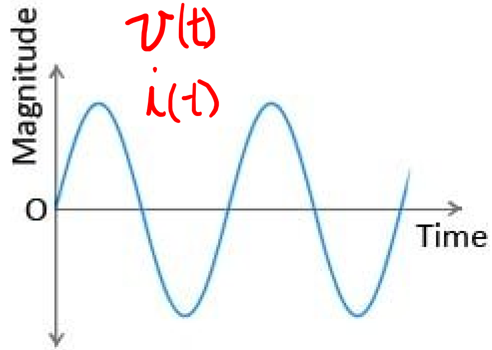
電子電工學
Lecture 11

Recap: AC circuits

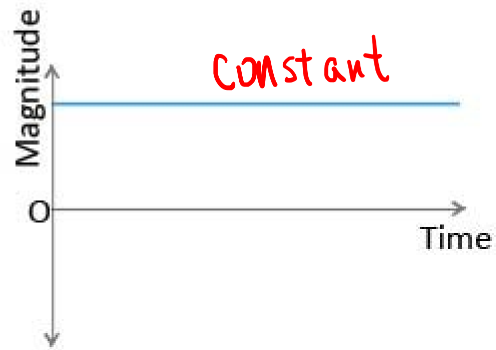


$v(t)$

$i(t)$

Alternating Current

AC



constant

Direct Current

DC

Sinusoidal Wave

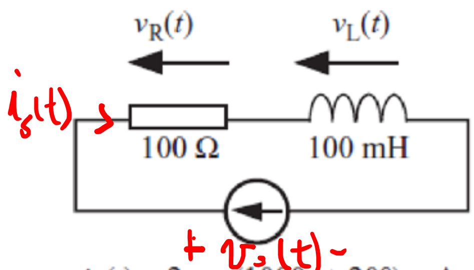
$$\cos(\omega t) \longrightarrow -\sin(\omega t) = \cos(\omega t + 90^\circ)$$

Recap: basic AC sinusoidal wave

KVL : $v_s(t) = v_R(t) + v_L(t)$

$= \cdots\cdot$

$= 200\left[\cos(\omega t + 20^\circ) + \cos(\omega t + 110^\circ)\right]$



$i_S(t) = 2\cos(1000t + 20°)$ mA

Figure 9.10 The circuit discussed in Example 9.2

Given $v_s(t) \rightarrow$ Find $i_s(t) = ?$??

Differential Equation ?

- Laplace Transform
- Fourier Transform

$\downarrow$ Algebraic

✱ Phasor diagram

Transform

Sum to Product Formula

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

Product to Sum Formula

$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$
$$-2\sin x\sin y = \cos(x+y) - \cos(x-y)$$
$$2\sin x\cos y = \sin(x+y) + \sin(x-y)$$
$$2\cos x\sin y = \sin(x+y) - \sin(x-y).$$

Euler's formula

Let $j = \sqrt{-1}$

$$\boxed{e^{jx} = \cos x + j \sin x}$$

2nd ODE :

e.g. $y''(x) + \omega^2 y(x) = 0$

Sol 1. $y(x) = A\cos\omega x + B\sin\omega x$

$y'(x) = -A\omega\sin\omega x + B\omega\cos\omega x$

$y''(x) = -A\omega^2\cos\omega x - B\omega^2\sin\omega x$

Real Domain Sol'n

Sol 2. Try $y = e^{\lambda x} \rightarrow y' = \lambda e^{\lambda x} \rightarrow y'' = \lambda^2 e^{\lambda x}$

$\Rightarrow \lambda^2 e^{\lambda x} + \omega^2 e^{\lambda x} = 0$

$\Rightarrow \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = +j\omega, -j\omega$

Complex Domain Sol'n

$\Rightarrow y_1 = e^{j\omega x} \qquad y_2 = e^{-j\omega x}$

Euler's formula for ODE

Taylor's expansion

$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos x \qquad + j \sin x$$

Euler's formula

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$

Conjugate

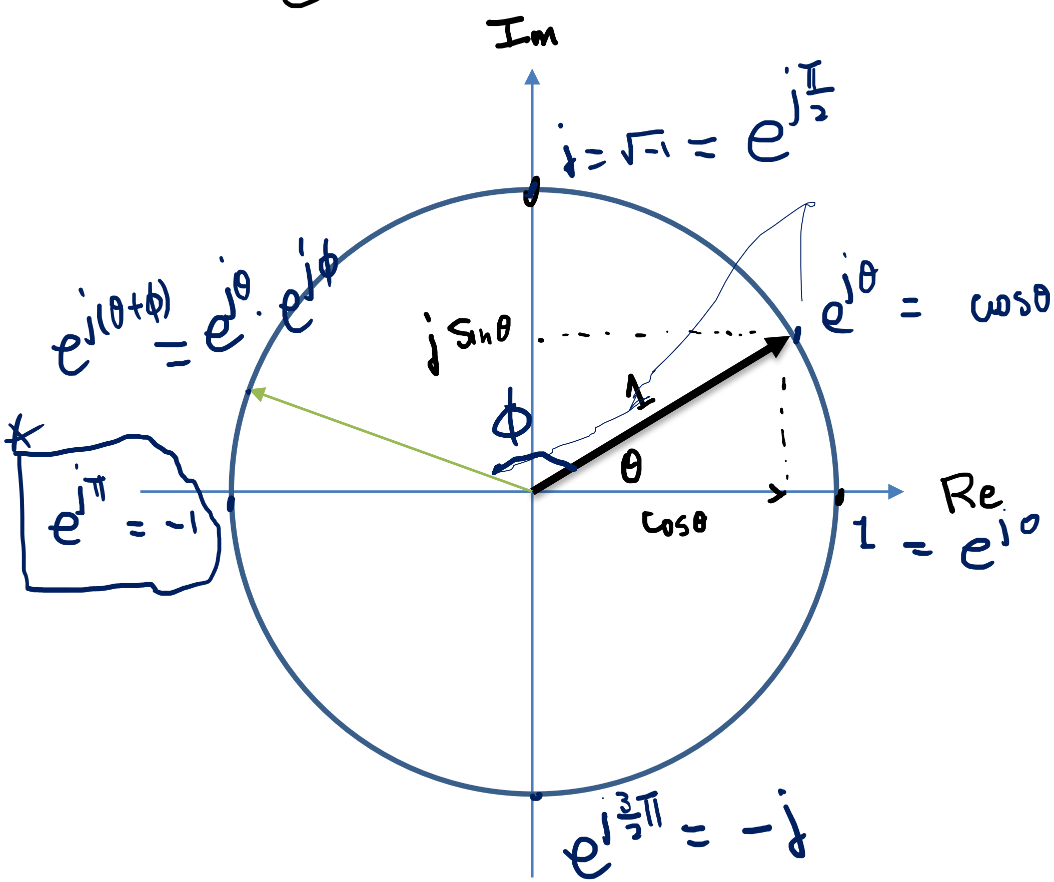
$$\Rightarrow \quad \cos x = \frac{1}{2}\left[e^{jx} + e^{-jx}\right] = \mathrm{Re}\left[e^{jx}\right]$$
$$\sin x = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right] = \mathrm{Im}\left(e^{jx}\right]$$

Euler's formula

$$e^{jx} = \cos x + j \sin x$$

[Unit Circle in Complex Pane]

$$e^{j\theta} = \cos\theta + j\sin\theta$$



$$\boxed{e^{j\pi} + 1 = 0}$$

"...the most remarkable formula in mathematics."

