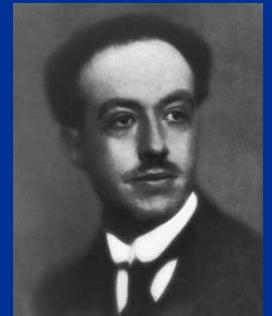


# Wave Properties of Particles



## De Broglie waves

A moving body behaves in certain way as though it has a wave nature.

Louis de Broglie

A photon of light of frequency  $\nu$  has the momentum : 
$$\mathbf{p} = \frac{\hbar \nu}{c} = \frac{\hbar}{\lambda} \quad (\lambda \nu = c)$$

The wavelength of a photon can therefore be specified by its momentum :

$$\lambda = \frac{\hbar}{p} \quad (3.1)$$

**De Broglie:** the above equation is a completely general one that applies to other particles as well as photons

**De Broglie wavelength:**

$$\lambda = \frac{\hbar}{p} = \frac{\hbar}{m v}, \text{ here } m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (3.2)$$

**A moving body has both particle and wave properties, which set of properties is most obvious depends on how its de Broglie wavelength compares with its dimensions and dimensions of whatever it interacts with.**

## Example

Find the de Broglie wavelengths of (a) a 46-g golf ball with a velocity of 30 m/s, and (b) an electron with a velocity of  $10^7$  m/s.

### Solution

(a) Since  $v \ll c$ , we can let  $m = m_0$ . Hence

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(0.046 \text{ kg})(30 \text{ m/s})} = 4.8 \times 10^{-34} \text{ m}$$

The wavelength of the golf ball is so small compared with its dimensions that we would not expect to find any wave aspects in its behavior.

(b) Again  $v \ll c$ , so with  $m = m_0 = 9.1 \times 10^{-31} \text{ kg}$ , we have

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.1 \times 10^{-31} \text{ kg})(10^7 \text{ m/s})} = 7.3 \times 10^{-11} \text{ m}$$

The dimensions of atoms are comparable with this figure—the radius of the hydrogen atom, for instance, is  $5.3 \times 10^{-11} \text{ m}$ . It is therefore not surprising that the wave character of moving electrons is the key to understanding atomic structure and behavior.

## WAVES OF WHAT?

### *Waves of probability*

In water waves, the quantity that varies periodically is the height of the water surface. In sound waves, it is pressure. In light waves, electric and magnetic fields vary. What is it that varies in the case of matter waves?

The quantity whose variations make up matter waves is called the **wave function**, symbol  $\Psi$  (the Greek letter psi). The value of the wave function associated with a moving body at the particular point  $x, y, z$  in space at the time  $t$  is related to the likelihood of finding the body there at the time.

The wave function  $\Psi$  itself, however, has no direct physical significance. There is a simple reason why  $\Psi$  cannot be interpreted in terms of an experiment. The probability that something be in a certain place at a given time must lie between 0 (the object is definitely not there) and 1 (the object is definitely there). An intermediate probability, say 0.2, means that there is a 20% chance of finding the object. But the amplitude of any wave is negative as often as it is positive, and a negative probability, say  $-0.2$ , is meaningless. Hence  $\Psi$  by itself cannot be an observable quantity.

This objection does not apply to  $|\Psi|^2$ , the square of the absolute value of the wave function, which is known as **probability density**:

The probability of experimentally finding the body described by the wave function  $\Psi$  at the point  $x, y, z$ , at the time  $t$  is proportional to the value of  $|\Psi|^2$  there at  $t$ .

## DESCRIBING A WAVE

### *A general formula for waves*

How fast do de Broglie waves travel? Since we associate a de Broglie wave with a moving body, we expect that this wave has the same velocity as that of the body. Let us see if this is true.

If we call the de Broglie wave velocity  $v_p$ , we can apply the usual formula

$$v_p = \nu\lambda$$

to find  $v_p$ . The wavelength  $\lambda$  is simply the de Broglie wavelength  $\lambda = h/mv$ . To find the frequency, we equate the quantum expression  $E = h\nu$  with the relativistic formula for total energy  $E = mc^2$  to obtain

$$h\nu = mc^2$$

$$\nu = \frac{mc^2}{h}$$

The de Broglie wave velocity is therefore

**De Broglie phase  
velocity**

$$v_p = \nu\lambda = \left(\frac{mc^2}{h}\right)\left(\frac{h}{mv}\right) = \frac{c^2}{v} \quad (3.3)$$

Because the particle velocity  $v$  must be less than the velocity of light  $c$ , the de Broglie waves always travel faster than light! In order to understand this unexpected result, we must look into the distinction between **phase velocity** and **group velocity**. (Phase velocity is what we have been calling wave velocity.)

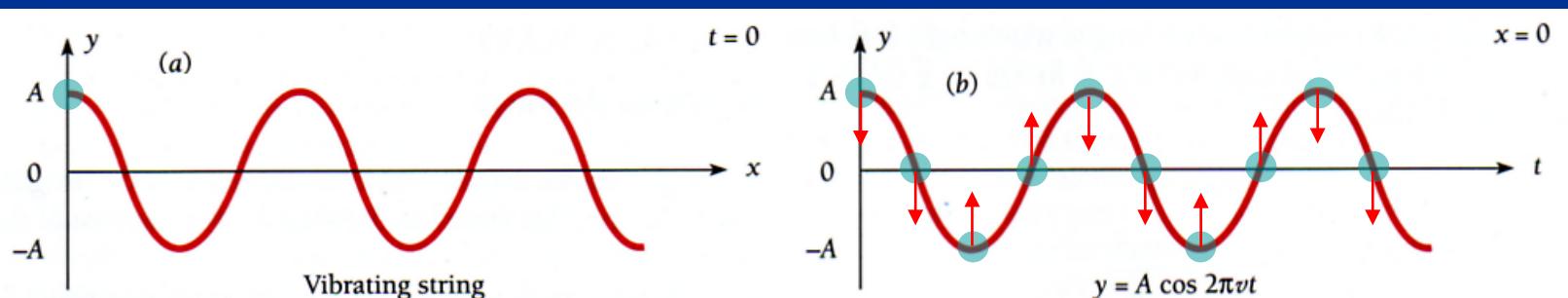
## Wave formula

Let us begin by reviewing how waves are described mathematically. For simplicity we consider a string stretched along the  $x$  axis whose vibrations are in the  $y$  direction, as in Fig. 3.1, and are simple harmonic in character. If we choose  $t = 0$  when the displacement  $y$  of the string at  $x = 0$  is a maximum, its displacement at any future time  $t$  at the same place is given by the formula

$$y = A \cos 2\pi\nu t \quad (3.4)$$

where  $A$  is the amplitude of the vibrations (that is, their maximum displacement on either side of the  $x$  axis) and  $\nu$  their frequency.

Equation (3.4) tells us what the displacement of a single point on the string is as a function of time  $t$ . A complete description of wave motion in a stretched string, however, should tell us what  $y$  is at *any* point on the string at *any* time. What we want is a formula giving  $y$  as a function of both  $x$  and  $t$ .



**Figure 3.1** (a) The appearance of a wave in a stretched string at a certain time. (b) How the displacement of a point on the string varies with time.

To obtain such a formula, let us imagine that we shake the string at  $x = 0$  when  $t = 0$ , so that a wave starts to travel down the string in the  $+x$  direction (Fig. 3.2). This wave has some speed  $v_p$  that depends on the properties of the string. The wave travels the distance  $x = v_p t$  in the time  $t$ , so the time interval between the formation of the wave at  $x = 0$  and its arrival at the point  $x$  is  $x/v_p$ . Hence the displacement  $y$  of the string at  $x$  at any time  $t$  is exactly the same as the value of  $y$  at  $x = 0$  at the earlier time  $t - x/v_p$ . By simply replacing  $t$  in Eq. (3.4) with  $t - x/v_p$ , then, we have the desired formula giving  $y$  in terms of both  $x$  and  $t$ :

