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# **Sage Reference Manual: Groups**

***Release 6.3***

**The Sage Development Team**

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# EXAMPLES OF GROUPS

The `groups` object may be used to access examples of various groups. Using tab-completion on this object is an easy way to discover and quickly create the groups that are available (as listed here).

Let `<tab>` indicate pressing the tab key. So begin by typing `groups.<tab>` to see primary divisions, followed by (for example) `groups.matrix.<tab>` to access various groups implemented as sets of matrices.

- **Permutation Groups** (`groups.permutation.<tab>`)

- `groups.permutation.Symmetric`
- `groups.permutation.Alternating`
- `groups.permutation.KleinFour`
- `groups.permutation.Quaternion`
- `groups.permutation.Cyclic`
- `groups.permutation.Dihedral`
- `groups.permutation.DiCyclic`
- `groups.permutation.Mathieu`
- `groups.permutation.Suzuki`
- `groups.permutation.PGL`
- `groups.permutation.PSL`
- `groups.permutation.PSp`
- `groups.permutation.PSU`
- `groups.permutation.PGU`
- `groups.permutation.Transitive`
- `groups.permutation.RubiksCube`

- **Matrix Groups** (`groups.matrix.<tab>`)

- `groups.matrix.QuaternionGF3`
- `groups.matrix.GL`
- `groups.matrix.SL`
- `groups.matrix.Sp`
- `groups.matrix.GU`

- `groups.matrix.SU`
- `groups.matrix.GO`
- `groups.matrix.SO`
- **Finitely Presented Groups** (`groups.presentation.<tab>`)
  - `groups.presentation.Alternating`
  - `groups.presentation.Cyclic`
  - `groups.presentation.Dihedral`
  - `groups.presentation.DiCyclic`
  - `groups.presentation.FGAbelian`
  - `groups.presentation.KleinFour`
  - `groups.presentation.Quaternion`
  - `groups.presentation.Symmetric`
- **Affine Groups** (`groups.affine.<tab>`)
  - `groups.affine.Affine`
  - `groups.affine.Euclidean`
- **Miscellaneous Groups** (`groups.misc.<tab>`)
  - `groups.misc.AdditiveAbelian`
  - `groups.misc.AdditiveCyclic`
  - `groups.misc.Braid`
  - `groups.misc.CoxeterGroup`
  - `groups.misc.Free`
  - `groups.misc.RightAngledArtin`
  - `groups.misc.SemimonomialTransformation`
  - `groups.misc.WeylGroup`

# BASE CLASS FOR GROUPS

Base class for groups

```
class sage.groups.group.AbelianGroup
    Bases: sage.groups.group.Group
```

Generic abelian group.

```
is_abelian()
    Return True.
```

EXAMPLES:

```
sage: from sage.groups.group import AbelianGroup
sage: G = AbelianGroup()
sage: G.is_abelian()
True
```

```
class sage.groups.group.AlgebraicGroup
    Bases: sage.groups.group.Group
```

The Python constructor

TESTS:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.category()
Category of groups
sage: G = Group(category=Groups()) # todo: do the same test with some subcategory of Groups when
sage: G.category()
Category of groups
sage: G = Group(category = CommutativeAdditiveGroups())
Traceback (most recent call last):
...
ValueError: (Category of commutative additive groups,) is not a subcategory of Category of group
sage: G._repr_option('element_is_atomic')
False
```

Check for #8119:

```
sage: G = SymmetricGroup(2)
sage: h = hash(G)
sage: G.rename('S2')
sage: h == hash(G)
True
```

**class** sage.groups.group.**FiniteGroup**

Bases: sage.groups.group.Group

Generic finite group.

**cayley\_graph** (connecting\_set=None)

Return the Cayley graph for this finite group.

INPUT:

- connecting\_set – (optional) list of elements to use for edges, default is the stored generators

OUTPUT:

The Cayley graph as a Sage DiGraph object. To plot the graph with with different colors

EXAMPLES:

```
sage: D4 = DihedralGroup(4); D4
```

Dihedral group of order 8 as a permutation group

```
sage: G = D4.cayley_graph()
```

```
sage: show(G, color_by_label=True, edge_labels=True)
```

```
sage: A5 = AlternatingGroup(5); A5
```

Alternating group of order  $5!/2$  as a permutation group

```
sage: G = A5.cayley_graph()
```

```
sage: G.show3d(color_by_label=True, edge_size=0.01, edge_size2=0.02, vertex_size=0.03)
```

```
sage: G.show3d(vertex_size=0.03, edge_size=0.01, edge_size2=0.02, vertex_colors={(1,1,1):G.v
```

```
sage: G.num_edges()
```

120

```
sage: G = A5.cayley_graph(connecting_set=[A5.gens()[0]])
```

```
sage: G.num_edges()
```

60

```
sage: g=PermutationGroup([(i+1,j+1) for i in range(5) for j in range(5) if j!=i])
```

```
sage: g.cayley_graph(connecting_set=[(1,2),(2,3)])
```

Digraph on 120 vertices

```
sage: s1 = SymmetricGroup(1); s = s1.cayley_graph(); s.vertices()
```

[()]

AUTHORS:

- Bobby Moretti (2007-08-10)

- Robert Miller (2008-05-01): editing

**is\_finite**()

Return True.

EXAMPLES:

```
sage: from sage.groups.group import FiniteGroup
```

```
sage: G = FiniteGroup()
```

```
sage: G.is_finite()
```

True

**class** sage.groups.group.**Group**

Bases: sage.structure.parent.Parent

Base class for all groups

TESTS:

```
sage: from sage.groups.group import Group
```

```
sage: G = Group()
```

```
sage: TestSuite(G).run(skip = ["_test_an_element",
```

["\_test\_



**is\_abelian()**

Test whether this group is abelian.

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.is_abelian()
Traceback (most recent call last):
...
NotImplementedError
```

**is\_commutative()**

Test whether this group is commutative.

This is an alias for `is_abelian`, largely to make groups work well with the Factorization class.

(Note for developers: Derived classes should override `is_abelian`, not `is_commutative`.)

EXAMPLE:

```
sage: SL(2, 7).is_commutative()
False
```

**is\_finite()**

Returns True if this group is finite.

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

**is\_multiplicative()**

Returns True if the group operation is given by `*` (rather than `+`).

Override for additive groups.

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.is_multiplicative()
True
```

**order()**

Returns the number of elements of this group, which is either a positive integer or infinity.

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.order()
Traceback (most recent call last):
...
NotImplementedError
```

**quotient( $H$ )**

Return the quotient of this group by the normal subgroup  $H$ .

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.quotient(G)
Traceback (most recent call last):
...
NotImplementedError
```

**random\_element** (*bound=None*)

Return a random element of this group.

EXAMPLES:

```
sage: from sage.groups.group import Group
sage: G = Group()
sage: G.random_element()
Traceback (most recent call last):
...
NotImplementedError
```

**sage.groups.group.is\_Group** (*x*)

Return whether *x* is a group object.

INPUT:

- *x* – anything.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: F.<a,b> = FreeGroup()
sage: from sage.groups.group import is_Group
sage: is_Group(F)
True
sage: is_Group("a string")
False
```

# LIBGAP-BASED GROUPS

## LibGAP-based Groups

This module provides helper class for wrapping GAP groups via `libgap`. See `free_group` for an example how they are used.

The parent class keeps track of the libGAP element object, to use it in your Python parent you have to derive both from the suitable group parent and `ParentLibGAP`

```
sage: from sage.groups.libgap_wrapper import ElementLibGAP, ParentLibGAP
sage: from sage.groups.group import Group
sage: class FooElement(ElementLibGAP):
...     pass
sage: class FooGroup(Group, ParentLibGAP):
...     Element = FooElement
...     def __init__(self):
...         lg = libgap(libgap.CyclicGroup(3))      # dummy
...         ParentLibGAP.__init__(self, lg)
...         Group.__init__(self)
```

Note how we call the constructor of both superclasses to initialize `Group` and `ParentLibGAP` separately. The parent class implements its output via `LibGAP`:

```
sage: FooGroup()
<pc group of size 3 with 1 generators>
sage: type(FooGroup().gap())
<type 'sage.libs.gap.element.GapElement'>
```

The element class is a subclass of `MultiplicativeGroupElement`. To use it, you just inherit from `ElementLibGAP`

```
sage: element = FooGroup().an_element()
sage: element
f1
```

The element class implements group operations and printing via `LibGAP`:

```
sage: element._repr_()
'f1'
sage: element * element
f1^2
```

## AUTHORS:

- Volker Braun

```
class sage.groups.libgap_wrapper.ElementLibGAP
    Bases: sage.structure.element.MultiplicativeGroupElement
```

A class for LibGAP-based Sage group elements

INPUT:

- parent – the Sage parent
- libgap\_element – the libgap element that is being wrapped

EXAMPLES:

```
sage: from sage.groups.libgap_wrapper import ElementLibGAP, ParentLibGAP
sage: from sage.groups.group import Group
sage: class FooElement(ElementLibGAP):
...     pass
sage: class FooGroup(Group, ParentLibGAP):
...     Element = FooElement
...     def __init__(self):
...         lg = libgap(libgap.CyclicGroup(3))      # dummy
...         ParentLibGAP.__init__(self, lg)
...         Group.__init__(self)
sage: FooGroup()
<pc group of size 3 with 1 generators>
sage: FooGroup().gens()
(f1,)
```

**gap()**

Returns a LibGAP representation of the element

OUTPUT:

A GapElement

EXAMPLES:

```
sage: G.<a,b> = FreeGroup('a, b')
sage: x = G([1, 2, -1, -2])
sage: x
a*b*a^-1*b^-1
sage: xg = x.gap()
sage: xg
a*b*a^-1*b^-1
sage: type(xg)
<type 'sage.libs.gap.element.GapElement'>
```

**inverse()**

Return the inverse of self.

TESTS:

```
sage: G = FreeGroup('a, b')
sage: x = G([1, 2, -1, -2])
sage: y = G([2, 2, 2, 1, -2, -2, -2])
sage: x.__invert__()
b*a*b^-1*a^-1
sage: y.__invert__()
b^3*a^-1*b^-3
sage: ~x
b*a*b^-1*a^-1
sage: x.inverse()
b*a*b^-1*a^-1
```

**is\_one()**

Test whether the group element is the trivial element.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: G.<a,b> = FreeGroup('a, b')
sage: x = G([1, 2, -1, -2])
sage: x.is_one()
False
sage: (x * ~x).is_one()
True
```

**class** sage.groups.libgap\_wrapper.**ParentLibGAP**(libgap\_parent, ambient=None)

Bases: sage.structure.sage\_object.SageObject

A class for parents to keep track of the GAP parent.

This is not a complete group in Sage, this class is only a base class that you can use to implement your own groups with LibGAP. See `libgap_group` for a minimal example of a group that is actually usable.

Your implementation definitely needs to supply

- `__reduce__()`: serialize the LibGAP group. Since GAP does not support Python pickles natively, you need to figure out yourself how you can recreate the group from a pickle.

INPUT:

- `libgap_parent` – the libgap element that is the parent in GAP.
- `ambient` – A derived class of `ParentLibGAP` or None (default). The ambient class if `libgap_parent` has been defined as a subgroup.

EXAMPLES:

```
sage: from sage.groups.libgap_wrapper import ElementLibGAP, ParentLibGAP
sage: from sage.groups.group import Group
sage: class FooElement(ElementLibGAP):
...     pass
sage: class FooGroup(Group, ParentLibGAP):
...     Element = FooElement
...     def __init__(self):
...         lg = libgap(libgap.CyclicGroup(3))      # dummy
...         ParentLibGAP.__init__(self, lg)
...         Group.__init__(self)
sage: FooGroup()
<pc group of size 3 with 1 generators>
```

**ambient()**

Return the ambient group of a subgroup.

OUTPUT:

A group containing `self`. If `self` has not been defined as a subgroup, we just return `self`.

EXAMPLES:

```
sage: G = FreeGroup(3)
sage: G.ambient() is G
True
```

**gap()**

Returns the gap representation of self

OUTPUT:

A GapElement

EXAMPLES:

```
sage: G = FreeGroup(3); G
Free Group on generators {x0, x1, x2}
sage: G.gap()
<free group on the generators [ x0, x1, x2 ]>
sage: G.gap().parent()
C library interface to GAP
sage: type(G.gap())
<type 'sage.libs.gap.element.GapElement'>
```

This can be useful, for example, to call GAP functions that are not wrapped in Sage:

```
sage: G = FreeGroup(3)
sage: H = G.gap()
sage: H.DirectProduct(H)
<fp group on the generators [ f1, f2, f3, f4, f5, f6 ]>
sage: H.DirectProduct(H).RelatorsOfFpGroup()
[ f1^-1*f4^-1*f1*f4, f1^-1*f5^-1*f1*f5, f1^-1*f6^-1*f1*f6, f2^-1*f4^-1*f2*f4,
  f2^-1*f5^-1*f2*f5, f2^-1*f6^-1*f2*f6, f3^-1*f4^-1*f3*f4, f3^-1*f5^-1*f3*f5,
  f3^-1*f6^-1*f3*f6 ]
```

**gen(*i*)**Return the *i*-th generator of self.**Warning:** Indexing starts at 0 as usual in Sage/Python. Not as in GAP, where indexing starts at 1.

INPUT:

- *i* – integer between 0 (inclusive) and `ngens()` (exclusive). The index of the generator.

OUTPUT:

The *i*-th generator of the group.

EXAMPLES:

```
sage: G = FreeGroup('a, b')
sage: G.gen(0)
a
sage: G.gen(1)
b
```

**generators()**

Returns the generators of the group.

EXAMPLES:

```
sage: G = FreeGroup(2)
sage: G.gens()
(x0, x1)
sage: H = FreeGroup('a, b, c')
sage: H.gens()
(a, b, c)
```

```

generators() is an alias for gens()
sage: G = FreeGroup('a, b')
sage: G.generators()
(a, b)
sage: H = FreeGroup(3, 'x')
sage: H.generators()
(x0, x1, x2)

```

**gens()**

Returns the generators of the group.

**EXAMPLES:**

```

sage: G = FreeGroup(2)
sage: G.gens()
(x0, x1)
sage: H = FreeGroup('a, b, c')
sage: H.gens()
(a, b, c)

```

```

generators() is an alias for gens()
sage: G = FreeGroup('a, b')
sage: G.generators()
(a, b)
sage: H = FreeGroup(3, 'x')
sage: H.generators()
(x0, x1, x2)

```

**is\_subgroup()**

Return whether the group was defined as a subgroup of a bigger group.

You can access the containing group with `ambient()`.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```

sage: G = FreeGroup(3)
sage: G.is_subgroup()
False

```

**ngens()**

Return the number of generators of self.

**OUTPUT:**

Integer.

**EXAMPLES:**

```

sage: G = FreeGroup(2)
sage: G.ngens()
2

```

**TESTS:**

```

sage: type(G.ngens())
<type 'sage.rings.integer.Integer'>

```

**one()**

Returns the identity element of self

EXAMPLES:

**sage:** `G = FreeGroup(3)`**sage:** `G.one()`

1

**sage:** `G.one() == G([])`

True

**sage:** `G.one().Tietze()``()`**subgroup**(*generators*)

Return the subgroup generated.

INPUT:

- *generators* – a list/tuple/iterable of group elements.

OUTPUT:

The subgroup generated by *generators*.

EXAMPLES:

**sage:** `F.<a,b> = FreeGroup()`**sage:** `G = F.subgroup([a^2*b]); G``Group([ a^2*b ])`**sage:** `G.gens()``(a^2*b,)`



# MISCELLANEOUS GENERIC FUNCTIONS

A collection of functions implementing generic algorithms in arbitrary groups, including additive and multiplicative groups.

In all cases the group operation is specified by a parameter ‘operation’, which is a string either one of the set of multiplication\_names or addition\_names specified below, or ‘other’. In the latter case, the caller must provide an identity, inverse() and op() functions.

```
multiplication_names = ( 'multiplication', 'times', 'product', '*' )
addition_names       = ( 'addition', 'plus', 'sum', '+' )
```

Also included are a generic function for computing multiples (or powers), and an iterator for general multiples and powers.

## EXAMPLES:

Some examples in the multiplicative group of a finite field:

- Discrete logs:

```
sage: K = GF(3^6, 'b')
sage: b = K.gen()
sage: a = b^210
sage: discrete_log(a, b, K.order()-1)
210
```

- Linear relation finder:

```
sage: F.<a>=GF(3^6, 'a')
sage: a.multiplicative_order().factor()
2^3 * 7 * 13
sage: b=a^7
sage: c=a^13
sage: linear_relation(b,c, '*')
(13, 7)
sage: b^13==c^7
True
```

- Orders of elements:

```
sage: k.<a> = GF(5^5)
sage: b = a^4
sage: order_from_multiple(b, 5^5-1, operation='*')
781
```

```
sage: order_from_bounds(b, (5^4, 5^5), operation='*')
781
```

Some examples in the group of points of an elliptic curve over a finite field:

- Discrete logs:

```
sage: F=GF(37^2, 'a')
sage: E=EllipticCurve(F, [1,1])
sage: F.<a>=GF(37^2, 'a')
sage: E=EllipticCurve(F, [1,1])
sage: P=E(25*a + 16, 15*a + 7)
sage: P.order()
672
sage: Q=39*P; Q
(36*a + 32 : 5*a + 12 : 1)
sage: discrete_log(Q,P,P.order(), operation='+')
39
```

- Linear relation finder:

```
sage: F.<a>=GF(3^6, 'a')
sage: E=EllipticCurve([a^5 + 2*a^3 + 2*a^2 + 2*a, a^4 + a^3 + 2*a + 1])
sage: P=E(a^5 + a^4 + a^3 + a^2 + a + 2, 0)
sage: Q=E(2*a^3 + 2*a^2 + 2*a, a^3 + 2*a^2 + 1)
sage: linear_relation(P,Q, '+')
(1, 2)
sage: P == 2*Q
True
```

- Orders of elements:

```
sage: k.<a> = GF(5^5)
sage: E = EllipticCurve(k, [2,4])
sage: P = E(3*a^4 + 3*a, 2*a + 1)
sage: M = E.cardinality(); M
3227
sage: plist = M.prime_factors()
sage: order_from_multiple(P, M, plist, operation='+')
3227
sage: Q = E(0,2)
sage: order_from_multiple(Q, M, plist, operation='+')
7
sage: order_from_bounds(Q, Hasse_bounds(5^5), operation='+')
7
```

`sage.groups.generic.bsgs(a, b, bounds, operation='*', identity=None, inverse=None, op=None)`

Totally generic discrete baby-step giant-step function.

Solves  $na = b$  (or  $a^n = b$ ) with  $lb \leq n \leq ub$  where `bounds == (lb, ub)`, raising an error if no such  $n$  exists.

$a$  and  $b$  must be elements of some group with given identity, inverse of  $x$  given by `inverse(x)`, and group operation on  $x, y$  by `op(x, y)`.

If operation is `*` or `+` then the other arguments are provided automatically; otherwise they must be provided by the caller.

INPUT:

- $a$  - group element
- $b$  - group element

- `bounds` - a 2-tuple of integers (`lower`, `upper`) with  $0 \leq \text{lower} \leq \text{upper}$
- `operation` - string: `'*'`, `'+'`, `'other'`
- `identity` - the identity element of the group
- `inverse()` - function of 1 argument `x` returning inverse of `x`
- `op()` - function of 2 arguments `x`, `y` returning `x*y` in group

OUTPUT:

An integer  $n$  such that  $a^n = b$  (or  $na = b$ ). If no such  $n$  exists, this function raises a `ValueError` exception.

NOTE: This is a generalization of discrete logarithm. One situation where this version is useful is to find the order of an element in a group where we only have bounds on the group order (see the elliptic curve example below).

ALGORITHM: Baby step giant step. Time and space are soft  $O(\sqrt{n})$  where  $n$  is the difference between upper and lower bounds.

EXAMPLES:

```
sage: b = Mod(2, 37); a = b^20
sage: bsgs(b, a, (0, 36))
20
```

```
sage: p=next_prime(10^20)
sage: a=Mod(2,p); b=a^(10^25)
sage: bsgs(a, b, (10^25-10^6, 10^25+10^6)) == 10^25
True
```

```
sage: K = GF(3^6, 'b')
sage: a = K.gen()
sage: b = a^210
sage: bsgs(a, b, (0, K.order()-1))
210
```

```
sage: K.<z>=CyclotomicField(230)
sage: w=z^500
sage: bsgs(z, w, (0, 229))
40
```

An additive example in an elliptic curve group:

```
sage: F.<a> = GF(37^5)
sage: E = EllipticCurve(F, [1, 1])
sage: P = E.lift_x(a); P
(a : 28*a^4 + 15*a^3 + 14*a^2 + 7 : 1) # 32-bit
(a : 9*a^4 + 22*a^3 + 23*a^2 + 30 : 1) # 64-bit
```

This will return a multiple of the order of `P`:

```
sage: bsgs(P, P.parent()(0), Hasse_bounds(F.order()), operation='+')
69327408
```

AUTHOR:

- John Cremona (2008-03-15)

```
sage.groups.generic.discrete_log(a, base, ord=None, bounds=None, operation='*', identity=None, inverse=None, op=None)
```

Totally generic discrete log function.

INPUT:

- `a` - group element
- `base` - group element (the base)
- `ord` - integer (multiple of order of base, or None)
- `bounds` - a priori bounds on the log
- `operation` - string: `'*'`, `'+'`, `'other'`
- `identity` - the group's identity
- `inverse()` - function of 1 argument `x` returning inverse of `x`
- `op()` - function of 2 arguments `x, y` returning `x*y` in group

`a` and `base` must be elements of some group with identity given by `identity`, inverse of `x` by `inverse(x)`, and group operation on `x, y` by `op(x, y)`.

If operation is `'*'` or `'+'` then the other arguments are provided automatically; otherwise they must be provided by the caller.

OUTPUT: Returns an integer  $n$  such that  $b^n = a$  (or  $nb = a$ ), assuming that `ord` is a multiple of the order of the base  $b$ . If `ord` is not specified, an attempt is made to compute it.

If no such  $n$  exists, this function raises a `ValueError` exception.

**Warning:** If `x` has a `log` method, it is likely to be vastly faster than using this function. E.g., if `x` is an integer modulo  $n$ , use its `log` method instead!

ALGORITHM: Pohlig-Hellman and Baby step giant step.

EXAMPLES:

```
sage: b = Mod(2, 37); a = b^20
sage: discrete_log(a, b)
20
sage: b = Mod(2, 997); a = b^20
sage: discrete_log(a, b)
20

sage: K = GF(3^6, 'b')
sage: b = K.gen()
sage: a = b^210
sage: discrete_log(a, b, K.order()-1)
210

sage: b = Mod(1, 37); x = Mod(2, 37)
sage: discrete_log(x, b)
Traceback (most recent call last):
...
ValueError: No discrete log of 2 found to base 1
sage: b = Mod(1, 997); x = Mod(2, 997)
sage: discrete_log(x, b)
Traceback (most recent call last):
...
ValueError: No discrete log of 2 found to base 1

See trac\#2356:
sage: F.<w> = GF(121)
sage: v = w^120
```

```
sage: v.log(w)
0
```

```
sage: K.<z>=CyclotomicField(230)
sage: w=z^50
sage: discrete_log(w,z)
50
```

An example where the order is infinite: note that we must give an upper bound here:

```
sage: K.<a> = QuadraticField(23)
sage: eps = 5*a-24          # a fundamental unit
sage: eps.multiplicative_order()
+Infinity
sage: eta = eps^100
sage: discrete_log(eta,eps,bounds=(0,1000))
100
```

In this case we cannot detect negative powers:

```
sage: eta = eps^(-3)
sage: discrete_log(eta,eps,bounds=(0,100))
Traceback (most recent call last):
...
ValueError: No discrete log of -11515*a - 55224 found to base 5*a - 24
```

But we can invert the base (and negate the result) instead:

```
sage: - discrete_log(eta^-1,eps,bounds=(0,100))
-3
```

An additive example: elliptic curve DLOG:

```
sage: F=GF(37^2,'a')
sage: E=EllipticCurve(F,[1,1])
sage: F.<a>=GF(37^2,'a')
sage: E=EllipticCurve(F,[1,1])
sage: P=E(25*a + 16 , 15*a + 7 )
sage: P.order()
672
sage: Q=39*P; Q
(36*a + 32 : 5*a + 12 : 1)
sage: discrete_log(Q,P,P.order(),operation='+')
39
```

An example of big smooth group:

```
sage: F.<a>=GF(2^63)
sage: g=F.gen()
sage: u=g**123456789
sage: discrete_log(u,g)
123456789
```

AUTHORS:

- William Stein and David Joyner (2005-01-05)
- John Cremona (2008-02-29) rewrite using `dict()` and make generic

```
sage.groups.generic.discrete_log_generic(a, base, ord=None, bounds=None, operation='*', identity=None, inverse=None, op=None)
```

Alias for `discrete_log`.

```
sage.groups.generic.discrete_log_lambda(a, base, bounds, operation='*', hash_function=<built-in function hash>)
```

Pollard Lambda algorithm for computing discrete logarithms. It uses only a logarithmic amount of memory. It's useful if you have bounds on the logarithm. If you are computing logarithms in a whole finite group, you should use Pollard Rho algorithm.