--Richard Feynman

Euler's formula for R-L circuits

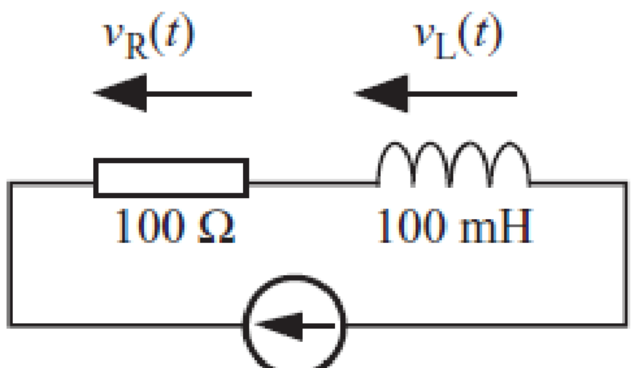
Real $i_s(t) = 2\cos(1000t + 20^\circ)$

$+ 2j\sin(1000t + 20^\circ)$ $\longrightarrow$ Complex $\hat{i}_s(t) = 2e^{j(1000t + 20^\circ)}$

KVL: $\hat{v}_s(t) = R\,\hat{i}_s(t) + L\frac{d}{dt}\hat{i}_s(t)$

$= 200e^{j(1000t+20^\circ)} + 0.1 \cdot 2j\,1000\,e^{j(1000t+20^\circ)}$

$= 200(1+j)\,e^{j(1000t+20^\circ)}$



$+\ v_s(t)\ -$

$i_S(t) = 2\cos(1000t + 20^\circ)$ mA

Figure 9.10 The circuit discussed in Example 9.2

Euler's formula for R-L circuits

$$\hat{v}_s(t) = 200(1+j)\, e^{j(1000t + 20^\circ)}$$

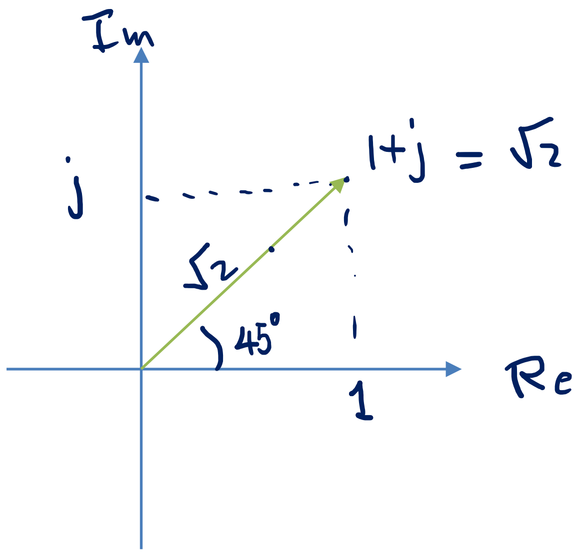
$$= 200\sqrt{2}\, e^{j45^\circ}\, e^{j(1000t + 20^\circ)}$$

$$= 200\sqrt{2}\, e^{j(1000t + 65^\circ)}$$

Re $\hookrightarrow$ $v_s(t) = 200\sqrt{2}\cos(1000t + 65^\circ)$

No sum-product rule!!

$1+j = \sqrt{2}\, e^{j45^\circ}$



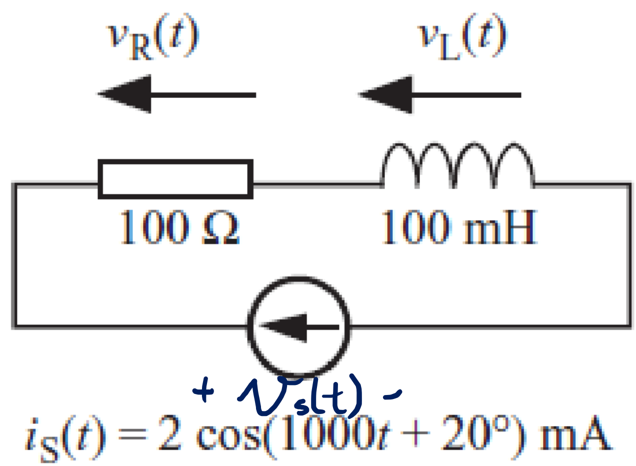


Figure 9.10 The circuit discussed in Example 9.2

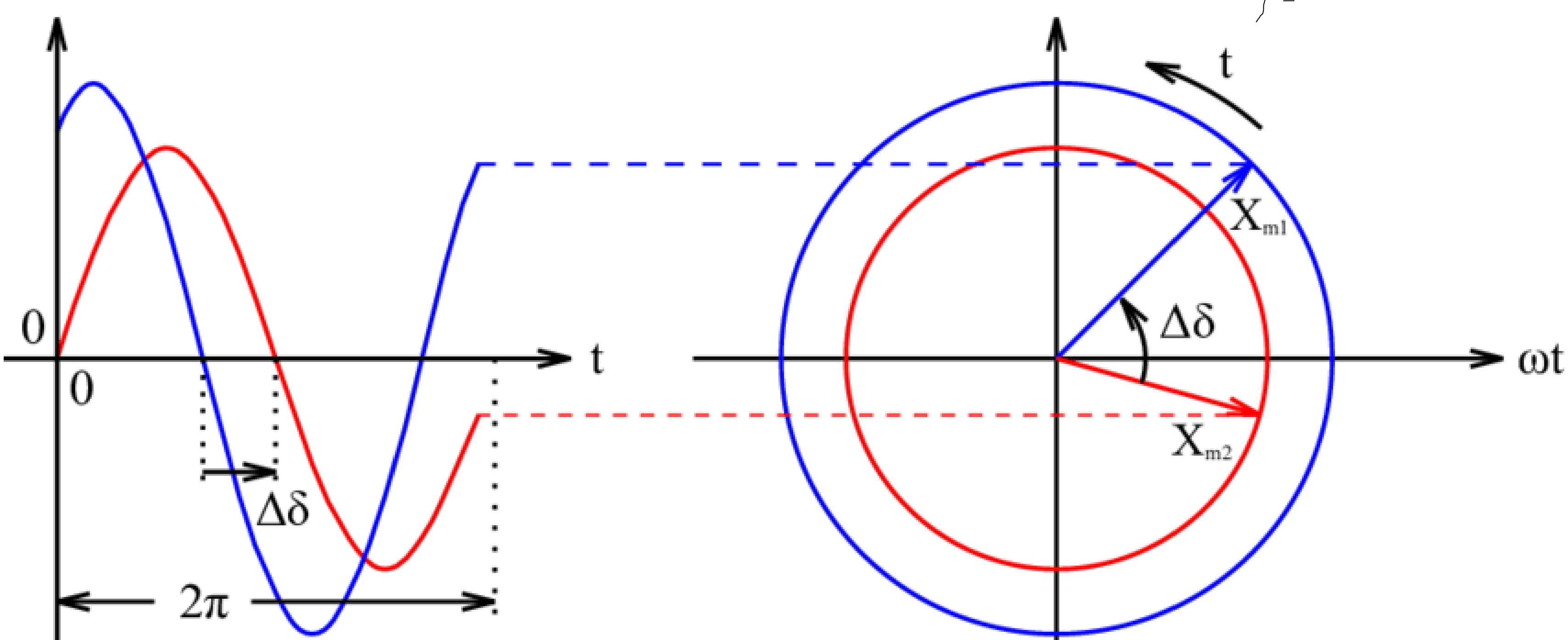
Phasor $\longrightarrow$ Phase Vector

相量

Real Voltage, Current $\xrightarrow{\text{Transform}}$ Complex Voltage, current

$v(t) = V\cos(\omega t) \longrightarrow V(t) = V e^{j\omega t}$

$i(t) = I\cos(\omega t) \longrightarrow I(t) = I e^{j\omega t}$



Sinusoidal waveform

Phasor representation

Phasor transform

Time-domain

Phasor

$$f(t) = Re\left[F_{(\omega)} e^{j\omega t}\right] \xrightarrow{\text{transform}} F_{(\omega)}$$

$$A \cos \omega t = Re\left[A e^{j\omega t}\right] \longrightarrow A$$

Complex impedance - capacitor

Phasor

$$v(t) = V\cos(\omega t) \xrightarrow[\text{Transform}]{\text{Complex}} \mathbb{V}(t) = V e^{j\omega t}$$

$$i(t) = C\frac{dv(t)}{dt} \longrightarrow \mathbb{I}(t) = C\frac{d}{dt}\left(V e^{j\omega t}\right)$$

$$\mathbb{I}(t) = j\omega C V e^{j\omega t}$$

$$i(t) = \mathrm{Re}\left[j\omega C V e^{j\omega t}\right] \xleftarrow{\mathrm{Re}[\,]}$$

$$= \omega C V \,\mathrm{Re}\left[e^{j\frac{\pi}{2}} e^{j\omega t}\right]$$

$$= \omega C V \cos\left(\omega t + \frac{\pi}{2}\right)$$

$$j = e^{j\frac{\pi}{2}} = e^{j90^\circ}$$

$$\boxed{\mathbb{I}(t) = j\omega C\, \mathbb{V}(t)}$$

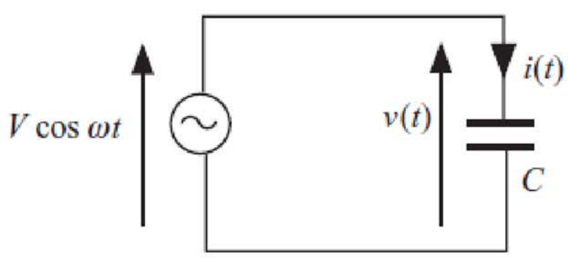


Figure 9.2 A capacitor with its voltage defined by a sinusoidal voltage source of radian frequency ω