### Wave formula

$$y = A \cos 2\pi\nu \left( t - \frac{x}{v_p} \right) \quad (3.5)$$

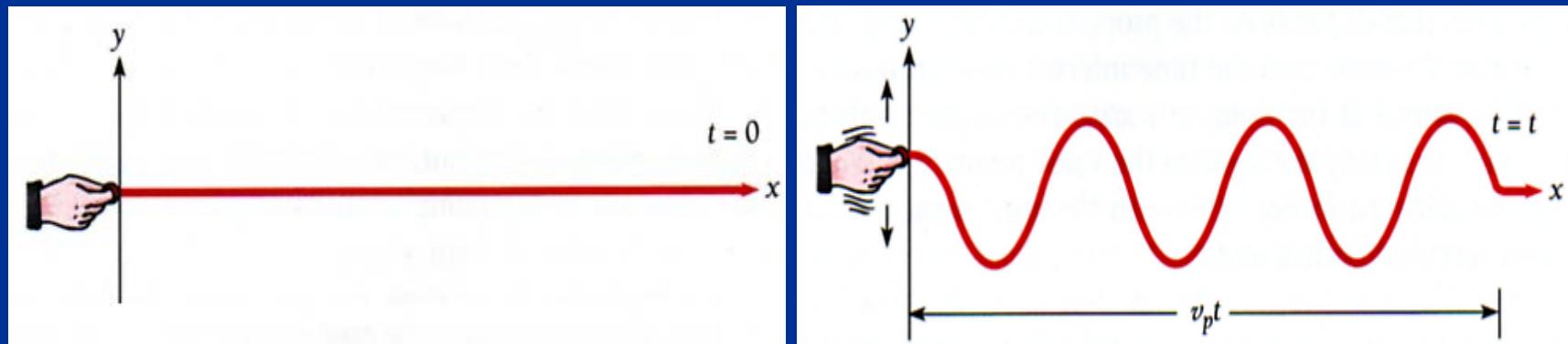


Fig. 3.2 Wave propagation

Equation (3.5) may be rewritten  $y = A \cos 2\pi \left( \nu t - \frac{\nu x}{v_p} \right)$

Since the wave speed  $v_p$  is given by  $v_p = \nu \lambda$  we have

**Wave formula**  $y = A \cos 2\pi \left( \nu t - \frac{x}{\lambda} \right)$  (3.6)

Equation (3.6) is often more convenient to use than Eq. (3.5).

Perhaps the most widely used description of a wave, however, is still another form of Eq. (3.5). The quantities **angular frequency**  $\omega$  and **wave number**  $k$  are defined by the formulas

**Angular frequency**

$$\omega = 2\pi\nu \quad (3.7)$$

**Wave number**

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p} \quad (3.8)$$

The unit of  $\omega$  is the radian per second and that of  $k$  is the radian per meter. Angular frequency gets its name from uniform circular motion, where a particle that moves around a circle  $\nu$  times per second sweeps out  $2\pi\nu$  rad/s. The wave number is equal to the number of radians corresponding to a wave train 1 m long, since there are  $2\pi$  rad in one complete wave.

In terms of  $\omega$  and  $k$ , Eq. (3.5) becomes

**Wave formula**

$$y = A \cos (\omega t - kx) \quad (3.9)$$

In three dimensions  $k$  becomes a vector  $\mathbf{k}$  normal to the wave fronts and  $x$  is replaced by the radius vector  $\mathbf{r}$ . The scalar product  $\mathbf{k} \cdot \mathbf{r}$  is then used instead of  $kx$  in Eq. (3.9).

$$u(\mathbf{r}, t) = \cos (\omega t - \mathbf{k} \cdot \mathbf{r})$$

## PHASE AND GROUP VELOCITIES

*A group of waves need not have the same velocity as the waves themselves*

The amplitude of the de Broglie waves that correspond to a moving body reflects the probability that it will be found at a particular place at a particular time. It is clear that de Broglie waves cannot be represented simply by a formula resembling Eq. (3.9), which describes an indefinite series of waves all with the same amplitude  $A$ . Instead, we expect the wave representation of a moving body to correspond to a **wave packet**, or **wave group**, like that shown in Fig. 3.3, whose waves have amplitudes upon which the likelihood of detecting the body depends.

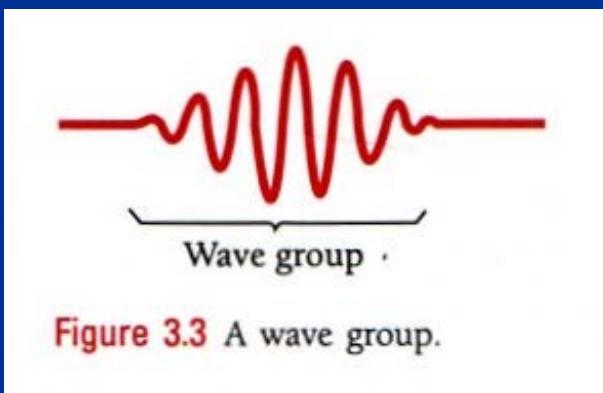
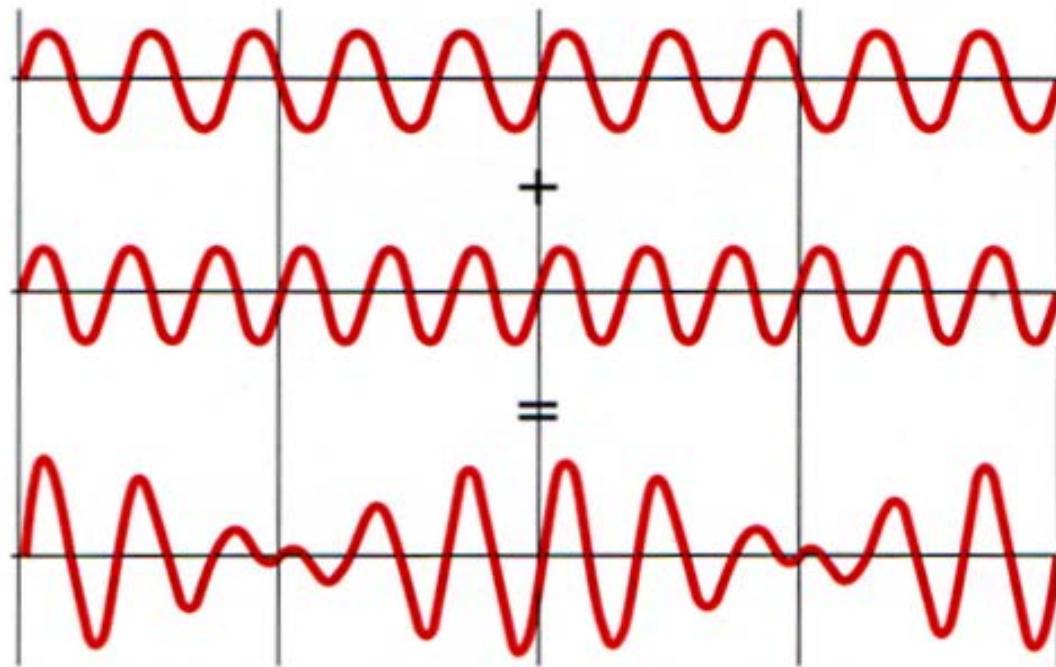


Figure 3.3 A wave group.

A familiar example of how wave groups come into being is the case of **beats**. When two sound waves of the same amplitude but of slightly different frequencies are produced simultaneously, the sound we hear has a frequency equal to the average of the two original frequencies and its amplitude rises and falls periodically. The amplitude fluctuations occur as many times per second as the difference between the two original frequencies. If the original sounds have frequencies of, say, 440 and 442 Hz, we will hear a fluctuating sound of frequency 441 Hz with two loudness peaks, called beats, per second. The production of beats is illustrated in Fig. 3.4.



**Figure 3.4** Beats are produced by the superposition of two waves with different frequencies.

## PHASE AND GROUP VELOCITIES

A way to mathematically describe a wave group, then, is in terms of a superposition of individual waves of different wavelengths whose interference with one another results in the variation in amplitude that defines the group shape. If the velocities of the waves are the same, the velocity with which the wave group travels is the common phase velocity. However, if the phase velocity varies with wavelength, an effect called dispersion, the different individual waves do not proceed together. As a result the wave group has a velocity different from the phase velocities of the waves that make it up. This is the case with de Broglie waves.

It is not hard to find the velocity  $v_g$  with which a wave group travels. Let us suppose that the wave group arises from the combination of two waves that have the same amplitude  $A$  but differ by an amount  $\Delta\omega$  in angular frequency and an amount  $\Delta k$  in wave number. We may represent the original waves by the formulas

$$y_1 = A \cos(\omega t - kx)$$

$$y_2 = A \cos[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

The resultant displacement  $y$  at any time  $t$  and any position  $x$  is the sum of  $y_1$  and  $y_2$ . With the help of the identity

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$$

and the relation

$$\cos(-\theta) = \cos \theta$$

we find that

$$y = y_1 + y_2$$

$$= 2A \cos \frac{1}{2}[(2\omega + \Delta\omega)t - (2k + \Delta k)x] \cos \frac{1}{2}(\Delta\omega t - \Delta k x)$$

Since  $\Delta\omega$  and  $\Delta k$  are small compared with  $\omega$  and  $k$  respectively,

$$2\omega + \Delta\omega \approx 2\omega$$

$$2k + \Delta k \approx 2k$$

and so

**Beats**

$$y = 2A \cos(\omega t - kx) \cos \left( \frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x \right) \quad (3.10)$$

**Beats**

$$y = 2A \cos(\omega t - kx) \cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \quad (3.10)$$

Equation (3.10) represents a wave of angular frequency  $\omega$  and wave number  $k$  that has superimposed upon it a modulation of angular frequency  $\frac{1}{2}\Delta\omega$  and of wave number  $\frac{1}{2}\Delta k$ .

