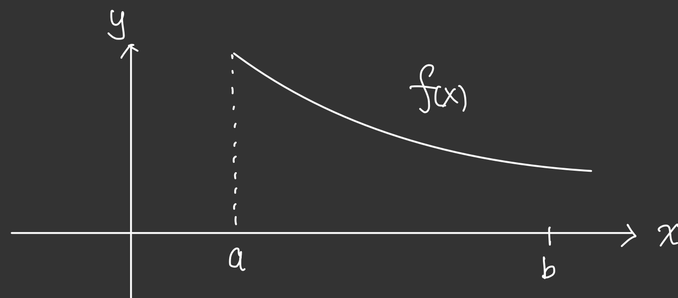
Calculus II

4/7 . 5/19 期中考 , 6/23 期末考

兩次小考 (時間再公佈)

Improper integrals (瑕積分).

Type 1:



Let $f(x)$ be a continuous function defined on $[a, \infty)$.

For any $b > a$, we can compute

$$\int_a^b f(x)\,dx := g(b).$$

$g(b)$ is a continuous function defined on $[a, \infty)$

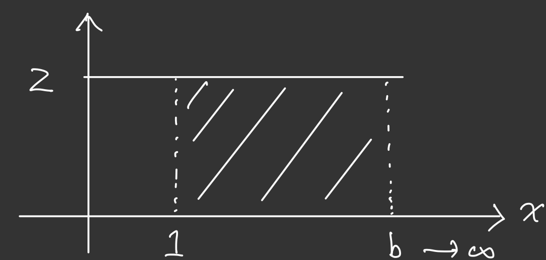
(with $g(a) = 0$)

Def: We define the improper integral:

$$\int_a^{\infty} f(x)\,dx := \lim_{b\to\infty} g(b).$$

(Therefore, the improper integral exists iff $\lim_{b\to\infty} g(b)$ exists)

Example: (1) Let $f(x) = 2$, $a = 1$



$$\int_1^b f(x)\,dx = 2(b-1)$$

$$\lim_{b\to\infty} 2(b-1) = \infty \quad (\text{doesn't exist})$$

$\Rightarrow \int_1^{\infty} f(x)\,dx$ doesn't exist.

Example : (2) Let $f(x) = 2^{-x}$, $x \geq 0$ $(a=0)$.

$2^{-x} = e^{-(\ln 2)x}$. For any $b > 0$, $(\ln 2)x := y$

$$\int_0^b f(x)\,dx = \int_0^b e^{-(\ln 2)x}\,dx = \frac{1}{\ln 2}\int_0^{(\ln 2)b} e^{-y}\,dy$$

$$= \frac{-1}{\ln 2} e^{-y} \Big|_0^{\ln 2\, b} = \frac{1-2^{-b}}{\ln 2} := g(b).$$

$$\lim_{b\to\infty} g(b) = \frac{1}{\ln 2} \quad \text{exists !}$$

So, $\displaystyle\int_0^\infty 2^{-x}\,dx = \frac{1}{\ln 2}$ exists.

Example: (3) Let $f(x) = x^{-2}$, $a = 1$.

For any $b > 1$,

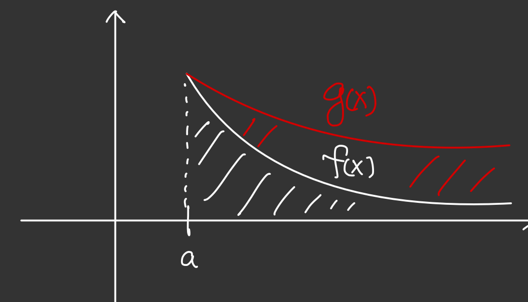
$$\int_1^b f(x)\,dx = \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b = (1 - b^{-1}) := g(b).$$

$\lim_{b\to\infty} g(b) = 1$. exists. So $\int_1^\infty x^{-2}\,dx = 1$. exists.

Remark: Notice that, in these cases we have

$$\int_a^\infty f(x)\,dx \text{ exists} \Rightarrow \lim_{x\to\infty} f(x) = 0$$

However, $\lim_{x \to \infty} f(x) = 0 \not\Rightarrow \int_a^\infty f(x)\,dx$ exists.



Example: (4) Let $f(x) = x^{-1}$; $a = 1$.

For any $b > 1$,

$$\int_1^b x^{-1}\,dx = \ln|x| \Big|_1^b$$

$$= \ln b - 0 = \ln b. := g(b)$$

$\lim_{b \to \infty} g(b) = \infty$ (limit doesn't exist)

$\Rightarrow \int_1^\infty x^{-1}\,dx$ doesn't exist.

Remark: We can also define

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{a \to -\infty} g(a).$$

$$g(a) := \int_a^b f(x)\,dx.$$ ($f(x)$ is continuous function defined on $(-\infty, b]$)

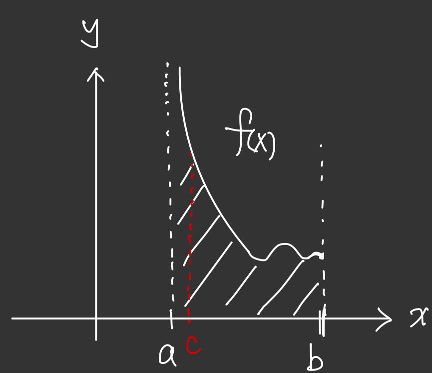
Type 2: Let $f(x)$ be a continuous function defined on $(a, b]$.

For any c, $a < c < b$,

(c is near a).

we can compute

$$\int_c^b f(x)\,dx := g(c)$$



<u>Def</u>: We define the improper integral

$$\int_a^b f(x)\,dx := \lim_{c \to a^+} g(c).$$

<u>Example</u>: (1) For any continuous function $f(x)$ defined $[a,b]$

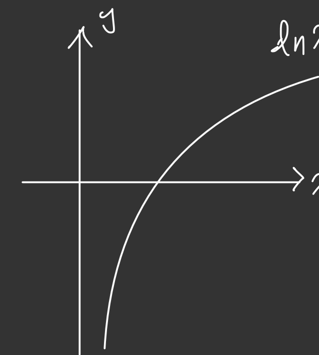
Then improper integral = definite integral

(2) Let $f(x) = x^{-\frac{1}{2}}$ defined on $(0,1]$.

$$\int_c^1 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_c^1 = 2(1-\sqrt{c}) := g(c)$$

$\lim_{c\to 0^+} g(c) = 2$ exists. So $\int_0^1 x^{-\frac{1}{2}}\,dx = 2$.

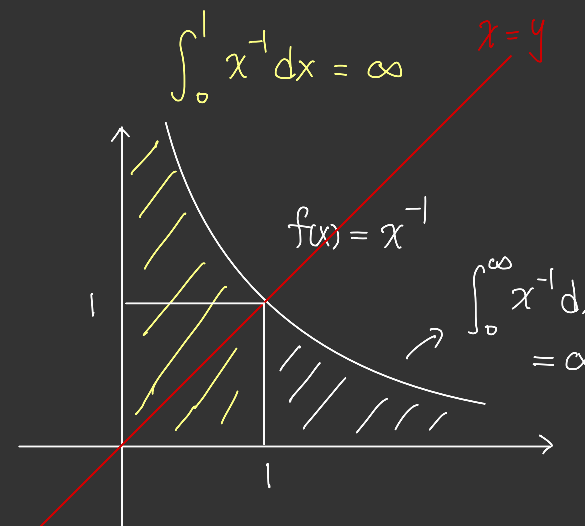
Example: (3) Let $f(x) = x^{-1}$ defined on $(0, 1]$



$$\int_c^1 x^{-1}\,dx = \ln|x|\Big|_c^1 = -\ln c := g(c).$$

$$\lim_{c\to 0^+} g(c) = \infty$$ doesn't exist.

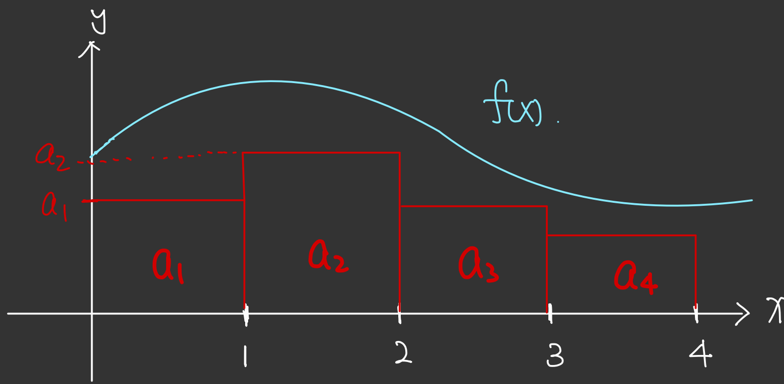
$$\Rightarrow \int_0^1 x^{-1}\,dx$$ doesn't exist.



(Notice that:

$$\int_0^1 x^{-1}\,dx = \infty \Leftrightarrow \int_1^\infty x^{-1}\,dx = \infty \;)$$

Applications of improper integrals



Let $f(x)$ be a continuous function defined on $[0, \infty)$.

$$\int_0^\infty f(x)\,dx \geq \sum_{n=1}^{\infty} a_n$$

Prop: Let $f(x)$ be a continuous, nonnegative function defined on $[0,\infty)$. $0 \leq a_n \leq f(x)$ for all $x \in [n-1, n]$, $n \in \mathbb{N}$.

Then we have

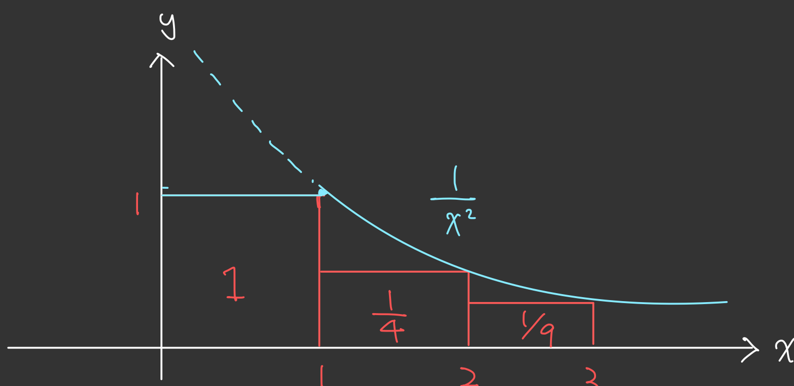
$$\sum_{n=1}^{\infty} a_n \leq \int_0^\infty f(x)\,dx.$$

Moreover, when $\sum_{n=1}^{\infty} a_n = \infty \Rightarrow \int_0^{\infty} f(x)\,dx = \infty$ (doesn't exist);

when $\int_0^{\infty} f(x)\,dx$ exists $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

We usually call this inequality the integral comparison test.

Example: (1) Let $\{a_n\}_{n\in\mathbb{N}} = \{\frac{1}{n^2}\}_{n\in\mathbb{N}}$



$$f(x) = \begin{cases} \frac{1}{x^2} & \text{when } x \geq 1 \\ 1 & \text{when } x \leq 1. \end{cases}$$

is a continuous function

$n=1, \quad \frac{1}{n^2} = 1 \leq 1 = f(x) \quad \text{when} \quad x \in [0, 1]$

$n>1, \quad \frac{1}{n^2} \leq \frac{1}{x^2} = f(x) \quad \text{when} \quad x \in [n-1, n].$

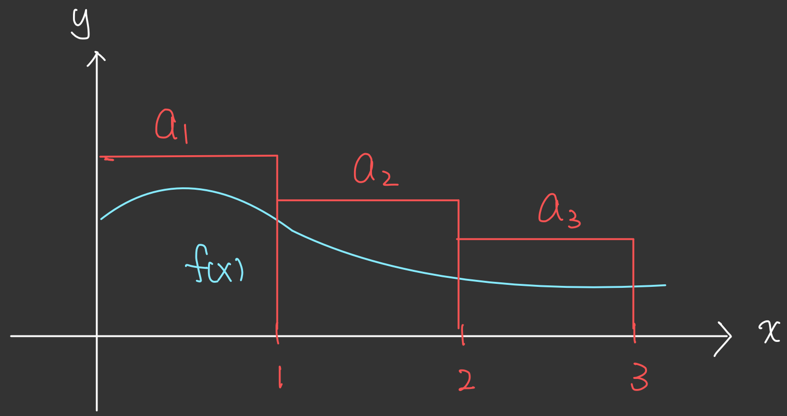
$$\int_0^b f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^b f(x)\,dx = 1 + \int_1^b x^{-2}\,dx = 1 - x^{-1}\Big|_1^b$$

$$= 1 + (1 - b^{-1})$$

$$:= g(b)$$

$$\lim_{b \to \infty} g(b) = 2 \Rightarrow \int_0^\infty f(x)\,dx = 2 < \infty$$

By integral comparison test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$



Prop: Let $f(x)$ be a continuous function with $0 \le f(x) \le a_n$ when $x \in [n-1, n]$. Then we have

$$\int_0^\infty f(x)\,dx \le \sum_{n=1}^\infty a_n.$$

This implies that

(1) $\sum_{n=1}^\infty a_n$ converges $\Rightarrow$ $\int_0^\infty f(x)\,dx$ exists.

(2) $\int_0^\infty f(x)\,dx$ doesn't exist $\Rightarrow$ $\sum_{n=1}^\infty a_n$ diverges.

<u>Example</u>: Let $p < 0$. $f(x) = x^p$ defined on $[1, \infty)$

$$\int_1^\infty x^p \, dx = ?$$

$$\int_1^b x^p \, dx = \begin{cases} \dfrac{1}{1+p} x^{p+1} \Big|_1^b = \dfrac{1}{1+p} \left(b^{p+1} - 1 \right) & (p \neq -1) \\ \ln|x| \Big|_1^b = \ln b. & (p = -1) \end{cases} := g(b).$$

When $p > -1$, $\lim\limits_{b \to \infty} g(b) = \infty \Rightarrow \int_1^\infty x^p \, dx$ doesn't exist.

When $p = -1$, $\lim\limits_{b \to \infty} g(b) = \lim\limits_{b \to \infty} \ln b = \infty \Rightarrow \int_1^\infty x^p \, dx$ doesn't exist

When $p < -1$, $\lim\limits_{b \to \infty} g(b) = \dfrac{-1}{1+p} \Rightarrow \int_1^\infty x^p \, dx$ exist.

Conclusion: When $p \geq -1$, $\int_1^\infty x^p \, dx$ doesn't exist.

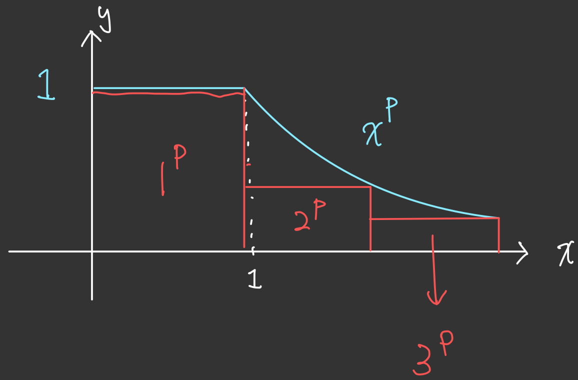
When $p < -1$, $\int_1^\infty x^p \, dx$ exists.

Exercise: Check: (1) When $p \geq -1$, $\sum_{n=1}^\infty n^p$ doesn't exist.

(2) when $p < -1$, $\sum_{n=1}^\infty n^p$ exists.

Sol: (2) Let

$$f(x) = \begin{cases} x^p & \text{if } x \geq 1 \\ 1 & \text{if } x < 1. \end{cases}$$



By integral comparison test,

$$\int_0^\infty f(x)\,dx = \int_0^1 1\,dx + \int_1^\infty x^p\,dx$$

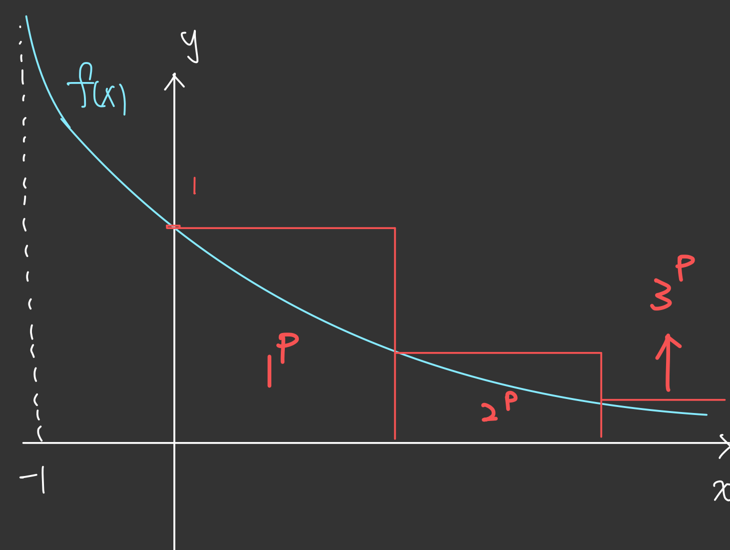
$$= 1 + \frac{-1}{1+p} \quad \text{conv.}$$

$$\Rightarrow \sum_{n=1}^{\infty} n^p \quad \text{conv.}$$

(1) Let $f(x) = (x+1)^p$ for $x \geq 0$.

By integral comparison test,

$$\int_0^\infty f(x)\,dx = \int_1^\infty x^p\,dx \quad \text{div.}$$



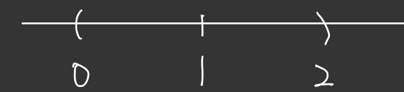
$$\sum_{n=1}^{\infty} n^p \geq \int_1^{\infty} x^p \Rightarrow \sum_{n=1}^{\infty} n^p \text{ also div.}$$

Taylor Series:

Can any function be a polynomials? **Ans:** No.

approximated by

逼近



Example: Let $f(x) = \dfrac{1}{2-x}$.

$$\frac{1}{2-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n \quad \text{when} \quad |x-1| < 1$$

$$\sum_{n=0}^{\infty} (x-1)^n \sim \sum_{n=0}^{M} (x-1)^n \quad \text{by taking } M \text{ "large enough".}$$

Therefore, $f(x)$ can be "approximated" by polynomials on $x \in (0, 2)$.

Def: (Power Series) A power series centered at c has the form.

$$\sum_{n=0}^{\infty} a_n \cdot (x-c)^n \quad \text{for some } a_n \in \mathbb{R}.$$

(We write it formally)

Recall: (1) $\{b_n\}_{n=0}^{\infty}$ be a bounded sequence iff there exists $M > 0$ such that $|b_n| \le M$ for all $n = 0, 1, 2 \ldots$

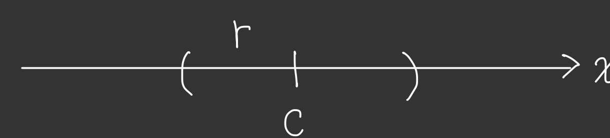
(2) Let $\sum_{n=0}^{\infty} b_n$ be a series. We call it absolutely convergent iff $\sum_{n=0}^{\infty} |b_n|$ conv.

(3) $\sum_{n=0}^{\infty} b_n$ conv. if $\sum_{n=0}^{\infty} |b_n|$ conv.

Theorem: Suppose that $\{|a_n| \cdot r^n\}$ is a bounded seq for some $r > 0$. Then the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is absolutely convergent for all $x \in (c-r, c+r)$



Remark: If $\{|a_n|r^n\}$ is not a bounded seq, then

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{divergences for all } |x-c| \geq r$$

$$\left(\text{Since } \left| a_n (x-c)^n \right| \geq |a_n| \cdot |x-c|^n \geq |a_n| r^n \right)$$

Proof of the Theorem: For any $x \in (c-r, c+r)$,

$|x-c| < r$. By taking $|x-c| := s$, $s/r < 1$

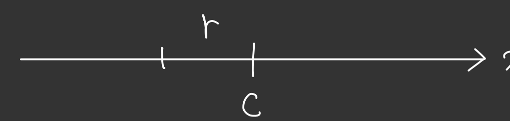
$$\sum_{n=0}^{\infty} |a_n||x-c|^n = \sum_{n=0}^{\infty} |a_n| s^n = \sum_{n=0}^{\infty} (|a_n| \cdot r^n)\left(\frac{s^n}{r^n}\right)$$

$$\leq \sum_{n=0}^{\infty} M \left(\frac{s}{r}\right)^n \qquad \underset{M}{\overset{\wedge|}{}} \quad (\{|a_n|r^n\} \text{ bdd})$$

$$= M \sum_{n=0}^{\infty} \left(\frac{s}{r}\right)^n = M\left(\frac{1}{1-(s/r)}\right) < \infty$$

Therefore, $\sum_{n=0}^{\infty} a_n (x-c)^n$ absolutely conv. $\blacksquare$

$S := \{ r > 0 \mid \{|a_n| r^n\} \text{ is a bdd seq} \}$

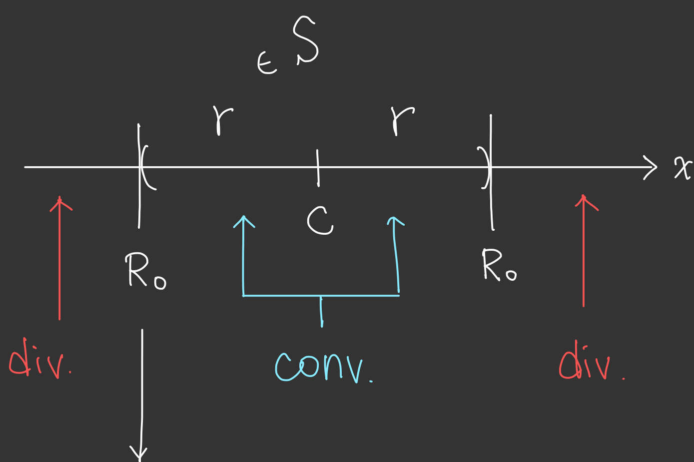


if there exists $R_0 = \max S$ or there exists R_0 such that

(1) $\{|a_n| R_0^n\}$ is not bdd,

(2) $r \in S$ for all $r < R_0$.

Then we call R_0 the radius of convergence of this power series.



$\{|a_n|R_0^n\}$ is bdd $\Rightarrow$ Case 1

$\{|a_n|R_0^n\}$ is not bdd $\Rightarrow$ Case 2.

Remark: It is possible that $R_0 = 0$ or $R_0 = \infty$.

When $R_0 = 0 \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n$ div. for all $x \neq c$

When $R_0 = \infty \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n$ conv. for all $x \in \mathbb{R}$.

Let $f(x)$ be a smooth function. $\left(\left(\frac{d}{dx}\right)^n f(x) \text{ exists for all } n \right)$

Case 1: $f(x)$ is a polynomial.

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

$$= a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots + a_n (x-c)^n$$

Then (1) $f(c) = a_0$.

(2) $f'(c) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + n a_n (x-c)^{n-1} \Big|_{x=c}$

$= a_1$

(3) $f''(c) = 2a_2 + 6a_3(x-c) + \cdots + n(n-1) a_n (x-c)^{n-2} \Big|_{x=c}$

$= 2a_2$

We can check that

$$f^{(k)}(c) = k! \cdot a_k. \qquad (0! = 1)$$

for all k.

$$\Rightarrow a_k = \frac{f^{(k)}(c)}{k!} \quad \text{for all } k.$$

Namely, we have

$$f(x) = \sum_{k=0}^{n} a_k (x-c)^k = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

<u>Case 2:</u> (General Case). We hope

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

is true for any smooth function.

Def: We define $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ the Taylor series of f centered at c.

Q1: What's the radius of convergence of the Taylor Series? (How to find it?)

Q2: When the Taylor Series converges, do we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Example: Let $f(x) = \sin x$. $c = 0$.

$f(0) = 0$; $f'(0) = \cos 0 = 1$; $f''(0) = -\sin 0 = 0$

$f'''(0) = -\cos 0 = -1$; $f^{(4)}(0) = \underline{\sin 0} = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad f(0).$

So, the Taylor Series of $\sin x$ centered at 0 is

$$\frac{0}{0!}\cdot 1 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 - \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$$

Example: Let $f(x) = \cos x$; $c = 0$

$f(0) = \cos 0 = 1$; $f'(0) = -\sin 0 = 0$; $f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$; $f^{(4)}(0) = \cos 0 = 1$.

Taylor Series of $\cos x$ centered at 0 is

$$\frac{1}{0!} \cdot 1 + \frac{0}{1!} x - \frac{1}{2!} x^2 + \frac{0}{3!} x^3 + \frac{1}{4!} x^4 + \cdots$$

$$= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} \qquad (x^0 := 1)$$

<u>Example</u>: Let $f(x) = e^x$; $c = 0$.

$$f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = e^0 = 1 \quad \text{for all } k.$$

$\Rightarrow$ Taylor Series of e^x centered at 0 is

$$\frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Ans of Q1: ratio test or root test.

Example: Let us consider Taylor Series of $\sin x$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} a_k$$

By ratio test, we consider

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\left(\frac{|x|^{2k+3}}{(2k+3)!}\right)}{\left(\frac{|x|^{2k+1}}{(2k+1)!}\right)} = \lim_{k\to\infty} \frac{|x|^2}{(2k+3)(2k+2)}$$

$$= 0 \quad \text{for all } x$$

So, by ratio test, $\sum_{k=0}^{\infty} a_k$ conv. for all x.

Therefore, $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ conv. for all x.

$R_0 := \infty$ (radius of convergence).

