Calculus I **Midterm II** **November. 25, 2020**

Class : ________ **Student ID**: ________ **Name**: ________ **Score**: ________

1. (10 points) Find $\frac{dy}{dx}$ by implicit differentiation.

   (a) $x^2 - 4xy + y^2 = 9$.

   $\frac{d}{dx}(x^2-4xy+y^2)=\frac{d}{dx}(9)$

   $\Rightarrow 2x-4y-4xy'+2yy'=0$

   $\Rightarrow 2x-4y+(-4x+2y)y'=0$

   $\Rightarrow y'=\frac{2x-4y}{4x-2y}=\frac{x-2y}{2x-y}$

   (b) $\cos(x^2 + 2y^2) = xe^y$

   $\frac{d}{dx}(\cos(x^2+2y^2))=\frac{d}{dx}(xe^y)$

   $\Rightarrow -\sin(x^2+2y^2)\cdot(2x+4yy')=e^y+xe^y y'$

   $\Rightarrow -2x\sin(x^2+2y^2)-e^y=(4y\sin(x^2+2y^2)+xe^y)\cdot y'$

   $\Rightarrow y'=\frac{-2x\sin(x^2+2y^2)-e^y}{4y\sin(x^2+2y^2)+xe^y}$

2. (16 points) Find the absolute maximum and absolute minimum values of f on the given interval.

(a) $f(x) = x^3 - 3x^2 - 9x + 1$, $[-4, 4]$.

$\because f$ is a polynomial $\therefore f'(x)$ exists for all $x \in (-4, 4)$

$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) \Rightarrow x = -1, 3$ are the only critical numbers

Compare $f(-1) = 6$, $f(3) = -26$, $f(-4) = -75$, $f(4) = -19$,

Absolute maximum is 6 and absolute minimum is -75

(b) $f(x) = \dfrac{\sqrt{x}}{1 + x^2}$, $[0, 2]$.

$\because \sqrt{x}$ is differentiable on $(0,2)$ and $\frac{1}{1+x^2}$ is differentiable on $(0,2)$

and $\frac{1}{1+x^2} \neq 0$ for all $x \in (0,2)$

$\therefore \frac{\sqrt{x}}{1+x^2}$ is differentiable on $(0,2)$

$f'(x) = \dfrac{\frac{1}{2\sqrt{x}}(1+x^2) - \sqrt{x}(2x)}{(1+x^2)^2} = \dfrac{1}{2\sqrt{x}(1+x^2)^2}(1 + x^2 - 4x^2)$

The only critical numbers are $x = \frac{1}{\sqrt{3}}$

Compare $f(\frac{1}{\sqrt{3}}) = \frac{1}{4} 3^{\frac{3}{4}}$, $f(0) = 0$, $f(2) = \frac{\sqrt{2}}{5}$

Absolute maximum is $\frac{1}{4} 3^{\frac{3}{4}}$ and absolute minimum is 0

3. (16 points) Determine whether the following limits are indeterminate forms and compute the limit if they exist.

(a) $\displaystyle\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin\theta}{1 + \cos(10\theta)}$.

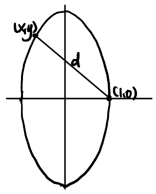
$$\lim_{\theta \to \pi/2} \frac{1-\sin\theta}{1+\cos(10\theta)} \left(\frac{0}{0}\right) \overset{\text{LiH}}{=} \lim_{\theta \to \pi/2} \frac{\sin\theta}{-10\sin(10\theta)} \left(\frac{0}{0}\right) \overset{\text{LiH}}{=} \lim_{\theta \to \pi/2} \frac{\cos\theta}{-100\cos(10\theta)} = \frac{1}{100}$$

(b) $\displaystyle\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{4x^3 + 2x^4}$.

$$\lim_{x \to 0} \frac{e^{2x}-1-2x-2x^2}{4x^3+2x^4} \left(\frac{0}{0}\right) \overset{\text{LiH}}{=} \lim_{x \to 0} \frac{2e^{2x}-2-4x}{12x^2+8x^3} \left(\frac{0}{0}\right) \overset{\text{LiH}}{=} \lim_{x \to 0} \frac{4e^{2x}-4}{24x+24x^2} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{8e^{2x}}{24+48x} = \frac{1}{3}$$

4. (16 points) Let $y^2 + 4x^2 = 4$ be a ellipse.

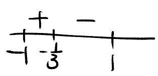
(a) Find the points which are farthest away from the point $(1, 0)$.



$\because y^2=4-4x^2 \therefore d^2=(x-1)^2+y^2=(x-1)^2+4-4x^2=-3x^2-2x+5$

Let $D(x)=-3x^2-2x+5, -1\le x\le 1$

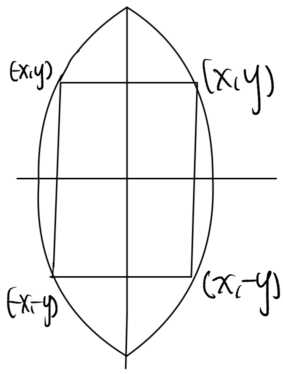
$D'(x)=-6x-2 \Rightarrow x=-\frac{1}{3}$



$\because$ D(x) increases on $(-1,-\frac{1}{3})$ decreases on $(-\frac{1}{3},1)$

$\therefore$ $D(-\frac{1}{3})$ is farest distance and $(-\frac{1}{3},\frac{\sqrt{32}}{3})$, $(-\frac{1}{3},-\frac{\sqrt{32}}{3})$ are farthest points

(b) Find the area of the largest rectangle that can be inscribed in this ellipse.