The effect of the modulation is thus to produce successive wave groups, as in Fig. 3.4. The phase velocity  $v_p$  is

**Phase velocity**

$$v_p = \frac{\omega}{k} \quad (3.11)$$

and the velocity  $v_g$  of the wave groups is

**Group velocity**

$$v_g = \frac{\Delta\omega}{\Delta k} \quad (3.12)$$

When  $\omega$  and  $k$  have continuous spreads instead of the two values in the preceding discussion, the group velocity is instead given by

**Group velocity**

$$v_g = \frac{d\omega}{dk} \quad (3.13)$$

Depending on how phase velocity varies with wave number in a particular situation, the group velocity may be less or greater than the phase velocities of its member waves. If the phase velocity is the same for all wavelengths, as is true for light waves in empty space, the group and phase velocities are the same.

The angular frequency and wave number of the de Broglie waves associated with a body of rest mass  $m_0$  moving with the velocity  $v$  are

**Angular frequency  
of de Broglie  
waves**

$$\begin{aligned}\omega &= 2\pi\nu = \frac{2\pi mc^2}{h} \\ &= \frac{2\pi m_0 c^2}{h\sqrt{1 - v^2/c^2}}\end{aligned} \quad (3.14)$$

**Wave number of  
de Broglie waves**

$$\begin{aligned}k &= \frac{2\pi}{\lambda} = \frac{2\pi mv}{h} \\ &= \frac{2\pi m_0 v}{h\sqrt{1 - v^2/c^2}}\end{aligned} \quad (3.15)$$

Both  $\omega$  and  $k$  are functions of the body's velocity  $v$ .

The group velocity  $v_g$  of the de Broglie waves associated with the body is

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

Now  $\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h(1 - v^2/c^2)^{3/2}}$        $\frac{dk}{dv} = \frac{2\pi m_0}{h(1 - v^2/c^2)^{3/2}}$

and so the group velocity turns out to be

**De Broglie group  
velocity**

$$v_g = v \quad (3.16)$$

The de Broglie wave group associated with a moving body travels with the same velocity as the body.

The phase velocity  $v_p$  of de Broglie waves is, as we found earlier,

**De Broglie phase  
velocity**

$$v_p = \frac{\omega}{k} = \frac{c^2}{v} \quad (3.3)$$

This exceeds both the velocity of the body  $v$  and the velocity of light  $c$ , since  $v < c$ . However,  $v_p$  has no physical significance because it is the motion of the wave group, not the motion of the individual waves that make up the group, that corresponds to the motion of the body, and  $v_g < c$  as it should be. The fact that  $v_p > c$  for de Broglie waves therefore does not violate special relativity.

# PARTICLE DIFFRACTION

*An experiment that confirms the existence of de Broglie waves*

A wave effect with no analog in the behavior of newtonian particles is diffraction. In 1927 Clinton Davisson and Lester Germer in the United States and G. P. Thomson in England independently confirmed de Broglie's hypothesis by demonstrating that electron beams are diffracted when they are scattered by the regular atomic arrays of crystals.

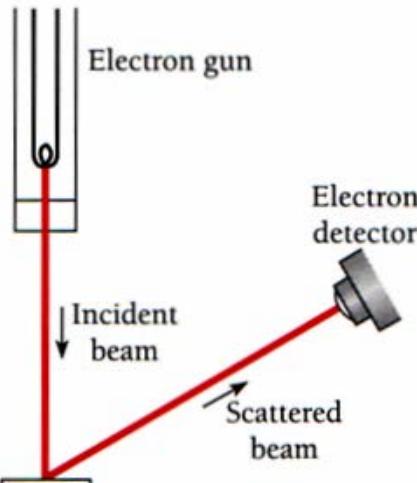


Figure 3.5

The Davisson-Germer experiment.

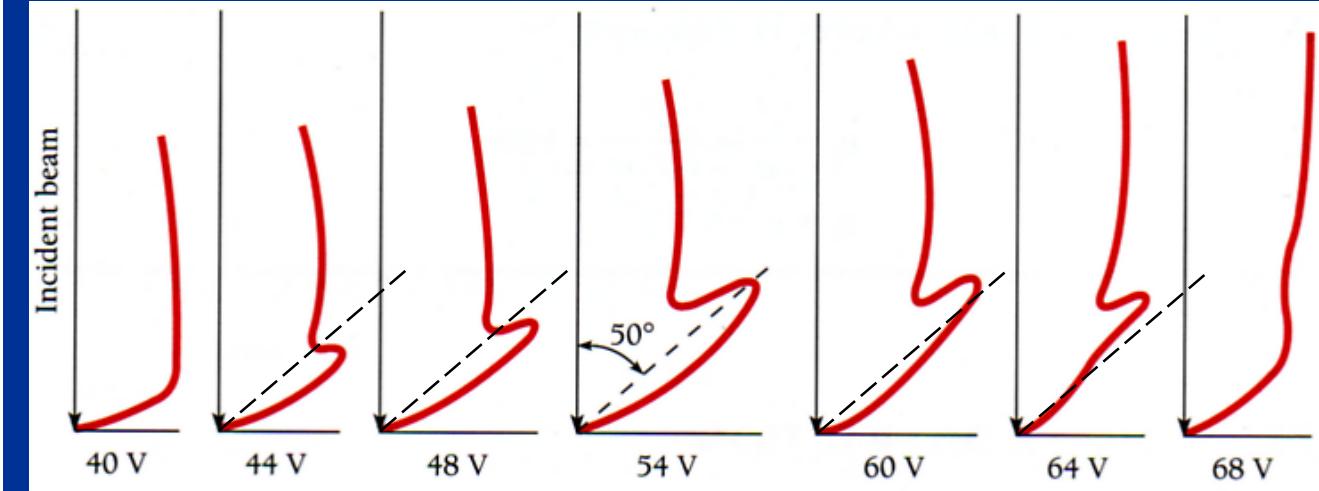


Figure 3.6 Results of the Davisson-Germer experiment. (after baking Ni)

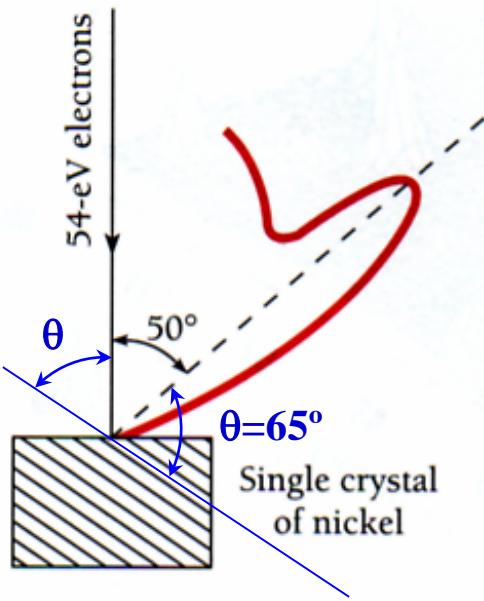
Nickel

Initially, continuous variations of intensity were observed, but after heating Ni, peaks appeared.

What is the reason for this new effect?

Why did it not appear until after the nickel target was baked?

De Broglie's hypothesis suggested that electron waves were being diffracted by the target, much as x-rays are diffracted by planes of atoms in a crystal. This idea received support when it was realized that heating a block of nickel at high temperature causes the many small individual crystals of which it is normally composed to form into a single large crystal, all of whose atoms are arranged in a regular lattice.



**Figure 3.7** The diffraction of the de Broglie waves by the target is responsible for the results of Davisson and Germer.

The Bragg equation for maxima in the diffraction pattern is

$$n\lambda = 2d \sin \theta \quad (2.13)$$

Here  $d = 0.091 \text{ nm}$  (measured from x-ray diffraction) and  $\theta = 65^\circ$ .

