# 電子電工學
# Lecture 13

# Recap: AC Circuit Analysis

DC ← Complex Voltage, Current, Impedance ← AC

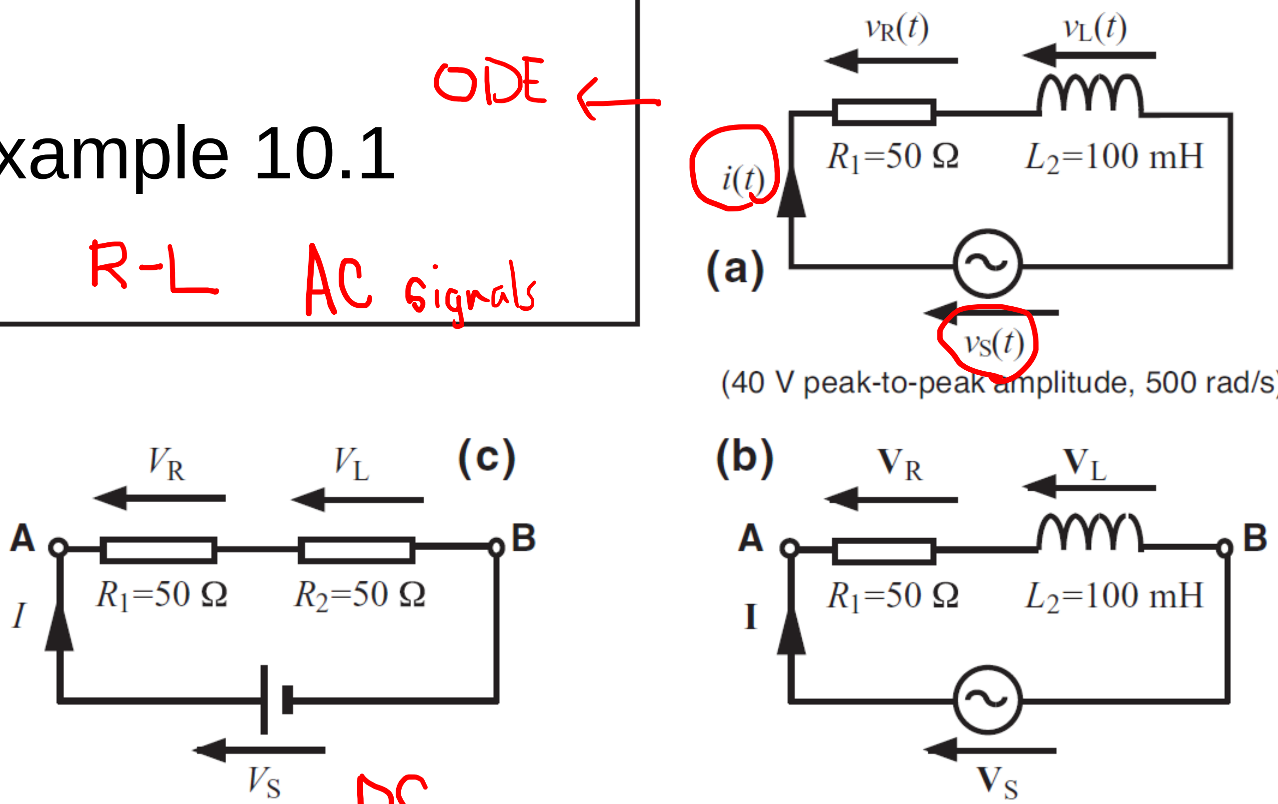
**Table 10.1** Similarities between the component relations and connection constraints in DC circuits and AC circuits

| DC Currents and voltages: representation | DC Currents and voltages: relation | | AC Currents and voltages: relation | AC Currents and voltages: representation |
|---|---|---|---|---|
| | | **Connections** | | |
| | $\sum_{node} I = 0$ | *at node* | $\sum_{node} \mathbf{I} = 0$ | |
| | $\sum_{loop} V = 0$ | *around loop* | $\sum_{loop} \mathbf{V} = 0$ | |
| | | **Components** | | |
| $R$ | $V = R\,I$ | *resistor* | $\mathbf{V} = R\,\mathbf{I}$ | $R$ |
| | | *capacitor* | $\mathbf{I} = j\omega C\,\mathbf{V}$ | $C$ |
| | | *inductor* | $\mathbf{V} = j\omega L\,\mathbf{I}$ | $L$ |
| | $V$ = constant | *voltage source* | $\mathbf{V}$ = constant | |
| | $I$ = constant | *current source* | $\mathbf{I}$ = constant | |

# Example 10.1

ODE ←

R-L AC signals




DC

Figure 10.4 Relevant to a demonstration of the similarities between DC and AC analysis

Known $\mathbb{V}_s$

Algebra ←

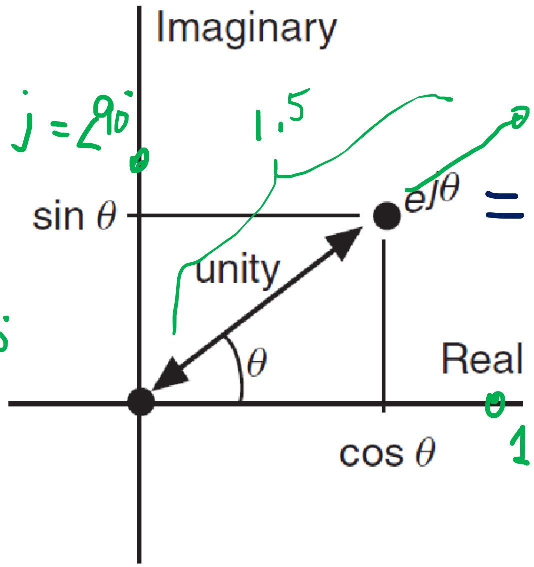
$$\mathbb{I} = \frac{\mathbb{V}_s}{\mathbb{Z}_{R_1} + \mathbb{Z}_{L_2}}$$

↳ $i(t) = \mathrm{Re}\left[\mathbb{I}\, e^{j\omega t}\right]$

# Recap: Simplified phasor notations

$\angle\theta \equiv e^{j\theta}$ (polar)

$1.5\angle\theta = 1.5(\cos\theta + j\sin\theta)$




$j = \angle 90°$

$1.5$

$e^{j\theta} = \cos\theta + j\sin\theta = 1\angle\theta$

$-1 = \angle 180°$

$1 + j0 = 1\angle 0°$

$(\alpha\angle\theta)(\beta\angle\phi) = \alpha\beta\angle(\theta+\phi)$

$\dfrac{\alpha\angle\theta}{\beta\angle\phi} = \dfrac{\alpha}{\beta}\angle(\theta-\phi)$

**Figure 10.1** Representation of $e^{j\theta}$ in the complex rane

$-j = \angle -90° = \angle 270°$

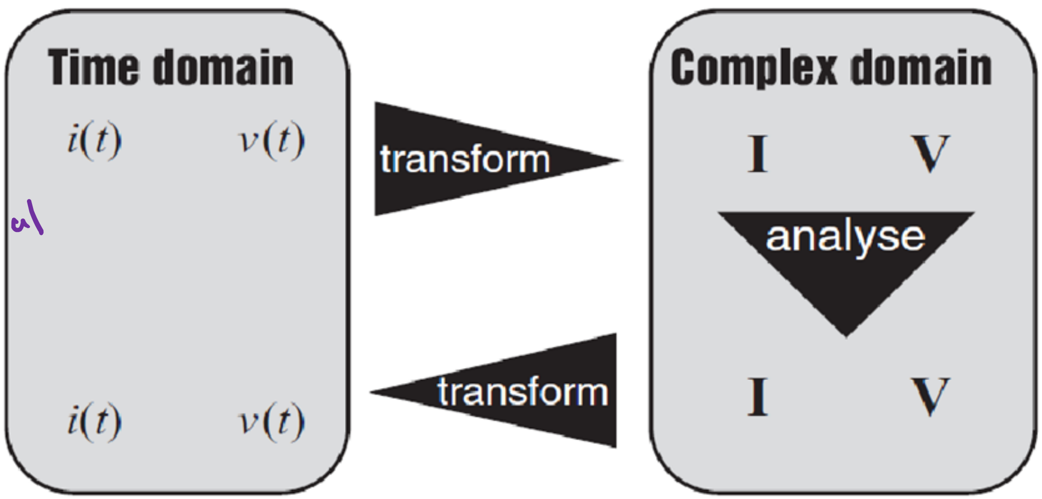
# Recap: Simplified AC analysis

Impedance $\leftarrow Z = R + jX$

$$\downarrow \; Y = 1/Z$$

Admitance $\leftarrow Y = G + jB$

Differential Equation




Algebra

**Figure 10.8** Illustrating the approach to AC analysis based on the use of complex currents and voltages

# Example 10.2 (a)

Find the impedance and admittance between the terminals A and B at a frequency of 1000 rad/s.

$\omega$

(a) $$Z_{AB} = 1\,k\Omega + \frac{1}{j\cdot 1000\cdot 10^{-6}}$$

Impedance

$$= 1\,k\Omega - j\,1\,k\Omega = (1-j)\,k\Omega$$

$\frac{1}{(\ )}$

$$Y_{AB} = \frac{1}{(1-j)\,k\Omega} = \left(\frac{1}{2} + j\frac{1}{2}\right) mS$$

Admittance

R $\qquad \frac{1}{j\omega C}$

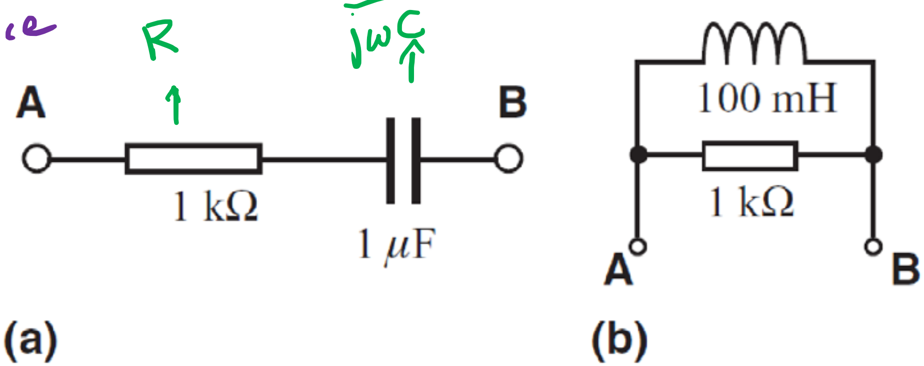



**Figure 10.9** Relevant to Example 10.2

# Example 10.2 (b)

Find the impedance and admittance between the terminals A and B at a frequency of 1000 rad/s.