INPUT:

- `a` - a group element
- `base` - a group element
- `bounds` - a couple (lb,ub) representing the range where we look for a logarithm
- `operation` - string: '+', '\*' or 'other'
- `hash_function` – having an efficient hash function is critical for this algorithm

OUTPUT: Returns an integer  $n$  such that  $a = base^n$  (or  $a = n * base$ )

**ALGORITHM: Pollard Lambda, if bounds are (lb,ub) it has time complexity  $O(\sqrt{ub-lb})$  and space complexity  $O(\log(ub-lb))$**

EXAMPLES:

```
sage: F.<a> = GF(2^63)
sage: discrete_log_lambda(a^1234567, a, (1200000,1250000))
1234567
```

```
sage: F.<a> = GF(37^5)
sage: E = EllipticCurve(F, [1,1])
sage: P = E.lift_x(a); P
(a : 9*a^4 + 22*a^3 + 23*a^2 + 30 : 1) # 32-bit
(a : 28*a^4 + 15*a^3 + 14*a^2 + 7 : 1) # 64-bit
```

This will return a multiple of the order of P:

```
sage: discrete_log_lambda(P.parent()(0), P, Hasse_bounds(F.order()), operation='+')
69327408
```

```
sage: K.<a> = GF(89**5)
sage: hs = lambda x: hash(x) + 15
sage: discrete_log_lambda(a**(89**3 - 3), a, (89**2, 89**4), operation = '*', hash_function = hs)
704966
```

AUTHOR:

– Yann Laigle-Chapuy (2009-01-25)

```
sage.groups.generic.discrete_log_rho(a, base, ord=None, operation='*', hash_function=<built-in function hash>)
```

Pollard Rho algorithm for computing discrete logarithm in cyclic group of prime order. If the group order is very small it falls back to the baby step giant step algorithm.

INPUT:

- `a` – a group element
- `base` – a group element

- `ord` – the order of `base` or `None`, in this case we try to compute it
- `operation` – a string (default: `'*'`) denoting whether we are in an additive group or a multiplicative one
- `hash_function` – having an efficient hash function is critical for this algorithm (see examples)

OUTPUT: an integer  $n$  such that  $a = \text{base}^n$  (or  $a = n * \text{base}$ )

ALGORITHM: Pollard rho for discrete logarithm, adapted from the article of Edlyn Teske, ‘A space efficient algorithm for group structure computation’.

EXAMPLES:

```
sage: F.<a> = GF(2^13)
sage: g = F.gen()
sage: discrete_log_rho(g^1234, g)
1234

sage: F.<a> = GF(37^5)
sage: E = EllipticCurve(F, [1,1])
sage: G = (3*31*2^4)*E.lift_x(a)
sage: discrete_log_rho(12345*G, G, ord=46591, operation='+')
12345
```

It also works with matrices:

```
sage: A = matrix(GF(50021), [[10577, 23999, 28893], [14601, 41019, 30188], [3081, 736, 27092]])
sage: discrete_log_rho(A^1234567, A)
1234567
```

Beware, the order must be prime:

```
sage: I = IntegerModRing(171980)
sage: discrete_log_rho(I(2), I(3))
Traceback (most recent call last):
...
ValueError: for Pollard rho algorithm the order of the group must be prime
```

If it fails to find a suitable logarithm, it raises a `ValueError`:

```
sage: I = IntegerModRing(171980)
sage: discrete_log_rho(I(31002), I(15501))
Traceback (most recent call last):
...
ValueError: Pollard rho algorithm failed to find a logarithm
```

The main limitation on the hash function is that we don’t want to have  $\text{hash}(x * y) = \text{hash}(x) + \text{hash}(y)$ :

```
sage: I = IntegerModRing(next_prime(2^23))
sage: def test():
....:     try:
....:         discrete_log_rho(I(123456), I(1), operation='+')
....:     except Exception:
....:         print "FAILURE"
sage: test() # random failure
FAILURE
```

If this happens, we can provide a better hash function:

```
sage: discrete_log_rho(I(123456), I(1), operation='+', hash_function=lambda x: hash(x*x))
123456
```

AUTHOR:

•Yann Laigle-Chapuy (2009-09-05)

`sage.groups.generic.linear_relation(P, Q, operation='+', identity=None, inverse=None, op=None)`

Function which solves the equation  $a * P = m * Q$  or  $P^a = Q^m$ .

Additive version: returns  $(a, m)$  with minimal  $m > 0$  such that  $aP = mQ$ . Special case: if  $\langle P \rangle$  and  $\langle Q \rangle$  intersect only in  $\{0\}$  then  $(a, m) = (0, n)$  where  $n$  is `Q.additive_order()`.

Multiplicative version: returns  $(a, m)$  with minimal  $m > 0$  such that  $P^a = Q^m$ . Special case: if  $\langle P \rangle$  and  $\langle Q \rangle$  intersect only in  $\{1\}$  then  $(a, m) = (0, n)$  where  $n$  is `Q.multiplicative_order()`.

ALGORITHM:

Uses the generic `bsgs()` function, and so works in general finite abelian groups.

EXAMPLES:

An additive example (in an elliptic curve group):

```
sage: F.<a>=GF(3^6, 'a')
sage: E=EllipticCurve([a^5 + 2*a^3 + 2*a^2 + 2*a, a^4 + a^3 + 2*a + 1])
sage: P=E(a^5 + a^4 + a^3 + a^2 + a + 2, 0)
sage: Q=E(2*a^3 + 2*a^2 + 2*a, a^3 + 2*a^2 + 1)
sage: linear_relation(P,Q, '+')
(1, 2)
sage: P == 2*Q
True
```

A multiplicative example (in a finite field's multiplicative group):

```
sage: F.<a>=GF(3^6, 'a')
sage: a.multiplicative_order().factor()
2^3 * 7 * 13
sage: b=a^7
sage: c=a^13
sage: linear_relation(b,c, '**')
(13, 7)
sage: b^13==c^7
True
```

`sage.groups.generic.merge_points(P1, P2, operation='+', identity=None, inverse=None, op=None, check=True)`

Returns a group element whose order is the lcm of the given elements.

INPUT:

- $P1$  – a pair  $(g_1, n_1)$  where  $g_1$  is a group element of order  $n_1$
- $P2$  – a pair  $(g_2, n_2)$  where  $g_2$  is a group element of order  $n_2$
- `operation` – string: '+' (default) or '\*' or other. If other, the following must be supplied:
  - `identity`: the identity element for the group;
  - `inverse()`: a function of one argument giving the inverse of a group element;
  - `op()`: a function of 2 arguments defining the group binary operation.

OUTPUT:

A pair  $(g_3, n_3)$  where  $g_3$  has order  $n_3 = \text{lcm}(n_1, n_2)$ .

EXAMPLES:



```

sage: F.<a>=GF(3^6,'a')
sage: b = a^7
sage: c = a^13
sage: ob = (3^6-1)//7
sage: oc = (3^6-1)//13
sage: merge_points((b,ob),(c,oc),operation='*')
(a^4 + 2*a^3 + 2*a^2, 728)
sage: d,od = merge_points((b,ob),(c,oc),operation='*')
sage: od == d.multiplicative_order()
True
sage: od == lcm(ob,oc)
True

sage: E=EllipticCurve([a^5 + 2*a^3 + 2*a^2 + 2*a, a^4 + a^3 + 2*a + 1])
sage: P=E(2*a^5 + 2*a^4 + a^3 + 2, a^4 + a^3 + a^2 + 2*a + 2)
sage: P.order()
7
sage: Q=E(2*a^5 + 2*a^4 + 1, a^5 + 2*a^3 + 2*a + 2)
sage: Q.order()
4
sage: R,m = merge_points((P,7),(Q,4), operation='+')
sage: R.order() == m
True
sage: m == lcm(7,4)
True

```

`sage.groups.generic.multiple(a, n, operation='*', identity=None, inverse=None, op=None)`

Returns either  $na$  or  $a^n$ , where  $n$  is any integer and  $a$  is a Python object on which a group operation such as addition or multiplication is defined. Uses the standard binary algorithm.

INPUT: See the documentation for `discrete_logarithm()`.

EXAMPLES:

```

sage: multiple(2,5)
32
sage: multiple(RealField()('2.5'),4)
39.06250000000000
sage: multiple(2,-3)
1/8
sage: multiple(2,100,'+') == 100*2
True
sage: multiple(2,100) == 2**100
True
sage: multiple(2,-100,) == 2**-100
True
sage: R.<x>=ZZ[]
sage: multiple(x,100)
x^100
sage: multiple(x,100,'+')
100*x
sage: multiple(x,-10)
1/x^10

```

Idempotence is detected, making the following fast:

```

sage: multiple(1,10^1000)
1

sage: E=EllipticCurve('389a1')

```

```

sage: P=E(-1,1)
sage: multiple(P,10,'+')
(645656132358737542773209599489/22817025904944891235367494656 : 52553217612428119288123181864417
sage: multiple(P,-10,'+')
(645656132358737542773209599489/22817025904944891235367494656 : -5289787576294984409495297030291

```

**class** `sage.groups.generic.multiples` (*P*, *n*, *P0=None*, *indexed=False*, *operation='+'*, *op=None*)  
 Return an iterator which runs through  $P0+i\cdot P$  for  $i$  in  $\text{range}(n)$ .

*P* and *P0* must be Sage objects in some group; if the operation is multiplication then the returned values are instead  $P0\cdot P^{**i}$ .

#### EXAMPLES:

```

sage: list(multiples(1,10))
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
sage: list(multiples(1,10,100))
[100, 101, 102, 103, 104, 105, 106, 107, 108, 109]

sage: E=EllipticCurve('389a1')
sage: P=E(-1,1)
sage: for Q in multiples(P,5): print Q, Q.height()/P.height()
(0 : 1 : 0) 0.0000000000000000
(-1 : 1 : 1) 1.0000000000000000
(10/9 : -35/27 : 1) 4.0000000000000000
(26/361 : -5720/6859 : 1) 9.0000000000000000
(47503/16641 : 9862190/2146689 : 1) 16.0000000000000000

sage: R.<x>=ZZ[]
sage: list(multiples(x,5))
[0, x, 2*x, 3*x, 4*x]
sage: list(multiples(x,5,operation='*'))
[1, x, x^2, x^3, x^4]
sage: list(multiples(x,5,indexed=True))
[(0, 0), (1, x), (2, 2*x), (3, 3*x), (4, 4*x)]
sage: list(multiples(x,5,indexed=True,operation='*'))
[(0, 1), (1, x), (2, x^2), (3, x^3), (4, x^4)]
sage: for i,y in multiples(x,5,indexed=True): print "%s times %s = %s"%(i,x,y)
0 times x = 0
1 times x = x
2 times x = 2*x
3 times x = 3*x
4 times x = 4*x

sage: for i,n in multiples(3,5,indexed=True,operation='*'): print "3 to the power %s = %s"%(i,n)
3 to the power 0 = 1
3 to the power 1 = 3
3 to the power 2 = 9
3 to the power 3 = 27
3 to the power 4 = 81

```

**next** ()

Returns the next item in this multiples iterator.