<u>Example</u>: Let us consider the Taylor Series of e^x:

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} a_k$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}|x|^{k+1}}{\frac{1}{k!}|x|^k} = \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0 \quad (<1)$$

for any $x \in \mathbb{R}$. By ratio test, $R_0 = \infty$

Example: Let $f(x)=\sqrt{x}$; $c=1$.

$$f(1)=\sqrt{1}=1 \quad ; \quad f'(1)=\frac{1}{2}x^{-\frac{1}{2}}\Big|_{x=1}=\frac{1}{2}$$

$$f''(1)=\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}}\Big|_{x=1}=\frac{1}{2}\cdot\left(-\frac{1}{2}\right)=-\frac{1}{4}$$

$$f'''(1)=\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot x^{-\frac{5}{2}}\Big|_{x=1}=\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)=\frac{3}{8}$$

$$f^{(4)}(1)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)=\frac{-15}{16}$$

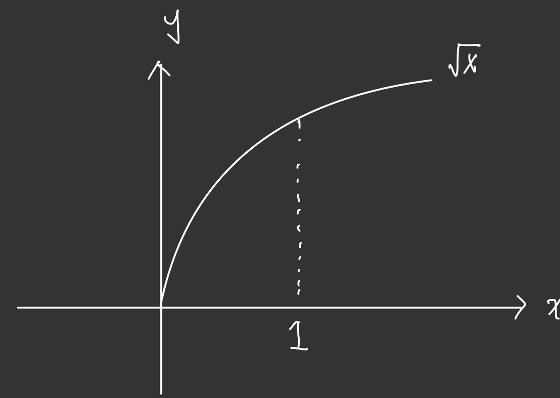
$$f^{(k)}(1)=-\left(\frac{-1}{2}\right)^{k}\cdot(2k-3)(2k-5)\cdots 1$$

$$=-\left(\frac{-1}{2}\right)^{k}\frac{(2k-3)!}{\underset{\substack{\| \\ 2(k-2)\,2(k-3)\cdots 2\cdot 1}}{(2k-4)\cdot(2k-6)}\cdots 2}=\frac{(-1)^{k-1}}{2^{k}}\frac{(2k-3)!}{2^{k-2}(k-2)!}$$

$$= \frac{(-1)^{k-1}}{\underbrace{2^{2k-2}}_{= 4^{k-1}}} \cdot \frac{(2k-3)!}{(k-2)!} = \left(\frac{-1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!} \quad \text{when } k \geq 2$$

Therefore, the Taylor Series of $\sqrt{x}$ at 1 is :

$$1 + \frac{1}{2}(x-1) + \sum_{k=2}^{\infty} \left(\frac{-1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!\,k!} (x-1)^k = \sum_{k=0}^{\infty} a_k$$



Radius of convergence :

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\left(\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!\,(k+1)!} |x-1|^{k+1}}{\left(\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!\,k!} |x-1|^{k}}$$

$$= \lim_{k\to\infty} \frac{(2k-1)(2k-2)}{4(k-1)(k+1)} |x-1| = |x-1| \lim_{k\to\infty} \frac{(2k-1)(2k-2)}{4(k-1)(k+1)}$$

$$= |x-1|$$

By ratio test, when $|x-1| < 1$, $\sum_{k=0}^{\infty} a_k$ conv.

i.e. $1 + \frac{1}{2}(x-1) + \sum_{k=2}^{\infty} \left(\frac{-1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!\,k!}(x-1)^k$ conv.

when $|x-1| > 1$, $\sum_{k=0}^{\infty} a_k$ div.

i.e. $1 + \frac{1}{2}(x-1) + \sum_{k=2}^{\infty} \left(\frac{-1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!\,k!}(x-1)^k$ div.

$\Rightarrow R_0 = 1.$

Ans of Q2: ① We assume : There exists $M > 0$ large such that

$$|f^{(n)}(x)| < M \quad \text{for any } n \in \mathbb{N}, \qquad \text{—— } (*)$$
$$x \in (c - R_0, c + R_0)$$

Then $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{for all } x \in (c-R_0, c+R_0)$$

② If $R_0 = \infty$ and for any $R > 0$, there exists $M_R > 0$

$$|f^{(n)}(x)| < M \quad \text{for any } n \in \mathbb{N},$$
$$x \in (c - R, c + R)$$

then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ for all $x \in \mathbb{R}$.

Let $f(x)$ be a smooth function.

<u>Def</u>: We define the Taylor's polynomial

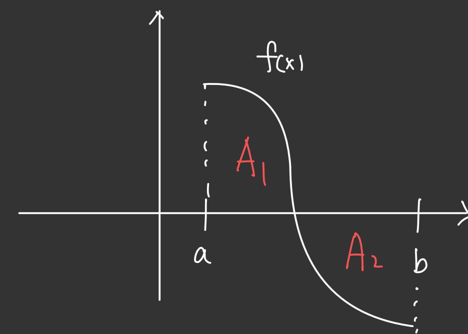
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

<u>Remark</u>.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = \lim_{n\to\infty} P_n(x).$$

Lemma (引理): For any continuous function,

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\left|\int_a^b f(x)\,dx\right| = |A_1 - A_2|$$

$$\wedge\!|$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Theorem: Assuming $(*)$, we have

$$|f(x) - P_n(x)| \leq \frac{|x-c|^{n+1}}{(n+1)!} \cdot M \quad \text{for all } n \in \mathbb{N}$$

<u>Remark</u>: Since $\lim_{n\to\infty} \frac{|x-c|^{n+1}}{(n+1)!} = 0 \Rightarrow 0 \le \lim_{n\to\infty} |f(x) - P_n(x)| \le 0$

by comparison test

$$\Rightarrow \lim_{n\to\infty} |f(x) - P_n(x)| = 0.$$

$$\Rightarrow f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

<u>Proof of theorem</u>: Fix $n \in \mathbb{N}$, we define

$$g(x) := f(x) - P_n(x),$$

$$\left[P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \right]$$

n-th

$$\Rightarrow g(c) = f(c) - P_n(c) = f(c) - f(c) = 0$$

$$g'(c) = (f(x) - P_n(x))'\Big|_{x=c} = f'(c) - P_n'(c) = f'(c) - f'(c) = 0$$

One can check that :

$$g^{(k)}(c) = 0 \quad \text{for all} \quad k \leq n.$$

$$g(x) = g(x) - g(c) = \int_c^x g'(s_1)\, ds_1 = \int_c^x \boxed{g'(s_1) - g'(c)}\, ds_1$$

$$= \int_c^x \boxed{\int_c^{s_1} g''(s_2)\, ds_2}\, ds_1 = \cdots$$

(s_1 is between x and c, s_2 is between s_1 and c.)

$$= \int_c^x \int_c^{s_1} \int_c^{s_2} \cdots \int_c^{s_n} g^{(n+1)}(s_{n+1})\, ds_{n+1} \cdots ds_1 \quad \text{—— } (**)$$

$$g^{(n+1)}(x) = f^{(n+1)}(x) - P_n^{(n+1)}(x) = f^{(n+1)}(x)$$

$(**)$ implies

$$g(x) = \int_c^x \int_c^{s_1} \cdots \int_c^{s_n} f^{(n+1)}(s_{n+1})\, ds_{n+1} \cdot ds_n \cdots ds_1$$

$$\Rightarrow |g(x)| = \left| \int_c^x \cdots\cdots ds_1 \right| \leq \int_c^x \left| \cdots\cdots \right| ds_1$$

by Lemma

$$\Rightarrow \quad |g(x)| \leq \int_c^x \int_c^{s_1} \cdots \int_c^{s_n} \underbrace{|f^{(n+1)}(s_{n+1})|}_{\leq M \text{ by } (*).} \, ds_{n+1} \cdots ds_1$$

$$\leq \int_c^x \int_c^{s_1} \cdots \int_c^{s_n} M \, ds_{n+1} \cdots ds_1$$

$$= \int_c^x \int_c^{s_1} \cdots \int_c^{s_{n-1}} \left(M s_{n+1} \right) \Big|_{s_{n+1}=c}^{s_{n+1}=s_n} ds_n \cdots ds_1$$

$$= \int_c^x \int_c^{s_1} \cdots \int_c^{s_{n-1}} M(s_n - c) \, ds_n \cdots ds_1$$

$$= \cdots = M \cdot \frac{|x-c|^{n+1}}{(n+1)!}$$

$$\Rightarrow \ |g(x)| = |f(x) - P_n(x)| \leq M \cdot \frac{|x-c|^{n+1}}{(n+1)!}$$

<u>Theorem</u>: Let $f(x)$ be a smooth function, $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ conv. on $(c-R_0, c+R_0)$ where R_0 is the radius of convergence.

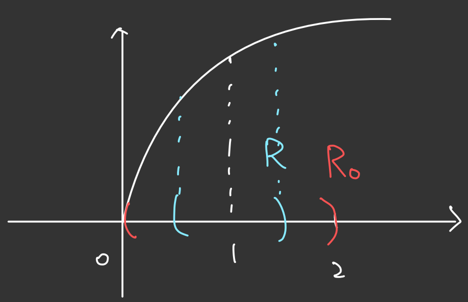
Suppose that for any $R < R_0$, there exists $M_R > 0$ such that

$$|f^{(n)}(x)| < M_R \quad \text{for all} \quad x \in (c-R, c+R)$$

Then we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{for all} \quad x \in (c-R_0, c+R_0)$$

$f(x) = \sqrt{x}$ centered at 1



Introduction to Differential Equations

An Ordinary differential equation has the form :

ODE

$$F\left(f^{(n)}(x),\ f^{(n-1)}(x),\ f^{(n-2)}(x), \ldots, f^{(1)}(x),\ f(x),\ x\right) = 0$$

(We call it n-th order ODE)

Example (1) $f'(x) - 6x = 0$ is a 1st order ODE.

(2) $f'(x) - 2f(x) = 0$ is a 1st order ODE.

(3) $f''(x) + 4f(x) = 0$ is a 2nd order ODE.

(4) $f''(x) - 9f(x) = 0$ is a 2nd order ODE.

$$\text{ODE Equations} \longmapsto \text{ODE Solutions}$$

~~Formula.~~

Solutions of ODE's :

Let $f'(x) - 6x = 0 \Rightarrow f'(x) = 6x \Rightarrow f(x) = 3x^2$. is a solution.

Let $f''(x) + 4f(x) = 0$. We consider $\sin(2x)$,

$$\left(\frac{d}{dx}\right)^2 \sin(2x) = \frac{d}{dx}\left(2\cos(2x)\right) = -4\sin(2x)$$

$\Rightarrow \sin(2x)$ is a solution (Similarly, $\cos(2x)$ is also a solution.)

Def: Let $F(f^{(n)}(x), \dots, f(x), x) = 0$ be an ODE. Then we call

$$S := \{ f(x) \mid f \text{ satisfies } F(f^{(n)}(x), \dots, f(x), x) = 0 \}$$

the solution space of this ODE.

In the following, we consider 1st order ODEs:

Case 1: $f'(x) - g(x) = 0$; $g(x)$ is given.

$$\Rightarrow f'(x) = g(x) \Rightarrow \int f'(x)\,dx = \int g(x)\,dx \quad (\text{by F.T.C})$$

$$\int f'(x)\,dx = f(x) + C$$

$$\Rightarrow f(x) = \int g(x)\,dx + C$$

Case 2: $f'(x) - a f(x) = 0$; $a \in \mathbb{R}$ is given.

$$g(x) = e^{-ax} ; \quad g'(x) = -a e^{-ax} = -a g(x).$$

$$\Rightarrow g(x) f'(x) - a g(x) f(x) = 0 \Rightarrow g(x) f'(x) + g'(x) f(x) = 0$$

$$\Rightarrow (f(x) g(x))' = 0$$

$$\Rightarrow f(x) g(x) = C \Rightarrow f(x) = \frac{C}{g(x)}.$$

Therefore,

$$S = \{ C \cdot e^{ax} \mid \text{for some } C \in \mathbb{R} \}$$

Remark: $f'(x) = a f(x)$ is a model of population growth.

$x = t$: time variable

$f(t)$: population at time t.

$$f'(t) \propto f(t) \Rightarrow f'(t) = a f(t) \; ; \quad 0 < a < 1$$

$$\Rightarrow f(t) = C \cdot e^{at} \qquad C > 0$$

Case 3: (Seperation of variables)

Let $f(x) = y$; $f'(x) = \frac{dy}{dx}$.

$$\underbrace{\frac{dy}{dx} + h(x)\cdot g(y) = 0}_{\substack{\parallel \\ F(y',\, y,\, x)}}$$

Suppose $g(y) \neq 0$. Then $\displaystyle\int \frac{-1}{g(y)} \cdot \frac{dy}{dx}\, dx = \int h(x)\, dx$

$$\Rightarrow \int \frac{-1}{g(y)}\, dy = \int h(x)\, dx \quad \text{——— solvable.}$$

<u>Example</u>: $\displaystyle \frac{dy}{dx} + x^2 \cdot \frac{1}{y} = 0$

$$\Rightarrow \int -y \frac{dy}{dx} dx = \int x^2 dx \Rightarrow -\int y\, dy = \int x^2 dx.$$

$$\Rightarrow -\frac{y^2}{2} + C_1 = \frac{1}{3}x^3 + C_2$$

$$\Rightarrow y^2 = C - \frac{2}{3}x^3$$

$$y = f(x) \Rightarrow f(x) = \pm\sqrt{C - \frac{2}{3}x^3}$$

Solution space

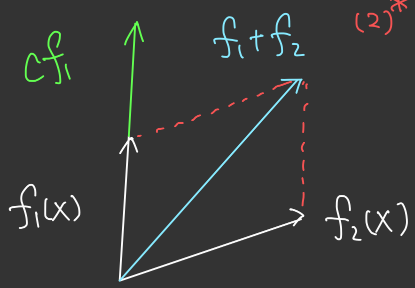
$$S = \left\{ \pm\sqrt{C - \frac{2}{3}x^3} \;\middle|\; C \in \mathbb{R} \right\}$$

Case 4: (First order linear ODE).

"linear" means that for any $f_1(x)$, $f_2(x)$ solutions of $F(f'(x), f(x), x) = 0$, we have

(1) $f_1(x) + f_2(x)$ is also a solution.

(2) $cf_1(x)$ is also a solution for any $c \in \mathbb{R}$.



(2)* $f(x)$ is a solution $\Rightarrow$ $Cf(x)$ is also a solution.

A First order linear ODE is of the form :

$$\underbrace{f'(x) + p(x)f(x)}_{F(f', f, x)} = q(x).$$

<u>Part 1</u> : Suppose $p(x) = p$ is a constant.

$$e^{px}\left(f'(x) + p f(x)\right) = e^{px} q(x)$$

$$\|$$

$$e^{px} f'(x) + (e^{px})' \cdot f(x) = \frac{d}{dx}\left(e^{px} \cdot f(x)\right)$$

$$e^{px} f(x) = \int e^{px} \cdot q(x)\, dx \Rightarrow f(x) = e^{-px} \int e^{px} q(x)\, dx .$$

is the solution of equation.

Example: Let $f'(x) + 2f(x) = x$

$$\Rightarrow e^{2x} \cdot f'(x) + 2e^{2x} f(x) = e^{2x} \cdot x \quad (\text{multiplies by } e^{2x})$$

$$\Rightarrow \frac{d}{dx}\left(e^{2x} f(x) \right) = e^{2x} x .$$

$$\Rightarrow e^{2x} f(x) = \int \underbrace{e^{2x}}_{\left(\frac{1}{2} e^{2x}\right)'} \cdot x \, dx = -\int \frac{1}{2} e^{2x} \cdot 1 + \left(\frac{1}{2} e^{2x} \cdot x \right)$$

$$= \frac{-1}{4} e^{2x} + \frac{1}{2} e^{2x} \cdot x + C .$$

$$\Rightarrow f(x) = -\frac{1}{4} + \frac{1}{2}x + Ce^{-2x}. \quad \text{is the solution of eq.}$$

Part 2: If $p(x)$ is not a constant, we consider

$$e^{\int p(x)\,dx} := r(x).$$

Notice that $r'(x) = p(x) \cdot r(x)$.

$$r(x)\left(f'(x) + p(x)\cdot f(x)\right) = r(x)\,q(x)$$

$$\Rightarrow \left(r(x)\cdot f(x)\right)' = r(x)\cdot q(x) \Rightarrow r(x)\cdot f(x) = \int r(x)\,q(x)\,dx.$$

$$\Rightarrow f(x) = \frac{1}{r(x)} \int r(x)\, q(x)\, dx \quad : \text{ solution of eq.}$$

Remark : We can replace $r(x)$ by any $r(x)$ satisfying

$$r'(x) = p(x) \cdot r(x)$$

Example: $f'(x) + x f(x) = 2$,

$$r'(x) = x \cdot r(x) \Rightarrow r(x) = e^{\frac{1}{2}x^2}$$

$$\Rightarrow f(x) = e^{-\frac{1}{2}x^2} \int e^{\frac{1}{2}x^2} \cdot 2\, dx = 2 e^{-\frac{1}{2}x^2} \cdot \int e^{\frac{1}{2}x^2} dx.$$

is a solution of our eq.

Initial data / Initial condition:

Let $F(f'(x), f(x), x) = 0$ be a 1st order ODE.

An initial data is a constraint as the following

$$f(p) = C_0 \quad ; \; p \text{ is in the domain (interval) of } f$$

$$C_0 \in \mathbb{R}$$

Thm: For any $F(f'(x), f(x), x) = 0$ and initial data $f(p) = C_0$, there exists an unique solution.

Example: $\begin{cases} f'(x) = x & \text{.... ODE} \\ f(0) = 1 & \text{.... initial condition.} \end{cases}$

$$\Rightarrow f(x) = \int x\,dx = \frac{1}{2}x^2 + C, \quad f(0) = C = 1.$$

$$\Rightarrow f(x) = \frac{1}{2}x^2 + 1$$

Example: $\begin{cases} f'(x) = \frac{1}{2}f(x) & \text{.... ODE} \\ f(0) = 100 & \text{.... initial condition.} \end{cases}$

$$\Rightarrow f(x) = C \cdot e^{\frac{1}{2}x}, \quad f(0) = C = 100$$

$$\Rightarrow f(x) = 100\, e^{\frac{1}{2}x}.$$

Example:
$$\begin{cases} \dfrac{dy}{dx} + \dfrac{x^2}{y} = 0 & \text{..... ODE} \\ y(0) = 1 & \text{..... initial condition.} \end{cases}$$

$$\Rightarrow \quad y(x) = \pm\sqrt{C - \frac{2}{3}x^3}\ , \quad y(0) = \underbrace{\pm\sqrt{C}}_{\text{取正}} = 1 \Rightarrow C = 1$$

$$\Rightarrow \quad y(x) = \sqrt{1 - \frac{2}{3}x^3}$$

Curves:

Let $\vec{p}$ be a function defined on $\mathbb{R}^1$ with values in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

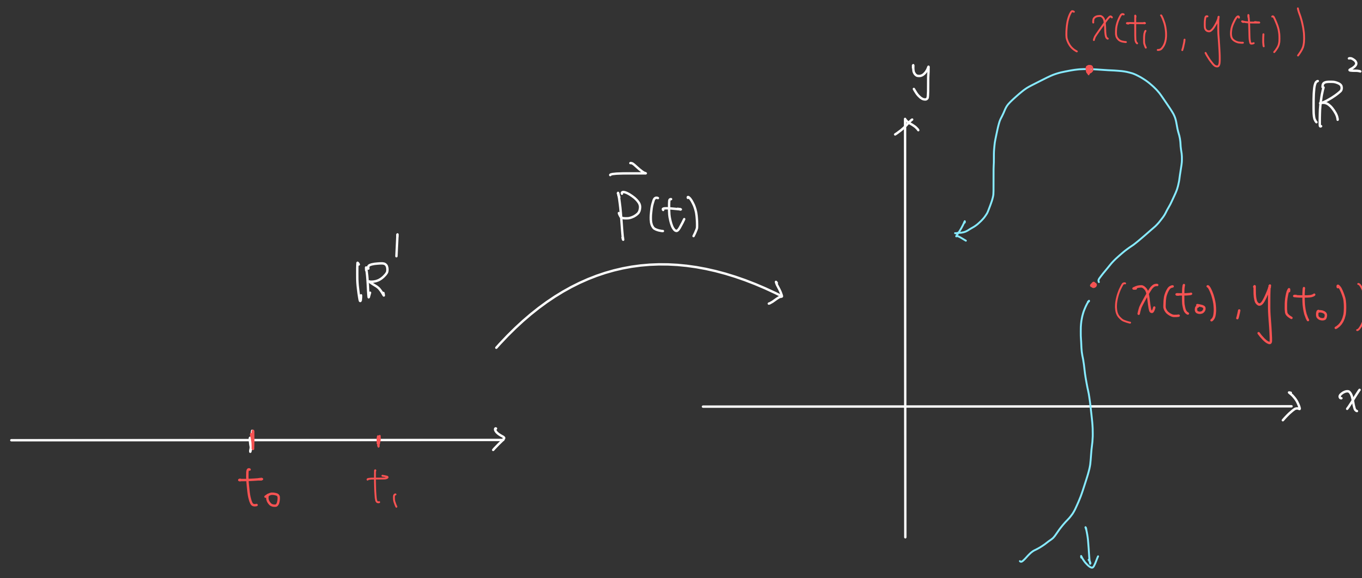
$$\vec{p}(t) = (x(t), y(t)), \quad t \in \mathbb{R}. \quad \left(\vec{p}(t) = (x(t), y(t), z(t))\right)$$

We call $\vec{p}$ a vector-valued function.

<u>Def</u>: If $x(t)$, $y(t)$ are (smooth) continuous, then we call

$$\{\vec{p}(t) \mid t \in \mathbb{R}\}$$

a curve in $\mathbb{R}^2$.



$$\{\vec{P}(t) \mid t \in \mathbb{R}\} = \mathcal{C}$$

This is a curve.

Def: Let $\vec{p}(t)$ be a vector-valued function. We define

$$\frac{d\vec{p}}{dt}(s) := \left(\frac{dx}{dt}(s), \frac{dy}{dt}(s) \right)$$

Prop: Let $\vec{p}, \vec{q}$ be two vector-valued function. Then

(1) $$\frac{d}{dt}(r \cdot \vec{p}) = r \frac{d\vec{p}}{dt}$$ for any $r \in \mathbb{R}$.

(2) $$\frac{d}{dt}\left(f(t) \cdot \vec{p}(t)\right) = \frac{df}{dt} \cdot \vec{p} + f \cdot \frac{d\vec{p}}{dt}.$$; $f(t)$ is a 1-variable function.

(3) $$\frac{d}{dt}(\vec{p} + \vec{q}) = \frac{d\vec{p}}{dt} + \frac{d\vec{q}}{dt}.$$

(4) $\dfrac{d}{dt}(\vec{p}\cdot\vec{q}) = \dfrac{d\vec{p}}{dt}\cdot\vec{q} + \vec{p}\cdot\dfrac{d\vec{q}}{dt}$

(5) $\dfrac{d}{dt}(\vec{p}\times\vec{q}) = \dfrac{d\vec{p}}{dt}\times\vec{q} + \vec{p}\times\dfrac{d\vec{q}}{dt}$ (注意順序)

(6) $\dfrac{d}{ds}(\vec{p}\circ r) = \dfrac{d}{ds}\big(x(r(s)),\, y(r(s))\big) = r'(s)\cdot\dfrac{d\vec{p}}{dt}$

$$\begin{tikzcd}
s \arrow[d, phantom, sloped, "\in"] & t \arrow[d, phantom, sloped, "\in"] & \\
\mathbb{R}^1 \arrow[r, "r"] \arrow[rr, bend right, "\vec{p}\circ r"'] & \mathbb{R}^1 \arrow[r, "\vec{p}"] & \mathbb{R}^2 \text{ or } \mathbb{R}^3
\end{tikzcd}$$

Let us call $\vec{p}$ a <u>position function.</u>

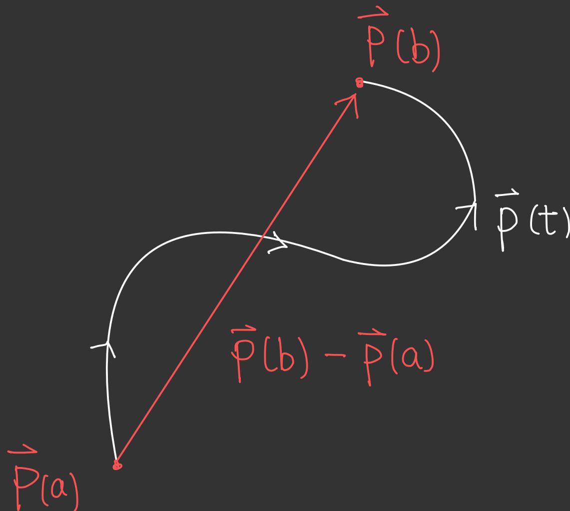
$\frac{d\vec{p}}{dt}$: velocity ; $\left|\frac{d\vec{p}}{dt}\right|$: speed.

By Fundamental Theorem of Calculus,

$$\int_a^b \frac{d\vec{p}}{dt}\,dt = \int_a^b (x'(t), y'(t))\,dt := \left(\int_a^b x'(t)\,dt, \int_a^b y'(t)\,dt\right)$$

$$= \left(x(b)-x(a), y(b)-y(a)\right)$$

$$= (x(b), y(b)) - (x(a), y(a)) \quad : \text{vector}$$



Denote by $\vec{v}$ the vector-valued function $\dfrac{d\vec{p}}{dt}$.

