For some reason it is popular to write the solution of the system of equations $AX = B$ in this form, and it is often this form that is called *Cramer's Rule*. However, this expression does not simplify computation. The main thing to remember is expression (5.8) for the inverse of a matrix in terms of its adjoint; the other formulas follow from this expression.

As with the complete expansion of the determinant (4.10), formulas (5.8–5.11) have theoretical as well as practical significance, because the answers A^{-1} and X are exhibited explicitly as quotients of polynomials in the variables $\{a_{ij}, b_i\}$, with integer coefficients. If, for instance, a_{ij} and b_j are all continuous functions of t, so are the solutions x_i.

A general algebraical determinant in its developed form
may be likened to a mixture of liquids seemingly homogeneous,
but which, being of differing boiling points, admit of being separated
by the process of fractional distillation.

James Joseph Sylvester

EXERCISES

1. The Basic Operations

1. What are the entries a_{21} and a_{23} of the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 0 & 9 & 4 \end{bmatrix}$?

2. Compute the products AB and BA for the following values of A and B.

   (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} -8 & -4 \\ 9 & 5 \\ -3 & -2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, B = [1 \quad 2 \quad 1]$

3. Let $A = (a_1, \dots, a_n)$ be a row vector, and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector. Compute the products AB and BA.

4. Verify the associative law for the matrix product

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Notice that this is a self-checking problem. You have to multiply correctly, or it won't come out. If you need more practice in matrix multiplication, use this problem as a model.

5. Compute the product $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}$.

6. Compute $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}^n$.

7. Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

8. Compute the following matrix products by block multiplication:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right], \quad \left[\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right]\left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 2 & 3 \\ 5 & 0 & 4 \end{array}\right].$$

9. Prove rule (1.20) for block multiplication.

10. Let A, B be square matrices.
(a) When is $(A + B)(A - B) = A^2 - B^2$?
(b) Expand $(A + B)^3$.

11. Let D be the diagonal matrix

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

and let $A = (a_{ij})$ be any $n \times n$ matrix.
(a) Compute the products DA and AD.
(b) Compute the product of two diagonal matrices.
(c) When is a diagonal matrix invertible?

12. An $n \times n$ matrix is called *upper triangular* if $a_{ij} = 0$ whenever $i > j$. Prove that the product of two upper triangular matrices is upper triangular.

13. In each case, find all real 2×2 matrices which commute with the given matrix.
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$

14. Prove the properties $0 + A = A$, $0A = 0$, and $A0 = 0$ of zero matrices.

15. Prove that a matrix which has a row of zeros is not invertible.

16. A square matrix A is called *nilpotent if* $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible.

17. **(a)** Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

(b) Prove that there is no matrix C such that $AC = I_3$.

18. Write out the proof of Proposition (1.18) carefully, using the associative law to expand the product $(AB)(B^{-1}A^{-1})$.

19. The *trace* of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}.$$

(a) Show that $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$, and that $\operatorname{tr} AB = \operatorname{tr} BA$.

(b) Show that if B is invertible, then $\operatorname{tr} A = \operatorname{tr} BAB^{-1}$.

20. Show that the equation $AB - BA = I$ has no solutions in $n \times n$ matrices with real entries.

2. Row Reduction

1. (a) For the reduction of the matrix M (2.10) given in the text, determine the elementary matrices corresponding to each operation.

(b) Compute the product P of these elementary matrices and verify that PM is indeed the end result.

2. Find all solutions of the system of equations $AX = B$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & -2 \end{bmatrix}$$

and B has the following value:

$$\text{(a)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

3. Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.

4. Determine the elementary matrices which are used in the row reduction in Example (2.22) and verify that their product is A^{-1}.

5. Find inverses of the following matrices:

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

6. Make a sketch showing the effect of multiplication by the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$ on the plane $\mathbb{R}^2$.

7. How much can a matrix be simplified if both row and column operations are allowed?

8. (a) Compute the matrix product $e_{ij}e_{k\ell}$.

(b) Write the identity matrix as a sum of matrix units.

(c) Let A be any $n \times n$ matrix. Compute $e_{ii}Ae_{jj}$.

(d) Compute $e_{ij}A$ and Ae_{ij}.

9. Prove rules (2.7) for the operations of elementary matrices.

10. Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A$ either is the identity or has its bottom row zero.

11. Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

12. Prove that if a product AB of $n \times n$ matrices is invertible then so are the factors A, B.

13. A matrix A is called symmetric if $A = A^t$. Prove that for any matrix A, the matrix AA^t is symmetric and that if A is a square matrix then $A + A^t$ is symmetric.

14. (a) Prove that $(AB)^t = B^tA^t$ and that $A^{tt} = A$.
(b) Prove that if A is invertible then $(A^{-1})^t = (A^t)^{-1}$.

15. Prove that the inverse of an invertible symmetric matrix is also symmetric.

16. Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

17. Let A be an $n \times n$ matrix. Prove that the operator "left multiplication by A" determines A in the following sense: If $AX = BX$ for very column vector X, then $A = B$.

18. Consider an arbitrary system of linear equations $AX = B$ where A and B have real entries.
(a) Prove that if the system of equations $AX = B$ has more than one solution then it has infinitely many.
(b) Prove that if there is a solution in the complex numbers then there is also a real solution.

*19. Prove that the reduced row echelon form obtained by row reduction of a matrix A is uniquely determined by A.

