Graph Decomposition*

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*Special Thanks is given to Prof. Yijia Chen for sharing his teaching materials.

Outline

Exploring Graphs

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Algorithm 1: EXPLORE(G, v)

Input: $G = (V, E)$ is a graph; $v \in V$

Output: VISITED(u) = *true* for all nodes u **reachable** from v

```
1 VISITED(v) = true;
2 PREVISIT(v);
3 foreach edge (v, u) ∈ E do
4     if not VISITED(u) then
5         EXPLORE(G, u);
6 POSTVISIT(v);
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- PREVISIT, POSTVISIT procedures are optional.
- work on a vertex when first discovered and left for the last time.

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EXPLORE moves from node to their neighbors; it can never jump to a region not reachable from v.

Every node reachable from v must be visited:
If $\exists u$ that EXPLORE misses, choose a path from v to u. Let z be the last vertex on that path that EXPLORE visited. Let w be the node immediately after it on this path.

So z was visited but w was not. This is a contradiction: while EXPLORE was at node z, it would have noticed w and moved on to it.

Depth-First Search

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Algorithm 2: DFS(G)

Input: $G = (V, E)$ is a graph

Output: VISITED(v) is set to *true* for all nodes $v \in V$

```
1 foreach v ∈ V do
2     VISITED(v) = false;
3 foreach v ∈ V do
4     if not VISITED(v) then
5         EXPLORE(G, v);
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- A loop in which adjacent edges are scanned, to see if they lead somewhere new.

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Thus the depth-first search has a running time of $O(|V| + |E|)$.

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$\text{CCNUM}[v] = cc$

Initially, $cc = 0$, will increment each time DFS calls EXPLORE.

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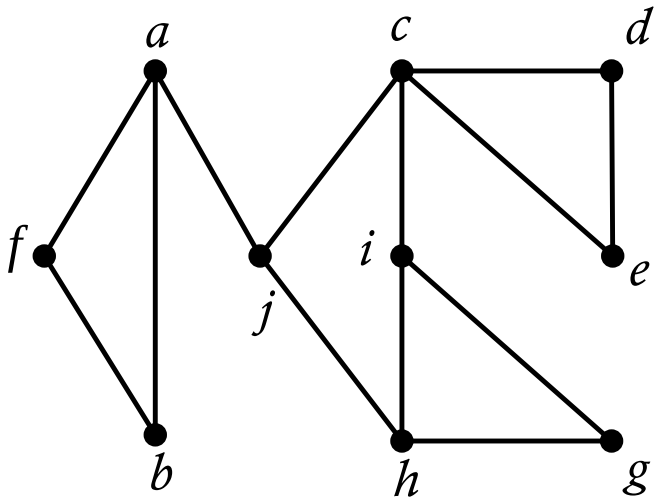
Lemma: $\forall u, v \in V$, intervals $[\text{PRE}(u), \text{POST}(u)]$, $[\text{PRE}(v), \text{POST}(v)]$ are either *disjoint* or *one is contained within the other*.

An executing example

Assume we use alphabetical order to explore G:



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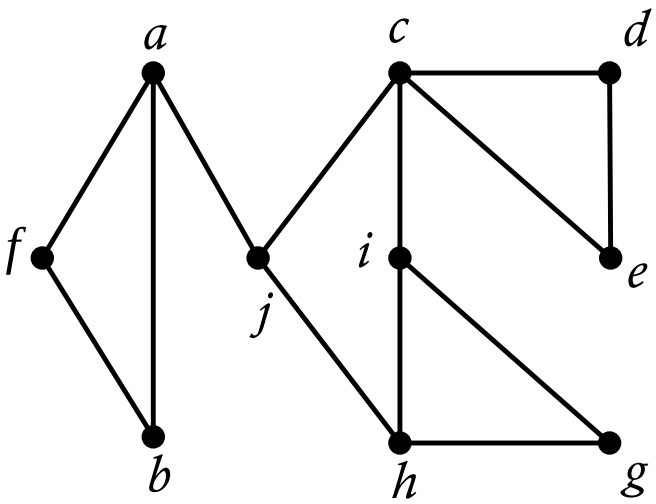