Generalized Ohm's law - capacitor

$$\mathbb{I}(t) = j\omega C \, \mathbb{V}(t) \quad *$$

$$\mathbb{V}(t) = \frac{1}{j\omega C} \, \mathbb{I}(t)$$

$\downarrow$

$$\mathbb{V}(t) = \underset{\substack{\downarrow \\ \text{Impedance of capacitor}}}{Z_c} \, \mathbb{I}(t)$$

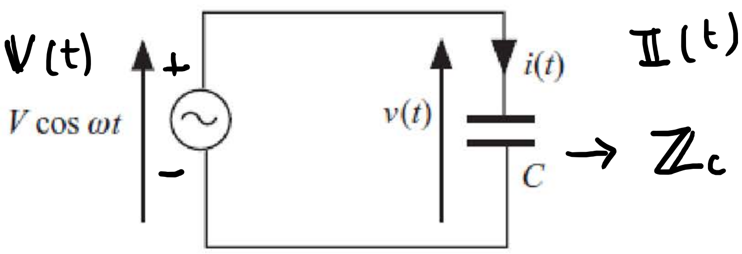


Figure 9.2 A capacitor with its voltage defined by a sinusoidal voltage source of radian frequency ω

Phasor diagram - capacitor

$\mathbb{I}(t) = j\omega C \, \mathbb{V}(t)$

$= e^{j\frac{\pi}{2}} \omega C \, V e^{j\omega t}$

$= \omega C V e^{j(\omega t + \frac{\pi}{2})}$

$\mathbb{V}(t) = V e^{j\omega t}$

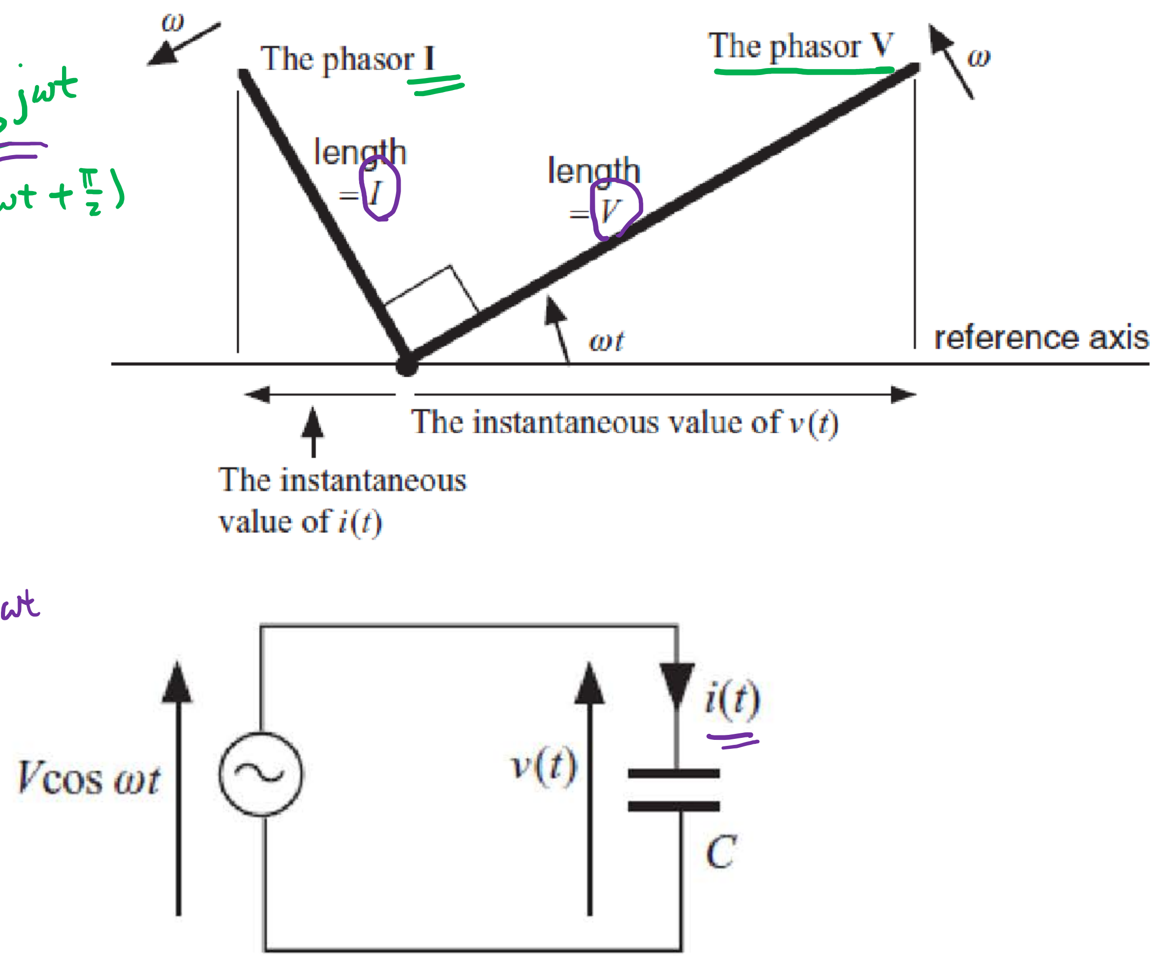


Figure 9.11 A capacitor with its voltage defined by a sinusoidal voltage source of radian frequency ω

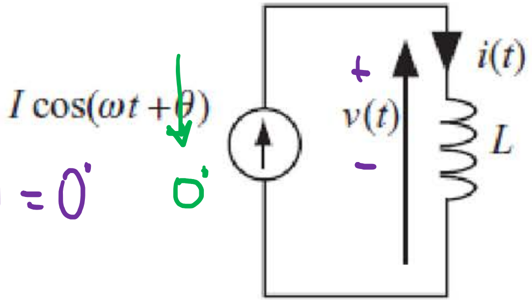
Complex impedance - inductor

Source $\mathbb{I}(t) = I e^{j\omega t}$

$$\mathbb{V}(t) = L \frac{d\,\mathbb{I}(t)}{dt}$$

$$= j\omega L \underbrace{I e^{j\omega t}}$$

$$\mathbb{V}(t) = j\omega L\, \mathbb{I}(t)$$



Consider $\theta = 0°$

Figure 9.7 An inductor with its current defined by a sinusoidal current source of radian frequency ω

Generalized Ohm's law - inductor

$$\mathbb{V}(t) = \underbrace{j\omega L}_{\downarrow} \, \mathbb{I}(t)$$

$\mathbb{Z}_L$: Impedance of inductor L

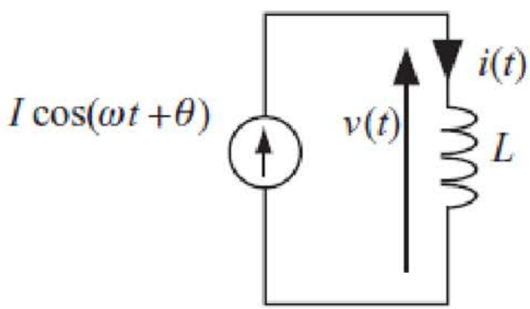


Figure 9.7 An inductor with its current defined by a sinusoidal current source of radian frequency ω

