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Advanced Engineering Mathmatics II

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In this semester, we will try to cover the following topics :

1. Fourier Series and Fourier transform (Harmonic Analysis)
2. Partial differential equation (PDE)
3. Complex analysis (complex variables and functions)
4. Vector analysis

In order to study Fourier series, we need some knowledge on orthogonal functions. Let's say that a function $f(x)$ can be expressed in terms of sine and cosine functions as follows :

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)$$

⇨ Can you try to evalute the following integrals's

$$\int_{-l}^{l} \sin \frac{m\pi}{l} x \sin \frac{n\pi}{l} x \, dx = \begin{cases} ? & , m \neq n \\ ? & , m = n \end{cases}$$

$$\int_{-l}^{l} \cos \frac{m\pi}{l} x \cos \frac{n\pi}{l} x \, dx = \begin{cases} ? & , m \neq n \\ ? & , m = n \end{cases}$$

$$\int_{-l}^{l} \sin \frac{m\pi}{l} x \cos \frac{n\pi}{l} x \, dx = ? \Rightarrow 0$$

↳ Sine and cosine are orthogonal functions

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In fact, quite a few functions are orthogonal. They mainly come from the solution of a special ODE

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0$$

which is commonly referred to as the Sturm-Liouville equation.

We will examine three special ones:

1. $(1-x^2)y'' - 2xy' + \lambda y = 0$ $\quad ((1-x^2)y'' - 2xy' = [(1-x^2)y']')$
   ⇨ Legendre equation ⇨ Legendre polynomials
2. $x^2y'' + xy' + (x^2-\nu^2)y = 0$ $\quad (x^2y'' + xy' = x(xy'' + y') = x(xy')')$
   ⇨ Bessel equation ⇨ Bessel functions
3. $y'' + \lambda^2 y = 0 \ (\lambda > 0)$ $\quad (y'' = (y')')$
   ⇨ Fourier series

Let's try to solve it by an infinite series. The question then is how to assume a series form? Let's use two simple ODEs to see what we can do.

1. $y' - \alpha y = 0$, $\alpha > 0$ $\quad (y = ce^{\alpha x})$
2. $xy' + y = 0$

How can we represent $y = ce^{\alpha t}$ as a series?

⇨ $e^{\alpha x} = (\alpha x)^0 + (\alpha x)^1/1! + (\alpha x)^2/2! + \cdots$

$= 1 + \alpha x + \alpha^2 x^2/2! + \cdots\cdots$

i.e. $y = C\sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!}$

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∴ Can you try to assume $y = \sum_{n=0}^{\infty} a_n x^n$, then use the ODE to obtain a_n?

∴ For $y' - \alpha y = 0$,

Assume $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$

⇨ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

⇨ $\sum_{n=1}^{\infty} n a_n x^{n-1} - \alpha \sum_{n=0}^{\infty} a_n x^n = 0$

Let $n-1 = m$, $\begin{cases} n=1, & m=0 \\ n=\infty, & m=\infty \end{cases}$, $n = m+1$

Then $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$

$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

⇨ $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} \alpha a_n x^n = 0$

⇨ $\sum_{n=0}^{\infty} [(n+1)a_{n+1} - \alpha a_n] x^n = 0$

⇓

$(n+1)a_{n+1} - \alpha a_n = 0$; $n = 0, 1, 2, 3 \cdots\cdots$

⇨ $a_{n+1} = \frac{\alpha}{n+1} a_n$ (recursion)

$n=0$, $a_1 = \alpha a_0$

$n=1$, $a_2 = \frac{\alpha}{2} a_1 = \frac{\alpha^2}{2} a_0$

$n=2$, $a_3 = \frac{\alpha}{3} a_2 = \frac{\alpha^3}{3!} a_0$ ⇨ $a_n = \frac{\alpha^n}{n!} a_0$

⇨ $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_0 \left(\frac{\alpha^n}{n!}\right) x^n = \sum_{n=0}^{\infty} a_0 \frac{(\alpha x)^n}{n!}$

$= a_0 e^{\alpha x}$

If use $y = \sum_{n=0}^{\infty} a_n x^n$ to solve $xy' + y = 0$, what will happen? ⇨ $y = 0$ trivial solution

⇨ why?

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example : $y' - y = 1$

Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 1,$$

$$\sum_{n=0}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 1$$

$$\Rightarrow (a_1 - a_0) - 1 + \sum_{n=1}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 0$$

or $\underbrace{(a_1 - a_0 - 1)}_{0} + \underbrace{(2a_2 - a_1)}_{0} x^1 + \underbrace{(3a_3 - a_2)}_{0} x^2 + \cdots\cdots = 0$

$$\Rightarrow \begin{cases} a_1 - a_0 = 1 \Rightarrow (a_0 = a_1 - 1) \\ (n+1) a_{n+1} - a_n = 0, \quad n = 1, 2, 3 \end{cases}$$

$$a_{n+1} = \frac{a_n}{n+1}, \quad n = 1, 2, \cdots \qquad \left(\begin{array}{l} a_2 = \frac{a_1}{2} \\ a_3 = \frac{a_2}{3} = \frac{a_1}{3!} \\ a_4 = \frac{a_3}{4} = \frac{a_1}{4!} \\ \cdots \end{array} \right)$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots$$

$$= a_0 + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \cdots$$

$$= a_1 - 1 + a_1 x + \frac{a_1}{2!} x^2 + \frac{a_1}{3!} x^3 + \cdots$$

$$= -1 + a_1 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right)$$

$$= -1 + a_1 e^x$$

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example : $y'' + \beta^2 y = 0$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \beta^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

i.e, $\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + \beta^2 a_n] x^n = 0$

$$\Rightarrow a_{n+2} = \left[\frac{-\beta^2}{(n+1)(n+2)}\right] a_n, \quad n = 0, 1, 2 \cdots$$

$n = 0$, $a_2 = \left(\frac{-\beta^2}{2}\right) a_0$

$n = 1$, $a_3 = \left(\frac{-\beta^2}{3 \cdot 2}\right) a_1$

$n = 2$, $a_4 = \left(\frac{-\beta^2}{4 \cdot 3}\right) a_2 = \frac{(-\beta^2)^2}{4!} a_0$

$n = 3$, $a_5 = \left(\frac{-\beta^2}{5 \cdot 4}\right) a_3 = \frac{(-\beta^2)^2}{5!} a_1$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x - \frac{\beta^2}{2!} a_0 x^2 - \frac{\beta^2}{3!} a_1 x^3 + \frac{(-\beta^2)^2}{4!} a_0 x^4 + \frac{(-\beta^2)^2}{5!} a_1 x^5 + \cdots$$

$$= a_0 \left[1 - \frac{(\beta x)^2}{2!} + \frac{(\beta x)^4}{4!} + \cdots\cdots\right] + a_1 \frac{\beta}{\beta} \left[x - \frac{\beta^2 x^3}{3!} + \frac{\beta^4 x^5}{5!} + \cdots\right]$$

$$= a_0 \cos \beta x + \frac{a_1}{\beta} \sin \beta x$$

$$= a_0 \cos \beta x + \gamma \sin \beta x$$

$$\Rightarrow y'' + \beta^2 y = 0, \quad \text{sol} \Rightarrow y = a \cos \beta x + c \sin \beta x$$

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Legendre ODE

$$(1-x^2)y'' - 2xy' + \eta y = 0$$

For convenience, let's change η into $l(l+1)$ + constant

$$\Rightarrow (1-x^2)y'' - 2xy' + l(l+1)y = 0$$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$

$$y'' - x^2 y'' - 2xy' + l(l+1)y = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} l(l+1)a_n x^n = 0$$

$$\Rightarrow 1\cdot 2a_2 + 2\cdot 3a_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$
$$- 2a_1 x - \sum_{n=2}^{\infty} 2na_n x^n + l(l+1)a_0 + l(l+1)a_1 x + \sum_{n=2}^{\infty} l(l+1)a_n x^n = 0$$

$$\Rightarrow \overset{0}{\overset{\shortparallel}{[2a_2 + l(l+1)a_0]}}x^0 + \overset{0}{\overset{\shortparallel}{[2\cdot 3a_3 - 2a_1 + l(l+1)a_1]}}x^1$$
$$+ \sum_{n=2}^{\infty}\underbrace{[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + l(l+1)a_n]}_{0}x^n = 0$$

$$\left(\begin{aligned}&[-n(n-1) - 2n + l(l+1)]a_n \\ &= [-n^2 + n - 2n + l^2 + l]a_n = -[n^2 + n - l^2 - l]a_n \\ &= -[(n^2 - l^2) + (n-l)]a_n = -(n-l)(n+l+1)a_n\end{aligned}\right)$$

$$[(n+1)(n+2)a_{n+2} - (n-l)(n+l+1)a_n]$$

$$\therefore\ 2a_2 + l(l+1)a_0 = 0 \Rightarrow a_2 = \frac{-l(l+1)}{2}a_0$$

$$2\cdot 3a_3 - 2a_1 + l(l+1)a_1 = 0 \Rightarrow a_3 = \frac{2 - l(l+1)}{2\cdot 3}a_1$$

$$a_{n+2} = \frac{(n-l)(n+l+1)}{(n+1)(n+2)}a_n, \quad n = 2, 3, 4, \ldots$$

$$\left\{\begin{aligned}&\text{if } n=0,\ a_2 = \frac{-l(l+1)}{2}a_0 \\ &n=1,\ a_3 = \frac{(1-l)(2+l)}{2\cdot 3}a_1 = \frac{2-l^2-l}{2\cdot 3}a_1\end{aligned}\right)$$

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$$\therefore\ y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \left(-\frac{l(l+1)}{2}a_0 x^2\right) + \frac{(1-l)(2+l)}{2\cdot 3}a_1 x^3 + \ldots$$

$$\begin{aligned}= \ & a_0 && + a_1 x \\ & - \frac{l(l+1)}{2}a_0 x^2 && + \frac{(1-l)(2+l)}{3!}a_1 x^3 \\ & - \frac{l(2-l)(l+1)(l+3)}{4!}a_0 x^4 && + \frac{(1-l)(3-l)(2+l)(4+l)}{5!}a_1 x^5 \\ & + \ldots\ldots\end{aligned}$$

$$= a_0 y_1 + a_1 y_2$$

$$y_1 = 1 - \frac{l(l+1)}{2!}x^2 - \frac{l(2-l)(l+1)(l+3)}{4!}x^4 + \ldots$$

$$y_2 = x + \frac{(1-l)(2+l)}{3!}x^3 + \frac{(1-l)(3-l)(2+l)(4+l)}{5!}x^5 + \ldots$$

What will happen if we choose $l = 0, 1, 2, 3, \ldots$?

$l=0$, $y_1 = 1$, y_2 = Infinite Series

$l=1$, y_1 = Infinite series, $y_2 = x$

$l=2$, $y_1 = 1 - 3x^2$, y_2 = Infinite Series

$l=3$, y_1 = Infinite series, $y_2 = x - \frac{5}{3}x^3$

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Legendre polynomial $P_n(x)$, $P_n(1) = 1$

$\Rightarrow$ $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = (\frac{3}{2}x^3 - \frac{3}{2}x)$

$$\int_{-1}^{1} P_1 P_2 \, dx = \int_{-1}^{1} x \cdot \frac{1}{2}(3x^2-1)\, dx = \int_{-1}^{1} (\frac{3}{2}x^3 - \frac{1}{2}x)\, dx = 0$$

Two functions $f(x)$, $g(x)$, $a \le x \le b$

$$\langle f, g \rangle \equiv \int_a^b w(x) f(x) g(x)\, dx = 0 \quad \text{when } f(x) \neq g(x)$$

$\Rightarrow$ $f(x)$ and $g(x)$ are orthogonal functions

The orthogonality of functions is an extension of the orthogonality (正交性) of vectors.

eg., 3-D space $\vec{a}$, $\vec{b}$;

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i b_i$$

Weighting function $W(x)$ ($W(x)$ of Legendre $= 1$)

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad m \neq n, \quad W(x) = 1,$$

$n = 0, 1, 2 \cdots$

$$\begin{cases} (1-x^2) P_m'' - 2x P_m' + m(m+1) P_m = 0 \\ (1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0 \end{cases} \quad \triangleright \; ([(1-x^2) P_n']')$$

$$\Rightarrow \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad m \neq n$$

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Legender ODE:

$$(1-x^2) y'' - 2xy' + l(l+1) y = 0$$

$$\Rightarrow [(1-x^2) \cdot y']' + l(l+1) y = 0$$

Two solutions, y_m, y_n $(m \neq n)$

$$\Rightarrow \begin{cases} [(1-x^2) y_m']' + m(m+1) y_m = 0 \\ [(1-x^2) y_n']' + n(n+1) y_n = 0 \end{cases}$$

To show their orthogonality

$$\Rightarrow y_n [(1-x^2) y_m']' - y_m [(1-x^2) y_n']' + \underbrace{m(m+1) y_m y_n - n(n+1) y_n y_m}_{[m(m+1) - n(n+1)] y_m y_n} = 0$$

* Note: $x = \pm 1$, singular point (奇異点, 簡稱奇點)

$$\int_{-1}^{1} \{ y_n [(1-x^2) y_m'] - y_m [(1-x^2) y_n'] \}\, dx + \int_{-1}^{1} [m(m+1) - n(n+1)] y_m y_n\, dx = 0$$

Goal: $\int_{-1}^{1} W(x) y_m(x) y_n(x)\, dx = 0$? $m \neq n$

The question is whether $\int_{-1}^{1} y_m(x) y_n(x)\, dx = 0$, $m \neq n$

$$\int_{-1}^{1} \overset{u}{y_n}\, d\overset{dv}{[(1-x^2) y_m']} - \int_{-1}^{1} y_m\, d[(1-x^2) y_n']$$

$$= \cancel{y_n [(1-x^2) y_m']\Big|_{-1}^{1}}^{=0} - \int_{-1}^{1} (1-x^2) y_m' \underbrace{dy_n'}_{\left(\frac{dy_n'}{dx}\right) dx = y_n'' dx} - \cancel{y_m [(1-x^2) y_n']\Big|_{-1}^{1}}^{=0} + \int_{-1}^{1} (1-x^2) y_n' \underbrace{dy_m'}_{= y_m'' dx}$$

$$= -\int_{-1}^{1} (1-x^2) y_m' y_n'\, dx + \int_{-1}^{1} (1-x^2) y_n' y_m'\, dx = 0$$

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What about $y'' + \beta^2 y = 0$? $(y(-l) = y(l) = 0)$ $-l \le x \le l$?

$\Rightarrow (y')' + \beta^2 y = 0$, $\sin\beta x$ & $\cos\beta x$

$$\begin{cases} (y_m')' + m^2 y_m = 0 \\ (y_n')' + n^2 y_n = 0 \end{cases} \Rightarrow \int_a^b y_m y_n \, dx = 0 \ ?$$

$$\Rightarrow y_n (y_m')' + m^2 y_m y_n - y_m (y_n')' + n^2 y_n y_m = 0,$$

$$\Rightarrow \int_a^b [y_n (y_m')' - y_m (y_n')'] dx + \int_a^b (m^2 - n^2) y_m y_n = 0$$

$$\Rightarrow y_n y_m' \Big|_a^b - y_m y_n' \Big|_a^b + \underbrace{\left(\int_a^b y_n' dy_m - \int_a^b y_m' dy_n\right)}_{=0} + \int_a^b (m^2 - n^2) y_m y_n dx = 0$$

$$\Rightarrow \underbrace{[y_n y_m' - y_m y_n']\Big|_a^b}_{\text{specified by bounding conditions}} + \underbrace{\int_a^b (m^2 - n^2) y_m y_n \, dx}_{\int_{-l}^{l} y_m y_n dx = 0,\ m \ne n} = 0$$

$$[p(x) y']' + q(x) y + \lambda w(x) y = 0$$

$$\Rightarrow \begin{cases} [p(x) y_m']' + q(x) y_m + \lambda w(x) y_m = 0 \\ [p(x) y_n'] + q(x) y_n + \lambda w(x) y_n = 0 \end{cases}$$

when $\int_a^b w(x) y_m(x) y_n(x) dx = 0$, $m \ne n$

What's the purpose of finding orthogonal functions?

Hint: In vector space, what's the purpose of finding orthonormal vectors?

Even	Odd	
$P_0(x) = 1$	$P_1(x) = x$	$f(x)$
$P_2(x) = \cdots$	$P_3(x) = \cdots$	$= \frac{1}{2}[f(x) + f(-x)]$
$P_4(x) = \cdots$	$P_5(x) = \cdots$	$+ \frac{1}{2}[f(x) - f(-x)]$

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Given Legendre polynomial

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = a_0 P_0 + a_1 P_1 + a_2 P_2 + \cdots$$

$$\text{V.S. } f(x) = \sum_{n=0}^{\infty} a_0 (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$\int_{-1}^{1} f(x) P_0 \, dx = a_0 \int_{-1}^{1} P_0^2 \, dx + a_1 \underbrace{\int_{-1}^{1} P_0 P_1 \, dx}_{=0} + a_2 \underbrace{\int_{-1}^{1} P_0 P_2 \, dx}_{=0} + \cdots$$

(orthogonal)

$$= a_0 \int_{-1}^{1} P_0^2 \, dx$$

$$a_0 = \frac{\int_{-1}^{1} f(x) P_0 \, dx}{\int_{-1}^{1} P_0^2(x) \, dx} \Rightarrow a_n = \frac{\int_{-1}^{1} f(x) P_n(x) \, dx}{\int_{-1}^{1} P_n^2(x) \, dx}$$

special function

Last time, we mentioned the expansion of function $f(x)$ in terms of either a Taylor series or Legendre polynomials.

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ (power series)

or $f(x) = a_0 P_0 + a_1 P_1 + a_2 P_2 + \cdots$

In which $P_0, P_1, P_2, \cdots$ are Legendre polynomials

What is the difference between these two expansions?

In these expansions, we need to find the coefficients $a_0, a_1, a_2, \cdots$ from the given function $f(x)$.

