# Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

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1. Prove that for any integer $n > 2$, there is a prime $p$ satisfying $n < p < n!$. (Hint: consider a prime factor $p$ of $n! - 1$ and prove by contradiction)

   **Proof.** Assume that there is no prime satisfying $n < p < n!$.Then $\forall m$ which satisfies $n < m < n!$ , m is not a prime.

   Consider $m = n! - 1$ which follows $n < m < n!$. $\because \forall\ p <= n, \frac{n!-1}{p} = \frac{n!}{p} - \frac{1}{p}, \frac{n!}{p}$ is an integer,$\frac{1}{p}$ is a fraction, So $\frac{n!-1}{p}$ is not an interger.

   $\therefore n! - 1$ is not divisible by primes smaller than $n$.

   $\therefore \exists$ prime $p_0 \in (n, n!)$,by which $n! - 1$ is divisible,which is contradictive to the assumption.

   $\therefore$ the former statement makes sense.

   □

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   **Proof.** Assume that $\exists$ a smallest $n_0 > 17$,there do not exist integers $i_{n_0} \geq 0$ and $j_{n_0} \geq 0$, such that $n_0 = i_{n_0} \times 4 + j_{n_0} \times 7$.

   Then $\exists i_{n_0-1} \geq 0, j_{n_0-1} \geq 0$, such that $n_0 - 1 = i_{n_0-1} \times 4 + j_{n_0-1} \times 7$.

   (a) $j_{n_0-1} = 0$
   $\because n_0 - 1 > 17 \therefore i_{n_0-1} \geq 5$.
   then $n_0 = (i_{n_0-1} - 5) \times 4 + 1 \times 7$, namely $\exists i_{n_0} = i_{n_0-1} - 5, j_{n_0} = 1$, such that $n_0 = i_{n_0} \times 4 + j_{n_0} \times 7$, which is objective to the assumption.

   (b) $j_{n_0-1} > 0$

   then $n_0 = (i_{n_0-1} + 2) \times 4 + (j_{n_0-1} - 1) \times 7$, namely $\exists i_{n_0} = i_{n_0-1} + 2, j_{n_0} = j_{n_0-1} - 1$, such that $n_0 = i_{n_0} \times 4 + j_{n_0} \times 7$,which is also objective to the assumption.

   $\therefore$ for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   □

3. Let $P = \{p_1, p_2, \cdots\}$ the set of all primes. Suppose that $\{p_i\}$ is monotonically increasing, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Please prove: $p_n < 2^{2^n}$. (Hint: $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n.$)

   **Proof.** (a) When n=1, $p_1 = 2 < 2^{2^1} = 4$. The former statement is true.

   (b) Assumes that when $n = k$, the former statement is true, namely $p_k < 2^{2^k}$

   (c) Then $\prod_{j=1}^{n} p_j < \prod_{j=1}^{n} 2^{2^j} = 2^{2^{n+1}-2} < 2^{2^{n+1}}$
   $\because p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n.$
   $\therefore \exists i_0 > n$, which satisfies that $p_{i_0} \mid (1 + \prod_{j=1}^{n} p_j)$
   $\because p_{n+1} \leq p_{i_0}$
   $\therefore p_{n+1} < (1 + \prod_{j=1}^{n} p_j) < 2^{2^{n+1}}$
   From (a),(b) and (c), we can know that the former statement is true.

   □

4. Prove that a plane divided by $n$ lines can be colored with only 2 colors, and the adjacent regions have different colors.

**Proof.** To prove the statement,we first need to prove that $n$ lines can divide a plain into at most $\frac{n^2+n+1}{2}$ areas.

(a) When $n = 1$, one line can divide a plain into at most $2 = \frac{1^2+1+1}{2}$ areas.

(b) Assumes that when $n = k$, $k$ lines can divide a plain into at most $\frac{k^2+k+1}{2}$ areas.

(c) Then when $n = k + 1$, line No.$(k + 1)$ can be divided into $(k + 1)$ parts by the former $k$ lines. So it can add at most $(k+1)$ areas to the plain. So $(k+1)$ lines can divide a plain into at most $\frac{k^2+k+1}{2} + k + 1 = \frac{(k+1)^2+(k+1)+1}{2}$ areas. Therefore, it can be proved that $n$ lines can divide a plain into at most $\frac{n^2+n+1}{2}$ areas.

Then we need to prove that to make sure adjacent areas have different colors, at most $\frac{n^2+n}{3}$ can be filled with the same color.

It does not matter to assume that the lines are not parrallel. Assumes that there are $m_k$ areas with $k$ line segments or half-lines as their boundaries.

Then $\sum_{j=1}^{n} m_j \leq n^2$.

$\because m_2 < n \therefore \sum_{j=1}^{n} m_j \leq \frac{m_2}{3} + \frac{\sum_{j=2}^{n} jm_j}{3} \leq \frac{n^2+n}{3}$

Considering that we have 2 colors, there should be not more than $\frac{2n^2+2n}{3}$ areas to be filled.

Because $n$ lines can divide a plain into at most $\frac{n^2+n+1}{2}$ areas, which is less than $\frac{2n^2+2n}{3}$, therefore we can prove that $n$ lines can be colored with only 2 colors, and the adjacent regions have different colors.

□

**Remark:** You need to include your .pdf and .tex files in your uploaded .rar or .zip file.