Admittance

$$Y_{AB} = Y_R + Y_L = \left(\frac{1}{1K} + \frac{1}{j\cdot 1000 \cdot 100mH}\right) S$$

$$\frac{1}{(\,)}$$

$$= \left(\frac{1}{1K} + \frac{1}{j\cdot 100}\right) S = (1 - j10)\ mS$$

$$Z_{AB} = \frac{1}{(1-j\cdot 10)}\ k\Omega = \frac{1+j10}{101}\ k\Omega$$

$$\simeq 0.0099 + j0.099\ k\Omega$$

Impedance

$$\simeq (1k\Omega \,//\, j\omega 100m)$$

$$\simeq (9.9 + j99)\ \Omega$$

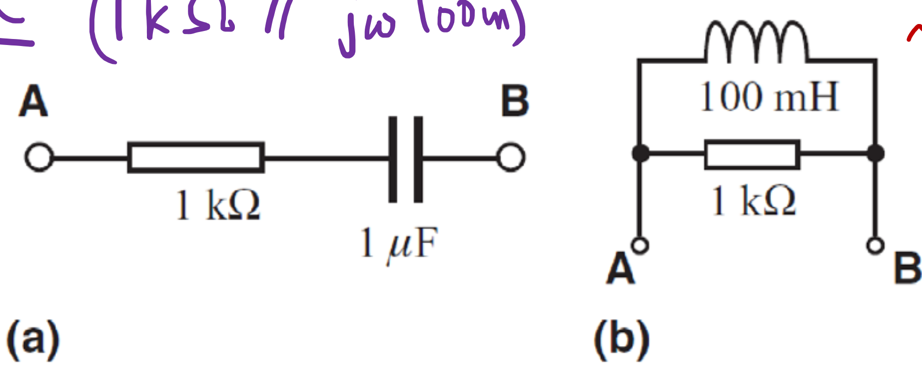



**Figure 10.9** Relevant to Example 10.2

# Example 10.3

$$Z_{XY} = R \parallel \frac{1}{j\omega C} = \frac{1}{\frac{1}{R} + j\omega C} = \frac{R}{1 + j\omega RC}$$

$$= \frac{1k}{1 + j \cdot 1000 \cdot 1000 \cdot 10^{-6}} = \frac{1k}{1+j} = (0.5 - j\,0.5)\ k\Omega$$

$$Z_{AB} = Z_{XY} + j\omega L$$

$$= (0.5 - j\,0.5)k + j \cdot 1000 \cdot 1$$

$$= (0.5 + j\,0.5)\ K$$

$$I = V_s / Z_{AB}$$

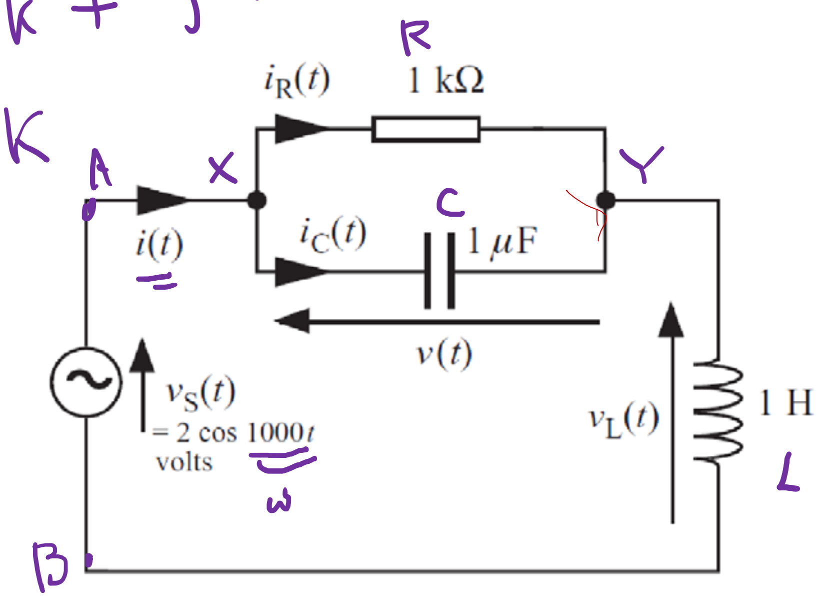
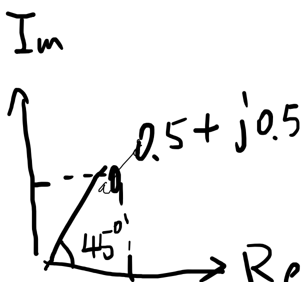



**Figure 10.10** The circuit to be analysed in Example 10.3

# Example 10.4




$$Z_{AB} = (0.5 + j0.5)\ k\Omega$$

$$= 0.5\sqrt{2}\angle 45^\circ\ k\Omega$$

$$I = V_s / Z_{AB}$$

$$= \frac{2\angle 0^\circ}{0.5\sqrt{2}\angle 45^\circ}$$

$$= 2\sqrt{2}\angle -45^\circ \text{ (mA)}$$

$$i(t) = 2\sqrt{2}\cos(\underbrace{1000}_{\omega}t - 45)$$

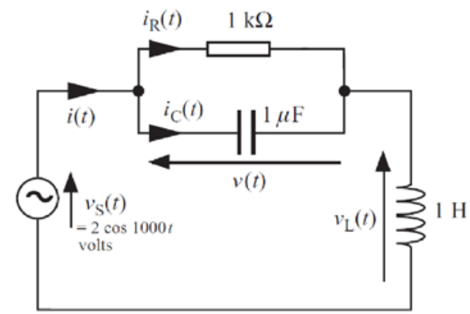



**Figure 10.10** The circuit to be analysed in Example 10.3

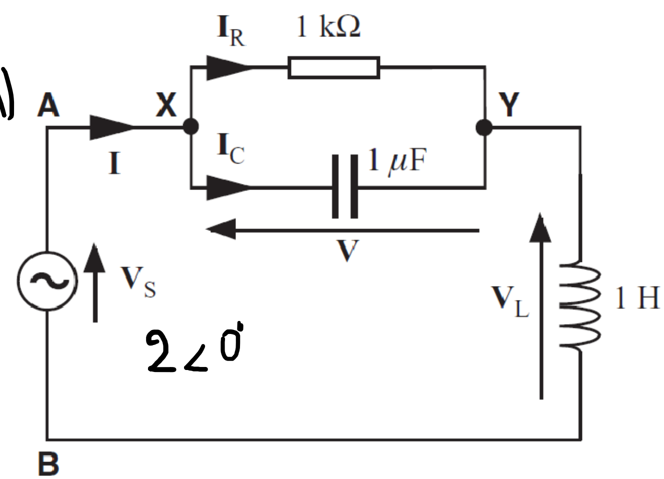



**Figure 10.11** The circuit of Figure 10.10 with currents and voltages represented by complex quantities

# Summary

AC ckt $\xrightarrow{\text{Complex } V, I,}$ DC analysis

Diff. eqn's (phasor $A\angle\theta$) Algebra

Basi

# Circuit analysis using Laplace transform

t-domain $f(t)$ $\xrightarrow{\mathcal{L}}$

Laplace Transform

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

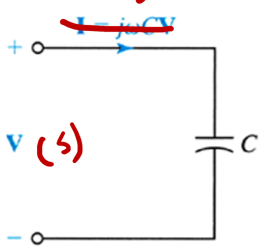
s-domain

$e^{at}$ $\qquad$ $\dfrac{1}{s-a}$

$\cos(\omega t)$ $\qquad$ $\dfrac{s}{s^2+\omega^2}$

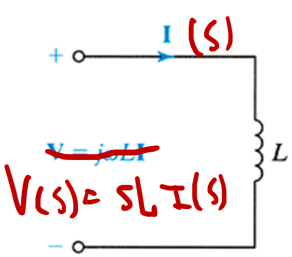
$\ldots$ $\qquad$ $\ldots$

$I = sC$




$I(s) = sC \cdot V(s)$

$Z_C = \dfrac{1}{sC}$




$V(s) = sL\; I(s)$

$Z_L = sL$

# Circuit analysis using Laplace transform

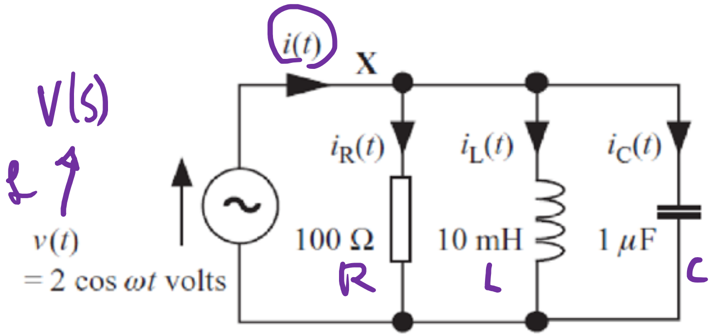



**Figure 9.18** A resonant circuit

$$I(s) = \frac{V(s)}{R \,//\, sL \,//\, \frac{1}{sC}}$$

$$= V(s)\underbrace{\left[\frac{1}{R} + \frac{1}{sL} + sC\right]}_{\mathcal{L}^{-1}}$$