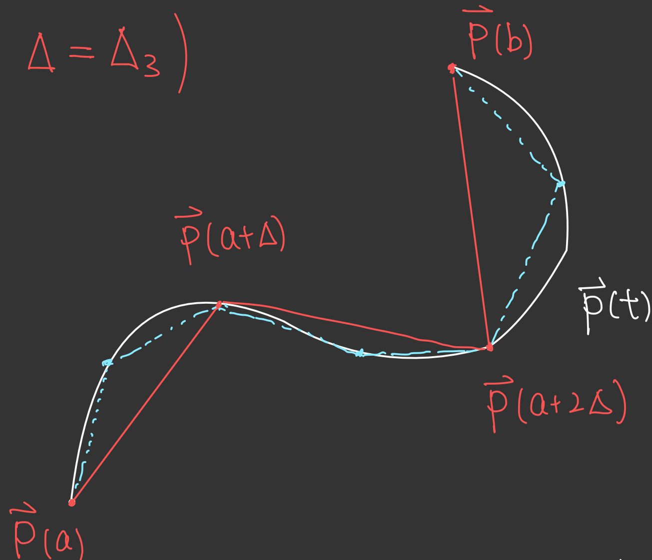
$$\int_a^b |\vec{v}(t)|\,dt = ?$$

<u>Recall that</u>: $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}$ $\quad (\vec{p}: \mathbb{R} \to \mathbb{R}^2)$

Let $\Delta_n = \dfrac{b-a}{n}$,

$$\int_a^b |\vec{v}(t)|\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} |\vec{v}(t_k)|\,\Delta_n \quad , \quad t_k \in [a+(k-1)\Delta_n,\ a+k\Delta_n]$$

$(\Delta = \Delta_3)$



$$\lim_{n\to\infty}\sum_{k=1}^{n}\left|\vec{p}(a+k\Delta_n)-\vec{p}(a+(k-1)\Delta_n)\right|$$

$$= \text{length of curve.}$$

Meanwhile, by mean value theorem,

$$\left|\vec{p}(\underbrace{a+k\Delta_n}_{\alpha+\Delta_n}) - \vec{p}(\underbrace{a+(k-1)\Delta_n}_{\alpha})\right|$$

$$\approx \left|\underbrace{\frac{d\vec{p}}{dt}}_{=\vec{v}}(t_k)\right| \cdot \Delta_n, \quad t_k \in [a+(k-1)\Delta_n,\ a+k\Delta_n]$$

So $\displaystyle\sum_{k=1}^{n} |\vec{p}(a+k\Delta_n) - \vec{p}(a+(k-1)\Delta_n)|$ = length of curve (from $\vec{p}(a)$ to $\vec{p}(b)$)

$$\approx \sum_{k=1}^{n} |\vec{v}(t_k)|\, \Delta_n .$$

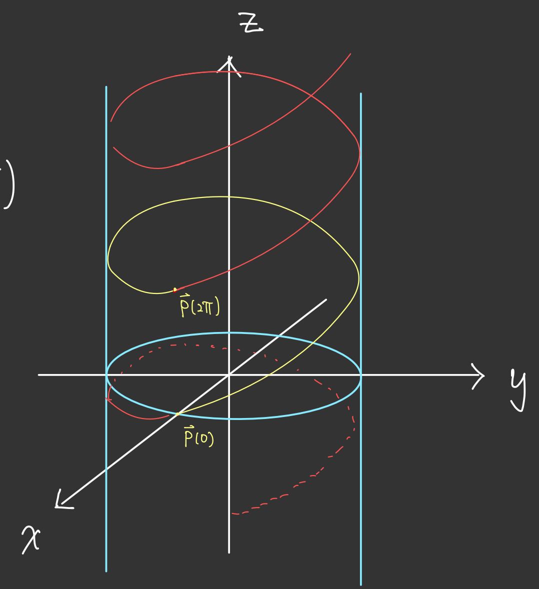
$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} |\vec{p}(a+k\Delta_n) - \vec{p}(a+(k-1)\Delta_n)|$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} |\vec{v}(t_k)|\, \Delta_n = \int_a^b |\vec{v}(t)|\, dt .$$

Prop: If we define $l_a^b :=$ length of curve $\{\vec{p}(t) \mid t\in[a,b]\}$, then $\int_a^b |\vec{v}(t)|\, dt = \ell_a^b$.

Example: (Helix) Let $\vec{p}(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$.

Find $\ell_0^{2\pi}$.

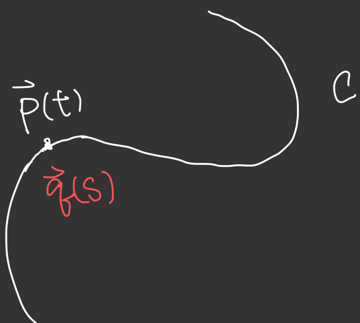


$$\vec{p}\,'(t) = (-\sin t, \cos t, 1) = \vec{v}(t)$$

$$\Rightarrow |\vec{v}(t)| = \sqrt{2}$$

$$\ell_0^{2\pi} = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Let $C := \{ \vec{p}(t) \mid t \in \mathbb{R} \}$ be a curve. We call t a parameter of C.



Suppose $\exists\ \vec{q}(s)$ also parametrizes C

i.e. $\{ \vec{q}(s) \mid s \in \mathbb{R} \} = C$.

Here we assume $\vec{p}, \vec{q}$ are one to one.

Then there exists an one to one and onto maps

$$s \longmapsto t$$

We can write $s = s(t)$.

$$\Rightarrow \quad \vec{q}(s) = \vec{q}(s(t)) = \vec{p}(t) . \quad \Rightarrow \quad \left| \frac{d\vec{q}}{ds} \right| \cdot \frac{ds}{dt} = \left| \frac{d\vec{p}}{dt} \right|$$

$$\int_a^b \left|\frac{d\vec{q}}{ds}\right| \frac{ds}{dt}\, dt = \int_a^b \left|\frac{d\vec{p}}{dt}\right| dt \ : \text{ length of } C \text{ from } \vec{p}(a) \text{ to } \vec{p}(b)$$

$$\parallel$$

$$\int_{s(a)}^{s(b)} \left|\frac{d\vec{q}}{ds}\right| ds$$

<u>Remark:</u> the length of C is invariant under parametrizations.

<u>Def:</u> Suppose that C is parametrized by $\vec{q}(s)$ and

$$\left|\frac{\vec{q}(s)}{ds}\right| = 1 \quad \text{for all } s \in \mathbb{R}.$$

Then we call C is parametrized by arc length.

Remark: Let $C = \{ \vec{q}(s) \mid s \in \mathbb{R} \}$ be a curve parametrized by arc length.

Then
$$\int_{s_1}^{s_2} \left| \frac{\vec{q}(s)}{ds} \right| ds = (s_2 - s_1)$$

Plane curves and Curvature:

A plane curve is $C = \{ \vec{p}(t) \mid t \in \mathbb{R} \}$ $\vec{p} : \mathbb{R} \to \mathbb{R}^2$.

Let $C = \{ \vec{q}(s) \mid s \in \mathbb{R} \}$ be a parametrization with.

$$\left| \frac{d\vec{q}}{ds} \right| = 1. \qquad \text{for all } s \in \mathbb{R}.$$

Denote by $\vec{v}(s)$ the vector $\dfrac{d\vec{q}}{ds}$.

Def: The curvature of C at $\vec{q}(s)$ is

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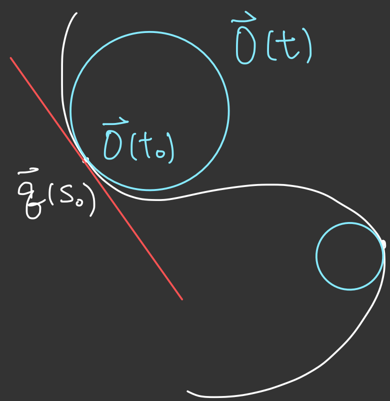
$$\kappa(s) := \left| \frac{d\vec{v}}{ds} \right|$$

Geometric meaning of curvatures



$C = \{ \vec{q}(s) \mid s \in \mathbb{R} \}$

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Def: (Osculating circles)

An osculating circle

$\{ \vec{O}(t) \mid t \in \mathbb{R} \}$

of C at $\vec{q}(s_0)$, is a circle satisfying

① There exists $t_0 \in \mathbb{R}$ such that

$$\vec{q}(s_0) = \vec{O}(t_0)$$

② $\vec{q}\,'(s_0) \,/\!/\, \vec{O}\,'(t_0)$.

③ $\left.\begin{array}{l}\text{Let } \vec{O}(t) \text{ parametrizes the circle by its arclength.} \\ \vec{q}(s) \text{ parametrizes } C \text{ by its arclength.}\end{array}\right\} \cdots\cdots (*)$

Then $\vec{O}\,''(t_0) = \vec{q}\,''(s_0)$.

Assuming $(*)$, we need to check $\begin{cases} \vec{q}(s_0) = \vec{O}(t_0) \\ \vec{q}\,'(s_0) = \vec{O}\,'(t_0) \\ \vec{q}\,''(s_0) = \vec{O}\,''(t_0) \end{cases}$

Let $\vec{O}(t) = \big(R\cos(\alpha t+\beta),\ R\sin(\alpha t+\beta) \big) + \underbrace{(x_0, y_0)}_{\text{center of the circle.}}$

be an osculating circle of C at $\vec{q}(s_0)$.

$$\vec{O}'(t) = \big(-R\sin(\alpha t+\beta)\cdot\alpha,\ R\cos(\alpha t+\beta)\cdot\alpha \big)$$

$$\Rightarrow |\vec{O}'(t)| = R\cdot|\alpha| = 1 \quad . \Rightarrow |\alpha| = \frac{1}{R}$$

$$\begin{cases} \vec{q}(s_0) = \vec{O}(t_0) \\ \vec{q}'(s_0) = \vec{O}'(t_0). \\ \boxed{\vec{q}''(s_0) = \vec{O}''(t_0)} \end{cases} \longrightarrow \quad |\vec{q}''(s_0)| = |\vec{O}''(t_0)|$$

$$|\vec{O}''(t_0)| = |(-R\cos(\alpha t+\beta)\cdot\alpha^2, -R\sin(\alpha t+\beta)\cdot\alpha^2)|$$

$$= R|\alpha|^2 = \frac{1}{R}.$$

Meanwhile, $\left|\dfrac{d^2\vec{q}}{ds^2}\right| = \left|\dfrac{d\vec{v}}{ds}\right| = k(s)$

Theorem:

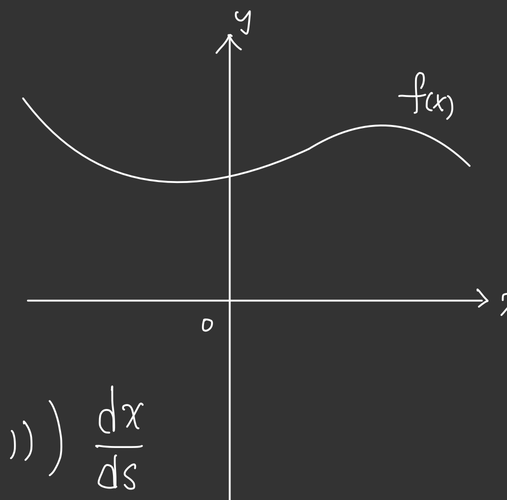
$$k(s) = \frac{1}{R}$$

where R is the radius of the osculating circle at $\vec{q}(s)$.

Remark: The formula $\underset{\text{kappa}}{\kappa}(s) = \left|\dfrac{d\vec{v}}{ds}\right|$; $\vec{v}(s) = \dfrac{d\vec{q}}{ds}$

holds only if the condition $\left|\dfrac{d\vec{q}}{ds}\right| = 1$. for all s.

Example: Let $C = \{(x, f(x)) \mid x \in \mathbb{R}\}$ be a curve. (f · smooth).



Let $x = x(s)$. $\vec{q}(s) = (x(s), f(x(s)))$

$$\vec{q}\,'(s) = \left(\frac{dx}{ds}, f'(x(s)) \cdot \frac{dx}{ds}\right) = (1, f'(x(s)))\,\frac{dx}{ds}$$

$$|\vec{q}'(s)| = 1 = \left|\frac{dx}{ds}\right| \cdot \sqrt{1+f'(x(s))^2} \quad ; \quad \text{So} \quad \frac{dx}{ds} = \frac{\pm 1}{\sqrt{1+f'(x(s))^2}}$$

We can take

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+f'(x(s))^2}},$$

then

$$\vec{q}'(s) = (1, f'(x(s))) \cdot \frac{1}{\sqrt{1+f'(x(s))^2}} = \left(\frac{1}{\sqrt{1+f'(x(s))^2}}, \frac{f'(x(s))}{\sqrt{1+f'(x(s))^2}} \right)$$

So,

$$\frac{d\vec{v}(s)}{ds} = \left(-\frac{1}{\cancel{2}} \frac{\cancel{2} f'(x(s)) f''(x(s))}{(1+f'(x(s))^2)^{3/2}} \frac{dx}{ds}, \; \frac{-\cancel{2} f'(x(s))^2 f''(x(s))}{\cancel{2}\,(1+f'(x(s))^2)^{3/2}} \cdot \frac{dx}{ds} + \frac{f''(x(s))}{(1+f'(x(s))^2)^{1/2}} \frac{dx}{ds} \right)$$

$$= \left(-\frac{f'(x(s))\, f''(x(s))}{\left(1+f'(x(s))^2\right)^2},\ -\frac{f'(x(s))^2 f''(x(s))}{\left(1+f'(x(s))^2\right)^2} + \frac{f''(x(s))}{\left(1+f'(x(s))^2\right)} \right)$$

$$= \left(-\frac{f'(x(s))\, f''(x(s))}{\left(1+f'(x(s))^2\right)^2},\ \frac{\cancel{-f'(x(s))^2 f''(x(s))} + f''(x(s)) + \cancel{f''(x(s))\, f'(x(s))^2}}{\left(1+f'(x(s))^2\right)^2} \right)$$

$$\Rightarrow \left| \frac{d\vec{v}}{ds} \right| = \sqrt{\frac{\left(f'(x(s))\, f''(x(s))\right)^2 + f''(x(s))^2}{\left(1+f'(x(s))^2\right)^4}} = \sqrt{\frac{f''(x(s))^2}{\left(1+f'(x(s))^2\right)^3}}$$

$$= \frac{|f''(x)|}{\left(1+f'(x)^2\right)^{3/2}} := K(x).$$

Recall: ① When the curve C parametrized by its arc-length, say

$$C = \{ \vec{q}(s) \mid s \in \mathbb{R} \},$$

the curvature

$$K(s) := |\vec{q}\,''(s)| = \left| \frac{d\vec{v}}{ds}(s) \right|$$

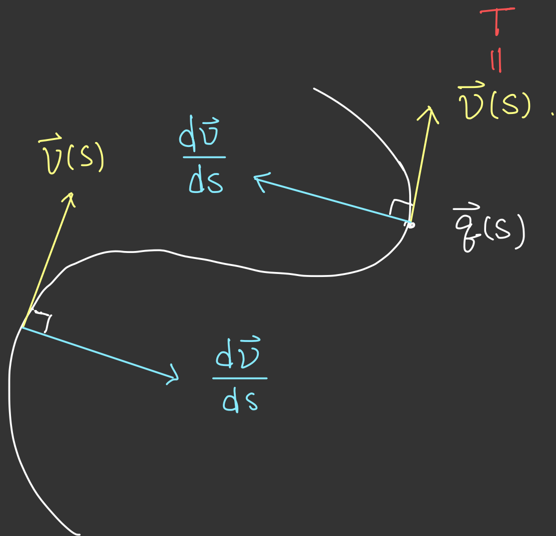
② When $C = \{ (x, f(x)) \mid x \in \mathbb{R} \}$, the curvature

$$K(x) := \frac{|f''(x)|}{\sqrt{1 + f'(x)^2}}$$

Let $\vec{v}(s) := \dfrac{d\vec{q}(s)}{ds}$. Since $\vec{v}(s) \cdot \vec{v}(s) = 1$ for all $s \in \mathbb{R}$,

we have
$$\frac{d}{ds}(\vec{v} \cdot \vec{v}) = 0 = \frac{d\vec{v}}{ds} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{ds} = 2\vec{v} \cdot \frac{d\vec{v}}{ds}.$$

So $\vec{v}(s) \cdot \frac{d\vec{v}}{ds}(s) = 0 \Rightarrow \frac{d\vec{v}}{ds}$ is perpendicular to $\vec{v}$



Let us denote by $\begin{cases} T(s) := \vec{v}(s) \\ N(s) := \frac{d\vec{v}}{ds} \Big/ \left|\frac{d\vec{v}}{ds}\right| \end{cases}$

Then $T(s)$ is the tangent vector of the curve C at $\vec{q}(s)$. ; $N(s)$ is the normal vector of C at $\vec{q}(s)$.

Moreover,

$$\frac{d\vec{v}}{ds} = \left|\frac{d\vec{v}}{ds}\right| N \Rightarrow \boxed{\frac{dT}{ds} = K \cdot N} \quad \text{———} \quad (*)$$

Remark: We can write

$$K(s) = \left|\frac{dT}{ds}\right| .$$

In general, if $T(t)$ is not parametrized by arc-length,

$$\frac{dT}{ds} = \frac{dT}{dt} \cdot \frac{dt}{ds} = \frac{dT}{dt} \Big/ \frac{ds}{dt}$$

$$\Rightarrow \quad K(t) = |T'(t)| \Big/ \frac{ds}{dt}$$

Let $\vec{P}(t) = (x(t), y(t))$ parametrizes C.

$$\frac{d\vec{p}}{dt} = (x'(t), y'(t)) . \Rightarrow T(t) = \left(\frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right)$$

$$\Rightarrow K(t) = \left| \frac{dT}{dt} \right| \Big/ \frac{ds}{dt} = \left| \frac{dT}{dt} \right| \cdot \frac{dt}{ds} .$$

Suppose $t = t(s)$. , $\vec{q}(s) = \vec{p}(t(s)) = (x(t(s)), y(t(s)))$

$$\Rightarrow \frac{d\vec{q}}{ds} = \left(x'(t(s)) \frac{dt}{ds}, y'(t(s)) \cdot \frac{dt}{ds} \right)$$

$$= (x'(t), y'(t)) \frac{dt}{ds} \Rightarrow \frac{dt}{ds} = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$\frac{dT}{dt} = \left(\frac{-\cancel{2}\left(x'(t)^2 \cdot x''(t) + x'(t)y'(t)y''(t)\right)}{\cancel{2}\left(x'(t)^2 + y'(t)^2\right)^{3/2}} + \frac{x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{1/2}}, \right.$$

$$\left. \frac{-\cancel{2}\left(x'(t)y'(t)x''(t) + y'(t)^2 \cdot y''(t)\right)}{\cancel{2}\left(x'(t)^2 + y'(t)^2\right)^{3/2}} + \frac{y''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{1/2}} \right)$$

$$= \left(\frac{-x'y'y'' + x''y'^2}{\left(x'^2 + y'^2\right)^{3/2}}, \frac{-x'y'x'' + x'^2y''}{\left(x'^2 + y'^2\right)^{3/2}} \right)$$

$$= \left(\frac{-y'\left(x''y' - x'y''\right)}{\left(x'^2 + y'^2\right)^{3/2}}, \frac{-x'\left(x''y' - x'y''\right)}{\left(x'^2 + y'^2\right)^{3/2}} \right) \Rightarrow \left|\frac{dT}{dt}\right| = \frac{|x''y' - x'y''|}{\left(x'^2 + y'^2\right)}$$

$$\Rightarrow \quad K(t) = \frac{|x''(t)\,y'(t) - y''(t)\,x'(t)|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

Functions of Several variables :

Recall: $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff

$\exists\ \delta : \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ such that

$|f(x) - f(a)| < \varepsilon$ when $|x - a| < \delta(\varepsilon)$.

($\lim\limits_{x \to a} f(x) = f(a)$ 極限值 = 函數值)

$|x-a| = \sqrt{(x-a)^2}$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of n-variables.

$$\begin{array}{c} \mathbb{R}^n \\ \Psi \\ (x_1, x_2, \dots, x_n) = \vec{x} \end{array}$$

Denote:

$$|\vec{x} - \vec{y}| := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

$\vec{x} = (x_1, \dots, x_n)$

$\vec{y} = (y_1, \dots, y_n)$

Def: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a n-variable function. For any $\vec{a} \in \mathbb{R}^n$, $L \in \mathbb{R}$, we call $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ iff

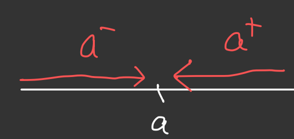
$\exists\, \delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|f(\vec{x}) - L| < \varepsilon \quad \text{when} \quad 0 < |\vec{x} - \vec{a}| < \delta(\varepsilon).$$

Def: We call f continuous at $\vec{a}$ iff

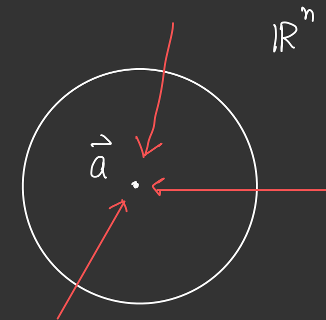
$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$

Remark: $f : \mathbb{R} \to \mathbb{R}$. f is cont. at a.



$$\Leftrightarrow \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

$f : \mathbb{R}^n \to \mathbb{R}$. f is cont. at $\vec{a}$



$$\Rightarrow \lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \in L_1}} f(\vec{x}) = \lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \in L_2}} f(\vec{x}) = f(\vec{a}).$$

where L_1, L_2 are two lines passing through $\vec{a}$.

<u>Def</u>: Let L be a line passing through $\vec{a}$.

$$L = \{ \vec{a} + t\vec{v} \mid t \in \mathbb{R} \}, \quad \vec{v} \in \mathbb{R}^n$$

$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \in L}} f(\vec{x}) := \lim_{t \to 0} f(\vec{a} + t\vec{v})$$

Prop: If L_1, L_2 are two lines passing through $\vec{a}$ and

$$\lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\in L_1}} f(\vec{x}) \neq \lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\in L_2}} f(\vec{x})$$

Then f is not continuous at $\vec{a}$.

Example: Let

$$f(x,y) := \begin{cases} \dfrac{x^2}{x^2-y^2} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let $L_1 = y\text{-axis} = \{ (0,t) \mid t \in \mathbb{R} \}$.

$$\lim_{\substack{\vec{x}\to\vec{0}\\ \vec{x}\in L_1}} f(\vec{x}) = \lim_{t\to 0} \frac{0^2}{0^2-t^2} = 0$$

Let $L_2 = x\text{-axis} = \{(t,0) \mid t \in \mathbb{R}\}$

$$\lim_{\substack{\vec{x}\to\vec{0} \\ \vec{x}\in L_2}} f(\vec{x}) = \lim_{t\to 0} \frac{t^2}{t^2 - 0^2} = 1$$

So $f(\vec{x})$ is <u>not</u> continuous at $\vec{0}$.

Partial differentiations:

<u>Def</u>: Let $f(x_1, \dots, x_n)$ be a function of n-variables.

$$\frac{\partial f}{\partial x_k}(\vec{a}) := \lim_{h\to 0} \frac{f(a_1 \dots a_k + h, \dots a_n) - f(a_1, \dots, a_k, \dots a_n)}{h}$$

$\vec{a} = (a_1 \dots a_n)$

Example: Let $f(x,y) = 2x^2 + xy + ye^x$.

$$\frac{\partial f}{\partial x} = ?\ , \quad \frac{\partial f}{\partial x}(0,0) = ? . \quad \frac{\partial f}{\partial y} = ?$$

Sol: $$\frac{\partial f}{\partial x} = 4x + y + ye^x \ , \quad \frac{\partial f}{\partial x}(0,0) = 0 .$$

$$\frac{\partial f}{\partial y} = 0 + x + e^x = x + e^x$$

Prop: (1) $$\frac{\partial}{\partial x}(f_1 + f_2) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} .$$

(2) $$\frac{\partial}{\partial x}(cf) = c \cdot \frac{\partial f}{\partial x} .$$

(3) $$\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x} .$$

Prop: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ ← 先微 x, 再微 y.

↑ 先微 y, 再微 x

Example: ① $f(x,y) = 2x^2 + xy + ye^x$.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x + e^x\right) = 1 + e^x.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(4x + y + ye^x\right) = 1 + e^x$$

② $f(x,y,z) = z^3 + 3xy^2z + \cos x \cdot e^y \cdot z^2$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(3z^2 + 3xy^2 + 2\cos x\, e^y z\right)$$

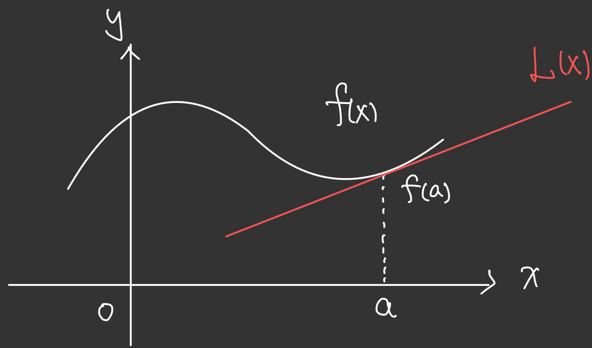
$$= 3y^2 - 2\sin x\, e^y z$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(0 + 3y^2 z - \sin x\, e^y z^2\right)$$

$$= 3y^2 - 2\sin x\, e^y z.$$

Total differentiation:

Recall: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$. be a differentiable function



Tangent line:

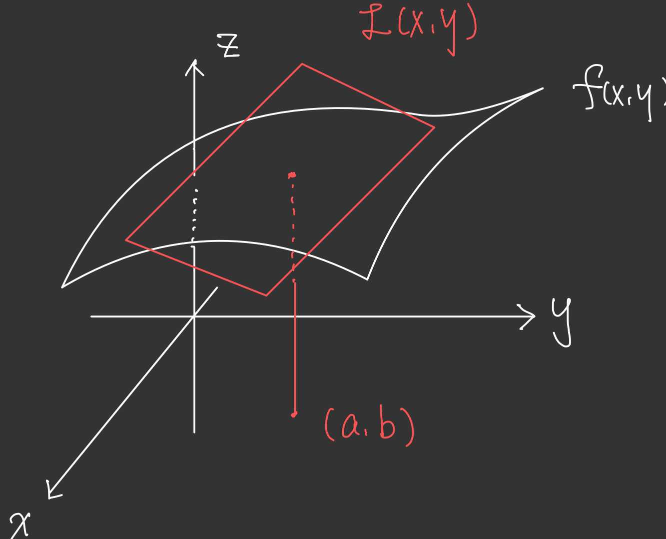
$$y = L(x) = f'(a)\cdot(x-a) + f(a)$$

In fact, we have

Prop: f is differentiable at a if and only if there exists an affine function $y = L(x)$ such that

$$\lim_{x\to a} \frac{f(x) - L(x)}{x-a} = 0$$

For n-variable function, say $f(x_1, \dots, x_n)$,



<u>Tangent Plane (Space):</u>

$$y = \mathcal{L}(x_1, \dots, x_n)$$

$$= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \beta$$

for some $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{R}$.