For  $n = 1$  the de Broglie wavelength  $\lambda$  of the diffracted electrons is

$$\lambda = 2d \sin \theta = (2)(0.091 \text{ nm})(\sin 65^\circ) = 0.165 \text{ nm}$$

**Now we use the de Broglie formula  $\lambda=h/mv$  to find the expected wavelength of the electrons.**

**The electron kinetic energy of 54 eV in this experiment is small compared to its rest energy  $m_0c^2$  (which is 0.51 MeV), so we can ignore relativistic effect and the kinetic energy of electron can be written as:**

$$KE = \frac{1}{2}mv^2$$

the electron momentum  $mv$  is

$$\begin{aligned}mv &= \sqrt{2mKE} \quad (KE = \frac{1}{2}mv^2) \\&= \sqrt{(2)(9.1 \times 10^{-31} \text{ kg})(54 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})} \\&= 4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s}\end{aligned}$$

The electron wavelength is therefore

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s}} = 1.66 \times 10^{-10} \text{ m} = \underline{\underline{0.166 \text{ nm}}}$$

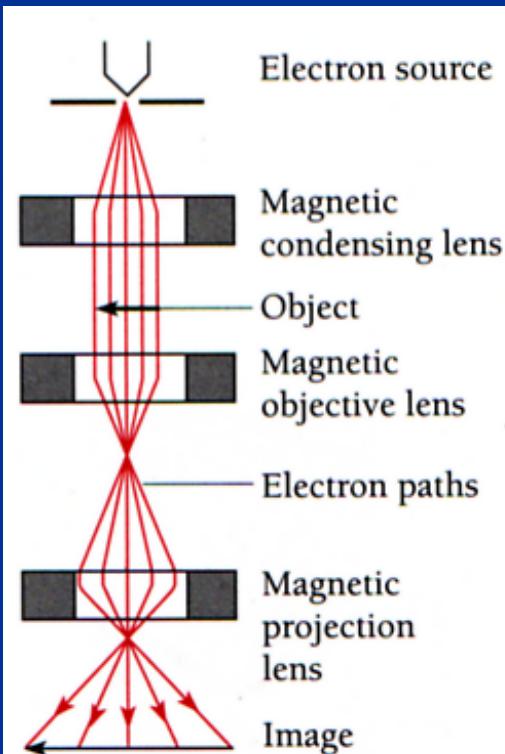
which agrees well with the observed wavelength of 0.165 nm. The Davisson-Germer experiment thus directly verifies de Broglie's hypothesis of the wave nature of moving bodies.

## Electron Microscopes

The wave nature of moving electrons is the basis of the electron microscope, the first of which was built in 1932. The resolving power of any optical instrument, which is limited by diffraction, is proportional to the wavelength of whatever is used to illuminate the specimen. In the case of a good microscope that uses visible light, the maximum useful magnification is about  $500 \times$ ; higher magnifications give larger images but do not reveal any more detail. Fast electrons, however, have wavelengths very much shorter than those of visible light and are easily controlled by electric and magnetic fields because of their charge. X-rays also have short wavelengths, but it is not (yet?) possible to focus them adequately.

In an electron microscope, current-carrying coils produce magnetic fields that act as lenses to focus an electron beam on a specimen and then produce an enlarged image on a fluorescent screen or photographic plate (Fig. 3.8). To prevent the beam from being scattered and thereby blurring the image, a thin specimen is used and the entire system is evacuated.

The technology of magnetic “lenses” does not permit the full theoretical resolution of electron waves to be realized in practice. For instance, 100-keV electrons have wavelengths of 0.0037 nm, but the actual resolution they can provide in an electron microscope may be only about 0.1 nm. However, this is still a great improvement on the  $\sim 200$ -nm resolution of an optical microscope, and magnifications of over  $1,000,000 \times$  have been achieved with electron microscopes.

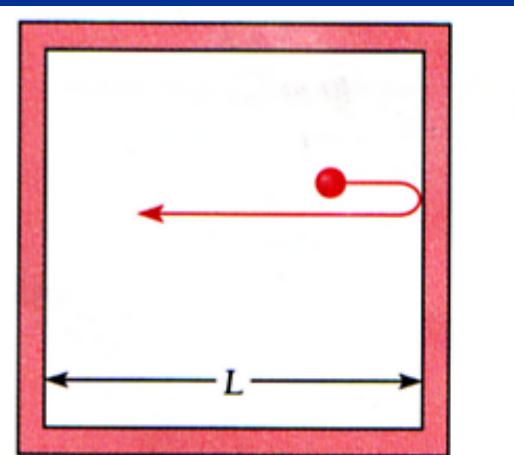


**Figure 3.8** Because the wavelengths of the fast electrons in an electron microscope are shorter than those of the light waves in an optical microscope, the electron microscope can produce sharp images at higher magnifications. The electron beam in an electron microscope is focused by magnetic fields.

## PARTICLE IN A BOX *Why the energy of a trapped particle is quantized*

The wave nature of a moving particle leads to some remarkable consequences when the particle is restricted to a certain region of space instead of being able to move freely.

The simplest case is that of a particle that bounces back and forth between the walls of a box, as in Fig. 3.9. We shall assume that the walls of the box are infinitely hard, so the particle does not lose energy each time it strikes a wall, and that its velocity is sufficiently small so that we can ignore relativistic considerations. Simple as it is, this model situation requires fairly elaborate mathematics in order to be properly analyzed, as we shall learn in Chap. 5. However, even a relatively crude treatment can reveal the essential results.

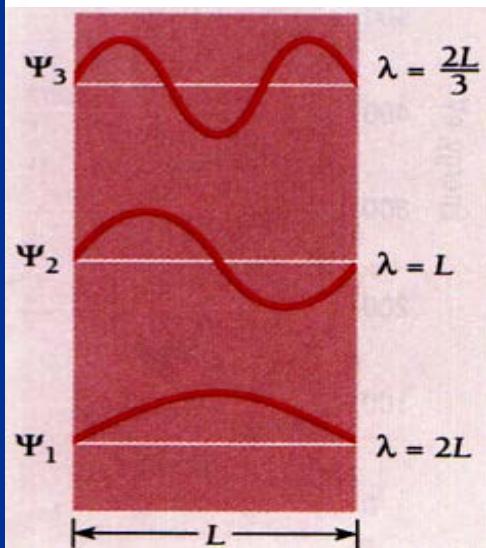


**Figure 3.9** A particle confined to a box of width  $L$ .

From a wave point of view, a particle trapped in a box is like a standing wave in a string stretched between the box's walls. In both cases the wave variable (transverse displacement for the string, wave function  $\Psi$  for the moving particle) must be 0 at the walls, since the waves stop there. The possible de Broglie wavelengths of the particle in the box therefore are determined by the width  $L$  of the box, as in Fig. 3.10. The longest wavelength is specified by  $\lambda = 2L$ , the next by  $\lambda = L$ , then  $\lambda = 2L/3$ , and so forth. The general formula for the permitted wavelengths is

**De Broglie  
wavelengths of  
trapped particle**

$$\lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, \dots \quad (3.17)$$



**Figure 3.10** Wave functions of a particle trapped in a box  $L$  wide.

Because  $mv = h/\lambda$ , the restrictions on de Broglie wavelength  $\lambda$  imposed by the width of the box are equivalent to limits on the momentum of the particle and, in turn, to limits on its kinetic energy. The kinetic energy of a particle of momentum  $mv$  is

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{2m\lambda^2}$$

The permitted wavelengths are  $\lambda_n = 2L/n$ , and so, because the particle has no potential energy in this model, the only energies it can have are

**Particle in a box**

$$E_n = \frac{n^2h^2}{8mL^2} \quad n = 1, 2, 3, \dots \quad (3.18)$$

Each permitted energy is called an energy level, and the integer  $n$  that specifies an energy level  $E_n$  is called its **quantum number**.

We can draw three general conclusions from Eq. (3.18). These conclusions apply to any particle confined to a certain region of space (even if the region does not have a well-defined boundary), for instance an atomic electron held captive by the attraction of the positively charged nucleus.

- 1 A trapped particle cannot have an arbitrary energy, as a free particle can. The fact of its confinement leads to restrictions on its wave function that allow the particle to have only certain specific energies and no others. Exactly what these energies are depends on the mass of the particle and on the details of how it is trapped.
- 2 A trapped particle cannot have zero energy. Since the de Broglie wavelength of the particle is  $\lambda = h/mv$ , a speed of  $v = 0$  means an infinite wavelength. But there is no way to reconcile an infinite wavelength with a trapped particle, so such a particle must have at least some kinetic energy. The exclusion of  $E = 0$  for a trapped particle, like the limitation of  $E$  to a set of discrete values, is a result with no counterpart in classical physics, where all non-negative energies, including zero, are allowed.
- 3 Because Planck's constant is so small—only  $6.63 \times 10^{-34} \text{ J} \cdot \text{s}$ —quantization of energy is conspicuous only when  $m$  and  $L$  are also small. This is why we are not aware of energy quantization in our own experience. Two examples will make this clear.