**sage.groups.generic.order\_from\_bounds** (*P*, *bounds*, *d=None*, *operation='+'*, *identity=None*,  
*inverse=None*, *op=None*)

Generic function to find order of a group element, given only upper and lower bounds for a multiple of the order (e.g. bounds on the order of the group of which it is an element)

INPUT:

- **P** - a Sage object which is a group element
- **bounds** - a 2-tuple  $(lb, ub)$  such that  $m \cdot P = 0$  (or  $P \cdot m = 1$ ) for some  $m$  with  $lb \leq m \leq ub$ .
- **d** - (optional) a positive integer; only  $m$  which are multiples of this will be considered.
- **operation** - string: '+' (default) or '\*' or other. If other, the following must be supplied:
  - **identity**: the identity element for the group;
  - **inverse()**: a function of one argument giving the inverse of a group element;
  - **op()**: a function of 2 arguments defining the group binary operation.

---

**Note:** Typically  $lb$  and  $ub$  will be bounds on the group order, and from previous calculation we know that the group order is divisible by  $d$ .

---

EXAMPLES:

```
sage: k.<a> = GF(5^5)
sage: b = a^4
sage: order_from_bounds(b, (5^4, 5^5), operation='*')
781
sage: E = EllipticCurve(k, [2, 4])
sage: P = E(3*a^4 + 3*a, 2*a + 1)
sage: bounds = Hasse_bounds(5^5)
sage: Q = E(0, 2)
sage: order_from_bounds(Q, bounds, operation='+')
7
sage: order_from_bounds(P, bounds, 7, operation='+')
3227

sage: K.<z> = CyclotomicField(230)
sage: w = z^50
sage: order_from_bounds(w, (200, 250), operation='*')
23
```

```
sage.groups.generic.order_from_multiple(P, m, plist=None, factorization=None,
                                         check=True, operation='+')
```

Generic function to find order of a group element given a multiple of its order.

INPUT:

- **P** - a Sage object which is a group element;
- **m** - a Sage integer which is a multiple of the order of  $P$ , i.e. we require that  $m \cdot P = 0$  (or  $P \cdot m = 1$ );
- **check** - a Boolean (default: True), indicating whether we check if  $m$  really is a multiple of the order;
- **factorization** - the factorization of  $m$ , or None in which case this function will need to factor  $m$ ;
- **plist** - a list of the prime factors of  $m$ , or None - kept for compatibility only, prefer the use of factorization;
- **operation** - string: '+' (default) or '\*'.

---

**Note:** It is more efficient for the caller to factor  $m$  and cache the factors for subsequent calls.

---

EXAMPLES:

```
sage: k.<a> = GF(5^5)
sage: b = a^4
sage: order_from_multiple(b, 5^5-1, operation='*')
781
sage: E = EllipticCurve(k, [2, 4])
sage: P = E(3*a^4 + 3*a, 2*a + 1)
sage: M = E.cardinality(); M
3227
sage: F = M.factor()
sage: order_from_multiple(P, M, factorization=F, operation='+')
3227
sage: Q = E(0, 2)
sage: order_from_multiple(Q, M, factorization=F, operation='+')
7

sage: K.<z>=CyclotomicField(230)
sage: w=z^50
sage: order_from_multiple(w, 230, operation='*')
23

sage: F=GF(2^1279, 'a')
sage: n=F.cardinality()-1 # Mersenne prime
sage: order_from_multiple(F.random_element(), n, factorization=[(n, 1)], operation='*')==n
True

sage: K.<a> = GF(3^60)
sage: order_from_multiple(a, 3^60-1, operation='*', check=False)
42391158275216203514294433200
```

# FREE GROUPS

Free groups and finitely presented groups are implemented as a wrapper over the corresponding GAP objects.

A free group can be created by giving the number of generators, or their names. It is also possible to create indexed generators:

```
sage: G.<x,y,z> = FreeGroup(); G
Free Group on generators {x, y, z}
sage: FreeGroup(3)
Free Group on generators {x0, x1, x2}
sage: FreeGroup('a,b,c')
Free Group on generators {a, b, c}
sage: FreeGroup(3,'t')
Free Group on generators {t0, t1, t2}
```

The elements can be created by operating with the generators, or by passing a list with the indices of the letters to the group:

EXAMPLES:

```
sage: G.<a,b,c> = FreeGroup()
sage: a*b*c*a
a*b*c*a
sage: G([1,2,3,1])
a*b*c*a
sage: a * b / c * b^2
a*b*c^-1*b^2
sage: G([1,1,2,-1,-3,2])
a^2*b*a^-1*c^-1*b
```

You can use call syntax to replace the generators with a set of arbitrary ring elements:

```
sage: g = a * b / c * b^2
sage: g(1,2,3)
8/3
sage: M1 = identity_matrix(2)
sage: M2 = matrix([[1,1],[0,1]])
sage: M3 = matrix([[0,1],[1,0]])
sage: g([M1, M2, M3])
[1 3]
[1 2]
```

AUTHORS:

- Miguel Angel Marco Buzunariz

- Volker Braun

`sage.groups.free_group.FreeGroup` (*n=None*, *names='x'*, *index\_set=None*, *abelian=False*,  
\*\**kws*)

Construct a Free Group.

INPUT:

- *n* – integer or `None` (default). The number of generators. If not specified the *names* are counted.
- *names* – string or list/tuple/iterable of strings (default: `'x'`). The generator names or name prefix.
- *index\_set* – (optional) an index set for the generators; if specified then the optional keyword *abelian* can be used
- *abelian* – (default: `False`) whether to construct a free abelian group or a free group

---

**Note:** If you want to create a free group, it is currently preferential to use `Groups().free(...)` as that does not load GAP.

---

EXAMPLES:

```
sage: G.<a,b> = FreeGroup(); G
Free Group on generators {a, b}
sage: H = FreeGroup('a, b')
sage: G is H
True
sage: FreeGroup(0)
Free Group on generators {}
```

The entry can be either a string with the names of the generators, or the number of generators and the prefix of the names to be given. The default prefix is `'x'`

```
sage: FreeGroup(3)
Free Group on generators {x0, x1, x2}
sage: FreeGroup(3, 'g')
Free Group on generators {g0, g1, g2}
sage: FreeGroup()
Free Group on generators {x}
```

We give two examples using the *index\_set* option:

```
sage: FreeGroup(index_set=ZZ)
Free group indexed by Integer Ring
sage: FreeGroup(index_set=ZZ, abelian=True)
Free abelian group indexed by Integer Ring
```

TESTS:

```
sage: G1 = FreeGroup(2, 'a,b')
sage: G2 = FreeGroup('a,b')
sage: G3.<a,b> = FreeGroup()
sage: G1 is G2, G2 is G3
(True, True)
```

**class** `sage.groups.free_group.FreeGroupElement` (*parent, x*)  
Bases: `sage.groups.libgap_wrapper.ElementLibGAP`

A wrapper of GAP's Free Group elements.

INPUT:

- `x` – something that determines the group element. Either a `GapElement` or the Tietze list (see `Tietze()`) of the group element.
- `parent` – the parent `FreeGroup`.

**EXAMPLES:**

```

sage: G = FreeGroup('a', 'b')
sage: x = G([1, 2, -1, -2])
sage: x
a*b*a^-1*b^-1
sage: y = G([2, 2, 2, 1, -2, -2, -2])
sage: y
b^3*a*b^-3
sage: x*y
a*b*a^-1*b^2*a*b^-3
sage: y*x
b^3*a*b^-3*a*b*a^-1*b^-1
sage: x^(-1)
b*a*b^-1*a^-1
sage: x == x*y*y^(-1)
True

```

**Tietze()**

Return the Tietze list of the element.

The Tietze list of a word is a list of integers that represent the letters in the word. A positive integer  $i$  represents the letter corresponding to the  $i$ -th generator of the group. Negative integers represent the inverses of generators.

**OUTPUT:**

A tuple of integers.

**EXAMPLES:**

```

sage: G.<a,b> = FreeGroup()
sage: a.Tietze()
(1,)
sage: x = a^2 * b^(-3) * a^(-2)
sage: x.Tietze()
(1, 1, -2, -2, -2, -1, -1)

```

**TESTS:**

```

sage: type(a.Tietze())
<type 'tuple'>
sage: type(a.Tietze()[0])
<type 'sage.rings.integer.Integer'>

```

**fox\_derivative(gen)**

Return the Fox derivative of self with respect to a given generator.

**INPUT:**

- `gen` : the generator with respect to which the derivative will be computed.

**OUTPUT:**

An element of the group algebra with integer coefficients.

**EXAMPLES:**

```

sage: G = FreeGroup(5)
sage: G.inject_variables()
Defining x0, x1, x2, x3, x4
sage: (~x0*x1*x0*x2*~x0).fox_derivative(x0)
-B[x0^-1] + B[x0^-1*x1] - B[x0^-1*x1*x0*x2*x0^-1]
sage: (~x0*x1*x0*x2*~x0).fox_derivative(x1)
B[x0^-1]
sage: (~x0*x1*x0*x2*~x0).fox_derivative(x2)
B[x0^-1*x1*x0]
sage: (~x0*x1*x0*x2*~x0).fox_derivative(x3)
0

```

**syllables()**

Return the syllables of the word.

Consider a free group element  $g = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ . The uniquely-determined subwords  $x_i^{e_i}$  consisting only of powers of a single generator are called the syllables of  $g$ .

OUTPUT:

The tuple of syllables. Each syllable is given as a pair  $(x_i, e_i)$  consisting of a generator and a non-zero integer.

EXAMPLES:

```

sage: G.<a,b> = FreeGroup()
sage: w = a^2 * b^-1 * a^3
sage: w.syllables()
((a, 2), (b, -1), (a, 3))

```

**class** sage.groups.free\_group.**FreeGroup\_class**(generator\_names, libgap\_free\_group=None)  
 Bases: sage.structure.unique\_representation.UniqueRepresentation,  
 sage.groups.group.Group, sage.groups.libgap\_wrapper.ParentLibGAP

A class that wraps GAP's FreeGroup

See `FreeGroup()` for details.

TESTS:

```

sage: G = FreeGroup('a, b')
sage: TestSuite(G).run()

```

**Element**

alias of `FreeGroupElement`

**abelian\_invariants()**

Return the Abelian invariants of self.

The Abelian invariants are given by a list of integers  $i_1 \dots i_j$ , such that the abelianization of the group is isomorphic to

$$\mathbf{Z}/(i_1) \times \cdots \times \mathbf{Z}/(i_j)$$

EXAMPLES:

```

sage: F.<a,b> = FreeGroup()
sage: F.abelian_invariants()
(0, 0)

```

**quotient(relations)**

Return the quotient of self by the normal subgroup generated by the given elements.



This quotient is a finitely presented groups with the same generators as `self`, and relations given by the elements of `relations`.

INPUT:

- `relations` – A list/tuple/iterable with the elements of the free group.

OUTPUT:

A finitely presented group, with generators corresponding to the generators of the free group, and relations corresponding to the elements in `relations`.

EXAMPLES:

```
sage: F.<a,b> = FreeGroup()
sage: F.quotient([a*b^2*a, b^3])
Finitely presented group < a, b | a*b^2*a, b^3 >
```

Division is shorthand for `quotient()`

```
sage: F / [a*b^2*a, b^3]
Finitely presented group < a, b | a*b^2*a, b^3 >
```

Relations are converted to the free group, even if they are not elements of it (if possible)

```
sage: F1.<a,b,c,d>=FreeGroup()
sage: F2.<a,b>=FreeGroup()
sage: r=a*b/a
sage: r.parent()
Free Group on generators {a, b}
sage: F1/[r]
Finitely presented group < a, b, c, d | a*b*a^-1 >
```

**rank()**

Return the number of generators of `self`.

Alias for `ngens()`.

OUTPUT:

Integer.