<u>Def:</u> Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a n-variable function and $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$. (denote by $\vec{x} = (x_1, \dots, x_n)$).

Then f is differentiable at $\vec{a}$ iff $\exists$ affine function $\mathcal{L}(\vec{x})$ s.t

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - \mathcal{L}(\vec{x})}{|\vec{x}-\vec{a}|} = 0$$

Q: Suppose $f(\vec{x})$ is differentiable at $\vec{a}$. i.e.,

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - \mathcal{L}(\vec{x})}{|\vec{x}-\vec{a}|} = 0 \quad \cdots\cdots (*)$$

for some $\mathcal{L}(\vec{x}) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \beta$

What are $\alpha_1, \ldots, \alpha_n, \beta$?

$$(*) \Rightarrow \lim_{\vec{x}\to\vec{a}} f(\vec{x}) - \mathcal{L}(\vec{x}) = 0 \Rightarrow f(\vec{a}) = \mathcal{L}(\vec{a})$$

i.e., we have

$$\mathcal{L}(\vec{a}) = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n + \beta = f(\vec{a})$$

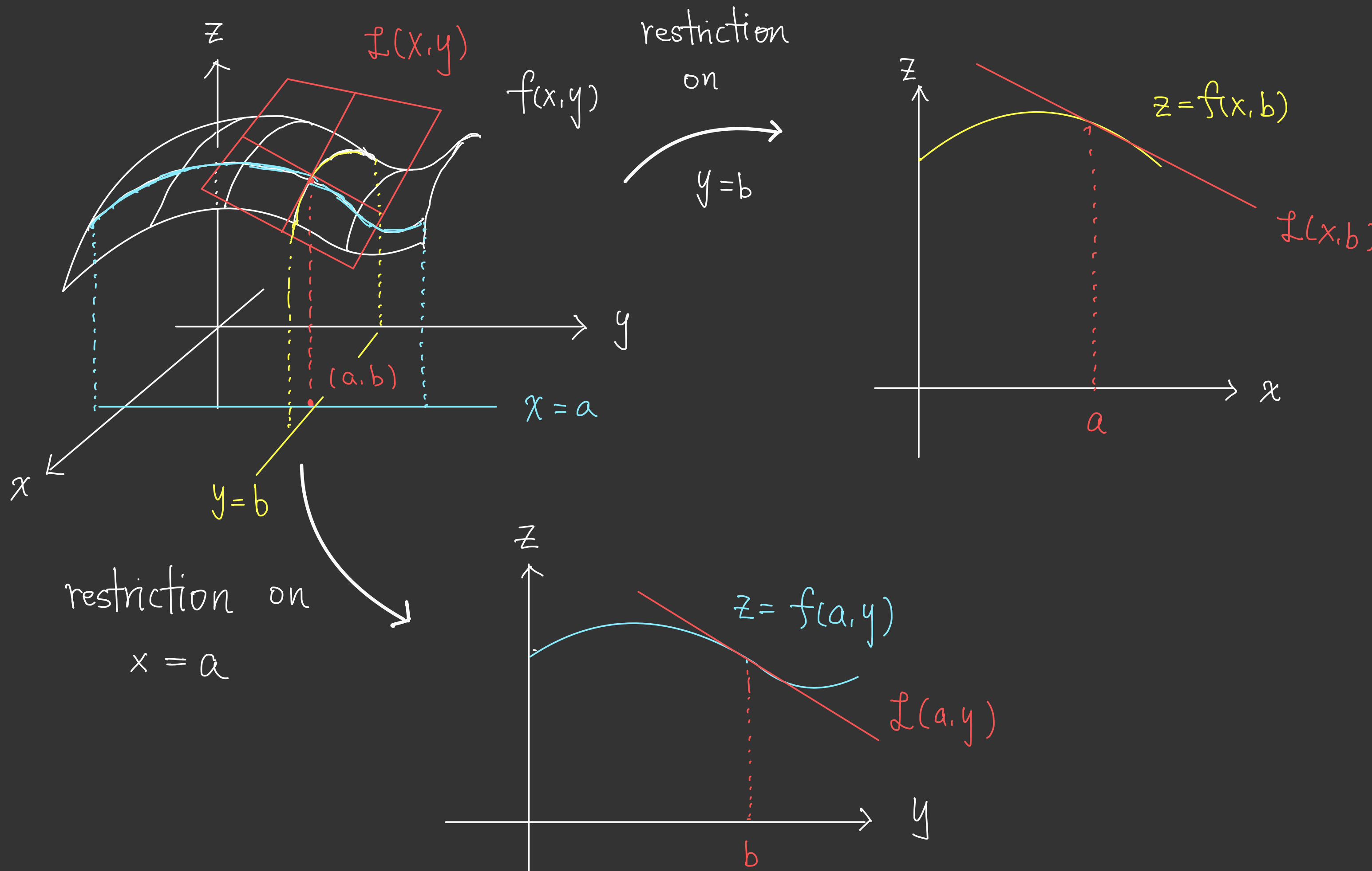
$$\mathcal{L}(\vec{x}) = \underbrace{\mathcal{L}(\vec{x}) - \mathcal{L}(\vec{a})} + \mathcal{L}(\vec{a})$$

$$= \underbrace{\alpha_1 (x_1 - a_1) + \alpha_2 (x_2 - a_2) + \cdots + \alpha_n (x_n - a_n)}$$

$$+ f(\vec{a})$$

$$= \vec{\alpha} \cdot (\vec{x} - \vec{a}) + f(\vec{a})$$

$\vec{\alpha} = ?$


z
$\mathcal{L}(x,y)$
$f(x,y)$
y
x
(a,b)
$x=a$
$y=b$
restriction on $y=b$
z
$z=f(x,b)$
$\mathcal{L}(x,b)$
x
a
restriction on $x=a$
z
$z=f(a,y)$
$\mathcal{L}(a,y)$
y
b

Notice that $\mathcal{L}(a,y)$ is the tangent line of $\underbrace{f(a,y)}_{g(y)}$ at b.

$$\mathcal{L}(a,y) = g'(b)(y-b) + g(b)$$

$$\Rightarrow \mathcal{L}(a,y) = \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b).$$

Similarly, we have

$$\mathcal{L}(x,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + f(a,b).$$

In general, for any $k = 1, 2, \ldots, n$,

$$\mathcal{L}(a_1, a_2 \ldots, x_k, \ldots a_n) = \underbrace{\frac{\partial f}{\partial x_k}(\vec{a})}_{\parallel} \cdot (x_k - a_k) + f(\vec{a}).$$

$$= \underline{\alpha_k} \cdot (x_k - a_k) + f(\vec{a})$$

Therefore

$$\mathcal{L}(\vec{x}) = \frac{\partial f}{\partial x_1}(\vec{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\vec{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\vec{a})(x_n - a_n) + f(\vec{a}).$$

<u>Remark</u>: We define

$$\nabla f(\vec{a}) := \left(\frac{\partial f}{\partial x_1}(\vec{a}), \frac{\partial f}{\partial x_2}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

the gradient of f at $\vec{a}$.

$$\Rightarrow \mathcal{L}(\vec{x}) = \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + f(\vec{a}).$$

Example: Let $f(x,y,z) = 2x^2y + z^3e^x + 3y^2z$

Find the tangent space of f at $(1,1,0)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(4xy + z^3e^x,\ 2x^2 + 6yz,\ 3z^2e^x + 3y^2\right)$$

$$\nabla f(1,1,0) = (4, 2, 3)$$

$$f(1,1,0) = 2$$

$$L(x,y,z) = 4\cdot(x-1) + 2(y-1) + 3z + 2$$

$$= 4x + 2y + 3z - 4.$$

Def: The total differentiation of f at $\vec{a}$ is

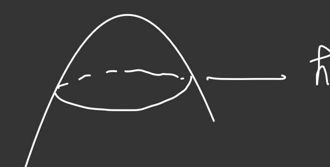
$$df = \frac{\partial f}{\partial x_1}(\vec{a})\cdot dx_1 + \frac{\partial f}{\partial x_2}(\vec{a})\cdot dx_2 + \cdots + \frac{\partial f}{\partial x_n}(\vec{a})\cdot dx_n .$$

Level sets:

Let $f(x,y)$ be a two variable function.

Def: For any $k \in \mathbb{R}$, we define the level set (curve)

$$S_k := \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = k \}$$



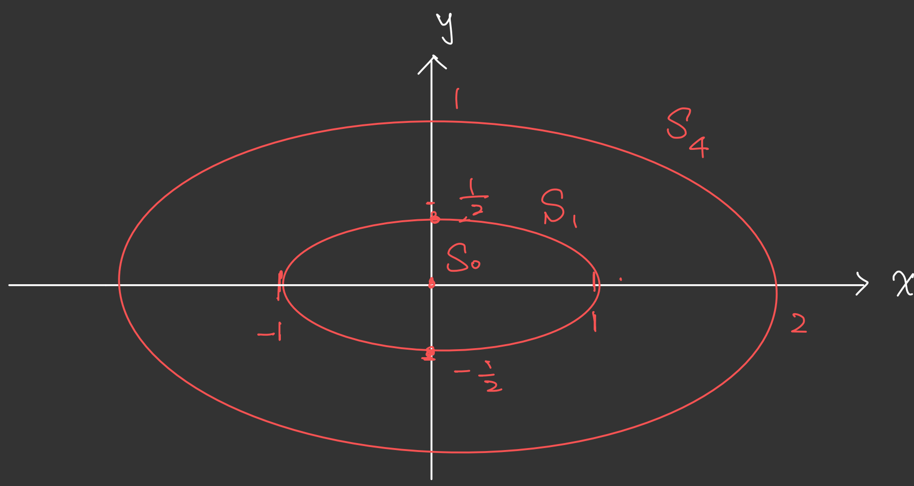
S_k is a curve in general.

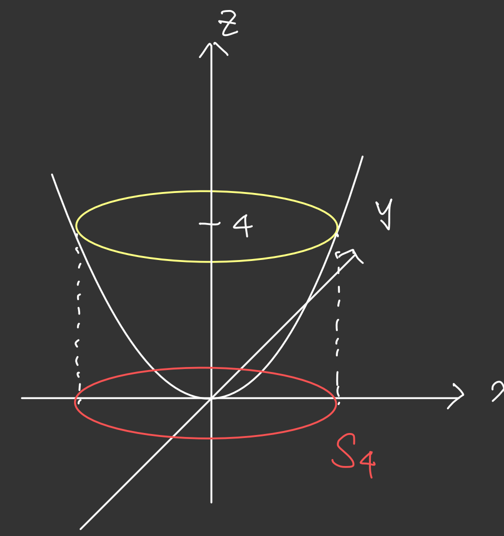
Example: Let $f(x,y) = x^2 + 4y^2$

$$S_0 = \{(x,y) \mid x^2+4y^2 = 0\} = \{(0,0)\}$$

$$S_1 = \{(x,y) \mid x^2+4y^2 = 1\} \rightarrow \text{ellipse.}$$

$$S_4 = \{(x,y) \mid x^2+4y^2 = 4\}$$

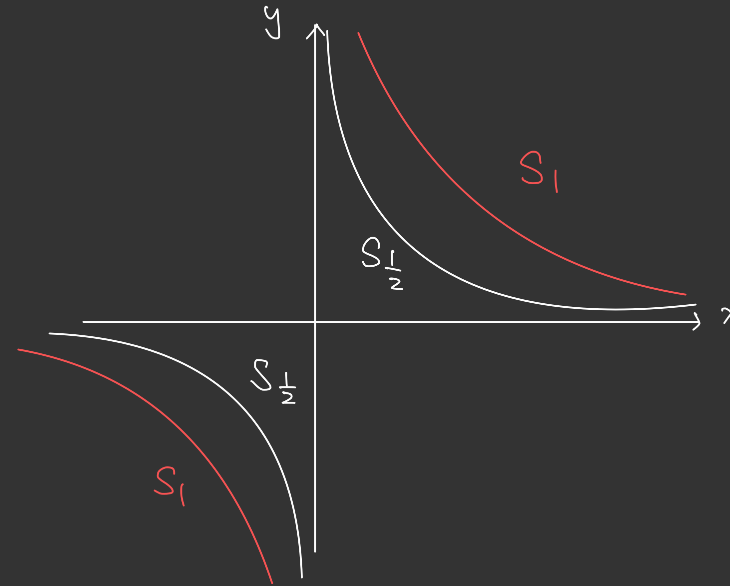




Example: $f(x,y) = xy$.

$$S_1 = \{(x,y) \mid xy = 1\}$$

$$S_{\frac{1}{2}} = \{(x,y) \mid xy = \tfrac{1}{2}\}$$



When $n = 3$:

Def: Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a diff. function.

For any $k \in \mathbb{R}$, we define

$$S_k = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = k\}$$

S_k is a surface in general.

Chain rule: Let $\vec{p}: \mathbb{R} \longrightarrow \mathbb{R}^2$ be a vector-valued function.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a 2-variable function.

$$f \circ \vec{p} : \mathbb{R} \xrightarrow{\vec{p}} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$t \longmapsto (x(t), y(t)) \longmapsto f(x(t), y(t))$$

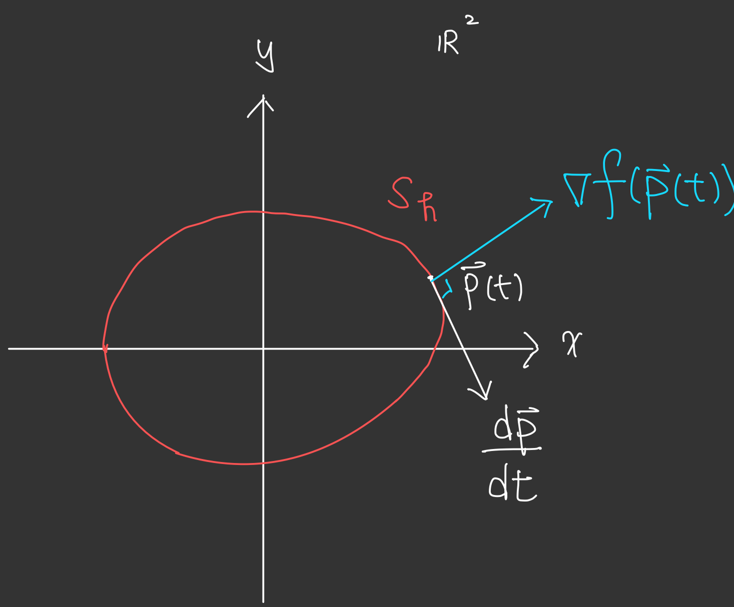
$$\boxed{\frac{d}{dt}(f \circ \vec{p}) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}.}$$

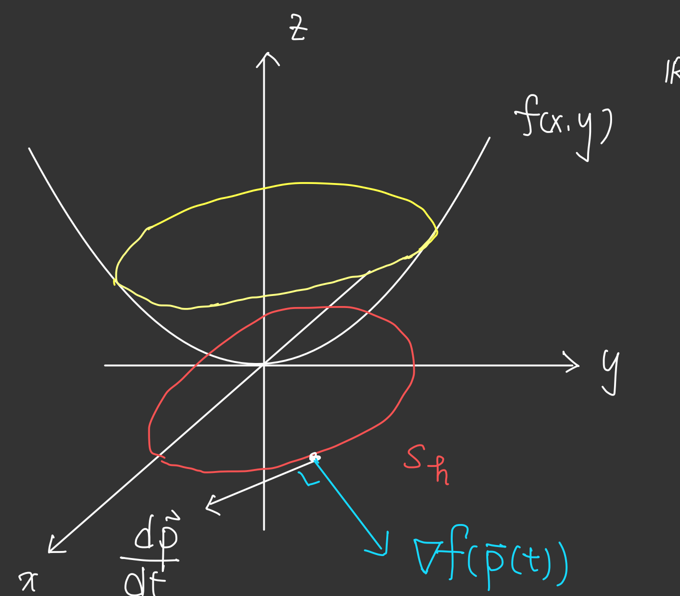
$$= \nabla f(x(t), y(t)) \cdot \frac{d\vec{p}}{dt}$$

Let $\vec{p}$ parametrizes a level set S_h of f.

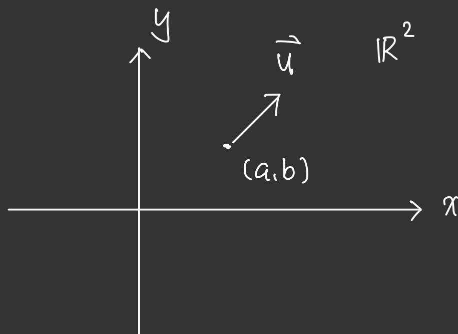
$$f \circ \vec{p} = f(\vec{p}(t)) = h \quad \text{for all } t.$$

$$\Rightarrow \frac{d}{dt}(f \circ \vec{p}) = 0 = \nabla f(\vec{p}(t)) \cdot \frac{d\vec{p}}{dt} \Rightarrow \nabla f(\vec{p}(t)) \perp \frac{d\vec{p}}{dt}.$$





Directional derivatives:



Let $\vec{u}$ be an unit vector in $\mathbb{R}^2$.

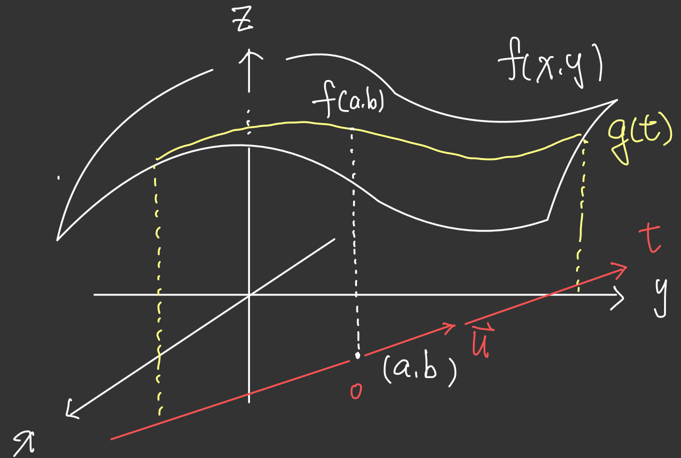
$$\vec{u} = (u_1, u_2) \; ; \; \sqrt{u_1^2 + u_2^2} = 1.$$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function.

We define

$$D_{\vec{u}} f(a,b) := \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

$$\left[\; f(a+tu_1, b+tu_2) := g(t), \quad D_{\vec{u}} f(a,b) = \frac{dg}{dt}(0) \; \right]$$



$D_{\vec{u}}f$: the dirational derivative of f along $\vec{u}$.

Remark: We usually call $D_{\vec{u}}f$ dirational derivative iff $\vec{u}$ is an unit vector.

$$\lim_{h\to 0}\frac{f(a+hu_1,b+hu_2)-f(a,b)}{h}=\lim_{h\to 0}\frac{f(a+hu_1,b+hu_2)-f(a,b+hu_2)}{hu_1}\cdot u_1$$

$$+\lim_{h\to 0}\frac{f(a,b+hu_2)-f(a,b)}{hu_2}\cdot u_2$$

$$= \lim_{h\to 0} \partial_x f(a+\xi_1 u_1, b+h u_2) u_1 + \lim_{h\to 0} \partial_y f(a, b+\xi_2 u_2) u_2$$ by Mean-value thm

where $\xi_1, \xi_2 \in (0, h)$. i.e., $\lim_{h\to 0} \xi_1 = \lim_{h\to 0} \xi_2 = 0$

$$= \partial_x f(a,b) u_1 + \partial_y f(a,b) u_2 = \nabla f(a,b) \cdot \vec{u}$$

Max. holds when $\vec{u} = \frac{\nabla f}{|\nabla f|}$

<u>Prop</u>:

$$\boxed{D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}}$$

Min holds when $\vec{u} = -\frac{\nabla f}{|\nabla f|}$

$\Uparrow$

$$\max_{\vec{u}\text{: unit vector}} D_{\vec{u}} f(a,b) = \max_{\vec{u}\text{: unit vector}} \nabla f(a,b) \cdot \vec{u} = \max_{\vec{u}\text{: unit vector}} |\nabla f(a,b)| \cdot |\cos\theta|$$

Example. $g(x,y) = x^2 y + 3xy^4$. $\vec{p}(t) = (\sin 2t, \cos t)$

$$\frac{d}{dt}(g \circ \vec{p}) = \frac{d}{dt}\left(\sin^2(2t) \cdot \cos t + 3\sin 2t \cdot \cos^4 t\right)$$

By Chain rule:

$$\frac{d}{dt}(g \circ \vec{p}) = \nabla g(\sin 2t, \cos t) \cdot \frac{d\vec{p}}{dt}$$

$$= \left(2xy + 3y^4,\ x^2 + 12xy^3\right)\Big|_{\substack{x=\sin 2t \\ y=\cos t}} \cdot (2\cos 2t, -\sin t)$$

$$= \left(2\sin 2t \cos t + 3\cos^4 t\right)(2\cos 2t)$$

$$- \left(\sin^2 2t + 12 \sin 2t \cos^3 t\right) \cdot \sin t$$

Example. Let $f(x,y,z) = x\sin(yz)$. Find ∇f and $D_{\frac{\vec{v}}{|\vec{v}|}} f(1,3,0)$

$$\vec{v} = (1, 2, -1)$$

Sol: $\nabla f = (\sin(yz),\ x\cdot\cos(yz)z,\ x\cos(yz)y)$

$$\frac{\vec{v}}{|\vec{v}|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$D_{\frac{\vec{v}}{|\vec{v}|}} f(1,3,0) = (\sin 0,\ 1\cdot\cos 0\cdot 0,\ 1\cos 0\cdot 3)$$

$$\cdot\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = \frac{-3}{\sqrt{6}}$$

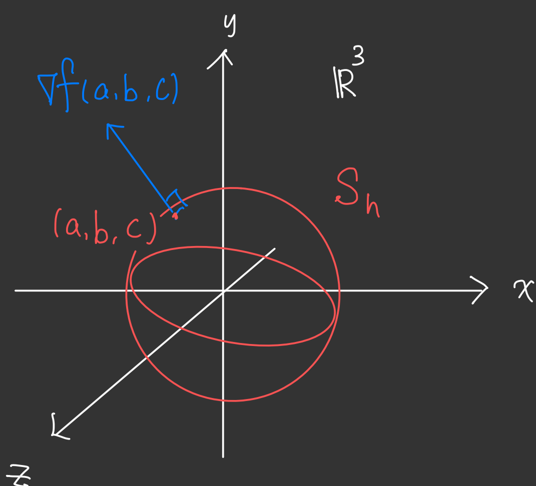
$$\nabla f(1,3,0) = (0, 0, 3).$$

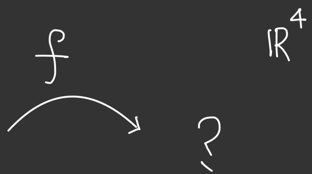
3-dimensional cases:

When $f(x,y,z)$ is a 3-variable function, we still define

$$S_h = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = h \}.$$

S_h is a surface for general $h \in \mathbb{R}$.





We also have

$$\nabla f(a,b,c) \perp S_k, \quad \text{for all } (a,b,c) \in \mathbb{R}^3,$$
$$k = f(a,b,c)$$

i.e., $\nabla f(a,b,c)$ is perpendicular to the tangent plane of S_k at (a,b,c)

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ holds for $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\vec{u}$: unit vector in $\mathbb{R}^3$

$$\text{Max } D_{\vec{u}} f \Rightarrow \vec{u} = \frac{\nabla f}{|\nabla f|} \; ; \quad \text{Min } D_{\vec{u}} f \Rightarrow \vec{u} = \frac{-\nabla f}{|\nabla f|}$$

Chain rule:

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. $\vec{p} : \mathbb{R} \longrightarrow \mathbb{R}^2$. We consider $f \circ \vec{p} : \mathbb{R} \longrightarrow \mathbb{R}$.

Prop:

$$\frac{d}{dt}(f \circ \vec{p}) = \frac{\partial f}{\partial x}(\vec{p}(t))\, x'(t) + \frac{\partial f}{\partial y}(\vec{p}(t))\, y'(t)$$

$$= \nabla f(\vec{p}(t)) \cdot \vec{p}\,'(t) \qquad (\vec{p}(t) = (x(t), y(t)))$$

Proof:

$$\lim_{h \to 0} \frac{f(\vec{p}(t+h)) - f(\vec{p}(t))}{h} = \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \lim_{h \to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{h} + \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{h} + \lim_{h\to 0} \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{x(t+h) - x(t)} \cdot \frac{x(t+h) - x(t)}{h}$$

$$+ \lim_{h\to 0} \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{y(t+h) - y(t)} \cdot \frac{y(t+h) - y(t)}{h}$$

$$= \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t)$$

__Chain rule:__ Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $x, y : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$x = x(s,t), \quad y = y(s,t).$$

$$f(x(s,t), y(s,t)) := g(s,t).$$

If we fix t,

$$\frac{\partial}{\partial s} g(s,t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

If we fix s,

$$\frac{\partial}{\partial t} g(s,t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

__Example:__ Let $f(x,y) = 2x^2 + 2y^2$, $x(r,\theta) = r\cos\theta$, $y(r,\theta) = r\sin\theta$

Clearly, $g(r,\theta) = 2r^2$. $\Rightarrow \frac{\partial g}{\partial r} = 4r, \quad \frac{\partial g}{\partial \theta} = 0$

By Chain rule:

$$\frac{\partial f}{\partial x}(x(r,\theta), y(r,\theta))\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}(x(r,\theta), y(r,\theta)) \cdot \frac{\partial y}{\partial r}$$

$$= 4r\cos\theta \cdot \cos\theta + 4r\sin\theta \cdot \sin\theta = 4r$$

$$\frac{\partial f}{\partial x}(x(r,\theta), y(r,\theta)) \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}(x(r,\theta), y(r,\theta)) \cdot \frac{\partial y}{\partial \theta}$$

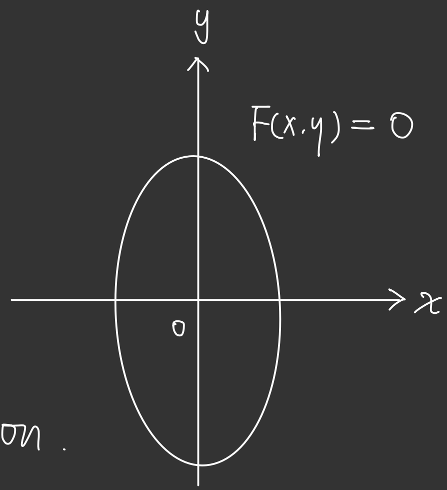
$$= 4r\cos\theta \cdot (-r\sin\theta) + 4r\sin\theta\,(r\cos\theta) = 0$$

Differentiation of Implicit functions:

An implicit function in general has the form:

$$F(x,y) = 0$$

A function $y = f(x) \Rightarrow \underbrace{f(x) - y}_{\overset{\shortparallel}{F(x,y)}} = 0$



Example: $F(x,y) = x^2 + \frac{1}{4}y^2 - 1 = 0$

is an implicit function, but not a function.