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For Taylor series, $a_n = f^{(n)}(n) / n!$

For Legendre polynomials, $a_n = \dfrac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx}$

we can rearrange the Taylor series as

$$f(x) = \underbrace{a_0 + a_2 x^2 + a_4 x^4 \cdots}_{\text{Even}} + \underbrace{a_1 x + a_3 x^3 + \cdots}_{\text{Odd}}$$

$f(x)$ can also be represented as :

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)$$

if $f(x) = f(x+2l) = f(x+4l) = \cdots\cdots$

(i.e., $f(x)$ is a periodic function with $2l$ as its period)

This expression is referred to as :

※ — Fourier Series

$$\begin{cases} f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right), \\ a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx \end{cases}$$

or

$$\begin{cases} f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right), \\ a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx \end{cases}$$

$$a_n = \frac{\int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\, dx}{\int_{-l}^{l} \cos^2 \frac{n\pi}{l} x\, dx}$$

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$a_0 = l$.

$$f(x) = a_0 + a_1 \cos \frac{\pi}{l} x + a_2 \cos \frac{2\pi}{l} x \cdots\cdots + b_1 \sin \frac{\pi}{l} x + b_2 \sin \frac{2\pi}{l} x \cdots\cdots$$

$$\Rightarrow \int_{-l}^{l} f(x)\, dx = a_0 \int_{-l}^{l} dx + a_1 \int_{-l}^{l} \cos \frac{\pi}{l} x\, dx + a_2 \int_{-l}^{l} \cos \frac{2\pi}{l} x\, dx + \cdots + b_1 \int_{-l}^{l} \sin \frac{\pi}{l} x\, dx + \cdots$$

$$= a_0 (2l)$$

$$\Rightarrow \int_{-l}^{l} f(x) \cos \frac{\pi x}{l}\, dx = a_0 \int_{-l}^{l} \cos \frac{\pi}{l} x\, dx + a_1 \int_{-l}^{l} \cos^2 \frac{\pi}{l} x\, dx$$

(積化和差)

$$+ a_2 \int_{-l}^{l} \cos \frac{\pi}{l} \cos \frac{2\pi}{l} x\, dx + \cdots + b_1 \int_{-l}^{l} \sin \frac{\pi}{l} x \cos \frac{\pi}{l} x\, dx$$

$$+ b_2 \int_{-l}^{l} \sin \frac{2\pi}{l} \cos \frac{\pi}{l} x\, dx + \cdots\cdots$$

i.e., $a_1 = \dfrac{\int_{-l}^{l} f(x) \cos \frac{\pi}{l} x\, dx}{\int_{-l}^{l} \cos^2 \frac{\pi}{l} x\, dx}$ (Legendre, $a_n = \dfrac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx}$)

$$\Rightarrow a_n = \frac{\int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\, dx}{\int_{-l}^{l} \cos^2 \frac{n\pi}{l} x\, dx}$$

($\int_{-l}^{l} \cos^2 \frac{n\pi}{l} x\, dx = l$)

($= \int_{-l}^{l} \sin^2 \frac{n\pi}{l} x\, dx$)

Similarly $b_n = \dfrac{\int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x\, dx}{\int_{-l}^{l} \sin^2 \frac{n\pi}{l} x\, dx}$

$\because \cos^2 \frac{n\pi}{l} = \frac{1}{2}\left(1 + \cos \frac{2n\pi x}{l}\right)$

$\sin^2 \frac{n\pi x}{l} = \frac{1}{2}\left(1 - \sin \frac{2n\pi x}{l}\right)$

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$\cos\theta$, $\theta \cdots ?$

$\theta = \omega t = 2\pi f t = 2\pi \frac{t}{T}$ (時間)

$\theta = kx = 2\pi \frac{x}{\lambda}$, $k = \frac{2\pi}{\lambda}$ (波數) (空間)

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} t + b_n \sin \frac{n\pi}{l} t\right), \quad l = \frac{T}{2}$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi n \frac{t}{T} + b_n \sin 2\pi n \frac{t}{T}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

$$= a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin\omega t + b_2 \sin 2\omega t \cdots$$

◎ Solving an ODE by assuming

Power Series (Taylor series)

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{v.s.} \quad y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

For $xy' + y = 0$,

If I assume $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\Rightarrow x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 ,$$

i.e, $\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n = 0$,

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) a_n x^n + a_0 = 0$$

$$\Rightarrow a_0 = 0, \ (n+1) \neq 0 \ (n = 1, 2, \cdots)$$

$$\Rightarrow a_n = 0 ; \ y \equiv 0 \ \text{(trivial Solution)}$$

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General Solution :

$xy' + y = 0$, $\Rightarrow x\,dy + y\,dx = 0$,

$d(xy) = 0$, $y = Cx^{-1}$

Modification

$$y = x^{\beta} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\beta}$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} (n+\beta) a_n x^{n+\beta-1}$$

Back to $xy' + y = 0$,

$$x \sum_{n=0}^{\infty} (n+\beta) a_n x^{n+\beta-1} + \sum_{n=0}^{\infty} a_n x^{n+\beta} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+\beta) a_n x^{n+\beta} + \sum_{n=0}^{\infty} a_n x^{n+\beta} = 0$$

$$\Rightarrow \therefore \sum_{n=0}^{\infty} (n+\beta+1) a_n x^n = 0$$

$(n+\beta+1) a_n = 0$, $n = 0, 1, 2, \cdots$

$n = 0$, $(\beta+1) a_0 = 0$

$n = 1$, $(\beta+2) a_1 = 0$

$\vdots$

$n = k$, $(k+1+\beta) a_k = 0$

① $a_0 = 0$

② $a_0 \neq 0$

$\beta = -1$, $a_1, a_2, a_3, \cdots = 0$

$xy' + y = 0$, $y = Cx^{-1}$

$$\Rightarrow y = a_0 x^{-1}$$

(Singular Point (2nd order ODE))

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Method of Frobenius

For a 2nd order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0$$

For example, Bessel's ODE

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + \frac{1}{x^2}(x^2 - \nu^2)y = 0$$

then $p(x) = \frac{1}{x}$, $q(x) = \frac{1}{x^2}(x^2 - \nu^2)$

A series solution can be written as

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad y = \sum_{n=0}^{\infty} a_n x^{n+\beta} \qquad (x_0 = 0)$$

A slight extension

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \; ; \; x_0 : \text{Ordinary point}$$

or

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\beta} \; ; \; x_0 : \text{Regular Singular Point}$$

If $p(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ and $q(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, then x_0 is an ordinary point.

If not, then it is referred to as a singular point.

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For example, $(1-x^2)y'' - 2xy' + l(l+1)y = 0$

$$\begin{cases} p(x) = -2x/1-x^2 \\ q(x) = l(l+1)/1-x^2 \end{cases}, \quad x_0 = 0 \text{ and } x_0 = 1$$

$x_0 = 0$:

$$p(x) = -\frac{2x}{1-x^2} = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \frac{d^n\left(-\frac{2x}{1-x^2}\right)}{n!\, dx^n}\Bigg|_{x=0} x^n$$

$$q(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dx^n}\left(\frac{l(l+1)}{1-x^2}\right)\Bigg|_{x=0} x^n$$

On the other hand, $x_0 = 1$

$$\frac{d^n}{dx^n}\left(-\frac{2x}{1-x^2}\right)\Bigg|_{x=1} \quad \text{or} \quad \frac{d^n}{dx^n}\left[\frac{l(l+1)}{1-x^2}\right]\Bigg|_{x=1}$$

$\rightarrow \infty$, i.e, They can't be expanded by Taylor series.

i.e, $x = 1$ is a singular point.

◎ Singular point $\begin{cases} \text{Regular singular point} \\ \text{Irregular singular point} \end{cases}$

$$(x - x_0)P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{and}$$

$$(x - x_0)^2 q(x) = \sum_{n=0}^{\infty} r_n (x - x_0)^n$$

Then x_0 is a regular singular point.

If not, then x_0 is an irregular singular point.

When x_0 is a regular singular point,

then $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\beta}$

For Legendre ODE, $x_0 = \pm 1$ are singular points

$$y'' - \frac{2x}{1-x^2}y' + \frac{l(l+1)}{1-x^2}y = 0 \quad \Rightarrow \text{Regular Singular points}$$

$$\left((1-x^2) = (1+x)(1-x) \right)$$

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Back to Bessel's ODE

From $p(x) = \frac{1}{x} \Rightarrow xp(x) = 1$,

$q(x) = \frac{1}{x^2}(x^2 - \nu^2) \Rightarrow x^2 q(x) = -\nu^2 + x^2$

Taylor series: $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 \cdots\cdots$

$f(x) = 1$, $f(x) = a_0 + a_1 x + a_2 x^2 \cdots\cdots$

$$\begin{cases} a_0 = f(0) = 1 \\ a_1 = df/dx \big|_{x=0} = 0 \end{cases}, \quad \therefore a_0 = 1,\ a_1 = 0$$

$y'' + \frac{1}{x^2} y' + y = 0$, $p(x) = \frac{1}{x^2} \Rightarrow xp(x) = \frac{1}{x}$

$x_0 = 0$ is an irregular singular point

$y'' + p(x) y' + q(x) y = 0$

If x_0 is a regular singular point, than we can assume $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\beta}$

$$\Rightarrow \begin{cases} (x-x_0)p(x) = \text{T.S expansion} \\ (x-x_0)^2 q(x) = \text{T.S expansion} \end{cases}$$

$\Rightarrow (x-x_0)^2 y'' + (x-x_0)^2 p(x) y' + (x-x_0)^2 q(x) y = 0$

$\Rightarrow (x-x_0)^2 \sum_{n=0}^{\infty} (n+\beta)(n+\beta-1) a_n (x-x_0)^{n+\beta-2} + (x-x_0)(x-x_0) p(x) \sum_{n=0}^{\infty} (n+\beta) a_n (x-x_0)^{n+\beta-1}$
$+ (x-x_0)^2 q(x) \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\beta} = 0$

$$\begin{cases} (x-x_0)p(x) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 \cdots \\ (x-x_0)^2 q(x) = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 \cdots \end{cases}$$

refer to Bessel's ODE

$xp(x) = 1 \Rightarrow x_0 = 0,\ p_0 = 1,\ p_1 = p_2 \cdots p_n = 0$

$x^2 q(x) = -\nu^2 + x^2 \Rightarrow q_0 = -\nu^2,\ q_1 = 0,\ q_2 = 1,\ q_3 = 0 \cdots\cdots$

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Look for an equation of β.

"$\beta(\beta-1) + p_0\beta + q_0 = 0$", x^{β} $(n=0)$

If x_0 is a regular singular point, for a 2nd order ODE

$y'' + p(x) y' + q(x) y = 0$

i.e., $(x-x_0)p(x) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \cdots\cdots$

$(x-x_0)^2 q(x) = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \cdots\cdots$

where p_i, q_i, $i = 0, 1, 2, \cdots$ are constant coefficient to be determined by $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$,

and we can assume a series solution

"$y = (x-x_0)^{\beta} \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\beta}$; $a_0 \neq 0$"

where β is a constant to be determined from the ODE.

In typical engineering applications, $x_0 = 0$,

$(x-x_0)^2 [y'' + p(x) y' + q(x) y] = 0$

$\Rightarrow (x-x_0)^2 y'' + (x-x_0) p(x) (x-x_0) y' + (x-x_0)^2 q(x) y = 0$

$\Rightarrow y' = \sum_{n=0}^{\infty} (n+\beta) a_n (x-x_0)^{n+\beta-1}$

(vs. $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$; $y' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{m=0}^{\infty} (m+1) a_m (x-x_0)^m$)

$\Rightarrow \sum_{n=0}^{\infty} (n+\beta)(n+\beta-1) a_n (x-x_0)^{n+\beta}$
$+ [p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \cdots] \sum_{n=0}^{\infty} (n+\beta) a_n (x-x_0)^{n+\beta}$
$+ [q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \cdots] \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\beta} = 0$

$n=0$, x^{β}: $\beta(\beta-1) a_0 + p_0 \beta a_0 + q_0 a_0 = 0$

Next!

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$\because a_0 \neq 0$, $\therefore \beta(\beta-1) + p_0\beta + q_0 = 0$ ← Indicial equation

The root of β has three possibilities:

a. $\beta_1 \neq \beta_2$, $\beta_i \in \mathbb{R}$ { β_1 is the larger root; $\beta_1 \to y_1$, $\beta_2 \to y_2$
It's possible that y_1 & y_2 may not be linearly independent when $\beta_1 - \beta_2$ is an integer.

b. $\beta_1 = \beta_2$, $\beta_i \in \mathbb{R}$

c. $\beta_1 = \overline{\beta_2}$, $\beta_i \in \mathbb{C}$

So what can we do to obtain a second linearly independent solution if y_1 and y_2 are not linearly independant?

When $\beta \in \mathbb{R}$, then the solutions have the following three situations.

a. $\beta_1 - \beta_2$ is not an integer, the general solution is: $y = c_1 y_1 + c_2 y_2$

b. $\beta_1 = \beta_2$ then $y_2 = y_1 \ln(x-x_0) + \sum_{n=1}^{\infty} b_n (x-x_0)^{n+\beta}$

c. When $\beta_1 - \beta_2$ is an integer

$$y_2 = \alpha y_1 \ln(x-x_0) + \sum_{n=1}^{\infty} b_n (x-x_0)^{n+\beta_2}$$

(α may be zero)

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Let's consider a special case of Bessel equation

Bessel equation $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$, $\nu \geq 0$

$x = 0$ is a regular singular point.

The example we will examine is for $\nu = \frac{1}{2}$

Let's start from ground zero by assuming $y = \sum_{n=0}^{\infty} a_n x^{n+\beta}$ and substitude it into the ODE, we can obtain:

$$\sum_{n=0}^{\infty} (n+\beta)(n+\beta-1) a_n x^{n+\beta} + \sum_{n=0}^{\infty} (n+\beta) a_n x^{n+\beta} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+\beta} + x^2 \sum_{n=0}^{\infty} a_n x^{n+\beta} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+\beta)(n+\beta-1) + (n+\beta) - \nu^2] a_n x^{n+\beta} + \sum_{n=0}^{\infty} a_n x^{n+\beta+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+\beta)^2 - \nu^2] a_n x^{n+\beta} + \sum_{n=2}^{\infty} a_{n-2} x^{n+\beta} = 0$$

The coefficient of x^β term: $\beta^2 - \nu^2 = 0$, $\beta = \pm \frac{1}{2}$

Note: The indicial equation of Bessel equation of order ν is:

$$\beta^2 - \nu^2 = 0$$

vs. $\beta(\beta-1) + p_0\beta + q_0 = 0$,

$xp(x) = p_0 + \cdots$; $x^2 q(x) = q_0 + \cdots$

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For Bessel equation,

$p(x) = 1/x \Rightarrow xp(x) = 1 \;,\; = P_0 + P_1 x + \cdots$

$\Rightarrow P_0 = 1 \;,\; P_1 = P_2 = \cdots P_n = 0$

$q(x) = (x^2 - \nu^2)/x^2 \Rightarrow x^2 q(x) = (x^2 - \nu^2) = q_0 + q_1 + \cdots$

$\Rightarrow q_0 = -\nu^2,\; q_1 = 0,\; q_2 = 1,\; q_3 \cdots = 0$

$\left(P_0 = 1,\; q_0 = -\nu^2 \text{ 代入 } \beta(\beta-1) + P_0\beta + q_0 = 0 \right)$

$\Rightarrow \beta(\beta-1) + \beta - \nu^2 = 0 \;,\; \beta^2 - \nu^2 = 0$

$\Rightarrow (\beta^2 - \nu^2) a_0 x^\beta + [(1+\beta)^2 - \nu^2] a_1 x^{1+\beta}$

$+ \sum_{n=2}^{\infty} \{ [(n+\beta)^2 - \nu^2] a_n + a_{n-2} \} x^{n+\beta} = 0$

$\Rightarrow (\beta^2 - \nu^2) a_0 = 0 \;,\; a_0 \neq 0 \Rightarrow \beta^2 - \nu^2 = 0 \Rightarrow \beta^2 = \nu^2$

$\Rightarrow [(1+\beta)^2 - \nu^2] a_1 = 0 \;,\; (1+2\beta) a_1 = 0 \begin{cases} a_1 = 0 & (\beta = \frac{1}{2} \text{ !!}) \\ \beta = -\frac{1}{2} \end{cases}$

From $\beta^2 - \nu^2 = 0 \Rightarrow \beta = \pm\frac{1}{2}$, when $\nu^2 = \frac{1}{4} = (\frac{1}{2})^2$

Let's consider $\nu = \frac{1}{2}$ first,

$[(1+\frac{1}{2})^2 - \frac{1}{4}] a_1 = 0 \Rightarrow a_1 = 0$

$[(n+\beta)^2 - \nu^2] a_n + a_{n-2} = 0 \;,\; n = 2, 3, 4, \ldots$

$\Rightarrow$ Recursive formula:

$$a_n = -\frac{a_{n-2}}{(n+\beta)^2 - \nu^2} = \frac{-a_{n-2}}{n^2 + 2n\beta + \beta^2 - \nu^2} = \frac{-a_{n-2}}{n^2 + 2\beta n}$$

$$= -\frac{a_{n-2}}{n(n+2\beta)} \;;\; \beta = \pm\nu$$

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for $\beta = \frac{1}{2}$, $a_n = \frac{-a_{n-2}}{n(n+1)}$, $n = 2, 3 \cdots$

$a_2 = \frac{-a_0}{2 \cdot 3} = (-1)^1 \frac{a_0}{3!}$

$a_3 = \frac{-a_1}{3 \cdot 4} = 0 \Rightarrow a_3 = a_5 = a_7 = \cdots\cdots = 0$ (奇数項=0)

$a_4 = \frac{-a_2}{5 \cdot 4} = (-1)^2 \frac{a_0}{5 \cdot 4 \cdot 3 \cdot 2} = (-1)^2 \frac{a_0}{5!}$

$a_6 = \frac{-a_4}{7 \cdot 6} = (-1)^3 \frac{a_0}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = (-1)^3 \frac{a_0}{7!}$

$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^{n+\beta} = x^{\frac{1}{2}} [a_0 + a_1 x + a_2 x^2 + \cdots]$

$= x^{\frac{1}{2}} [a_0 - \frac{a_0}{3!} x^3 + \frac{a_0}{5!} x^4 - \frac{a_0}{7!} x^6 \cdots]$

$= \frac{a_0}{\sqrt{x}} [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots] = \frac{a_0}{\sqrt{x}} \sin x$

$\Rightarrow y = \frac{1}{\sqrt{x}} \sin x$

If we assume $y = \frac{1}{\sqrt{x}} u(x)$, (unknown function)

If y is a solution of $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$

What's the equation of u?

$\Rightarrow y = \frac{1}{\sqrt{x}} u \;,\; y' = -\frac{1}{2} x^{-\frac{3}{2}} u + \frac{1}{\sqrt{x}} u' \;,\; y'' = \frac{3}{4} x^{-\frac{5}{2}} u - x^{-\frac{3}{2}} u' + \frac{1}{\sqrt{x}} u''$

$\Rightarrow x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0 \Rightarrow y'' + p(x) y' + q(x) y = 0$

by $y = \frac{u}{\sqrt{x}} \Rightarrow u'' + u = 0 \quad (u = c_1 \cos x + c_2 \sin x)$

$y = \square u \;,\; \Rightarrow u'' + (\;) u = 0$

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The question we asked last time for Bessel equation with $\nu = \frac{1}{2}$ was

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0$$

If we transform the equation by

$$y = \frac{u(x)}{\sqrt{x}},$$

where $u(x)$ is an unknown function.

then we can obtain: $u'' + u = 0$

By comparing $\begin{cases} x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \\ \text{vs.} \\ u'' + u = 0 \end{cases}$

What is your observation?

⇨ The main difference is the 1st derivative term!