3. Determinants

1. Evaluate the following determinants:

(a) $\begin{bmatrix} 1 & i \\ 2-i & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 0 & 9 & 7 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 4 & 1 & 3 \\ 2 & 3 & 5 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

2. Prove that $\det \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 1 & 7 & 7 \\ 0 & 0 & 2 & 3 \\ 4 & 2 & 1 & 5 \end{bmatrix} = -\det \begin{bmatrix} 2 & 1 & 5 & 1 \\ 1 & 3 & 7 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 4 & 1 & 4 \end{bmatrix}$.

3. Verify the rule $\det AB = (\det A)(\det B)$ for the matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix}$. Note that this is a self-checking problem. It can be used as a model for practice in computing determinants.

4. Compute the determinant of the following $n \times n$ matrices by induction on n.

(a) $\begin{bmatrix} & & & 1 \\ & & 1 & \\ & ⋰ & & \\ 1 & & & \\ 1 & & & \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$

5. Evaluate $\det \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & & \vdots \\ 3 & 3 & 3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ n & & \cdots & & n \end{bmatrix}$

*6. Compute det $\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$.

7. Prove that the determinant is linear in the rows of a matrix, as asserted in (3.6).
8. Let A be an $n \times n$ matrix. What is $\det(-A)$?
9. Prove that $\det A^t = \det A$.
10. Derive the formula $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ from the properties (3.5, 3.6, 3.7, 3.9).
11. Let A and B be square matrices. Prove that $\det(AB) = \det(BA)$.
12. Prove that $\det\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks.

*13. Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold when $AC \neq CA$.

4. Permutation Matrices

1. Consider the permutation p defined by $\mathbf{1} \rightsquigarrow \mathbf{3}$, $\mathbf{2} \rightsquigarrow \mathbf{1}$, $\mathbf{3} \rightsquigarrow \mathbf{4}$, $\mathbf{4} \rightsquigarrow \mathbf{2}$.
   (a) Find the associated permutation matrix P.
   (b) Write p as a product of transpositions and evaluate the corresponding matrix product.
   (c) Compute the sign of p.
2. Prove that every permutation matrix is a product of transpositions.
3. Prove that every matrix with a single 1 in each row and a single 1 in each column, the other entries being zero, is a permutation matrix.
4. Let p be a permutation. Prove that $\operatorname{sign} p = \operatorname{sign} p^{-1}$.
5. Prove that the transpose of a permutation matrix P is its inverse.
6. What is the permutation matrix associated to the permutation $\mathbf{i} \rightsquigarrow \mathbf{n-i}$?
7. (a) The complete expansion for the determinant of a 3×3 matrix consists of six triple products of matrix entries, with sign. Learn which they are.
   (b) Compute the determinant of the following matrices using the complete expansion, and check your work by another method:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

8. Prove that the complete expansion (4.12) defines the determinant by verifying rules (3.5–3.7).
9. Prove that formulas (4.11) and (4.12) define the same number.

5. Cramer's Rule

1. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with determinant 1. What is A^{-1}?

2. (self-checking) Compute the adjoints of the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 4 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$, and $\begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, and verify Theorem (5.7) for them.

3. Let A be an $n \times n$ matrix with integer entries a_{ij}. Prove that A^{-1} has integer entries if and only if $\det A = \pm 1$.

4. Prove that expansion by minors on a row of a matrix defines the determinant function.

Miscellaneous Problems

1. Write the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices, using as few as you can. Prove that your expression is as short as possible.

2. Find a representation of the complex numbers by real 2×2 matrices which is compatible with addition and multiplication. Begin by finding a nice solution to the matrix equation $A^2 = -I$.

3. (*Vandermonde determinant*) **(a)** Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

   ***(b)** Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column cleverly.

*4. Consider a general system $AX = B$ of m linear equations in n unknowns. If the coefficient matrix A has a left inverse A', a matrix such that $A'A = I_n$, then we may try to solve the system as follows:

$$AX = B$$
$$A'AX = A'B$$
$$X = A'B.$$

But when we try to check our work by running the solution backward, we get into trouble:

$$X = A'B$$
$$AX = AA'B$$
$$AX \stackrel{?}{=} B.$$

We seem to want A' to be a right inverse: $AA' = I_m$, which isn't what was given. Explain. (Hint: Work out some examples.)

5. **(a)** Let A be a real 2×2 matrix, and let A_1, A_2 be the rows of A. Let P be the parallelogram whose vertices are $0, A_1, A_2, A_1 + A_2$. Prove that the area of P is the absolute value of the determinant $\det A$ by comparing the effect of an elementary row operation on the area and on $\det A$.

***(b)** Prove an analogous result for $n \times n$ matrices.

***6.** Most invertible matrices can be written as a product $A = LU$ of a lower triangular matrix L and an upper triangular matrix U, where in addition all diagonal entries of U are 1.

(a) Prove uniqueness, that is, prove that there is at most one way to write A as a product.

(b) Explain how to compute L and U when the matrix A is given.

(c) Show that every invertible matrix can be written as a product LPU, where L, U are as above and P is a permutation matrix.

7. Consider a system of n linear equations in n unknowns: $AX = B$, where A and B have *integer* entries. Prove or disprove the following.

(a) The system has a rational solution if $\det A \neq 0$.

(b) If the system has a rational solution, then it also has an integer solution.

***8.** Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.