However, we have $\{(x,y) \mid F(x,y) = 0\}$ = union of two graphs

$$\{(x,y)\in\mathbb{R}^2 \mid y=\sqrt{4-x^2}\} \cup \{(x,y)\in\mathbb{R}^2 \mid y=-\sqrt{4-x^2}\}$$

$$\underset{\text{a function}}{\uparrow} \qquad\qquad \underset{\text{a function.}}{\uparrow}$$

Suppose this is the case: We can write $F(x,y)=0$ locally as a function, say $y=y(x)$. Then

$$F(x,y(x))=0$$

$$\frac{d}{dx}F(x,y(x)) = \frac{\partial F}{\partial x}\cdot \cancel{\frac{dx}{dx}}^{\,1} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

Therefore, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

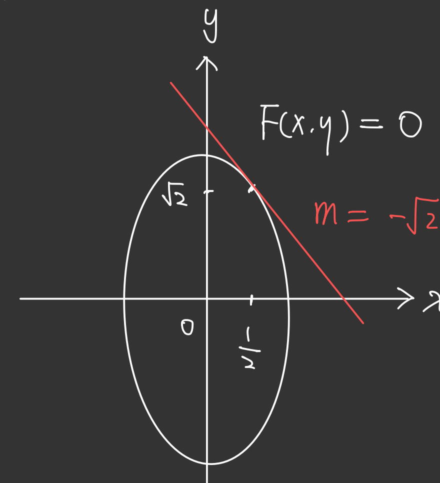
Theorem: In particular, if $\frac{\partial F}{\partial y} \neq 0$, we have

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\partial_x F}{\partial_y F}}$$

Example: $F(x,y) = x^2 + \frac{1}{4}y^2 - 1$, Find $\frac{dy}{dx}$ at $(\frac{1}{2}, \sqrt{2})$.

$$\partial_x F = 2x \,,\quad \partial_y F = \frac{1}{2} y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\partial_x F}{\partial_y F}\left(\frac{1}{2}, \sqrt{2}\right) = -\frac{1}{\frac{\sqrt{2}}{2}} = -\sqrt{2}$$



Example: Let $F(x,y) = x^2 - 3xy + y^3 - 7 = 0$

Find $\dfrac{dy}{dx}$ at (4.3)

$$\partial_x F = 2x - 3y\,, \quad \partial_y F = -3x + 3y^2$$

Check: $\partial_y F(4,3) = -12 + 27 = 15 \neq 0$

$\partial_x F(4,3) = 8 - 9 = -1$

Therefore,

$$\frac{dy}{dx}(4.3) = \frac{-\partial_x F(4.3)}{\partial_y F(4.3)} = \frac{1}{15}$$

Extreme values for multivariable functions (2-variable)

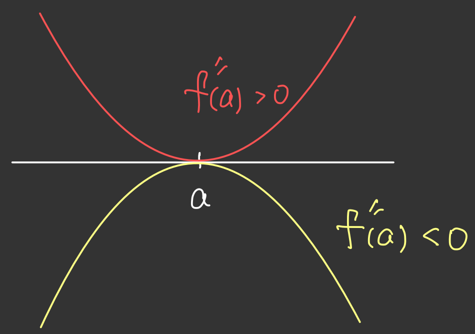
Recall: Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.

Then ① a is a critical point of $f \Leftrightarrow f'(a) = 0$

② a is a critical and $f''(a) > 0 \Rightarrow f$ has a local minimum at a.

$f''(a) < 0 \Rightarrow f$ has a local maximum at a.

$f''(a) = 0 \Rightarrow$ no conclusion.



Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function.

Def: We call (a,b) is a critical point of f if and only if

$$\nabla f(a,b) = \vec{0} \quad (\partial_x f(a,b) = 0,\ \partial_y f(a,b) = 0)$$

Def: For any $(a,b) \in \mathbb{R}^2$, we define $\mathrm{Hess}(f): \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$\mathrm{Hess}(f)(a,b) = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix} = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

Remark: Notice that this is a symmetric metric.

Theorem: Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be smooth and (a,b) is a critical point of f. Then

① If $\mathrm{Hess}(f)(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$

$\Rightarrow f$ has a local min. at (a,b)

② If $\mathrm{Hess}(f)(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$

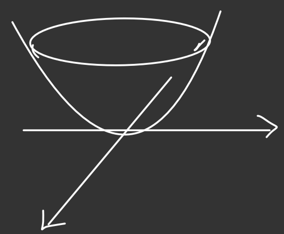
$\Rightarrow f$ has a local maximum at (a,b).

③ If $\mathrm{Hess}(f)(a,b) < 0 \Rightarrow f$ has a saddle point at (a,b)

④ If $\mathrm{Hess}(f)(a,b) = 0 \Rightarrow$ no conclusion.

Example: $f(x,y) = x^2 + y^2$, $\nabla f(0,0) = (2x, 2y)\Big|_{(x,y)=(0,0)} = 0$

$$\text{Hess}(f) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 > 0, \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$$



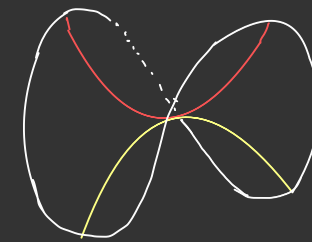
Example: $f(x,y) = -x^2 - y^2$

(exercise). $\nabla f(0,0) = 0$, $\text{Hess}(f) = 4$

$$\frac{\partial^2 f}{\partial x^2}(0,0) < 0.$$

Example: $f(x,y) = x^2 - y^2$; $\nabla f(0,0) = (2x, -2y)\Big|_{(x,y)=(0,0)} = 0$

$$\text{Hess}(f) = \det\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4 < 0 \Rightarrow \text{saddle point.}$$



Sketch of proof:

If (a,b) is a critical point of f and for any unit vector $v \in \mathbb{R}^2$, $D^2_{\vec{v}} f(a,b) > 0$, then f has a local minimum at (a,b).

Lemma: Let M be an 2×2 symmetric matrix and $\vec{v}^t M \vec{v} > 0$ for any unit vector $\vec{v} \in \mathbb{R}^2 \setminus \{0\}$

Then $\det(M) > 0$, $a > 0$.

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x^2 + y^2 = 1.$$

$$\vec{v}^t M \vec{v} = (x \ \ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (ax+by \quad bx+cy)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= ax^2 + bxy + bxy + cy^2 = ax^2 + 2bxy + cy^2$$

$$> 0$$

When $y \neq 0$, we have

$$az^2 + 2bz + c > 0 \qquad \text{for } z = \frac{x}{y} \quad \text{i.e., } z \in \mathbb{R} - \{0\}$$

$$\Leftrightarrow \ a > 0, \ 4b^2 - 4ac < 0$$

$$\Leftrightarrow \ a > 0, \ ac - b^2 > 0 \Leftrightarrow a > 0, \ \det(M) > 0.$$

By using this Lemma, we take $\vec{p} = (a, b)$. $\mathcal{L} = \{\vec{p} + t\vec{v}\}$

$$D_{\vec{v}}^2 f = \frac{d^2}{dt^2}\left(f(\vec{p} + t\vec{v})\right)\Big|_{t=0} = \frac{d}{dt}\left(\nabla f(\vec{p} + tv) \cdot \vec{v}\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{\partial f}{\partial x}(\vec{p} + t\vec{v}) \cdot v_1 + \frac{\partial f}{\partial y}(\vec{p} + t\vec{v}) \cdot v_2\right)\Big|_{t=0}$$

$$= \underbrace{\frac{\partial^2 f}{\partial x^2}(\vec{p})}_{a}\, \underbrace{V_1^2}_{x^2} + \underbrace{\frac{\partial^2 f}{\partial y \partial x}(\vec{p})}_{b} \cdot \underbrace{V_1 V_2}_{xy} + \underbrace{\frac{\partial^2 f}{\partial x \partial y}(\vec{p})}_{b} \cdot \underbrace{V_1 V_2}_{xy} + \underbrace{\frac{\partial^2 f}{\partial y^2}(\vec{p})}_{c} \cdot \underbrace{V_2^2}_{y^2}$$

$$= \vec{v}^{\,t} M \vec{v} > 0 .$$

We have $a > 0$, $\det(M) > 0$

$$\Leftrightarrow \frac{\partial^2 f}{\partial x^2}(a,b) > 0 \quad \text{and} \quad \mathrm{Hess}(f)(a,b) > 0 .$$

i.e.,

$$D^2_{\vec{v}} f(a,b) > 0 \text{ for all unit vectors } \vec{v} \iff \frac{\partial^2 f}{\partial x^2}(a,b) > 0, \quad \mathrm{Hess}(f)(a,b) > 0 .$$

Remark 1: The following two conditions are equivalent:

$$\mathrm{Hess}(f)(a,b) > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(a,b) > 0$$

$$\Leftrightarrow \quad \mathrm{Hess}(f)(a,b) > 0 \text{ and } \frac{\partial^2 f}{\partial y^2}(a,b) > 0$$

$$\left(\mathrm{Hess}(f)(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b) \cdot \frac{\partial^2 f}{\partial y^2}(a,b) - \left(\frac{\partial^2 f}{\partial x \partial y}(a,b)\right)^2 > 0 \right.$$

$$\left. \Rightarrow \frac{\partial^2 f}{\partial x^2}(a,b) \ \& \ \frac{\partial^2 f}{\partial y^2}(a,b) \text{ have same sign} \right)$$

<u>Example</u>: $f(x,y) = 2x^2y - xy^2 - xy$ be a function defined on

$$\Omega = \{ (x,y) \in \mathbb{R}^2 \mid |x| \leq 2, |y| \leq 2 \}$$

$$\nabla f = (\partial_x f, \partial_y f) = (4xy - y^2 - y, \ 2x^2 - 2xy - x)$$

$$\partial_x f = 0 \Leftrightarrow (4x - y - 1)y = 0 \Leftrightarrow y = 0 \text{ or } 4x - y = 1$$

$$\partial_y f = 0 \Leftrightarrow (2x - 2y - 1)x = 0 \Leftrightarrow x = 0 \text{ or } 2x - 2y = 1$$

<u>Critical points</u>: (in the interior of Ω)

$$(a,b) = (0,0) \text{ or } (0,-1) \text{ or } (\tfrac{1}{2}, 0) \text{ or } (\tfrac{1}{6}, \tfrac{-1}{3})$$

$\partial_x^2 f = 4y$, $\partial_x \partial_y f = 4x - 2y - 1$

$\partial_y^2 f = -2x$.

$\text{Hess}(f) = -8xy - (4x-2y-1)^2$

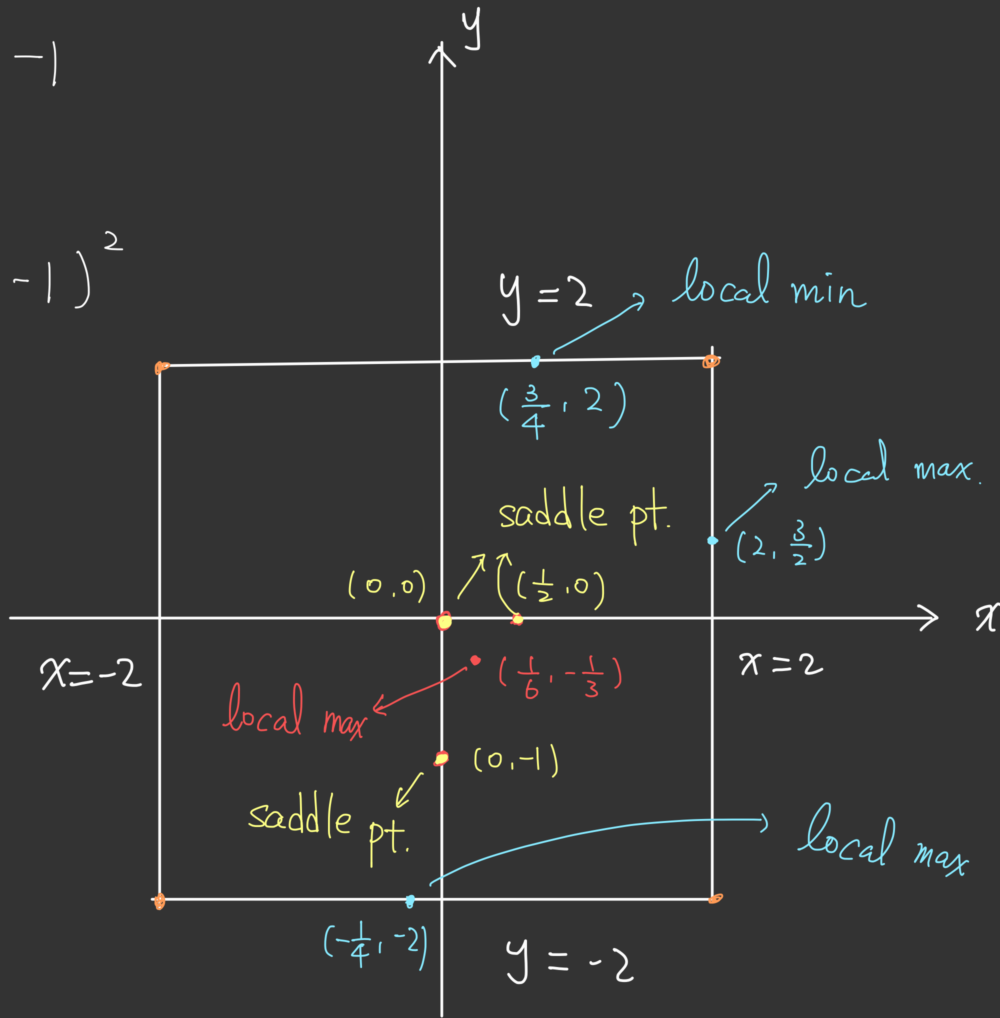
When $(a,b) = (0,0)$:

$\text{Hess}(f)(0,0) = -1 < 0$

$\Rightarrow (0,0)$ is a saddle point

When $(a,b) = (0,-1)$

$\text{Hess}(f)(0,-1) = -1 < 0$



When $(a,b) = (\frac{1}{2}, 0)$, $\text{Hess}(f)(\frac{1}{2}, 0) = -1 < 0 \Rightarrow$ saddle pt

When $(a,b) = (\frac{1}{6}, -\frac{1}{3})$, $\text{Hess}(f)(\frac{1}{6}, -\frac{1}{3}) = \frac{4}{9} - (\frac{2}{3} + \frac{2}{3} - 1)^2$

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0$$

and $\partial_x^2 f(\frac{1}{6}, -\frac{1}{3}) = -\frac{4}{3} < 0$

$\Rightarrow$ local maximum.

<u>extreme values on the bdy :</u>

When $x = 2$, $f|_{x=2} = 8y - 2y^2 - 2y = 6y - 2y^2 := g(y)$

$g : [-2, 2] \longrightarrow \mathbb{R}$. $\frac{d}{dy} g = 6 - 4y \Rightarrow$ critical pt at $y = \frac{3}{2}$

$$\frac{d^2}{dy^2} g\left(\frac{3}{2}\right) = -4 < 0 \Rightarrow \text{local max}$$

When $x = -2$, $f\big|_{x=-2} = 10y + 2y^2 = g(y)$. $g'(y) = 0 \Rightarrow 4y + 10 = 0$

$$\Rightarrow y = \frac{-5}{2} < -2$$

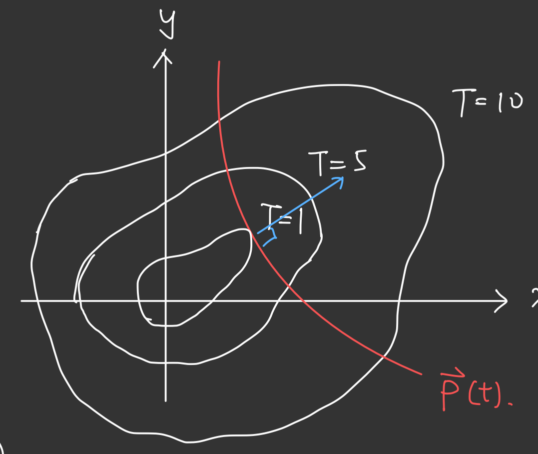
When $y = 2$, $f\big|_{y=2} = 4x^2 - 6x = h(x)$. $h'(x) = 0 \Rightarrow 8x - 6 = 0$

$$\Rightarrow x = \frac{3}{4}. \quad \text{and} \quad h''\left(\frac{3}{4}\right) = 8 > 0 \Rightarrow \text{local min}$$

When $y = -2$, $f\big|_{y=-2} = -4x^2 - 2x = h(x)$, $h'(x) = 0 \Rightarrow -8x - 2 = 0$

$$\Rightarrow x = -\frac{1}{4} \quad \text{and} \quad h''\left(-\frac{1}{4}\right) = -8 < 0 \Rightarrow \text{local max}$$

Lagrange multipliers:



Let $\vec{p}(t) = (x(t), y(t))$ be a plane curve.

$$\frac{d}{dt}\vec{p} = (x'(t), y'(t)).$$

Suppose we have $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$. $(z = f(x,y))$

$$\frac{d}{dt} f(x(t), y(t)) = \nabla f(\vec{p}(t)) \cdot \underbrace{\frac{d}{dt}\vec{p}}_{}$$

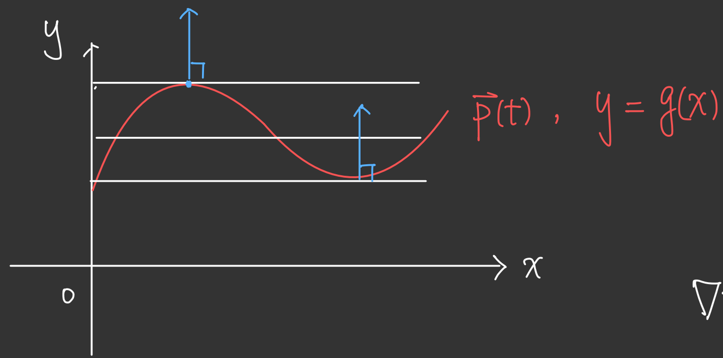
$\downarrow$

tangent vector of the curve at $\vec{p}(t)$

When $f(\vec{p}(t))$ has a local max/min at t, we have.

$$\nabla f(\vec{p}(t)) \perp \frac{d\vec{p}}{dt}$$

Example: Let $\vec{p}(t) = (t, g(t))$, $f(x,y) = y$.



$$\max\{f(\vec{p}(t))\} = \max\{g(x)\}$$

$$\min\{f(\vec{p}(t))\} = \min\{g(x)\}$$

$$\nabla f = (0, 1)$$

Suppose we have $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function,

$$\mathcal{C} = \{\vec{p}(t) \mid t \in \mathbb{R}\} = \{g(x,y) = 0\}$$

Theorem: f has a local extrema subject to $\mathcal{C}$ at $\vec{p}(t)$ only if

$$\nabla f(\vec{p}(t)) = \lambda \nabla g(\vec{p}(t))$$

To see this, consider

$$\mathscr{C} = S_0 = \{ g(x,y) = 0 \}$$

$$\Rightarrow g(\vec{p}(t)) = 0 \Rightarrow \frac{d}{dt} g(\vec{p}(t)) = \nabla g(\vec{p}(t)) \cdot \frac{d\vec{p}}{dt}$$

$$\Rightarrow \nabla g(\vec{p}(t)) \perp \frac{d\vec{p}}{dt}$$

So, $\nabla f(\vec{p}(t)) \,/\!/\, \nabla g(\vec{p}(t))$

$\Rightarrow \exists \lambda \in \mathbb{R}$ s.t.

$$\nabla f(\vec{p}(t)) = \lambda \nabla g(\vec{p}(t))$$

$f: \mathbb{R}^2 \to \mathbb{R}$, $\mathcal{C} = \{ g(x,y) = 0 \}$

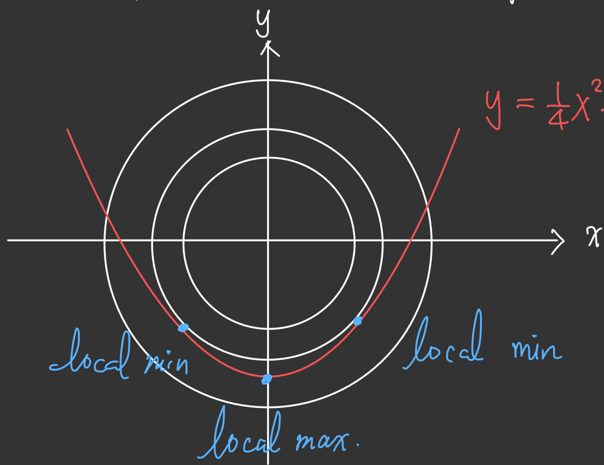
<u>Theorem</u>: f has a local extrema subject to $\mathcal{C}$ at (a,b) only if

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

<u>Example</u>: Let $f(x,y) = x^2 + y^2$, $\mathcal{C} = \{ y - \frac{1}{4}x^2 + 6 = 0 \}$

$\Updownarrow$

$y = \frac{1}{4}x^2 - 6$



$$\nabla f = (2x, 2y)$$

$$\nabla g = \left(-\frac{1}{2}x, 1\right)$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = -\frac{\lambda}{2}x & \text{①} \\ 2y = \lambda & \text{②} \\ y = \frac{1}{4}x^2 - 6 & \text{③} \end{cases} \Leftarrow \nabla f = \lambda \nabla g$$

① $\Rightarrow 4x + \lambda x = 0 \Rightarrow x(\lambda + 4) = 0 \Rightarrow x = 0$ or $\lambda = -4$.

when $x = 0$, ③ $\Rightarrow y = -6$, ② $\Rightarrow \lambda = -12$

$(x, y, \lambda) = (0, -6, -12)$ is a solution.

when $\lambda = -4$, ② $\Rightarrow y = -2$, ③ $\Rightarrow 4 = \frac{1}{4}x^2 \Rightarrow x = \pm 4$.

$(x, y, \lambda) = (4, -2, -4)$ and $(-4, -2, -4)$ are two solutions

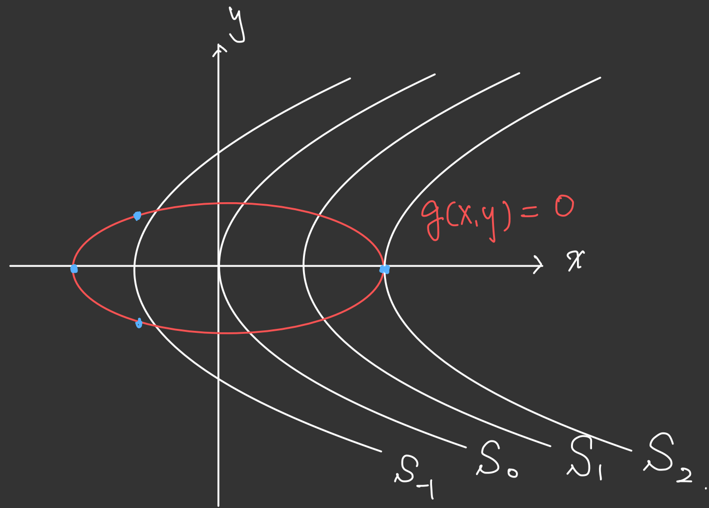
When $(x,y)=(0,-6)$, $f(x,y)=36$ is a local max of f subject to $y=\frac{1}{4}x^2-6$.

when $(x,y)=(4,-2)$, $(x,y)=(-4,-2)$, $f(x,y)=20$ is a local min of f subject to $y=\frac{1}{4}x^2-6$. (They are absolute minimum, too).

