Turing Machine*

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Outline

1. Effective Procedures
   - Basic Concepts
   - Computable Function
2. Turing Machine
   - Introduction
   - One-Tape Turing Machine
   - Multi-Tape Turing Machine
3. TM Variation and TM-Computability
   - TM Variations
   - Computable and Decidable

Outline

What is Effective Procedure

- Methods for addition, multiplication $\cdots$
  - Given n, finding the nth prime number.
  - Differentiating a polynomial.
  - Finding the highest common factor of two numbers $HCF(x, y) \rightarrow$ Euclidean algorithm
  - Given two numbers x, y, deciding whether x is a multiple of y.
- Their implementation requires no ingenuity, intelligence, inventiveness.

Intuitive Definition

An *algorithm* or *effective procedure* is a mechanical rule, or automatic method, or programme for performing some mathematical operations.

Blackbox: input $\longrightarrow \blacksquare \longrightarrow$ output

What is "effective procedure"?

An Example: Consider the function $g(n)$ defined as follows:

$$g(n) = \begin{cases} 1, & \text{if there is a run of exactly } n \text{ consecutive } 7'\text{s} \\ & \text{in the decimal expansion of } \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Question: Is $g(n)$ effective?

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▷ The answer is unknown $\neq$ the answer is negative.

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Other Examples:

- *Theorem Proving* is in general not effective/algorithmic.
- *Proof Verification* is effective/algorithmic.

Algorithm

An algorithm is a procedure that consists of a finite set of *instructions* which, given an *input* from some set of possible inputs, enables us to obtain an *output* through a systematic execution of the instructions that *terminates* in a finite number of steps.

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Computable Function

When an algorithm or effective procedure is used to calculate the value of a numerical function then the function in question is effectively calculable (or algorithmically computable, effectively computable, computable).

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Examples:

- $HCF(x, y)$ is computable;
- $g(n)$ is non-computable.

Development of Computation Models

1. Gödel-Kleene (1936): Partial recursive functions.
2. Turing (1936): Turing machines.
3. Church (1936): λ-terms.
4. Post (1943): Post systems.
5. Markov (1951): Variants of the Post systems.
6. Shepherdson-Sturgis (1963): URM-computable functions.

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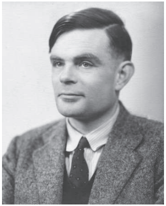
6. Shepherdson-Sturgis (1963): URM-computable functions.

Church-Turing Thesis: Each of the above proposals for a characterization of the notion of effective computability gives rise to the same class of functions.

Outline

Alan Turing (23 Jun. 1912 - 7 Jun. 1954)

- An English student of Church
- Introduced a machine model for effective calculation in "On Computable Numbers, with an Application to the Entscheidungsproblem", Proc. of the London Mathematical Society, 42:230-265, 1936.
- Turing Machine, Halting Problem, Turing Test



Motivation

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- The machine must be able to operate and manipulate numbers according to a set of predefined instructions

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- The machine should be able to interpret numbers
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and

- The input number has to be stored in an accessible place
- The output number has to be put in an accessible place
- There should be an accessible place for the machine to store intermediate results

Outline

One-Tape Turing Machine

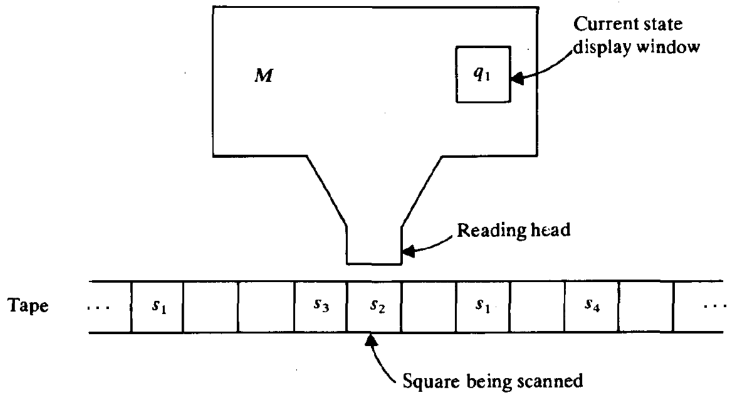
A Turing machine has five components:

1. A finite set $\{s_1, \ldots, s_n\} \cup \{\triangleright, \triangleleft\} \cup \{\square\}$ of symbols.
2. A tape consists of an infinite number of cells, each cell may store a symbol.

$$\cdots \square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square \cdots$$

3. A reading head that scans and writes on the cells.
4. A finite set $\{q_S, q_1, \ldots, q_m, q_H\}$ of states.
5. A finite set of instructions (specification).