Phasor diagram - inductor

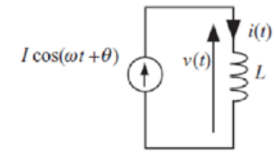


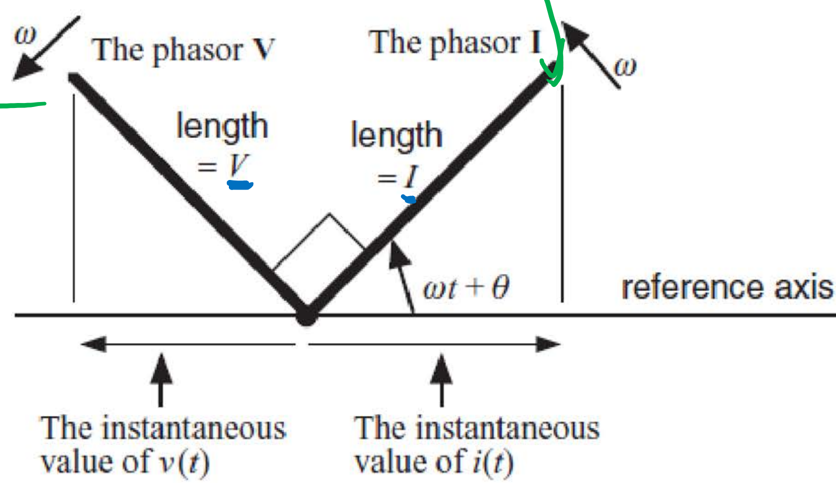
Figure 9.7 An inductor with its current defined by a sinusoidal current source of radian frequency ω

$$v(t) = \omega L I \cos\left(\omega t + \theta + \frac{\pi}{2}\right)$$

Re[·]

$$\mathbb{V}(t) = j\omega L \cdot \mathbb{I}(t)$$

$$\mathbb{V}(t) = j\omega L \, \underbrace{I e^{j(\omega t+\theta)}}$$



$$V = \omega L I$$

Figure 9.13 A Phasor diagram showing phasors representing inductor voltage and current

Phasor diagram - resistor

complex

$$v(t) = R\, i(t)$$

$$\mathbb{V}(t) = R\, \mathbb{I}(t)$$

$$\mathbb{V} = R \cdot \mathbb{I}$$

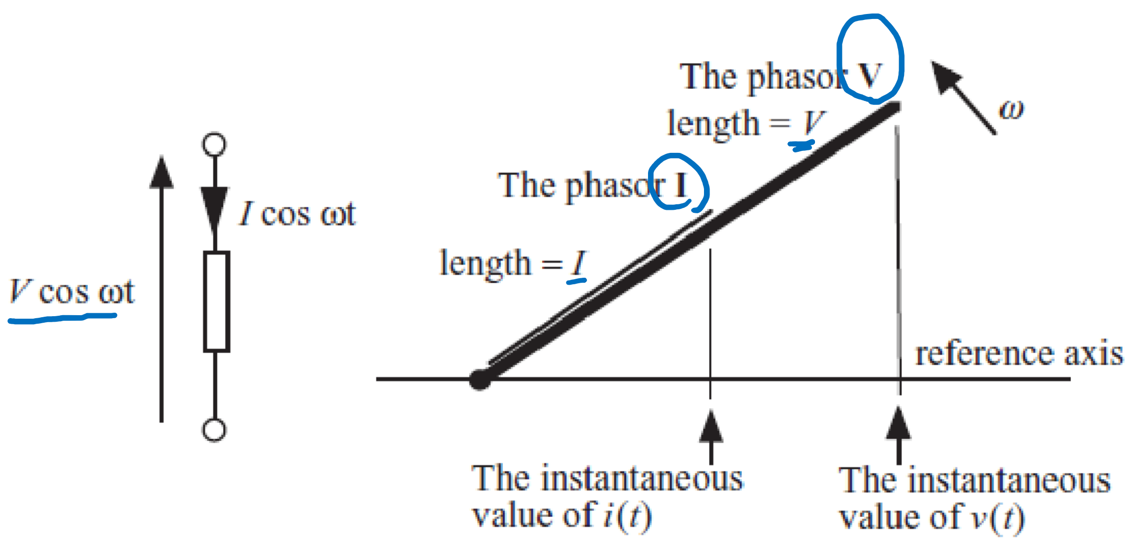


Figure 9.14 The phasors representing the sinusoidal voltage and current of a resistor are in phase. The current phasor has been offset slightly for clarity

Phasor diagram

Z_C

$$\mathbb{V}(t) = \boxed{\frac{1}{j\omega C}}\, \mathbb{I}(t)$$

Z_L

$$\mathbb{V}(t) = \boxed{j\omega L}\cdot \mathbb{I}(t)$$

$Z_R = R$

$$\mathbb{V}(t) = \boxed{R}\, \mathbb{I}(t)$$

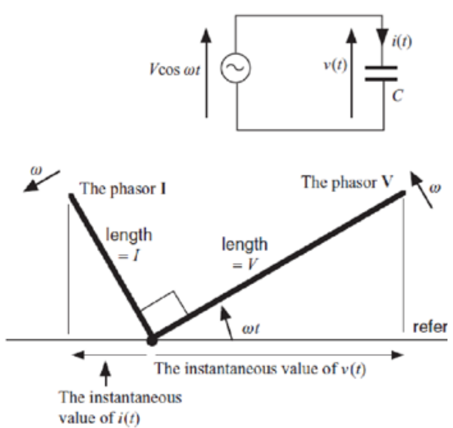


Figure 9.12 A phasor diagram showing phasors representing capacito
Note that the phasors rotate at an angular frequency ω and it is th
reference axis that determines the actual scalar instantaneous values o

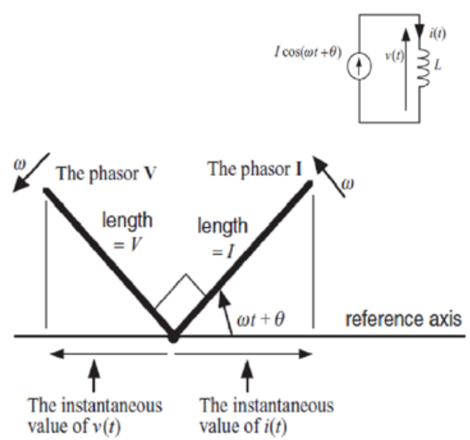


Figure 9.13 A Phasor diagram showing phasors representing inductor voltage and current