⇨ Let's consider a second order ODE

$$y'' + p(x)y' + q(x)y = 0,$$

assume $y = u(x)v(x)$

$$\begin{cases} y' = u'v + uv' \\ y'' = u''v + 2u'v' + uv'' \end{cases}$$

substituting these equation into the ODE, then

⇨ $u''v + 2u'v' + uv'' + p(x)[u'v + uv'] + quv = 0$

⇨ $u''v + (2v' + pv)u' + (v'' + pv' + qv)u = 0$

That is, if we let

$$2v' + pv = 0$$

⇨ $u''v + (v'' + pv' + qv)u = 0$

(Cornical Form)

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By $2v' + pv = 0$

$\frac{dv}{v} = -\frac{1}{2}p\,dx$ ⇨ $v = \cancel{C}^{=1} e^{-\frac{1}{2}\int p\,dx}$

Back to Bessel equation.

$p(x) = 1/x$,

$\because v = e^{-\frac{1}{2}\int \frac{dx}{x}} = e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$!!!

Back to Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Note:

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

(Modified Bessel equation)

Q: Can you see the relation between them ...? Think!

$x = 0$ ⇨ regular singular point

by Forbenius method,

we can assume $y = \sum_{n=0}^{\infty} a_n x^{n+\beta}$

$$\underbrace{[\beta(\beta-1) + \beta - \nu^2]}_{\beta^2 - \nu^2 \text{, Indicial equation}} a_0 x^{\beta} + [(\beta+1)\beta + (\beta+1) - \nu^2] a_1 x^{\beta+1} + \sum_{n=2}^{\infty} \{[(n+\beta)(n+\beta-1) + (n+\beta) - \nu^2] a_n + a_{n-2}\} = 0$$

The Indicial equation

$(\beta^2 - \nu^2)a_0 = 0$, ⇨ $\beta = \pm\nu$ ($\because a_0 \neq 0$)

$$a_n = -\frac{a_{n-2}}{(n+\beta)^2 - \nu^2}, \quad n = 2, 3, \cdots$$

turn~!

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$$\Rightarrow a_n = -\frac{a_{n-2}}{(n+\beta)^2-\nu^2}, \quad n=2,3,\cdots$$

Let consider the case that $\beta = +\nu \ (\nu \geq 0)$

$\Rightarrow a_1 = 0$;

$$a_n = -\frac{a_{n-2}}{(n+\nu)^2-\nu^2} = -\frac{a_{n-2}}{n^2+2n\nu} = -\frac{a_{n-2}}{n(n+2\nu)}, \quad n=2,3,4,\cdots$$

$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{2^2(1+\nu)}$$

$$a_4 = -\frac{a_2}{4(4+2\nu)} = -\frac{a_2}{2^3(2+\nu)} = (-1)^2\frac{a_0}{2^5(2+\nu)(1+\nu)}$$

$$\Rightarrow (-1)^2\frac{a_0}{2^4\cdot 2\cdot(2+\nu)(1+\nu)}$$

$$a_6 = -\frac{a_4}{6(6+2\nu)} = -\frac{a_4}{2^2\cdot 3(3+\nu)} = (-1)^3\frac{a_0}{2^4\cdot 3(3+\nu)(2+\nu)(1+\nu)}$$

$$\Rightarrow (-1)^3\frac{a_0}{2^6\cdot \underbrace{3\cdot 2}_{3!}(3+\nu)(2+\nu)(1+\nu)}$$

$\vdots$

$a_1 = a_3 = a_7 = \cdots = 0$

$$a_{2k} = (-1)^k\frac{a_0}{2^{2k}\cdot k!\,(k+\nu)(k+\nu-1)\cdots(1+\nu)}$$

$$\therefore\ y(x) = \sum_{n=0}^{\infty} a_n x^{n+\beta} = \sum_{n=0}^{\infty} a_{2k} x^{2k+\nu} = \sum_{k=1}^{\infty}\frac{(-1)^k a_0}{2^{2k}\cdot k!\,(k+\nu)(k+\nu-1)\cdots(1+\nu)}x^{2k+\nu}$$

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If $\nu = m$, ($m \geq 0$, integer)

$$y(x) = \sum_{k=1}^{\infty}\frac{(-1)^k\cdot a_0\cdot m!}{2^{2k}\cdot k!\,(m+k!)}\cdot x^{2k+m}$$

(Gamma function : $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$

recall $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$

$\Gamma(x+1) = x\Gamma(x)$ (自己證))

$$y(x) = \sum_{k=1}^{\infty}\frac{(-1)^k a_0}{2^{2k}\cdot k!\,(k+\nu)(k+\nu-1)\cdots(1+\nu)}x^{2k+\nu}$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^k a_0\,\Gamma(\nu+1)}{2^{2k}\cdot k!\,\Gamma(k+\nu+1)}x^{2k+\nu}$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^k a_0\,\Gamma(\nu+1)\cdot 2^{\nu}}{2^{2k+\nu}\,k!\,\Gamma(k+\nu+1)}x^{2k+\nu}$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^k a_0\cdot 2^{\nu}\,\Gamma(\nu+1)}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

$$= a_0\,2^{\nu}\,\Gamma(\nu+1)\,J_\nu(x)$$

— The first kind, Bessel Function (

(the Bessel Function of first kind.)

If ν = Integer $= m$, $J_m(x) = (-1)^m J_m(x)$

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Bessel equation and Bessel functions

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Bessel's equation of order ν

At least, we can obtain an independent solution $J_\nu(x)$; (Bessel function of the first kind.)

$$J_\nu(x) = \sum \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

when ν is an integer, say $\nu = n$ (positive integer)

$\Rightarrow J_{-n}(x) = (-1)^n J(x)$ (i.e., they are linearly dependent)

If $\nu_1 - \nu_2$ is not an integer, the general solution is $y(t) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$

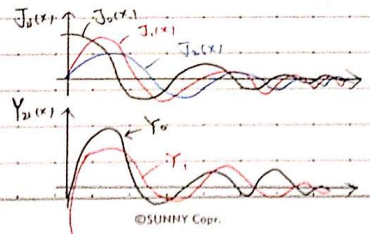
So, mathematicians defined

$$Y_\nu(x) \equiv \frac{\cos \pi\nu \, J_\nu(x) - J_{-\nu}(x)}{\sin \pi\nu}$$

Bessel's function of the 2nd kind.

Then the general solution of the Bessel equation is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$



Try to use Google to see what are the modified Bessel functions of the 1st & 2nd kind: $I_n(x)$ & $K_n(x)$

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Fourier Series and Transformation

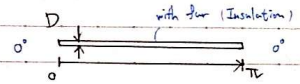
—— A method to represent a "function"

Fourier series : periodic function (period $= 2l$)

Fourier transform : $2l \to \infty$, so it can treat both periodic and non-periodic functions.

The Problem Fourier tried to solve was the heat conduction problem of a thin rod

Two keywords : Heat conduction.
thin rod.



Thin rod : diameter ≪ length

$T = T(x,t)$; Initial condition (time $= 0$) :

$T(x, 0) = f(x)$, } known conditions

Boundary Conditions :

$T(0,t) = T(\pi,t) = 0°C$

What is the equation for the temperature distribution? Heat conduction : governing equation

$$\oint \delta W = \oint \delta Q \Rightarrow \oint (\underbrace{\delta W - \delta Q}_{dE}) = 0$$

dE — energy

$\delta Q - \delta W$ (heat original)

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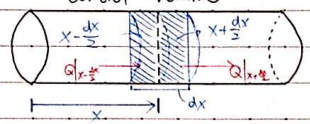
control volume



at $x-\frac{1}{2}dx$: $Q\big|_{x-\frac{1}{2}dx}$
$x+\frac{1}{2}dx$: $Q\big|_{x+\frac{1}{2}dx}$, $Q\big|_{x-\frac{1}{2}dx} - Q\big|_{x+\frac{1}{2}dx} = dE$

$= m s \Delta T = m \underset{\text{capacity}}{c} \Delta T$ 熱容量 ceramic

specific heat $\begin{cases} \text{constant volume} & C.V. \\ \text{constant pressure} & C.P. \end{cases}$

Fouries Law of conduction

cross sectional area A

$\begin{cases} Q \propto \Delta T & \text{spatial change (空間相關)} \\ Q_x \propto \frac{\partial T}{\partial x} \Rightarrow Q_x = -K_x \frac{\partial T}{\partial x} A \Delta t \end{cases}$

K_x: thermal conductivity

$$\Rightarrow \underbrace{-K_x \frac{\partial T}{\partial x} A \Delta t\Big|_{x-\frac{1}{2}dx}}_{Q_{x-\frac{1}{2}dx}} - \underbrace{\left(-K_x \frac{\partial T}{\partial x} A \Delta t\right)\Big|_{x+\frac{1}{2}dx}}_{Q_{x+\frac{1}{2}dx}} = m c \frac{\Delta T}{A \cdot \Delta t}$$

$$\Rightarrow K_x\left[-\frac{\partial T}{\partial x}\Big|_{x-\frac{1}{2}dx} - \left(-\frac{\partial T}{\partial x}\Big|_{x+\frac{1}{2}dx}\right)\right] = \frac{mc}{A}\frac{\Delta T}{\Delta t}$$

Taylor expansion $f(x) = f(x_0) + f'(x_0)(x-x_0)$

$$\Rightarrow K_x\left\{-\frac{\partial T}{\partial x}\Big|_{x-\frac{1}{2}dx} + \left(\frac{\partial T}{\partial x}\Big|_{x-\frac{1}{2}dx} + \frac{d}{dx}\left(\frac{\partial T}{\partial x}\right)\Big|_{x-\frac{1}{2}dx}\left[(x+\tfrac{1}{2}dx) - (x-\tfrac{1}{2}dx)\right]\right)\right\}$$

$$= K_x\left(\frac{d^2T}{dx^2}\Big|_{x-\frac{1}{2}dx}\right) = \frac{mc}{A}\frac{\Delta T}{\Delta t}$$

$$\therefore \Rightarrow \frac{mc}{A}\frac{\Delta T}{\Delta t} = K\frac{d^2T}{dx^2}\cdot dx = \frac{(\rho A dx)}{A} c \frac{\Delta T}{\Delta t}$$

when $\Delta t \to 0$, $\frac{\Delta T}{\Delta t} \to \frac{dT}{dt}$; $\therefore \rho c \frac{dT}{dt} = K\frac{d^2T}{dx^2}$

But T is $T(x,t)$, so $\Rightarrow \rho c \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} K$

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{K}{\rho c}\frac{\partial^2 T}{\partial x^2} = \kappa \frac{\partial^2 T}{\partial x^2}$$

Thermal diffusivity 熱擴散係數

$$\left(\begin{array}{l} \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \\ \frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \end{array}\right) \Rightarrow \frac{\partial u}{\partial t} = \rho \frac{\partial^2 u}{\partial x^2};$$

$$\begin{cases} \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, & 0<x<\pi, t>0 \\ T(0,t) = T(\pi,t) = 0^\circ C \\ T(x,0) = f(x) \end{cases}$$

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Fourier analytic theory of heat conduction

By the 1st law of thermodynamics and Fourier law of heat conduction, we obtained a partial differential equation for the one-dimensional, transient heat conduction situation as

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \pi$$

(Isotropic material)

subjected to the following condition :

a. Bounding conditions. $T(0,t) = T(\pi,t) = 0$

b. Initial conditions. $T(x,0) = f(x)$

A common method to solve PDEs is to convert them to ODEs. One of the methods is "separation of variable". "assumption"

That is by assuming $T(x,t) = h(x)\,g(t)$

$$\Rightarrow \frac{1}{K g(t)}\frac{dg(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = \text{Constant}$$

We have three choices for this "constant":

a. "Constant" < 0

b. "Constant" $= 0$ (×)

c. "Constant" > 0 (×)

let "constant" $= -\beta^2$

i.e., $\frac{1}{K g(t)}\frac{dg(t)}{dt} = \frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = -\beta^2$

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So we have two ODEs :

$$\begin{cases} \frac{1}{K}\frac{1}{g(t)}\frac{dg(t)}{dt} = -\beta^2 \ (K>0) \\ \frac{1}{h(x)}\frac{d^2h(x)}{dt^2} = -\beta^2 \quad \left(\frac{d^2h(x)}{dx^2}+\beta^2 h(x)=0\right) \end{cases}$$

$$\therefore \Rightarrow \begin{cases} g(t) = Ce^{-K\beta^2 t} \\ h(t) = C_1\cos\beta x + C_2\sin\beta x \end{cases}$$

$$\Rightarrow T(x,t) = \underline{(CC_1\cos\beta x + CC_2\sin\beta x)\, e^{-K\beta^2 t}}, \ \Delta$$

Before we combine them together

Let's use the boundary conditions to see what can we obtain first.

$$\begin{cases} T(0,t) = h(0)g(t) = 0 \\ T(\pi,t) = h(\pi)g(t) = 0 \end{cases} \Rightarrow h(0) = h(\pi) = 0$$

From $h(0) = h(\pi) = 0$ & $h(x) = C_1\cos\beta x + C_2\sin\beta x$

$\Rightarrow h(0) = C_1 = 0$, $\therefore h(x) = C_2\sin\beta x$

$h(\pi) = C_2\sin(\pi\beta) = 0$,

$$\Rightarrow \begin{cases} C_2 = 0 \Rightarrow h(\pi)\equiv 0,\ T(x,t)\equiv 0 \ (X) \\ \sin\beta\pi = 0 \Rightarrow \beta\pi = n\pi,\ n = \pm1, \pm2, \cdots \end{cases}$$

i.e., $\beta = n$

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$$\therefore T(x,t) = Ce^{-K\beta^2 t}C_2\sin nx, \quad n = \pm1, \pm2, \cdots$$
$$= b_n e^{-K\beta^2 t}\sin nx, \quad n = \pm1, \pm2, \cdots$$

$$\Rightarrow T(x,t) = \sum_{n=-\infty}^{\infty} b_n e^{-K\beta^2 t}\sin nx$$

But $\sin(-x) = -\sin x$ → linearly dependent (不獨立的解)

$$\therefore T(x,t) = \sum_{n=1}^{\infty} b_n e^{-K\beta^2 t}\sin nx$$

We need to know b_n which must be obtained from the initial condition $T(x,0) = f(x)$

$$\Rightarrow T(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

If $\begin{cases} \frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}, \quad 0<x<l \\ T(0,t) = T(l,t) = 0 \\ T(x,0) = f(x) \end{cases}$

$$\left.\begin{aligned} T(0,t) &= h(0)g(t) = 0 \\ T(l,t) &= h(l)g(t) = 0 \end{aligned}\right\} h(0) = h(l) = 0$$

From $h(0) = h(l) = 0$ & $h(x) = C_1\cos\beta x + C_2\sin\beta x$

$$\begin{cases} h(0) = C_1 = 0, \ \therefore h(x) = C_2\sin\beta x \\ h(\pi) = C_2\sin\beta l = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \Rightarrow h(x)=0,\ T(x,t)=0 \ (X) \\ \sin\beta l = 0 \Rightarrow \beta l = n\pi,\ n = \pm1, \pm2, \cdots \end{cases}$$

$$\Rightarrow \beta = \frac{n\pi}{l}$$

$$\therefore T(x,t) = Ce^{-K\frac{n^2\pi^2}{l^2}t}C_2\sin\frac{n\pi}{l}x, \quad n = \pm1, \pm2, \cdots$$
$$= b_n e^{-K\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x, \quad n = \pm1, \pm2, \cdots$$

i.e., $T(x,t) = \sum_{n=1}^{\infty} b_n e^{-K\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x$

From initial condition : $T(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$

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For a period function $f(x)$ of period $2l$,

We want to find

known

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$$f(x) = \underbrace{\tfrac{1}{2}[f(x)+f(-x)]}_{\text{even function}} + \underbrace{\tfrac{1}{2}[f(x)-f(-x)]}_{\text{odd function}}$$

(任何函数都可用奇函数&偶函数表示)

compare~ Taylor : $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$

$$a_n = \frac{1}{n!}\frac{d^n f(x)}{dx^n}\Big|_{x=x_0}$$

If we convert x to t, $2l$ to T

$2l = T$, $l = T/2$,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{n2\pi}{T}t\right)$$

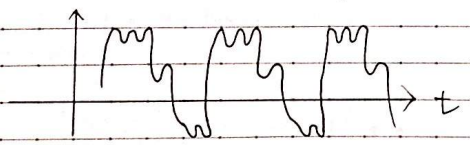
$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \underbrace{a_0}_{DC} + a_1 \cos\omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin\omega t + b_2 \sin 2\omega t \cdots$$

看該頻率之振幅, 可作訊號分析

$\omega = \frac{2\pi}{T}$ = fundamartal radian frequency (基頻)

$f(t) = \frac{1}{2} + \cos t + 3\cos 2t$, Draw it!



How to find it $f(x)$?!

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For a periodic function $f(x)$, $f(x) = f(x+2l)$, $2l$ = period, its Fourier series expansion can be expressed as :

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$, $\theta = \omega t$

or $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$ → physical meaning

or $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nkx + b_n \sin nkx)$; $k = \frac{2\pi}{\lambda}$ = wave number

Note : in material science → Microscopic
strength of materials / statics } Macroscopic

periodic function
complex representation

How can we convert it to a complex form ?

$i = \sqrt{-1}$; $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$; $\begin{cases}\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})\end{cases}$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{i\frac{n\pi}{l}x} + e^{-i\frac{n\pi}{l}x}\right) + \frac{b_n}{2i}\left(e^{i\frac{n\pi}{l}x} - e^{-i\frac{n\pi}{l}x}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\underbrace{\left(\frac{a_n}{2} - \frac{ib_n}{2}\right)}_{C_n} e^{i\frac{n\pi}{l}x} + \left(\frac{a_n}{2} + \frac{ib_n}{2}\right)e^{-i\frac{n\pi}{l}x}\right]$$

$$C_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\underbrace{\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx}_{a_n} - \underbrace{\frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx}_{ib_n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi}{l}x - i\sin\frac{n\pi}{l}x\right)dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{n\pi}{l}x}\,dx$$

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In contrast, $\frac{a_n}{2} + \frac{ib_n}{2} = \frac{1}{2l}\int_{-l}^{l} f(x) e^{+i\frac{n\pi}{l}x} dx = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-i(\frac{n\pi}{l})x} dx$

$= C_{-n}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[C_n e^{\frac{i\pi n x}{l}} + C_{-n} e^{-\frac{i\pi n x}{l}} \right]$$

$$= a_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{i\frac{n\pi}{l}x}$$

But $C_0 = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{0\pi}{l}x} dx = \frac{1}{2l}\int_{-l}^{l} f(x)\, dx = a_0$

$$\therefore f(x) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{i\frac{n\pi}{l}x} = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{l}x}$$

What do you think about the meaning of

$\frac{1}{2l}\int_{-l}^{l} f^2(x)\, dx$? ⇒ $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V^2(t)\, dt$, a mean power

$$f^2(x) = a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 \cos^2 \frac{n\pi}{l}x + b_n^2 \sin^2 \frac{n\pi}{l}x\right) +$$

$$\boxed{\begin{aligned} &2(a_0 a_1 \cos\tfrac{n\pi}{l}x) + 2a_0 a_2 \cos\tfrac{2\pi}{l}x \cdots + \\ &2 a_1 b_1 \cos\tfrac{\pi}{l}x \sin\tfrac{\pi}{l}x + \cdots\cdots \end{aligned}}$$

cross product terms

$$\Rightarrow \frac{1}{2l}\int_{-l}^{l} f^2(x)\, dx = \frac{a_0^2}{2l}\int_{-l}^{l} dx + \frac{1}{2l}\int_{-l}^{l} \sum_{n=1}^{\infty} \left[a_n^2 \cos^2 \frac{n\pi}{l}x + b_n^2 \sin^2 \frac{n\pi}{l}x\right] dx$$

$$+ \frac{1}{2l}\int_{-l}^{l} \underbrace{\cancel{(\text{cross product terms})}}_{=0}\, dx$$

(orthogonal property of sine and cosine functions)

and
$$\begin{cases} \frac{1}{2l}\int_{-l}^{l} a_n^2 \cos^2 \frac{n\pi}{l}x\, dx = \frac{a_n^2}{2l}\int_{-l}^{l} \cos^2 \frac{n\pi}{l}x\, dx = \frac{a_n^2}{2l}\cdot l = \frac{a_n^2}{2} \\ \frac{1}{2l}\int_{-l}^{l} b_n^2 \sin^2 \frac{n\pi}{l}x\, dx = \frac{b_n^2}{2} \end{cases}$$

⇒ i.e. $\frac{1}{2l}\int_{-l}^{l} f^2(x)\, dx = a_0^2 + \sum_{n=1}^{\infty} \left(\frac{a_n^2 + b_n^2}{2}\right)$

v.s. $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l}x + b_n \sin \frac{n\pi}{l}x\right)$

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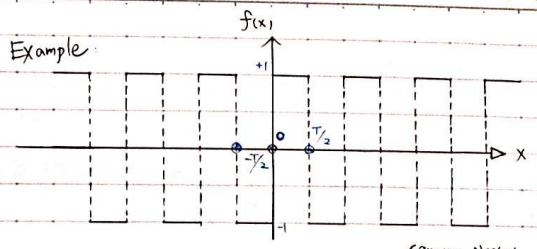
Example:



square wave.