One-Tape Turing Machine



One-Tape Turing Machine

The input data

$$\triangleright s_1^1 \dots s_{i_1}^1 \square \dots \square s_1^k \dots s_{i_k}^k \triangleleft \square \cdots$$

The reading head may write a symbol, move left, move right.

An instruction is of the form:

$$\langle q_i, s_j \rangle \to \langle q_l, s_k, L/R/S \rangle,$$

which means when reads s_j with state q_i, the machine will turn to state q_l, replace s_j with s_k, and turn one cell to the left.

The direction can be L, R, or S, meaning move to left, right, or stay at the current position.

An Example

Given a Turing machine M with the alphabet $\{0,1\} \cup \{\triangleright, \Box, \triangleleft\}$.

$$
\begin{aligned}
\langle q_S, \triangleright\rangle &\to \langle q_1, \triangleright, R\rangle \\
\langle q_1, 0\rangle &\to \langle q_1, 0, R\rangle \\
\langle q_1, 1\rangle &\to \langle q_2, 0, S\rangle \\
\langle q_2, 0\rangle &\to \langle q_2, 0, R\rangle \\
\langle q_2, 1\rangle &\to \langle q_1, 1, R\rangle \\
\langle q_1, \triangleleft\rangle &\to \langle q_3, \triangleleft, L\rangle \\
\langle q_2, \triangleleft\rangle &\to \langle q_3, \triangleleft, L\rangle \\
\langle q_3, 0\rangle &\to \langle q_3, 0, L\rangle \\
\langle q_3, 1\rangle &\to \langle q_3, 1, L\rangle \\
\langle q_3, \triangleright\rangle &\to \langle q_H, \triangleright, R\rangle
\end{aligned}
$$

An Example

Given a Turing machine M with the alphabet $\{0, 1\} \cup \{\triangleright, \square, \triangleleft\}$.

$\triangleright$	1	1	0	1	$\triangleleft$

$\widehat{q_S}$

$$\langle q_S, \triangleright \rangle \rightarrow \langle q_1, \triangleright, R \rangle$$
$$\langle q_1, 0 \rangle \rightarrow \langle q_1, 0, R \rangle$$
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$$\langle q_2, \triangleleft \rangle \rightarrow \langle q_3, \triangleleft, L \rangle$$
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$$\langle q_3, \triangleright \rangle \rightarrow \langle q_H, \triangleright, R \rangle$$

An Example

Given a Turing machine M with the alphabet $\{0, 1\} \cup \{\triangleright, \Box, \triangleleft\}$.

$\triangleright$	1	1	0	1	$\triangleleft$
$\widehat{q_S}$					

$\triangleright$	**1**	1	0	1	$\triangleleft$
	$\widehat{q_1}$				

$$
\begin{aligned}
\langle q_S, \triangleright\rangle &\rightarrow \langle q_1, \triangleright, R\rangle \\
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\langle q_2, \triangleleft\rangle &\rightarrow \langle q_3, \triangleleft, L\rangle \\
\langle q_3, 0\rangle &\rightarrow \langle q_3, 0, L\rangle \\
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\end{aligned}
$$

$\triangleright$	1	1	0	1	$\triangleleft$
$\widehat{q_S}$					

$\triangleright$	**1**	1	0	1	$\triangleleft$
	$\widehat{q_1}$				

$\triangleright$	**0**	1	0	1	$\triangleleft$
	$\widehat{q_2}$				

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Given a Turing machine M with the alphabet $\{0,1\} \cup \{\triangleright, \Box, \triangleleft\}$.