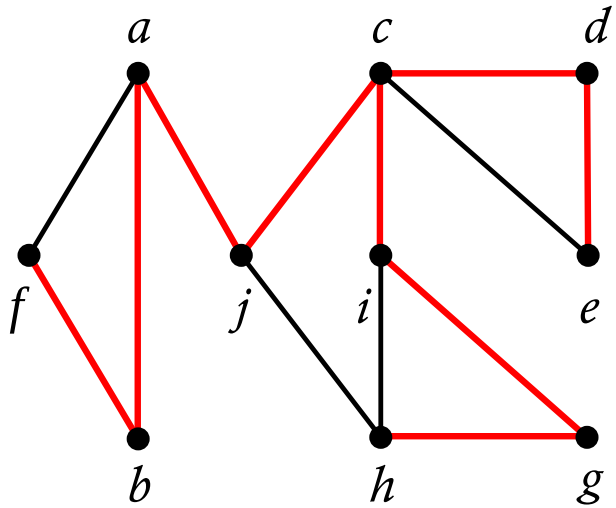

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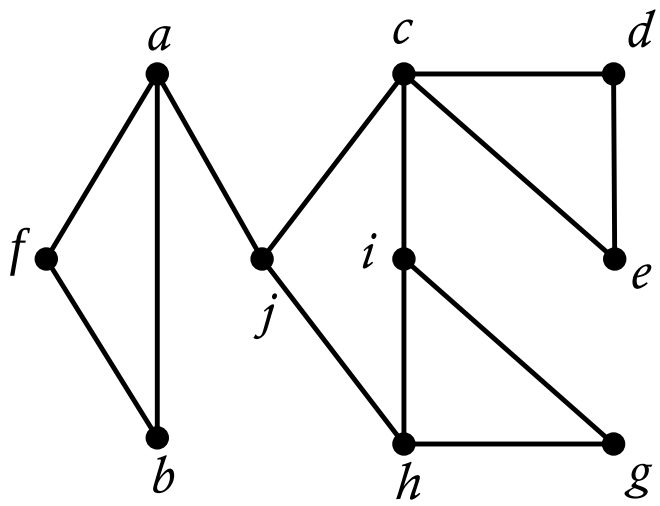