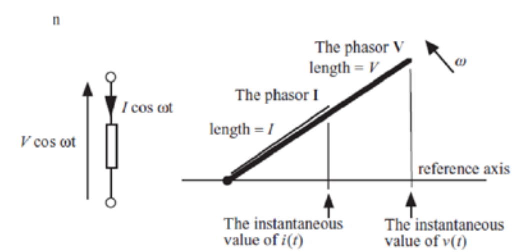


Figure 9.14 The phasors representing the sinusoidal voltage and current of a resistor ar
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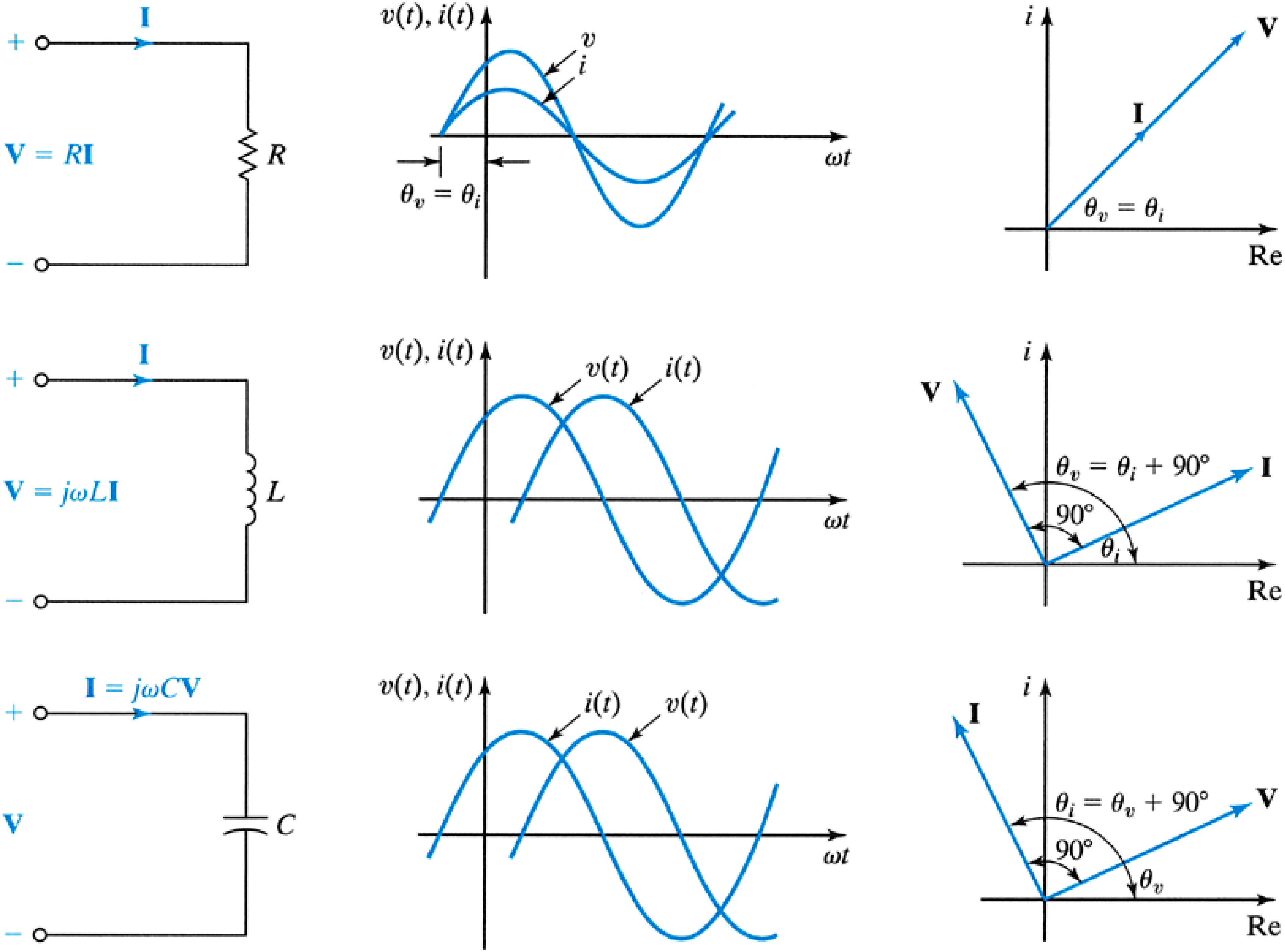


FIGURE 4.6 Characteristics of the passive elements *R*, *L*, and *C*.

Kirchhoff's laws – phasor addition

KVL holds at any time t

$$v(t) = v_1(t) + v_2(t)$$
$$= V_1 \cos(\omega t + \theta_1) + V_2 \cos(\omega t + \theta_2)$$

Transform

$$\mathbb{V}(t) = V_1 e^{j(\omega t + \theta_1)} + V_2 e^{j(\omega t + \theta_2)}$$
$$= \mathbb{V}_1(t) + \mathbb{V}_2(t)$$

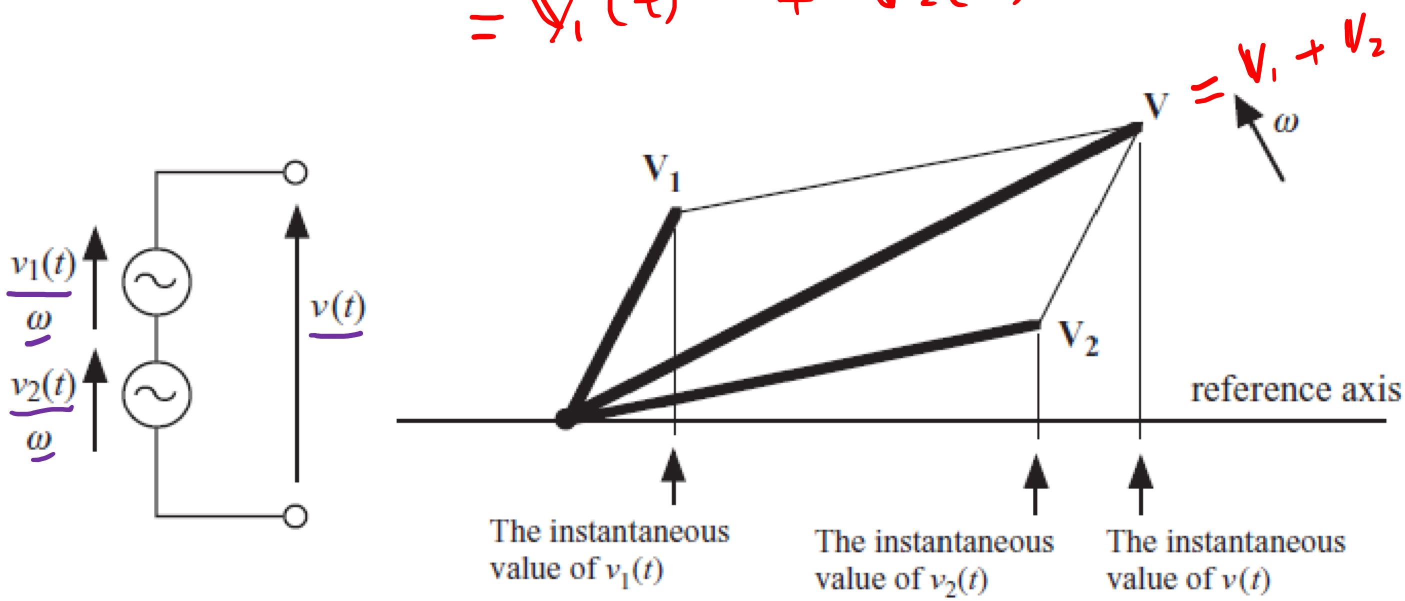


Figure 9.15 A Phasor diagram showing that the phasor addition of voltages obeys KVL. Hairlines are used to indicate construction

