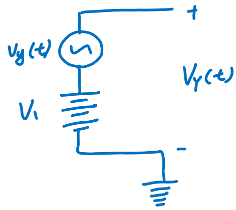
# 13




$V_Y(t) = V_Y + V_y(t)$

Large signal   DC Bias   Small-signal

# Small-signal Analysis

In the last chapter we saw how we could predict the effect of changes in a linear circuit. We now extend our discussion to accommodate two new features. First, many useful circuits contain *nonlinear* components such as diodes, so that the approach to change analysis we have developed, based on the property of *linearity*, is no longer applicable without modification. Second, whereas the examples we have discussed referred to 'unwanted changes', the changes in current and voltage may often be extremely useful in that they represent information. For example, the varying voltage from a CD player may represent an orchestral concert, and our circuit may be an amplifier that accepts these changes in voltage and generates larger voltage changes – but of the same waveform – to apply to a loudspeaker. There is therefore a need to be able to predict the performance of a circuit that contains one or more nonlinear components and whose currents and voltages are varying in response to a source of varying voltage.

In this chapter we also take the opportunity to consider circuits in which the voltage at any node may be composed of two parts: a constant voltage to which is added a time-varying voltage, the latter usually being a useful signal of some sort. To distinguish the various components of a voltage at node Y we typically denote the actual voltage as $v_Y(t)$, its (constant) average value as $V_Y$ and its time-varying component of average value zero by $v_y(t)$, so that

$$v_Y(t) = V_Y + v_y(t) \tag{13.1}$$

## 13.1 The Extension of Change Analysis

If we are to handle the presence of nonlinear components, only a simple modification is required to the method of change analysis developed in Chapter 12. To take

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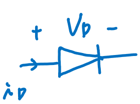



Ideal diode

If $V_D > 0$ Forward bias $i_D > 0$

If $V_D < 0$ Reverse bias $i_D = 0$

an example, Figure 13.1(a) shows the voltage–current characteristic of a diode. It is certainly nonlinear, but we could take the view that, if changes in current and voltage are *sufficiently small,* then there will be an approximately linear relation between them. However, it is clear from Figure 13.1(a) that the value of $\Delta I/\Delta V$ depends upon the values of current and voltage around which the changes occur, values we call the *quiescent*, *bias* or *average* values. For example, if the diode is operating at point **A** the ratio $\Delta I/\Delta V$ is smaller than at point **B**. By contrast, for a linear resistor (Figure 13.1b), the ratio $\Delta I/\Delta V$ is independent of the current and voltage around which the changes occur. Thus, the method of change analysis developed in Chapter 12 is perfectly valid for circuits containing nonlinear components provided that we: (1) restrict the magnitude of the changes so that we can use a linear approximation to part of a nonlinear function; and (2) employ as a change model for each nonlinear component a resistance corresponding to the value of $\Delta I/\Delta V$ at the operating point. Thus, the right-hand column of Table 12.1 undergoes the minor, but important addition highlighted in Figure 13.2. When changes are small the value of $\Delta I/\Delta V$ is called the *incremental conductance* and denoted by $g_{\mathrm{d}}$: its reciprocal $r_{\mathrm{d}}$ is called the *incremental resistance*.

It is because we have to restrict the magnitude of the changes that we refer to *small-signal operation*. The term 'signal' is used because the small variations in current and voltage usually represent a signal of some sort, such as speech or music. The obvious question 'How small is *small*?' will be addressed later.

## 13.2 The Calculation of Incremental Resistance

Before the small-signal analysis of a circuit can be carried out, the incremental resistance of each nonlinear component must be found. There are many nonlinear components that could appear in a circuit, so to illustrate the method of analysis we shall assume that the circuit contains one *exponential diode*, so-called in view of the relation it imposes between voltage and current.

The symbol for an exponential diode and the general nature of its current∼voltage relation were shown in Figure 13.1(a). The relation between the diode current $i_{\mathrm{D}}$ and diode voltage $v_{\mathrm{D}}$ can be derived from a knowledge of semiconductor physics and is

$V_T \simeq 25mA$

$I_D(t) = I_s\left(e^{\frac{V_D(t)}{V_T}} - 1\right)$

$\simeq I_s e^{\frac{V_D(t)}{V_T}}$

forword bias
$V_D(t)$ sufficient
Large

$$i_{\mathrm{D}} = I_{\mathrm{S}}[\exp(v_{\mathrm{D}}/v_{\mathrm{T}}) - 1] \tag{13.2}$$

real-world Diode
exponential diode

where $v_{\mathrm{T}}$ is called the 'thermal voltage' and has a value of 25 mV at room temperature. When the diode is in its 'forward' bias condition ($v_{\mathrm{D}}$ is positive) the current rises rapidly with increase in voltage: when reverse-biased by more than 100 mV the reverse current is approximately $I_{\mathrm{S}}$ in magnitude. $I_{\mathrm{S}}$ is usually very small and can be as low as $10^{-8}$A.