What is its Fourier Series ?

Observations : 1. periodic function

2. odd function

3. mean value over a period = 0 (平均)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l}x + b_n \sin \frac{n\pi}{l}x\right)$$

$2l = T$, $l = T/2$

⇒ i.e., $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{T}x + b_n \sin \frac{2n\pi}{T}x\right)$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\, dx = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, dx = \frac{1}{T}\left[\int_{-T/2}^{0} (-1)\, dx + \int_{0}^{T/2} (1)\, dx\right]$$

$= \frac{1}{T}\left[-T/2 + T/2\right] = 0$ (mean value over a period = 0)

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi}{l}x\, dx = \frac{2}{T}\int_{-T/2}^{T/2} f(x) \cos \frac{2n\pi}{T}x\, dx$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} (-1) \cos \frac{2n\pi}{T}x\, dx + \int_{0}^{T/2} (1) \cos \frac{2n\pi}{T}x\, dx\right]$$

$$= \frac{2}{T}\frac{T}{2n\pi}\left[-\sin \frac{2n\pi}{T}x \Big|_{-T/2}^{0} + \sin \frac{2n\pi}{T}x \Big|_{0}^{T/2}\right] = 0, \quad n = 1, 2, 3 \cdots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi}{l}x\, dx = \frac{2}{T}\int_{-T/2}^{T/2} f(x) \sin \frac{2n\pi}{T}x\, dx$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} (-1) \sin \frac{2n\pi}{T}x\, dx + \int_{0}^{T/2} (1) \cos \frac{2n\pi}{T}x\, dx\right]$$

$$= \frac{2}{T}\frac{T}{2n\pi}\left[-\cos \frac{2n\pi}{T}x \Big|_{-T/2}^{0} + \cos \frac{2n\pi}{T}x \Big|_{0}^{T/2}\right]$$

$$= \frac{1}{n\pi}\left[-(1 - \cos n\pi) + (\cos n\pi - 1)\right] = \frac{2}{n\pi}(1 - \cos n\pi), \quad n = 1, 2, 3 \cdots$$

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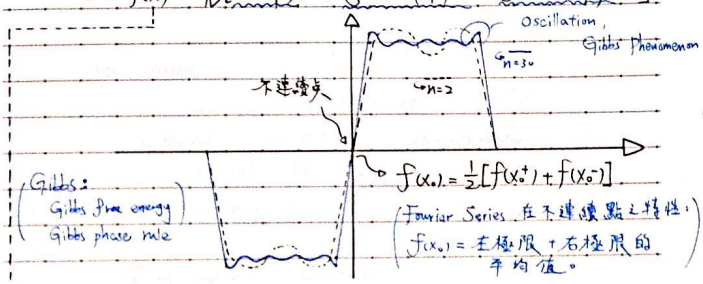
$$\Rightarrow b_n = \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}[1-(-1)^n] = \begin{cases} \frac{4}{n\pi}, & n=1,3,5\cdots \\ 0, & n=2,4,6\cdots \end{cases}$$

$$\therefore f(x) = \underbrace{\frac{4}{\pi}\left[\sin\frac{2\pi}{T}x\right.}_{b_1} + \underbrace{\frac{1}{3}\sin 3\left(\frac{2\pi}{T}\right)x}_{b_3} + \underbrace{\frac{1}{5}\sin 5\left(\frac{2\pi}{T}\right)x}_{b_5}\cdots\Big]$$



---Mean energy $\propto \frac{1}{2}\left(\frac{4}{\pi}\right)^2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\cdots\right]$

如只在基頻的地方有能量

For this example, energy is contained in

Fundamental frequency (b_1) : $\frac{1}{2}b_1^2 = \frac{1}{2}\left(\frac{4}{\pi}\right)^2$

3rd harmonics (b_3) : $\frac{1}{2}b_3^2 = \frac{1}{2}\left(\frac{4}{\pi}\right)^2\left(\frac{1}{9}\right)$

5rd harmonics (b_5) : $\frac{1}{2}b_5^2 = \frac{1}{2}\left(\frac{4}{\pi}\right)^2\left(\frac{1}{25}\right)$

⋮

From this example, we know some simplifications:

For an even function : $f(x) = f(-x)$, $b_n = 0$;

$$\begin{cases} f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x, \\ a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx \\ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \end{cases}$$

兩者誤差最小時, a_n, b_n 之值, 就是 Fourier 的係數

方法 $\Rightarrow$ $f(x) \approx a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$

無窮級數

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use me.

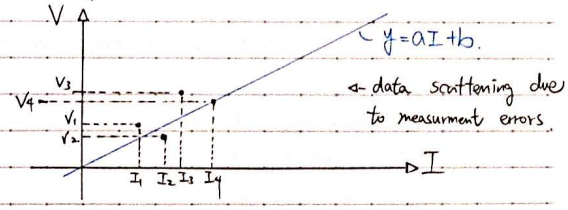
If we express a periodic function $f(x)$ by $a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$, under what conditions, the difference between them will be minimum? This kind of problem is commonly referred as the best curve fit for a given order.

For example

$V = RI$ (ohm's law)

How can we obtain R from ohm's law?

By measurements! How?



How can we obtain this approximated curve? assume $y = aI + b$ (need $a \neq 0$, $b \neq 0$)

$V_i - y_i = V_i - (aI_i + b)$

measurment voltage — linear curve voltage

$\sum_{i=1}^{k}(V_i - y_i) \approx 0$ (有正誤差&負誤差)

$$\Rightarrow \sqrt{\frac{\sum_{i=1}^{k}(V_i - y_i)^2}{K}}$$

r m s

root mean square

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$$E = \frac{\sum_{n=1}^{k}(v_i - y_i)^2}{k} = \frac{\sum_{n=1}^{k}[v_i - (aI_i + b)]^2}{K}$$

minimize $E = E(a,b)$

$\partial E/\partial a = \partial E/\partial b = 0$ to obtain a & b

error :

$$\frac{1}{2l}\int_{-l}^{l}\left\{f(x) - \left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)\right]\right\}dx$$

Not what we need!!

$$\Rightarrow E = \frac{1}{2l}\int_{-l}^{l}\left\{f(x) - \left[a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right]\right\}^2 dx$$

i.e., we can use following conditions :

$\partial E/\partial a_0 = 0 \Rightarrow a_0 = ?$

$\partial E/\partial a_n = 0 \Rightarrow a_n = ?$

$\partial E/\partial b_n = 0 \Rightarrow b_n = ?$

$$\partial E/\partial a_0 = 0$$

$$= \frac{\partial}{\partial a_0}\frac{1}{2l}\int_{-l}^{l}\left\{f(x) - \left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)\right]\right\}^2 dx$$

$$= \frac{1}{2l}\int_{-l}^{l}\frac{\partial}{\partial a_0}\left\{f(x) - \left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)\right]\right\}^2 dx$$

$$= \frac{1}{2l}\int_{-l}^{l} 2\left\{f(x) - \left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)\right]\right\}(-1)dx$$

$$= \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{l}\int_{-l}^{l} a_0 dx + \frac{1}{l}\int_{-l}^{l}\sum\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)dx$$

$$= \frac{a_0}{l}(2l) = 2a_0$$

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$$\therefore\ a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx \ !!!$$

Simlarly

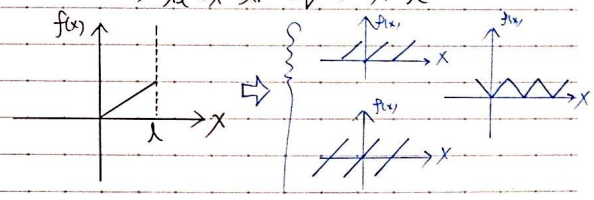
$$\begin{cases} a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx \\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx \end{cases}$$

Sometimes, in engineering applications, we only know $f(x)$ $x_l < x < x_r$, $x_r - x_l$ as a portion of a period

The most commonly adopted method is to treat $x_r - x_l$ as half a period.

i.e., $x_r - x_l = l$

$\hookrightarrow x_l < x < x_r \Rightarrow 0 < x < l$



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For a periodic function $f(t)$ with a period of $2l$, its Fourier Series can be written as

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)$$

$$\begin{cases} a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx, \\ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\, dx \end{cases} \qquad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x\, dx$$

What can we do when $f(x)$ is not a periodic function?

Can we extend Fourier Series to cover non-periodic functions?

A possible thinking is the period is infinite!

For x as a spatial coordinates, we can use $2l = \lambda$ to replace l (i.e., $l = \lambda/2$)

wavelength

$\frac{2\pi}{\lambda} = k$ (wave number)

$\frac{2\pi}{T} = \omega$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nkx + b_n \sin nkx \right)$$

$\Rightarrow$ If we define $k_n = nk$,

then $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos k_n x + b_n \sin k_n x \right)$

In contrast, if we consider a time varying function $f(t)$ where t is time,

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin \underbrace{n\omega}_{\omega_n} t \right)$$

(i.e., $2l = T$ = period)

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$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos W_n t + b_n \sin W_n t)$$

$$\begin{cases} a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos W_n t\,dt \\ b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin W_n t\,dt \end{cases}$$

$$\Rightarrow f(t) = \underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt}_{a_0} + \sum_{n=1}^{\infty}\left\{\underbrace{\left[\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos W_n t\,dt\right]}_{a_n}\cos W_n t + \underbrace{\left[\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin W_n t\,dt\right]}_{b_n}\sin W_n t\right\}$$

For $W_n = nW = n\frac{2\pi}{T}$,

$\Rightarrow \Delta W_n = W_{n+1} - W_n = (n+1)\frac{2\pi}{T} - n\frac{2\pi}{T} = 2\pi/T$

difference

$\Rightarrow \frac{1}{T} = \frac{\Delta W_n}{2\pi}$

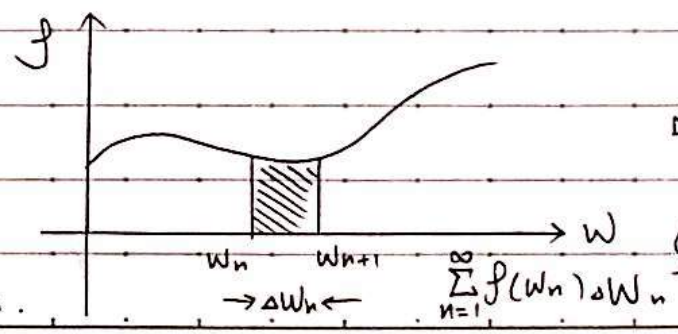
$$\Rightarrow f(t) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\left[\int_{-T/2}^{T/2} f(t)\cos W_n t\,dt\right]\cos W_n t + \left[\int_{-T/2}^{T/2} f(t)\sin W_n t\,dt\right]\sin W_n t\right\}\Delta W_n$$

when $T \to \infty$, what happens to the above formula?

$\Rightarrow \lim_{T\to\infty}\frac{1}{T} = \frac{dW}{2\pi}$!

In general, we have $\int_{-\infty}^{\infty}|f(t)|\,dt$ = finite ($<\infty$)

then $\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \to 0$



when $\Delta W_n \to 0$

$\Rightarrow \lim_{\Delta\to 0}\Delta W_n = dW_n$

$\sum_{n=1}^{\infty} f(W_n)\Delta W_n$. $\Rightarrow \int f(W)\,dW$!

Calculus.

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$$\therefore f(t) = \frac{1}{\pi}\int_0^\infty \left\{ \left[\int_{-\infty}^{\infty} f(t)\cos wt\,dt \right]\cos wt + \left[\int_{-\infty}^{\infty} f(t)\sin wt\,dt\right]\sin wt \right\} dw$$

(哑变量!)

$$\left(\Rightarrow \quad f(t) = \frac{1}{\pi}\int_0^\infty \left\{ \left[\int_{-\infty}^{\infty} f(\xi)\cos w\xi\,d\xi \right]\cos wt + \cdots \right. \right)$$

$$\therefore f(t) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)\cos w\xi\,d\xi\,\cos wt\,dw$$

$$+ \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)\sin w\xi\,d\xi\,\sin wt\,dw$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)[\cos w\xi\cos wt + \sin w\xi\sin wt]\,d\xi\,dw$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)\cos w(\xi - t)\,d\xi\,dw \quad , \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)\left[\underbrace{e^{iw(\xi-t)}}_{\text{let } w=-\eta} + e^{-iw(\xi-t)}\right] d\xi\,dw$$

i.e $$f(t) = \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)e^{i(-\eta)(\xi-t)}\,d\xi\,d(-\eta)$$

$$+ \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\,d\xi\,dw$$

$$= \frac{-1}{2\pi}\int_0^{-\infty}\int_{-\infty}^{\infty} f(\xi)\underbrace{e^{-i\eta(\xi-t)}\,d\xi\,d\eta}_{\text{let } \eta = w} + \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\,d\xi\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\,d\xi\,dw + \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\,d\xi\,dw$$

$$\Rightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\,d\xi\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{-iw\xi}\,d\xi\right]e^{iwt}\,dw$$

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Fourier Series

$$\begin{cases} f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x\right) \\ a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \quad , \quad \begin{cases} a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\,dx \\ b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x\,dx \end{cases} \end{cases}$$

Two situations :

x : spatial coordinate

period $2l \to \lambda$ (wave length)

From λ , we have the wave number

$k = 2\pi/\lambda$,

Then $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nkx + b_n \sin nkx)$

If we let $nk = k_n$,

Then $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos k_n x + b_n \sin k_n x)$

⇨ multiple dimension issue

In contrast , x : time coordinate ⇨ $x \to t$

Then the corresponding Fourier series

is : $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n t + b_n \sin w_n t)$

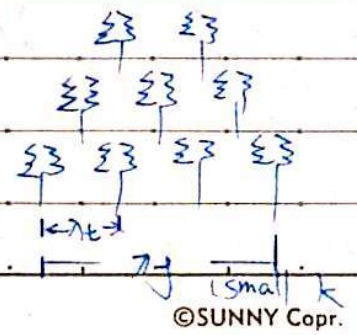
where $w_n = nw = n\frac{2\pi}{T}$

high frequency (small wave length)

見樹不見林

low frequency

見林不見樹 (large wave length)



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when the period approaches infinite) (in terms of t)

$a_n : w_n,\ n=1,2,\cdots \Rightarrow a(w)$

$$\Rightarrow f(t) = \frac{1}{\pi}\int_0^\infty \underbrace{\int_{-\infty}^{\infty} f(\xi)\cos w\xi\, d\xi}\cos wt\, dw + \frac{1}{\pi}\int_0^\infty \underbrace{\int_{-\infty}^{\infty} f(\xi)\sin w\xi\, d\xi}_{b_n \Rightarrow b(w)}\sin wt\, dt$$

$$\begin{cases} a(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\, dt \\ b(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt \end{cases}$$

$$\Rightarrow f(t) = \int_0^\infty [a(w)\cos wt + b(w)\sin wt]\, dw$$ Fourier Integral

$$\Rightarrow \underline{\underline{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-t)}\, d\xi\, dw}}$$

Fourier Transform !