### Example 3.4

An electron is in a box 0.10 nm across, which is the order of magnitude of atomic dimensions. Find its permitted energies.

#### Solution

Here  $m = 9.1 \times 10^{-31}$  kg and  $L = 0.10$  nm =  $1.0 \times 10^{-10}$  m, so that the permitted electron energies are

$$E_n = \frac{(n^2)(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(8)(9.1 \times 10^{-31} \text{ kg})(1.0 \times 10^{-10} \text{ m})^2} = 6.0 \times 10^{-18} n^2 \text{ J}$$
$$= 38n^2 \text{ eV}$$

$n = 1$	$E_1 = 38 \text{ eV}$
$n = 2$	$E_2 = 152 \text{ eV}$
$n = 3$	$E_3 = 342 \text{ eV}$
$n = 4$	$E_4 = 608 \text{ eV}$

Energy quantization is prominent and it is indeed also the case for an atomic electron

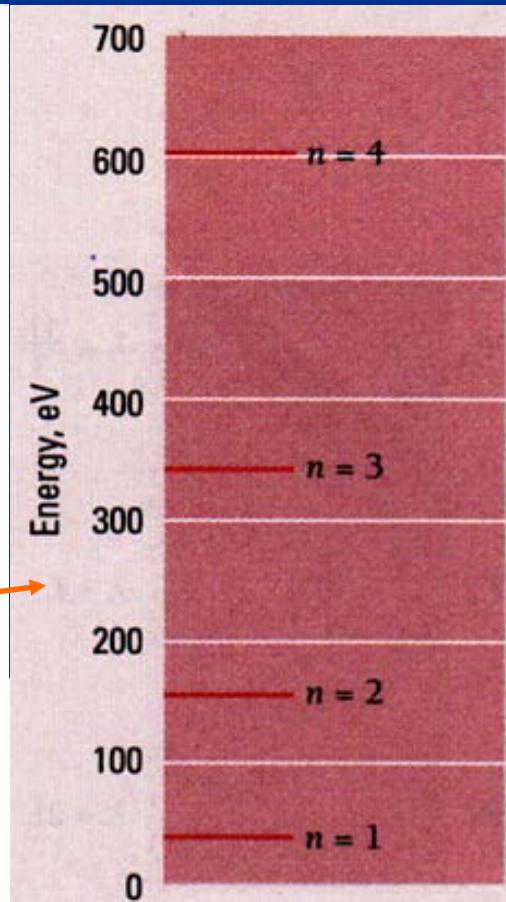


Figure 3.11 Energy levels of an electron confined to a box 0.1 nm wide.

## Example 3.5

A 10-g marble is in a box 10 cm across. Find its permitted energies.

### Solution

With  $m = 10 \text{ g} = 1.0 \times 10^{-2} \text{ kg}$  and  $L = 10 \text{ cm} = 1.0 \times 10^{-1} \text{ m}$ ,

$$E_n = \frac{(n^2)(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(8)(1.0 \times 10^{-2} \text{ kg})(1.0 \times 10^{-1} \text{ m})^2}$$
$$= 5.5 \times 10^{-64} n^2 \text{ J} = 3.4 \times 10^{-45} n^2 \text{ eV}$$

The minimum energy the marble can have is  $5.5 \times 10^{-64} \text{ J}$ , corresponding to  $n = 1$ . A marble with this kinetic energy has a speed of only  $3.3 \times 10^{-31} \text{ m/s}$  and therefore cannot be experimentally distinguished from a stationary marble. A reasonable speed a marble might have is, say,  $\frac{1}{2} \text{ m/s}$ —which corresponds to the energy level of quantum number  $n = 10^{30}$ ! The permissible energy levels are so very close together, then, that there is no way to determine whether the marble can take on only those energies predicted by Eq. (3.18) or any energy whatever. Hence in the domain of everyday experience, quantum effects are imperceptible, which accounts for the success of newtonian mechanics in this domain.

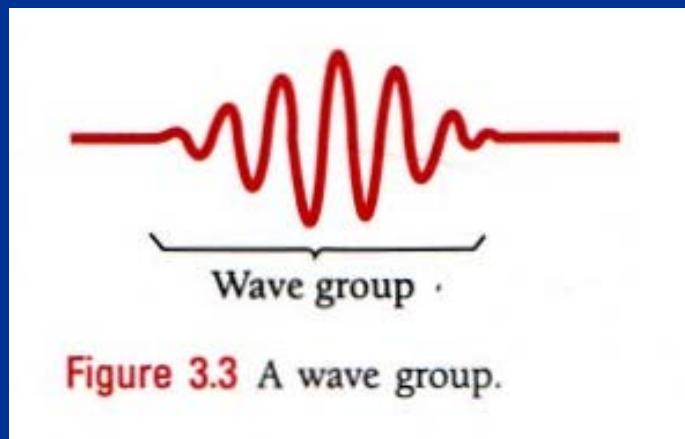
$$E_{n+1}/E_n = (n+1)^2/n^2 = 1 + (2/n) + (1/n^2); \text{ if } n=1000, E_{n+1}/E_n = 1.002$$

## UNCERTAINTY PRINCIPLE 1

*We cannot know the future because we cannot know the present*

To regard a moving particle as a wave group implies that there are fundamental limits to the accuracy with which we can measure such “particle” properties as position and momentum.

To make clear what is involved, let us look at the wave group of Fig. 3.3. The particle that corresponds to this wave group may be located anywhere within the group at a given time. Of course, the probability density  $|\Psi|^2$  is a maximum in the middle of the group, so it is most likely to be found there. Nevertheless, we may still find the particle anywhere that  $|\Psi|^2$  is not actually 0.



**Figure 3.3** A wave group.

The narrower its wave group, the more precisely a particle's position can be specified (Fig. 3.12a). However, the wavelength of the waves in a narrow packet is not well defined; there are not enough waves to measure  $\lambda$  accurately. This means that since  $\lambda = h/mv$ , the particle's momentum  $mv$  is not a precise quantity. If we make a series of momentum measurements, we will find a broad range of values.

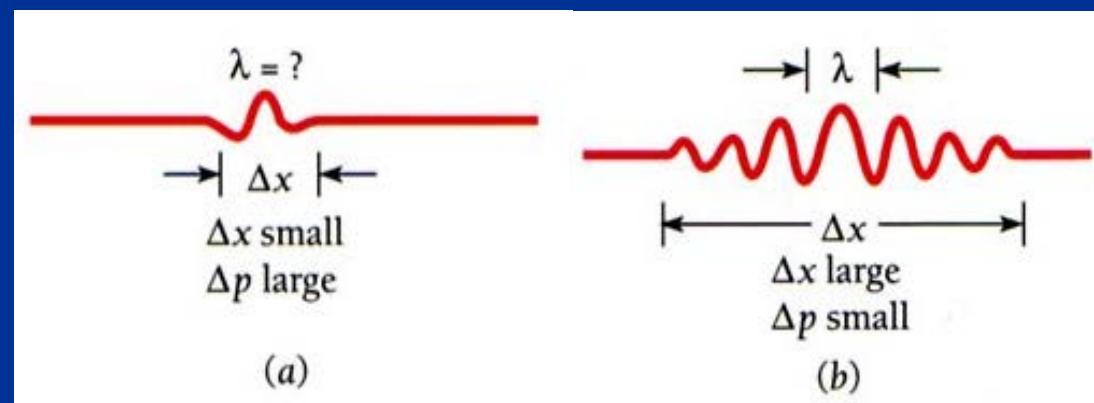
On the other hand, a wide wave group, such as that in Fig. 3.12b, has a clearly defined wavelength. The momentum that corresponds to this wavelength is therefore a precise quantity, and a series of measurements will give a narrow range of values. But where is the particle located? The width of the group is now too great for us to be able to say exactly where it is at a given time.

Thus we have the **uncertainty principle**:

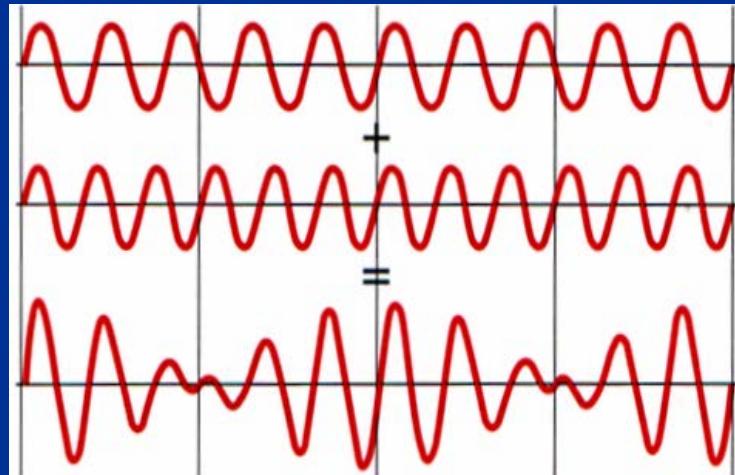
It is impossible to know both the exact position and exact momentum of an object at the same time.