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$\triangleright$	1	1	0	1	$\triangleleft$
$\widehat{q_S}$					

$\triangleright$	**1**	1	0	1	$\triangleleft$
	$\widehat{q_1}$				

$\triangleright$	**0**	1	0	1	$\triangleleft$
	$\widehat{q_2}$				

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$\triangleright$	1	1	0	1	$\triangleleft$
$\widehat{q_S}$					

$\triangleright$	**1**	1	0	1	$\triangleleft$
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$\triangleright$	**0**	1	0	1	$\triangleleft$
	$\widehat{q_2}$				

$\triangleright$	0	**1**	0	1	$\triangleleft$
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$\triangleright$	0	1	**0**	1	$\triangleleft$
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$\widehat{q_S}$					

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	$\widehat{q_1}$				

$\triangleright$	**0**	1	0	1	$\triangleleft$
	$\widehat{q_2}$				

$\triangleright$	0	**1**	0	1	$\triangleleft$
		$\widehat{q_2}$			

$\triangleright$	0	1	**0**	1	$\triangleleft$
			$\widehat{q_1}$		

$\triangleright$	0	1	0	**1**	$\triangleleft$
				$\widehat{q_1}$	

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▷	1	1	0	1	◁
$\widehat{q_S}$					

▷	1	1	0	1	◁
	$\widehat{q_1}$				

▷	0	1	0	1	◁
	$\widehat{q_2}$				

▷	0	1	0	1	◁
		$\widehat{q_2}$			

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Given a Turing machine M with the alphabet $\{0,1\} \cup \{\triangleright, \Box, \triangleleft\}$.

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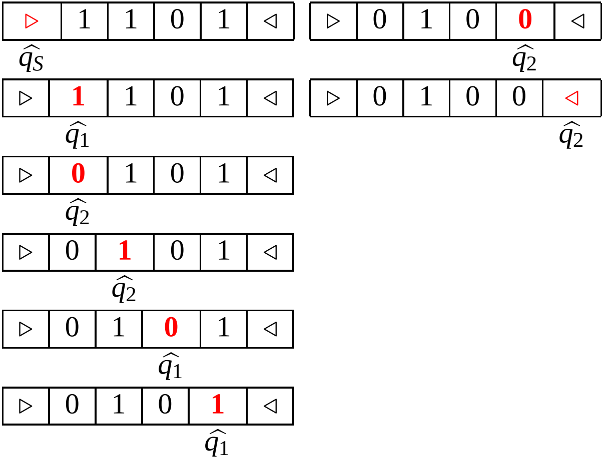

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$\triangleright$	1	1	0	1	$\triangleleft$	$\widehat{q_S}$
$\triangleright$	**1**	1	0	1	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	**0**	1	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	**1**	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	1	**0**	1	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	0	1	0	**1**	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	0	1	0	**0**	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	1	0	0	**$\triangleleft$**	$\widehat{q_2}$
$\triangleright$	0	1	0	**0**	$\triangleleft$	$\widehat{q_3}$

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\end{aligned}
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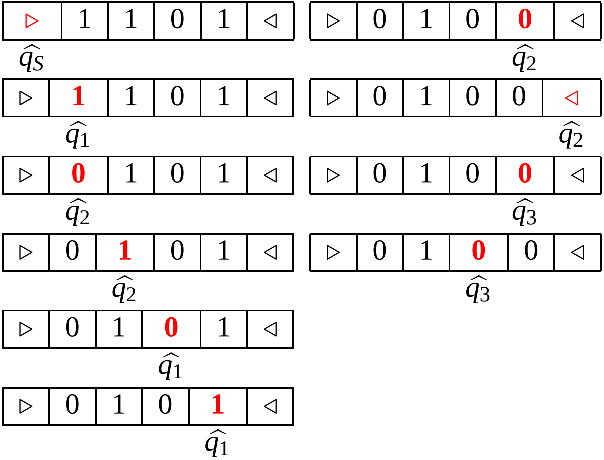

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$$\langle q_3, 0 \rangle \rightarrow \langle q_3, 0, L \rangle$$
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$$\langle q_3, \triangleright \rangle \rightarrow \langle q_H, \triangleright, R \rangle$$

