Discrete Mathematics

Homework 4

5-1 2.

a)

$\{(1,2)\}$

$\{(1,2),(1,4),(1,5)\}$

$\{(1,2),(2,4),(3,5)\}$

b)

$\{(1,1), (1,2), (1,3)\}$

$\{(2,1), (2,2), (2,3)\}$

$\{(3,1), (3,2), (3,3)\}$

5-1 4.

1. When A=B, $A \times B = B \times A$

Let A=(a1, a2, a3,…, an), B=(b1, b2, b3,…, bm)

Then $A \times B = \{(a1, b1),(a1, b2), \cdots (a1, bm),(a2, b1),(a2, b2), \cdots,$
$(a2, bm), \cdots (an, b1),(an, b2), \cdots (an, bm)\}$

Then $B \times A = \{(b1, a1),(b1, a2), \cdots (b1, an),(b2, a1),(b2, a2), \cdots,$
$(b2, an), \cdots (bm, a1),(bm, a2), \cdots (bm, an)\}$

If $A \times B = B \times A$,

$\{(a1, b1),(a1, b2), \cdots (an, bn)\} == \{(b1, a1),(b1, a2), \cdots (bn, an)\}$

By definition, we know if above is true, $(a1, b1) = (b1, a1), (a1, b2) = (b1, a2), \dots, (ai, bj) = (bi, aj), \dots, (an, bm) = (bm, an)$ for all $1 \le i, j \le \max(m, n)$

Get $ai = bi$ for all $1 \le i \le m$ and $m = n$, which means $(a1, a2, a3, \dots, an) = (b1, b2, b3, \dots, bm)$, so A=B

2. When A or B is $\emptyset$, Because $X \times \emptyset = \emptyset$ for all set X.

5-2 2.

$f(x) = \frac{1}{(x^2-2)}$, if $f(x) \in R$, $(x^2 - 2) \neq 0, x \neq \pm\sqrt{2}$

$\pm\sqrt{2} \in R$, so the domain of $f(x)$ is $R - \{\pm\sqrt{2}\}$, not R

a)

$R - \{\pm\sqrt{2}\} \subset R, f$ can't define $f: R \to R$

b)

However, $Z \subset R - \{\pm\sqrt{2}\}$, so we can define $f: Z \to R$

5-2 18.

$f(x) = x^2, A = R, A_1 = [0, +\infty), A_2 = (-\infty, 0]$

$A_1 \cap A_2 = \{0\}, f(A_1 \cap A_2) = 0$

$f(A_1) = [0, +\infty), f(A_2) = [0, +\infty), f(A_1) \cap f(A_2) = [0, +\infty)$

$f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$

5-3 2.

a)

one-to-one, onto

b)

one-to-one

However, we can't find $x \in Z$ s.t. $f(x) = 2$, not onto

Range $f(z) =$ all odd integer.

c)

one-to-one, onto

d)

$f(x) = x^2, f(-x) = f(x)$, not one-to-one

We can't find any $x \in Z$ s.t. $f(x) = -1$, not onto

Range $f(z) = \{1,4,9,16 \cdots\}$

e)

$f(x) = x^2 + x, f(0) = f(-1) = 0$,not one-to-one

We can't find any $x \in Z$ s.t. $f(x) = -1$, not onto

Range $f(z) = \{0, 2, 6, 12, 20, \cdots\}$

f)

one-to-one

We can't find any $x \in Z$ s.t. $f(x) = 2$, not onto

Range $f(z) = \{\cdots, -64, -27, -8, -1, 0, 1, 8, 27, 64 \cdots\}$

5-3 4.

a)

Total functions: $6 \times 6 \times 6 \times 6 = 6^4 = 1296$

One-to-one functions: $6 \times 5 \times 4 \times 3 = 360$

Onto functions: No, we could inject one element in domain to one element in range, and # of range elements is bigger than # of domain

elements.
b)
Total functions: $4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6 = 4096$
Onto functions: $\sum_{k=0}^{4} (-1)^k C_{4-k}^{4} (4-k)^6 = 4!\,S(6,4)$

One-to-one functions: 0, 原因和 a 的 onto 很像。

5-4 6.
$A \times A = \{(x,x),(x,a),(x,b),(x,c),(x,d),\ldots,(d,d)\}$
a)
We have totally 25 elements need to inject on A, and one of them ((a,b)) is selected.
We left 24, each have 5 kinds to choose, so 5^{24} closed binary operation satisfy $f(a,b) = c$.
b)
If we want x as an identity, then (x,y) and (y,x) should be x $(y \in A)$, we have 9 is selected, and don't forget (a,b), totally 10 is selected.
We left 15, each have 5 kinds to choose, so 5^{15} closed binary operation satisfy the situation.
c)
if we have an identity, we should choose an identity from c,d,x (a, b can't be identity).
So we have 3×5^{15} functions to choose.
d)
if $f(a,b) = f(b,a)$, it satisfied commutative law. We have 3 identity could choose (set we choose from k1, k2, k3, and we choose k3as identity), and we could choose (a,a), (a,k1), (a,k2), (b,b), (b,k1), (b,k2), (k1,a), (k1,k2), (k2,k2).
So we have 3×5^9 kinds of functions.

5-4 8.
Yes, we can expressed A as $A = \{2, 2^2, 2^3, 2^4, 2^5\}$.
Then $f(x,32) = \gcd(x,2^5) = x$,

and $\gcd(2^i, 2^5) = 2^i = \gcd(2^5, 2^i)$ for $1 \le i \le 5$,

so $x = 2^5 = 32$ is identity element for f

5-5 2.

There have 7 days a week, and know we have 8 people, just like we have 7 pigeonholes and 8 pigeons.
By Pigeonhole Principle, there will at least 2 people have birthday occurs on the same day of the week.

5-5 4.
110=103+7=101+9=…=55+55, so we can separate set S into {3}, {7,103}, {11,99},…, {51,59}, {55} , totally 14 subset (pigeonholes).
By Pigeonhole Principle, we need to choose at least 15 elements (pigeons) in set S such that at least two whose sum is 110.

5-6.2
a)

$g(x) = \frac{2x^2-8}{x+2} = \frac{2(x+2)(x-2)}{x+2}$

for all $x \in A = (-2,7]$, g(x) could be simplify as $g(x) = 2(x-2)$
Then, $g(x) = 2x - 4 = f(x)$
b)

Yes, because there is undefined in $g(x)$ when $x = -2$ $\left(g(x) = \frac{0}{0}\right)$,

but $f(-2) = -8$.

5-6.22
The function is invertible only when it is one-to-one and onto, and to let $f: A \to B$ is one-to-one and onto, there has $5! = 120$ ways.