$i_s(t) \rightarrow$ diode, $+ V_s(t) -$ equivalent $\rightarrow$ resistor, $+ V_s(t) -$, $i_s(t)$

$r_d = \frac{V_T}{I_D}$

$i_D \rightarrow$ diode, $+ V_D -$

$$\begin{cases} i_D(t) = I_D + i_s(t) \\ v_D(t) = V_D + V_s(t) \end{cases}$$

$V_s(t) \xrightarrow{?} i_s(t)$

small-signal analysis

$i_D(t) \simeq I_s \, e^{\frac{V_D(t)}{V_T}}$

Assume $\frac{V_s}{V_T} \ll 1$ (25mA)

$$\cancel{I_D} + i_s(t) = I_s \, e^{\frac{V_D + V_s(t)}{V_T}}$$

$$= I_s \, e^{\frac{V_D}{V_T}} \cdot e^{\frac{V_s(t)}{V_T}}$$

$$= I_D \left( 1 + \frac{V_s(t)}{V_T} + \frac{1}{2}\cancel{\left(\frac{V_s(t)}{V_T}\right)^2} + \dots \right)$$

0

$$= \cancel{I_D} + \frac{I_D}{V_T} V_s(t) \Rightarrow \therefore \boxed{i_s(t) \simeq \frac{I_D}{V_T} V_s(t)}$$

$I_D = I_s \, e^{\frac{V_D}{V_T}}$

$I_D(t) \rightarrow$ diode, $+ V_D(t) -$

$I_D \rightarrow$ diode, $+ V_D -$

different Bias Average

$I_D$

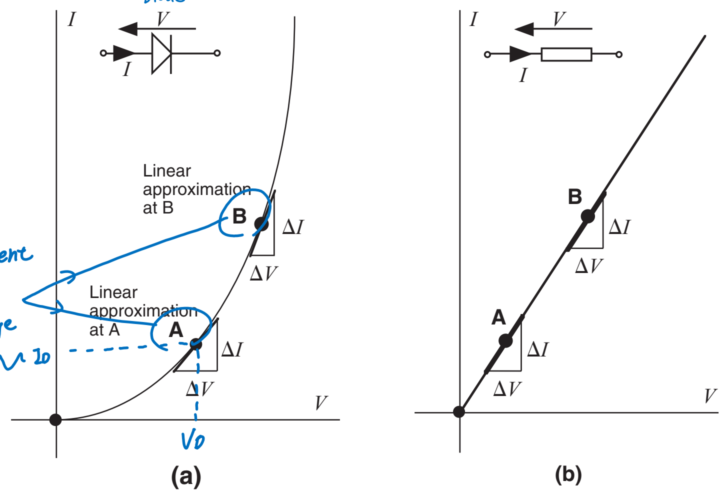



**Figure 13.1** (a) The general form of the characteristic of a diode, showing the variation of $\Delta I/\Delta V$ with quiescent condition; (b) a reminder that for a linear resistor the slope $\Delta I/\Delta V$ is constant

Alternative view (change analysis)

$$\left.\begin{aligned} I_D &\rightarrow I_D + \Delta I \\ V_D &\rightarrow V_D + \Delta V \end{aligned}\right\} \text{relation of } \Delta V \sim \Delta I = ?$$

[Incremental conductance]

define $g_d \equiv \frac{I_D}{V_T}$

$$\frac{\Delta I}{\Delta V} \simeq \frac{d i_D}{d v_D} = \frac{d}{d v_D}\left(\overset{i_D}{I_s \cdot e^{\frac{V_D}{V_T}}}\right) = \frac{I_s}{V_T} e^{\frac{V_D}{V_T}} \simeq \frac{i_D}{V_T} \simeq \frac{I_D}{V_T}$$