$\triangleright$	1	1	0	1	$\triangleleft$	$\widehat{q_S}$
$\triangleright$	**1**	1	0	1	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	**0**	1	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	**1**	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	1	**0**	1	$\triangleleft$	$\widehat{q_1}$
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\langle q_2, \triangleleft\rangle &\rightarrow \langle q_3, \triangleleft, L\rangle \\
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\end{aligned}
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$\triangleright$	1	1	0	1	$\triangleleft$	$\widehat{q_S}$
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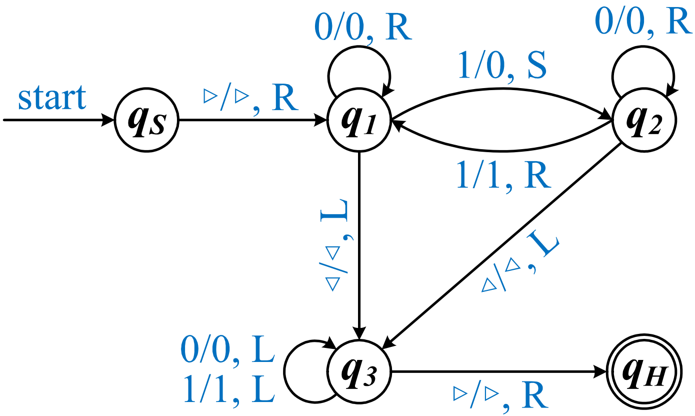
$\triangleright$	**0**	1	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	**1**	0	1	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	1	**0**	1	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	0	1	0	**1**	$\triangleleft$	$\widehat{q_1}$
$\triangleright$	0	1	0	**0**	$\triangleleft$	$\widehat{q_2}$
$\triangleright$	0	1	0	0	**$\triangleleft$**	$\widehat{q_2}$
$\triangleright$	0	1	0	**0**	$\triangleleft$	$\widehat{q_3}$
$\triangleright$	0	1	**0**	0	$\triangleleft$	$\widehat{q_3}$
$\triangleright$	0	1	0	0	$\triangleleft$	$\widehat{q_3}$
$\triangleright$	**0**	1	0	0	$\triangleleft$	$\widehat{q_H}$

State Transition Diagram



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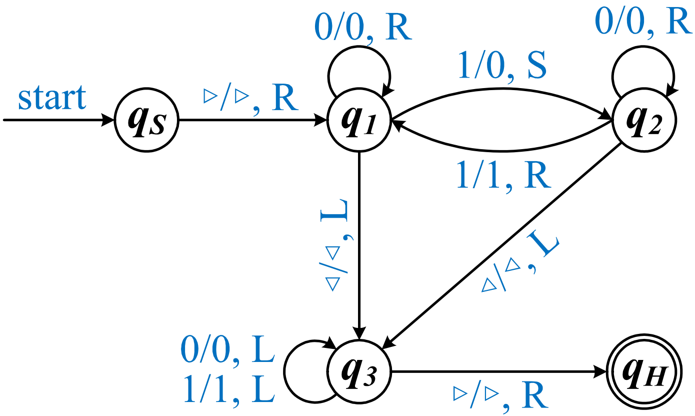

M's action is to work from left to right along the tape, replacing alternate 1's by the symbol 0's.

Outline

Multi-Tape Turing Machine

A multi-tape TM is described by a tuple (Γ, Q, δ) containing

- A finite set Γ called alphabet, of symbols. It contains a blank symbol $\Box$, a start symbol $\triangleright$, and the digits 0 and 1.
- A finite set Q of states. It contains a start state q_{start} and a halting state q_{halt}.
- A transition function $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^{k-1} \times L, S, R^k$, describing the rules of each computation step.

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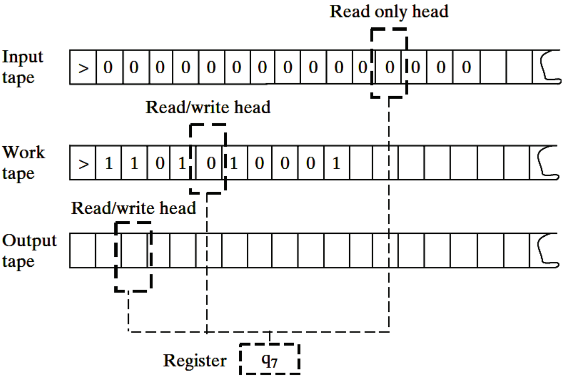
Example: A 2-Tape TM will have transition function (also named as specification) like follows:

$$\begin{array}{lcl} \langle q_s, \triangleright, \triangleright \rangle & \to & \langle q_1, \triangleright, R, R \rangle \\ \langle q_1, 0, 1 \rangle & \to & \langle q_2, 0, S, L \rangle \end{array}$$

Computation and Configuration



Computation, configuration, initial/final configuration

A 3-Tape TM for the Palindrome Problem

A palindrome is a word that reads the same both forwards and backwards. For instance:

ada, anna, madam, and nitalarbralatin.

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Requirement: Give the specification of M with $k = 3$ to recognize palindromes on symbol set $\{0, 1, \triangleright, \triangleleft, \Box\}$.