$\mathcal{L}^{-1}$

$$i(t) = v(t) \circledast h(t)$$

convolution

Impulse Response

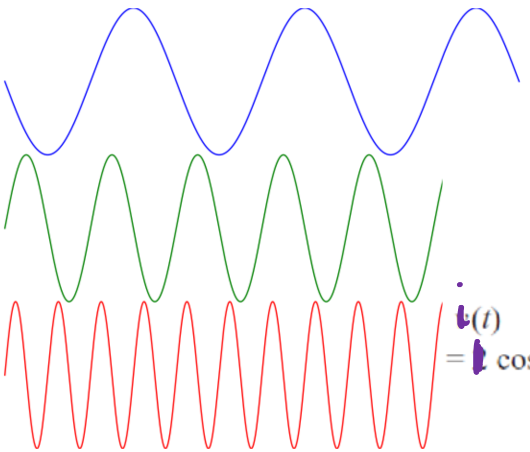
$v(t) \rightarrow \boxed{h(t)} \rightarrow i(t)$

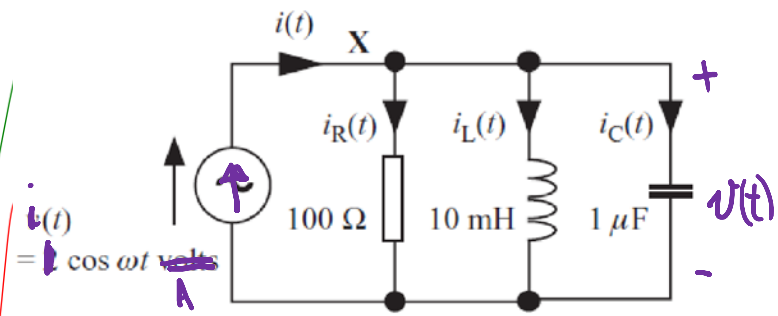
$V(s) \qquad H(s) \qquad I(s)$

# Frequency Domain Behaviour

Chapter 11

# Review: resonance





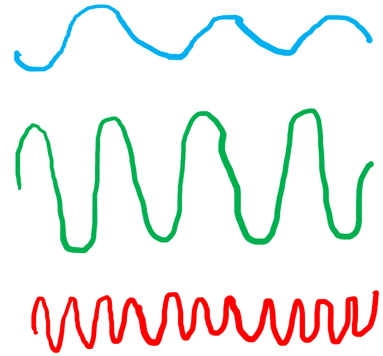



**Figure 9.18** A resonant circuit

R-L-C
Frequency
Response
⟸

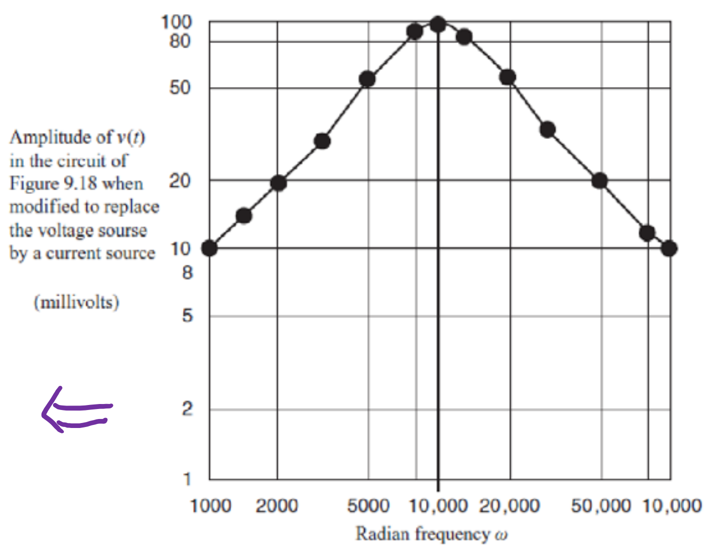



**Figure 9.20** Pertinent to the circuit of Figure 9.18 in which the sinusoidal voltage source is replaced by a sinusoidal current source of 1 mA amplitude. The sketch shows the variation of the ampitude of the voltage $v(t)$ as the source frequency $\omega$ varies

# Frequency behaviour

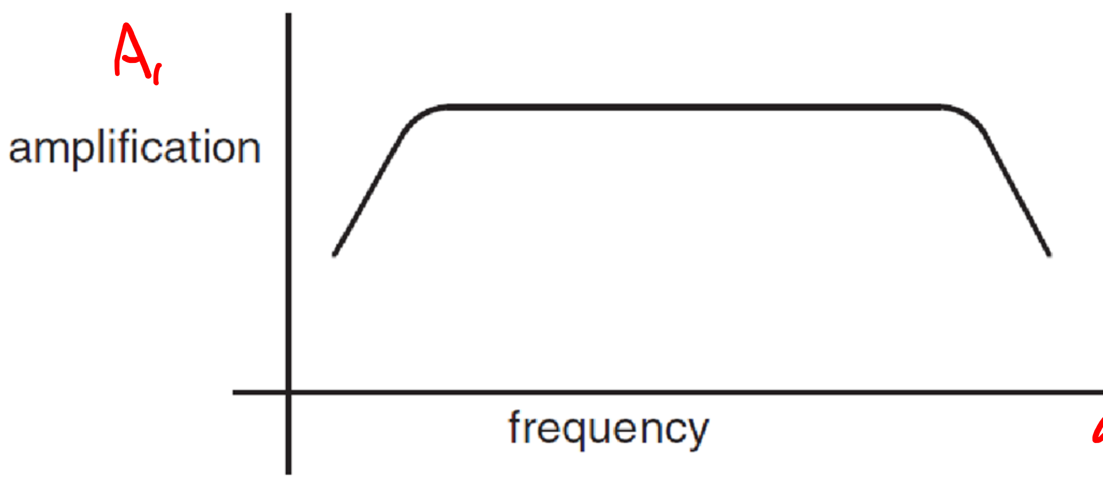
$1 \cos(\omega t)$


$A_\omega \cos(\omega t + \phi_\omega)$

$A_\omega \angle \phi_\omega$

Frequency Response

$A_\omega$




$\omega$

**Figure 11.1** Typical form of the frequency dependence of an amplifier's gain

# RC circuit

Voltage Divider

$$\mathbb{V}_c = \mathbb{V}_s \cdot \frac{Z_c}{R + Z_c} = \mathbb{V}_s \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \mathbb{V}_s \frac{1}{1 + j\omega RC}$$

$$\frac{\mathbb{V}_c}{\mathbb{V}_s} = \frac{1}{1+j\omega RC} = A_\omega \angle \phi_\omega$$

$$\begin{cases} A_\omega = \left|\frac{\mathbb{V}_c}{\mathbb{V}_s}\right| = \frac{1}{\sqrt{1+\omega^2R^2C^2}} \\ \phi_\omega = -\tan^{-1}(\omega RC) \end{cases}$$

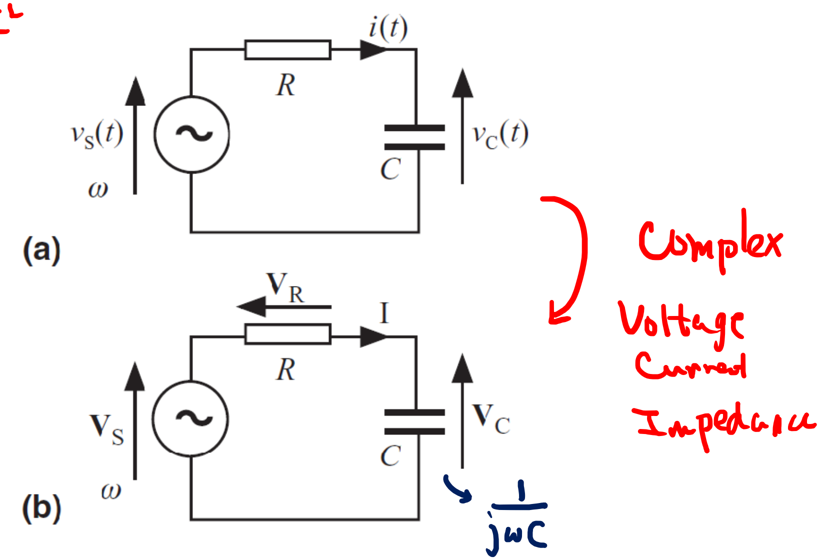



**Figure 11.2** (a) The circuit whose currents and voltages are of interest; (b) the same circuit with currents and voltages represented by complex values

# Frequency response

$$\underline{\underline{\left|\frac{V_c}{V_s}\right|}} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

Gain

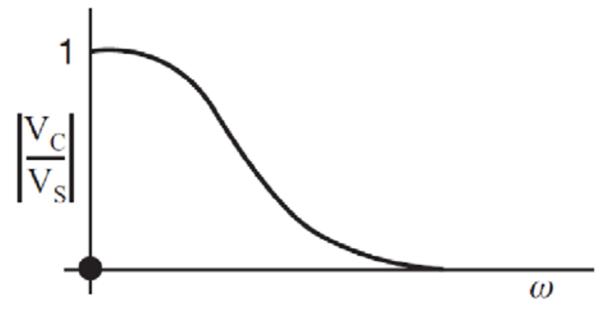



**Figure 11.3** A plot of $|V_C/V_S|$ for the circuit of Figure 12.2 (linear scales)

$\omega \to 0 \Rightarrow \left|\frac{V_c}{V_s}\right| = 1$

$\omega \downarrow$ Gain $\uparrow$

$\omega \uparrow$ Gain $\downarrow$

"Low-pass Filter"