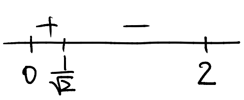


$\because$ the largest rectangle can be inscribed in this ellipse must have the vertices (x,y), $(-x,y)$, $(x,-y)$, $(-x,-y)$

$\therefore$ Area $=4xy$. Consider $(4xy)^2=16x^2y^2=16x^2(4-4x^2)$

Let $D(x)=16x^2(4-4x^2)=64x^2-64x^4$, $0\le x\le 2$

$D'(x)=128x-256x^3=128x(1-2x^2)=128x(1-\sqrt{2}x)(1+\sqrt{2}x)$



$\because$ D(x) increases on $(0,\frac{1}{\sqrt{2}})$ decreases on $(\frac{1}{\sqrt{2}},2)$

$\therefore$ $D(\frac{1}{\sqrt{2}})=16$ is absolute maximum value of D(x)

Therefore $4=\sqrt{16}$ is absolute maximum of the area.

5. (16 points) Determine whether the limit of the following sequences converges. Find the limits if they exist.

(a) $a_n = \dfrac{6n}{4+2n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6n}{4+2n} = \lim_{n\to\infty} \frac{6}{\frac{4}{n}+2} = 3$$

(b) $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2+a_n}$.

We want to use M.S.T to say a_n converges.

- $\{a_n\}$ is increasing

  For $n=1$, $a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$

  Suppose $a_{n-1} > a_{n-2}$, we have $a_n = \sqrt{2+a_{n-1}} > \sqrt{2+a_{n-2}} = a_{n-1}$

  By induction, $\{a_n\}$ is increasing.

- $\{a_n\}$ is bouned above by 2

  For $n=1$, $a_1 = \sqrt{2} \leq 2$.

  Suppose $a_{n-1} \leq 2$, we have $a_n = \sqrt{2+a_{n-1}} \leq \sqrt{2+2} = 2$

  By induction, $\{a_n\}$ is bounded above bi 2.

By M.S.T, a_n converges. Let $L = \lim_{n\to\infty} a_n$

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+\lim_{n\to\infty} a_n} = \sqrt{2+L} \Rightarrow L^2 = 2+L \Rightarrow (L-2)(L+1) = 0$

$\because a_n > 0 \therefore L = 2$

6. (16 points) Determine whether the following series are convergent or not. Explain your reasons.

(a) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt[2020]{k}}$.

$\because \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt[2020]{k}}$ is a alternating series

We need to verify $\frac{1}{\sqrt[2020]{k}}$ decreases and $\lim_{k\to\infty}\frac{1}{\sqrt[2020]{k}}=0$

Let $f(x)=x^{-\frac{1}{2020}}$, $x>0$, $f'(x)=-\frac{1}{2020}x^{-\frac{2021}{2020}}<0$

$\therefore \frac{1}{\sqrt[2020]{k}}=f(k)$ is decreasing

$\lim_{k\to\infty}\frac{1}{\sqrt[2020]{k}}=0$ clearly.

By alternating series test, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt[2020]{k}}$ converges.

(b) $\sum_{n=1}^{\infty}(\sin 1125)^n$.

$\because 1125 \neq \frac{\pi}{2}+k\pi$ for some integer k

$\therefore -1<\sin 1125<1$ and $|\sin 1125|<1$

Let $S_n=\sum_{k=1}^{n}(\sin(1125))^k=\frac{\sin 1125(1-(\sin 1125)^n)}{1-\sin 1125}$

$\because |\sin 1125|<1 \therefore \lim_{n\to\infty}(\sin 1125)^n=0$

and $\sum_{n=1}^{\infty}(\sin 1125)^n=\lim_{n\to\infty}S_n=\frac{\sin 1125}{1-\sin 1125}$

$\therefore \sum_{n=1}^{\infty}(\sin 1125)^n$ converges.

7. (10 points) Let f be a differentiable function satisfying $f''(x) = f(x)$ and $f(0) = 1$, $f'(0) = 1$.

(a) Show that $f'^2 = f^2$ (Recall: by Mean Value Theorem, if $g' = h'$, $g(x) = h(x) + C$ for some constant C).

$$\frac{d}{dx} f'(x)^2 = 2f'(x)f''(x) = 2f'(x)f(x) = \frac{d}{dx} f(x)^2$$

By M.V.T, $f'(x)^2 = f(x)^2 + C$ where C is a constant.

$\because f(0)=1, f'(0)=1 \therefore 1=1+C$ and $f'(x)^2 = f(x)^2$

(b) What can we conclude if $f(x), f'(x) > 0$ for all x? Can we write down the formula for f?

$\because f(x)>0, f'(x)>0$ for all x

$\therefore f'(x) = f(x)$

$f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1 \Rightarrow \frac{d}{dx}(\ln f(x)) = 1$

$\because \frac{d}{dx}(x) = 1 \therefore$ By M.V.T, $\ln f(x) = x + C$

$\because f(0) = 1 \therefore C=0$ and $\ln f(x) = x \Rightarrow f(x) = e^x$