Lab06-Linear Programming

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

∗ If there is any problem, please contact TA Yiming Liu.

∗ Name:Zehao Wang   Student ID: 518021910976   Email: davidwang200099@sjtu.edu.cn

1. **Controlling Air Pollution.** The three main types of pollutants in an airshed are particulate matter, sulfur oxides, and hydrocarbons. The new standards require that the steelworks reduce its annual emission of these pollutants by the amounts shown in the following table:

Pollutant	**Required Reduction in Annual Emission Rate (Million Pounds)**
Particulates	60
Sulfur oxides	150
Hydrocarbons	125

The steelworks has two primary sources of pollution, namely, the blast furnaces for making pig iron and the open-hearth furnaces for changing iron into steel. In both cases the engineers have decided that the most effective types of abatement methods are (1) increasing the height of the smokestacks, (2) using filter devices (including gas traps) in the smokestacks, and (3) including cleaner, high-grade materials among the fuels for the furnaces. Note that each of these methods has a technological limit on how heavily it can be used (e.g., a maximum feasible increase in the height of the smokestacks), but there also is considerable flexibility for using the method at a fraction of its technological limit.

The following table shows how much emission (in millions of pounds per year) can be eliminated from each type of furnace by fully using any abatement method to its technological limit. For purposes of analysis, it is assumed that each method also can be used less fully to achieve any fraction of the emission-rate reductions shown in this table. Furthermore, the fractions can be different for blast furnaces and for open-hearth furnaces. For either type of furnace, the emission reduction achieved by each method is not substantially affected by whether the other methods also are used.

	Taller Smokestacks		**Filters**		**Better Fuels**	
Pollutant	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**	**Blast Furnaces**	**Open-Hearth Furnaces**
Particulates	12	9	25	20	17	13
Sulfur oxides	35	42	18	31	56	49
Hydrocarbons	37	53	28	24	29	20

The total annual cost from the maximum feasible use of an abatement method (in millions of dollars) was shown in the following table. The board of directors wants to figure out how to achieve these reductions with minimum annual cost. Please design a scheme for them.

Abatement Method	**Blast Furnaces**	**Open-Health Furnaces**
Taller smokestacks	8	10
Filters	7	6
Better fuels	11	9

(a) Formulate a linear programming with necessary explanations.

(b) Transform your LP into its standard form.

(c) Transform your LP into its dual form.

(d) Assume that the clean air standards have been relaxed. The steelworks only needs to meet any two of the three pollutants emission standards. Please update your LP in (a) to satisfy the relaxed clean air standards. (Hint: You can refer to Reference14-ModelFormulation.pdf)

Solution.

(a) Assume that the table below denote how full each method is used on the two furnaces.

Abatement Method	Blast Furnaces	Open-Health Furnaces
Taller smokestacks	x_1	x_2
Filters	x_3	x_4
Better fuels	x_5	x_6

Now that x_i's mean how full a method is used, they are numbers not smaller than 0 and not larger than 1.

This problem requires to minimize the total cost. So according to table (1), we must minimize this formula:

$$8x_1 + 10x_2 + 7x_3 + 6x_4 + 11x_5 + 9x_6 \tag{1}$$

We are required to meet the requirements for pollutant reduction on all of Particulates, Sulfur oxides and Hydrocarbons.

So the requirements can be formulated as follows.

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 = 60 \tag{2}$$

$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 = 150 \tag{3}$$

$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 = 125 \tag{4}$$

Therefore, this linear programming problem can be formulated as follows:

$$\min \quad 8x_1 + 10x_2 + 7x_3 + 6x_4 + 11x_5 + 9x_6$$

s.t.

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 = 60$$
$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 = 150$$
$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 = 125$$
$$0 \leq x_i \leq 1 \ (1 \leq i \leq 6, i \in \mathbb{Z})$$

(b) The LP problem above is not a standard form. It is a minimization problem and has both equality constraints and inequality constrains with $\geq$.