Kirchhoff's laws

DC / AC circuits

complex / phasor

KCL: $\sum_{node} I_i(t) = 0$

KVL: $\sum_{loop} V_n(t) = 0$

Generalized Ohm's law

$V(t) = Z \, I(t)$

Nodal Analysis
Superposition
Thevenin' / Norton Equiv. ckts

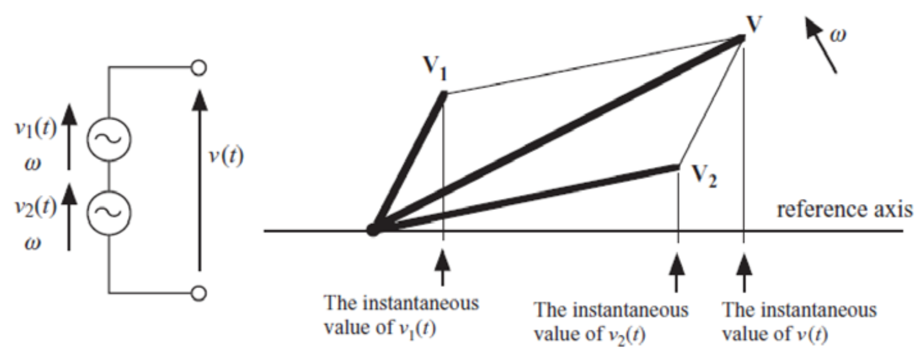


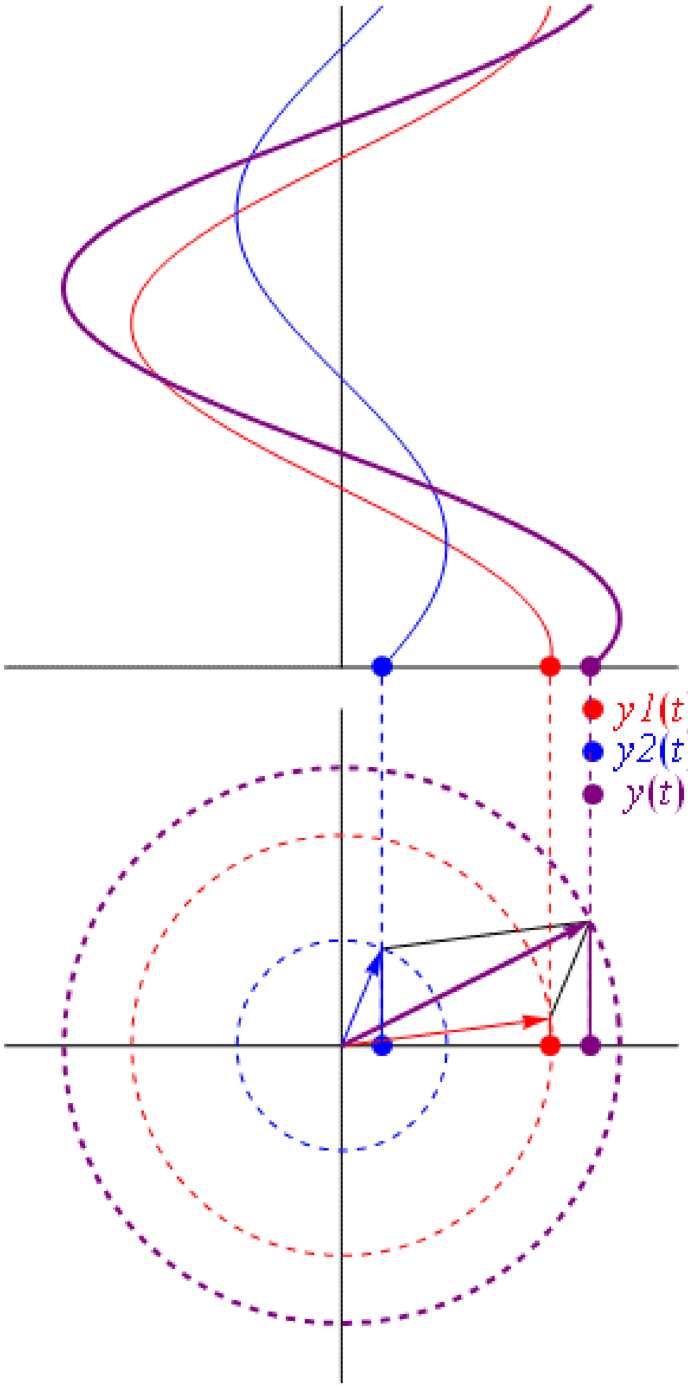
Figure 9.15 A Phasor diagram showing that the phasor addition of voltages obeys KVL. Hairlines are used to indicate construction

Sum of phasors

$$y(t) = y_1(t) + y_2(t)$$

phasor $\downarrow$

$$Y(t) = Y_1(t) + Y_2(t)$$



Constructing a phasor diagram

$$v_s(t) = 2\cos(1000t) \longrightarrow \mathbb{V}_s(t) = 2e^{j1000t}$$

KCL @ X :

$$\underline{\mathbb{I}(t)} = \mathbb{I}_R(t) + \mathbb{I}_C(t)$$

$$= \frac{\mathbb{V}(t)}{R} + (j\omega C)\mathbb{V}(t)$$

$$= \left(\frac{1}{R} + j\omega C\right)\mathbb{V}(t)$$

$$= 10^{-3}(1+j)\,\mathbb{V}(t)$$

$\omega = 1000$

$R = 1000\,\Omega$

$C = 10^{-6}$ F

$L = 1$ H

$$\underline{\underline{\mathbb{V}_L(t)}} = (j\omega L)\mathbb{I}(t)$$

$$= j(1+j)\mathbb{V}(t)$$

$$= (-1+j)\,\underline{\underline{\mathbb{V}(t)}}$$

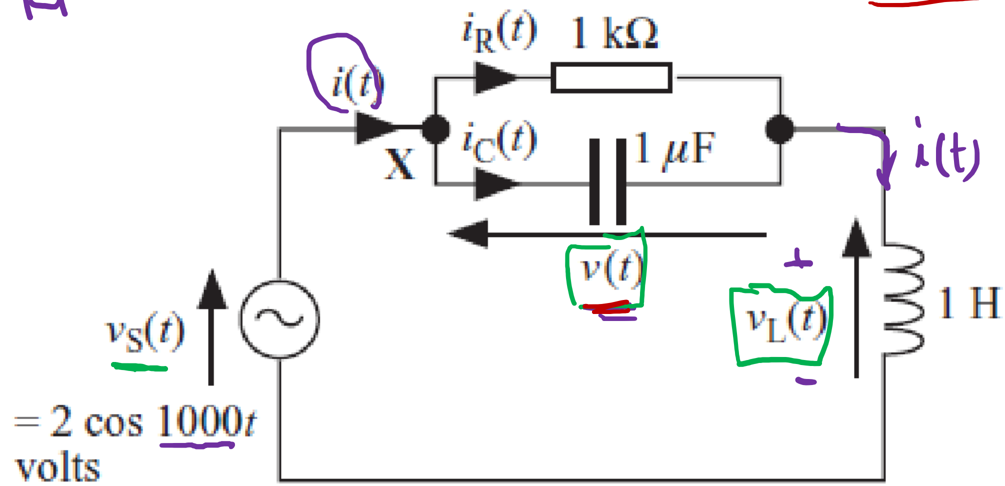


Figure 9.16 The circuit whose phasor diagram is to be constructed

Constructing a phasor diagram

KVL: $\mathbb{V}_s(t) = \mathbb{V}(t) + \mathbb{V}_L(t)$

$= \mathbb{V}(t) + (-1+j)\mathbb{V}(t)$

$= j\mathbb{V}(t)$

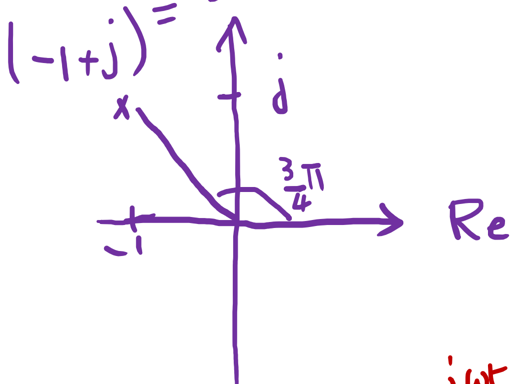
$\Rightarrow \mathbb{V}(t) = \frac{1}{j}\mathbb{V}_s(t) = -j\,\mathbb{V}_s(t) = -e^{j\frac{\pi}{2}} \cdot 2e^{j1000t}$

$= -2e^{j(1000t+\frac{\pi}{2})} = e^{j(1000t-\frac{\pi}{2})}$

Re[·]

$v(t) = -2\cos(1000t+\frac{\pi}{2}) = 2\cos(1000t-\frac{\pi}{2})$

$(-1+j) = \sqrt{2}\,e^{j\frac{3}{4}\pi}$



$\mathbb{V}_L(t) = \underline{(-1+j)}\,\mathbb{V}(t)$

$= \underline{\sqrt{2}\,e^{j\frac{3}{4}\pi}}\,\mathbb{V}(t)$

Re[·]

