to determine the number of comparisons made between pairs of elements in S.
  - Let a_n denote the number of needed comparisons.

    $n = 2$, $|S| = 2^2 = 4$, $S = \{x_1, x_2, y_1, y_2\} = S_1 \cup S_2$,

    $S_1 = \{x_1, x_2\}$, $S_2 = \{y_1, y_2\}$

    $\mathbf{a_{n+1} = 2a_n + 2}$, $n \geq 1$.

    $a_n^{(h)} = c(2^n)$, $a_n^{(p)} = A$

    $a_1 = 1 \Rightarrow a_n = (3/2)(2^n) - 2$

The Nonhomogeneous Recurrence Relation

- **Ex 10.31** : For the alphabet = {0,1,2,3}, how many strings of length n contains an even number of 1's.
  - Let a_n count those strings among the 4^n strings.
    Consider the nth symbol of a string of length n
    1. The nth symbol is 0, 2, 3 $\Rightarrow 3a_{n-1}$ 前面n-1個symbol為偶數個1，最後一個可為0、2、3
    2. The nth symbol is 1 $\Rightarrow$ there must be an odd number of 1's among the first n -1 symbols $\Rightarrow 4^{n-1} - a_{n-1}$

    $a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}$ 前面n-1個symbol為奇數個1，最後一個只能為1（全部 - 偶數個1）

    $a_n^{(h)} = c(2^n)$, $a_n^{(p)} = A(4^{n-1})$

    $a_1 = 3 \rightarrow a_n = 2^{n-1} + 2(4^{n-1})$

The Nonhomogeneous Recurrence Relation

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\underline{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$= e^x \cdot \left(\frac{e^x + e^{-x}}{2}\right) \cdot e^x \cdot e^x$$

$$= \left(\frac{1}{2}\right) e^{4x} + \left(\frac{1}{2}\right) e^{2x}$$

$$= \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} + \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Here a_n = the coefficient of $\frac{x^n}{n!}$ in $f(x) = \left(\frac{1}{2}\right) 4^n + \left(\frac{1}{2}\right) 2^n = 2^{n-1} + 2(4^{n-1})$, as above.

The Nonhomogeneous Recurrence Relation

- **Ex 10.32** : Snowflake curve shown in Figure 10.12.
- Let a_n denote the area of the polygon P_n obtained from the original equilateral triangle after we apply n transformations.

$$a_0 = \sqrt{3}/4$$

$$a_1 = (\sqrt{3}/4) + (3)(\sqrt{3}/4)(1/3)^2 = \sqrt{3}/3$$

$$a_2 = a_1 + (4)(3)(\sqrt{3}/4)[(1/3)^2]^2 = 10\sqrt{3}/27$$

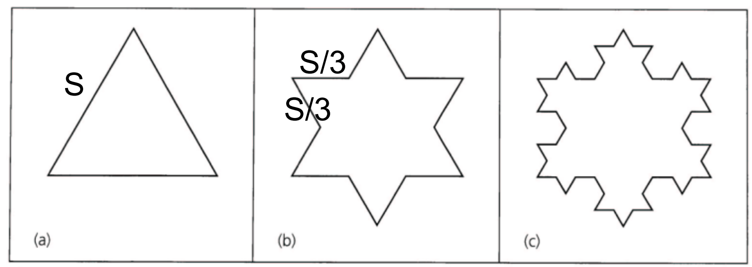


Figure 10.12

A special kind of fractal curves *1904, Helge von Koch*

http://en.wikipedia.org/wiki/Koch_snowflake

#segment in each side

$$a_{n+1} = a_n + (4^n(3))(\sqrt{3}/4)(1/3^{n+1})^2 = a_n + (1/(4\sqrt{3}))(4/9)^n$$

$$a_n = a_n^{(h)} + a_n^{(p)} = A(1)^n + B(4/9)^n = (6/(5\sqrt{3})) - (1/(5\sqrt{3}))(4/9)^{n-1} \approx 6/(5\sqrt{3})$$

The Nonhomogeneous Recurrence Relation

- **Ex 10.34** : Solve the recurrence relation $a_{n+2} - 4a_{n+1} + 3a_n = -200$ for $n \geq 0$ and $a_0 = 3000$ and $a_1 = 3300$.
  - **Solution**

    $a_n^{(h)} = c_1(3^n) + c_2(\mathbf{1^n})$.