$$r_d = \frac{1}{g_d} = \frac{V_T}{I_D} \simeq \frac{25\,mA}{I_D}$$

$\Delta V = r_d \, \Delta I$

$V_Y(t) = V_Y + V_y(t)$

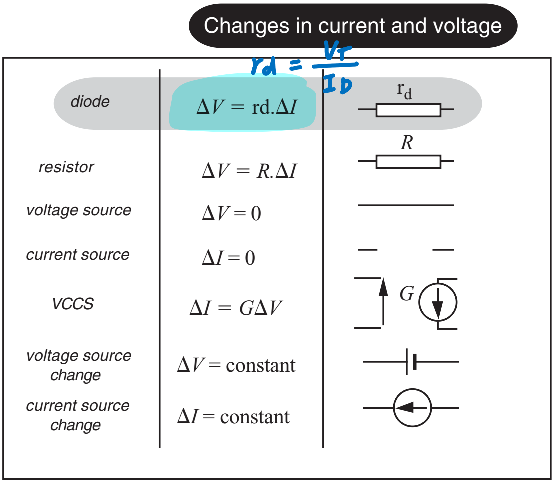



**Figure 13.2** Modification (highlighted) required to Table 12.1 to account for the small-signal operation of a nonlinear component. For a nonlinear component the value of the incremental resistance $r_d$ will vary with quiescent condition

In order to carry out a small-signal analysis of a circuit containing an exponential diode we must be able to calculate the value of $\mathrm{d}i_D/\mathrm{d}v_D$ for any operating point. Differentiating Equation (13.2) with respect to $v_D$ we obtain

$$\mathrm{d}i_D/\mathrm{d}v_D = (I_S/v_T)\exp(v_D/v_T) \tag{13.3}$$

If the diode voltage $v_D$ exceeds 100 mV (i.e., $v_D/v_T \gg 1$) then, to a good approximation, Equation (13.2) can be rewritten as $i_D = I_S\exp(v_D/v_T)$ so that Equation (13.3) becomes

$$\mathrm{d}i_D/\mathrm{d}v_D = I_D/v_T \tag{13.4}$$

The term on the left of Equation (13.4) has the dimensions of conductance, so we can say that the incremental conductance of the diode when the direct current through it is $I_D$ is given by

$$g_d = I_D/v_T \tag{13.5}$$

If, as pointed out at the beginning of this chapter, we denote changes in currents and voltages around their average values by lowercase $i$ and $v$, both with lowercase subscripts, we can rewrite Equation (13.4) as:

$$i_d = g_d\, v_d \tag{13.6}$$

or, if one's preference is for working in terms of resistance,

$$v_d = r_d \, i_d \tag{13.7}$$

where $r_d = 1/g_d$.

The advantage of the relation shown as Equation (13.6) is, of course, that it is linear. It is also useful to ask whether Eqaution (13.5) appears reasonable. Examination of Figure 13.1(a) will show that it is, because at the lower quiescent condition (A) – i.e., the lower value of $I_D$ – the slope is smaller.

From this, we see that, when applying change analysis to a circuit containing an exponential diode, that diode must be represented in the change model by a resistance having a value $r_d$ equal to the thermal voltage $v_T$ (= 25 mV) divided by the quiescent diode current $I_D$ .

## How small is 'small'?

We have suggested – and illustrated in Figure 13.1(a) – that it might be acceptable to approximate a limited region of a nonlinear characteristic by a linear segment. How small must this linear segment be for the approximation to be acceptable? The answer can be obtained by taking the expression for diode current (Equation 13.2), by assuming that $v_D$ is sufficiently high for the minus one term to be negligible, and by expressing the diode voltage $v_D$ as the sum of its quiescent and signal components:

$$i_D(t) = I_S \exp[\{V_D + v_d(t)\}/v_T] = I_S \exp(V_D/v_T)\exp(v_d(t)/v_T)$$

Recalling the series expression for $e^x$:

$$e^x = 1 + x + x^2/2 + x^3/6 + \dots.$$

we can write that, if $v_d(t)/v_T \ll 1$,

$$i_D(t) = I_S \exp(v_D/v_T)[1 + v_d(t)/v_T]$$

Since, to a good appoximation, $I_S \exp(v_D/v_T) = I_D$, the quiescent value of the diode current, the signal component of $i_D(t)$ is

$$i_d(t) = I_D \, v_d(t)/v_T$$

which is in agreement with Equation (13.6). Thus we see that a linear approximation is valid if $v_d(t) \ll v_T$.

This result has been derived for an exponential diode, and is not necessarily valid for other nonlinear components.