EXAMPLES:

```
sage: G = FreeGroup('a, b'); G
Free Group on generators {a, b}
sage: G.rank()
2
sage: H = FreeGroup(3, 'x')
sage: H
Free Group on generators {x0, x1, x2}
sage: H.rank()
3
```

`sage.groups.free_group.is_FreeGroup(x)`

Test whether `x` is a `FreeGroup_class`.

INPUT:

- `x` – anything.

OUTPUT:

Boolean.

## EXAMPLES:

```
sage: from sage.groups.free_group import is_FreeGroup
sage: is_FreeGroup('a string')
False
sage: is_FreeGroup(FreeGroup(0))
True
sage: is_FreeGroup(FreeGroup(index_set=ZZ))
True
```

```
sage.groups.free_group.wrap_FreeGroup(libgap_free_group)
```

Wrap a LibGAP free group.

This function changes the comparison method of `libgap_free_group` to comparison by Python id. If you want to put the LibGAP free group into a container (set, dict) then you should understand the implications of `_set_compare_by_id()`. To be safe, it is recommended that you just work with the resulting Sage `FreeGroup_class`.

## INPUT:

- `libgap_free_group` – a LibGAP free group.

## OUTPUT:

A Sage `FreeGroup_class`.

## EXAMPLES:

First construct a LibGAP free group:

```
sage: F = libgap.FreeGroup(['a', 'b'])
sage: type(F)
<type 'sage.libs.gap.element.GapElement'>
```

Now wrap it:

```
sage: from sage.groups.free_group import wrap_FreeGroup
sage: wrap_FreeGroup(F)
Free Group on generators {a, b}
```

## TESTS:

Check that we can do it twice (see [trac ticket #12339](#))

```
sage: G = libgap.FreeGroup(['a', 'b'])
sage: wrap_FreeGroup(G)
Free Group on generators {a, b}
```

# FINITELY PRESENTED GROUPS

Finitely presented groups are constructed as quotients of `free_group`:

```
sage: F.<a,b,c> = FreeGroup()
sage: G = F / [a^2, b^2, c^2, a*b*c*a*b*c]
sage: G
Finitely presented group < a, b, c | a^2, b^2, c^2, (a*b*c)^2 >
```

One can create their elements by multiplying the generators or by specifying a Tietze list (see `Tietze()`) as in the case of free groups:

```
sage: G.gen(0) * G.gen(1)
a*b
sage: G([1,2,-1])
a*b*a^-1
sage: a.parent()
Free Group on generators {a, b, c}
sage: G.inject_variables()
Defining a, b, c
sage: a.parent()
Finitely presented group < a, b, c | a^2, b^2, c^2, (a*b*c)^2 >
```

Notice that, even if they are represented in the same way, the elements of a finitely presented group and the elements of the corresponding free group are not the same thing. However, they can be converted from one parent to the other:

```
sage: F.<a,b,c> = FreeGroup()
sage: G = F / [a^2,b^2,c^2,a*b*c*a*b*c]
sage: F([1])
a
sage: G([1])
a
sage: F([1]) is G([1])
False
sage: F([1]) == G([1])
False
sage: G(a*b/c)
a*b*c^-1
sage: F(G(a*b/c))
a*b*c^-1
```

Finitely presented groups are implemented via GAP. You can use the `gap()` method to access the underlying LibGAP object:

```
sage: G = FreeGroup(2)
sage: G.inject_variables()
Defining x0, x1
sage: H = G / (x0^2, (x0*x1)^2, x1^2)
sage: H.gap()
<fp group on the generators [ x0, x1 ]>
```

This can be useful, for example, to use GAP functions that are not yet wrapped in Sage:

```
sage: H.gap().LowerCentralSeries()
[ Group(<fp, no generators known>), Group(<fp, no generators known>) ]
```

The same holds for the group elements:

```
sage: G = FreeGroup(2)
sage: H = G / (G([1, 1]), G([2, 2, 2]), G([1, 2, -1, -2])); H
Finitely presented group < x0, x1 | x0^2, x1^3, x0*x1*x0^-1*x1^-1 >
sage: a = H([1])
sage: a
x0
sage: a.gap()
x0
sage: a.gap().Order()
2
sage: type(_)      # note that the above output is not a Sage integer
<type 'sage.libs.gap.element.GapElement_Integer'>
```

You can use call syntax to replace the generators with a set of arbitrary ring elements. For example, take the free abelian group obtained by modding out the commutator subgroup of the free group:

```
sage: G = FreeGroup(2)
sage: G_ab = G / [G([1, 2, -1, -2])]; G_ab
Finitely presented group < x0, x1 | x0*x1*x0^-1*x1^-1 >
sage: a,b = G_ab.gens()
sage: g = a * b
sage: M1 = matrix([[1,0],[0,2]])
sage: M2 = matrix([[0,1],[1,0]])
sage: g(3, 5)
15
sage: g(M1, M1)
[1 0]
[0 4]
sage: M1*M2 == M2*M1      # matrices do not commute
False
sage: g(M1, M2)
Traceback (most recent call last):
...
ValueError: the values do not satisfy all relations of the group
```

**Warning:** Some methods are not guaranteed to finish since the word problem for finitely presented groups is, in general, undecidable. In those cases the process may run until the available memory is exhausted.

#### REFERENCES:

- [Wikipedia article Presentation\\_of\\_a\\_group](#)
- [Wikipedia article Word\\_problem\\_for\\_groups](#)

AUTHOR:

- Miguel Angel Marco Buzunariz

**class** `sage.groups.finitely_presented.FinitelyPresentedGroup` (*free\_group, relations*)  
 Bases: `sage.groups.libgap_mixin.GroupMixinLibGAP`, `sage.structure.unique_representation.UniqueRepresentation`,  
`sage.groups.group.Group`, `sage.groups.libgap_wrapper.ParentLibGAP`

A class that wraps GAP's Finitely Presented Groups.

**Warning:** You should use `quotient()` to construct finitely presented groups as quotients of free groups.

EXAMPLES:

```
sage: G.<a,b> = FreeGroup()
sage: H = G / [a, b^3]
sage: H
Finitely presented group < a, b | a, b^3 >
sage: H.gens()
(a, b)

sage: F.<a,b> = FreeGroup('a, b')
sage: J = F / (F([1]), F([2, 2, 2]))
sage: J is H
True

sage: G = FreeGroup(2)
sage: H = G / (G([1, 1]), G([2, 2, 2]))
sage: H.gens()
(x0, x1)
sage: H.gen(0)
x0
sage: H.ngens()
2
sage: H.gap()
<fp group on the generators [ x0, x1 ]>
sage: type(_)
<type 'sage.libs.gap.element.GapElement'>
```

**Element**

alias of `FinitelyPresentedGroupElement`

**abelian\_invariants()**

Return the abelian invariants of `self`.

The abelian invariants are given by a list of integers  $(i_1, \dots, i_j)$ , such that the abelianization of the group is isomorphic to  $\mathbf{Z}/(i_1) \times \dots \times \mathbf{Z}/(i_j)$ .

EXAMPLES:

```
sage: G = FreeGroup(4, 'g')
sage: G.inject_variables()
Defining g0, g1, g2, g3
sage: H = G.quotient([g1^2, g2*g1*g2^(-1)*g1^(-1), g1*g3^(-2), g0^4])
sage: H.abelian_invariants()
(0, 4, 4)
```

**ALGORITHM:**

Uses GAP.

**alexander\_matrix()**

Return the Alexander matrix of the group.

This matrix is given by the fox derivatives of the relations with respect to the generators.

OUTPUT:

A group algebra-valued matrix. It depends on the (fixed) choice of presentation.

EXAMPLES:

```
sage: G.<a,b,c> = FreeGroup()
```

```
sage: H = G.quotient([a*b/a/b, a*c/a/c, c*b/c/b])
```

```
sage: H.alexander_matrix()
```

```
[      B[1] - B[a*b*a^-1] B[a] - B[a*b*a^-1*b^-1]      0]
[      B[1] - B[a*c*a^-1]      0 B[a] - B[a*c*a^-1*c^-1]]
[      0 B[c] - B[c*b*c^-1*b^-1]      B[1] - B[c*b*c^-1]]
```

```
sage: G.<a,b,c,d,e> = FreeGroup()
```

```
sage: H = G.quotient([a*b/a/b, a*c/a/c, a*d/a/d, b*c*d/(c*d*b), b*c*d/(d*b*c)])
```

```
sage: H.alexander_matrix()
```

```
[      B[1] - B[a*b*a^-1]      B[a] - B[a*b*a^-1*b^-1]
[      B[1] - B[a*c*a^-1]      0      B[a] - B[a*c*a^-1*c^-1]
[      B[1] - B[a*d*a^-1]      0
[      0      B[1] - B[b*c*d*b^-1] B[b] - B[b*c*d*b^-1*d^-1]
[      0      B[1] - B[b*c*d*c^-1*b^-1]      B[b] - B[b*c*d*c^-1*b^-1]
```

**as\_permutation\_group(limit=4096000)**

Return an isomorphic permutation group.

The generators of the resulting group correspond to the images by the isomorphism of the generators of the given group.

INPUT:

- `limit` – integer (default: 4096000). The maximal number of cosets before the computation is aborted.

OUTPUT:

A Sage `PermutationGroup()`. If the number of cosets exceeds the given limit, a `ValueError` is returned.

EXAMPLES:

```
sage: G.<a,b> = FreeGroup()
```

```
sage: H = G / (a^2, b^3, a*b~a~b)
```

```
sage: H.as_permutation_group()
```

```
Permutation Group with generators [(1,2)(3,5)(4,6), (1,3,4)(2,5,6)]
```

```
sage: G.<a,b> = FreeGroup()
```

```
sage: H = G / [a^3*b]
```

```
sage: H.as_permutation_group(limit=1000)
```

```
Traceback (most recent call last):
```

```
...
```

```
ValueError: Coset enumeration exceeded limit, is the group finite?
```

ALGORITHM:

Uses GAP's coset enumeration on the trivial subgroup.

**Warning:** This is in general not a decidable problem (in fact, it is not even possible to check if the group is finite or not). If the group is infinite, or too big, you should be prepared for a long computation that consumes all the memory without finishing if you do not set a sensible `limit`.

**cardinality** (*limit=4096000*)

Compute the cardinality of `self`.

INPUT:

- `limit` – integer (default: 4096000). The maximal number of cosets before the computation is aborted.

OUTPUT:

Integer or `Infinity`. The number of elements in the group.

EXAMPLES:

```
sage: G.<a,b> = FreeGroup('a', 'b')
sage: H = G / (a^2, b^3, a*b*a*b)
sage: H.cardinality()
6

sage: F.<a,b,c> = FreeGroup()
sage: J = F / (F([1]), F([2, 2, 2]))
sage: J.cardinality()
+Infinity
```

ALGORITHM:

Uses GAP.

**Warning:** This is in general not a decidable problem, so it is not guaranteed to give an answer. If the group is infinite, or too big, you should be prepared for a long computation that consumes all the memory without finishing if you do not set a sensible `limit`.

**direct\_product** (*H, reduced=False, new\_names=True*)

Return the direct product of `self` with finitely presented group `H`.

Calls GAP function `DirectProduct`, which returns the direct product of a list of groups of any representation.

From [JohnsonPG90] (pg 45, proposition 4): If  $G, H$  are groups presented by  $\langle X \mid R \rangle$  and  $\langle Y \mid S \rangle$  respectively, then their direct product has the presentation  $\langle X, Y \mid R, S, [X, Y] \rangle$  where  $[X, Y]$  denotes the set of commutators  $\{x^{-1}y^{-1}xy \mid x \in X, y \in Y\}$ .

INPUT:

- `H` – a finitely presented group
- `reduced` – (default: `False`) boolean; if `True`, then attempt to reduce the presentation of the product group
- `new_names` – (default: `True`) boolean; If `True`, then lexicographical variable names are assigned to the generators of the group to be returned. If `False`, the group to be returned keeps the generator names of the two groups forming the direct product. Note that one cannot ask to reduce the output and ask to keep the old variable names, as they may change meaning in the output group if its presentation is reduced.

OUTPUT:

The direct product of `self` with `H` as a finitely presented group.