    Let $a_n^{(p)} = A\mathbf{n} \Rightarrow A(n+2) - 4A(n+1) + 3An = -200$

    $\Rightarrow a_n^{(p)} = 100n.$

    $a_n = a_n^{(h)} + a_n^{(p)} = c_1(3^n) + c_2(1^n) + 100n$

    $\Rightarrow a_n = 100(3^n) + 2900 + 100n$

The Nonhomogeneous Recurrence Relation

- Two procedures of computing the nth Fibonacci number in Figure 10.15. Which one is more efficient?
- $a_n = a_{n-1} + a_{n-2}$ **+ 1**

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \qquad n \geq 0.$$

$$a_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} - 1.$$

recursive

```
procedure FibNum2(n: nonnegative integer)
begin
  if n = 0 then
    fib := 0
  else if n = 1 then
    fib := 1
  else
    fib := FibNum2(n - 1) + FibNum2(n - 2)
end                                           (b)
```

iterative

```
procedure FibNum1(n: nonnegative integer)
begin
  if n = 0 then
    fib := 0
  else if n = 1 then
    fib := 1
  else
    begin
      last := 1
      next_to_last := 0
      for i := 2 to n do
        begin
          temp := last
          last := last + next_to_last
          next_to_last := temp
        end
      fib := last
    end
end                                           (a)
```

Figure 10.15

Particular Solutions to Nonhomogeneous Recurrence Relation

- $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k} = f(n)$

(1) If $f(n)$ is a constant multiple of one of the forms in the first column of Table 10.2 $\Rightarrow a_n^{(p)}$ in the second column .

(2) When $f(n)$ comprises a sum of constant multiples of terms.

- E.g., $f(n) = n^2 + 3\sin 2n \Rightarrow a_n^{(p)} = (A_2 n^2 + A_1 n + A_0) + (A \sin 2n + B\cos 2n)$

(3) If a summand $f_1(n)$ of $f(n)$ is a solution of the associated homogeneous relation.

- If $f_1(n)$ causes this problem, we multiply the trial solution $(a_n^{(p)})_1$ corresponding to $f_1(n)$ by the smallest power of n, say n^s, for which no summand of $n^s f_1(n)$ is a solution of the associated homogeneous relation. Thus, $n^s(a_n^{(p)})_1$ is the corresponding part of $a_n^{(p)}$.

Table 10.2

$f(n)$	$a_n^{(p)}$
c, a constant	A, a constant
n	$A_1n + A_0$
n^2	$A_2n^2 + A_1n + A_0$
$n^t,\ t \in \mathbf{Z}^+$	$A_tn^t + A_{t-1}n^{t-1} + \cdots + A_1n + A_0$
$r^n,\ r \in \mathbf{R}$	Ar^n
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n(A_tn^t + A_{t-1}n^{t-1} + \cdots + A_1n + A_0)$
$r^n \sin \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$
$r^n \cos \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$

Particular Solutions to Nonhomogeneous Recurrence Relation

- **Ex 10.36** : For n people at a party, each of them shakes hands with others. C(n, 2)
  - a_n counts the total number of handshakes:

    $$a_{n+1} = a_n + n,\ n \geq 2,\ a_2 = 1$$
  - $a_n^{(h)} = c(1^n) = c.$
  - Let $a_n^{(p)} = A_1 n + A_0$
  - By the third remark stated above, multiplying $a_n^{(p)}$ by n^1, then $a_n^{(p)} = A_1 n^2 + A_0 n$
  - $A_1 = ½$, $A_0 = -½ \Rightarrow a_n^{(p)} = (½)n^2 + (-½)n.$
  - $a_n = a_n^{(h)} + a_n^{(p)} = c + (½)n^2 + (-½)n \Rightarrow c = 0$
  - $a_n = (½)n(n-1)$