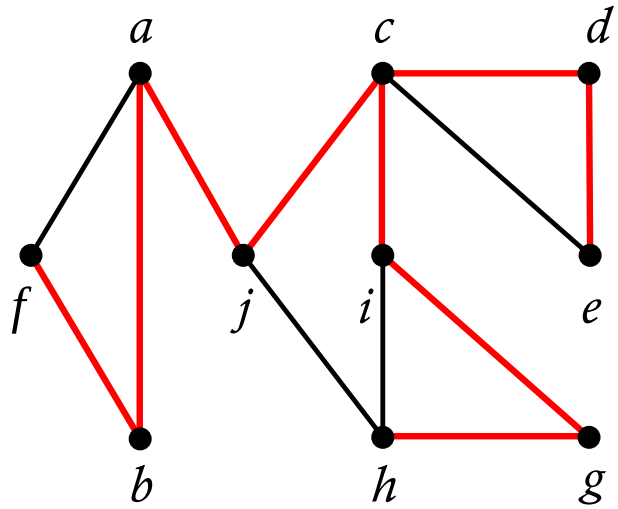

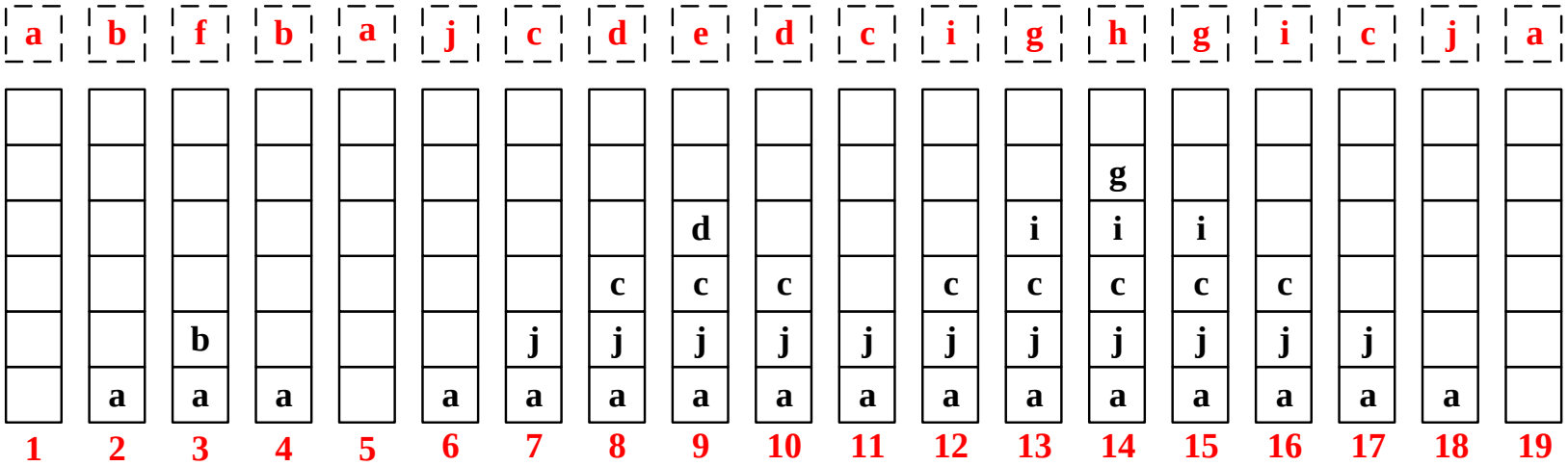

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PRE/POST ordering for (u, v)	Edge type
$[u \quad [v \quad]v \quad]u$	Tree/forward
$[v \quad [u \quad]u \quad]v$	Back
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All the other v_j on the cycle are reachable from it and will therefore be its descendants in the search tree.

In particular, the edge $v_{i-1} \to v_i$ (or $v_k \to v_0$ if $i = 0$) is a back edge.

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The guaranteed existence of a source suggests an alternative approach to linearization:

① Find a source, output it, and delete it from the graph.
② Repeat until the graph is empty.

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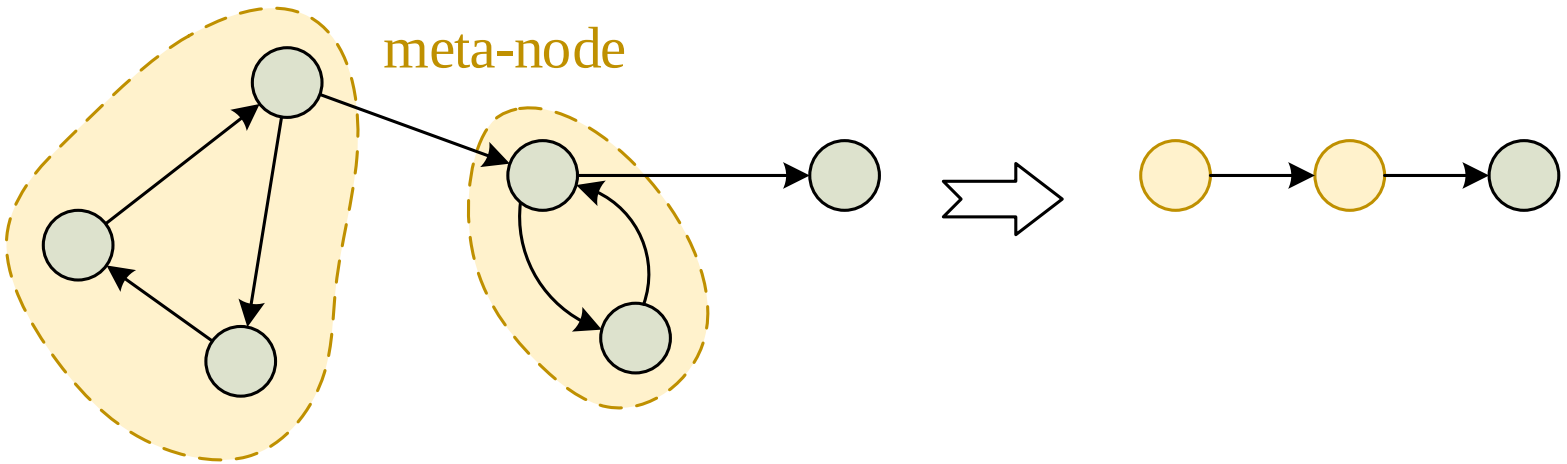
Lemma: Every directed graph is a dag of its strongly connected components.