So to change the problem above into standard form, we need to change minimization into maximization:

$$\max \quad -8x_1 - 10x_2 - 7x_3 - 6x_4 - 11x_5 - 9x_6$$

And then change equality constrains into inequality:

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \leq 60$$
$$-12x_1 - 9x_2 - 25x_3 - 20x_4 - 17x_5 - 13x_6 \leq -60$$
$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \leq 150$$
$$-35x_1 - 42x_2 - 18x_3 - 31x_4 - 56x_5 - 49x_6 \leq -150$$
$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \leq 125$$
$$-37x_1 - 53x_2 - 28x_3 - 29x_4 - 29x_5 - 20x_6 \leq -125$$
$$x_i \leq 1 (1 \leq i \leq 6,\ i \in \mathbb{Z})$$

Then constrains on x_i's which ensure they are non-negative:

$$x_i \geq 0 (1 \leq i \leq 6,\ i \in \mathbb{Z})$$

Therefore the standard form of this problem is:

$$\max \quad -8x_1 - 10x_2 - 7x_3 - 6x_4 - 11x_5 - 9x_6$$

s.t.

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \leq 60$$
$$-12x_1 - 9x_2 - 25x_3 - 20x_4 - 17x_5 - 13x_6 \leq -60$$
$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \leq 150$$
$$-35x_1 - 42x_2 - 18x_3 - 31x_4 - 56x_5 - 49x_6 \leq -150$$
$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \leq 125$$
$$-37x_1 - 53x_2 - 28x_3 - 24x_4 - 29x_5 - 20x_6 \leq -125$$
$$x_i \leq 1 (1 \leq i \leq 6,\ i \in \mathbb{Z})$$
$$x_i \geq 0 (1 \leq i \leq 6,\ i \in \mathbb{Z})$$

(c) According to the matrix-vector form of LP, we can know:

$$\mathbf{c} = \begin{bmatrix} -8 & -10 & -7 & -6 & -11 & -9 \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} 12 & 9 & 25 & 20 & 17 & 13 \\ -12 & -9 & -25 & -20 & -17 & -13 \\ 35 & 42 & 18 & 31 & 56 & 49 \\ -35 & -42 & -18 & -31 & -56 & -49 \\ 37 & 53 & 28 & 24 & 29 & 20 \\ -37 & -53 & -28 & -24 & -29 & -20 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 60 & -60 & 150 & -150 & 125 & -125 & 1 & \cdots & 1 \end{bmatrix}^T$$

Its dual form is:

$$min \quad \mathbf{y}^T\mathbf{b}$$

s.t.

$$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$$
$$\mathbf{y} \geq \mathbf{0}$$

Namely

$$min \quad 60y_1 - 60y_2 + 150y_3 - 150y_4 + 125y_5 - 125y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}$$

s.t.

$$12y_1 - 12y_2 + 35y_3 - 35y_4 + 37y_5 - 37y_6 + y_7 \geq -8$$
$$9y_1 - 9y_2 + 42y_3 - 42y_4 + 53y_5 - 53y_6 + y_8 \geq -10$$
$$25y_1 - 25y_2 + 18y_3 - 18y_4 + 28y_5 - 28y_6 + y_9 \geq -7$$
$$20y_1 - 20y_2 + 31y_3 - 31y_4 + 24y_5 - 24y_6 + y_{10} \geq -6$$
$$17y_1 - 17y_2 + 56y_3 - 56y_4 + 29y_5 - 29y_6 + y_{11} \geq -11$$
$$13y_1 - 13y_2 + 49y_3 - 49y_4 + 20y_5 - 20y_6 + y_{12} \geq -9$$
$$y_i \geq 0 (1 \leq i \leq 12, i \in \mathbb{Z})$$

(d) We can introduce binary variables and a very large number to omit some constrains.