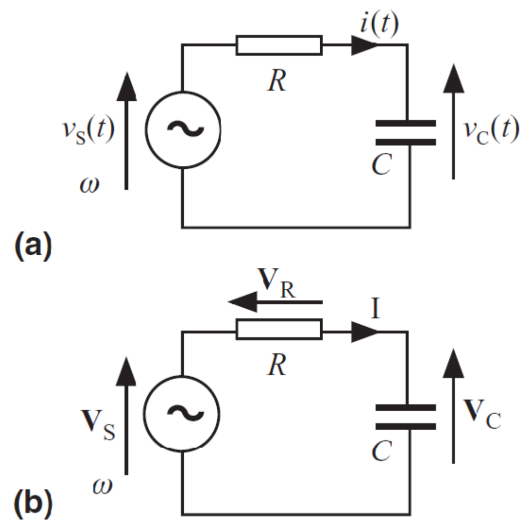



**Figure 11.2** (a) The circuit whose currents and voltages are of interest; (b) the same circuit with currents and voltages represented by complex values

# Asymptotic behavior (1)

$$\log\left|\frac{V_c}{V_s}\right| = \log \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$= -\frac{1}{2}\log\left(1+\omega^2R^2C^2\right)$$

For $\omega \gg \frac{1}{RC}$

$$\underbrace{\log\left|\frac{V_c}{V_s}\right|}_{y\text{-axis}} \simeq -\frac{1}{2}\log\left(\omega^2R^2C^2\right)$$

$$= -\underbrace{\log(\omega)}_{x\text{-axis}} - \underbrace{\log(RC)}_{\text{constant}}$$

$$y \simeq -x - \text{constant}$$

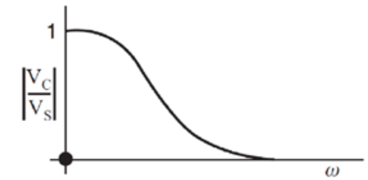
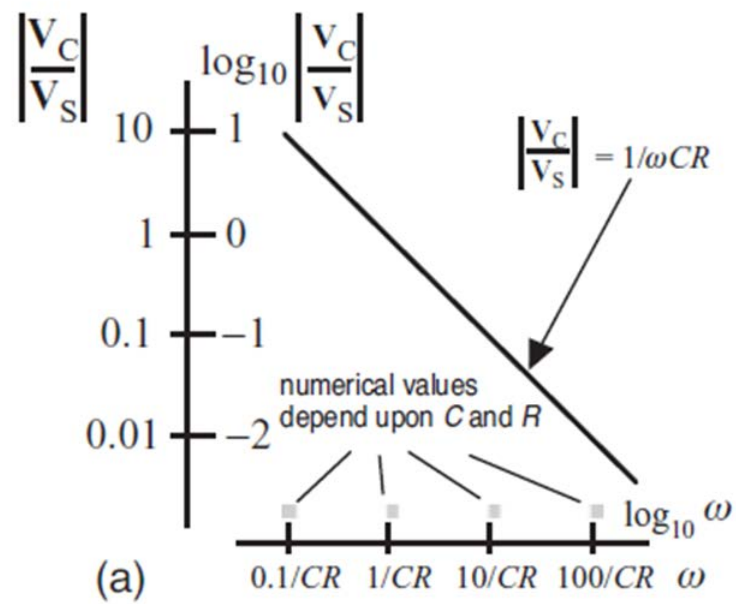



**Figure 11.3** A plot of $|V_C/V_S|$ for the circuit of Figure 12.2 (linear scales)



# Asymptotic behavior (2)

For $\omega \ll 1/RC$

$$\log \left|\frac{V_c}{V_s}\right| = -\frac{1}{2}\log\left(1+\omega^2R^2C^2\right)$$

$$\cong -\frac{1}{2}\log 1$$

$$= 0$$

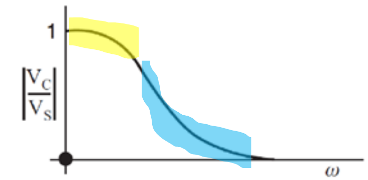
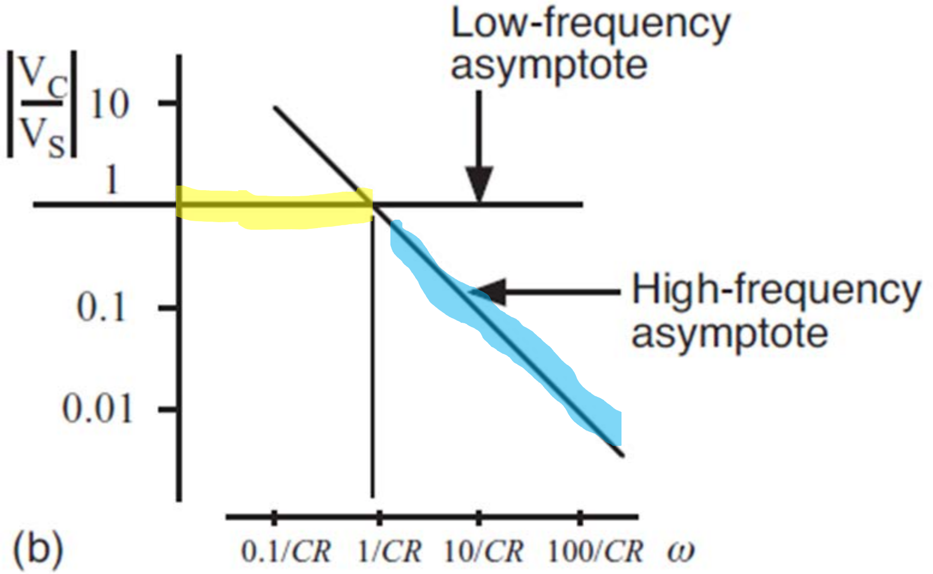



**Figure 11.3** A plot of $|V_C/V_S|$ for the circuit of Figure 12.2 (linear scales)

Log-Log scale

"Bode" Plot



# Asymptotic behavior (3)

If $\omega \simeq 1/RC$

$$\log\left|\frac{V_c}{V_s}\right| = \log\frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$\simeq \log\frac{1}{\sqrt{2}} = -\frac{1}{2}\log 2 \simeq -0.15$$

$$\Updownarrow$$

$$\left|\frac{V_c}{V_s}\right| \simeq \frac{1}{\sqrt{2}} = 0.707$$

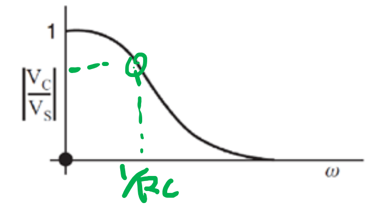
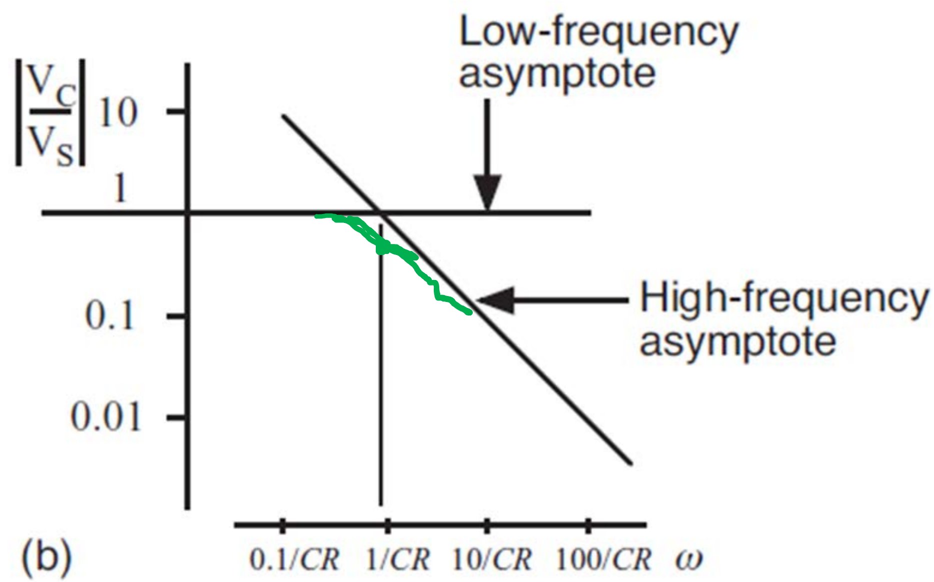



**Figure 11.3** A plot of $|V_C/V_S|$ for the circuit of Figure 12.2 (linear scales)



# Frequency behaviour

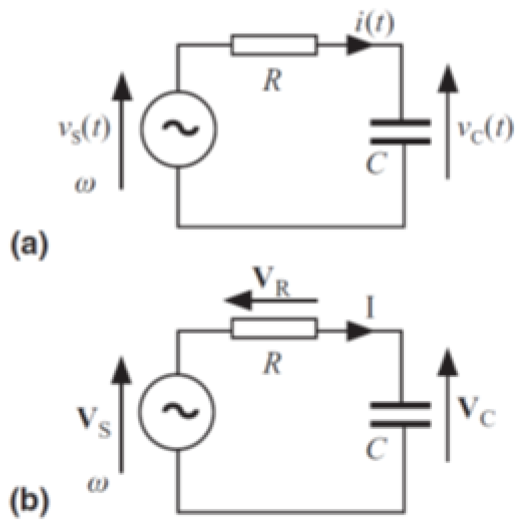
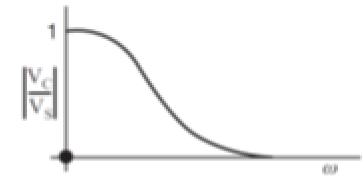