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2\int_{-\infty}^{\infty}\Big(\underbrace{\int_{-\infty}^{\infty} f(\xi)e^{-iw\xi}\, d\xi}_{w}\Big)e^{iwt}\, dw$$

$$\left(\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\, dt\right)$$

$$\begin{cases} F(w) \equiv \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}\, dt \\ f(t) \equiv \mathcal{F}^{-1}[F(w)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)e^{iwt}\, dw \end{cases}$$

Fourier Transform Pair

Remind that $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right)$

$$= \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{l}x}$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\frac{n\pi}{l}x}\, dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-ik_n x}\, dx$$

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$\frac{x}{\lambda}$: 在波長中佔的比例
$\frac{t}{T}$ = 在週期中佔的比例

For 1-D wave

$$u(x,t) = \cos(kx - wt + \phi)$$
$$= \cos\left(2\pi \cdot \frac{x}{\lambda} - 2\pi \cdot \frac{t}{T} + \phi\right)$$

$\frac{w}{k} = \frac{2\pi\nu}{2\pi/\lambda}$ ... frequency
$= \nu\lambda$
= phase velocity
wave velocity
phase group

let $\phi = 0$, $u(x,t) = \cos(kx - wt)$
$$= \cos k\left(x - \frac{w}{k}t\right)$$

$\Rightarrow u(x,y,z,t) = \cos(k_x x + k_y y + k_z z \pm wt)$
$$u(\vec{r},t) = \cos(\vec{k}\cdot\vec{r} \pm wt)$$
$\Rightarrow \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{k} = k_x\vec{i} + k_y\vec{j} + k_z\vec{k}$

If $f(t)$ is an even function,
$$f(t) = f(-t);$$

$F_c(w) = \mathcal{F}_c[f(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\cos\xi\, d\xi$
$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(w)\cos wt\, dw$
(FT pair of cosine)

Fourier Integral

$$f(t) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(\xi)\cos w\xi\, d\xi\right)\cos wt\, dw$$

$\Rightarrow$ Fourier cosine transform

In contrast, $f(t)$ is an odd function,
$$f(t) = -f(-t)$$

then $$f(t) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(\xi)\sin w\xi\, d\xi\right)\sin wt\, dw$$

$\Rightarrow$ Fourier sine transform

$F_s(w) = \mathcal{F}_s[f(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\sin\xi\, d\xi$
$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(w)\sin wt\, dw$
(FT pair of sine)

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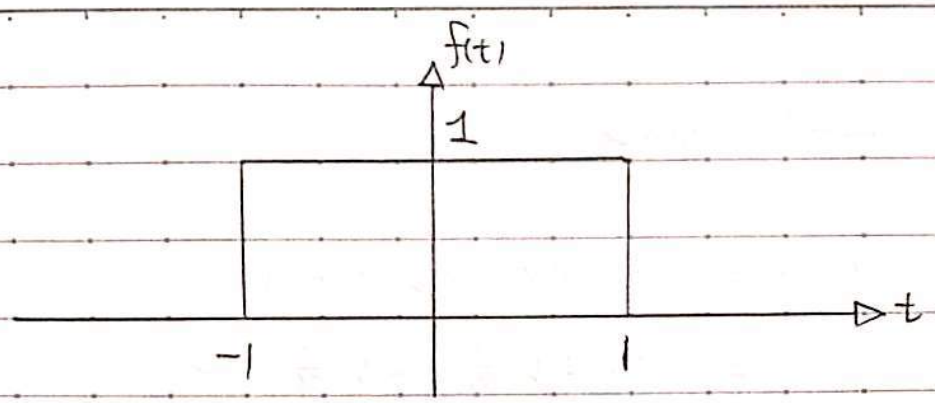


Fourier Trasnform

$$\mathcal{F}[f(t)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-1}^{1} f(t) e^{-i\omega t} dt$$

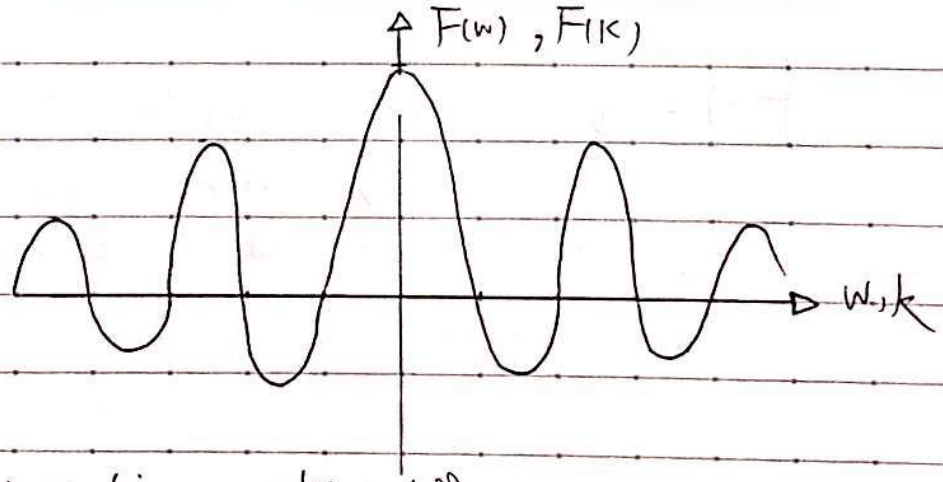
$$= \sqrt{\frac{1}{2\pi}} \left(\frac{1}{-i\omega}\right) \left[e^{-i\omega t} \Big|_{-1}^{1} \right]$$

$$\left(\sin\theta = \frac{1}{2i} e^{i\theta} - e^{-i\theta} \right)$$

$$= \sqrt{\frac{1}{2\pi}} \left(-\frac{1}{i\omega}\right) \left[e^{-i\omega} - e^{i\omega} \right] = \sqrt{\frac{1}{2\pi}} \left(-\frac{1}{i\omega}\right) (-2i \sin\omega)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

⇨



⇒ optics : slit diffraction

(Interference) ⇨ Fourier optics

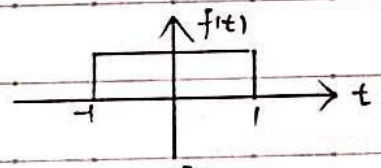
Fourier Integral

$$f(t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(\xi)\cos\xi\, d\xi \cos wt + \int_{-\infty}^{\infty} f(\xi)\sin\xi\, d\xi \sin wt \right] dw$$

Fourier transform pair

$$\begin{cases} \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt = F(w) \\ f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dt \end{cases}$$

Last time, we evaluate



$$\Rightarrow \mathcal{F}[f(t)] = \sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}$$

exercise: $\mathcal{F}[e^{-t^2/\sigma^2}] = ?$ $\sigma =$ Constant

What's the corresponding Fourier Integral?

$$f(t) = \frac{1}{\pi}\int_0^\infty \int_{-1}^{1} \cos w\xi\, d\xi \cos wt\, dw + \frac{1}{\pi}\int_0^\infty \int_{-1}^{1} \sin w\xi\, d\xi \sin wt\, dw$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin w\xi}{w}\Big|_{\xi=-1}^{1} \cos wt\, dw + \frac{1}{\pi}\int_0^\infty \frac{-\cos w\xi}{w}\Big|_{\xi=-1}^{1} \sin wt\, dw$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wt\, dw + 0$$

i.e., $f(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wt\, dw$

what is $\int_0^\infty \frac{\sin w}{w}\, dw$?

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cdot 1\, dw, \quad 1 = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\, dw$$

$$\Rightarrow \int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}!$$

$$f(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w} \cos wt \, dw \quad \text{v.s.} \quad f(t) = \sum_{n=1}^{\infty} (a_n \cos w_n t + b_n \sin w_n t)$$

$$f_\beta(t) = \frac{2}{\pi}\int_0^\beta \frac{\sin w}{w} \cos wt \, dw \ , \quad f(t) = \lim_{\beta\to\infty} f_\beta(t)$$

$$\text{Sinc}(wt) = \int_0^t \frac{\sin \tau}{\tau} d\tau = Si(t) \ ,$$

Can you use $Si(t)$ to represent $f_\beta(t)$?

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi) e^{-iw\xi} d\xi \ e^{iwt} dw$$

For convenience, let's change $f(t)$ to $g(t)$

$$\Rightarrow \quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi) e^{-iw\xi} d\xi \, e^{iwt} dw$$

$H(t) = \begin{cases} 1, & t>0 \\ 0 & t<0 \end{cases}$

$$\text{if } g(t) = f(t) e^{-\alpha t} H(t) \ , \quad \alpha \in \mathbb{R} \ (\alpha > 0)$$

$$\Rightarrow f(t) e^{-\alpha t} H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\xi) e^{-\alpha\xi} e^{-iw\xi} d\xi \, e^{iwt} dw$$

$$\Rightarrow f(t) H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\xi) e^{-(\alpha+iw)\xi} d\xi \, e^{(\alpha+iw)t} dw$$

If we let $\alpha + iw = s$ ($\Rightarrow i\,dw = ds$, $dw = \frac{ds}{i}$)

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{\int_0^{\infty} f(\xi) e^{-s\xi} d\xi} \, e^{st} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}[f(t)] e^{st} dw$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \mathcal{L}[f(t)] e^{st} ds$$

$$\text{i.e., } \mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

complex variable

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(s)\, e^{st}\, ds$$

Contour integration
Residue theorem } to obtain the integration

$f(t) = e^t$, $\mathcal{L}[f(t)] = F(s) = \frac{1}{s-1}$

$= 2\pi i \sum \text{Residues} = 2\pi i \sum_{\text{Residues}} \left(\frac{1}{2\pi i}\frac{e^{st}}{s-1}\right)$

(until we learn Complex functions) $= \lim_{s\to 1}\left[(s-1)\frac{e^{st}}{s-1}\right] = e^t$

Some properties of Fourier Transform

(a) Fourier transform is a Linear Operator

If $\mathcal{F}[f_1(t)] = F_1(w)$, $\mathcal{F}[f_2(t)] = F_2(w)$

$\Rightarrow$ then $\mathcal{F}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(w) + c_2 F_2(w)$

Similarly,

$\mathcal{L}^{-1}[c_1 F_1(w) + c_2 F_2(w)] = c_1 f_1(t) + c_2 f_2(t)$

(b) differentiation :

$\mathcal{F}[\frac{df}{dt}] = ?$ if $\mathcal{F}[f(t)] = F(w)$

$$\Rightarrow \mathcal{F}\left[\frac{df}{dt}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df}{dt}e^{-iwt}dt = iw\,\mathcal{F}[f(t)] = iwF(w)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwt}\,df = \frac{1}{\sqrt{2\pi}} f e^{-iwt}\Big|_{-\infty}^{\infty} - \frac{(-iw)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}dt$$

$$\Rightarrow \mathcal{F}\left[\frac{d^n f}{dt^n}\right] = ? = (iw)^n F(w)$$

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$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega(\xi - t)}\, d\xi\, d\omega$$

Fourier transform of derivatives.

$$\mathcal{F}[f(t)] = F(\omega),$$

$$\mathcal{F}\left[\frac{df(t)}{dt}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t}\, dt = i\omega F(\omega)$$

$$\mathcal{F}\left[\frac{d^2 f(t)}{dt}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^2 f}{dt^2} e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{df}{dt} e^{-i\omega t}\Big|_{-\infty}^{\infty}$$

$$- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dt}(-i\omega)\, e^{-i\omega t}\, dt$$

From $\lim_{t\to\pm\infty} \frac{df}{dt} \to 0$,

$$\mathcal{F}\left[\frac{d^2 f}{dt^2}\right] = (i\omega)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t}\, dt = (i\omega)^2 F(\omega)$$

$$\Rightarrow \mathcal{F}\left[\frac{d^n f}{dt^n}\right] = (i\omega)^n F(\omega)$$

for $\frac{d^k f}{dt^k} \to 0$ as $t \to \pm\infty$ with $k = 0, 1, 2, \cdots$

Try to compare this with the results of Laplace transform.

what will be $\mathcal{F}[t f(t)]$??

$\Rightarrow \mathcal{F}[t^n f(t)] = ?$ $n = 1, 2, 3 \cdots$

$\because \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega)$

what happends when $F(\omega)$ is differentiated with respect to ω ?

$$dF(w)/dw = \frac{d}{dw}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

Note:

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \Leftrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dt$$

Different between two type!

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \Leftrightarrow f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dt$$

what's the relation between $\int_{-\infty}^{\infty} f(t)^2\, dt$ & $\int_{-\infty}^{\infty} F(w)^2\, dw$?

'Parsaval theorem'

$$dF(w)/dw = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial}{\partial w}\left[f(t)\, e^{-iwt}\right] dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-it)\, f(t)\, e^{-iwt}\, dt$$

$$= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t f(t)\, e^{-iwt}\, dt = -i\,\mathcal{F}[t f(t)]$$

i.e., $\mathcal{F}[t f(t)] = -\frac{1}{i}\frac{dF(w)}{dt} = i\,\frac{dF(w)}{dw}$

$\Rightarrow \mathcal{F}[t^n f(t)] = ?$

Scaling

$$\mathcal{F}[f(at)] = ? \Rightarrow = \frac{1}{|a|} F\left(\frac{w}{a}\right)$$

$$\begin{cases} t \to t \pm t_0 & \leftarrow \text{translation} \\ t \to at & \leftarrow \text{scaling} \end{cases}$$

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt,$$

$a>0$, let $at=\tau$, $dt=\frac{1}{a}d\tau$

$$\Rightarrow \mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\, e^{-i\frac{\omega}{a}\tau}\, \frac{d\tau}{a} = \underbrace{\frac{1}{a}F\left(\frac{\omega}{a}\right)}_{a>0}$$

$a<0$, let $a=-b$, $b>0$

$$\Rightarrow \mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-bt)\, e^{-i\omega t}\, dt,$$

$\Rightarrow$ let $-bt=\tau$, $dt=\frac{-\tau}{b}$

$(t(-\infty,\infty) \to \tau:(\infty,-\infty))$

$$\Rightarrow = -\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(\tau)\, e^{-i\left(\frac{\omega}{-b}\right)\tau}\, \frac{d\tau}{b} = \frac{1}{b}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\, e^{-i\frac{\omega}{a}t}\, d\tau$$

$$= \frac{1}{b}F\left(\frac{\omega}{a}\right) = \underbrace{\frac{1}{-a}F\left(\frac{\omega}{a}\right)}_{a<0}$$

Two kinds of translation.

t translation & ω translation

$f(t) \to F(\omega)$

$\hookrightarrow f(t-a) \to ?$ $\quad \mathcal{F}[f(t-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-a)\, e^{-i\omega t}\, dt$

$\left\{\begin{array}{l} \text{let } (t-a)=\tau,\ a=\text{const} \\ dt=d\tau,\ t=a+\tau \end{array}\right\}$ $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega(a+\tau)}\, d\tau$

$$= e^{-i\omega a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= e^{-i\omega a}F(\omega)$$

$\mathcal{F}[e^{iat}f(t)] = ?$ $\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iat} f(t)\, e^{-i\omega t}\, dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-a)t}\, dt = F(\omega-a)$$

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Convolution $f.g.$

Laplace : $f * g \equiv \int_0^t f(\tau) g(t-\tau)\, d\tau$

Fourier : $f * g \equiv \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau$

Correlation.

$$\mathcal{F}[f * g(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f * g(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau\, e^{-i\omega t}\, dt$$

let $\underbrace{t-\tau}_{\text{parameter}} = y$, $d\tau = dy$

$$\mathcal{F}[f * g(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(t) e^{-i\omega(\tau+y)}\, d\tau\, dy$$

$$= \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\omega y}\, dy \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) e^{-i\omega t}\, d\tau$$

$$= \sqrt{2\pi}\, F(\omega)\, G(\omega)$$

$$\Rightarrow \mathcal{F}^{-1}[F(\omega) G(\omega)] = \frac{1}{\sqrt{2\pi}} f * g(t)$$

Indefinite integral

$$\mathcal{F}\left[\int f(t)\,dt\right] = ?$$

$$\int f(t)\,dt = g(t) + C \quad , \quad f(t) = g'(t)$$

$$\mathcal{F}[f(t)] = \mathcal{F}[g'(t)] \Rightarrow F(w) = iw\,G(w)$$

$$\mathcal{F}\left[\int f(t)\,dt\right] = \mathcal{F}[g(t)+C] = \mathcal{F}[g(t)] + \mathcal{F}[C]$$

$$= G(w) + \mathcal{F}(C)$$

$$\therefore \mathcal{F}\left[\int f(t)\,dt\right] = \frac{F(w)}{iw} + \mathcal{F}[C]$$

By direct integration

$$\mathcal{F}[C] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C e^{-iwt}\,dt = \frac{-C}{iw\sqrt{2\pi}}\, e^{-iwt}\Big|_{-\infty}^{\infty} = ???$$

$f(t) \to 0$ as $t \to \pm\infty$, GG ; But ; Dirac

Can we say something about delta function from this equation?

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) = f(t_0)$$

$$\begin{cases} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} e^{-i\omega(\xi - t)}\, d\omega\, d\xi \\ f(t_0) = \int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0) \end{cases}$$

$\Rightarrow = t \quad \xi =$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) \underbrace{\int_{-\infty}^{\infty} e^{-i\omega(\xi - t)}\, d\omega}\, d\xi = \int_{-\infty}^{\infty} f(\xi)\, \underbrace{\delta(\xi - t)}\, d\xi = f(t)$$

$$\Rightarrow \delta(\xi - t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(\xi - t)}\, d\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega} e^{-i\tau(\xi - t)}\, d\tau$$

令 $\omega = \xi$, $t = \omega_0$

$$\Rightarrow \delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)\tau}\, d\tau = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega_0 t} e^{-i\omega t}\, dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}[e^{i\omega t_0}] \quad (!)$$

i.e., $\mathcal{F}[e^{i\omega_0 t}] = \sqrt{2\pi}\,\delta(\omega - \omega_0)$

$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(\omega)$; $\mathcal{F}[c] = \sqrt{2\pi} \cdot c \cdot \delta(\omega)$;

$\cos \omega_0 t = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$;

$$\Rightarrow \mathcal{F}[\cos \omega_0 t] = \frac{\sqrt{2\pi}}{2}\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$$

Modulation !

$$\mathcal{F}[e^{i\omega_0 t}] = \sqrt{2\pi}\,\delta(\omega-\omega_0),$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{\ell}t} \Rightarrow f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{2\pi}{T}t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}$$

$$\text{or } f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{\ell}x} = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{2\pi}{\lambda}x} = \sum_{n=-\infty}^{\infty} C_n e^{ink x}$$

$$\mathcal{F}[f(t)] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}\right] = \sum_{n=-\infty}^{\infty} \mathcal{F}[C_n e^{i\omega_n t}]$$

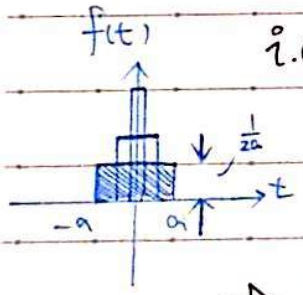
$$= \sum_{n=-\infty}^{\infty} C_n\,\delta(\omega-\omega_n)$$

Back to Dirac delta function

$\delta(t-a)$,

$$\mathcal{F}[\delta(t-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t-a)e^{-i\omega t}dt = e^{-i\omega a}/\sqrt{2\pi}$$

$(= \Delta(\omega))$



i.e., $\mathcal{F}[\delta(t)] = 1/\sqrt{2\pi}$, $(= \Delta(\omega))$

Heaviside unit step function

$$\delta(t) = \lim_{a\to 0}\delta(t-a) = \lim_{a\to 0}\frac{H(t+a)-H(t-a)}{2a}$$

$$\Rightarrow \mathcal{F}[\delta(t)] = \mathcal{F}\left[\lim_{a\to 0}\frac{H(t+a)-H(t-a)}{2a}\right] = \lim_{a\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{H(t+a)-H(t-a)}{2a}e^{-i\omega t}dt$$

$$= \lim_{a\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{2a}\int_{-a}^{a}e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\lim_{a\to 0}\frac{(-1)}{2a(i\omega)}e^{-i\omega t}\Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{a\to 0}\frac{(-1)}{2ai\omega}\left[e^{-i\omega a}-e^{i\omega a}\right] = \frac{1}{\sqrt{2\pi}}\lim_{a\to 0}\frac{\sin(a\omega)}{(a\omega)}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot 1 = \frac{1}{\sqrt{2\pi}}$$

What would you say about

$$\int_{-\infty}^{\infty} f^2(t)\,dt \quad \text{vs.} \quad \int_{-\infty}^{\infty} F^2(w)\,dw \;;$$

➡ Are they equal ?

⇨ physically, YES !

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$\overline{F(w)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{iwt}\, dt$$

$$\Rightarrow f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dw \quad ; \quad f^2(t) = f(t)\,\overline{f(t)}$$

$$\Rightarrow \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} f(t)\cdot\overline{f(t)}\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w)\, e^{iwt}\, dw\, \overline{f}(t)\,dt$$

$$= \int_{-\infty}^{\infty} \frac{F(w)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{iwt}\, dt\, dw = \int_{-\infty}^{\infty} F(w) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{iwt}\, dt\right] dw$$

$$= \int_{-\infty}^{\infty} F(w)\,\overline{F(w)}\,dw = \int_{-\infty}^{\infty} F^2(w)\,dw$$

$$\text{i.e., } \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} F^2(w)\,dw$$

time domain :

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{iwt}\, dt$$

x domain :

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

If $f = f(x, y, z, t)$; what is $\mathcal{F}[f(x, y, z, t)]$?