Particular Solutions to Nonhomogeneous Recurrence Relation

- **Ex 10.37** : $a_{n+2} - 10a_{n+1} + 21a_n = f(n),\ n \geq 0$
- $a_n^{(h)} = c_1(3^n) + c_2(7^n)$.

Table 10.3

$f(n)$	$a_n^{(p)}$
5	A_0
$3n^2 - 2$	$A_3n^2 + A_2n + A_1$
$7(11^n)$	$A_4(11^n)$
$31(r^n),\ r \neq 3, 7$	$A_5(r^n)$
$6(3^n)$	A_6n3^n
$2(3^n) - 8(9^n)$	$A_7n3^n + A_8(9^n)$
$4(3^n) + 3(7^n)$	$A_9n3^n + A_{10}n7^n$

10.4 The Method of Generating Functions

- **Ex 10.38** : Solve the relation $a_n - 3a_{n-1} = n$, $n \geq 1$, $a_0 = 1$.
  - Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots, a_n$.

$$\sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n.$$

$$(f(x) - a_0) - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} n x^n \left(= \sum_{n=0}^{\infty} n x^n \right).$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots,$$

$$(f(x) - 1) - 3x f(x) = \frac{x}{(1-x)^2}, \quad \text{and} \quad f(x) = \frac{1}{(1-3x)} + \frac{x}{(1-x)^2(1-3x)}.$$

$$f(x) = \frac{1}{1-3x} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2} + \frac{(3/4)}{(1-3x)}$$
$$= \frac{(7/4)}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2}.$$

We find a_n by determining the coefficient of x^n in each of the three summands.

a) $(7/4)/(1-3x) = (7/4)[1/(1-3x)]$
$= (7/4)[1 + (3x) + (3x)^2 + (3x)^3 + \cdots]$, and the coefficient of x^n is $(7/4)3^n$.

b) $(-1/4)/(1-x) = (-1/4)[1 + x + x^2 + \cdots]$, and the coefficient of x^n here is $(-1/4)$.

c) $(-1/2)/(1-x)^2 = (-1/2)(1-x)^{-2}$
$= (-1/2)\left[\binom{-2}{0} + \binom{-2}{1}(-x) + \binom{-2}{2}(-x)^2 + \binom{-2}{3}(-x)^3 + \cdots\right]$
and the coefficient of x^n is given by $(-1/2)\binom{-2}{n}(-1)^n = (-1/2)(-1)^n\binom{2+n-1}{n} \cdot (-1)^n = (-1/2)(n+1)$.

Therefore $a_n = \underbrace{(7/4)3^n}_{a_n^{(h)}} \underbrace{- (1/2)n - (3/4)}_{a_n^{(p)}}, n \geq 0.$

The Method of Generating Functions

- **Ex 10.39** : Solve the relation
  $a_{n+2} - 5a_{n+1} + 6a_n = 2,\ n \geq 0,\ a_0 = 3,\ a_1 = 7.$
  - Let $f(x) = \sum^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots, a_n$

1) We first multiply this given relation by x^{n+2} because $n+2$ is the largest subscript that appears. This gives us

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}.$$

2) Then we sum all of the equations represented by the result in step (1) and obtain

$$\sum^{\infty} a_{n+2}x^{n+2} - 5\sum^{\infty} a_{n+1}x^{n+2} + 6\sum^{\infty} a_n x^{n+2} = 2\sum^{\infty} x^{n+2}.$$

4) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the solution. The equation in step (3) now takes the form

$$(f(x) - a_0 - a_1 x) - 5x(f(x) - a_0) + 6x^2 f(x) = \frac{2x^2}{1-x},$$

5) Solving for $f(x)$ we have

$$(1-5x+6x^2)f(x) = 3-8x+\frac{2x^2}{1-x} = \frac{3-11x+10x^2}{1-x},$$

from which it follows that

$$f(x) = \frac{3-11x+10x^2}{(1-5x+6x^2)(1-x)} = \frac{(3-5x)(1-2x)}{(1-3x)(1-2x)(1-x)} = \frac{3-5x}{(1-3x)(1-x)}.$$

A partial-fraction decomposition (by hand, or via a computer algebra system) gives us

$$f(x) = \frac{2}{1-3x} + \frac{1}{1-x} = 2\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty} x^n.$$

Consequently, $a_n = 2(3^n)+1,\ n \geq 0$.

The Method of Generating Functions

- **Ex 10.40** : Let $a(n, r)$ = the number of ways we can select, with repetitions allowed, r objects from a set of n distinct objects.
- Let $\{b_1, b_2, \ldots, b_n\}$ be the set, consider b_1
  - b_1 is never selected: the r objects from $\{b_2, \ldots, b_n\} \Rightarrow a(n - 1, r)$
  - b_1 is selected at least once: must select r -1 objects from $\{b_1, b_2, \ldots, b_n\} \Rightarrow a(n, r - 1)$
- Then $\boldsymbol{a(n, r) = a(n\text{-}1, r) + a(n, r\text{-}1)}$.
- Let $f_n = \sum_{r=0}^{\infty} a(n,r)x^r$ be the generating function for $a(n, 0)$, $a(n, 1)$, $a(n, 2), \ldots,$

$$a(n, r)x^r = a(n-1, r)x^r + a(n, r-1)x^r \quad \text{and}$$

$$\sum_{r=1}^{\infty} a(n, r)x^r = \sum_{r=1}^{\infty} a(n-1, r)x^r + \sum_{r=1}^{\infty} a(n, r-1)x^r.$$

Realizing that $a(n, 0) = 1$ for $n \geq 0$ and $a(0, r) = 0$ for $r > 0$, we write

$$f_n - a(n, 0) = f_{n-1} - a(n-1, 0) + x \sum_{r=1}^{\infty} a(n, r-1)x^{r-1}.$$

so $f_n - 1 = f_{n-1} - 1 + xf_n$. Therefore, $f_n - xf_n = f_{n-1}$, or $f_n = f_{n-1}/(1-x)$.

If $n = 5$, for example, then

$$f_5 = \frac{f_4}{(1-x)} = \frac{1}{(1-x)} \cdot \frac{f_3}{(1-x)} = \frac{f_3}{(1-x)^2} = \frac{f_2}{(1-x)^3} = \frac{f_1}{(1-x)^4}$$

$$= \frac{f_0}{(1-x)^5} = \frac{1}{(1-x)^5},$$

since $f_0 = a(0, 0) + a(0, 1)x + a(0, 2)x^2 + \cdots = 1 + 0 + 0 + \cdots$.

In general, $f_n = 1/(1-x)^n = (1-x)^{-n}$, so $a(n, r)$ is the coefficient of x^r in $(1-x)^{-n}$, which is $\binom{-n}{r}(-1)^r = \binom{n+r-1}{r}$.

10.5 A Special Kind of Nonlinear Recurrence Relation

- **Ex 10.42** : Let b_n denote the number of rooted ordered binary trees on n vertices.
- $b_3 = 5$ is shown in Figure 10.18.
- $b_{n+1} = b_0 b_n + b_1 b_{n-1} + \ldots + b_{n-1} b_1 + b_n b_0$
- Let $f(x) = \sum_{n=0}^{\infty} b_n x^r$ be the generating function for b_0, $b_1, \ldots, b_n$.