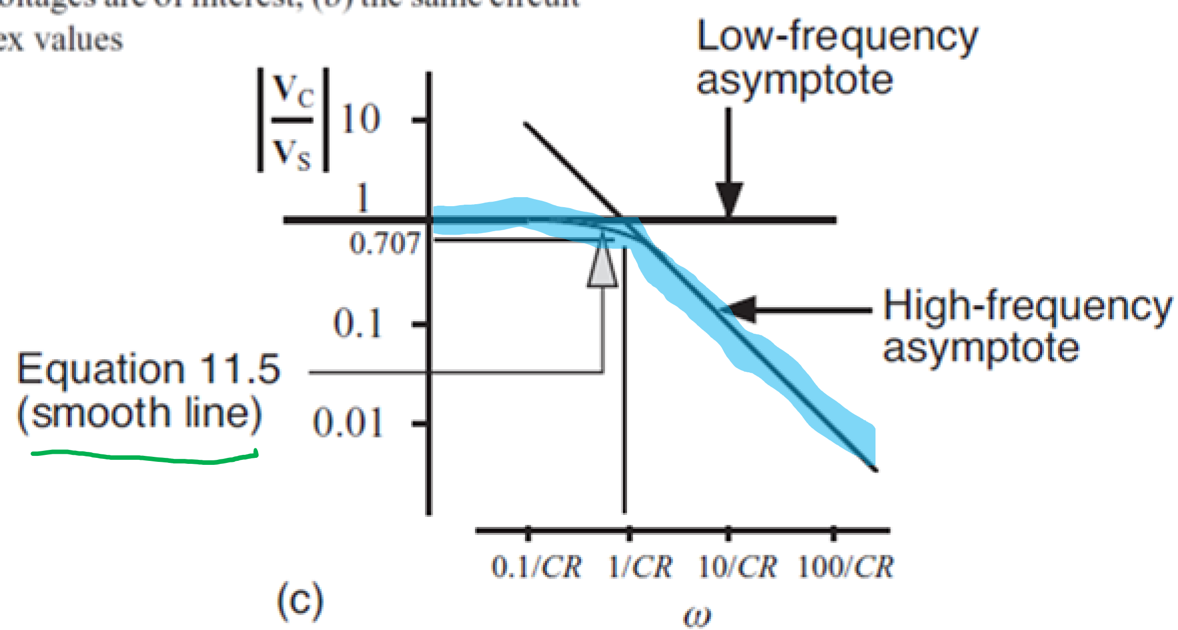

**Figure 11.2** (a) The circuit whose currents and voltages are of interest; (b) the same circuit with currents and voltages represented by complex values

Linear Scale




plot of $|V_C / V_S|$ for the circuit of Figure 12.2 (linear scales)

Log-log Scale



# Extreme frequencies

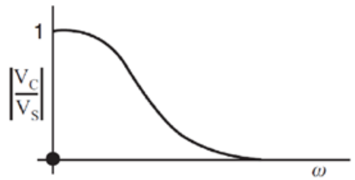



**Figure 11.3** A plot of $|V_C / V_S|$ for the circuit of Figure 12.2 (linear scales)

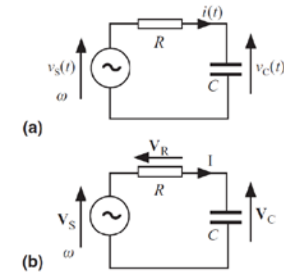



**Figure 11.2** (a) The circuit whose currents and voltages are of interest; (b) the same circuit with currents and voltages represented by complex values

$\omega \to 0$

$v_s(t) \cong V_s \cos(0) = V_s$

DC source

$|Z_c| = \left.\frac{1}{j\omega C}\right|_{\omega \to 0} \to \infty$

Open-ckt

$\left|\frac{V_c}{V_s}\right| \xrightarrow{\omega \to 0} \frac{\infty}{R+\infty} \Rightarrow 1$

$\omega \to \infty$

$|Z_c| = \left.\frac{1}{j\omega C}\right|_{\omega \to \infty} \to 0$

Short-ckt

$\left|\frac{V_c}{V_s}\right| \xrightarrow{\omega \to \infty} 0$

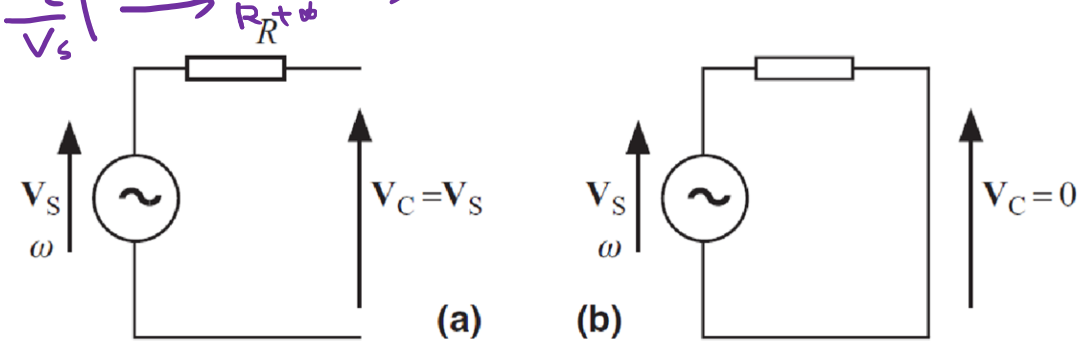



**Figure 11.7** (a) At zero frequency the impedance of a capacitor is infinite and the capacitor acts as an open-circuit; (b) at infinite frequency the capacitor acts as a short-circuit

# Example 11.1

R-L circuits

Voltage divider

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{j\omega L}{R + j\omega L}\right| = \left|\frac{1}{1 + \frac{R}{j\omega L}}\right|$$

$$= \frac{1}{\sqrt{1 + \left[\frac{1}{\omega(L/R)}\right]^2}}$$

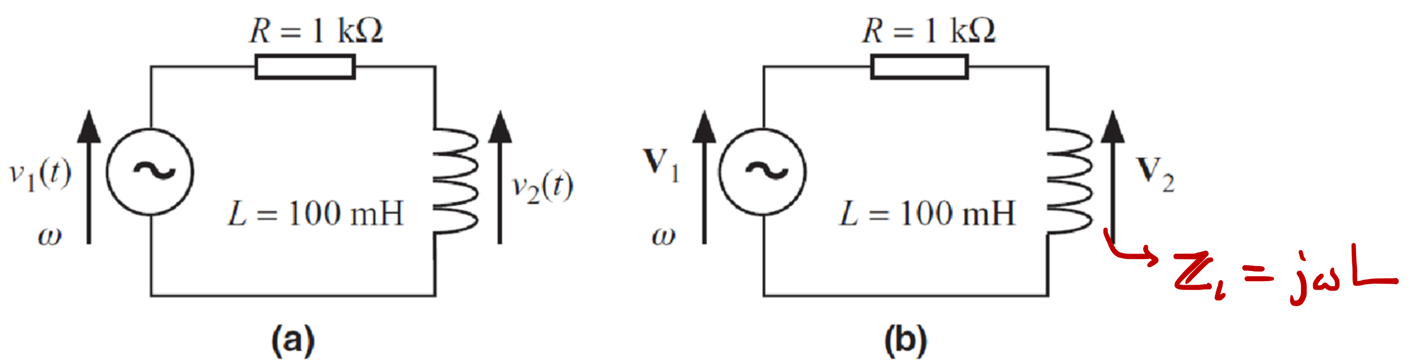



**Figure 11.5** (a) The circuit of Example 11.1; (b) the same circuit with sinusoidal voltages represented by complex voltages

# Example 11.1

$$\log\left|\frac{V_2}{V_1}\right| = -\frac{1}{2}\log\left[1+\frac{1}{(\omega L/R)^2}\right]$$