Preparation

To recognize palindrome we need to check the input string, output 1 if the string is a palindrome, and 0 otherwise.

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The head on each tape points the first symbol "$\rhd$" as the starting state, with state mark q_S.

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In the final state q_F, the output of the k^{th} tape should be "$\rhd$ 1 $\lhd$ $\Box$" if the input is a palindrome, and "$\rhd$ 0 $\lhd$ $\Box$" otherwise.

A 3-Tape TM for the Palindrome Problem

$Q = \{q_s, q_h, q_c, q_l, q_t, q_r\}$; $\Gamma = \{\square, \rhd, \lhd, 0, 1\}$; two work tapes.

A 3-Tape TM for the Palindrome Problem

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Start State:

$\langle q_s, \rhd, \rhd, \rhd \rangle \rightarrow \langle q_c, \rhd, \rhd, R, R, R \rangle$

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Begin to copy:

$\langle q_c, 0, \Box, \Box \rangle \rightarrow \langle q_c, 0, \Box, R, R, S \rangle$
$\langle q_c, 1, \Box, \Box \rangle \rightarrow \langle q_c, 1, \Box, R, R, S \rangle$
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$\langle q_c, \lhd, \square, \square \rangle \rightarrow \langle q_l, \square, \square, L, S, S \rangle$

Return back to the leftmost:

$\langle q_l, 0, \square, \square \rangle \rightarrow \langle q_l, \square, \square, L, S, S \rangle$
$\langle q_l, 1, \square, \square \rangle \rightarrow \langle q_l, \square, \square, L, S, S \rangle$
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$\langle q_l, \rhd, \Box, \Box \rangle \to \langle q_t, \Box, \Box, R, L, S \rangle$

Begin to compare:

$\langle q_t, \lhd, \rhd, \Box \rangle \to \langle q_r, \rhd, 1, S, S, R \rangle$
$\langle q_t, 0, 1, \Box \rangle \to \langle q_r, 1, 0, S, S, R \rangle$
$\langle q_t, 1, 0, \Box \rangle \to \langle q_r, 0, 0, S, S, R \rangle$
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Begin to copy:

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$\langle q_c, 1, \Box, \Box \rangle \to \langle q_c, 1, \Box, R, R, S \rangle$
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$\langle q_t, 1, 1, \Box \rangle \to \langle q_t, 1, \Box, R, L, S \rangle$

Ready to terminate:

$\langle q_r, \lhd, \rhd, \Box \rangle \to \langle q_h, \rhd, \lhd, S, S, S \rangle$
$\langle q_r, 0, 1, \Box \rangle \to \langle q_h, 1, \lhd, S, S, S \rangle$
$\langle q_r, 1, 0, \Box \rangle \to \langle q_r, 0, \lhd, S, S, S \rangle$

Outline

Language System

Let $\Sigma = \{a_1, \ldots, a_k\}$ be the set of symbols, called alphabet.

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For example, if $\Sigma = \{a, b\}$, we have

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Λ is the empty string, that has no symbols. (ε)

$\{0, 1, \Box, \rhd\}$ vs. Larger Alphabets

Fact: If $f : \{0, 1\}^* \to \{0, 1\}^*$ is computable in time $T(n)$ by a TM M using the alphabet set Γ, then it is computable in time $4 \log |\Gamma| T(n)$ by a TM $\widetilde{M}$ using the alphabet $\{0, 1, \Box, \rhd\}$.

$\{0, 1, \Box, \triangleright\}$ vs. Larger Alphabets

Suppose M has k tapes with the alphabet Γ.

A symbol of M is encoded in $\widetilde{M}$ by a string $\sigma \in \{0, 1\}^*$ of length $\log |\Gamma|$.

A state q in M is turned into a number of states in $\widetilde{M}$

- q,
- $\langle q, \sigma_1^1, \ldots, \sigma_1^k \rangle$ where $|\sigma_1^1| = \ldots = |\sigma_1^k| = 1$,
- $\cdots$,
- $\langle q, \sigma_{\log |\Gamma|}^1, \ldots, \sigma_{\log |\Gamma|}^k \rangle$, the size of $\sigma_{\log |\Gamma|}^1, \ldots, \sigma_{\log |\Gamma|}^k$ is $\log |\Gamma|$.