EXAMPLES:

```
sage: G = FreeGroup()
sage: C12 = ( G / [G([1,1,1,1])] ).direct_product( G / [G([1,1,1])] ); C12
Finitely presented group < a, b | a^4, b^3, a^-1*b^-1*a*b >
sage: C12.order(), C12.as_permutation_group().is_cyclic()
(12, True)
sage: klein = ( G / [G([1,1])] ).direct_product( G / [G([1,1])] ); klein
Finitely presented group < a, b | a^2, b^2, a^-1*b^-1*a*b >
sage: klein.order(), klein.as_permutation_group().is_cyclic()
(4, False)
```

We can keep the variable names from `self` and `H` to examine how new relations are formed:

```
sage: F = FreeGroup("a"); G = FreeGroup("g")
sage: X = G / [G.0^12]; A = F / [F.0^6]
sage: X.direct_product(A, new_names=False)
Finitely presented group < g, a | g^12, a^6, g^-1*a^-1*g*a >
sage: A.direct_product(X, new_names=False)
Finitely presented group < a, g | a^6, g^12, a^-1*g^-1*a*g >
```

Or we can attempt to reduce the output group presentation:

```
sage: F = FreeGroup("a"); G = FreeGroup("g")
sage: X = G / [G.0]; A = F / [F.0]
sage: X.direct_product(A, new_names=True)
Finitely presented group < a, b | a, b, a^-1*b^-1*a*b >
sage: X.direct_product(A, reduced=True, new_names=True)
Finitely presented group < | >
```

But we cannot do both:

```
sage: K = FreeGroup(['a', 'b'])
sage: D = K / [K.0^5, K.1^8]
sage: D.direct_product(D, reduced=True, new_names=False)
Traceback (most recent call last):
...
ValueError: cannot reduce output and keep old variable names
```

TESTS:

```
sage: G = FreeGroup()
sage: Dp = (G / [G([1,1])]).direct_product( G / [G([1,1,1,1,1,1])] )
sage: Dp.as_permutation_group().is_isomorphic(PermutationGroup(['(1,2)', '(3,4,5,6,7,8)']))
True
sage: C7 = G / [G.0**7]; C6 = G / [G.0**6]
sage: C14 = G / [G.0**14]; C3 = G / [G.0**3]
sage: C7.direct_product(C6).is_isomorphic(C14.direct_product(C3))
True
sage: F = FreeGroup(2); D = F / [F([1,1,1,1]), F([2,2]), F([1,2])**2]
sage: D.direct_product(D).as_permutation_group().is_isomorphic(
....: direct_product_permgroups([DihedralGroup(5), DihedralGroup(5)]))
True
```

AUTHORS:

•Davis Shurbert (2013-07-20): initial version

REFERENCES:



**free\_group()**

Return the free group (without relations).

OUTPUT:

A `FreeGroup()`.

EXAMPLES:

```
sage: G.<a,b,c> = FreeGroup()
sage: H = G / (a^2, b^3, a*b*a~b)
sage: H.free_group()
Free Group on generators {a, b, c}
sage: H.free_group() is G
True
```

**order** (*limit=4096000*)

Compute the cardinality of self.

INPUT:

- *limit* – integer (default: 4096000). The maximal number of cosets before the computation is aborted.

OUTPUT:

Integer or Infinity. The number of elements in the group.

EXAMPLES:

```
sage: G.<a,b> = FreeGroup('a, b')
sage: H = G / (a^2, b^3, a*b*a~b)
sage: H.cardinality()
6

sage: F.<a,b,c> = FreeGroup()
sage: J = F / (F([1]), F([2, 2, 2]))
sage: J.cardinality()
+Infinity
```

ALGORITHM:

Uses GAP.

**Warning:** This is in general not a decidable problem, so it is not guaranteed to give an answer. If the group is infinite, or too big, you should be prepared for a long computation that consumes all the memory without finishing if you do not set a sensible *limit*.

**relations()**

Return the relations of the group.

OUTPUT:

The relations as a tuple of elements of `free_group()`.

EXAMPLES:

```
sage: F = FreeGroup(5, 'x')
sage: F.inject_variables()
Defining x0, x1, x2, x3, x4
sage: G = F.quotient([x0*x2, x3*x1*x3, x2*x1*x2])
sage: G.relations()
(x0*x2, x3*x1*x3, x2*x1*x2)
```

```
sage: all(rel in F for rel in G.relations())
True
```

### **rewriting\_system()**

Return the rewriting system corresponding to the finitely presented group. This rewriting system can be used to reduce words with respect to the relations.

If the rewriting system is transformed into a confluent one, the reduction process will give as a result the (unique) reduced form of an element.

#### EXAMPLES:

```
sage: F.<a,b> = FreeGroup()
sage: G = F / [a^2,b^3,(a*b/a)^3,b*a*b*a]
sage: k = G.rewriting_system()
sage: k
Rewriting system of Finitely presented group < a, b | a^2, b^3, a*b^3*a^-1, (b*a)^2 >
with rules:
  a^2      --->      1
  b^3      --->      1
  (b*a)^2  --->      1
  a*b^3*a^-1 --->      1

sage: G([1,1,2,2,2])
a^2*b^3
sage: k.reduce(G([1,1,2,2,2]))
1
sage: k.reduce(G([2,2,1]))
b^2*a
sage: k.make_confluent()
sage: k.reduce(G([2,2,1]))
a*b
```

### **semidirect\_product(H, hom, check=True, reduced=False)**

The semidirect product of `self` with `H` via `hom`.

If there exists a homomorphism  $\phi$  from a group  $G$  to the automorphism group of a group  $H$ , then we can define the semidirect product of  $G$  with  $H$  via  $\phi$  as the cartesian product of  $G$  and  $H$  with the operation

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, \phi(g_2)(h_1) h_2).$$

#### INPUT:

- `H` – Finitely presented group which is implicitly acted on by `self` and can be naturally embedded as a normal subgroup of the semidirect product.
- `hom` – Homomorphism from `self` to the automorphism group of `H`. Given as a pair, with generators of `self` in the first slot and the images of the corresponding generators in the second. These images must be automorphisms of `H`, given again as a pair of generators and images.
- `check` – Boolean (default `True`). If `False` the defining homomorphism and automorphism images are not tested for validity. This test can be costly with large groups, so it can be bypassed if the user is confident that his morphisms are valid.
- `reduced` – Boolean (default `False`). If `True` then the method attempts to reduce the presentation of the output group.

#### OUTPUT:

The semidirect product of `self` with `H` via `hom` as a finitely presented group. See [PermutationGroup\\_generic.semidirect\\_product](#) for a more in depth explanation of a

semidirect product.

AUTHORS:

•Davis Shurbert (8-1-2013)

EXAMPLES:

Group of order 12 as two isomorphic semidirect products:

```
sage: D4 = groups.presentation.Dihedral(4)
sage: C3 = groups.presentation.Cyclic(3)
sage: alpha1 = ([C3.gen(0)], [C3.gen(0)])
sage: alpha2 = ([C3.gen(0)], [C3([1,1])])
sage: S1 = D4.semirect_product(C3, ([D4.gen(1), D4.gen(0)], [alpha1, alpha2]))
sage: C2 = groups.presentation.Cyclic(2)
sage: Q = groups.presentation.DiCyclic(3)
sage: a = Q([1]); b = Q([-2])
sage: alpha = (Q.gens(), [a,b])
sage: S2 = C2.semirect_product(Q, ([C2.0], [alpha]))
sage: S1.is_isomorphic(S2)
True
```

Dihedral groups can be constructed as semidirect products of cyclic groups:

```
sage: C2 = groups.presentation.Cyclic(2)
sage: C8 = groups.presentation.Cyclic(8)
sage: hom = (C2.gens(), [([C8([1])], [C8([-1])])])
sage: D = C2.semirect_product(C8, hom)
sage: D.as_permutation_group().is_isomorphic(DihedralGroup(8))
True
```

You can attempt to reduce the presentation of the output group:

```
sage: D = C2.semirect_product(C8, hom); D
Finitely presented group < a, b, c, d |
  a^2, b^-1*a^-1*b*a*d^-1*c^-1, c^-1*a^-1*c*a*d^-1, d^-1*a^-1*d*a,
  b^2*c^-1, c^-1*b^-1*c*b, d^-1*b^-1*d*b, c^2*d^-1, d^-1*c^-1*d*c, d^2 >
sage: D = C2.semirect_product(C8, hom, reduced=True); D
Finitely presented group < a, b | a^2, (a*b)^2, b^8 >

sage: C3 = groups.presentation.Cyclic(3)
sage: C4 = groups.presentation.Cyclic(4)
sage: hom = (C3.gens(), [(C4.gens(), C4.gens())])
sage: C3.semirect_product(C4, hom)
Finitely presented group < a, b, c |
  a^3, b^-1*a^-1*b*a, c^-1*a^-1*c*a, b^2*c^-1, c^-1*b^-1*c*b, c^2 >
sage: D = C3.semirect_product(C4, hom, reduced=True); D
Finitely presented group < a, b | a^3, b^4, b^-1*a^-1*b*a >
sage: D.as_permutation_group().is_cyclic()
True
```

You can turn off the checks for the validity of the input morphisms. This check is expensive but behavior is unpredictable if inputs are invalid and are not caught by these tests:

```
sage: C5 = groups.presentation.Cyclic(5)
sage: C12 = groups.presentation.Cyclic(12)
sage: hom = (C5.gens(), [(C12.gens(), C12.gens())])
sage: sp = C5.semirect_product(C12, hom, check=False); sp
Finitely presented group < a, b, c, d |
  a^5, b^-1*a^-1*b*a, c^-1*a^-1*c*a, d^-1*a^-1*d*a, b^2*d^-1,
  c^-1*b^-1*c*b, d^-1*b^-1*d*b, c^3, d^-1*c^-1*d*c, d^2 >
```

```
sage: sp.as_permutation_group().is_cyclic(), sp.order()
(True, 60)
```

TESTS:

The following was fixed in Gap-4.7.2:

```
sage: C5.semidirect_product(C12, hom) == sp
True
```

A more complicated semidirect product:

```
sage: C = groups.presentation.Cyclic(7)
sage: D = groups.presentation.Dihedral(5)
sage: id1 = ([C.0], [(D.gens(), D.gens())])
sage: Se1 = C.semidirect_product(D, id1)
sage: id2 = (D.gens(), [(C.gens(), C.gens()), (C.gens(), C.gens())])
sage: Se2 = D.semidirect_product(C, id2)
sage: Dp1 = C.direct_product(D);
sage: Dp1.is_isomorphic(Se1), Dp1.is_isomorphic(Se2)
(True, True)
```

Most checks for validity of input are left to GAP to handle:

```
sage: bad_aut = ([C.0], [(D.gens(), [D.0, D.0])])
sage: C.semidirect_product(D, bad_aut)
Traceback (most recent call last):
...
ValueError: images of input homomorphism must be automorphisms
sage: bad_hom = ([D.0, D.1], [(C.gens(), C.gens())])
sage: D.semidirect_product(C, bad_hom)
Traceback (most recent call last):
...
ValueError: libGAP: Error, <gens> and <imgs> must be lists of same length
```

**simplification\_isomorphism()**

Return an isomorphism from self to a finitely presented group with a (hopefully) simpler presentation.

EXAMPLES:

```
sage: G.<a,b,c> = FreeGroup()
sage: H = G / [a*b*c, a*b^2, c*b/c^2]
sage: I = H.simplification_isomorphism()
sage: I
Generic morphism:
  From: Finitely presented group < a, b, c | a*b*c, a*b^2, c*b*c^-2 >
  To:   Finitely presented group < b | >
sage: I(a)
b^-2
sage: I(b)
b
sage: I(c)
b
```

TESTS:

```
sage: F = FreeGroup(1)
sage: G = F.quotient([F.0])
sage: G.simplification_isomorphism()
Generic morphism:
  From: Finitely presented group < x | x >
```

To: Finitely presented group  $\langle \quad | \quad \rangle$

ALGORITHM:

Uses GAP.

**simplified()**

Return an isomorphic group with a (hopefully) simpler presentation.