$$12x_1 + 9x_2 + 25x_3 + 20x_4 + 17x_5 + 13x_6 \leq 60 + 2^{31}y_1$$
$$-12x_1 - 9x_2 - 25x_3 - 20x_4 - 17x_5 - 13x_6 \leq -60 + 2^{31}y_1$$
$$35x_1 + 42x_2 + 18x_3 + 31x_4 + 56x_5 + 49x_6 \leq 150 + 2^{31}y_2$$
$$-35x_1 - 42x_2 - 18x_3 - 31x_4 - 56x_5 - 49x_6 \leq -150 + 2^{31}y_2$$
$$37x_1 + 53x_2 + 28x_3 + 24x_4 + 29x_5 + 20x_6 \leq 125 + 2^{31}y_3$$
$$-37x_1 - 53x_2 - 28x_3 - 29x_4 - 29x_5 - 20x_6 \leq -125 + 2^{31}y_3$$
$$y_1 + y_2 + y_3 \leq 1$$
$$y_1, y_2, y_3 \in \{0, 1\}$$
$$x_i \leq 1 (1 \leq i \leq 6, \ i \in \mathbb{Z})$$
$$x_i \geq 0 (1 \leq i \leq 6, \ i \in \mathbb{Z})$$

2. **Factory Production.** An engineering factory makes seven products (PROD 1 to PROD 7) on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and two planer. Each product yields a certain contribution to profit (in £/unit). These quantities (in £/unit) together with the unit production times (hours) required on each process are given below. A dash indicates that a product does not require a process.

   There are marketing limitations on each product in each month, given in the following table:

   It is possible to store up to 100 of each product at a time at a cost of £0.5 per unit per month (charged at the end of each month according to the amount held at that time). There are no

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
Contribution to profit	10	6	8	4	11	9	3
Grinding	0.5	0.7	0.2	-	0.3	0.2	0.5
Vertical drilling	0.1	0.2	0	0.3	-	0.6	-
Horizontal drilling	0.2	-	0.8	-	-	-	0.6
Boring	0.05	0.03	-	0.07	0.1	-	0.08
Planing	-	-	0.01	-	0.05	0.02	0.04

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

stocks at present, but it is desired to have a stock of exactly 50 of each type of product at the end of June. The factory works six days a week with two shifts of 8h each day. It may be assumed that each month consists of only 24 working days. Each machine must be down for maintenance in one month of the six. No sequencing problems need to be considered.

When and what should the factory make in order to maximize the total net profit?

(a) Use *CPLEX Optimization Studio* to solve this problem. Describe your model in *Optimization Programming Language* (OPL). Remember to use a separate data file (.dat) rather than embedding the data into the model file (.mod).

(b) Solve your model and give the following results.

   i. For each machine:

      A. the month for maintenance.

   ii. For each product:

      A. The amount to make in each month.

      B. The amount to sell in each month.

      C. The amount to hold at the end of each month.

   iii. The total selling profit.

   iv. The total holding cost.

   v. The total net profit (selling profit minus holding cost).

Solution.
The total selling profit is 109330 £.
The total holding cost is 475£.
The total net profit is 108855£.

Table 1: The month for maintenance

	January	February	March	April	May	June
Grinding	0	0	0	4	0	0
Vertical drilling	0	1	0	1	0	0
Horizontal drilling	1	0	0	2	0	0
Boring	0	0	0	1	0	0
Planing	0	0	0	1	0	0

Table 2: The amount to make in each month

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	400	700	100	100	600	400	200
April	0	0	0	0	0	0	0
May	0	100	500	100	1000	300	0
June	550	550	150	350	1150	550	110

Table 3: The amount to sell in each month

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	100	100	100	100	100	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

Table 4: the amout to hold in each month

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
January	0	0	0	0	0	0	0
February	0	0	0	0	0	0	0
March	100	100	100	100	100	0	100
April	0	0	0	0	0	0	0
May	0	0	0	0	0	0	0
June	50	50	50	50	50	50	50