Lab00-Solution

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

$*$ If there is any problem, please contact TA Yiming Liu.

1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof.

   Claim 1: For any integer $n > 2$, $n! - 1 > n$.

   Since $n > 2$, $n! > 2n = n + (n-1) + > n + 1$, which is equivalent to $n! - 1 > n$.

   Claim 2: $\forall n \in \mathbb{N}$ with $n \geq 2$, n has prime factorizations.

   The proof for this can be found in Slide01-Prologue.pdf

   If $n! - 1$ is a prime, $p = n! - 1$ will satisfy $n < p < n!$. If $n! - 1$ is not a prime, assume it's prime factor p satisfies $2 \leq p \leq n$.

$$\begin{aligned} \frac{n! - 1}{p} &= \frac{1 \times 2 \times 3 \times \cdots \times (p-1) \times p \times (p+1) \times \cdots \times n - 1}{p} \\ &= 1 \times 2 \times 3 \times \cdots \times (p-1) \times (p+1) \times \cdots \times n - \frac{1}{p} \end{aligned}$$

   Since $\frac{n!-1}{p}$ is not an integer, p is not a factor of $n! - 1$, which is a contradiction. Therefore, $n < p < n! - 1 < n!$

   □

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   Proof. If $P(n) = n = i_n \times 4 + j_n \times 7$ is not true for every integer $n > 17$, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

   Since $P(18) = 18 = 1 \times 4 + 2 \times 7$, we have $k \geq 18$, and $k - 1 \geq 17$.

   Since k is the smallest value for which $P(k) = k = i_k \times 4 + j_k \times 7$ is false, $P(k-1) = k - 1 = i_{k-1} \times 4 + j_{k-1} \times 7$ is true.

   However, we have

$$\begin{aligned} k &= (k-1) + 1 \\ &= i_{k-1} \times 4 + j_{k-1} \times 7 + 1 \\ &= (i_{k-1} + 2) \times 4 + (j_{k-1} - 1) \times 7 \\ &= (i_{k-1} - 5) \times 4 + (j_{k-1} + 3) \times 7 \end{aligned}$$

   As $n > 17$,

   if $j_{k-1} \geq 1, k = (i_{k-1} + 2) \times 4 + (j_{k-1} - 1) \times 7$;

   if $j_{k-1} = 0, k = (i_{k-1} - 5) \times 4$, we have $20 = 5 \times 4$ which is the smallest $n = 20$ with $j_n = 0$.

   Thus, one of these two expressions must be true. We have derived a contradiction, which allows us to conclude that our original assumption is false.

   □

3. Let $P = \{p_1, p_2, \cdots\}$ the set of all primes. Suppose that $\{p_i\}$ is monotonically increasing, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Please prove: $p_n < 2^{2^n}$. (Hint: $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n.$)

   Proof. (a) $i = 1$: It can be easily verified that $p_i = 2 < 2^{2^2}$

   (b) Assume that $p_i < 2^{2^t}$ for $\forall i \leq k$, then we have

   $$\prod_{i=1}^{k} p_i < 2^{\Sigma_{i=1}^{k} 2^i} = 2^{2^{k+1}-2}$$

   hence

   $$\prod_{i=1}^{k} p_i + 1 < 2^{2^{k+1}-2} + 1 < 2^{2^{k+1}}$$

   since $p_i \nmid \left(1 + \prod_{j=1}^{k} p_j\right), i = 1, 2, \cdots, k$, we set p^* as the largest prime factor of $\left(1 + \prod_{i=1}^{k} p_i\right)$ and we have

   $$p_{k+1} \leq p^* \leq 1 + \prod_{i=1}^{k} p_i < 2^{2^{k+1}}$$

   $\square$

   Remark: The hint is obvious since $\forall i \in \{1, 2, \cdots, n\}$

   $$\frac{1 \backslash \Pi_{j=1}^{n} p_j}{p_i} = \frac{1}{p_i} + \prod_{j=1}^{i-1} p_j \prod_{k=i+1}^{n} p_k$$

   where the result is not an integer since $\frac{1}{p_i} \notin \mathbb{Z}$ but $\left(\prod_{j=1}^{i-1} p_j \prod_{k=i+1}^{n} p_k\right) \in \mathbb{Z}$.

4. Prove that a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors.

   Proof. Define P(n) as the statement that "a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors."

   Basis step. When $n = 1$, obviously the plane could be colored with 2 colors, $P(1)$ is true.

   Induction hypothesis. $P(k)$ is true for $k \geq 2$.

   Proof of induction step. When $n = k + 1$, the plane can be divided into 2 parts (lets denote them as A and B)by the $k + 1^{th}$ line. We simply swap colors of regions in A and keep colors of regions in B the same. And $P(k + 1)$ is true. $\square$

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.