OUTPUT:

A new finitely presented group. Use `simplification_isomorphism()` if you want to know the isomorphism.

EXAMPLES:

```
sage: G.<x,y> = FreeGroup()
sage: H = G / [x^5, y^4, y*x*y^3*x^3]
sage: H
Finitely presented group < x, y | x^5, y^4, y*x*y^3*x^3 >
sage: H.simplified()
Finitely presented group < x, y | y^4, y*x*y^-1*x^-2, x^5 >
```

A more complicate example:

```
sage: G.<e0, e1, e2, e3, e4, e5, e6, e7, e8, e9> = FreeGroup()
sage: rels = [e6, e5, e3, e9, e4*e7^-1*e6, e9*e7^-1*e0,
...          e0*e1^-1*e2, e5*e1^-1*e8, e4*e3^-1*e8, e2]
sage: H = G.quotient(rels); H
Finitely presented group < e0, e1, e2, e3, e4, e5, e6, e7, e8, e9 |
e6, e5, e3, e9, e4*e7^-1*e6, e9*e7^-1*e0, e0*e1^-1*e2, e5*e1^-1*e8, e4*e3^-1*e8, e2 >
sage: H.simplified()
Finitely presented group < e0 | e0^2 >
```

**class** sage.groups.finitely\_presented.**FinitelyPresentedGroupElement** (*parent*, *x*,  
*check=True*)

Bases: sage.groups.free\_group.FreeGroupElement

A wrapper of GAP's Finitely Presented Group elements.

The elements are created by passing the Tietze list that determines them.

EXAMPLES:

```
sage: G = FreeGroup('a, b')
sage: H = G / [G([1]), G([2, 2, 2])]
sage: H([1, 2, 1, -1])
a*b
sage: H([1, 2, 1, -2])
a*b*a*b^-1
sage: x = H([1, 2, -1, -2])
sage: x
a*b*a^-1*b^-1
sage: y = H([2, 2, 2, 1, -2, -2, -2])
sage: y
b^3*a*b^-3
sage: x*y
a*b*a^-1*b^2*a*b^-3
sage: x^(-1)
b*a*b^-1*a^-1
```

**Tietze()**

Return the Tietze list of the element.

The Tietze list of a word is a list of integers that represent the letters in the word. A positive integer  $i$  represents the letter corresponding to the  $i$ -th generator of the group. Negative integers represent the inverses of generators.

OUTPUT:

A tuple of integers.

EXAMPLES:

```
sage: G = FreeGroup('a, b')
sage: H = G / (G([1]), G([2, 2, 2]))
sage: H.inject_variables()
Defining a, b
sage: a.Tietze()
(1,)
sage: x = a^2*b^(-3)*a^(-2)
sage: x.Tietze()
(1, 1, -2, -2, -2, -1, -1)
```

**class** sage.groups.finitely\_presented.**RewritingSystem**( $G$ )

Bases: `object`

A class that wraps GAP's rewriting systems.

A rewriting system is a set of rules that allow to transform one word in the group to an equivalent one.

If the rewriting system is confluent, then the transformed word is a unique reduced form of the element of the group.

**Warning:** Note that the process of making a rewriting system confluent might not end.

INPUT:

- $G$  – a group

REFERENCES:

- [Wikipedia article Knuth-Bendix\\_completion\\_algorithm](#)

EXAMPLES:

```
sage: F.<a,b> = FreeGroup()
sage: G = F / [a*b/a/b]
sage: k = G.rewriting_system()
sage: k
Rewriting system of Finitely presented group < a, b | a*b*a^-1*b^-1 >
with rules:
  a*b*a^-1*b^-1  --->  1

sage: k.reduce(a*b*a*b)
(a*b)^2
sage: k.make_confluent()
sage: k
Rewriting system of Finitely presented group < a, b | a*b*a^-1*b^-1 >
with rules:
  b^-1*a^-1  --->  a^-1*b^-1
  b^-1*a     --->  a*b^-1
  b*a^-1     --->  a^-1*b
```

$$b*a \quad \text{--->} \quad a*b$$

```
sage: k.reduce(a*b*a*b)
a^2*b^2
```

### Todo

- Include support for different orderings (currently only shortlex is used).
- Include the GAP package kbmag for more functionalities, including automatic structures and faster compiled functions.

### AUTHORS:

- Miguel Angel Marco Buzunariz (2013-12-16)

### `finitely_presented_group()`

The finitely presented group where the rewriting system is defined.

#### EXAMPLES:

```
sage: F = FreeGroup(3)
sage: G = F / [ [1,2,3], [-1,-2,-3], [1,1], [2,2] ]
sage: k = G.rewriting_system()
sage: k.make_confluent()
sage: k
```

Rewriting system of Finitely presented group  $\langle x_0, x_1, x_2 \mid x_0*x_1*x_2, x_0^{-1}*x_1^{-1}*x_2^{-1}, x_0^2, x_1^2 \rangle$  with rules:

$x_0^{-1}$	--->	$x_0$
$x_1^{-1}$	--->	$x_1$
$x_2^{-1}$	--->	$x_2$
$x_0^2$	--->	1
$x_0*x_1$	--->	$x_2$
$x_0*x_2$	--->	$x_1$
$x_1*x_0$	--->	$x_2$
$x_1^2$	--->	1
$x_1*x_2$	--->	$x_0$
$x_2*x_0$	--->	$x_1$
$x_2*x_1$	--->	$x_0$
$x_2^2$	--->	1

```
sage: k.finitely_presented_group()
```

Finitely presented group  $\langle x_0, x_1, x_2 \mid x_0*x_1*x_2, x_0^{-1}*x_1^{-1}*x_2^{-1}, x_0^2, x_1^2 \rangle$

### `free_group()`

The free group after which the rewriting system is defined

#### EXAMPLES:

```
sage: F = FreeGroup(3)
sage: G = F / [ [1,2,3], [-1,-2,-3] ]
sage: k = G.rewriting_system()
sage: k.free_group()
Free Group on generators {x0, x1, x2}
```

### `gap()`

The gap representation of the rewriting system.

#### EXAMPLES:

```

sage: F.<a,b>=FreeGroup()
sage: G=F/[a*a,b*b]
sage: k=G.rewriting_system()
sage: k.gap()
Knuth Bendix Rewriting System for Monoid( [ a, A, b, B ] ) with rules
[ [ a^2, <identity ...> ], [ a*A, <identity ...> ],
  [ A*a, <identity ...> ], [ b^2, <identity ...> ],
  [ b*B, <identity ...> ], [ B*b, <identity ...> ] ]

```

**is\_confluent()**

Return True if the system is confluent and False otherwise.

**EXAMPLES:**

```

sage: F = FreeGroup(3)
sage: G = F / [F([1,2,1,2,1,3,-1]),F([2,2,2,1,1,2]),F([1,2,3])]
sage: k = G.rewriting_system()
sage: k.is_confluent()
False
sage: k
Rewriting system of Finitely presented group < x0, x1, x2 | (x0*x1)^2*x0*x2*x0^-1, x1^3*x0^2
with rules:
  x0*x1*x2      --->      1
  x1^3*x0^2*x1  --->      1
  (x0*x1)^2*x0*x2*x0^-1  --->      1

sage: k.make_confluent()
sage: k.is_confluent()
True
sage: k
Rewriting system of Finitely presented group < x0, x1, x2 | (x0*x1)^2*x0*x2*x0^-1, x1^3*x0^2
with rules:
  x0^-1      --->      x0
  x1^-1      --->      x1
  x0^2       --->      1
  x0*x1      --->      x2^-1
  x0*x2^-1   --->      x1
  x1*x0      --->      x2
  x1^2       --->      1
  x1*x2^-1   --->      x0*x2
  x1*x2      --->      x0
  x2^-1*x0    --->      x0*x2
  x2^-1*x1    --->      x0
  x2^-2      --->      x2
  x2*x0      --->      x1
  x2*x1      --->      x0*x2
  x2^2       --->      x2^-1

```

**make\_confluent()**

Applies Knuth-Bendix algorithm to try to transform the rewriting system into a confluent one.

Note that this method does not return any object, just changes the rewriting system internally.

**ALGORITHM:**

Uses GAP's MakeConfluent.

**EXAMPLES:**



```

sage: F.<a,b> = FreeGroup()
sage: G = F / [a^2,b^3,(a*b/a)^3,b*a*b*a]
sage: k = G.rewriting_system()
sage: k
Rewriting system of Finitely presented group < a, b | a^2, b^3, a*b^3*a^-1, (b*a)^2 >
with rules:
  a^2      --->    1
  b^3      --->    1
  (b*a)^2  --->    1
  a*b^3*a^-1 --->    1

sage: k.make_confluent()
sage: k
Rewriting system of Finitely presented group < a, b | a^2, b^3, a*b^3*a^-1, (b*a)^2 >
with rules:
  a^-1     --->    a
  a^2      --->    1
  b^-1*a   --->    a*b
  b^-2     --->    b
  b*a      --->    a*b^-1
  b^2      --->    b^-1

```

**reduce (element)**

Applies the rules in the rewriting system to the element, to obtain a reduced form.

If the rewriting system is confluent, this reduced form is unique for all words representing the same element.

**EXAMPLES:**

```

sage: F.<a,b> = FreeGroup()
sage: G = F/[a^2, b^3, (a*b/a)^3, b*a*b*a]
sage: k = G.rewriting_system()
sage: k.reduce(b^4)
b
sage: k.reduce(a*b*a)
a*b*a

```

**rules ()**

Return the rules that form the rewriting system.

**OUTPUT:**

A dictionary containing the rules of the rewriting system. Each key is a word in the free group, and its corresponding value is the word to which it is reduced.

**EXAMPLES:**

```

sage: F.<a,b> = FreeGroup()
sage: G = F / [a*a*a,b*b*a*a]
sage: k = G.rewriting_system()
sage: k
Rewriting system of Finitely presented group < a, b | a^3, b^2*a^2 >
with rules:
  a^3      --->    1
  b^2*a^2  --->    1

sage: k.rules()
{b^2*a^2: 1, a^3: 1}
sage: k.make_confluent()

```

```
sage: sorted(k.rules().items())
[(a^-2, a), (a^-1*b^-1, a*b), (a^-1*b, b^-1), (a^2, a^-1),
 (a*b^-1, b), (b^-1*a^-1, a*b), (b^-1*a, b), (b^-2, a^-1),
 (b*a^-1, b^-1), (b*a, a*b), (b^2, a)]
```

```
sage.groups.finitely_presented.wrap_FpGroup(libgap_fpgroup)
```

Wrap a GAP finitely presented group.

This function changes the comparison method of `libgap_free_group` to comparison by Python `id`. If you want to put the LibGAP free group into a container (`set`, `dict`) then you should understand the implications of `_set_compare_by_id()`. To be safe, it is recommended that you just work with the resulting Sage `FinitelyPresentedGroup`.

INPUT:

- `libgap_fpgroup` – a LibGAP finitely presented group

OUTPUT:

A Sage `FinitelyPresentedGroup`.

EXAMPLES:

First construct a LibGAP finitely presented group:

```
sage: F = libgap.FreeGroup(['a', 'b'])
sage: a_cubed = F.GeneratorsOfGroup()[0] ^ 3
sage: P = F / libgap([a_cubed]); P
<fp group of size infinity on the generators [ a, b ]>
sage: type(P)
<type 'sage.libs.gap.element.GapElement'>
```

Now wrap it:

```
sage: from sage.groups.finitely_presented import wrap_FpGroup
sage: wrap_FpGroup(P)
Finitely presented group < a, b | a^3 >
```

# NAMED FINITELY PRESENTED GROUPS

Construct groups of small order and “named” groups as quotients of free groups. These groups are available through tab completion by typing `groups.presentation.<tab>` or by importing the required methods. Tab completion is made available through Sage’s *group catalog*. Some examples are engineered from entries in [THOMAS-WOODS].