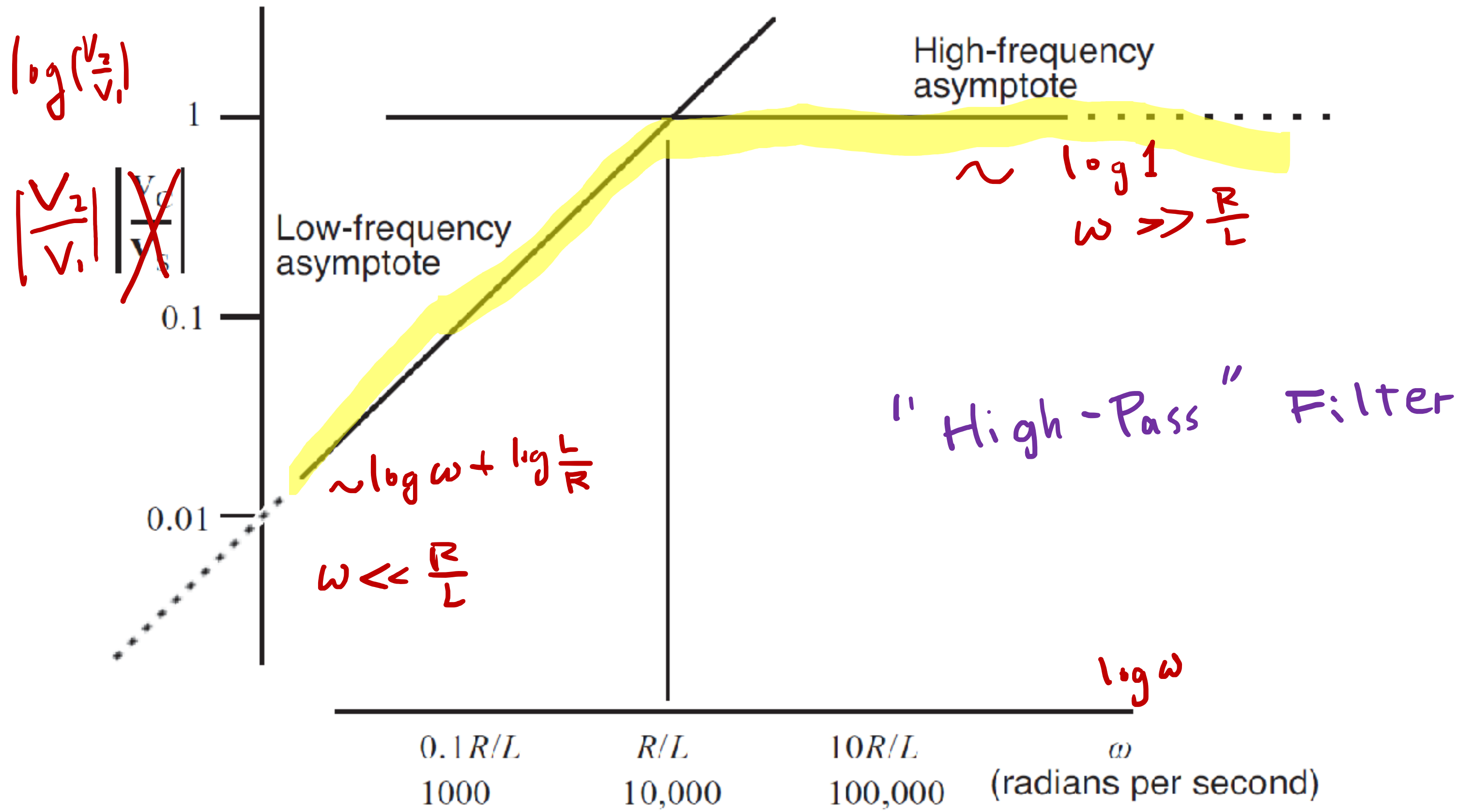



**Figure 11.6** The asymptotes associated with the voltage ratio $|V_1/V_2|$ in the circuit of Figure 11.5

# Extreme frequencies

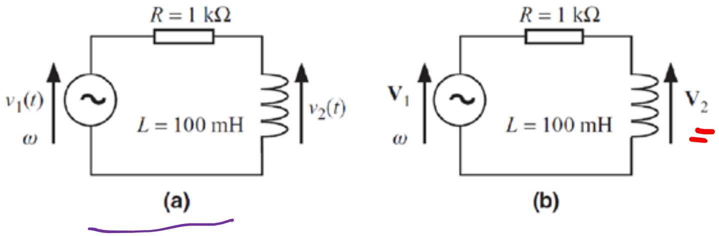



**Figure 11.5** (a) The circuit of Example 11.1; (b) the same circuit with sinusoidal voltages represented by complex voltages

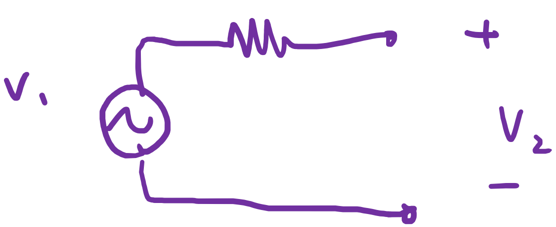
$\omega \to \infty$

$|Z_L| = |j\omega L| \to \infty$

Open-ckt

$V_2 \approx V_1$

$|V_2/V_1| \to 1$



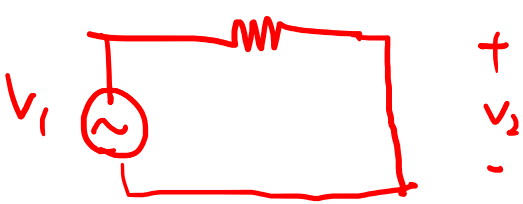

$\omega \to 0$ DC

$|Z_L| = |j\omega L| \to 0$

Short-ckt

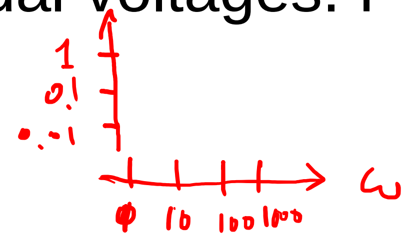
$V_2 \to 0$

$|V_2/V_1| \to 0$



# Quiz

For the following circuit sketch, on the log–log scale, the low- and high-frequency asymptotes of the magnitude of the voltage amplification $|V_2/V_1|$ where $V_1$ and $V_2$ are the complex voltages representing sinusoidal voltages. Put numerical values on the x-axis.



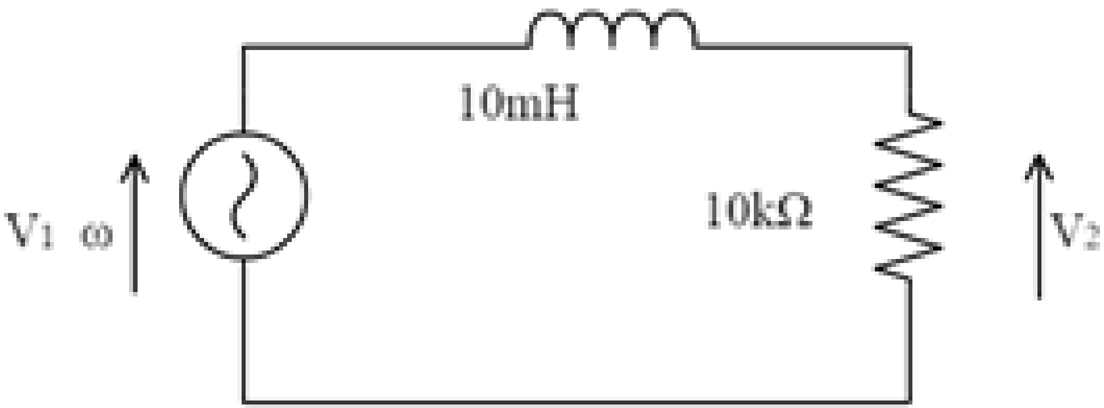




Figure 1

# Quiz Review

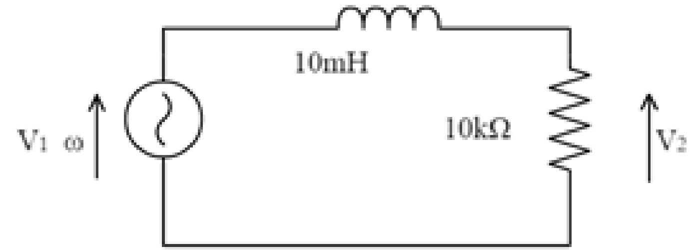



Figure 1

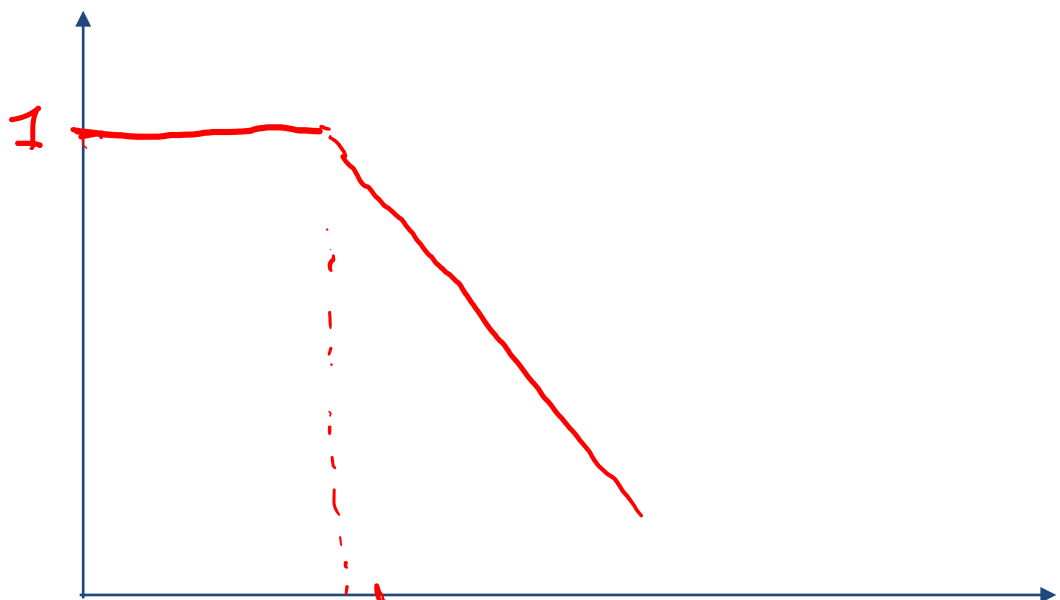
$$\frac{V_2}{V_1} = \frac{R}{R+j\omega L} = \frac{1}{1+j\omega (L/R)}$$