$v_L(t) = 2\sqrt{2}\cos(1000t+\frac{\pi}{4})$

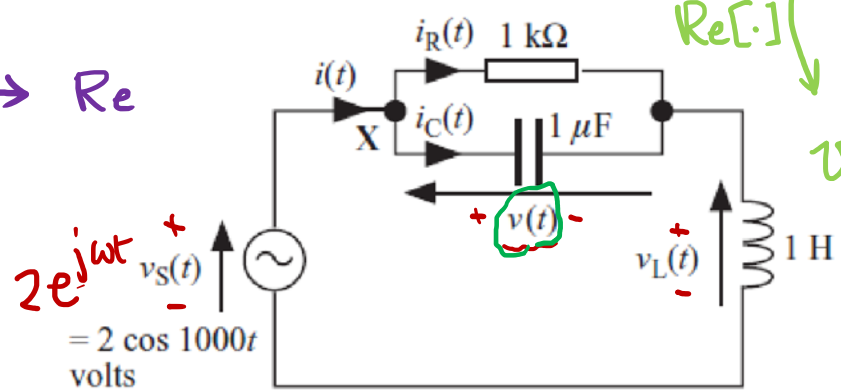


Figure 9.16 The circuit whose phasor diagram is to be constructed

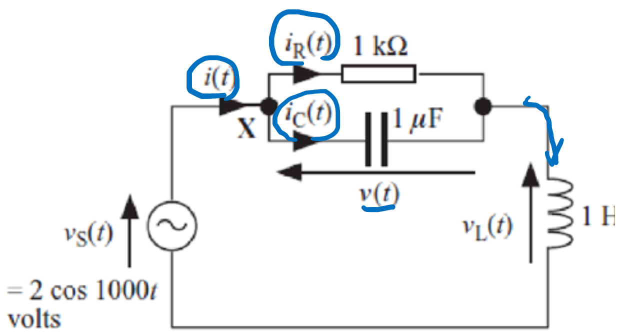


Figure 9.16 The circuit whose phasor diagram is to be cc

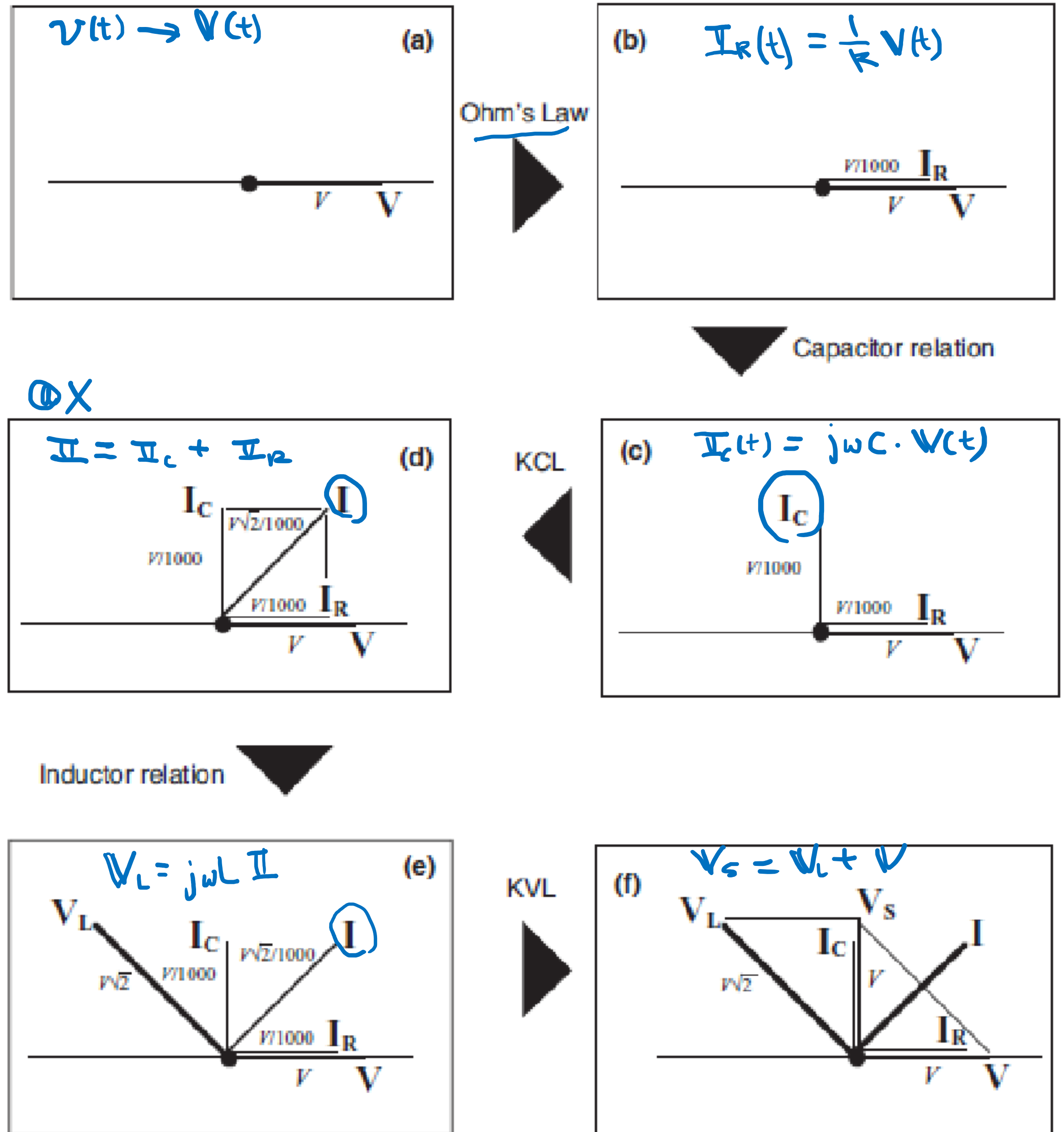
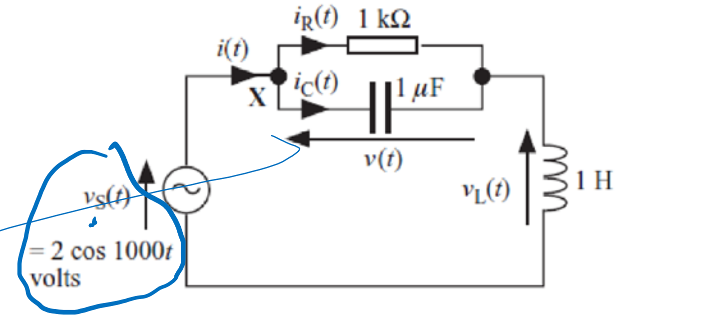
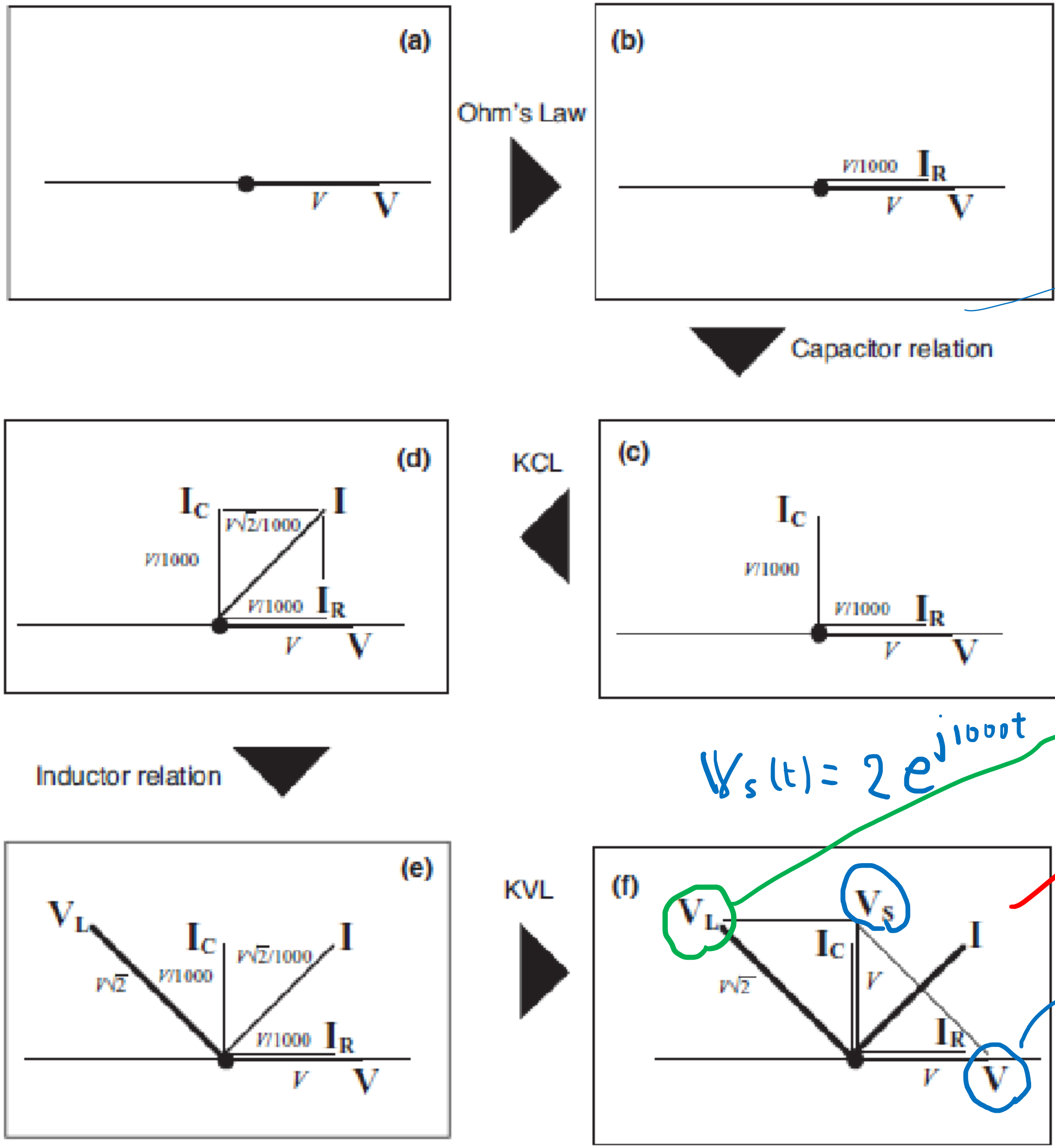