If $f = f(x,y,z)$;

$$\mathcal{F}[f(x,y,z)] = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, e^{-ik_x x} e^{-ik_y y} e^{-ik_z z}\, dx\,dy\,dz$$

$$\Rightarrow e^{-ik_x x} e^{-ik_y y} e^{-ik_z z} = e^{-i(k_x x + k_y y + k_z z)}$$

$$\begin{cases} \vec{k} = k_x \vec{i} + k_y \vec{j} + k_z \vec{k} \\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \end{cases} \qquad = e^{-i(\vec{k}\cdot\vec{r})}$$

$$\Rightarrow \mathcal{F}[f(\vec{r})] = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} f(\vec{r})\, e^{-i(\vec{k}\cdot\vec{r})}\, (d\vec{r})^3$$

$$\Rightarrow \mathcal{F}[f(\vec{r},t)] = \left(\frac{1}{\sqrt{2\pi}}\right)^4 \iiiint_{-\infty}^{\infty} f(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}+\omega t)}\, (d\vec{r})^3 dt$$

Fourier transform is basically for $a(-\infty,\infty)$ domains which is referred to as an infinite domain. Sometimes, the domain may be $[0,\infty)$ which is referred to as a semi-infinite domain.

For a semi-infinite domain what kinds of transforms $[0,\infty)$ can we use?

a. Laplace transform.
b. Fourier Sine/Cosine transform.

Fourier integral

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(w)\cos wt\, dw$$

Even function $f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\xi)\cos w\xi\, d\xi\right]\cos wt\, dw$

cosine

$$\Rightarrow F_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos wt\, dt$$

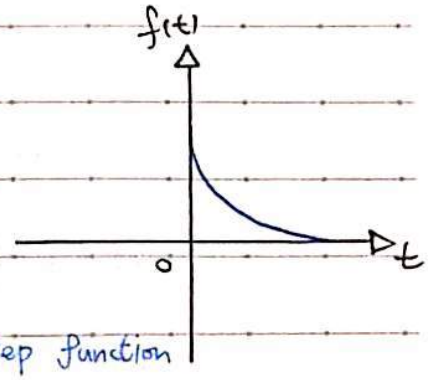
odd function $f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\xi)\sin w\xi\, d\xi\right]\sin wt\, dw$

$$\Rightarrow F_s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin wt\, dt$$

sine

Examples :

a. If $f(t) = \begin{cases} 0 & , t<0 \\ e^{-\beta t} & , t \geq 0 \, , \beta > 0 \end{cases}$



Heaviside unit step function

(i.e., $f(t) = H(t) e^{-\beta t}$, $\beta > 0$)

What is its Fourier Transform ?

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt = F(w)$$

$$\Rightarrow F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(t) e^{-\beta t} e^{-iwt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\beta + iw)t} dt = \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{\beta + iw} \right) e^{-(\beta + iw)t} \Big|_{0}^{\infty}$$

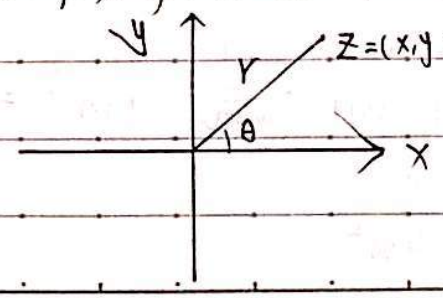
$$\left(\lim_{t \to \infty} e^{-(\beta + iw)t} = \lim_{t \to \infty} e^{-\beta t} (\cos wt - i \sin wt) = 0 \right)$$

$$\therefore F(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{\beta + iw} = \frac{1}{\sqrt{2\pi}} \frac{\beta - iw}{\beta^2 + w^2} ;$$

magnitude $|F(w)| = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\beta^2 + w^2}}$ $\left(|F(w)| = \sqrt{F(w) F^*(w)} \right)$

Engineering applications , when or where can we use complex variables ?

Complex plane (Z-plane) :



$Z = (x, y)$ Zplane, $Z \in C$;

magnitude

$Z = x + iy = r \angle \theta$ angle

$r = x^2 + y^2$

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$\begin{cases} A \text{ capacitor}, C \Rightarrow \frac{1}{j\omega C} \\ A \text{ inductor}, L \Rightarrow j\omega L \end{cases}$ > impedence ; $V_C = \frac{1}{C}\int i(t)\,dt$, $V_L = L \cdot di/dt$

$$\begin{cases} V_C = L\frac{di}{dt} = -LA\sin\omega t = LA\omega\cos(\omega t + \frac{\pi}{2}) \\ i(t) = A\cos\omega t \end{cases}$$

How can we "generalize" the ohm's law?

DC law $\qquad\qquad$ AC!

$V = RI \quad \Rightarrow \quad \mathbb{V} = Z\mathbb{I}$

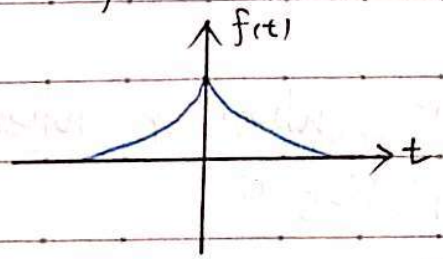
$V \propto I \qquad\qquad Z$ = Impedence

$\mathbb{I} = i = Ae^{\omega t}$;

$$\begin{cases} \text{For a resister (R)} : V_R = Ri = RAe^{j\omega t} \\ \text{For a capacitor (C)} : V_C = \frac{1}{C}\int i\,dt = \frac{A}{j\omega C}e^{j\omega t} \\ \text{For a inductor (L)} : V_L = L\frac{di}{dt} = LAj\omega e^{j\omega t} \end{cases}$$

$$\Rightarrow V = V_R + V_C + V_L = \left(R + \frac{1}{j\omega C} + j\omega L\right)Ae^{j\omega t} = Z\mathbb{I}$$

Example : $f(t) = e^{-\beta|t|}$, $-\infty < t < \infty$, $\beta > 0$



$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{\beta t}e^{-i\omega t}\,dt + \int_{0}^{\infty} e^{-\beta t}e^{-i\omega t}\,dt\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\beta - i\omega}\right) + \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\beta + i\omega}\right) = \frac{1}{\sqrt{2\pi}}\frac{2\beta}{\beta^2 + \omega^2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{\beta}{\beta^2 + \omega^2}$$

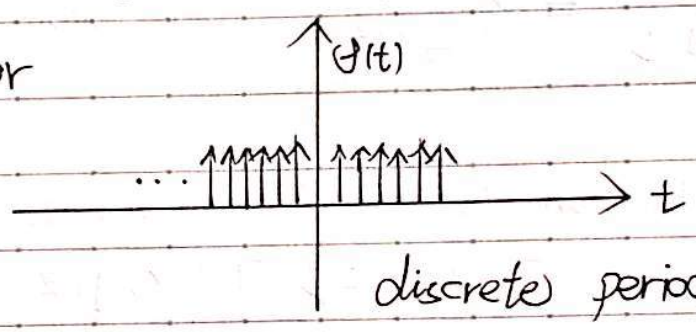
$\mathcal{F}[e^{-|t|}] = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$, $\mathcal{F}[e^{-2|t|}] = \sqrt{\frac{2}{\pi}} \frac{2}{4+w^2}$

If $F(w)\,G(w) = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2} \cdot \sqrt{\frac{2}{\pi}} \frac{2}{4+w^2}$

what's its inverse ? $\mathcal{F}^{-1}[F(w)\,G(w)] = ?$

Exercise) : $f(t) = e^{-t^2}$, $\mathcal{F}[f(t)] = ?$

or



$\Rightarrow \mathcal{F}[f(t)] = ?$

discrete) periodic function

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Partial differential equations

What did you encounter so far ?

Heat conduction $\begin{cases} 1^{st} \text{ of thermodynamic} \\ \text{Fourier law of heat conduction} \end{cases}$

$$\text{"}\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}\text{"}$$ 1D, unsteady heat conduction

a. extension to 3-d mensional space

$$T(x,t) \Rightarrow T(x,y,z,t)$$

b. extension to other geometrics
(different cordinate systems)

Maxwell equations :

Two divergence equations and two curl equations.

$\vec{B}$: magnetic flux intensity

$\vec{H}$: magnetic field (strength)

$\vec{E}$: electric field (strength)

$\vec{D}$: displacement current

$$\Rightarrow \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \cdot \vec{D} = \rho \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\partial \vec{B}/\partial t \end{cases}$$

charge density

current density

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A simplified situation

Permeability 導磁係數

$$\begin{cases} \nabla\cdot\vec{B}=0 \Rightarrow \nabla\cdot\vec{H}=0 \\ \nabla\cdot\vec{D}=0 \Rightarrow \nabla\cdot\vec{E}=0 \\ \nabla\times\vec{H}=\partial\vec{D}/\partial t=\frac{\partial}{\partial t}(\varepsilon\vec{E}) \\ \nabla\times\vec{E}=-\partial\vec{B}/\partial t=-\frac{\partial}{\partial t}(\mu\vec{H}) \end{cases}, \quad \begin{cases} \vec{B},\vec{H} \Rightarrow \vec{B}=\mu\vec{H} \\ \vec{D},\vec{E} \Rightarrow \vec{D}=\varepsilon\vec{E} \end{cases}$$

介电常数

Permittiveity
(dielectric constant)

Maxwell $\Rightarrow \begin{cases} \mu\to\mu_0 \\ \varepsilon\to\varepsilon_0 \end{cases}, \quad \left(\frac{1}{\mu_0\varepsilon_0}\right)^{\frac{1}{2}}=3\times10^8\ m/s$

Time dependent Maxwell equations

$$\begin{cases} \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H}=J+\frac{\partial\vec{D}}{\partial t} \end{cases}$$

$\vec{B}$: magnetic flux intensity

$\vec{D}$: displacement current

If $\frac{\partial\vec{B}}{\partial t}=\frac{\partial\vec{D}}{\partial t}=0 \Rightarrow$ Megneto-statics

$$\begin{cases} \vec{B}=\mu\vec{H} \\ \vec{D}=\varepsilon\vec{E} \end{cases} \Rightarrow \begin{array}{l} \nabla\times\vec{E}=-\frac{\partial\mu\vec{H}}{\partial t}=-\mu\frac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{H}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t} \end{array}$$

Let's consider a simplified case

From Gauss Law $\vec{J}$ (current density) $=0$

$\nabla\cdot\vec{D}=\rho \Rightarrow \nabla\cdot\vec{E}=0$ (The space is free of charge)

$$\nabla\cdot\vec{H}=0 \Rightarrow \begin{cases} \nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{H}=\varepsilon\frac{\partial E}{\partial t} \end{cases} \Leftarrow \text{coupled !!}$$

A system of partial differential equation.

decouple !

∇ (gradient operator) 梯度運算子

Let's suppose we have a scalar function in Cartesian coordinate $f = f(x, y, z)$

$$\nabla f \equiv \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) f(x,y,z)$$

$$= \partial f/\partial x\ \vec{i} + \partial f/\partial y\ \vec{j} + \partial f/\partial z\ \vec{k}$$

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

⇨ operates on scalar functions

operates on vector func.

$\nabla \cdot$ = divergence operator (散度運算子)

$\nabla \times$ = curl operator (旋量運算子)

$$ax + by + cz = \beta$$

Let's perform a total differentiation

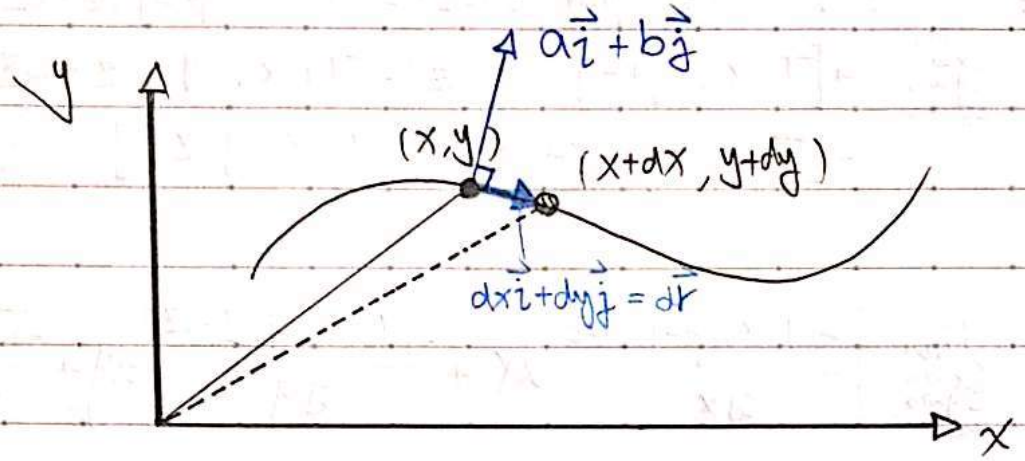
$$d(ax + by + cz = \beta)$$

$$\Rightarrow a\,dx + b\,dy + c\,dz = 0$$

$$\Rightarrow (a\vec{i} + b\vec{j} + c\vec{k}) \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}) = 0$$

Simple example :

$$(a\vec{i}+b\vec{j})(dx\,\vec{i}+dx\,\vec{j})=0$$



$$\left(\begin{cases} dx\,\vec{i}+dy\,\vec{j}=d\vec{r} \Rightarrow \text{tangent vector} \\ a\vec{i}+b\vec{j} \Rightarrow \text{normal vector} \end{cases}\right)$$

So, $\underbrace{(a\vec{i}+b\vec{j}+c\vec{k})}_{\text{normal vector}}\underbrace{(dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k})}_{\text{tangent to the surface}}=0$

If we have a 3D temperature distribution

$T=T(x,y,z)=C$ ⇨ isothermal curves

If we perform total differentiation to $T(x,y,z)=C$

$$\Rightarrow dT(x,y,z)=dC=0$$

$$\Rightarrow \frac{\partial T}{\partial x}dx+\frac{\partial T}{\partial y}dy+\frac{\partial T}{\partial z}dz=0$$

$$dT(x,y,z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \left[T(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - T(x,y,z) \right]$$

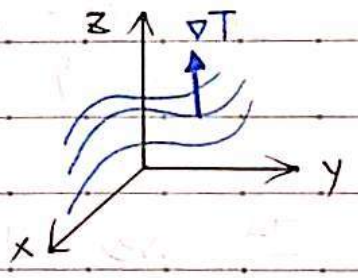
$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \left\{ \begin{array}{l} \left[T(x+\Delta x, y+\Delta y, z+\Delta z) - T(x, y+\Delta y, z+\Delta z)\right] \cdot dx/dx \\ +\left[T(x, y+\Delta y, z+\Delta z) - T(x, y, z+\Delta z)\right] \cdot dy/dy \\ +\left[T(x, y, z+\Delta z) - T(x,y,z)\right] \cdot dz/dz \end{array} \right\}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \left\{ \frac{\partial T(x, y+\Delta y, z+\Delta z)}{\partial x} dx + \frac{\partial T(x, y, z+\Delta z)}{\partial y} dy + \frac{\partial T(x,y,z)}{\partial z} dz \right\}$$

$$= \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \quad (*)$$

$$\Rightarrow = \left(\partial T/\partial x\ \vec{i} + \partial T/\partial y\ \vec{j} + \partial T/\partial z\ \vec{k} \right) \left(dx\vec{i} + dy\vec{j} + dz\vec{k} \right)$$

$$= \nabla T \cdot d\vec{r} = 0.$$



$m\vec{g}$ $(mg\vec{j}) \Rightarrow mgy$,

$$\nabla y = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right) y = \vec{j}$$

$m\vec{g} = mg\nabla y = \nabla mgy$ 保守力

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow \nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t} \right)$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \Rightarrow \nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \varepsilon \frac{\partial}{\partial t} (\nabla \times \vec{E}) = \varepsilon \frac{\partial}{\partial t} \left(-\mu \frac{\partial \vec{H}}{\partial t} \right)$$

$$= -\mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad *$$

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Maxwell equation

$$\begin{cases} \nabla \cdot \vec{D} = \rho \\ \nabla \cdot \vec{B} = 0 \end{cases} \qquad \begin{aligned} \vec{D} &= \varepsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \end{aligned}$$

$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{cases}$$

For a situation without ρ & $\vec{J}$

$$\Rightarrow \begin{aligned} \nabla \cdot \vec{D} &= 0 \\ \nabla \cdot \vec{B} &= 0 \end{aligned} \quad \begin{cases} \nabla \times \vec{E} = -\mu \, \partial \vec{H} / \partial t \\ \nabla \times \vec{H} = \varepsilon \, \partial \vec{E} / \partial t \end{cases}$$

To decouple $\vec{E}$ & $\vec{H}$, Let's try

$$\begin{cases} \nabla \times [\nabla \times \vec{E} = -\mu \, \partial \vec{H} / \partial t] \\ \nabla \times [\nabla \times \vec{H} = \varepsilon \, \partial \vec{E} / \partial t] \end{cases}, \quad \text{i.e.,}$$

$$\Rightarrow \begin{cases} \nabla \times (\nabla \times \vec{E}) = -\nabla \times (\mu \, \partial \vec{H} / \partial t) \\ \nabla \times (\nabla \times \vec{H}) = \nabla \times (\varepsilon \, \partial \vec{E} / \partial t) \end{cases}$$

Are μ and ε function of $\vec{r}$?

For an isotropic homogeneous material, μ and ε are constant.

Otherwise, $\mu = \mu(\vec{r})$, $\varepsilon = \varepsilon(\vec{r})$;

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (position vector)

Recall: The gradient operator is a manipulation of a scalar function with respect to spatial coordinates. Therefore, if $\mu = \mu(\vec{r})$, $\varepsilon = \varepsilon(\vec{r})$, then $\nabla \times \left(\varepsilon \frac{\partial E}{\partial t}\right) \neq \varepsilon \frac{\partial}{\partial t}(\nabla \times \vec{E})$

Note: $\begin{cases} \nabla \times [\varepsilon(\vec{r})\vec{E}(\vec{r},t)] = ? \\ \nabla \cdot [\varepsilon(\vec{r})\vec{E}(\vec{r},t)] = ? \end{cases}$

Also, what are the possible reasons to introduce the symbol ∇, $\nabla\cdot$, $\nabla\times$?

Let's consider the cases with μ, ε being constant, Then Maxwell equations are reduce to

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{cases}, \quad \begin{cases} \nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H}) \\ \nabla \times (\nabla \times \vec{H}) = \varepsilon \frac{\partial}{\partial t}(\nabla \times \vec{E}) \end{cases}$$

$$\nabla \times (\nabla \times \vec{H}) = ?$$

$$\nabla \times (\nabla \times \vec{E}) = ?$$

Let's consider a simpler case first.