This principle, which was discovered by Werner Heisenberg in 1927, is one of the most significant of physical laws.

Figure 3.12

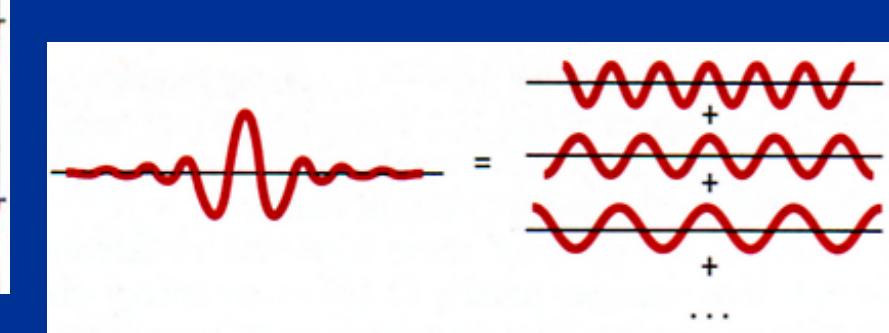


A formal analysis supports the above conclusion and enables us to put it on a quantitative basis. The simplest example of the formation of wave groups is that given in Sec. 3.4, where two wave trains slightly different in angular frequency  $\omega$  and wave number  $k$  were superposed to yield the series of groups shown in Fig. 3.4. A moving body corresponds to a single wave group, not a series of them, but a single wave group can also be thought of in terms of the superposition of trains of harmonic waves. However, an infinite number of wave trains with different frequencies, wave numbers, and amplitudes is required for an isolated group of arbitrary shape, as in Fig. 3.13.



**Figure 3.4**

Beats are produced by the superposition of two waves with different frequencies.

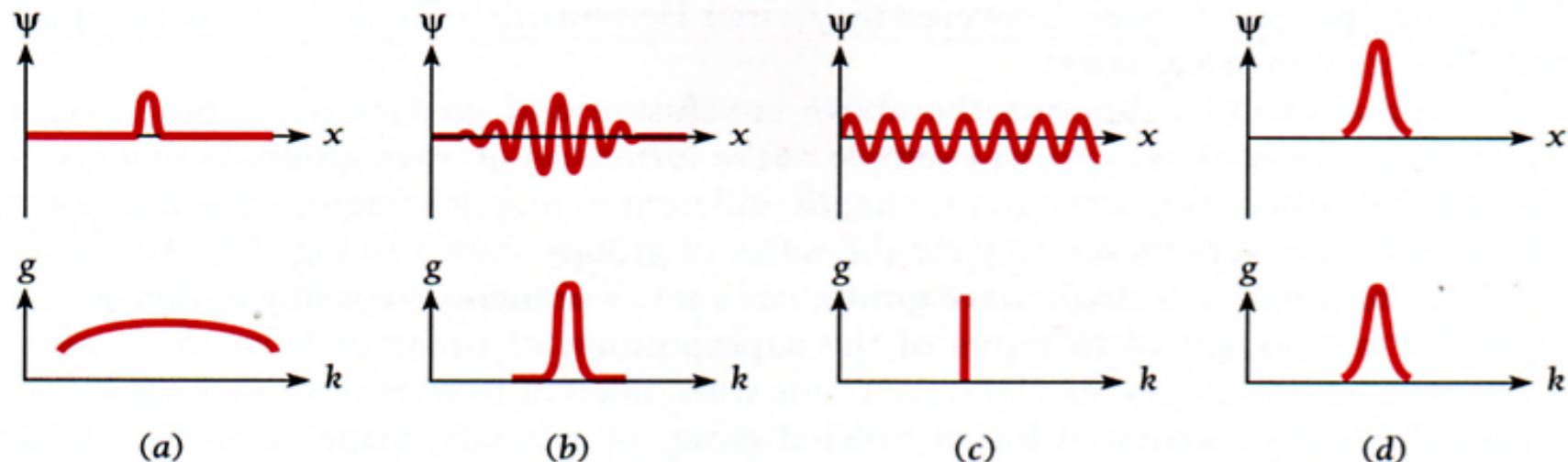


**Figure 3.13** An isolated wave group is the result of superposing an infinite number of waves with different wavelengths. The narrower the wave group, the greater the range of wavelengths involved. A narrow de Broglie wave group thus means a well-defined position ( $\Delta x$  smaller) but a poorly defined wavelength and a large uncertainty  $\Delta p$  in the momentum of the particle the group represents. A wide wave group means a more precise momentum but a less precise position.

At a certain time  $t$ , the wave group  $\Psi(x)$  can be represented by the **Fourier integral**

$$\Psi(x) = \int_0^{\infty} g(k) \cos kx \, dk \quad (3.19)$$

where the function  $g(k)$  describes how the amplitudes of the waves that contribute to  $\Psi(x)$  vary with wave number  $k$ . This function is called the **Fourier transform** of  $\Psi(x)$ , and it specifies the wave group just as completely as  $\Psi(x)$  does. Figure 3.14 contains graphs of the Fourier transforms of a pulse and of a wave group. For comparison, the Fourier transform of an infinite train of harmonic waves is also included. There is only a single wave number in this case, of course.



**Figure 3.14** The wave functions and Fourier transforms for (a) a pulse, (b) a wave group, (c) an wave train, and (d) a gaussian distribution. A brief disturbance needs a broader range of frequencies to describe it than a disturbance of greater duration. The Fourier transform of a gaussian function is also a gaussian function.

Strictly speaking, the wave numbers needed to represent a wave group extend from  $k = 0$  to  $k = \infty$ , but for a group whose length  $\Delta x$  is finite, the waves whose amplitudes  $g(k)$  are appreciable have wave numbers that lie within a finite interval  $\Delta k$ . As Fig. 3.14 indicates, the shorter the group, the broader the range of wave numbers needed to describe it, and vice versa.

The relationship between the distance  $\Delta x$  and the wave-number spread  $\Delta k$  depends upon the shape of the wave group and upon how  $\Delta x$  and  $\Delta k$  are defined. The minimum value of the product  $\Delta x \Delta k$  occurs when the envelope of the group has the familiar bell shape of a gaussian function. In this case the Fourier transform happens to be a gaussian function also. If  $\Delta x$  and  $\Delta k$  are taken as the standard deviations of the respective functions  $\Psi(x)$  and  $g(k)$ , then this minimum value is  $\Delta x \Delta k = \frac{1}{2}$ . Because wave groups in general do not have gaussian forms, it is more realistic to express the relationship between  $\Delta x$  and  $\Delta k$  as

$$\Delta x \Delta k \geq \frac{1}{2} \quad (3.20)$$

The de Broglie wavelength of a particle of momentum  $p$  is  $\lambda = h/p$  and the corresponding wave number is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

In terms of wave number the particle's momentum is therefore

$$p = \frac{hk}{2\pi}$$

Hence an uncertainty  $\Delta k$  in the wave number of the de Broglie waves associated with the particle results in an uncertainty  $\Delta p$  in the particle's momentum according to the formula

$$\Delta p = \frac{h \Delta k}{2\pi}$$

Since  $\Delta x \Delta k \geq \frac{1}{2}$ ,  $\Delta k \geq 1/(2\Delta x)$  and

Uncertainty principle

$$\boxed{\Delta x \Delta p \geq \frac{h}{4\pi}} \quad (3.21)$$

This equation states that the product of the uncertainty  $\Delta x$  in the position of an object at some instant and the uncertainty  $\Delta p$  in its momentum component in the  $x$  direction at the same instant is equal to or greater than  $h/4\pi$ .

## Uncertainty principle

$$\Delta x \Delta p \geq \frac{h}{4\pi} \quad (3.21)$$

If we arrange matters so that  $\Delta x$  is small, corresponding to a narrow wave group, then  $\Delta p$  will be large. If we reduce  $\Delta p$  in some way, a broad wave group is inevitable and  $\Delta x$  will be large.

These uncertainties are due not to inadequate apparatus but to the imprecise character in nature of the quantities involved. Any instrumental or statistical uncertainties that arise during a measurement only increase the product  $\Delta x \Delta p$ . Since we cannot know exactly both where a particle is right now and what its momentum is, we cannot say anything definite about where it will be in the future or how fast it will be moving then.

*We cannot know the future for sure because we cannot know the present for sure.* But our ignorance is not total: we can still say that the particle is more likely to be in one place than another and that its momentum is more likely to have a certain value than another.