$\{0, 1, \Box, \triangleright\}$ vs. Larger Alphabets

To simulate one step of M, the machine $\widetilde{M}$ will

1. use $\log|\Gamma|$ steps to read from each tape the $\log|\Gamma|$ bits encoding a symbol of Γ,
2. use its state register to store the symbols read,
3. use M's transition function to compute the symbols M writes and M's new state given this information,
4. store this information in its state register, and
5. use $\log|\Gamma|$ steps to write the encodings of these symbols on its tapes.

$\{0, 1, \Box, \triangleright\}$ vs. Larger Alphabets

Example: $\{0, 1, \Box, \triangleright\}$ vs. English Alphabets

M's tape:	>	m	a	c	h	i	n	e									
$\tilde{M}$'s tape:	>	0	1	1	0	1	0	0	0	0	1	0	0	0	1	1	

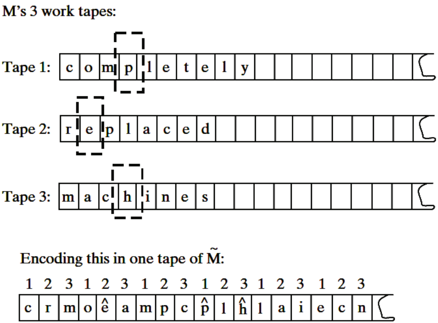
Single-Tape vs. Multi-Tape

Define a single-tape TM to be a TM that has one read-write tape.

Fact: If $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is computable in time $T(n)$ by a TM M using k tapes, then it is computable in time $5kT(n)^2$ by a single-tape TM $\widetilde{M}$.

Single-Tape vs. Multi-Tape

- The basic idea is to interleave k tapes into one tape.
- The first $n+1$ cells are reserved for the input.



- Every symbol a of M is turned into two symbols $a, \hat{a}$ in $\tilde{M}$, with $\hat{a}$ used to indicate head position.

Single-Tape vs. Multi-Tape

The outline of the algorithm:

The machine $\widetilde{M}$ places $\triangleright$ after the input string and then starts copying the input bits to the imaginary input tape. During this process whenever an input symbol is copied it is overwritten by $\triangleright$.

$\widetilde{M}$ marks the $n+2$-cell, . . ., the $n+k$-cell to indicate the initial head positions.

$\widetilde{M}$ Sweeps $kT(n)$ cells from the $(n+1)$-th cell to right, recording in the register the k symbols marked with the hat $\hat{\_}$.

$\widetilde{M}$ Sweeps $kT(n)$ cells from right to left to update using the transitions of M. Whenever it comes across a symbol with hat, it moves right k cells, and then moves left to update.

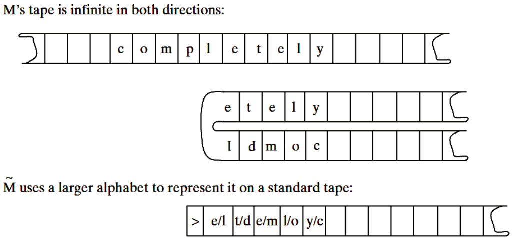
Unidirectional Tape vs. Bidirectional Tape

Define a bidirectional Turing Machine to be a TM whose tapes are infinite in both directions.

Fact: If $f : \{0,1\}^* \to \{0,1\}^*$ is computable in time $T(n)$ by a bidirectional TM M, then it is computable in time $4T(n)$ by a TM $\widetilde{M}$ with one-directional tape.

Unidirectional Tape vs. Bidirectional Tape

- The idea is that $\widetilde{M}$ makes use of the alphabet $\Gamma \times \Gamma$.



- Every state q of M is turned into $\bar{q}$ and $\underline{q}$.

Unidirectional Tape vs. Bidirectional Tape

Let H range over $\{L, S, R\}$ and let $-H$ be defined by

$$-H = \begin{cases} R, & \text{if} \quad H = L, \\ S, & \text{if} \quad H = S, \\ L, & \text{if} \quad H = R. \end{cases}$$

$\widetilde{M}$ contains the following transitions:

$$\langle \overline{q}, (\rhd, \rhd) \rangle \rightarrow \langle \underline{q}, (\rhd, \rhd), R \rangle$$
$$\langle \underline{q}, (\rhd, \rhd) \rangle \rightarrow \langle \overline{q}, (\rhd, \rhd), R \rangle$$

$$\langle \overline{q}, (a, b) \rangle \rightarrow \langle \overline{q'}, (a', b), H \rangle \text{ if } \langle q, a \rangle \rightarrow \langle q', a', H \rangle$$
$$\langle \underline{q}, (a, b) \rangle \rightarrow \langle \underline{q'}, (a, b'), -H \rangle \text{ if } \langle q, b \rangle \rightarrow \langle q', b', H \rangle$$

Outline

TM-Computable Function

What does it mean that a TM computes a (partial) n-ary function f?