If two vectors $\vec{V} = \vec{V}(\vec{r})$, $\vec{U} = \vec{U}(\vec{r})$,

$$\vec{U} \times \vec{V} = ? = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{U} \times (\vec{V} \times \vec{W}) = ?$$

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = ?$$

Do by self, Think!

$$\vec{U} \times (\vec{V} \times \vec{W}) = \alpha \vec{V} + \beta \vec{W}$$
$$= \pm(\vec{U} \cdot \vec{W})\vec{V} \pm (\vec{U} \cdot \vec{V})\vec{W}$$
choose $\Rightarrow$ $+(\vec{U} \cdot \vec{W})\vec{V} - (\vec{U} \cdot \vec{V})\vec{W}$

$$\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$
$$= -\mu\varepsilon \, \partial^2 \vec{E}/\partial t^2 \quad ;$$
$$\nabla \times (\nabla \times \vec{E}) = \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0}) - (\nabla \cdot \nabla)\vec{E}$$

$\nabla \cdot \nabla \equiv \nabla^2$ (Laplacian operators)

$$\Rightarrow \text{"}\nabla^2 \vec{E} = \mu\varepsilon \, \partial^2 \vec{E}/\partial t^2\text{"}$$

$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$, Similarly, $\text{"}\nabla^2 \vec{H} = \mu\varepsilon \, \partial^2 \vec{H}/\partial t^2\text{"}$

In Cartesian coordinates,
$$\nabla \equiv \vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z$$
$$\therefore \nabla^2 = \nabla \cdot \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Namely, In Cartesian coordinates, we have
$$\Rightarrow \partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 + \partial^2 E/\partial z^2 = \mu\varepsilon \, \partial^2 \vec{E}/\partial t^2$$

(Vector equation $\equiv$ 3 scalar functions in 3D)

$$\vec{E} = E_x \vec{i} + E_y \vec{j} + E_z \vec{k}$$
$$\Rightarrow \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

→ a simpler case

$$\vec{E} = E_x \vec{i} = E\vec{i} = E(x,y)\vec{i}$$

$$\Rightarrow \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

(2D, transiant Electric Field Strength)

Fourier conduction equation

$$\begin{cases} \rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \; ; \; T = T(x,t) \\ \mu\varepsilon \, \partial^2 E/\partial t^2 = \partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 \; ; \; E = E(x,y,t) \\ \mu\varepsilon \, \partial^2 E/\partial t^2 = \nabla^2 \vec{E} \; , \; \vec{E} = \vec{E}(x,y,z,t) = \vec{E}(\vec{r},t) \end{cases}$$

If $E = E(x,y)$ only $\Rightarrow \partial^2 E/\partial x^2 + \partial^2 E/\partial y^2 = 0$

or if $E = E(x,t)$

$$\Rightarrow \mu\varepsilon \, \partial^2 E/\partial t^2 = \partial^2 E/\partial x^2$$

Barrowing the terminology of ODEs, the above partial differential equation (PDEs) are:

a. Order: 2nd order

b. linear vs. nonlinear : linear

c. homogeneous v.s. inhomogeneous : homogeneous

In space, $\mu = \mu_0 = 4\pi \times 10^{-4}$ Henry/sec , $\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12}$ Farad/m

From $\mu\varepsilon \frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2}$, what is the unit of $\mu\varepsilon$? (t: sec, x: m)

$\Rightarrow (\mu\varepsilon) \cdot \frac{E}{s^2} = \frac{E}{x^2} \Rightarrow \frac{1}{\mu\varepsilon} : (\frac{m}{s})^2$; $1/\sqrt{\mu\varepsilon} : m/s$

Maxwell:
$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8$ m/s
⇒ light is a part of electromagnetic wave.

$n_1 \sin\theta_1 = n_2 \sin\theta_2$
$n = \frac{c}{v} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \times \frac{1}{\sqrt{\mu\varepsilon}}$
$= \sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\mu_r \varepsilon_r}$
($\mu_r = \frac{\mu}{\mu_0}$, $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$)

For a general 2nd linear PDE $\quad a = a(x,y)$

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + a_3 u = a_4$$

~~(similiar: $ax^2 + bxy + cy^2 + dx + ey + f = 0$)~~ $\quad b^2 - 4ac \begin{cases} > 0 \text{, hyperbolic} \\ = 0 \text{, parabolic} \\ < 0 \text{, elliptic} \end{cases}$

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Three commonly occurred PDEs:

a. Diffusion equation $\frac{\partial u}{\partial t} = K\frac{d^2u}{dx^2}$ ($\xrightarrow[3D]{}$ $\frac{\partial u}{\partial t} = \alpha\nabla^2 u$) parabolic type PDE, 拋物

b. Wave equation $\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2}$ ($\xrightarrow[3D]{}$ $\frac{\partial^2 u}{\partial t^2} = C^2\nabla^2 u$) hyperbolic type PDE, 雙曲

c. Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ($\xrightarrow[3D]{}$ $\nabla^2 u = 0$) elliptic type PDE, 橢圓

If we try to solve them by the method of Separation of variables, namely, by assuming

$$u(x,t) = X(x)\,T(t) \quad \text{or}$$

$$u(x,y) = X(x)\,Y(y)$$

Then the corresponding equations are:

a. $\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{K}{X(x)}\frac{d^2X(x)}{dx^2} = \text{constant}$

b. $\frac{1}{T(t)}\frac{d^2T(t)}{dt} = C^2\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \text{constant}$

c. $\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{-1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \text{constant}$

These constants have three possibilities.

- greater than zero
- equal to zero
- smaller than zero

which are determined by boundary conditions

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Three types of boundary conditions :

a. Dirichlet boundary condition :

The values of u are given on the boundaries.

b. Neumann boundary condition :

The values of $\partial u/\partial x$ etc are given on the boundaries.

c. Cauchy boundary condition :

The values of $c_1 u + c_2 \frac{\partial u}{\partial x}$ etc are given :

A simple comparison :

$$y = ce^t$$

$$y' = ce^t \Rightarrow y' = y \qquad c: \text{ a constant}$$

function

$$\begin{cases} u = x f(y) \\ \partial u/\partial x = f(y) \end{cases} \Rightarrow u = x \frac{\partial u}{\partial x}$$

$$u = x \, \partial u/\partial x \quad \text{v.s.} \quad u = x f(y)$$

e.g. $\begin{cases} f(y) = \sin y \\ f(y) = e^y \end{cases}$, The PDE $u = x \frac{\partial u}{\partial x}$ is satisfied

or for

$$\partial^2 u/\partial t^2 = c^2 \, \partial^2 u/\partial x^2 .$$

$$u(x,t) = f(x-ct) + g(x+ct)$$

f & g are arbitrary functions

(proof: $\frac{\partial u}{\partial t} = -cf + cg$, $\frac{\partial^2 u}{\partial t^2} = (-c)^2 f + c^2 g = c^2(f+g)$

$\frac{\partial u}{\partial x} = f + g = \frac{\partial^2 u}{\partial x^2}$; $\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ #)

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One of the key approaches is to change a PDE into ODEs

Let's start with the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial t^2} \; ; \; \dfrac{\partial u}{\partial t} \sim \dfrac{u}{t} \; ; \\ K\dfrac{\partial^2 u}{\partial t} \sim K\dfrac{u}{x^2} \end{cases}$$

For this PDE, how can we convert it into ODEs ?

K : diffusion constant (diffusion coefficient)

The dimension of K, $[K] = (\text{length})^2/\text{time}$

$$\begin{cases} \text{dimension : time, length, mass} \\ \quad \begin{cases} \text{primary dimension : time, length, mass} \\ \text{Derived dimension : speed, pressure} \end{cases} \\ \text{unit : sec, m (inch), kg (lbm), m/s} \end{cases}$$

i.e., $\dfrac{u}{t} \sim K\dfrac{u}{x^2}$; $x \sim \sqrt{Kt}$

$u = u(x,t) \Rightarrow u = u(\eta)$;

$\eta = x/\sqrt{Kt}$

Possible methods to convert $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}$ into ODEs

a. similarity transform, Let $\eta = x/\sqrt{Kt}$

b. separation of variable $u(x,t) = X(x)T(t)$

c. Integral transform : $\begin{cases} \text{Laplace} \\ \text{Fourier} \end{cases}$ transforms

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a. $\partial u/\partial t = K\, \partial^2 u/\partial x^2$, $u = u(x,t)$

$\eta = x/\sqrt{\nu t}$ $\Rightarrow$ $u = u(\eta)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial t} = \dfrac{du}{d\eta}\dfrac{\partial \eta}{\partial t} \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{du}{d\eta}\dfrac{\partial \eta}{\partial x} \end{cases}$$

One dimensional, transient diffusion equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

K: diffusion coefficient (diffusivity)

If the condition given are

$u(0,t) = u_0$, $u(\infty,t) = u_1$

($\because$ u in the x-coordinate is a 2nd order partial derivative, $\therefore$ it needs to condition)

These condition are called boundary conditions

$u(x,0) = u_1$ (Initial condition)

u_0, u_1 are constants.

If alternatively, we consider

$u(0,t) = u_2$, $u(\infty,t) = 0$, $u(x,0) = 0$

$\Rightarrow$ These two problems are equivalent!!

If we perform a simple transformation

as $\hat{u} = u + \underset{\text{constant}}{u_1}$ for $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ $\begin{cases} u(0,t) = u_0 \\ u(\infty,t) = u_1 \\ u(x,0) = u_1 \end{cases}$

$$\frac{\partial}{\partial t}(\hat{u} + u_1) = K \frac{\partial^2}{\partial x^2}(\hat{u} + u_1)$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} = K \frac{\partial^2 \hat{u}}{\partial x^2}$$

$$u(0,t) = \hat{u}(0,t) + u_1 = u_0$$

$$\therefore \hat{u}(0,t) = u_0 - u_1$$

$$u(\infty,t) = \hat{u}(\infty,t) + u_1 = u_1$$

$$\therefore \hat{u}(\infty,t) = 0$$

$$u(x,0) = \hat{u}(x,0) + u_1 = u_1$$

$$\therefore \hat{u}(x,0) = 0$$

Therefore, let's consider

$$\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} \quad \begin{cases} u(0,t) = u_2 = \text{Constant} \\ u(\infty,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Let's try the method of separation of variable.

That is, assume $u(x,t) = X(x)\,T(t)$

$$\frac{1}{K}\frac{\partial}{\partial t}(XT) = \frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow \frac{X}{K}\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

function of t function of X

$$\frac{1}{K\tau}\frac{d\tau}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{constant} = -\beta^2, \ \beta \in \mathbb{R}$$

So we have two ODEs :

$$\begin{cases} \frac{1}{K\tau}\frac{d\tau}{dt} = -\beta^2 \Rightarrow \tau = C_1 e^{-K\beta^2 t} \\ \frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2 \Rightarrow X = C_2 \cos\beta x + C_3 \sin\beta x \end{cases}$$

From $u(x,0) = 0$; $X(x)\tau(0) = 0$

$\Rightarrow \tau(0) = 0$;

From $\tau(0) = 0 = C_1 e^{-0} = C_1$

$\therefore \ C_1 = 0 \Rightarrow \tau(t) \equiv 0$

NOT ENGINEERING SOUND

$\Rightarrow$ we use a method which will not give us correct answer.

- If we invite Laplace and use Laplace transform

$$\mathcal{L}\left[\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\right] ; \ \mathcal{L}[u(x,t)] = \int_0^\infty u(x,t)e^{-st}dt$$

$$= U(x,s)$$ parameter

$$\Rightarrow \int_0^\infty \frac{1}{K}\frac{\partial u}{\partial t}e^{-st}dt = \int_0^\infty \frac{\partial^2 u}{\partial x^2}e^{-st}dt$$

$$\Rightarrow \frac{1}{K}\left[sU(x,s) - u(x,0)\right] = \frac{d^2}{dx^2}\int_0^\infty u e^{st}dt$$

$$= \frac{d^2U(x,s)}{dx^2}$$

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$$\therefore \frac{d^2U}{dx^2} - \frac{s}{K}U(x,s) = 0 \;;$$

$$U(x,s) = C_1 e^{\sqrt{\frac{s}{K}}x} + C_2 e^{-\sqrt{\frac{s}{K}}x}$$

$$\mathcal{L}[u(0,t)] = \mathcal{L}[u_2]$$

$$\Rightarrow U(0,s) = \frac{u_2}{s} \;;$$

$$\mathcal{L}[u(\infty,t)] = \mathcal{L}[0]$$

$$\Rightarrow U(\infty,s) = 0 \qquad \Rightarrow C_1 = 0$$

$\left(U(\infty,s) = C_1 e^{\sqrt{\frac{s}{K}}\infty} + 0 = 0 \Rightarrow C_1 = 0 \right)$

$$U(x,s) = C_2 e^{-\sqrt{\frac{s}{K}}x} \;;\; U(0,s) = C_2 e^{-0} = C_2 = \frac{u_2}{s}$$

$$\therefore U(x,s) = \frac{u_2}{s} e^{-\sqrt{\frac{s}{K}}x}$$

$$\text{i.e., } u(x,t) = \mathcal{L}^{-1}[U(x,s)] = \mathcal{L}^{-1}\left[\frac{u_0}{s} e^{-\frac{x}{\sqrt{K}}\sqrt{s}}\right]$$

$$= u_2 \mathcal{L}^{-1}\left[\frac{e^{-\frac{x}{\sqrt{K}}\sqrt{s}}}{s}\right] = u_2 \operatorname{erfc}\left(\frac{x}{2\sqrt{Kt}}\right) \;;$$

(Recall)

$$\mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

$$\operatorname{erf}(x) = \text{erro function} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$$ (鐘型曲線分布)

$$\operatorname{erfc}(x) = \text{coupli mentary error function}$$

$$= 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}dy$$

Remember $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$,

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-y^2}dy = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$$ (機率)

Similarity transformation

$$\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(x,t) \Rightarrow u(\eta)$: combine x and t together to become a single variable.

PDE $\Rightarrow$ ODE

Two method to obtain η :

a. mathmetically, assume $\eta = \alpha t^m x^n$ to obtain m and n from the PDE. So that the equation will contain u and η only, no t or x is present!

b. scale analysis

$$\frac{1}{K}\frac{\partial u}{\partial t} \sim \frac{1}{K}\frac{u}{t} \quad ; \quad \frac{\partial^2 u}{\partial x^2} \sim \frac{u}{x^2}$$

$\Rightarrow x \sim \sqrt{Kt}$,

adjusting constant

Try $\eta = \frac{x}{\sqrt{Kt}} \Rightarrow \eta = \frac{\alpha x}{\sqrt{Kt}}$

wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{u}{t^2} \sim c^2 \frac{u}{x^2}$$

$x^2 \sim (ct)^2 \Rightarrow x \sim \pm ct$

$\eta = x + ct \; ; \quad \xi = x - ct$

$\frac{\partial^2 u}{\partial \eta \partial \xi} = 0 \; , \quad u = u(x,t) \Rightarrow u = u(\xi, \eta)$

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$$\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t) \Rightarrow u = u(\eta)$$

① $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t},$$

$$\frac{\partial \eta}{\partial t} = \frac{\alpha x}{\sqrt{K}}\left(-\frac{1}{2}\right)\frac{1}{t^{3/2}} = \frac{\alpha x}{\sqrt{Kt}}\left(-\frac{1}{2t}\right) = -\frac{\eta}{2t}$$

$$\therefore \frac{\partial u}{\partial t} = -\frac{\eta}{2t}\frac{du}{d\eta}$$

② $$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$= \frac{du}{d\eta}\frac{\partial \eta}{\partial x} = \frac{du}{d\eta}\frac{\alpha}{\sqrt{Kt}} \quad \left(\eta = \frac{\alpha x}{\sqrt{Kt}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{du}{d\eta}\frac{\alpha}{\sqrt{Kt}}\right)$$

$$= \frac{\alpha}{\sqrt{Kt}}\frac{\partial}{\partial \eta}\left(\frac{du}{d\eta}\right)\frac{\partial \eta}{\partial x} = \frac{\alpha}{\sqrt{Kt}}\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}$$

$$= \frac{\alpha^2}{Kt}\frac{\partial^2 u}{\partial \eta^2}$$

① = ②

$$\Rightarrow \frac{1}{K}\left(-\frac{\eta}{2t}\frac{du}{d\eta}\right) = \frac{\alpha^2}{Kt}\frac{d^2u}{d\eta^2}$$

$$\therefore \frac{d^2u}{d\eta^2} + \frac{\eta}{2\alpha^2}\frac{du}{d\eta} = 0$$

Let $\frac{du}{d\eta} = P \Rightarrow \frac{dP}{d\eta} + \frac{\eta}{2\alpha^2}P = 0$

$$\Rightarrow \frac{dP}{P} + \frac{1}{2\alpha^2}\eta\, d\eta = 0$$

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By choosing $\alpha^2 = \frac{1}{4}$ (i.e., $\alpha = \frac{1}{2}$)

$$\eta = \frac{\alpha X}{\sqrt{Kt}} = \frac{X}{2\sqrt{Kt}} \quad ;$$

$$\frac{dP}{P} + d\eta^2 = 0 \Rightarrow P = C_1 e^{-\eta^2} \quad ;$$

$$\therefore \frac{du}{d\eta} = C_1 e^{-\eta^2}, \quad du = C_1 e^{-\eta^2} d\eta$$

$$\Rightarrow u = C_1 \int_0^{\eta} e^{-y^2} dy + C_2$$

$$\eta = \frac{X}{2\sqrt{Kt}}, \quad \begin{array}{l} t = 0 \Rightarrow \eta = \infty \Rightarrow u(\infty, t) = 0 \\ X = \infty \Rightarrow \eta = \infty \Rightarrow u(X, 0) = 0 \end{array}$$

$$u(0,t) = u_2, \quad \eta = 0, \quad C_1 \int_0^0 e^{-y^2} dy = 0$$

$$\Rightarrow u_2 = 0 + C_2$$

$$\Rightarrow \therefore u = C_1 \int_0^{\eta} e^{-y^2} dy + u_2$$

$$\eta = \infty \Rightarrow 0 = C_1 \underbrace{\int_0^{\infty} e^{-y^2} dy}_{\frac{\sqrt{\pi}}{2}} + u_2 \quad ;$$

$$C_1 = -\frac{2}{\sqrt{\pi}} u_2 \quad ,$$

$$\therefore u(\eta) = -\frac{2}{\sqrt{\pi}} u_2 \int_0^{\eta} e^{-y^2} dy + u_2 = u_2 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-y^2} dy \right]$$

$$= u_2 \left[\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2} dy - \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2} dy \right] = u_2 \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-y^2} dy$$

$$= u_2 \operatorname{erfc}(\eta) = u_2 \operatorname{erfc}\left(\frac{X}{2\sqrt{Kt}} \right)$$

#

Therefore, let's consider

$$\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{cases} u(0,t)=0 \\ u(\infty,t)=0 \\ u(x,0)=f(x) \leftarrow \text{given} \end{cases}$$

$$\mathcal{L}\left[\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\right] \Rightarrow \frac{1}{K}\left[sU - u(x,0)\right] = \frac{d^2U}{dx^2}$$

$$\Rightarrow \frac{d^2U}{dx^2} - \frac{s}{K}U = \frac{-u(x,0)}{K} = \frac{-f(x)}{K}$$

$$\Rightarrow \frac{d^2U}{dx^2} - \frac{s}{K}U = -\frac{f(x)}{K} \quad \begin{cases} U(0,s)=0 \\ U(l,s)=0 \end{cases}$$

Alternately, assume

$$u(x,t) = X(x)\,T(t),$$

$$\frac{1}{KT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2$$

$$T = Ge^{-\beta^2 Kt}, \quad X = C_2\cos\beta x + C_3\sin\beta x$$

$$\begin{cases} u(0,t) = 0 = X(0)T(t) = 0 \\ u(l,t) = 0 = X(l)T(t) = 0 \end{cases}$$

$$\Rightarrow X(0) = X(l) = 0$$

$$X(0) = C_2 + 0 = 0 \Rightarrow C_2 = 0$$

$$X(l) = C_3\sin\beta l = 0 \Rightarrow \sin\beta l = 0 \quad \text{or } \beta l = n\pi, \pm1, \pm2\ldots \quad \left(\beta = \frac{n\pi}{l}\right)$$

($C_3 = 0 \Rightarrow X(x) = 0 \Rightarrow u(x,t) = 0$ will not satisfy the initial condi.)