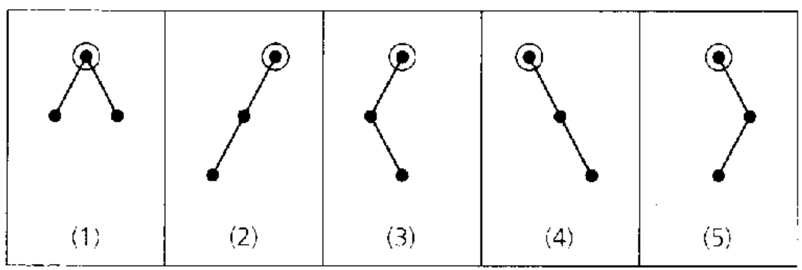


Figure 10.18

A Special Kind of Nonlinear Recurrence Relation

- $b_{n+1} = b_0 b_n + b_1 b_{n-1} + \ldots + b_{n-1} b_1 + b_n b_0$
  1. 0 vertices on the left, n vertices on the right $\Rightarrow b_0 b_n$
  2. 1 vertices on the left, n -1 vertices on the right $\Rightarrow b_1 b_{n-1}$
  3. i vertices on the left, n - i vertices on the right $\Rightarrow b_i b_{n-i}$
  4. n vertices on the left, *none* on the right $\Rightarrow b_n b_0$

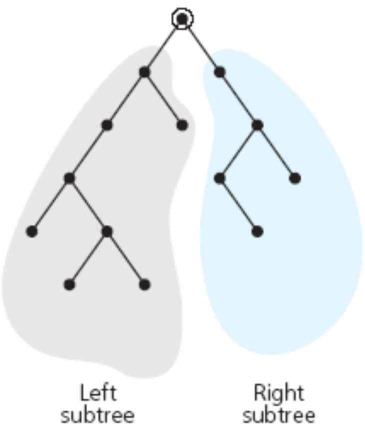


Figure 10.19

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + b_2 b_{n-2} + \cdots + b_{n-1} b_1 + b_n b_0.$$

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0) x^{n+1}.$$

$$(f(x) - b_0) = x \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0) x^n = x[f(x)]^2.$$

$$x[f(x)]^2 - f(x) + 1 = 0, \quad \text{so} \quad \underline{f(x) = [1 \pm \sqrt{1-4x}]/(2x).}$$

But $\sqrt{1-4x} = (1-4x)^{1/2} = \binom{1/2}{0} + \binom{1/2}{1}(-4x) + \binom{1/2}{2}(-4x)^2 + \cdots$, where the coefficient of x^n, $n \geq 1$, is

$$\binom{1/2}{n}(-4)^n = \frac{(1/2)((1/2)-1)((1/2)-2)\cdots((1/2)-n+1)}{n!}(-4)^n$$

$$= (-1)^{n-1}\frac{(1/2)(1/2)(3/2)\cdots((2n-3)/2)}{n!}(-4)^n$$

$$= \frac{(-1)2^n(1)(3)\cdots(2n-3)}{n!}$$

$$= \frac{(-1)2^n(n!)(1)(3)\cdots(2n-3)(2n-1)}{(n!)(n!)(2n-1)}$$

$$= \frac{(-1)(2)(4)\cdots(2n)(1)(3)\cdots(2n-1)}{(2n-1)(n!)(n!)} = \frac{(-1)}{(2n-1)}\binom{2n}{n}.$$

$$f(x) = \frac{1}{2x}\left[\cancel{1} - \left[1 - \sum_{n=1}^{\infty} \frac{1}{(2n-1)}\binom{2n}{n}x^n\right]\right],$$

b_n, the coefficient of x^n in $f(x)$, is half the coefficient of x^{n+1} in

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\binom{2n}{n}x^n.$$

$$b_n = \frac{1}{2}\left[\frac{1}{2(n+1)-1}\right]\binom{2(n+1)}{n+1} = \frac{(2n)!}{(n+1)!(n!)} = \frac{1}{(n+1)}\binom{2n}{n}$$