$\omega << R/L$

$$\left|\frac{V_2}{V_1}\right| \cong 1$$

$\omega >> R/L$

$$\left|\frac{V_2}{V_1}\right| = \frac{R}{\omega L}$$




$$R/L = \frac{10k}{10m} = 10^6 \text{ rad/s}$$

$\omega$

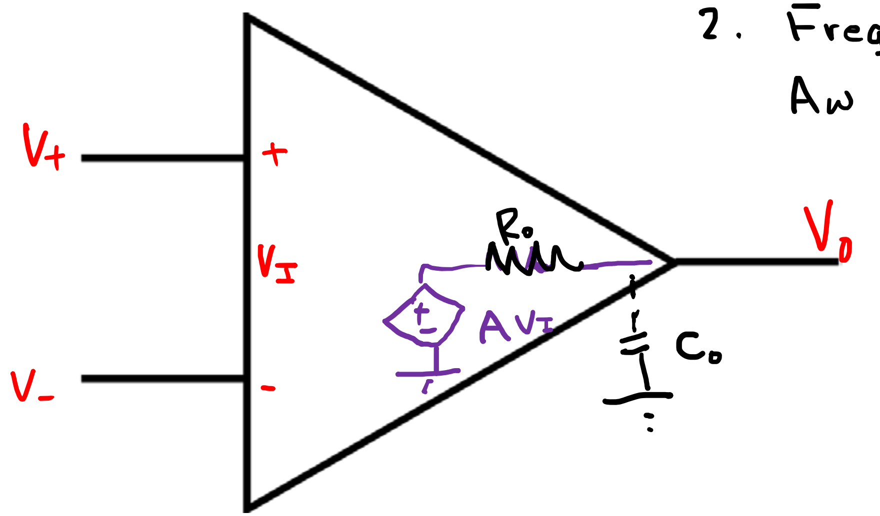
# Opamp limitations

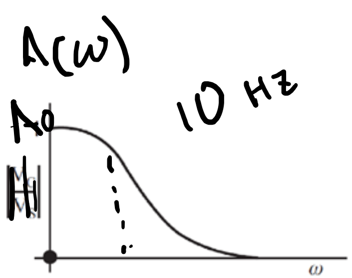
Ideal opamp $A \to \infty$

For any frequency

In real world

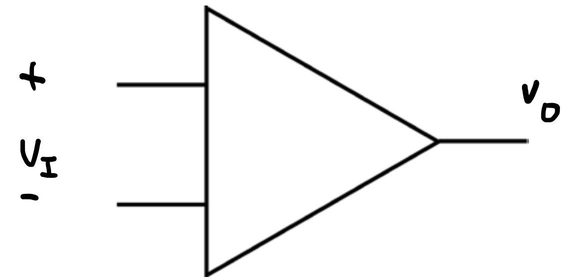
1. $A$ "finite"
2. Frequency dependent
   $A_\omega \downarrow$ as $\omega \uparrow$



# Opamp limitations

$$\left|\frac{V_O}{V_I}\right| = A_0 \cdot \frac{1}{1 + j\frac{\omega}{(\omega_0)}}$$




$\frac{\omega_0}{2\pi} = 10\,Hz$

"cut-off frequency"

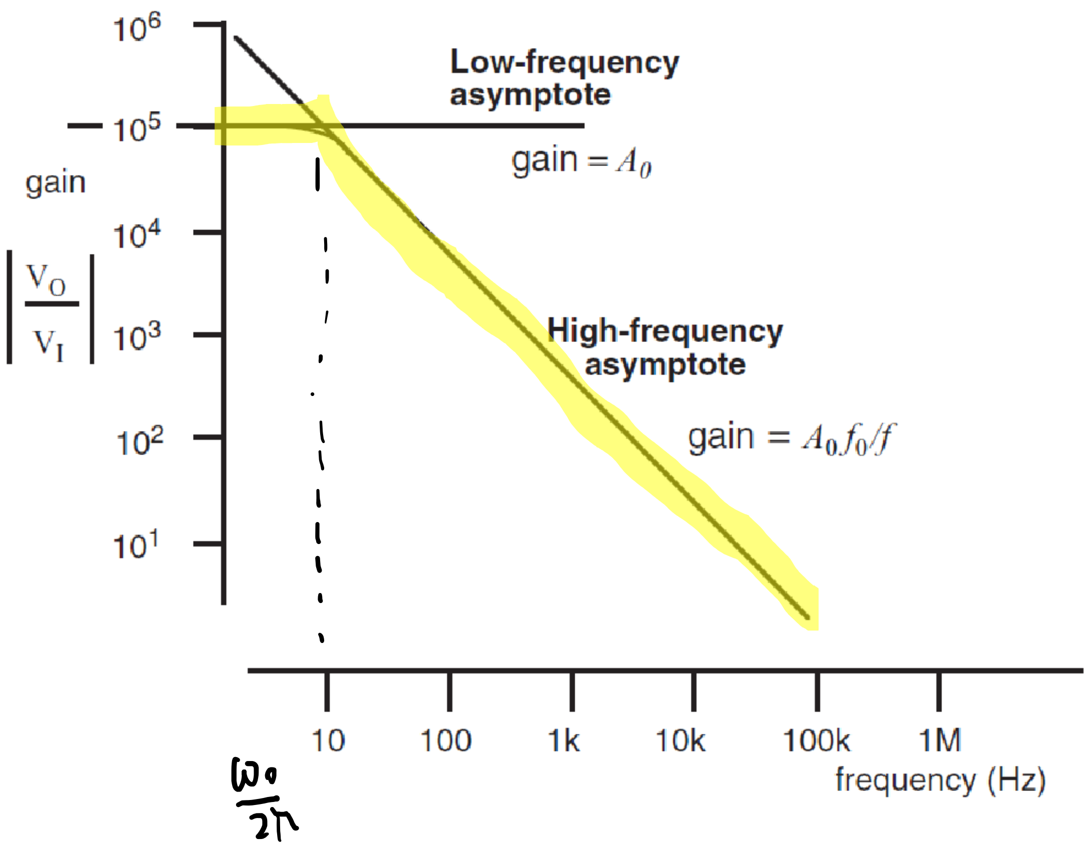



**Figure 11.9** Typical low- and high-frequency asymptotes of the gain of an opamp

# Inverter frequency response

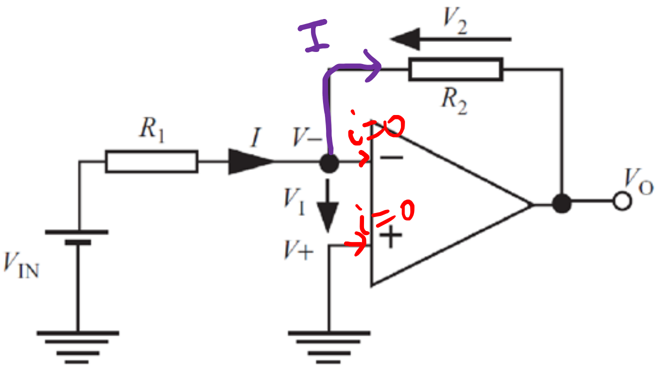



**Figure 11.8** The inverter circuit

$V_o \sim V_{IN}$ Relation

$$\begin{cases} I = \dfrac{V_{IN} - V_-}{R_1} = \dfrac{V_- - V_o}{R_2} \\ V_- = V_+ - V_I = -V_I \\ V_o = A_\omega V_I \end{cases}$$

$$\Rightarrow \frac{V_o}{V_{IN}} = -\frac{R_2}{R_1} \frac{1}{1 + \frac{1}{A_\omega}\left(\frac{R_2}{R_1} + 1\right)}$$