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$$\therefore X(x) = C_3 \sin \frac{n\pi}{l} x ,$$

$$u = C_1 C_3 e^{-(\frac{n\pi}{l})^2 Kt} \sin \frac{n\pi}{l} x , \quad n = \pm 1, \pm 2, \pm 3 \cdots$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\frac{n\pi}{l})^2 K} \sin \frac{n\pi}{l} x$$

given

$$\therefore f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} x$$

given

$$\therefore C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$\Rightarrow \therefore u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{2}{l} \int_0^l f(y) \sin \frac{n\pi}{l} y \, dy \right\} e^{-(\frac{n\pi}{l})^2 Kt} \sin \frac{n\pi}{l} x$$

Question : ① If $\begin{cases} u(0,t) = u_1 \text{ (constant)} \\ u(l,t) = u_2 \text{ (constant)} \\ u(x,0) = 0 \end{cases}$?

② $\nabla^2 u = \nabla \cdot \nabla u(r, \phi, z) \Rightarrow$ Bessel ODE

(圓柱 → Bessel
球 → Legendre)

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$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

$$\begin{cases} u(0,t) = u_1 \ , \ u(l,t) = u_2 \Rightarrow \text{Independent of } t \\ u(x,0) = f(x) \end{cases}$$

What can we do in order to convert $u(0,t) = u_1$ & $u(l,t) = u_2$ to make $u_1 = 0$ & $u_2 = 0$?

a. Take a wild guess

Say we assume a function v

$$v = \begin{cases} v(x) \ ? \Rightarrow \checkmark \\ v(t) \ ? \\ v(x,t) \ ? \end{cases} \ ; \ v(x) \Rightarrow \begin{cases} v(0) = u_1 \\ v(l) = u_2 \end{cases}$$

$$\Rightarrow v(x) = a + bx \ , \ \begin{cases} v(0) = a = u_1 \\ v(l) = u_2 = u_1 + bl \ , \ b = \frac{u_2 - u_1}{l} \end{cases}$$

Then by letting

$$u(x,t) = \hat{u}(x,t) + v(x) \ ,$$

$$\Rightarrow u(0,t) = \hat{u}(0,t) + v(0) \ ;$$

$$u_1 = \hat{u}(0,t) + u_1 \Rightarrow \hat{u}(0,t) = 0$$

$$\Rightarrow u(l,t) = \hat{u}(l,t) + v(l) \ ,$$

$$u_2 = \hat{u}(l,t) + u_2 \Rightarrow \hat{u}(l,t) = 0$$

$$\Rightarrow f(x) = \hat{u}(x,0) + u_1 + \frac{u_2 - u_1}{l} x \ ;$$

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i.e., $\begin{cases} \hat{u}(x,0) = f(x) - u_1 - \left(\frac{u_2-u_1}{l}\right)x \\ \hat{u}(0,t) = 0 \\ \hat{u}(l,t) = 0 \end{cases}$

$$\Rightarrow \frac{\partial}{\partial t}\left[\hat{u}(x,t) + v(x)\right] = K\frac{\partial^2}{\partial x^2}\left[\hat{u}(x,t) + v(x)\right]$$

$$\Rightarrow \frac{\partial \hat{u}(x,t)}{\partial t} + \cancel{\frac{\partial v(x)}{\partial t}} = K\frac{\partial^2 \hat{u}}{\partial x^2} + K\frac{\partial^2}{\partial x^2}\left[u_1 + \cancel{\frac{u_2-u_1}{l}x}\right]$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} = \frac{\partial \hat{u}^2}{\partial x^2}, \quad \hat{u} = \sum_{n=1}^{\infty}\left[\left(\frac{2}{l}\int_0^l \left[f(\xi) - u_1 - \left(\frac{u_2-u_1}{l}\right)\xi\right]\sin\frac{n\pi}{l}\xi\, d\xi\right) \sin\frac{n\pi}{l}x\right] e^{-\left(\frac{n\pi}{l}\right)^2 Kt}.$$

$$\therefore\ u = \hat{u} + v \quad *$$

b. mathematically, let's

$$u(x,t) = \hat{u}(x,t) + v(x)$$

and imposing the conditions

$v(0) = u_1$ and $v(l) = u_2$

So that $\begin{cases} \hat{u}(0,t) = 0 \\ \hat{u}(l,t) = 0 \end{cases}$,

$$\partial\hat{u}/\partial t = K\frac{d^2\hat{u}}{dx^2} + K\frac{d^2 v}{dx^2}$$

with the goal of $\partial\hat{u}/\partial t = K\,\partial^2\hat{u}/\partial x^2$

$$\Rightarrow \frac{d^2 v}{dx^2} = 0 \quad \text{with } v(0) = u_1,\ v(l) = u_2$$

$$\frac{d^2 v}{dx^2} = 0 \Rightarrow \frac{dv}{dx} = C_1, \quad \text{So, } v = C_1 x + C_2$$

$$\mathcal{V}(0) = C_1 \cdot 0 + C_2 \Rightarrow C_2 = \mathcal{V}(0) = U_2$$

$$\mathcal{V}(l) = C_1 \cdot l + C_2 \Rightarrow U_2 = C_1 l + U_1 \Rightarrow C_1 = \frac{U_2 - U_1}{l}$$

i.e., $\mathcal{V}(x) = U_1 + \frac{U_2 - U_1}{l} x$

Note: if u is not a function of t

$$\frac{\partial u}{\partial t} = 0 \Rightarrow K \frac{\partial^2 u}{\partial x^2} = 0$$

$$u = u(x,t) \Rightarrow u = u(t), \quad \therefore K \frac{d^2 u}{dx^2} = 0$$

$U_1 + \frac{U_2 - U_1}{l} x$ is the steady state solution of $u(x,t)$; ($\partial(\,)/\partial t = 0 \Rightarrow$ steady state! Not a function of time)

An extension to 2-D (Cartesian)

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Non linear, chaotic

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\frac{1}{\rho} \nabla p + \vec{b} + \nu \nabla^2 \vec{V}$$

(Navier - Stokes equation)

Come from $\vec{F} = m\vec{a}$ (牛頓)

If solved by Separation of variables

$$u(x,y,t) = X(x)\, Y(y)\, T(t)$$

$$XY \frac{dT}{dt} = \left[YT \frac{d^2 X}{dx^2} + XT \frac{d^2 Y}{dy^2} \right]$$

$$\Rightarrow \frac{1}{XT} \frac{dT}{dx} = \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2},$$

Consider a steady state case, i.e.,

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

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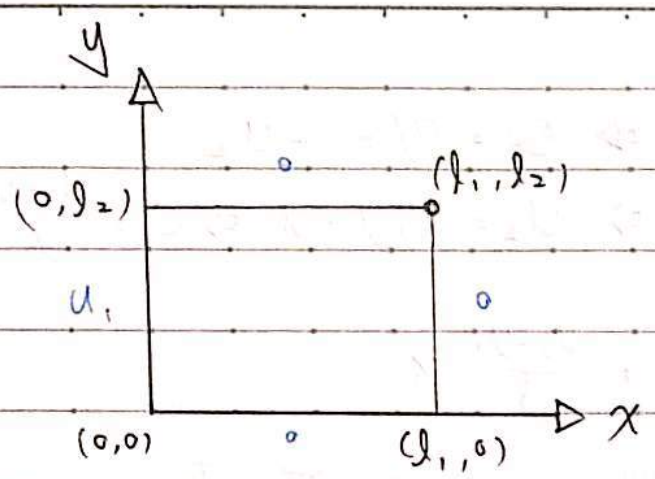


$$\begin{cases} u(0,y) = u_1 \\ u(l,y) = 0 \\ u(x,0) = 0 \\ u(x,l_2) = 0 \end{cases}$$

$\}$ y changes

$\}$ x changes

homogeneous boundary

$\Rightarrow u(x,0) = X(x)Y(0) = 0$

$u(x,l_2) = X(x)Y(l_2) = 0$

$\Rightarrow Y(0) = Y(l_2) = 0$

$\frac{1}{Y}\frac{d^2Y}{dy^2} = 0$, $Y = c_1 y + c_2 = 0 \Rightarrow c_1 = c_2 = 0$, No!

$\frac{1}{Y}\frac{d^2Y}{dy^2} = +\beta^2$, $Y = c_1 e^{\beta y} + c_2 e^{-\beta y} \Rightarrow c_1 = c_2 = 0$, No!!

$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2$, $Y = c_1 \cos\beta y + c_2 \sin\beta y$

$Y(0) = c_1 + c_2 0 = 0$; $\therefore c_1 = 0$

$Y(l_2) = c_2 \sin\beta l_2 \Rightarrow \beta l_2 = n\pi$, $n = \pm 1, \pm 2 \cdots$

$\therefore Y(y) = c_2 \sin\frac{n\pi}{l_2} y$, $n = \pm 1, \pm 2, \cdots\cdots$

$\because \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2$, $\therefore \frac{1}{X}\frac{d^2X}{dx^2} = \beta^2$

(既是 homogeneous 項)

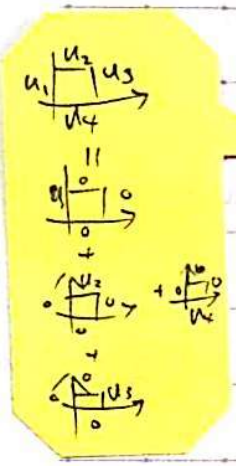
$\therefore X = c_1 e^{\beta x} + c_2 e^{-\beta x}$

$X(l_1) = c_1 e^{\beta l_1} + c_2 e^{-\beta l_2} = 0$, $\therefore c_2 = -c_1 e^{2\beta l_1}$

$\therefore X(x) = c_1 (e^{\beta x} - e^{2\beta l_1} e^{-\beta x})$

i.e., $u(x,y) = X(x)Y(y) = C_1\left(e^{\beta x} - e^{2\beta l_1} - e^{-\beta x}\right) C_2 \sin\frac{n\pi}{l_2}y$

$$= C_1C_2\left(e^{\frac{n\pi}{l_2}} - e^{2\frac{n\pi}{l_2}l_1} e^{-\frac{n\pi}{l_2}x}\right)\sin\frac{n\pi}{l_2}y, \quad n=\pm1,\pm2\ldots$$



$$u(x,y) = \sum C_n\left(e^{\frac{n\pi}{l_2}x} - e^{2\frac{n\pi}{l_2}l_1} - e^{\frac{n\pi}{l_2}x}\right)\sin\frac{n\pi}{l_2}y$$

$$u_1 = u(0,y) = \sum C_n\left(1 - e^{2\frac{n\pi}{l_2}l_1}\right)\sin\frac{n\pi}{l_2}y$$

Back to the diffusion equation in 2 dimensions (in Cartesian coordinate)

For the diffusion equation

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

K : diffusivity

This equation is obtained by treating the material as being homogeneous and isotropic.

Back to conducting of heat.

How can heat be transfered through a material via conduction?

In reference to electrical conduction

metal : free electrons

semeconductor : electrons + holes

Heat conduction

a. free electrons

b. phonons (声子, 声振子, 振子)

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$$\begin{cases} x \text{ - direction} : k_x \frac{\partial T}{\partial x} \\ y \text{ - direction} : k_y \frac{\partial T}{\partial y} \\ z \text{ - direction} : k_z \frac{\partial T}{\partial z} \end{cases}$$

T: temperature

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)$$

$\therefore$ only for isotropic homogeneous meterials such as metals, $k_x = k_y = k_z = k \neq k(\vec{r})$

$$\Rightarrow \rho c \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Last time, we examine a steady 2D case where the governing equation was

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let's get back to

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

If we use separation of variables

$$u = X(x)\,Y(y)\,T(t)$$

$$\frac{1}{K\tau}\frac{d\tau}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

steady state : $\frac{d\tau}{dt} = 0$

$$\frac{1}{K\tau}\frac{d\tau}{dt} = \text{Const} = \begin{cases} +\beta^2 \\ -\beta^2 \quad \checkmark \end{cases}$$ (不会無限增長)

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$$\begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = \begin{cases} +\beta_1^2 \\ -\beta_1^2 \end{cases} \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \begin{cases} +\beta_2^2 \\ -\beta_2^2 \end{cases} \end{cases}$$

Will determined by the homogeneous boundary conditions.

For the homogeneous boundary conditions, the constant is a negative value.

What happens to $\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ when the coordinate system is cylindrical or spherical?

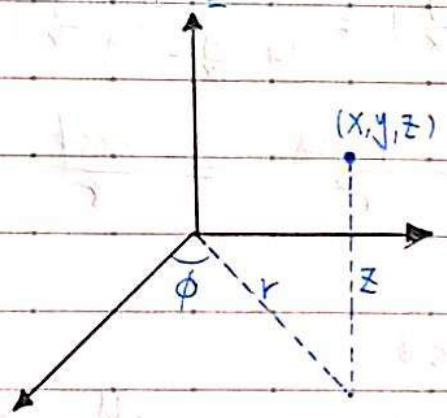
$$\partial u/\partial t = K\nabla\cdot(\nabla u) = \nabla^2 u$$

↳ Laplacian operator

divergence ($\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}$) gradient

A Li's suggestion : Change of variable

$(r, \phi, \underline{\underline{z}})$ vs. $(x, y, \underline{\underline{z}})$ z: the same



$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$u(x,y,z) \Downarrow u(r,\phi,z)$ $\begin{cases} \frac{\partial^2 u}{\partial x^2} \ ? \\ \frac{\partial^2 u}{\partial y^2} \ ? \\ \frac{\partial^2 u}{\partial z^2} \Rightarrow \text{(same)}. \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{\partial u(x,y,z)}{\partial x} = \dfrac{\partial u(r,\phi,z)}{\partial r}\dfrac{\partial r}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \phi}\dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial u(x,y,z)}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \phi}\dfrac{\partial \phi}{\partial y} + \dfrac{\partial u}{\partial z}\cancel{\dfrac{\partial z}{\partial \phi}}^{=0} \end{cases}$$

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$$\begin{cases} r^2 = x^2+y^2 \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\phi}{r} = \cos\phi$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\phi}{r} = \sin\phi$$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1+x^2}\left(\frac{dx}{dx}\right)$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2+y^2} = -\frac{r\sin\phi}{r^2} = -\frac{\sin\phi}{r}$$

$$\left(\begin{aligned} &\frac{\partial}{\partial x}\left[\frac{y}{x} = \tan\phi\right] \Rightarrow -\frac{y}{x^2} = \frac{\partial}{\partial x}\left(\frac{\sin\phi}{\cos\phi}\right) \\ &= \frac{\cos^2\phi+\sin^2\phi}{\cos^2\phi}\frac{\partial\phi}{\partial x} \Rightarrow -\frac{y}{r^2\cos^2\phi} = \frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial x} \\ &\because \partial\phi/\partial x = -y/r^2 = -\frac{r\sin\phi}{r^2} = -\frac{\sin\phi}{r} \end{aligned}\right)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\frac{1}{x}$$

$$= \frac{x}{x^2+y^2} = \frac{r\cos\phi}{r^2} = \frac{\cos\phi}{r}$$

$$\Rightarrow \begin{cases} \partial r/\partial x = \cos\phi \\ \partial r/\partial y = \sin\phi \\ \partial\phi/\partial x = -\sin\phi/r \\ \partial\phi/\partial y = \cos\phi/r \end{cases}, \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\Rightarrow \therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\phi + \frac{\partial u}{\partial \phi}\left(-\frac{\sin\phi}{r}\right) = \cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}$$

$$= \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right)u$$

$$\Rightarrow \frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}$$

※ $r\partial\phi$ 因次才會 $= \partial r = \partial x$!!

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me.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cos\phi - \dfrac{\partial u}{\partial \phi}\dfrac{\sin\phi}{r} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\sin\phi + \dfrac{\partial u}{\partial \phi}\dfrac{\cos\phi}{r} \end{cases};$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)$$

$$= \frac{\partial}{\partial r}\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)\frac{\partial \phi}{\partial x}$$

$$= \frac{\partial}{\partial r}\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)\cos\phi$$

$$+ \frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial u}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u}{\partial \phi}\right)\left(-\frac{\sin\phi}{r}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\sin\phi + \frac{\cos\phi}{r}\frac{\partial u}{\partial \phi}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\phi + \frac{\cos\phi}{r}\frac{\partial u}{\partial \phi}\right)\sin\phi$$

$$+ \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial r}\sin\phi + \frac{\cos\phi}{r}\frac{\partial u}{\partial \phi}\right)\frac{\cos\phi}{r}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

Two simplified cases

a. unsteady 1-D, i.e., $u = u(r,t)$

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

If we assume $u = R(r)\,T(t)$,

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\beta^2$$

i.e., $\frac{1}{kT}\frac{dT}{dt} = -\beta^2$

$$\Rightarrow \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\beta^2,$$

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} + r\beta^2 R = 0$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + r^2\beta^2 R = 0$$

Recall:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Bessel equation !!

$$\Rightarrow r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} + (r\beta)^2 y = 0 \quad (\nu = 0)$$

$$\Rightarrow \beta^2 r^2 \, d^2y/d\beta^2 r^2 + \beta r \, dy/d\beta r + (r\beta)^2 y = 0$$

The general solution of Bessel equation?

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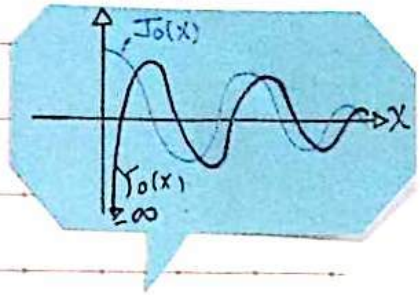
The general solution of Bassel equation of order ν :



$$y = C_1 J_\nu(x) = C_2 J_{-\nu}(x)$$

$$\Downarrow$$

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

$(J_{-n}(x) = (-1)^n J_n(x)$ when n is integer$)$

$(\because Y_0(0) \to \infty \quad \therefore C_2 = 0)$

$\Rightarrow R(r) = c_1 \cdot J_0(\beta r)$,

From $T(t) = C_3 e^{-\beta^2 k t}$

$\therefore u(r,t) = c\, J_0(\beta r)\, e^{-\beta^2 k t}$

If $u(r,0) = f(r)$, how can we determine c?

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