- b_n are called *Catalan numbers*

A Special Kind of Nonlinear Recurrence Relation

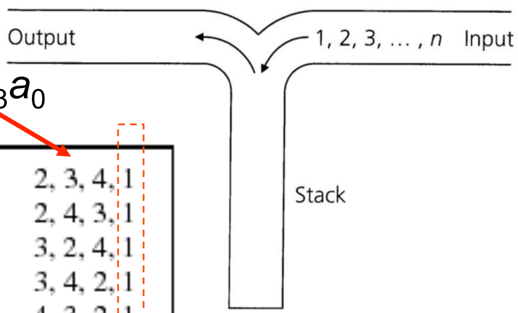
n=2, 12, 21
n=3, 123,132, 213, 321, 231 **312?**

- **Ex 10.43** : permute 1, 2, 3,…, n, which must be pushed onto the top of the stack in the order given.
  - $n = 0 \Rightarrow 1$
  - $n = 1 \Rightarrow 1$
  - $n = 2 \Rightarrow 2$
  - $n = 3 \Rightarrow 5$
  - $n = 4 \Rightarrow 14$

$$a_4 = a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0$$

Table 10.4

1, 2, 3, 4	2, 1, 3, 4	2, 3, 1, 4	2, 3, 4, 1
1, 2, 4, 3	2, 1, 4, 3	3, 2, 1, 4	2, 4, 3, 1
1, 3, 2, 4			3, 2, 4, 1
1, 3, 4, 2			3, 4, 2, 1
1, 4, 3, 2			4, 3, 2, 1



1) There are five permutations with 1 in the first position, because after 1 is pushed onto and popped from the stack, there are five ways to permute 2, 3, 4 using the stack.

2) When 1 is in the second position, 2 must be in the first position. This is because we pushed 1 onto the (empty) stack, then pushed 2 on top of it and then popped 2 and then 1. There are two permutations in column 2, because 3, 4 can be permuted in two ways on the stack.

A Special Kind of Nonlinear Recurrence Relation

3) For column 3 we have 1 in position three. We note that the only numbers that can precede it are 2 and 3, which can be permuted on the stack (with 1 on the bottom) in two ways. Then 1 is popped, and we push 4 onto the (empty) stack and then pop it.

4) In the last column we obtain five permutations: After we push 1 onto the top of the (empty) stack, there are five ways to permute 2, 3, 4 using the stack (with 1 on the bottom). Then 1 is popped from the stack to complete the permutation.

- $a_4 = a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0$
- $a_{n+1} = a_0a_n + a_1a_{n-1} + \ldots + a_{n-1}a_1 + a_na_0$
- $a_n = \dfrac{1}{n+1}\dbinom{2n}{n}$
- Push, pop permutation with limitation (Ex 1.43)

10.6 Divide-and-Conquer Algorithms

- In general, solve a given problem of size n by
  - Solving the problem for a small value of n directly.
  - Breaking the problem into a smaller problems of the same type and the same size $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$
- Divide-and-conquer algorithms

1) The time to solve the initial problem of size $n = 1$ is a constant $c \geq 0$, and
2) The time to break the given problem of size n into a smaller (similar) problems, together with the time to combine the solutions of these smaller problems to get a solution for the given problem, is $h(n)$, a function of n.

- Time complexity function $f(n)$
  - $f(1) = c$
  - $f(n) = af(n/b) + h(n)$ for $n = b^k$

Divide-and-Conquer Algorithms

- Theorem 10.1:

Let $a, b, c \in \mathbf{Z}^+$ with $b \geq 2$, and let $f: \mathbf{Z}^+ \rightarrow \mathbf{R}$. If

$$f(1) = c, \quad \text{and}$$
$$f(n) = af(n/b) + c, \qquad \text{for } n = b^k, \qquad k \geq 1,$$

then for all $n = 1, b, b^2, b^3, \ldots,$

1) $f(n) = c(\log_b n + 1)$, when $a = 1$, and
2) $f(n) = \dfrac{c(an^{\log_b a} - 1)}{a - 1}$, when $a \geq 2$.