Let $G = \dfrac{R_2}{R_1} \gg 1$

$$\frac{V_o}{V_{IN}}(\omega) \simeq -G \frac{1}{1 + \frac{G}{A_\omega}}$$

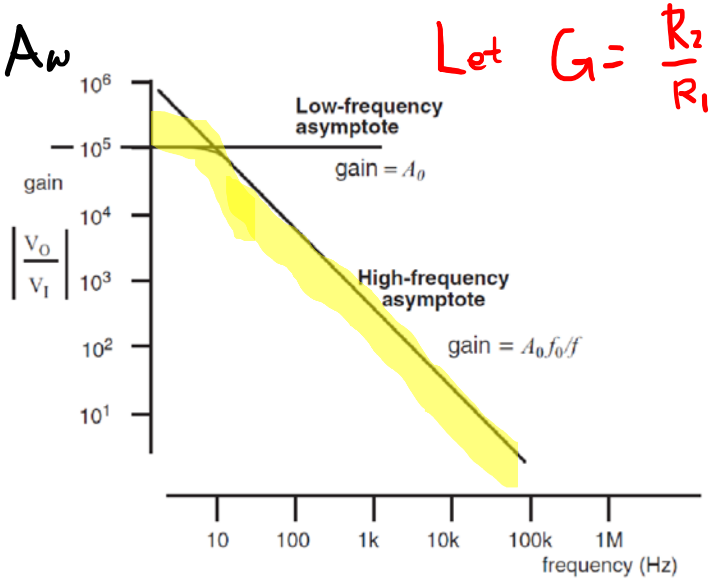



**Figure 11.9** Typical low- and high-frequency asymptotes of the gain of an opamp

# Inverter frequency response

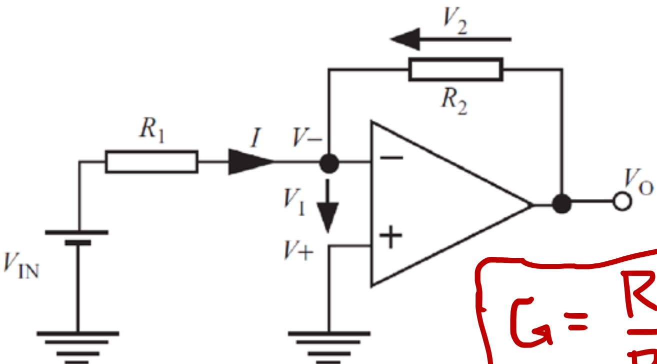



**Figure 11.8** The inverter circuit

$$G = \frac{R_2}{R_1}$$ DC-gain

$$\frac{V_O}{V_{IN}}(\omega) = -G\,\frac{1}{1+\frac{G}{A_\omega}}$$

$$A_\omega = A_0\,\frac{1}{1+j\frac{\omega}{\omega_0}}$$

$$\frac{V_O}{V_{IN}} \cong -G\,\frac{1}{1+\frac{1+j\omega/\omega_0}{A_0/G}}$$

$$= -G\,\frac{1}{1+\frac{1}{A_0/G}+j\frac{\omega}{\omega_0(A_0/G)}}$$

~0

$$\cong -G\,\frac{1}{1+j\frac{\omega}{\omega_0'}}$$

where

$$\omega_0' = \omega_0\,\frac{A_0}{G}$$

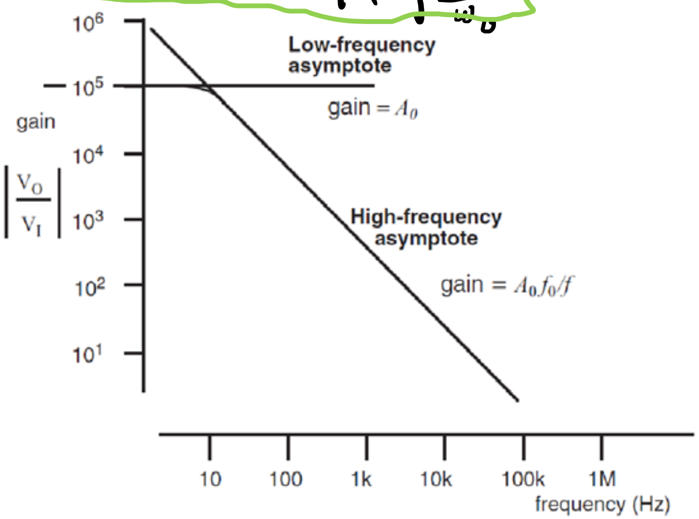



**Figure 11.9** Typical low- and high-frequency asymptotes of the gain of an opamp

# Inverter frequency response

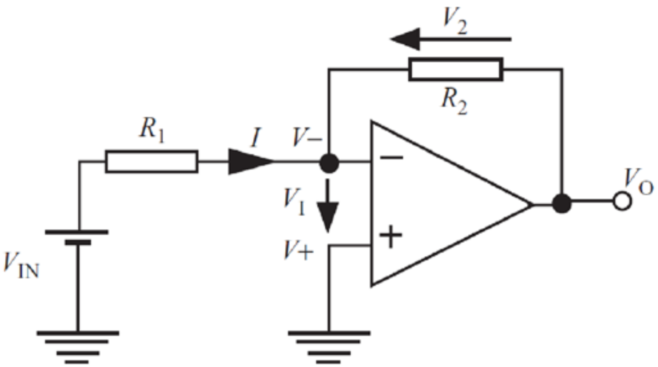



**Figure 11.8** The inverter circuit

$$\frac{V_O}{V_{IN}} \simeq -\frac{R_2}{R_1} \times \frac{1}{1 + j\frac{\omega}{\omega_0'}}$$

G : dc gain

$$\omega_0' = \omega_0 \cdot \frac{A_0}{(R_2/R_1)} = \omega_0 \frac{A_0}{G}$$

Opamp

$$A_\omega = A_0 \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

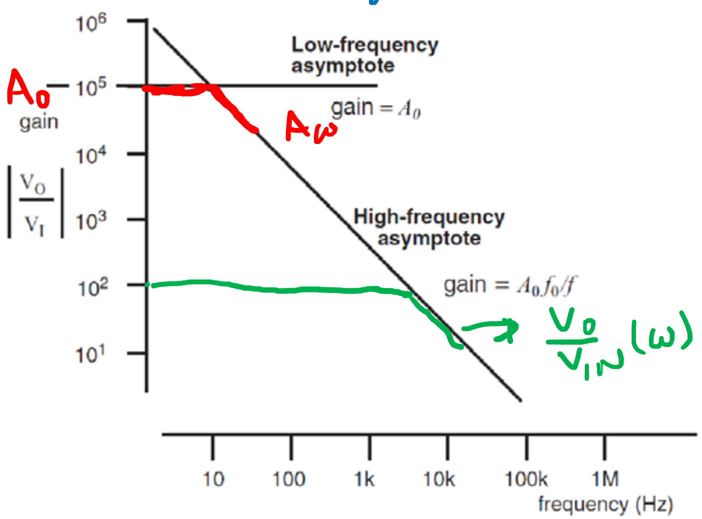



**Figure 11.9** Typical low- and high-frequency asymptotes of the gain of an opamp

If $G = R_2/R_1 = 100$

$A_0 = 10^5$

$$\Rightarrow \omega_0' = \omega_0 \cdot 1000$$

Opamp $\longrightarrow$ Inverter

Gain $10^5 \longrightarrow 100 \downarrow$

Cut-off freq $10\,\text{Hz} \longrightarrow 10000 \uparrow$

# Inverter frequency response

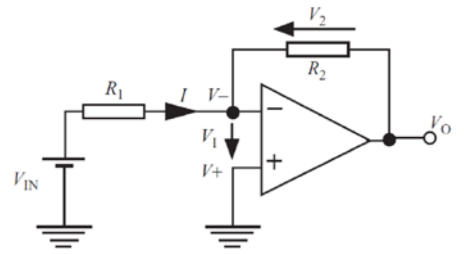
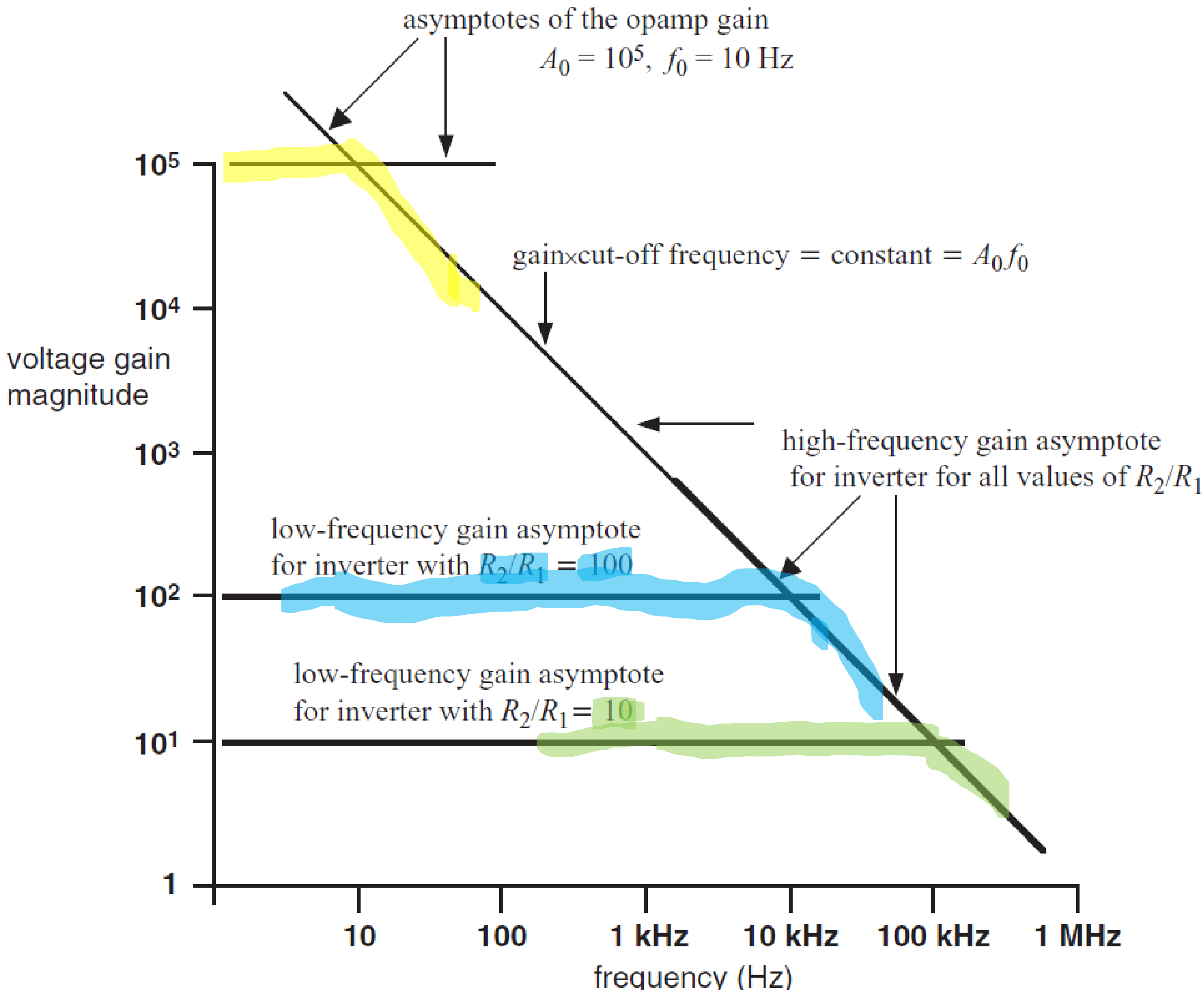
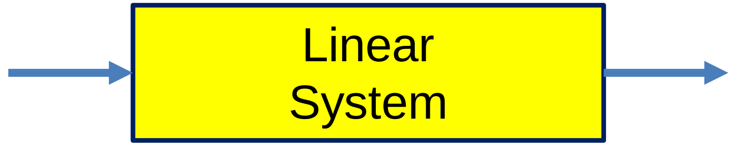



**Figure 11.8** The inverter circuit



# Summary

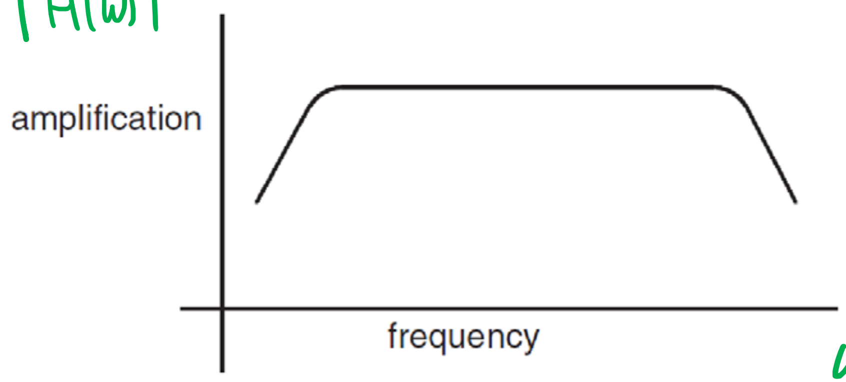



$H(\omega)$

Log-Log Scale

"Bode" plot

$|H(\omega)|$




$\omega$

**Figure 11.1** Typical form of the frequency dependence of an amplifier's gain