Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Since $n > 2$, n and 2 are two distinct factors of $n!$. Therefore, $n! \geq 2n = n+n > n+1$, and thus $n! - 1 > n$.

   Let us consider a prime factor p of $n! - 1$. (Since $n! - 1 > n > 2$, $n! - 1$ must have a prime factor.) Since p is a divisor of $n! - 1$, we have $p \leq n! - 1 < n!$.

   Suppose for the sake of contradiction that $p \leq n$. Then since p is one of the positive integers less than or equal to n, p is a factor of $n!$. Thus we have p is a factor of both $n!$ and $n! - 1$, but this can not be true.

   If p is a factor of both $n!$ and $n! - 1$, it will be a factor of 1, their difference, and this is impossible. Therefore the assumption that $p \leq n$ leads to a contradiction, now we may conclude that $n < p < n!$. □

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   Proof. Let $P(n)$ be the statement: For any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   if $P(n)$ is not true, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

   Since $P(18) = 1 \times 4 + 2 \times 7$, we have $k > 18$, and $k - 1 > 17$.

   Since k is the smallest value for which $P(k)$ is false, $P(k-1)$ is true. Thus there exist integers i_{k-1} and j_{k-1}, such that $k - 1 = i_{k-1} \times 4 + j_{k-1} \times 7$. Note that $i_{k-1} \geq 1$, and $j_{k-1} \geq 2$.

   However, we have

$$\begin{aligned} k &= (k-1) + 1 \\ &= (i_{k-1} \times 4 + j_{k-1} \times 7) + 1 \\ &= (i_{k-1} + 2) \times 4 + (j_{k-1} - 1) \times 7 \end{aligned}$$

   Since $i_{k-1} + 2 \geq 0$ and $j_{k-1} - 1 \geq 0$, we have $P(k)$ is true. We have derived a contradiction, which allows us to conclude that our original assumption is false. □

3. Let $P = \{p_1, p_2, \cdots\}$ the set of all primes. Suppose that $\{p_i\}$ is monotonically increasing, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Please prove: $p_n < 2^{2^n}$. (Hint: $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n$.)

   Proof. Let $P(n)$ be the statement: $p_n < 2^{2^n}$. We try to prove $P(n)$ is true for any integer $n \geq 1$.

   Basis step. $P(1)$ is the statement that $2 < 4$. This is obviously true. $P(2)$ is the statement that $3 < 16$. This is also true.

   Induction hypothesis. $k \geq 2$, and for every n with $1 \leq n \leq k$, $p_n < 2^{2^n}$.

Proof of induction step. We first proof that $p_{k+1} \leq 1 + \prod_{j=1}^{k} p_j$.

If $1 + \prod_{j=1}^{k} p_j$ is a prime, since $\prod_{j=1}^{k} p_j > p_k$, we have $1 + \prod_{j=1}^{k} p_j \geq p_{k+1}$.

If $1 + \prod_{j=1}^{k} p_j$ is not a prime, then it must have prime factors. Since $p_i \nmid (1 + \prod_{j=1}^{k} p_j), (i = 1, 2, \cdots, k.)$ we may conclude that all of its prime factors are greater than or equal to p_{k+1}. Thus we also have $1 + \prod_{j=1}^{k} p_j \geq p_{k+1}$.

Now we have

$$p_{k+1} \leq 1 + \prod_{j=1}^{k} p_j < 1 + \prod_{j=1}^{k} 2^{2^j} = 1 + 2^{2^{k+1}-2} < 2^{2^{k+1}}$$

$\square$

4. Prove that a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors.

Proof. Let $P(n)$ be the statement that a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors.

Basis step. when $n = 1$, the plane is divided into two regions, it can be colored with only 2 colors, and the adjacent regions have different colors. Thus $P(1)$ is true.

Induction hypothesis. $k \geq 1$, and for every n with $1 \leq n \leq k$, $P(n)$ is true.

Proof of induction step. when $n = k + 1$, based on the original, we add a new line l_{k+1}, the color on one side of l_{k+1} remains unchanged, and the color on the other side flips.

If two regions were different regions and were colored with different colors when $n = k$, after the process they are still colored with different colors, whether they are on the same side of l_{k+1} or not.

If two regions were one region when $n = k$, then they used to be the same color and now they are on the different sides of l_{k+1}, after the process they are colored with different colors.

Thus we may conclude that $P(k + 1)$ is true. $\square$