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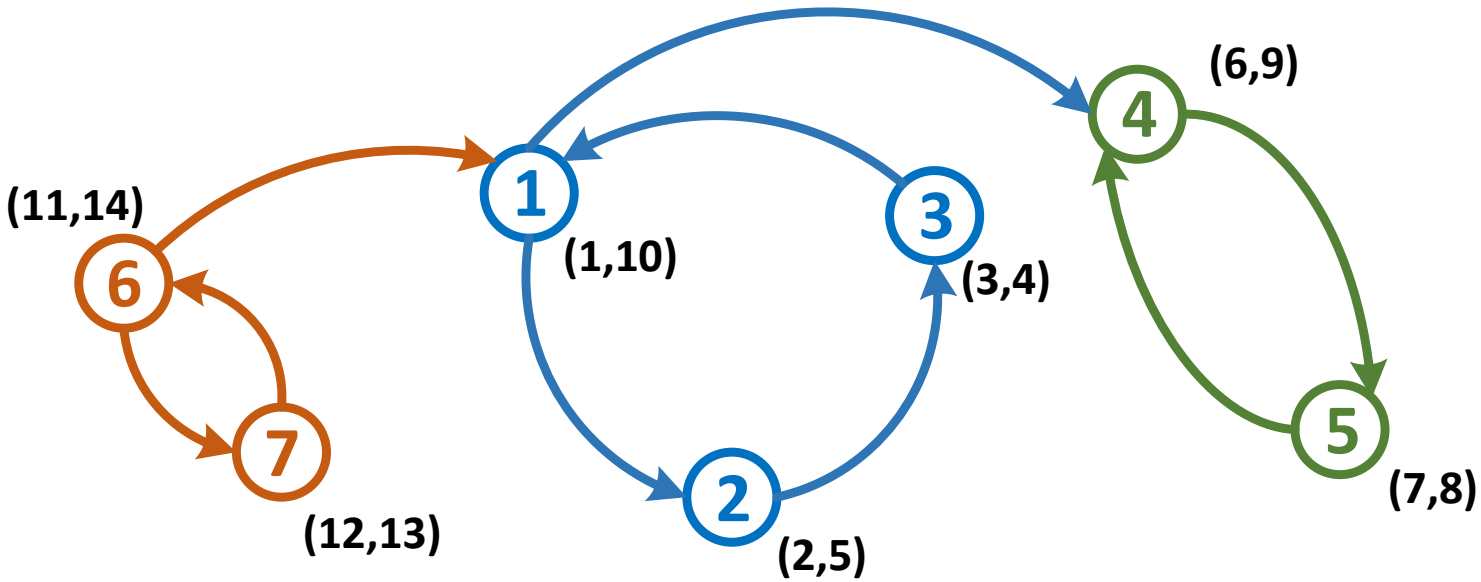
Lemma: The node that receives the highest POST number in a depth-first search must lie in a *source strongly connected component*.

Investigation (Cont')

Note: The smallest POST number in a depth-first search may NOT lie in a *sink strongly connected component*!

An Counter Example: (Node ID denotes the explore order)



The smallest POST number is Node 3, NOT in the sink strongly connected component (green).

An Efficient Algorithm

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To design a linear-time algorithm, we have two problems:

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Solving Problem A:

Consider the **reverse graph** G^R, the same as G but with all edges reversed (has exactly the same strongly connected components as G).

So, if we do a depth-first search of G^R, the node with the highest POST number will come from a source strongly connected component in G^R, which is a sink strongly connected component in G.

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Solving Problem B:

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The Linear-Time Algorithm:

1. Run depth-first search on G^R.
2. Run depth-first search on G, and process the vertices in decreasing order of their POST numbers from step 1.

Outline

Breadth-First Search

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Algorithm 3: BFS(G, s)

Input: Graph $G = (V, E)$, directed or undirected; vertex $s \in V$
Output: DIST(u) is set to the distance from s to all reachable u

```
1  foreach u ∈ V do
2      DIST(u) = ∞;
3  DIST(s) = 0;
4  Q = [s] (queue containing just s);
5  while Q is not empty do
6      u = EJECT(Q);
7      foreach edge (u, v) ∈ E do
8          if DIST(v) = ∞ then
9              INJECT(Q, v);
10             DIST(v) = DIST(u) + 1;
```

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Lemma: For each $d = 0, 1, 2, \ldots$, there is a moment at which

(1) all nodes at distance $\leq d$ from s have their distances correctly set;

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Lemma: BFS has a running time of $O(|V| + |E|)$.