__Example__: Let $f(x,y)=x-8y^2$, $g(x,y)=\frac{1}{4}x^2+y^2-1$

Find local extrema of f subject to $\mathscr{C}=\{(x,y)\mid g(x,y)=0\}$

$$\begin{cases}\nabla f=\lambda\nabla g\\ g(x,y)=0\end{cases}$$



$$\Rightarrow \begin{cases} 1 = \dfrac{\lambda}{2}x & \text{①} \\ -16y = 2\lambda y & \text{②} \\ \frac{1}{4}x^2 + y^2 - 1 = 0 & \text{③} \end{cases}$$

$$\text{②} \Rightarrow \lambda y + 8y = 0 \Rightarrow y(\lambda+8) = 0$$

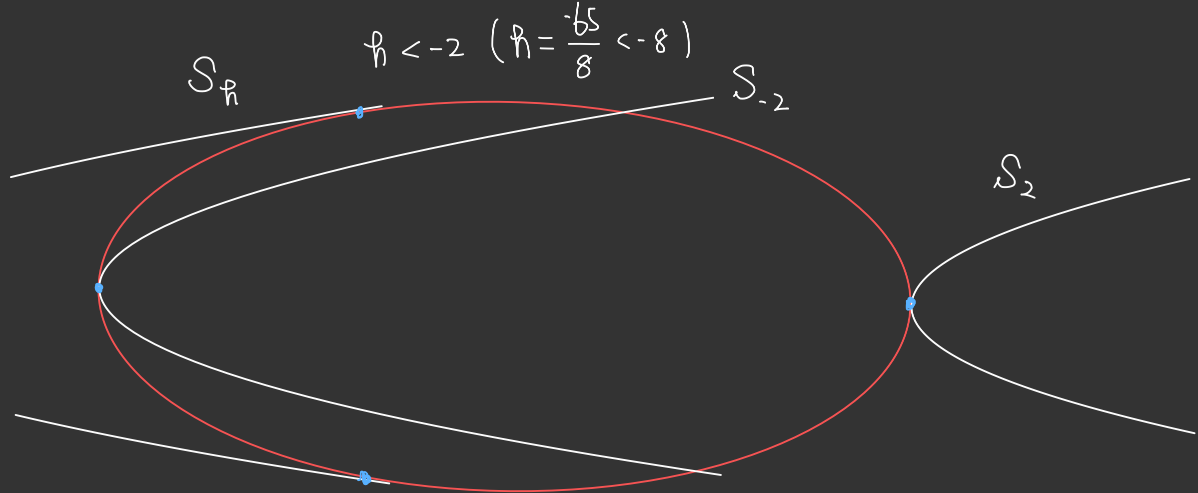
$$\Rightarrow y = 0 \quad \text{or} \quad \lambda = -8$$

when $y=0$, ③ $\Rightarrow \frac{1}{4}x^2 = 1 \Rightarrow x = \pm 2$, ① $\Rightarrow \lambda = \pm 1$

$(x, y, \lambda) = (2, 0, 1)$ and $(-2, 0, -1)$ are two sol.

when $\lambda = -8$, ① $\Rightarrow x = -\frac{1}{4}$, ③ $\Rightarrow y^2 = \frac{63}{64} \Rightarrow y = \pm\frac{\sqrt{63}}{8}$

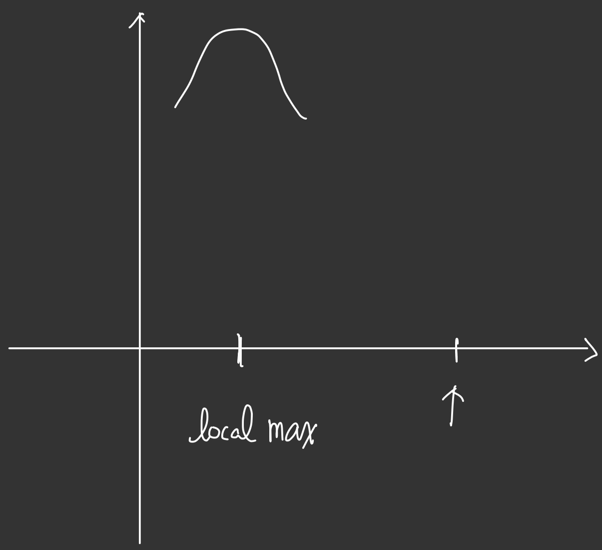
$(x, y, \lambda) = \left(-\frac{1}{4}, \frac{\sqrt{63}}{8}, -8\right)$ and $\left(-\frac{1}{4}, -\frac{\sqrt{63}}{8}, -8\right)$ are sol.



When $(x,y) = (2,0)$, $f(x,y) = 2$ is a local max. of f subject to $\frac{1}{4}x^2 + y^2 = 1$.

When $(x,y) = \left(-\frac{1}{4}, \frac{\sqrt{63}}{8}\right)$ or $\left(-\frac{1}{4}, -\frac{\sqrt{63}}{8}\right)$, $f(x,y) = -\frac{1}{4} - \frac{63}{8} = \frac{-65}{8}$ is a local min of f subject to $\frac{1}{4}x^2 + y^2 = 1$

When $(x,y)=(-2,0)$, $f(x,y)=-2$ is a local max. of f subject to $\frac{1}{4}x^2+y=1$



Lagrange multipliers:

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function. $\mathcal{C} = \{(x, y, z) \mid g(x, y, z) = 0\}$

$\mathcal{C}$ is a surface in general.

(For example $z = F(x, y)$, $\mathcal{C} = \{(x, y, F(x, y)) \mid x, y \in \mathbb{R}\}$)

<u>Thm</u>: Suppose f has a local extrema at $\vec{p}$ subject to $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 0\}$. Then

$$\nabla f(\vec{p}) = \lambda \nabla g(\vec{p}).$$

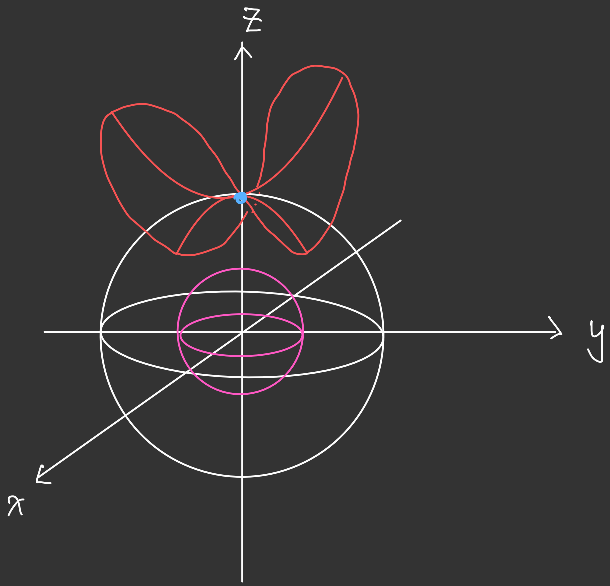
<u>Remark</u>: $\nabla f(\vec{p}) = \lambda \nabla g(\vec{p})$ doesn't imply that f has local extrema at $\vec{p}$. For example, $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = z - 10x^2 + 10y^2 - 1$. We have $\vec{p} = (0,0,1)$ satisfies

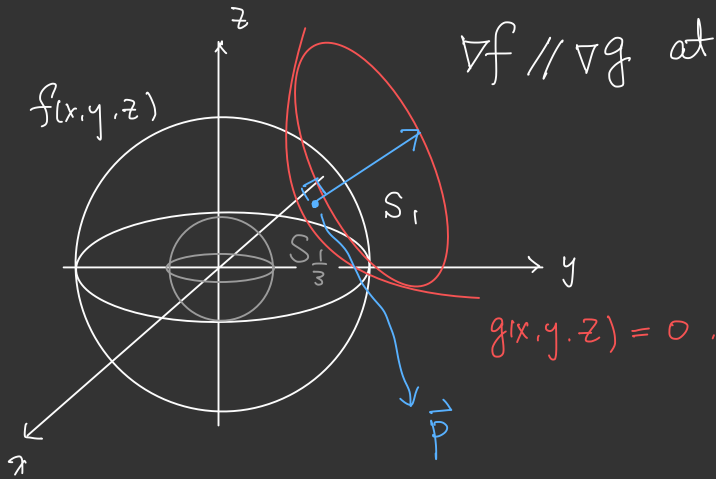
$$\begin{cases} 2x = \lambda(-20x) \\ 2y = \lambda(20y) \\ 2z = \lambda \\ z - 10x^2 + 10y^2 - 1 = 0 \end{cases}$$

(i.e., $(x,y,z,\lambda) = (0,0,1,2)$ is a sol.)

$$g(x,y,z) = 0 \Rightarrow z = \underbrace{10x^2 - 10y^2 + 1}_{\text{Hess} < 0} : \text{saddle pt.}$$

$f(0,0,1)$ is not a local extrema (max/min).


z
y
x



$\nabla f \parallel \nabla g$ at $\vec{p}$

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function.

$\mathcal{C} := \{ g(x,y,z) = 0 \} \cap \{ h(x,y,z) = 0 \}$ (This is a curve in general)

Thm: Suppose f has a local extrema at $\vec{p}$ subject to $\mathcal{C}$.

Then we have $\nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) + \mu \nabla h(\vec{p})$.

Example: Let $f(x,y,z) = x^2 + y^2 + z^2$, $\mathcal{C} = \{ z - (x^2 + y^2) + 4 = 0 \}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} 2x = -2\lambda x \\ 2y = -2\lambda y \\ 2z = \lambda \\ z - (x^2 + y^2) + 4 = 0 \end{cases}$$

① $\Rightarrow (1+\lambda) x = 0 \Rightarrow \lambda = -1$ or $x = 0$

② $\Rightarrow (1+\lambda) y = 0 \Rightarrow \lambda = -1$ or $y = 0$

when $\lambda = -1$, $x, y \neq 0 \Rightarrow z = -\frac{1}{2}$, $x^2+y^2 = \frac{7}{2}$

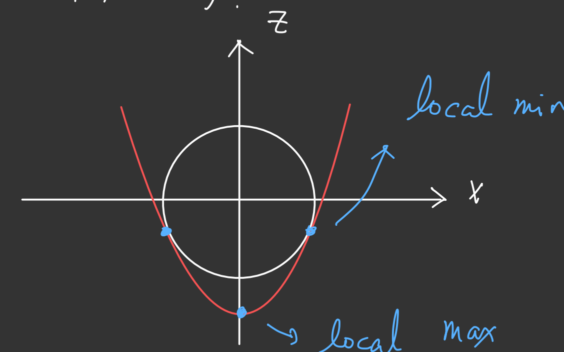
$$\Rightarrow (x, y, z, \lambda) = \left(x, y, -\frac{1}{2}, -1\right)$$

where $x^2+y^2 = \frac{7}{2}$.

when $\lambda = -1$, $x = 0 \Rightarrow (x, y, z, \lambda) = \left(0, \pm\sqrt{\frac{7}{2}}, -\frac{1}{2}, -1\right)$

$y = 0 \Rightarrow (x, y, z, \lambda) = \left(\pm\sqrt{\frac{7}{2}}, 0, -\frac{1}{2}, -1\right)$

when $x = y = 0 \Rightarrow (x, y, z, \lambda) = (0, 0, -4, -8)$.



Example: Let $f(x,y,z) = x^2 + 4y^2 + z^2$, $\mathcal{C} = \{ z - (x^2+y^2) + 4 = 0 \}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} 2x = -2\lambda x \\ 8y = -2\lambda y \\ 2z = \lambda \\ z - (x^2+y^2) + 4 = 0 \end{cases}$$

① $\Rightarrow (\lambda+1)x = 0 \Rightarrow \lambda = -1$ or $x = 0$

② $\Rightarrow (\lambda+4)y = 0 \Rightarrow \lambda = -4$ or $y = 0$

when $\lambda = -1,\ y = 0 \Rightarrow z = -\frac{1}{2} \Rightarrow x^2 = \frac{7}{2} \Rightarrow (x,y,z,\lambda) = (\pm\sqrt{\frac{7}{2}}, 0, -\frac{1}{2}, -1)$

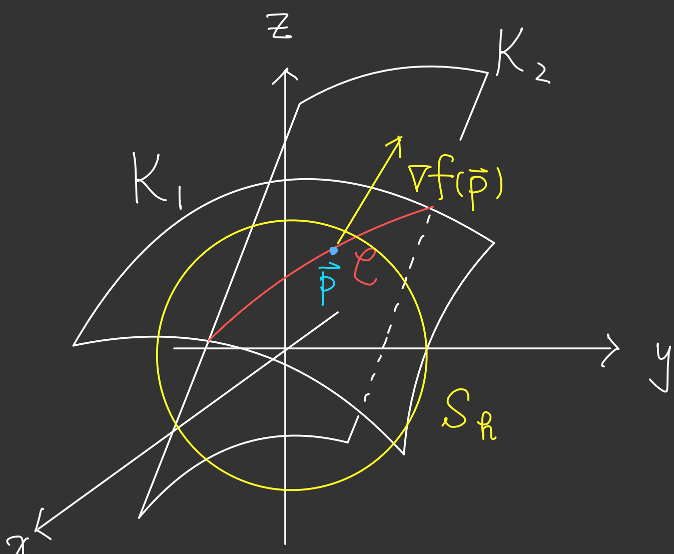
when $\lambda = -4,\ x = 0 \Rightarrow z = -2 \Rightarrow y^2 = 2 \Rightarrow (x,y,z,\lambda) = (0, \pm\sqrt{2}, -2, -4)$

when $x = 0,\ y = 0 \Rightarrow z = -4,\ \lambda = -8 \Rightarrow (x,y,z,\lambda) = (0, 0, -4, -8)$

Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$. $\quad \mathcal{C} = K_1 \cap K_2$

$K_1 = \{ (x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = 0 \}$: a surface. (曲面)

$K_2 = \{ (x,y,z) \in \mathbb{R}^3 \mid h(x,y,z) = 0 \}$: a surface



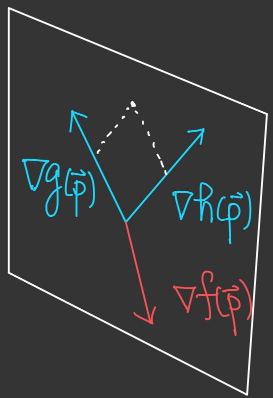
$$\begin{cases} \nabla g(\vec{p}) \perp K_1 \\ \nabla h(\vec{p}) \perp K_2 \end{cases}$$

$$\Rightarrow \begin{cases} \nabla g(\vec{p}) \perp \text{tangent vector of } \mathcal{C} \text{ at } \vec{p} \\ \nabla h(\vec{p}) \perp \text{tangent vector of } \mathcal{C} \text{ at } \vec{p} \end{cases}$$

Meanwhile, $\nabla f(\vec{p}) \perp$ tangent vector of $\mathcal{C}$ at $\vec{p}$.

$\Rightarrow$ $\nabla f(\vec{p})$, $\nabla g(\vec{p})$, $\nabla h(\vec{p})$ are on the same plane.

(We assume $\nabla g(\vec{p}) \not\parallel \nabla h(\vec{p})$)



$$\Rightarrow \quad \lambda \nabla g(\vec{p}) + \mu \nabla h(\vec{p}) = \nabla f(\vec{p})$$

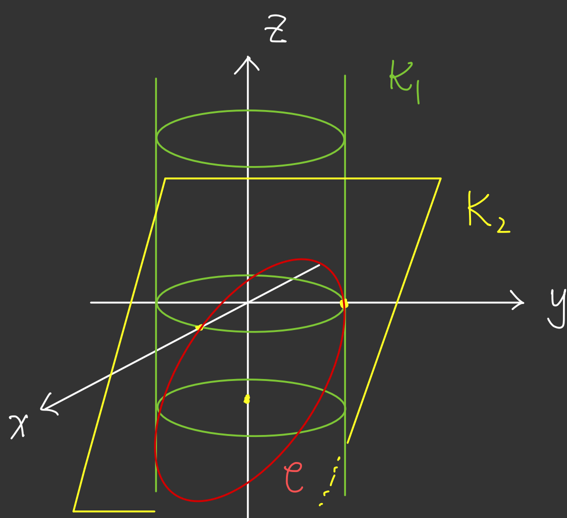
for some $\lambda, \mu \in \mathbb{R}$.

Example: Let $f(x,y,z) = x^2+y^2+z^2$, $\mathcal{C} = K_1 \cap K_2$

$$K_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-1 = 0\}$$

$$K_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y-z-1 = 0\}$$

Denote by $g(x,y,z) = x^2+y^2-1$, $h(x,y,z) = x+y-z-1$



To find a local extrema, we shall solve :

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2\lambda x + \mu \\ 2y = 2\lambda y + \mu \\ 2z = 0 - \mu \\ x^2 + y^2 = 1 \\ x + y - z = 1 \end{cases} \Rightarrow \begin{cases} 2(1-\lambda)x = \mu & \text{——— ①} \\ 2(1-\lambda)y = \mu & \text{——— ②} \\ z = -\frac{1}{2}\mu & \text{——— ③} \\ x^2 + y^2 = 1 & \text{——— ④} \\ x + y - z = 1 & \text{——— ⑤} \end{cases}$$

When $\lambda \neq 1 \Rightarrow x = y = \dfrac{\mu}{2(1-\lambda)}$ ——— by ① and ②

$$\left.\begin{aligned} &\Rightarrow 2\frac{\mu^2}{4(1-\lambda)^2} = 1 \Rightarrow \frac{\mu}{1-\lambda} = \pm\sqrt{2} && \text{——— by ④} \\ &\frac{\mu}{1-\lambda} + \frac{1}{2}\mu = 1 && \text{——— by ⑤} \end{aligned}\right\} \Rightarrow$$

$$\begin{cases} \mu = \pm\sqrt{2}\,(1-\lambda) \\[2ex] \mu = \dfrac{1}{\left(\frac{1}{1-\lambda}\right) + \frac{1}{2}} = \dfrac{2(1-\lambda)}{3-\lambda} \end{cases}$$

$$\Rightarrow \quad \pm\sqrt{2} = \frac{2}{3-\lambda} \Rightarrow 3-\lambda = \pm\sqrt{2} \Rightarrow \lambda = 3 \pm \sqrt{2}$$

$$\mu = \frac{2(2\pm\sqrt{2})}{\pm\sqrt{2}}$$

$$= \pm\sqrt{2}\,(2 \pm \sqrt{2})$$

i.e., $\mu = 2 \pm 2\sqrt{2}$

When $\lambda = 3+\sqrt{2}$, $\mu = 2+2\sqrt{2}$

$$\Rightarrow \begin{cases} x = y = \dfrac{-(2+2\sqrt{2})}{2(2+\sqrt{2})} = \dfrac{-1}{\sqrt{2}} \\[3ex] z = \dfrac{-2}{\sqrt{2}} - 1 \end{cases}$$

When $\lambda = 3-\sqrt{2}$, $\mu = 2-2\sqrt{2}$,

$$\begin{cases} x = y = \dfrac{-(2-2\sqrt{2})}{2(2-\sqrt{2})} = \dfrac{1}{\sqrt{2}} \\ z = \dfrac{2}{\sqrt{2}} - 1 \end{cases}$$

When $\lambda = 1$, $\mu = 0$ ———— by ① or ②

$\Rightarrow z = 0$ ———— by ③

$$\Rightarrow \begin{cases} x^2 + y^2 = 1 \\ x + y = 1 \end{cases} \quad \text{by ④ and ⑤} \quad \Rightarrow (x, y) = (1, 0) \text{ or } (0, 1)$$

Therefore, we have 4 critical pts:

$$(x, y, z) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{-2}{\sqrt{2}} - 1\right) \text{ ——— local max.}$$

$$\text{or } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}} - 1\right) \text{ ——— local max}$$

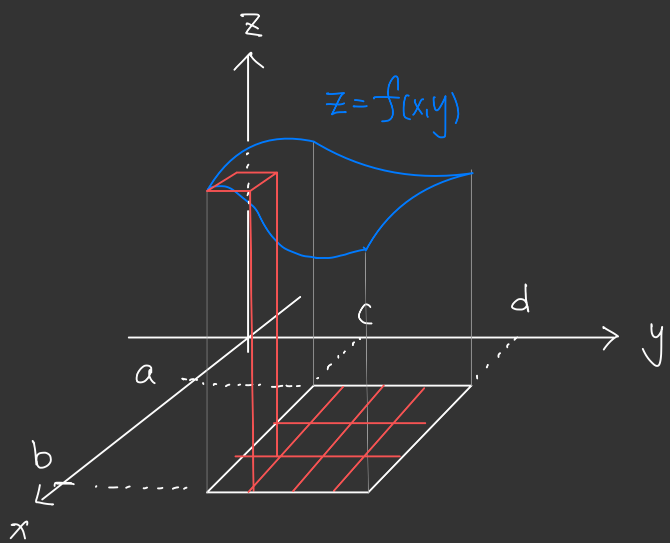
$$\text{or } (1, 0, 0) \text{ ——— local min}$$

$$\text{or } (0, 1, 0) \text{ ——— local min.}$$

Double Integrals and Triple Integrals:

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function.

$$[a,b] \times [c,d] \subset \mathbb{R}^2, \quad a<b, \; c<d.$$



Let $n, m \in \mathbb{N}$. Define.

$$\begin{cases} \dfrac{b-a}{n} := \Delta_x \\[2ex] \dfrac{d-c}{m} := \Delta_y \end{cases}$$

$$\text{Volume} := V \sim \sum_{k=1}^{n} \sum_{l=1}^{m} \underbrace{f(a+k\Delta_x,\ c+l\Delta_y)}_{\text{高度}} \underbrace{\Delta_x \Delta_y}_{\text{底面積}}$$

$$V = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{m} \underline{f(a+k\Delta_x,\ c+l\Delta_y)}\, \Delta_x \Delta_y$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\lim_{m\to\infty} \sum_{l=1}^{m} f(\underbrace{a+k\Delta_x}_{a_k},\ c+l\Delta_y)\, \Delta_y \right) \Delta_x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_c^d f(a+k\Delta_x,\ y)\, dy \right) \Delta_x$$

since $\Delta_x = \dfrac{b-a}{n}$ is indep. of m.

$$= \lim_{n\to\infty} \int_c^d \left(\sum_{k=1}^{n} f(a+k\Delta x, y)\Delta x \right) dy$$

$$= \int_c^d \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(a+k\Delta x, y)\Delta x \right) dy$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

<u>Def</u>: We define the Double integral

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \quad \text{by} \quad \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

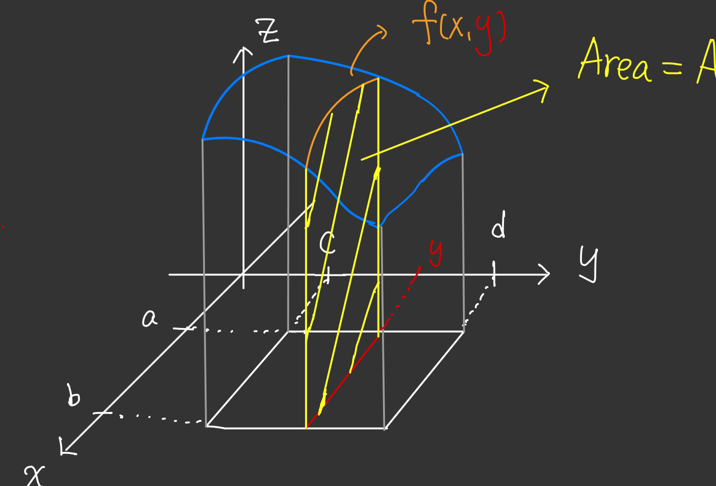
Thm: (Fubini's)

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Geometric meaning of Double integrals

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function.

Fix y, define $g(x) := f(x, \underset{\text{fixed}}{y})$.



$$\int_a^b g(x)\,dx = A$$

$$= \int_a^b f(x,y)\,dx := A(y)$$

We can approximate

$$\text{Volume} \sim \sum_{l=1}^{m} A(c + l\Delta y) \cdot \Delta y \qquad \text{where } \Delta y = \frac{d-c}{m}$$

By taking $m \to \infty$, we have

$$\text{Volume} = \lim_{m\to\infty} \sum_{l=1}^{m} A(c + l\Delta y) \cdot \Delta y$$

$$= \int_c^d A(y)\, dy = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

<u>Example</u>: Let $f(x,y) = x^2 + xy - y^2$ be a cont. function defined on $\mathbb{R}^2$.

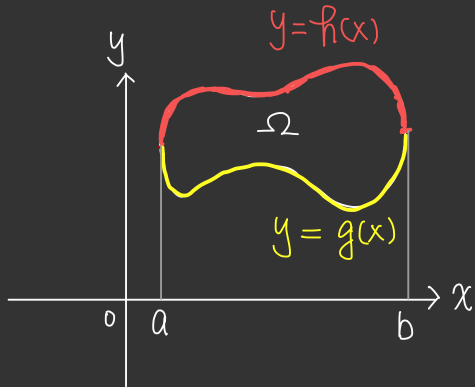
Find $\displaystyle\int_{[0,1]\times[2,3]} f(x,y)\ dx\,dy$

<u>Sol:</u> $\displaystyle\int_2^3\int_0^1 f(x,y)\,dx\ dy = \int_2^3\int_0^1 x^2+xy-y^2\ dx\ dy$

$$= \int_2^3 \left(\frac{1}{3}x^3+\frac{y}{2}\cdot x^2-y^2x\right)\Bigg|_{x=0}^{x=1}\ dy$$

$$= \int_2^3 \left[\frac{1}{3}+\frac{y}{2}-y^2-0\right] dy$$

$$= \left(\frac{y}{3}+\frac{y^2}{4}-\frac{y^3}{3}\right)\Bigg|_{y=2}^{y=3} = \left(1+\frac{9}{4}-9\right)-\left(\frac{2}{3}+1-\frac{8}{3}\right)$$



Let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function.

How do we compute $\iint_{\Omega} f(x,y)\, dx\, dy$?