## H-Bar

The quantity  $h/2\pi$  appears often in modern physics because it turns out to be the basic unit of angular momentum. It is therefore customary to abbreviate  $h/2\pi$  by the symbol  $\hbar$  ("h-bar"):

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \text{ J} \cdot \text{s}$$

In the remainder of this book  $\hbar$  is used in place of  $h/2\pi$ . In terms of  $\hbar$ , the uncertainty principle becomes

**Uncertainty  
principle**

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}} \quad (3.22)$$

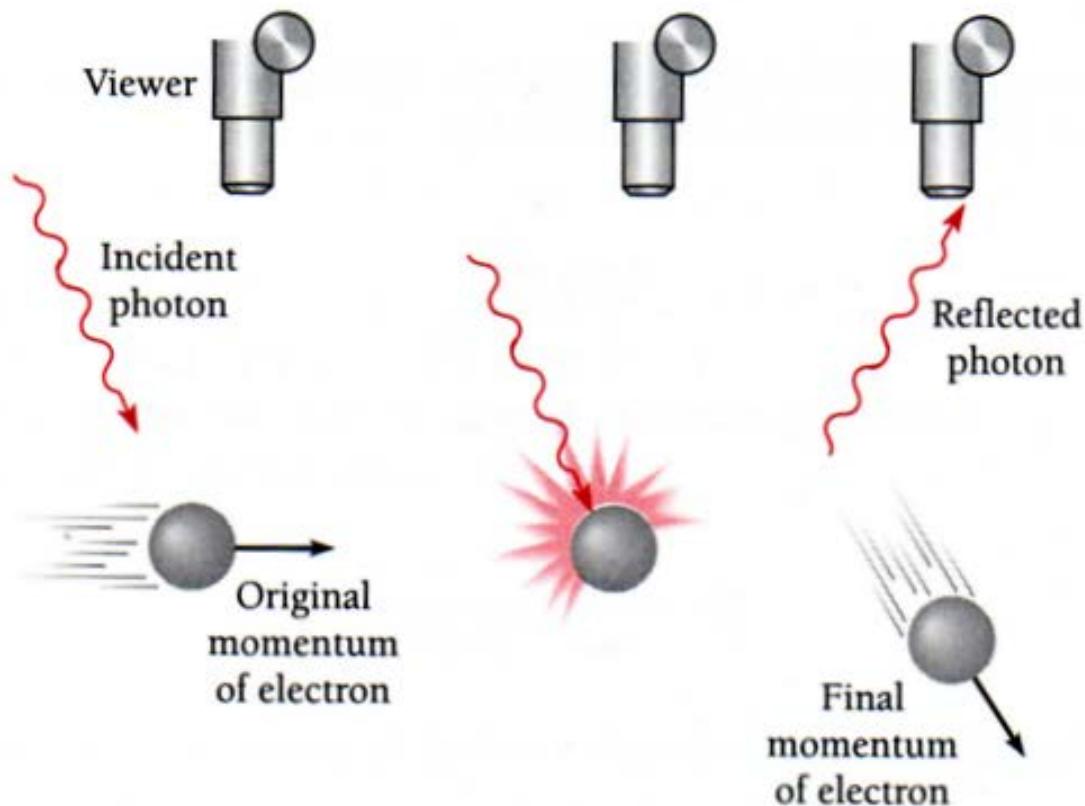
## **3.8 UNCERTAINTY PRINCIPLE II**

*A particle approach gives the same result*

The uncertainty principle can be arrived at from the point of view of the particle properties of waves as well as from the point of view of the wave properties of particles.

We might want to measure the position and momentum of an object at a certain moment. To do so, we must touch it with something that will carry the required information back to us. That is, we must poke it with a stick, shine light on it, or perform some similar act. The measurement process itself thus requires that the object be interfered with in some way. If we consider such interferences in detail, we are led to the same uncertainty principle as before even without taking into account the wave nature of moving bodies.

Suppose we look at an electron using light of wavelength  $\lambda$ , as in Fig. 3.16. Each photon of this light has the momentum  $h/\lambda$ . When one of these photons bounces off the electron (which must happen if we are to “see” the electron), the electron’s original momentum will be changed.



**Figure 3.16** An electron cannot be observed without changing its momentum.

The exact amount of the change  $\Delta p$  cannot be predicted, but it will be of the same order of magnitude as the photon momentum  $h/\lambda$ . Hence

$$\Delta p \approx \frac{h}{\lambda} \quad (3.23)$$

The longer the wavelength of the observing photon, the smaller the uncertainty in the electron's momentum.

Because light is a wave phenomenon as well as a particle phenomenon, we cannot expect to determine the electron's location with perfect accuracy regardless of the instrument used. A reasonable estimate of the minimum uncertainty in the measurement might be one photon wavelength, so that

$$\Delta x \geq \lambda \quad (3.24)$$

The shorter the wavelength, the smaller the uncertainty in location. If we use light of short wavelength to increase the accuracy of the position measurement, there will be a corresponding decrease in the accuracy of the momentum measurement because the higher photon momentum will disturb the electron's motion to a greater extent. Light of long wavelength will give a more accurate momentum but a less accurate position.

Combining Eqs. (3.23) and (3.24) gives

$$\Delta x \Delta p \geq h \quad (3.25)$$

This result is consistent with Eq. (3.22),  $\Delta x \Delta p \geq \hbar/2$ .

Arguments like the preceding one, although superficially attractive, must be approached with caution. The argument above implies that the electron can possess a definite position and momentum at any instant and that it is the measurement process that introduces the indeterminacy in  $\Delta x \Delta p$ . On the contrary, this **indeterminacy is inherent in the nature of a moving body**. The justification for the many “derivations” of this kind is first, they show it is impossible to imagine a way around the uncertainty principle; and second, they present a view of the principle that can be appreciated in a more familiar context than that of wave groups.

## APPLYING THE UNCERTAINTY PRINCIPLE

*A useful tool, not just a negative statement*

Planck's constant  $\hbar$  is so small that the limitations imposed by the uncertainty principle are significant only in the realm of the atom. On such a scale, however, this principle is of great help in understanding many phenomena. It is worth keeping in mind that the lower limit of  $\hbar/2$  for  $\Delta x \Delta p$  is rarely attained. More usually  $\Delta x \Delta p \geq \hbar$ , or even (as we just saw)  $\Delta x \Delta p \geq h$ .

## Example 3.7

A typical atomic nucleus is about  $5.0 \times 10^{-15}$  m in radius. Use the uncertainty principle to place a lower limit on the energy an electron must have if it is to be part of a nucleus.

### Solution

Letting  $\Delta x = 5.0 \times 10^{-15}$  m we have

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq \frac{1.054 \times 10^{-34} \text{ J} \cdot \text{s}}{(2)(5.0 \times 10^{-15} \text{ m})} \geq 1.1 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

If this is the uncertainty in a nuclear electron's momentum, the momentum  $p$  itself must be at least comparable in magnitude. An electron with such a momentum has a kinetic energy KE many times greater than its rest energy  $m_0 c^2$ . From Eq. (1.24) we see that we can let  $KE = pc$  here to a sufficient degree of accuracy. Therefore

$$KE = pc \geq (1.1 \times 10^{-20} \text{ kg} \cdot \text{m/s})(3.0 \times 10^8 \text{ m/s}) \geq 3.3 \times 10^{-12} \text{ J} \quad (= 20.6 \text{ MeV})$$

Since  $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$ , the kinetic energy of an electron must exceed 20 MeV if it is to be inside a nucleus. Experiments show that even the electrons associated with unstable atoms never have more than a fraction of this energy, and we conclude that nuclei do not contain electrons.

## Example 3.8

A hydrogen atom is  $5.3 \times 10^{-11}$  m in radius. Use the uncertainty principle to estimate the minimum energy an electron can have in this atom.

### Solution

Here we find that with  $\Delta x = 5.3 \times 10^{-11}$  m,

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq 9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s}$$

An electron whose momentum is of this order of magnitude behaves like a classical particle, and its kinetic energy is

$$KE = \frac{p^2}{2m} \geq \frac{(9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s})^2}{(2)(9.1 \times 10^{-31} \text{ kg})} \geq 5.4 \times 10^{-19} \text{ J}$$

which is 3.4 eV. The kinetic energy of an electron in the lowest energy level of a hydrogen atom is actually 13.6 eV.