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Let M be a TM and $a_1, \ldots, a_n, b \in \mathbb{N}$. When computation $M(a_1, \ldots, a_n)$ converges to b if $M(a_1, \ldots, a_n) \downarrow$ and $r_1 = b$ in the final configuration. We write $M(a_1, \ldots, a_n) \downarrow b$.

- M TM-computes f if, for all $a_1, \ldots, a_n, b \in \mathbb{N}$,

$$M(a_1, \ldots, a_n) \downarrow b \text{ iff } f(a_1, \ldots, a_n) = b$$

- Function f is TM-computable if there is a Turing Machine that TM-computes f.
- (We abbreviate "TM-computable" to "computable")

Function Defined by Program

Given any program P and $n \geq 1$, by thinking of the effect of P on initial configurations of the form $a_1, \cdots, a_n, 0, 0, \cdots$, there is a unique n-ary function that P computes, denoted by $f_P^{(n)}$.

$$f_P^{(n)}(a_1, \ldots, a_n) = \begin{cases} b, & \text{if } P(a_1, \ldots, a_n) \downarrow b, \\ \text{undefined}, & \text{if } P(a_1, \ldots, a_n) \uparrow. \end{cases}$$

Predicate and Decision Problem

The value of a predicate is either ‘true’ or ‘false’.

The answer of a *decision problem* is either ‘yes’ or ‘no’.

Example: Given two numbers x, y, check whether x is a multiple of y.
Input: x, y;
Output: ‘Yes’ or ‘No’.

The operation amounts to calculation of the function

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is a multiple of } y, \\ 0, & \text{if otherwise.} \end{cases}$$

Thus the property or predicate ‘x is a multiple of y’ is algorithmically or effectively decidable, or just decidable if function f is computable.

Decidable Predicate and Decidable Problem

Suppose that $P(x_1, \ldots, x_n)$ is an n-ary predicate of natural numbers. The characteristic function $c_P(\mathbf{x})$,

$$f_M^{(n)}(a_1, \ldots, a_n) = \begin{cases} 1, & \text{if } P(\mathbf{x}) \text{ holds}, \\ 0, & \text{if otherwise}. \end{cases}$$

The predicate $P(\mathbf{x})$ is decidable if c_P is computable; it is undecidable otherwise.

Computability on other Domains

Suppose D is an object domain. A coding of D is an explicit and effective injection $\alpha : D \to \mathbb{N}$. We say that an object $d \in D$ is coded by the natural number $\alpha(d)$.

A function $f : D \to D$ extends to a numeric function $f^* : \mathbb{N} \to \mathbb{N}$. We say that f is computable if f^* is computable.

$$f^* = \alpha \circ f \circ \alpha^{-1}$$

Example

Consider the domain $\mathbb{Z}$. An explicit coding is given by the function α where

$$\alpha(n) = \begin{cases} 2n, & \text{if } n \geq 0, \\ -2n-1, & \text{if } n < 0. \end{cases}$$

Then α^{-1} is given by

$$\alpha^{-1}(m) = \begin{cases} \frac{1}{2}m, & \text{if } m \text{ is even}, \\ -\frac{1}{2}(m+1), & \text{if } m \text{ is odd}. \end{cases}$$

Example (Continued)

Consider the function $f(x) = x - 1$ on $\mathbb{Z}$, then $f^* : \mathbb{N} \to \mathbb{N}$ is given by

$$f^*(x) = \begin{cases} 1 & \text{if } x = 0 \text{ (i.e. } x = \alpha(0)), \\ x - 2 & \text{if } x > 0 \text{ and } x \text{ is even } (\text{i.e. } x = \alpha(n), n > 0), \\ x + 2 & \text{if } x \text{ is odd } (\text{i.e. } x = \alpha(n), n < 0). \end{cases}$$

It is a routine exercise to write a program that computes f^*, hence $x - 1$ is a computable function on $\mathbb{Z}$.