## *Example 13.1*

Figure 13.3 shows a circuit containing one exponential diode. We are required to calculate the total voltage appearing across the diode, i.e., both average and time-varying (signal) component. Because the method of obtaining the solution is common to many problems involving small-signal analysis, we shall formalize the solution into five identifiable steps (*in italics*) and concurrently apply those solution steps to the circuit of Figure 13.3 (non-italic text). $V_D = ?$

$i_D = I_s e^{\frac{V_D}{V_T}}$

source $V_s(t) = 20 + 1\sin 100t$, $V_s$ = bias, $v_s(t)$ small signal

Step 1: Find DC bias (quiessant condition)

$V_s = 20V$ $V_D = 0.7$

$I_D = \frac{20 - V_D}{10k} = 1.93\,mA$

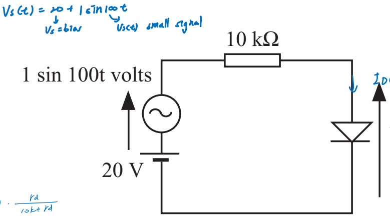
Step 2: Find small signal model

$r_d = \frac{1}{g_d} = \frac{V_T}{I_D} = \frac{25mV}{1.93mA} = 12.95\,\Omega$

Step 3: create small signal model

Step 4: Analyse small signal ckt $V_s(t) = V_s(t) \cdot \frac{r_d}{10k + r_d}$

$= 1.29 \sin(100t)$ (mV)




$I_D(t) = \underbrace{I_D}_{1.93mA} + I_s(t)$

$V_D(t) = \underbrace{V_D}_{0.7V} + V_s(t)$

**Figure 13.3** A circuit containing a nonlinear component (an exponential diode) and both a constant source and a small-signal source

### Step 1

*If the circuit contains one or more nonlinear components, find their quiescent conditions with signal amplitudes set to zero.*

We shall assume in this example that the direct current through the diode is sufficient to establish a voltage $v_D$ of 0.7 V across the diode. Then, by Ohm's law, the current through the resistor (and hence the diode) is (20 – 0.7)/10 = 1.93 mA. This is the value of $I_D$ required in Equation 13.5.

### Step 2

*Find the small-signal model of each nonlinear component at its quiescent condition.*

From Equation (13.5), $g_d = I_D/v_T = 1.93/25 = 0.077$ S, or $r_d = 12.95\ \Omega$

**Step 3**

*Create the small-signal equivalent of the actual circuit by replacing each component with its small-signal model.*

See Figure 13.4

$V_s(t) = V_s(t) \cdot \frac{r_d}{10k + r_d}$

$= 1.29 \sin(100t)$ (mV)

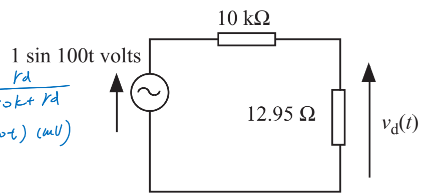



**Figure 13.4** The small-signal equivalent of the circuit of Figure 13.3

**Step 4**

*Analyse the small-signal equivalent circuit to find the signal components of the required voltages and currents.*

By the voltage divider principle, $v_d(t) = [12.95/(10000 + 12.95)] \times 1 \sin 100t = 1.29 \sin 100t$ mV, to a good approximation

As a reminder, we mention that none of the constant voltages in the actual circuit (e.g., the 20 V source and the 0.7 V across the diode) appear in the small-signal equivalent circuit.

**Step 5**

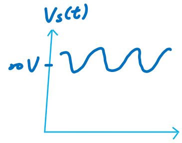
*Check that signal amplitudes are sufficiently small for the linearity*

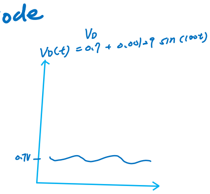
$\frac{V_s}{V_T} << 1$ $\quad \left|\frac{V_s(t)}{V_T}\right| \leq \frac{1.29\,mV}{25\,mV} << 1$

Total (large-signal) Voltage across diode

$V_D(t) = V_D + V_s(t)$

$= 0.7V + 1.29\,mV$



Summary

locally linearization

Nonlinear $\rightarrow$ approximation $\longrightarrow$ linear analysis

$I = f(V)$ $\quad I + \Delta I = f(V + \Delta V)$

$\Delta I \simeq \left(\frac{df}{dV}\right)\Delta V$

Diode : $\frac{1}{r_d} = g_d$ $\quad V_D$ : sufficient large

*assumption to be valid (e.g., $v_d/v_T \ll 1$ for an exponential diode).*

For the diode, $v_d(t)$ is always much less than 25 mV because its maximum value is 1.29 mV as found in step 4.