Suppose $\Omega = \{ (x,y) \in \mathbb{R} \mid x \in [a,b],$
$g(x) \leq y \leq h(x) \}$

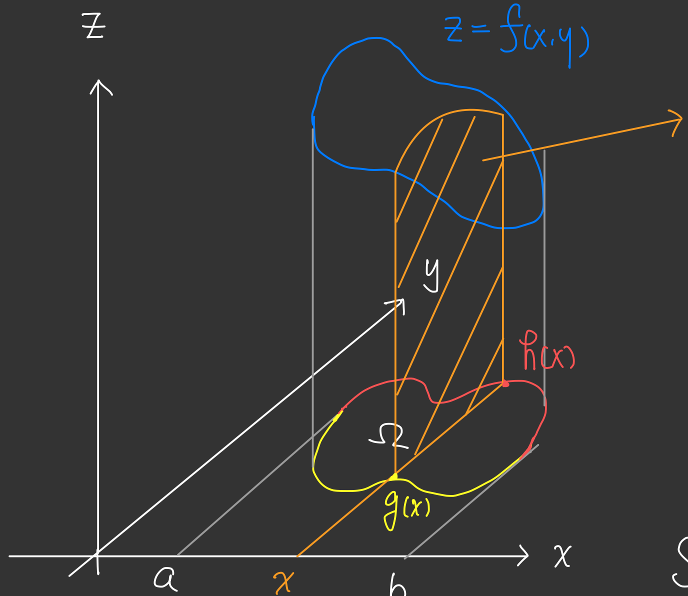
$= \{ (x,y) \in \mathbb{R} \mid y \leq h(x), \ x \in [a,b] \}$

$\cap \{ (x,y) \in \mathbb{R} \mid y \geq g(x), \ x \in [a,b] \}$

By the same argument,

$$\text{Volume} = \lim_{n\to\infty} \sum_{k=1}^{n} A(a+\Delta_x)\cdot\Delta x$$

$$= \int_a^b A(x)\,dx.$$



$$A(x) = \int_{g(x)}^{h(x)} f(x,y)\,dy$$

fixed

So we have the following formula

Thm: $$\iint_\Omega f(x,y)\,dx\,dy = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

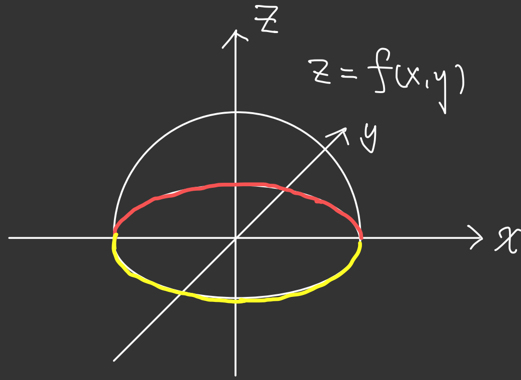
when $\Omega = \{ (x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ g(x) \leq y \leq h(x) \}$

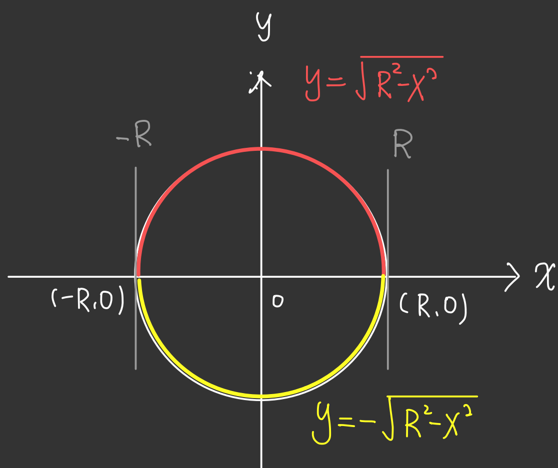
Example: Let $R > 0$ be a fixed number.

$f(x,y) := \sqrt{R^2 - x^2 - y^2}$ be a continuous function

defined on $\Omega = \{ (x,y) \in \mathbb{R}^2 \mid -\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$

$-R \leq x \leq R \}$





$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx$$

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy \qquad \left(\text{Let } \sqrt{R^2-x^2}=u\right)$$

$$= \int_{-u}^{u} \sqrt{u^2-y^2}\,dy \qquad \left(\text{Let } s=\frac{y}{u}\right)$$

$$\frac{ds}{dy}=\frac{1}{u} \Leftrightarrow dy = u\,ds$$

$$= \int_{-1}^{1} u\sqrt{1-s^2}\;u\cdot ds$$

$$= u^2\int_{-1}^{1}\sqrt{1-s^2}\,ds$$

Recall:

$$\int \sqrt{1-s^2}\,ds = \int \cos^2\theta\,d\theta \quad ; \; s=\sin\theta,\; \frac{ds}{d\theta}=\cos\theta$$

$$\left(s=-1 \Rightarrow \theta=-\frac{\pi}{2} \;;\; s=1,\ \theta=\frac{\pi}{2} \right)$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta \quad ; \; \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{1}{2}\int 1+\cos 2\theta\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

So

$$u^2 \int_{-1}^{1} \sqrt{1-s^2}\,ds = u^2 \left(\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \right)\Bigg|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{u^2}{2}\left(\pi + 0 + 0 \right) = \frac{\pi}{2}\cdot u^2$$

$$= \frac{\pi}{2}\left(R^2 - x^2\right)$$

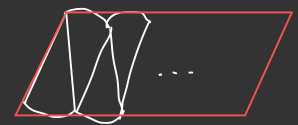
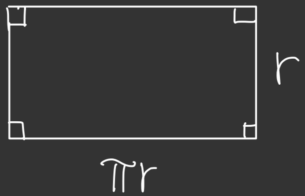
$$\Rightarrow \int_{-R}^{R} \frac{\pi}{2} \cdot (R^2 - x^2)\, dx = \frac{\pi}{2} \left(R^2 x - \frac{1}{3} x^3 \right) \Big|_{x=-R}^{R}$$

$$= \frac{\pi}{2} \left(2R^3 - \frac{2}{3} R^3 \right) = \frac{\pi}{2} R^3 \left(2 - \frac{2}{3} \right)$$

$$= \frac{2}{3} \pi R^3 \longrightarrow \text{半球体積.}$$

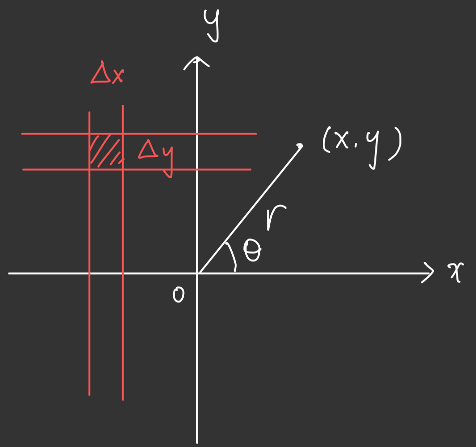
<u>Remark:</u> The volume of a 3-dim ball of radius R is

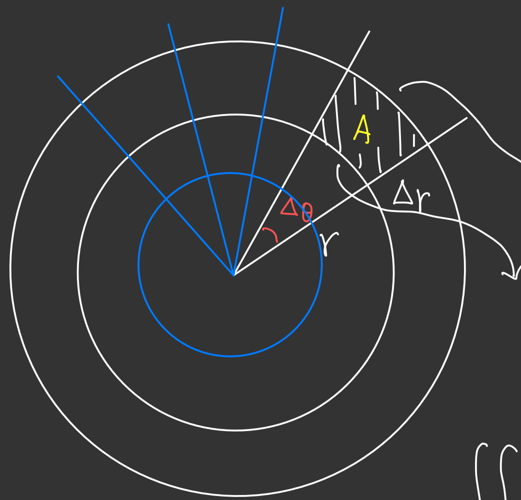
$$\frac{4}{3} \pi R^3$$

Vol  $=$  $\xrightarrow[\text{limit}]{\text{by taking}}$ 

Polar coordinates and change of variables:

We define: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$





$(r+\Delta_r)\cdot\Delta_\theta.$

$r\cdot\Delta_\theta$

The area of $A = \frac{1}{2}\Delta_\theta (r+\Delta_r)^2 - \frac{1}{2}\Delta_\theta r^2$

$$= \frac{1}{2}\Delta_\theta \left(2r\Delta_r + \Delta_r^2\right)$$

$$= r\Delta_r\Delta_\theta + \frac{1}{2}\Delta_r^2\cdot\Delta_\theta.$$

$$\iint_\Omega f(r,\theta) \sim \sum_{l=1}^{n}\sum_{k=1}^{m} f(r_k,\theta_l)\left(r_k\cdot\Delta_r\cdot\Delta_\theta + \frac{1}{2}\cdot\Delta_r^2\Delta_\theta\right)$$

$$\xrightarrow{m,n\to\infty} \iint_\Omega f(r,\theta)\, r\,dr\,d\theta + 0$$

Prop: Suppose that $f : \Omega \to \mathbb{R}$ is a continuous function.

$$\tilde{\Omega} := \{ (r, \theta) \mid (r\cos\theta, r\sin\theta) \in \Omega \}$$

$$g(r, \theta) = f(r\cos\theta, r\sin\theta).$$

Then $$\iint_{\Omega} f(x, y)\, dx dy = \iint_{\tilde{\Omega}} g(r, \theta) \cdot r \, dr d\theta.$$

Example: $f(x, y) := \sqrt{R^2 - x^2 - y^2}$

$$\Omega = \{ (x, y) \in \mathbb{R}^2 \mid -\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}, -R \leq x \leq R \}$$

$$\Rightarrow g(r, \theta) = \sqrt{R^2 - r^2}, \quad \tilde{\Omega} = \{ (r, \theta) \mid 0 \leq r \leq R, 0 \leq \theta < 2\pi \}$$

By Prop., $\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\tilde{\Omega}} \sqrt{R^2-r^2} \cdot r\,dr\,d\theta$

$$= \int_0^R \int_0^{2\pi} \sqrt{R^2-r^2} \cdot r\,d\theta\,dr$$

$$= 2\pi \int_0^R \sqrt{R^2-r^2}\, r\,dr \,, \qquad u = R^2-r^2$$

$$\frac{du}{dr} = -2r$$

$$= -\pi \int_{R^2}^0 u^{\frac{1}{2}}\,du$$

$$= -\pi \left(\frac{2}{3} \cdot u^{\frac{3}{2}}\right)\Bigg|_{u=R^2}^{0} = \pi \frac{2}{3} \cdot R^3$$

$$= \frac{2}{3}\pi R^3$$

In general, let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function.

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$ be an one-to-one map.

between $\tilde{\Omega}$ and Ω. Let $g(u,v) = f(x(u,v), y(u,v))$.

Define: We define the Jacobian

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Prop: $$\boxed{\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\tilde{\Omega}} g(u,v) \cdot J(u,v)\,du\,dv.}$$

Remark: When $x = r\cos\theta$, $y = r\sin\theta$.

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

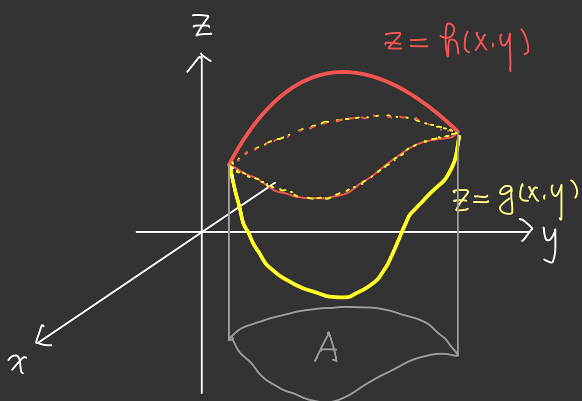
Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function. Then

Def: $$\iiint_{[a,b]\times[c,d]\times[e,g]} f(x,y,z)\, dx\,dy\,dz = \int_a^b \int_c^d \left(\int_e^g f(x,y,z)\, dz \right) dy\, dx.$$

When $f : \Omega \longrightarrow \mathbb{R}$, $\quad \Omega \subset \mathbb{R}^3$

$$\Omega = \{ (x, y, z) \in \mathbb{R}^3 \mid g(x,y) \leq z \leq h(x,y),\ (x,y) \in A \}$$

Here $A \subset \mathbb{R}^2$. Then



<u>Prop</u>: $$\iiint_{\Omega} f(x,y,z)\, dx\,dy\,dz$$

$$= \int_A \Big(\underbrace{\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz}_{\text{a two-variable function}} \Big)\, dx\,dy$$

Suppose $A = \{(x,y) \in \mathbb{R}^2 \mid l(y) \leq x \leq k(y),\ y \in [a,b]\}$

Then we also have

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\iint_{\Omega_y} f(x,y,z)\,dz\,dx \right) dy$$

$$\Omega_y = \{(x,z) \in \mathbb{R}^2 \mid (x,y,z) \in \Omega\}$$

Remark: We will always write $\iint\left(\int f\,dx\right)dy\,dz$ or $\int\left(\iint f\,dy\,dz\right)dx$

or $\iiint f\,dz\,dx\,dy$

Change of variables:

Let $f : \Omega \longrightarrow \mathbb{R}$. $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$ be a one-to-one map.

$$\tilde{\Omega} := \{ (u,v,w) \in \mathbb{R}^3 \mid (x(u,v,w), y(u,v,w), z(u,v,w)) \in \Omega \}$$

$$g(u,v,w) := f(x(u,v,w), y(u,v,w), z(u,v,w) \}$$

Then we have

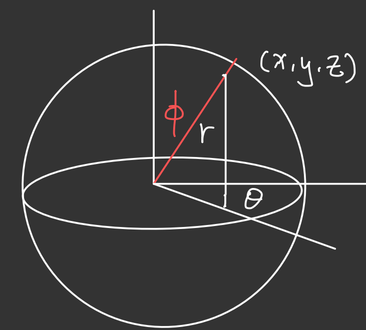
$$\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz = \iiint_{\tilde{\Omega}} g(u,v,w) \cdot J(u,v,w)\, du\, dv\, dw.$$

Here

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

<u>Example</u>: (Spherical coordinate)

$$\begin{cases} x = r\cos\theta\ \sin\phi \\ y = r\sin\theta\ \sin\phi \\ z = r\cos\phi \end{cases}$$



$$J(r,\phi,\theta) = \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

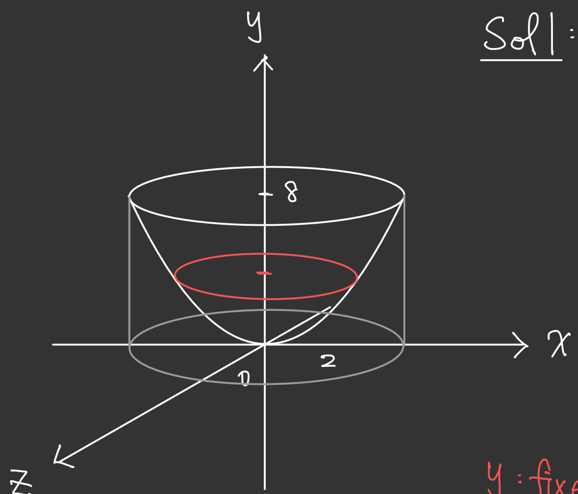
$$= \cos\phi \left(r^2 \sin\phi \cos\phi \right) + r^2 \sin^3\phi$$

$$= r^2 \left(\sin\phi \cdot \left(\cos^2\phi + \sin^2\phi \right) \right) = r^2 \sin\phi .$$

Therefore, we have

$$\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz = \iiint_{\tilde{\Omega}} g(r,\phi,\theta)\, r^2 \sin\phi \, dr\, d\phi\, d\theta .$$

Example: Let $f(x,y,z) = 3x^2 + 3z^2$ be a continuous function defined on $\Omega = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x^2 + 2z^2 \leq y \leq 8 \}$

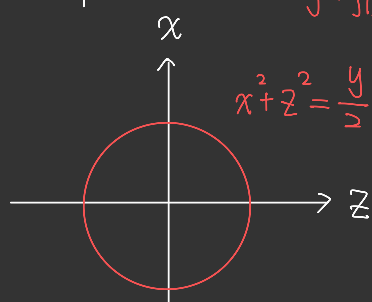


Sol:

$$\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz$$

$$= \int_0^8 \left(\iint_{\{x^2+z^2 \leq \frac{y}{2}\}} 3x^2 + 3z^2 \, dz\, dx \right) dy$$

y : fixed



$$= \int_0^8 \left(\int_0^{2\pi} \int_0^{\frac{y}{2}} 3r^2 \; r\, dr\, d\theta \right) dy$$

$$\left(\text{by taking } \begin{cases} z = r\cos\theta \\ x = r\sin\theta \end{cases} \right)$$

$$= \int_0^8 \int_0^{2\pi} \left(\frac{3}{4} r^4 \Big|_{r=0}^{\frac{y}{2}} \right) d\theta \, dy$$

$$= \int_0^8 \int_0^{2\pi} \frac{3}{4} \cdot \frac{y^4}{16} \, d\theta \, dy = \int_0^8 \frac{3\pi}{32} \cdot y^4 \, dy$$

$$= \frac{3\pi}{32} \frac{y^5}{5} \Big|_0^8$$

$$= 3\pi \cdot \frac{2^{15}}{2^5 \cdot 5} = \frac{3072\pi}{5}$$

Sol 2:

$$\iint_{\underset{A}{\overset{\shortparallel}{\{x^2+z^2 \le 4\}}}} \left(\int_{2x^2+2z^2}^{8} f(x,y,z) \, dy \right) dz \, dx$$

$$= \iint_A \left(\int_{2x^2+2z^2}^{8} (3x^2+3z^2)\, dy \right) dz\, dx$$

$$= \iint_A \left((3x^2+3z^2)\, y \Big|_{y=2x^2+2z^2}^{8} \right) dz\, dx$$

$$= \iint_A (3x^2+3z^2)\, 8 - 6(x^2+z^2)^2 \, dz\, dx .$$

$$= \int_0^{2\pi}\int_0^4 (24 \cdot r^2 - 6r^4)\, r\, dr\, d\theta = \int_0^{2\pi} \left(6 \cdot r^4 - \frac{6}{5} r^5 \right)\Big|_{r=0}^{4} d\theta$$

$$= \left(6 \cdot 2^8 - \frac{6}{5} \cdot 2^{10} \right) \cdot 2\pi = \left(\frac{6 \cdot 2^8}{5} \right) 2\pi = \frac{3072}{5}\pi .$$

<u>Example</u>: Let $f(x,y,z) = x^2+y^2+z^2$, $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 2\}$

$$\iiint_\Omega x^2+y^2+z^2 \, dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} r^2 \; r^2 \sin\phi \, dr\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{5} r^5 \sin\phi\right)\Big|_{r=0}^{r=2} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{32}{5} \cdot \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{32}{5}\cos\phi\right)\Big|_{\phi=0}^{\pi} d\theta = \frac{32}{5}\cdot 2\pi$$

$$= \frac{64}{5}\pi.$$

Conservative vector fields and Green's Theorem:

Let $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (or $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$) be a differentiable function.

i.e., $V(x,y) = \big(u(x,y),\ v(x,y) \big)$; $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

V is called a vector field

Suppose $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a 2-variable function.

$\nabla F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field.

Property of ∇F:

Let $\vec{\gamma} : [a, b] \longrightarrow \mathbb{R}^2$ be a plane curve.

$F \circ \vec{\gamma} : [a,b] \longrightarrow \mathbb{R}$ satisfies

$$\int_a^b \frac{d}{dt}(F \circ \vec{\gamma}) \, dt = F(\vec{\gamma}(b)) - F(\vec{\gamma}(a)) \quad (\text{by FTC})$$

$$\Rightarrow \underbrace{\int_a^b \nabla F(\vec{\gamma}(t)) \cdot \frac{d\vec{\gamma}}{dt} \, dt}_{\displaystyle \overset{\shortparallel}{\int_{\mathcal{C}} \nabla F \cdot d\vec{\gamma}}} = F(\vec{\gamma}(b)) - F(\vec{\gamma}(a))$$

indep. of the path.

$$\mathcal{C} := \{ \vec{\gamma}(t) \mid t \in [a,b] \}$$

Namely,

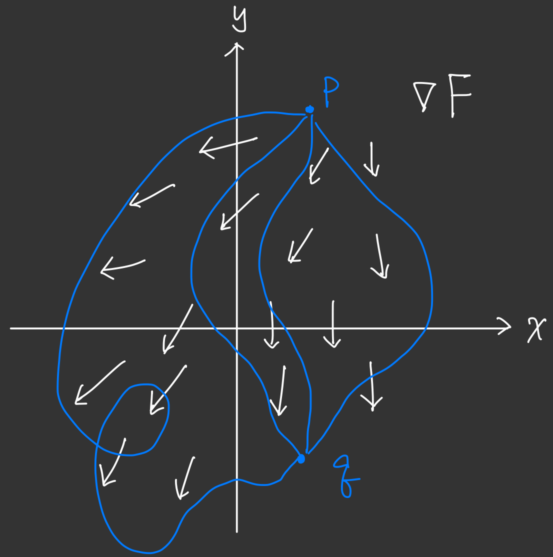
$$\int_{\mathscr{C}} \nabla F \cdot d\vec{\gamma} = \int_{\tilde{\mathscr{C}}} \nabla F \cdot d\vec{\beta}$$

when $\mathscr{C}$ and $\tilde{\mathscr{C}}$ have same end points.

In particular, when $\mathscr{C}$ is a closed curve (封閉的曲線),

$$\int_{\mathscr{C}} \nabla F \cdot d\vec{\gamma} = 0$$

(We write $\oint_{\mathscr{C}} \nabla F \cdot d\vec{\gamma}$ to denote the integral of a closed curve.)



In general, let $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a vector field. We can also define

Def:
$$\int_{\mathcal{C}} V \cdot d\vec{\gamma} := \int_a^b V(\vec{\gamma}(t)) \cdot \frac{d\vec{\gamma}}{dt}\, dt.$$

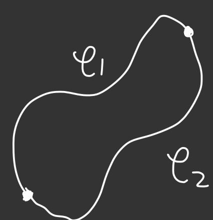
We call V a conservative vector field iff
$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = 0$$
for all closed curves.

Remark: The following two properties are equivalent:

① $\displaystyle\int_{\mathcal{C}_1} V \cdot d\vec{\gamma} = \int_{\mathcal{C}_2} V \cdot d\vec{\beta}$ whenever $\mathcal{C}_1, \mathcal{C}_2$ have the same end points

② $\displaystyle\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = 0$ whenever $\mathcal{C}$ is a closed curve.

Sketch of proof: ① holds $\Rightarrow$ Any closed curve $\mathcal{C}$ can be written as $\mathcal{C}_1 \cup \mathcal{C}_2$



$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = \int_{\mathcal{C}_1} V \cdot d\vec{\gamma} - \int_{\mathcal{C}_2} V \cdot d\vec{\gamma} = 0$$

$\Rightarrow$ ② holds

② $\Rightarrow$ ① .

<u>Thm</u>: Let $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a vector field, say $V = (P, Q)$.

Then V is conservative vector field iff

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Suppose V is <u>not</u> conservative. Then

①

$$\oint_{\mathcal{C}} V \cdot d\vec{r} \neq 0 \quad \text{for some closed curve } \mathcal{C}.$$

or

② $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \neq 0$ at some $(x, y) \in \mathbb{R}^2$

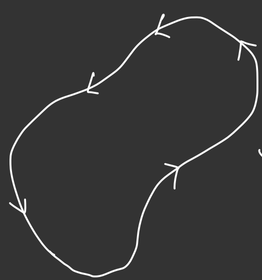
Green's Theorem :

Let $\vec{V} := (P, Q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be any vector field.

$\mathcal{C}$ is a closed curve, which enclose the region R.

$\vec{\gamma} : [a, b] \longrightarrow \mathbb{R}^2$: parametrization of $\mathcal{C}$,

Counterclockwisely.



$\mathcal{C}$

Theorem (Green's):

$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

(Recall:

$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = \int_a^b \left(V \cdot \frac{d\vec{\gamma}}{dt} \right) dt \qquad \left(\vec{\gamma}(t) = (x(t), y(t)) \right)$$

$$= \int_a^b \Big(P(x(t), y(t)), Q(x(t), y(t)) \Big) \cdot \left(x'(t), y'(t) \right) dt$$

$$= \int_a^b \Big(P(x(t), y(t))\, \underset{\substack{\parallel \\ \frac{dx}{dt}}}{x'(t)} + Q(x(t), y(t))\, \underset{\substack{\parallel \\ \frac{dy}{dt}}}{y'(t)} \Big)\, dt.$$)

(denote) $= \oint_{\mathcal{C}} P\,dx + Q\,dy$)

<u>Remark</u>: When V is conservative

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \oint_{\mathcal{C}} V \cdot d\vec{\gamma} = 0 \text{ for all closed curves } \mathcal{C}.$$

In particular, ∇F is coservative.