Proof: For $k \geq 1$ and $n = b^k$, we write the following system of k equations. [Starting with the second equation, we obtain each of these equations from its immediate predecessor by (i) replacing each occurrence of n in the prior equation by n/b and (ii) multiplying the resulting equation in (i) by a.]

$$\begin{aligned}
f(n) &= af(n/b) + c \\
af(n/b) &= a^2 f(n/b^2) + ac \\
a^2 f(n/b^2) &= a^3 f(n/b^3) + a^2 c \\
\vdots \qquad & \quad \vdots \qquad\qquad \vdots \\
a^{k-2} f(n/b^{k-2}) &= a^{k-1} f(n/b^{k-1}) + a^{k-2} c \\
a^{k-1} f(n/b^{k-1}) &= a^k f(n/b^k) + a^{k-1} c
\end{aligned}$$

$$f(n) = a^k f(n/b^k) + [c + ac + a^2c + \cdots + a^{k-1}c].$$

Since $n = b^k$ and $f(1) = c$, we have

$$\begin{aligned} f(n) &= a^k f(1) + c[1 + a + a^2 + \cdots + a^{k-1}] \\ &= c[1 + a + a^2 + \cdots + a^{k-1} + a^k]. \end{aligned}$$

1) If $a = 1$, then $f(n) = c(k+1)$. But $n = b^k \Leftrightarrow \log_b n = k$, so $f(n) = c(\log_b n + 1)$, for $n \in \{b^i | i \in \mathbf{N}\}$.

2) When $a \geq 2$, then $f(n) = \dfrac{c(1 - a^{k+1})}{1 - a} = \dfrac{c(a^{k+1} - 1)}{a - 1}$, from identity 4 of Table 9.2. Now $n = b^k \Leftrightarrow \log_b n = k$, so

$$a^k = a^{\log_b n} = (b^{\log_b a})^{\log_b n} = (b^{\log_b n})^{\log_b a} = n^{\log_b a},$$

and

$$f(n) = \frac{c(an^{\log_b a} - 1)}{(a - 1)}, \qquad \text{for } n \in \{b^i | i \in \mathbf{N}\}.$$

Divide-and-Conquer Algorithms

$$f(n) = \frac{c(an^{\log_b a} - 1)}{(a-1)}$$

- **Ex 10.45** :

  (a) $f(1) = 3$ and $f(n) = f(n/2) + 3$ for $n = 2^k$

  $c = 3$, $b = 2$, $a = 1$

  $f(n) = 3(\log_2 n + 1)$

  (b) $g(1) = 7$ and $g(n) = 4g(n/3) + 7$ for $n = 3^k$

  $c = 7$, $b = 3$, $a = 4$

  $g(n) = (7/3)(4n^{\log_3 4} - 1)$

  (c) $h(1) = 5$ and $h(n) = 7h(n/7) + 5$ for $n = 7^k$

  $c = 5$, $b = 7$, $a = 7$

  $h(n) = (5/6)(7n - 1)$

Changing variables

$$T(n) = 2T(\sqrt{n}) + \lg n$$

Let $m = \lg n$.

$$T(2^m) = 2T(2^{m/2}) + m$$

$$\boxed{T(n) = T(2^{\sqrt{\lg n}}) + c\,?}$$

$$\boxed{Ans: T(n) = O(\lg\lg\lg n)}$$

Suppose $S(m) = T(2^m)$,

Then $S(m) = 2S(m/2) + m$.

$$\Rightarrow S(m) = O(m \lg m)$$

$$\Rightarrow T(n) = T(2^m) = S(m) = O(m \lg m)$$

$$= O(\lg n \lg\lg n)$$