## Energy and Time

Another form of the uncertainty principle concerns energy and time. We might wish to measure the energy  $E$  emitted during the time interval  $\Delta t$  in an atomic process. If the energy is in the form of em waves, the limited time available restricts the accuracy with which we can determine the frequency  $\nu$  of the waves. Let us assume that the minimum uncertainty in the number of waves we count in a wave group is one wave. Since the frequency of the waves under study is equal to the number of them we count divided by the time interval, the uncertainty  $\Delta\nu$  in our frequency measurement is

$$\Delta\nu \geq \frac{1}{\Delta t}$$

The corresponding energy uncertainty is

$$\Delta E = h \Delta\nu$$

and so

$$\Delta E \geq \frac{h}{\Delta t} \quad \text{or} \quad \Delta E \Delta t \geq h$$

A more precise calculation based on the nature of wave groups changes this result to

**Uncertainties in  
energy and time**

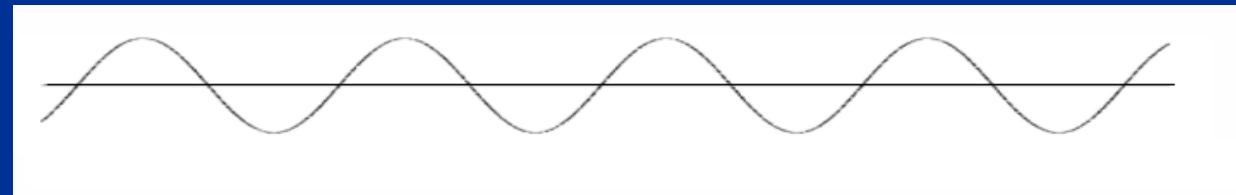
$$\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}} \quad (3.26)$$

Equation (3.26) states that the product of the uncertainty  $\Delta E$  in an energy measurement and the uncertainty  $\Delta t$  in the time at which the measurement is made is equal to or greater than  $\hbar/2$ . This result can be derived in other ways as well and is a general one not limited to em waves.

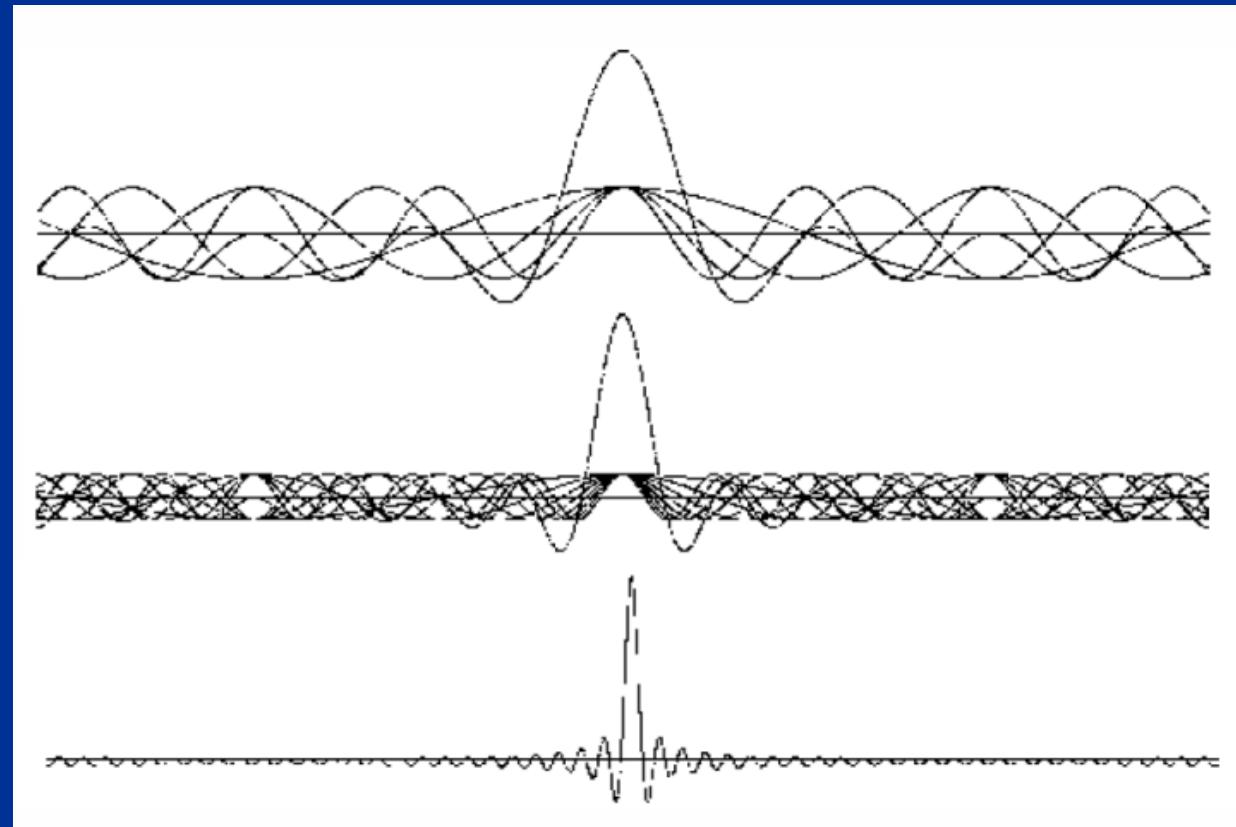




**A pure cosine wave extends infinitely in both directions with no change in the size of the peaks or valleys.**



**4 cosine waves and their sum**



**The sum of 32 cosine waves**

**An impulse has all frequencies**

## Gaussian Function

When a set of measurements is made of some quantity  $x$  in which the experimental errors are random, the result is often a gaussian distribution whose form is the bell-shaped curve shown in Fig. 3.15. The standard deviation  $\sigma$  of the measurements is a measure of the spread of  $x$  values about the mean of  $x_0$ , where  $\sigma$  equals the square root of the average of the squared deviations from  $x_0$ . If  $N$  measurements were made,

**Standard deviation**

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - x_0)^2}$$

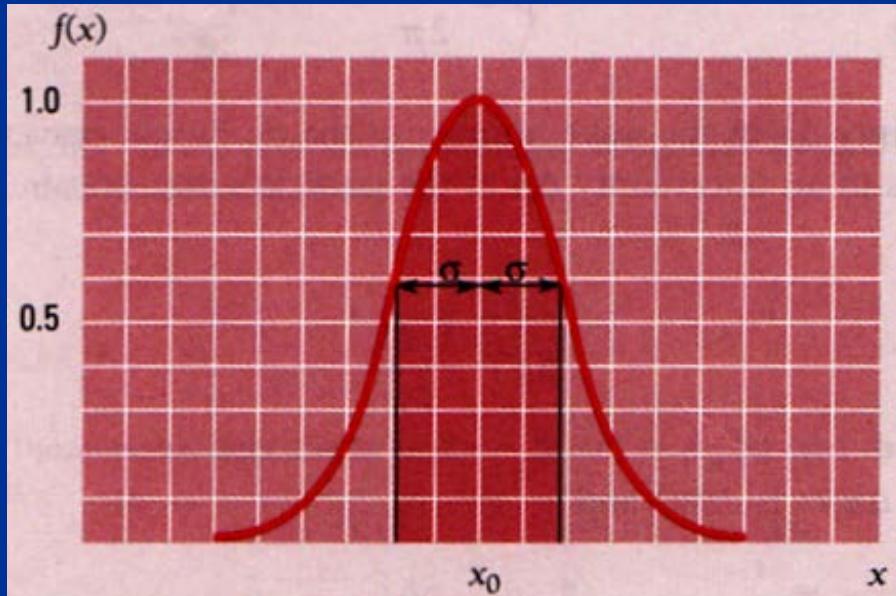
The width of a gaussian curve at half its maximum value is  $2.35\sigma$ .

The gaussian function  $f(x)$  that describes the above curve is given by

**Gaussian function**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_0)^2/2\sigma^2}$$

where  $f(x)$  is the probability that the value  $x$  be found in a particular measurement. Gaussian functions occur elsewhere in physics and mathematics as well.



**Figure 3.15** A gaussian distribution.

The probability that a measurement lie inside a certain range of  $x$  values, say between  $x_1$  and  $x_2$ , is given by the area of the  $f(x)$  curve between these limits. This area is the integral

$$P_{x_1 x_2} = \int_{x_1}^{x_2} f(x) dx$$

An interesting question is what fraction of a series of measurements has values within a standard deviation of the mean value  $x_0$ . In this case  $x_1 = x_0 - \sigma$  and  $x_2 = x_0 + \sigma$ , and

$$P_{x_0 \pm \sigma} = \int_{x_0 - \sigma}^{x_0 + \sigma} f(x) dx = 0.683$$

Hence 68.3 percent of the measurements fall in this interval, which is shaded in Fig. 3.15. A similar calculation shows that 95.4 percent of the measurements fall within two standard deviations of the mean value.