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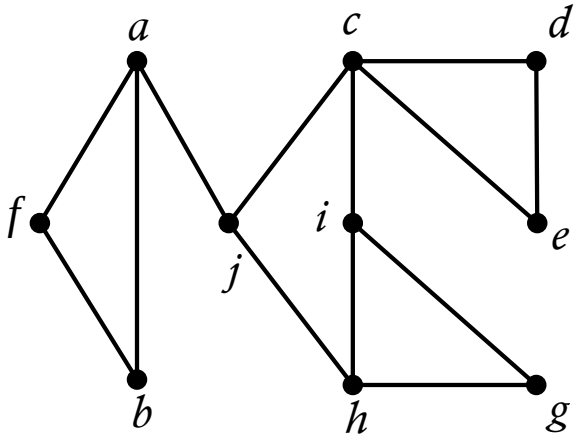

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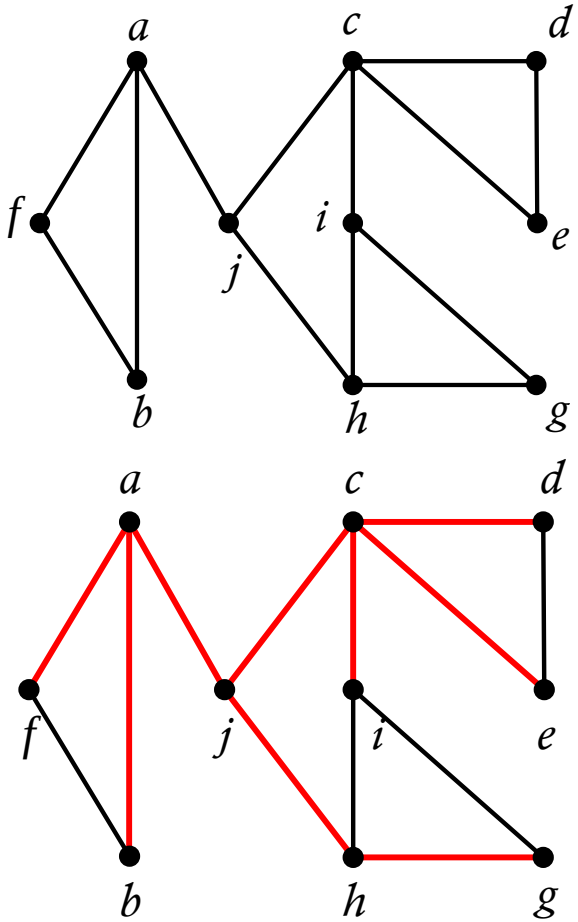

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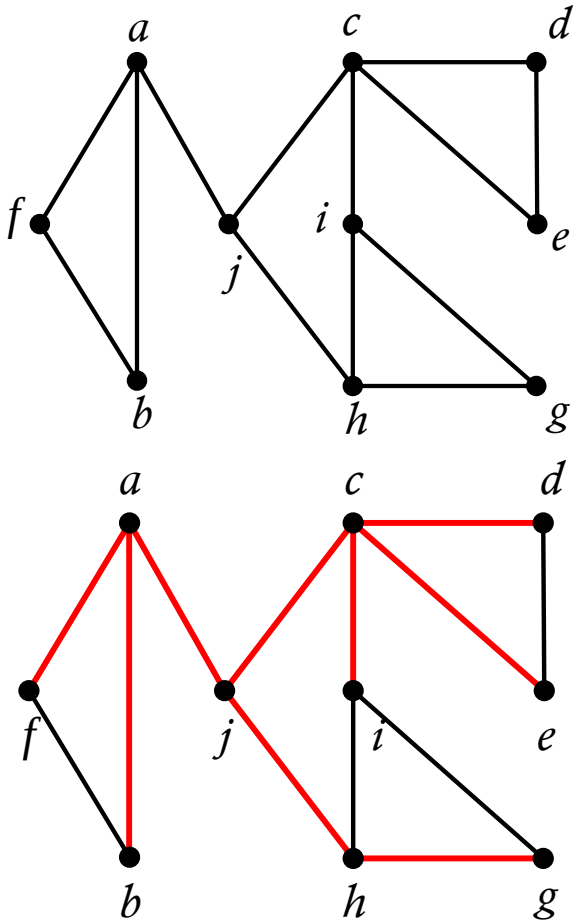

1		a					
2	a	b	f	j			
3	b	f	j				
4	f	j					
5	j	c	h				
6	c	h	d	e	i		
7	h	d	e	i	g		
8	d	e	i	g			
9	e	i	g				
10	i	g					
11	g						