The original problem was to find the voltage $v_D(t)$ across the diode. This is the sum of its quiescent and signal components, i.e.:

$$v_D(t) = 0.7 + 0.00129 \sin 100t \text{ V}$$

## 13.3 Problems

**Problem 13.1**

Find the incremental resistance of the exponential diode in the circuit of Figure P13.1 for the quiescent current determined by the current source. If the value of the current source is increased by 0.1 mA what change can be expected in the diode voltage $v_D$?

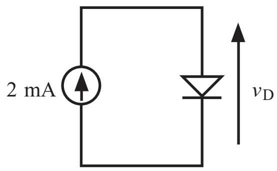



**Figure P13.1**

**Problem 13.2**

For the circuit shown in Figure P13.2 determine the quiescent current flowing in the diodes by making a reasonable assumption about the voltage across each diode.

For each diode determine its incremental resistance

Under the assumption that the capacitors possess negligible impedance at the frequency of the sinusoidal voltage source, draw the small-signal equivalent circuit of the actual circuit. Hence calculate the amplitude of the sinusoidal voltages $v_a(t)$ and $v_b(t)$.

If the amplitude of the sinusoidal voltage source is increased above its current value of 2 mV, what is the approximate maximum value it can assume without violating the assumption that both diodes are operating in an essentially linear manner?

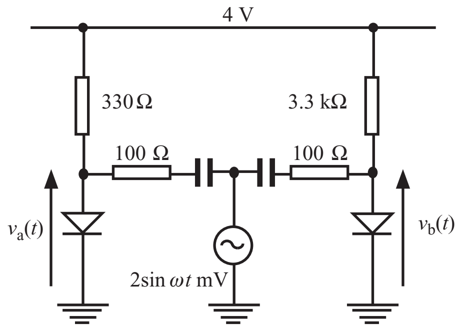



**Figure P13.2**

### Problem 13.3

The circuit shown in Figure P13.3 is a voltage-controlled voltage divider: the direct voltage $V$ determines the quiescent current through the exponential diodes, thereby affecting their incremental resistance. The terms $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ denote, respectively, the small-signal voltages at the input and output of the voltage divider.

Assume that the capacitor has negligible impedance at the frequency of the sinusoidal source, and that the voltage $V$ can range between 5 and 20 V.

For each of the extreme values of $V$ (5 and 20 V) calculate the small-signal voltage amplification $v_{\text{out}}/v_{\text{in}}$ and the maximum magnitude of $v_{\text{out}}$ for the calculation to be reasonably accurate. For each diode it may be assumed that, for $i_D > 0.1$ mA, $v_D = 0.7$ V.

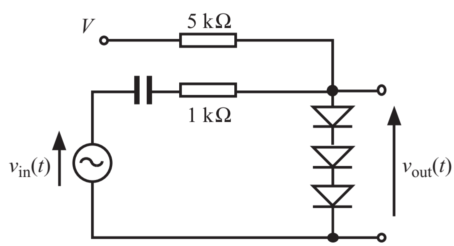



**Figure P13.3**

### Problem 13.4

In the circuit of Figure P13.4 the voltage across each exponential diode can be assumed to be approximately 0.7 V if the diode current exceeds 0.2 mA.

Apply Kirchhoff's current law at point X and hence determine the quiescent voltage at this point and the current in each of the diodes.

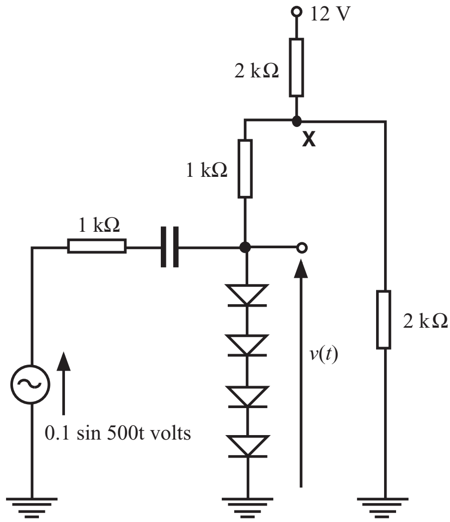



Figure P13.4

Determine the small-signal resistance of each diode.

Assuming that the impedance of the capacitor is negligible, draw the small-signal equivalent of the circuit and hence calculate the peak-to-peak amplitude of the sinusoidal voltage $v(t)$.