Figure 9.17 Steps in the construction of a phasor diagram for the circuit of Figure 9.16. Where possible phasor lengths have been indicated beside the phasor. Some overlapping phasors have been offset slightly for clarity. Hairlines are included to indicate constructions



?.16 The circuit whose phasor diagram is to be constructed



$V_s(t) = 2e^{j1000t}$

$V_L = 2\sqrt{2}\, e^{j(1000t + \frac{\pi}{4})}$

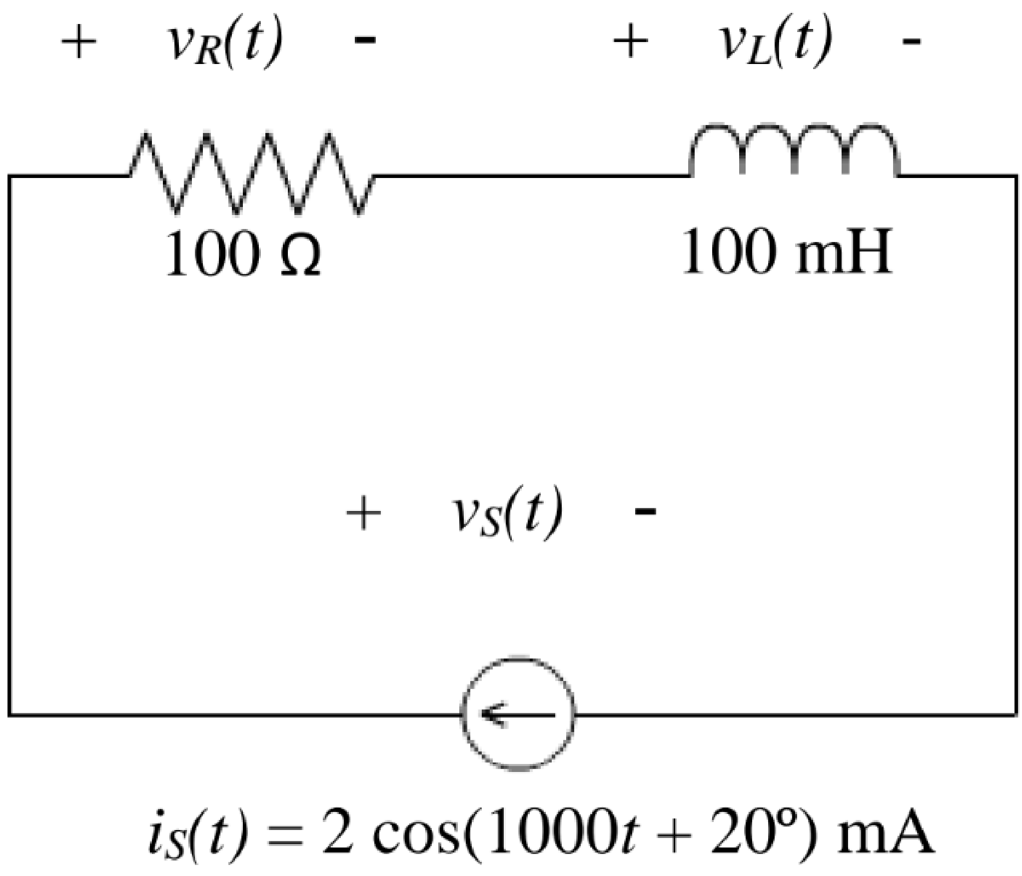
$I(t) = \frac{2\sqrt{2}}{1000} e^{j(1000t - \frac{\pi}{4})}$

$V(t) = 2e^{j(1000t - \pi/2)}$

Figure 9.17 Steps in the construction of a phasor diagram for the circuit of Figure 9.16. Where possible phasor lengths have been indicated beside the phasor. Some overlapping phasors have been offset slightly for clarity. Hairlines are included to indicate constructions

Quiz

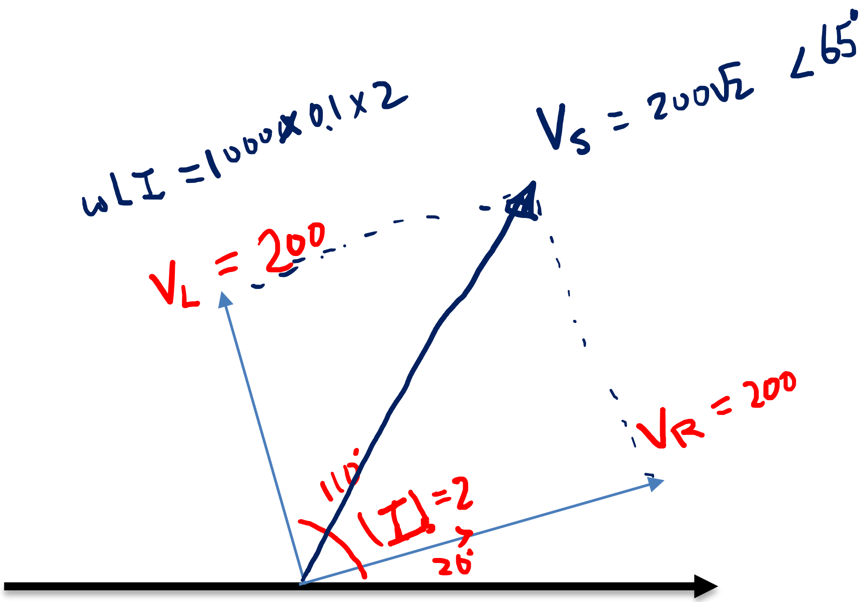
Sketch a dimensioned phasor diagram representing all the voltages v_R, v_L and v_S, as well as the current i_S, in the circuit of the following figure.

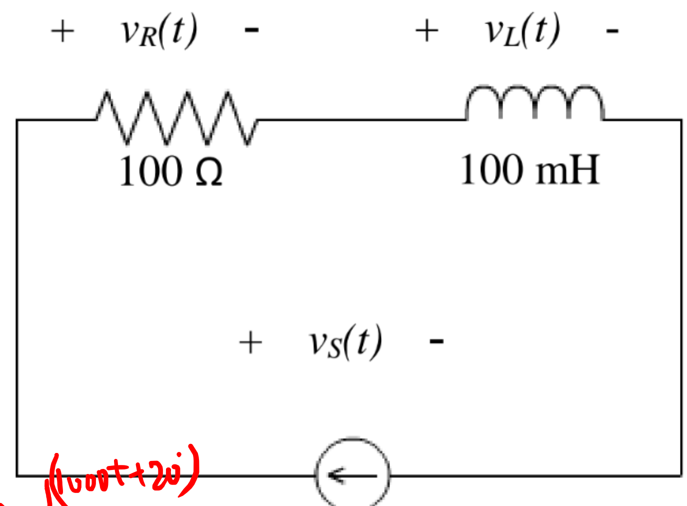


Quiz review

$v_S(t) = v_R(t) + v_L(t)$

$\omega = 1000$





$I = 2e^{j(1000t+20°)}$

$i_S(t) = 2\cos(1000t + 20°)$ mA