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\Rightarrow \oint_{\mathcal{C}} \nabla F \cdot d\vec{\gamma} = 0 \quad \forall \text{ closed curves } \mathcal{C}.$$

<u>Example</u>: Find $\oint_{\mathcal{C}} xy\,dx + (x+y)\,dy$; $V(x,y) = (xy,\ x+y)$

$$\mathcal{C} = \{ x^2 + y^2 = 4 \}$$

<u>Sol</u>: By Green's theorem.

$$\oint_{\mathcal{C}} xy\,dx + (x+y)\,dy = \iint_R (1-x)\,dx\,dy \quad ; \quad R = \{ x^2+y^2 \le 4 \}$$

$$= \int_0^{2\pi} \int_0^2 (1 - r\cos\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3}\cos\theta \right) \Big|_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} \left(2 - \frac{8}{3}\cos\theta\right) d\theta = 4\pi$$

Sol 2: Let $x = 2\cos\theta$, $y = 2\sin\theta$ be the parametrization of $\mathcal{C}$.

$$\oint_{\mathcal{C}} xy\,dx + (x+y)\,dy = \int_0^{2\pi} \Big(-8\cos\theta\sin^2\theta$$

$$+4(\cos\theta + \sin\theta)\cos\theta\Big)\,d\theta$$

$$= -8\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta + 4\int_0^{2\pi} \sin\theta\cos\theta\,d\theta + A$$

$$\sin\theta = u \; ; \; \frac{du}{d\theta} = \cos\theta$$

$$= -8\int_0^0 u^2\,du + 4\int_0^0 u\,du. + A$$

$$A = 4\int_0^{2\pi} \cos^2\theta \, d\theta = 4\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$2\theta = t$$

$$\cos 2\theta = 1 - 2\cos^2\theta$$

$$= 4\pi - \int_0^{2\pi} \cos 2\theta \cdot 2 d\theta$$

$$= 4\pi - \int_0^{4\pi} \cos t \, dt$$

$$= 4\pi$$

Example: Find $\oint_{\mathscr{C}} (x+y)\,dx - (x-y)\,dy$, $\mathscr{C} = \{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \}$

Sol: By Green's theorem : $(P, Q) = (x+y,\ y-x)$

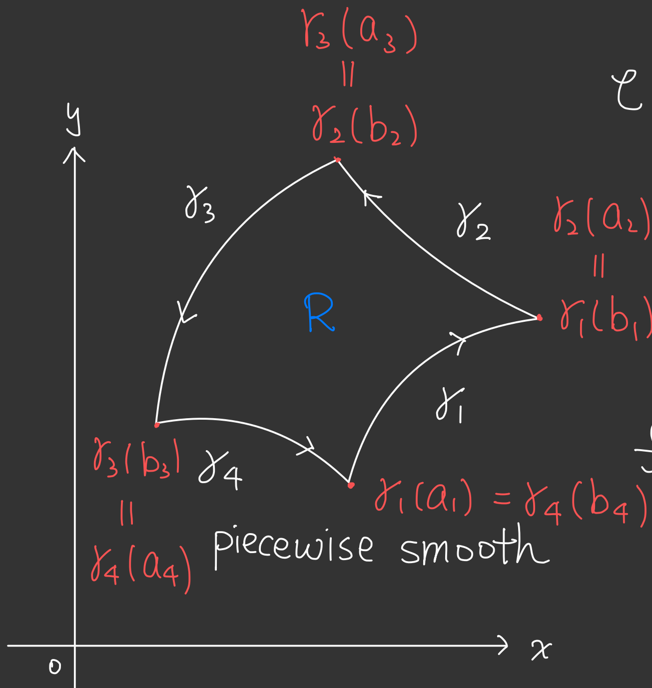
$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R (-1-1)\,dx\,dy = -2 \iint_R 1\,dx\,dy$$

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

$$= -2 \cdot |a||b| \cdot \pi$$

Remark: ① We allows the curve $\mathcal{C}$ to be piecewise smooth, i.e., (closed)



$$\mathcal{C} = \{\gamma_1(t) \mid t \in [a_1, b_1]\}$$
$$\cup \{\gamma_2(t) \mid t \in [a_2, b_2]\}$$
$$\cup \cdots \cup \{\gamma_n(t) \mid t \in [a_n, b_n]\}$$

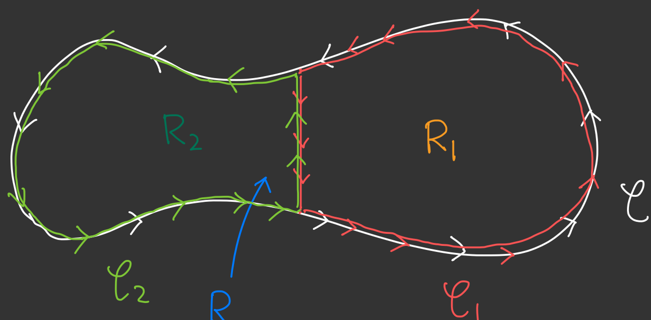
for some $\underset{\text{smooth}}{\gamma_i : [a_i, b_i] \longrightarrow \mathbb{R}^2}$.

$$\begin{cases} \gamma_i(b_i) = \gamma_{i+1}(a_{i+1}) & \forall i = 1, \ldots n-1 \\ \gamma_n(b_n) = \gamma_1(a_1) \end{cases}$$

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \sum_{i=1}^{n} \int_{\gamma_i} P\,dx + Q\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

② We can "cut" the region R and obtain Green's theorem.



$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\begin{array}{ccc} \| & & \| \end{array}$$

$$\left\{ \begin{array}{c} \displaystyle\oint_{\mathcal{C}_1} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \\ + \qquad\qquad\qquad\qquad + \\ \displaystyle\oint_{\mathcal{C}_2} P\,dx + Q\,dy = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \end{array} \right.$$

Stoke's theorem:

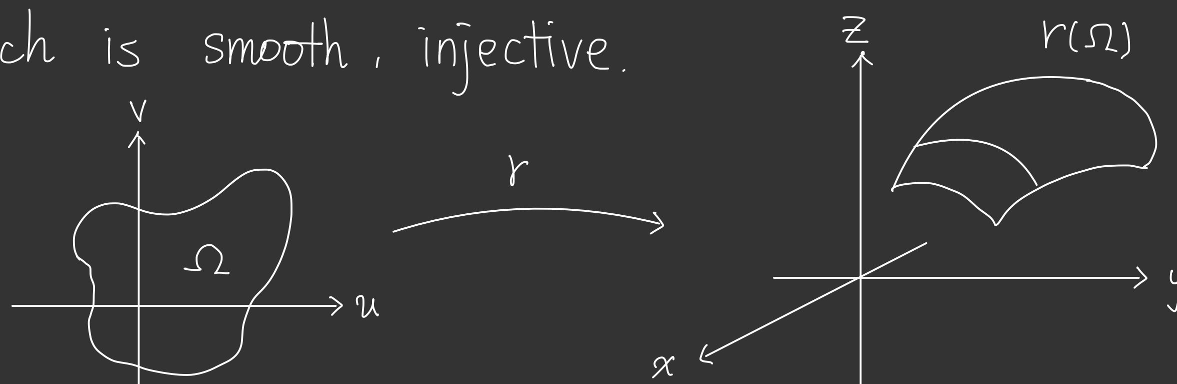
Surface integrals:

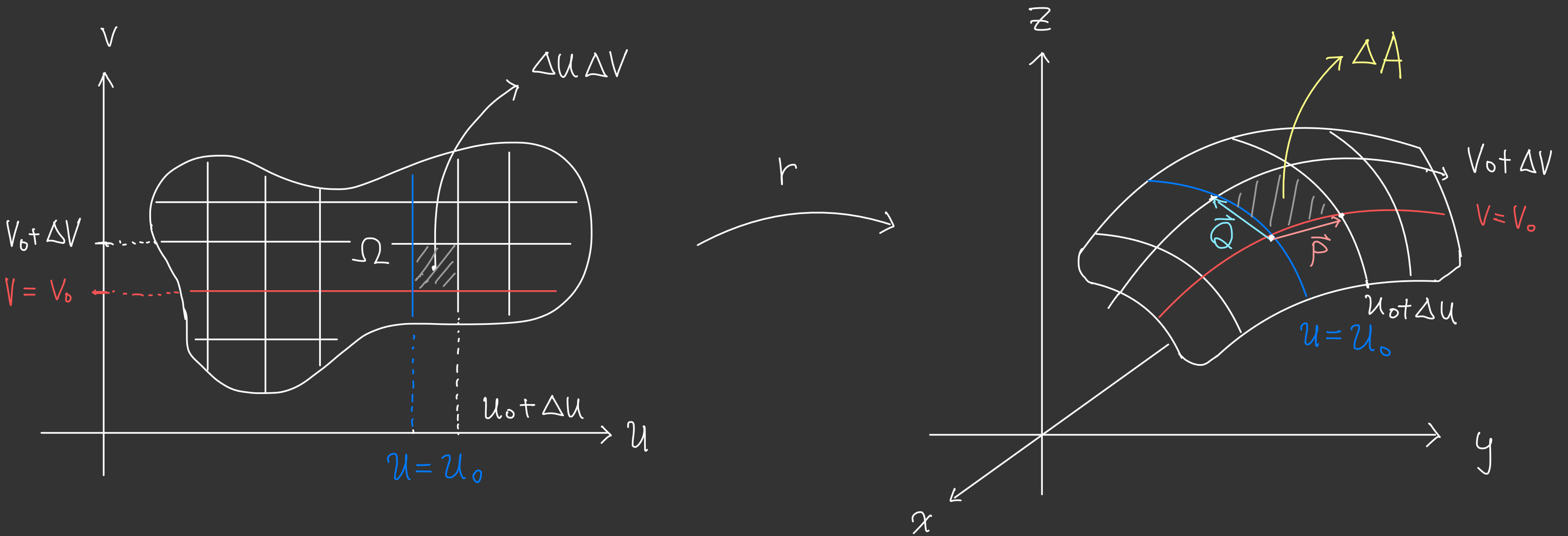
Let $\Omega \subset \mathbb{R}^2$ be a region in $\mathbb{R}^2$ with variables (u, v).

Def: A smooth surface S is the image of a map

$$r : \Omega \longrightarrow \mathbb{R}^3$$

which is smooth, injective.





$\underline{Q}$. How to compute the area of this surface?

$$\Delta A \sim |\vec{P} \times \vec{Q}| \quad \text{where} \begin{cases} \vec{P} = (r(u_0 + \Delta u, v_0) - r(u_0, v_0)) \\ \vec{Q} = (r(u_0, v_0 + \Delta v) - r(u_0, v_0)) \end{cases}$$

By mean value theorem:

$$\vec{P} = \frac{\partial r}{\partial u}(u_0+\xi, v_0)\,\Delta u. \quad \text{for some } \xi \in (0, \Delta u)$$

$$\vec{Q} = \frac{\partial r}{\partial v}(u_0, v_0+\eta)\,\Delta v \quad \text{for some } \eta \in (0, \Delta v)$$

So,

$$\Delta A \sim |\vec{P} \times \vec{Q}| = \left| \frac{\partial r}{\partial u}(u_0+\xi, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0+\eta) \right| \Delta u \Delta v.$$

Therefore, the surface area.

$$A = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum \Delta A = \iint_{\Omega} \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du dv.$$

In general, we will write

$$\iint_{\Omega} ds = \text{Area of surface}$$

where $ds := \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du dv$ (曲面的面積元)

<u>Remark</u>: Notice that,

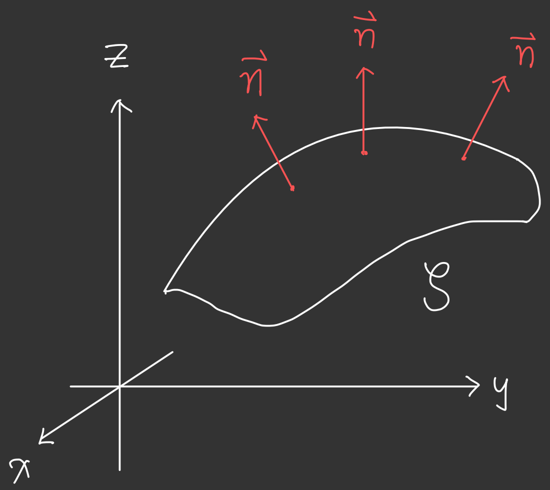
$\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}$ are tangent to the surface

$$S = \{ r(u,v) \mid (u,v) \in \Omega \}$$

Therefore, $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \perp S$.

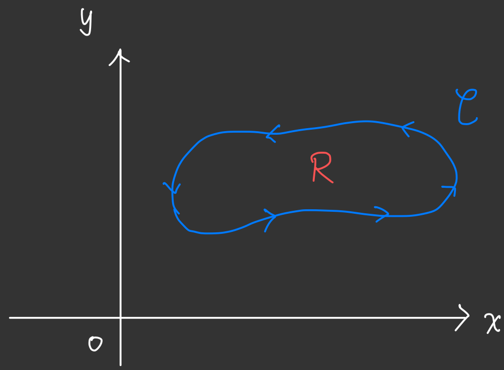
Def: We define the normal vector

$$\vec{n} := \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|} \qquad \text{(vecto field defined on } S \text{)}$$

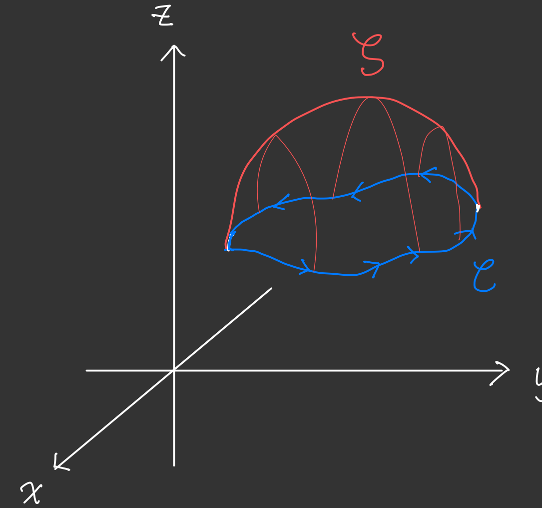


(unit normal vector)

Stoke's Theorem:



Generalize



Green's theorem

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \longrightarrow \quad ?$$

$\uparrow$

Stoke's theorem

Let $V = (P, Q, R)$ be a vector field defined on $\mathbb{R}^3$.

<u>Def</u>: Let $\mathcal{C}$ be a closed curve in $\mathbb{R}^3$,

$$\mathcal{C} = \{ \vec{\gamma}(t) = (x(t), y(t), z(t)) \mid t \in [a, b],\ \vec{\gamma}(a) = \vec{\gamma}(b) \}$$

for some $\vec{\gamma}$. Then we define

$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} := \int_a^b \left(V \cdot \frac{d\vec{\gamma}}{dt} \right) dt$$

$$= \int_a^b \left(P(\vec{\gamma}(t))\, x'(t) + Q(\vec{\gamma}(t))\, y'(t) + R(\vec{\gamma}(t))\, z'(t) \right) dt$$

Def: Let $V = (P, Q, R)$ be a vector field defined on $\mathbb{R}^3$.

We define

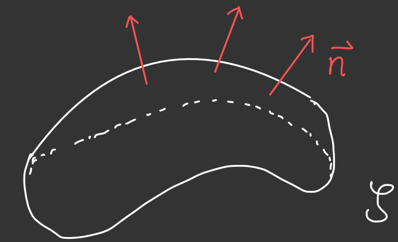
$$\operatorname{curl}(V) := (\nabla \times V) \quad \leftarrow \text{notation}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} , \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} , \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Theorem: (Stoke's theorem). Let S be a surface parametrized by $r : \Omega \longrightarrow S$. Then

$$\oint_{\mathcal{C}} V \cdot d\vec{\gamma} = \iint_{\Omega} \operatorname{curl}(V) \cdot \vec{n} \, ds \qquad \text{———} \quad (*)$$

$\mathcal{C}$: boundary of $\mathcal{S}$.



Remark : ① Since we have

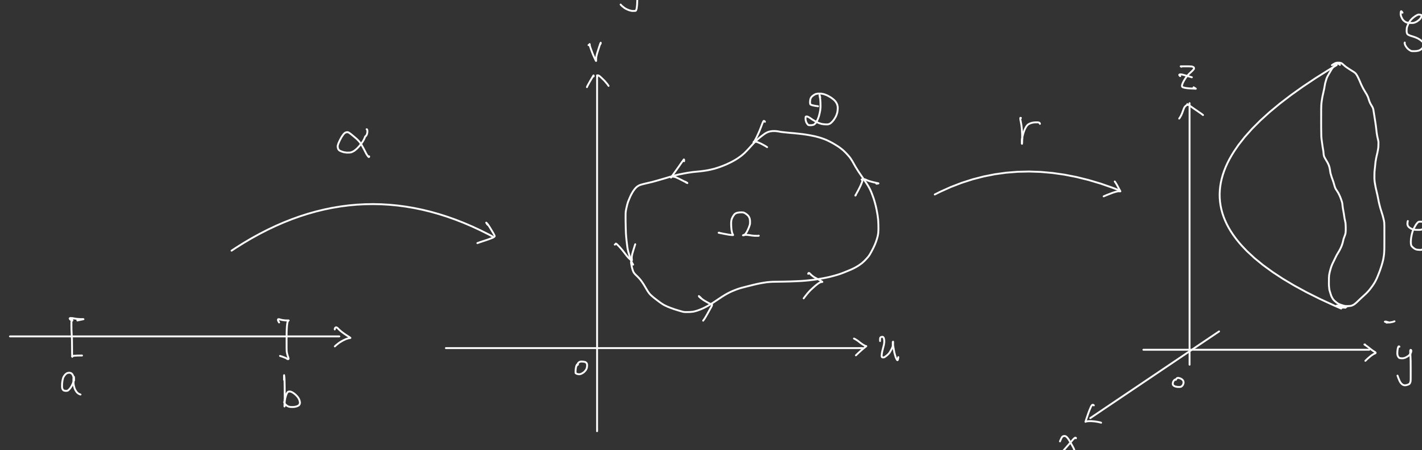
$$\vec{n} = \frac{\partial_u r \times \partial_v r}{|\partial_u r \times \partial_v r|} \qquad \text{and} \qquad ds = |\partial_u r \times \partial_v r| \, du \, dv ,$$

The right hand side of $(*)$ can be written as

$$\iint_{\Omega} \operatorname{curl}(V) \cdot (\partial_u r \times \partial_v r) \, du \, dv .$$

② The curve $\mathscr{C}$ should be parametrized by the boundary of Ω. Denote by $\mathscr{D}$ the bdy of Ω.

Let $\alpha : [a, b] \longrightarrow \mathscr{D}$ be a parametrization of $\mathscr{D}$ counterclockwisely.



Then we can define

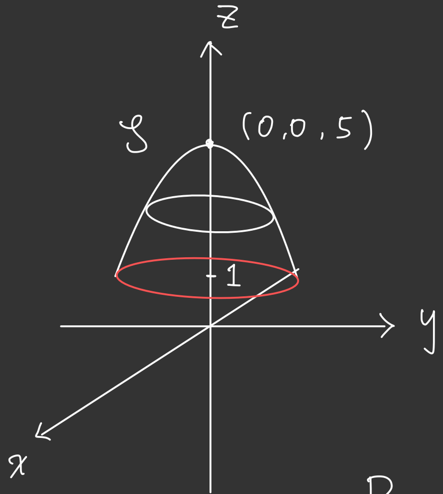
$$\vec{\gamma} := r \circ \alpha : [a,b] \longrightarrow \mathscr{C}.$$

be the parametrization for $\mathscr{C}$.

<u>Example</u>: Let $S = \{ (x, y, z) \in \mathbb{R}^3 \mid z = 5 - x^2 - y^2 , \ z \geq 1 \}$

$V = (z^2, -3xy, x^3y^3)$: vector field defined on $\mathbb{R}^3$.

Find $\iint_S \mathrm{curl}(V) \cdot \vec{n} \, ds = ?$



Let $u = x$, $v = y$

$$\Omega = \{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 4\}$$

$$r(u,v) = (u, v, 5 - u^2 - v^2)$$

By Stoke's theorem,

$$\iint_S \operatorname{curl}(V) \cdot \vec{n}\, ds = \iint_\Omega \operatorname{curl}(V) \cdot \vec{n}\, ds$$

$$= \oint_C V \cdot d\vec{r}$$

$\vec{\gamma} : [0, 2\pi] \longrightarrow \mathbb{R}^3$ defined by

$\vec{\gamma}(\theta) = (2\cos\theta, 2\sin\theta, 1)$

$$\int_0^{2\pi} (1, -12\cos\theta\sin\theta, 8\cos^3\theta + 8\sin^3\theta) \cdot (-2\sin\theta, 2\cos\theta, 0) \, d\theta$$

$$= \int_0^{2\pi} -2\sin\theta - 24\cos^2\theta\sin\theta \, d\theta$$

$$= -24\int_0^{2\pi} \cos^2\theta\sin\theta \, d\theta = 24\int_1^1 u^2 \, du = 0.$$

$u = \cos\theta, \quad \frac{du}{d\theta} = -\sin\theta$

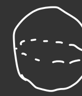
Divergence Theorem:

Let $V: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field defined on $\mathbb{R}^3$.

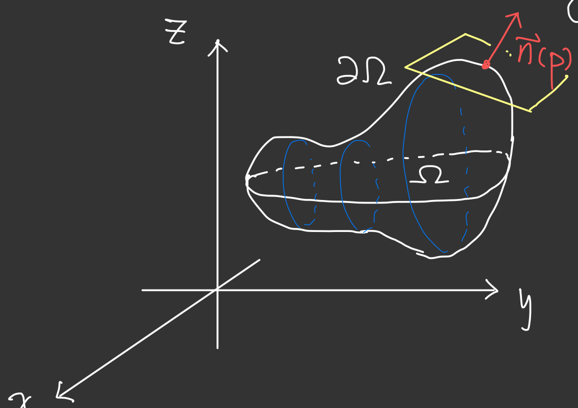
$$V = (P, Q, R)$$

Def: We call

$$\operatorname{div}(V) := \nabla \cdot V = \partial_x P + \partial_y Q + \partial_z Q.$$



Let $\Omega \subset \mathbb{R}^3$ be a region bounded by a closed surface.



Denote by $\partial\Omega :=$ bdy of Ω

$\partial\Omega$: closed surface

We define

$$\vec{n} : \partial\Omega \longrightarrow \mathbb{R}^3$$
$$p \longmapsto \vec{n}(p) : \text{"outer" unit normal vector of } \partial\Omega \text{ at } p$$

Suppose that $\partial\Omega = S_1 \cup S_2$

$$S_1 = \{ r_1(u,v) \mid (u,v) \in U_1 \}$$
$$S_2 = \{ r_2(u,v) \mid (u,v) \in U_2 \}$$

$U_1, U_2 \subset \mathbb{R}^2$

$$\begin{cases} r_1 : \text{parametrization of } S_1. \\ r_2 : \text{parametrization of } S_2. \end{cases}$$

Then $\quad \vec{n}(p) = \begin{cases} \pm \dfrac{\partial_u r_1 \times \partial_v r_1}{|\partial_u r_1 \times \partial_v r_1|}(u,v) & \text{when } p = r_1(u,v) \in S_1 \\ \pm \dfrac{\partial_u r_2 \times \partial_v r_2}{|\partial_u r_2 \times \partial_v r_2|}(u,v) & \text{when } p = r_2(u,v) \in S_2 \end{cases}$

<u>Theorem</u> (Divergence Theorem).

$$\oiint_{\partial\Omega} V \cdot \vec{n} \, ds = \iiint_{\Omega} \operatorname{div}(V) \, dx\,dy\,dz.$$

Remark:

$$\oiint_{\partial\Omega} V\cdot\vec{n}\, ds = \iint_{S_1} V\cdot\vec{n}\, ds + \iint_{S_2} V\cdot\vec{n}\, ds$$

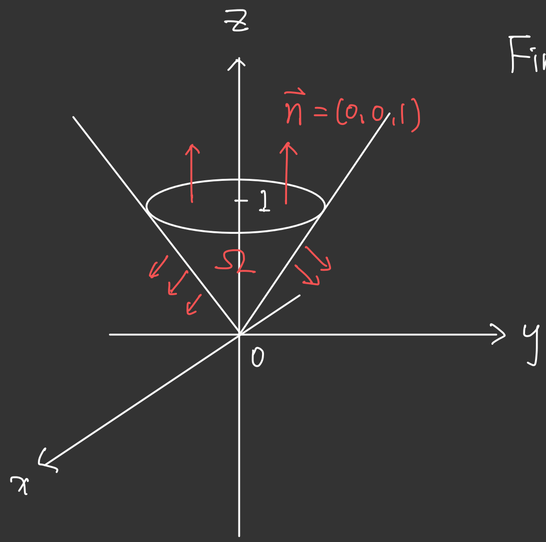
$$= \pm\iint_{U_1} V\cdot\frac{\partial_u r_1\times\partial_v r_1}{\cancel{|\partial_u r_1\times\partial_v r_1|}}\cdot\cancel{|\partial_u r_1\times\partial_v r_1|}\, du\,dv$$

$$+ \pm\iint_{U_2} V\cdot\frac{\partial_u r_2\times\partial_v r_2}{\cancel{|\partial_u r_2\times\partial_v r_2|}}\cdot\cancel{|\partial_u r_2\times\partial_v r_2|}\, du\,dv$$

$$= \pm\iint_{U_1} V\cdot(\partial_u r_1\times\partial_v r_1)\, du\,dv \pm \iint_{U_2} V\cdot(\partial_u r_2\times\partial_v r_2)\, du\,dv$$

Remark: It is not necessary to assume $\partial\Omega$ to be smooth.

Example: Let $V = (x^3, y^3, z^3)$, $\Omega = \{ (x,y,z) \mid z^2 \geq x^2 + y^2,\ 0 \leq z \leq 1 \}$



Find $\displaystyle\oiint_{\partial\Omega} V \cdot \vec{n}\, ds = ?$

By Divergence Theorem,

$$\oiint_{\partial\Omega} V \cdot \vec{n}\, ds = \iiint_{\Omega} \operatorname{div}(V)\, dx\, dy\, dz$$

$$= \iiint_{\Omega} (3x^2 + 3y^2 + 3z^2)\, dx\,dy\,dz$$

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

$$= \int_0^1 \left(\int_0^{2\pi} \int_0^{z} 3(z^2 + r^2)\, r\,dr\,d\theta \right) dz = 3 \int_0^1 \left(\int_0^{2\pi} \left(\frac{1}{2} z^2 r^2 + \frac{1}{4} r^4 \right) \Big|_{r=0}^{r=z} d\theta\, dz \right.$$

$$= 3 \int_0^1 \frac{3}{4} z^4 \cdot 2\pi\, dz = \frac{9\pi}{2} \frac{1}{5} z^5 \Big|_{z=0}^{z=1} = \frac{9}{10} \pi .$$

Notice that

$$\oiint_{\partial\Omega} V\cdot\vec{n}\, ds = \iint_B (x^3, y^3, 1)\cdot(0,0,1)\, dx\,dy$$

$$B = \{(x,y) \mid x^2+y^2 \le 1\}$$

$$+ \iint_B (x^3, y^3, 1)\cdot\vec{n}\, dx\,dy$$

$$= \iint_B 1\, dx\,dy + \iint_B (x^3, y^3, 1)\left(\frac{x}{\sqrt{2x^2+2y^2}}, \frac{y}{\sqrt{2x^2+2y^2}}, \frac{-1}{\sqrt{2}}\right) dx\,dy$$

$$= \iint_B 1\, dx\,dy + \iint_B \left(\frac{x^4+y^4}{\sqrt{2x^2+2y^2}} - \frac{1}{\sqrt{2}}\right) dx\,dy$$

$$r(x,y) = (x,\ y,\ \sqrt{x^2+y^2}\ )$$

$$\partial_x r = \left(1,\ 0,\ \frac{x}{\sqrt{x^2+y^2}}\right), \quad \partial_y r = \left(0,\ 1,\ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\partial_x r \times \partial_y r = \left(-\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1\right)$$

$$\vec{n} = \left(\frac{x}{\sqrt{2x^2+2y^2}},\ \frac{y}{\sqrt{2x^2+2y^2}},\ \frac{-1}{\sqrt{2}}\right)$$