Groups available as finite presentations:

- Alternating group,  $A_n$  of order  $n!/2$  – `groups.presentation.Alternating`
- Cyclic group,  $C_n$  of order  $n$  – `groups.presentation.Cyclic`
- Dicyclic group, nonabelian groups of order  $4n$  with a unique element of order 2 – `groups.presentation.DiCyclic`
- Dihedral group,  $D_n$  of order  $2n$  – `groups.presentation.Dihedral`
- Finitely generated abelian group,  $\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_k}$  – `groups.presentation.FGAbelian`
- Klein four group,  $C_2 \times C_2$  – `groups.presentation.KleinFour`
- Quaternion group of order 8 – `groups.presentation.Quaternion`
- Symmetric group,  $S_n$  of order  $n!$  – `groups.presentation.Symmetric`

AUTHORS:

- Davis Shurbert (2013-06-21): initial version

EXAMPLES:

```
sage: groups.presentation.Cyclic(4)
Finitely presented group < a | a^4 >
```

You can also import the desired functions:

```
sage: from sage.groups.finitely_presented_named import CyclicPresentation
sage: CyclicPresentation(4)
Finitely presented group < a | a^4 >
```

```
sage.groups.finitely_presented_named.AlternatingPresentation(n)
```

Build the Alternating group of order  $n!/2$  as a finitely presented group.

INPUT:

- $n$  – The size of the underlying set of arbitrary symbols being acted on by the Alternating group of order  $n!/2$ .

OUTPUT:

Alternating group as a finite presentation, implementation uses GAP to find an isomorphism from a permutation representation to a finitely presented group representation. Due to this fact, the exact output presentation may not be the same for every method call on a constant  $n$ .

EXAMPLES:

```
sage: A6 = groups.presentation.Alternating(6)
sage: A6.as_permutation_group().is_isomorphic(AlternatingGroup(6)), A6.order()
(True, 360)
```

TESTS:

```
sage: #even permutation test..
sage: A1 = groups.presentation.Alternating(1); A2 = groups.presentation.Alternating(2)
sage: A1.is_isomorphic(A2), A1.order()
(True, 1)
sage: A3 = groups.presentation.Alternating(3); A3.order(), A3.as_permutation_group().is_cyclic()
(3, True)
sage: A8 = groups.presentation.Alternating(8); A8.order()
20160
```

`sage.groups.finitely_presented_named.CyclicPresentation(n)`  
Build cyclic group of order  $n$  as a finitely presented group.

INPUT:

- $n$  – The order of the cyclic presentation to be returned.

OUTPUT:

The cyclic group of order  $n$  as finite presentation.

EXAMPLES:

```
sage: groups.presentation.Cyclic(10)
Finitely presented group < a | a^10 >
sage: n = 8; C = groups.presentation.Cyclic(n)
sage: C.as_permutation_group().is_isomorphic(CyclicPermutationGroup(n))
True
```

TESTS:

```
sage: groups.presentation.Cyclic(0)
Traceback (most recent call last):
...
ValueError: finitely presented group order must be positive
```

`sage.groups.finitely_presented_named.DiCyclicPresentation(n)`  
Build the dicyclic group of order  $4n$ , for  $n \geq 2$ , as a finitely presented group.

INPUT:

- $n$  – positive integer, 2 or greater, determining the order of the group ( $4n$ ).

OUTPUT:

The dicyclic group of order  $4n$  is defined by the presentation

$$\langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$$

**Note:** This group is also available as a permutation group via `groups.permutation.DiCyclic`.

#### EXAMPLES:

```
sage: D = groups.presentation.DiCyclic(9); D
Finitely presented group < a, b | a^18, b^2*a^-9, b^-1*a*b*a >
sage: D.as_permutation_group().is_isomorphic(groups.permutation.DiCyclic(9))
True
```

#### TESTS:

```
sage: Q = groups.presentation.DiCyclic(2)
sage: Q.as_permutation_group().is_isomorphic(QuaternionGroup())
True
sage: all([groups.presentation.DiCyclic(i).as_permutation_group(
....: ).is_isomorphic(groups.permutation.DiCyclic(i)) for i in [5,8,12,2^5]])
True
sage: groups.presentation.DiCyclic(1)
Traceback (most recent call last):
...
ValueError: input integer must be greater than 1
```

`sage.groups.finitely_presented_named.DihedralPresentation(n)`  
Build the Dihedral group of order  $2n$  as a finitely presented group.

#### INPUT:

- $n$  – The size of the set that  $D_n$  is acting on.

#### OUTPUT:

Dihedral group of order  $2n$ .

#### EXAMPLES:

```
sage: D = groups.presentation.Dihedral(7); D
Finitely presented group < a, b | a^7, b^2, (a*b)^2 >
sage: D.as_permutation_group().is_isomorphic(DihedralGroup(7))
True
```

#### TESTS:

```
sage: n = 9
sage: D = groups.presentation.Dihedral(n)
sage: D.ngens() == 2
True
sage: groups.presentation.Dihedral(0)
Traceback (most recent call last):
...
ValueError: finitely presented group order must be positive
```

`sage.groups.finitely_presented_named.FinitelyGeneratedAbelianPresentation(int_list)`  
Return canonical presentation of finitely generated abelian group.

#### INPUT:

- `int_list` – List of integers defining the group to be returned, the defining list is reduced to the invariants of the input list before generating the corresponding group.

#### OUTPUT:

Finitely generated abelian group,  $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$  as a finite presentation, where  $n_i$  forms the invariants of the input list.

**EXAMPLES:**

```
sage: groups.presentation.FGAbelian([2,2])
Finitely presented group < a, b | a^2, b^2, a^-1*b^-1*a*b >
sage: groups.presentation.FGAbelian([2,3])
Finitely presented group < a | a^6 >
sage: groups.presentation.FGAbelian([2,4])
Finitely presented group < a, b | a^2, b^4, a^-1*b^-1*a*b >
```

**You can create free abelian groups:**

```
sage: groups.presentation.FGAbelian([0])
Finitely presented group < a | >
sage: groups.presentation.FGAbelian([0,0])
Finitely presented group < a, b | a^-1*b^-1*a*b >
sage: groups.presentation.FGAbelian([0,0,0])
Finitely presented group < a, b, c | a^-1*c^-1*a*c, a^-1*b^-1*a*b, c^-1*b^-1*c*b >
```

**And various infinite abelian groups:**

```
sage: groups.presentation.FGAbelian([0,2])
Finitely presented group < a, b | a^2, a^-1*b^-1*a*b >
sage: groups.presentation.FGAbelian([0,2,2])
Finitely presented group < a, b, c | a^2, b^2, a^-1*c^-1*a*c, a^-1*b^-1*a*b, c^-1*b^-1*c*b >
```

**Outputs are reduced to minimal generators and relations:**

```
sage: groups.presentation.FGAbelian([3,5,2,7,3])
Finitely presented group < a, b | a^3, b^210, a^-1*b^-1*a*b >
sage: groups.presentation.FGAbelian([3,210])
Finitely presented group < a, b | a^3, b^210, a^-1*b^-1*a*b >
```

**The trivial group is an acceptable output:**

```
sage: groups.presentation.FGAbelian([])
Finitely presented group < | >
sage: groups.presentation.FGAbelian([1])
Finitely presented group < | >
sage: groups.presentation.FGAbelian([1,1,1,1,1,1,1,1,1])
Finitely presented group < | >
```

**Input list must consist of positive integers:**

```
sage: groups.presentation.FGAbelian([2,6,3,9,-4])
Traceback (most recent call last):
...
ValueError: input list must contain nonnegative entries
sage: groups.presentation.FGAbelian([2,'a',4])
Traceback (most recent call last):
...
TypeError: unable to convert x (=a) to an integer
```

**TESTS:**

```
sage: ag = groups.presentation.FGAbelian([2,2])
sage: ag.as_permutation_group().is_isomorphic(groups.permutation.KleinFour())
True
sage: G = groups.presentation.FGAbelian([2,4,8])
sage: C2 = CyclicPermutationGroup(2)
sage: C4 = CyclicPermutationGroup(4)
sage: C8 = CyclicPermutationGroup(8)
```

```

sage: gg = (C2.direct_product(C4)[0]).direct_product(C8)[0]
sage: gg.is_isomorphic(G.as_permutation_group())
True
sage: all([groups.presentation.FGAbelian([i]).as_permutation_group().is_isomorphic(groups.presentation.FGAbelian([i]).as_permutation_group()) for i in range(1, 4)])
True

```

sage.groups.finitely\_presented\_named.**KleinFourPresentation**()  
Build the Klein group of order 4 as a finitely presented group.

OUTPUT:

Klein four group ( $C_2 \times C_2$ ) as a finitely presented group.

EXAMPLES:

```

sage: K = groups.presentation.KleinFour(); K
Finitely presented group < a, b | a^2, b^2, a^-1*b^-1*a*b >

```

sage.groups.finitely\_presented\_named.**QuaternionPresentation**()  
Build the Quaternion group of order 8 as a finitely presented group.

OUTPUT:

Quaternion group as a finite presentation.

EXAMPLES:

```

sage: Q = groups.presentation.Quaternion(); Q
Finitely presented group < a, b | a^4, b^2*a^-2, a*b*a*b^-1 >
sage: Q.as_permutation_group().is_isomorphic(QuaternionGroup())
True

```

TESTS:

```

sage: Q = groups.presentation.Quaternion()
sage: Q.order(), Q.is_abelian()
(8, False)
sage: Q.is_isomorphic(groups.presentation.DiCyclic(2))
True

```

sage.groups.finitely\_presented\_named.**SymmetricPresentation**(n)  
Build the Symmetric group of order  $n!$  as a finitely presented group.

INPUT:

- $n$  – The size of the underlying set of arbitrary symbols being acted on by the Symmetric group of order  $n!$ .

OUTPUT:

Symmetric group as a finite presentation, implementation uses GAP to find an isomorphism from a permutation representation to a finitely presented group representation. Due to this fact, the exact output presentation may not be the same for every method call on a constant  $n$ .

EXAMPLES:

```

sage: S4 = groups.presentation.Symmetric(4)
sage: S4.as_permutation_group().is_isomorphic(SymmetricGroup(4))
True

```

TESTS:

```

sage: S = [groups.presentation.Symmetric(i) for i in range(1, 4)]; S[0].order()
1
sage: S[1].order(), S[2].as_permutation_group().is_isomorphic(DihedralGroup(3))
2, 6

```

```
(2, True)
sage: S5 = groups.presentation.Symmetric(5)
sage: perm_S5 = S5.as_permutation_group(); perm_S5.is_isomorphic(SymmetricGroup(5))
True
sage: groups.presentation.Symmetric(8).order()
40320
```



# BRAID GROUPS

Braid groups are implemented as a particular case of finitely presented groups, but with a lot of specific methods for braids.

A braid group can be created by giving the number of strands, and the name of the generators:

```
sage: BraidGroup(3)
Braid group on 3 strands
sage: BraidGroup(3, 'a')
Braid group on 3 strands
sage: BraidGroup(3, 'a').gens()
(a0, a1)
sage: BraidGroup(3, 'a,b').gens()
(a, b)
```

The elements can be created by operating with the generators, or by passing a list with the indices of the letters to the group:

```
sage: B.<s0,s1,s2> = BraidGroup(4)
sage: s0*s1*s0
s0*s1*s0
sage: B([1,2,1])
s0*s1*s0
```

The mapping class action of the braid group over the free group is also implemented, see [MappingClassGroupAction](#) for an explanation. This action is left multiplication of a free group element by a braid:

```
sage: B.<b0,b1,b2> = BraidGroup()
sage: F.<f0,f1,f2,f3> = FreeGroup()
sage: B.strands() == F.rank() # necessary for the action to be defined
True
sage: f1 * b1
f1*f2*f1^-1
sage: f0 * b1
f0
sage: f1 * b1
f1*f2*f1^-1
sage: f1^-1 * b1
f1*f2^-1*f1^-1
```

AUTHORS:

- Miguel Angel Marco Buzunariz

- Volker Braun
- Robert Lipshitz
- Thierry Monteil: add a `__hash__` method consistent with the word problem to ensure correct Cayley graph computations.

**class** `sage.groups.braid.Braid`(*parent, x, check=True*)  
 Bases: `sage.groups.finitely_presented.FinitelyPresentedGroupElement`

Class that models elements of the braid group.

It is a particular case of element of a finitely presented group.

EXAMPLES:

```
sage: B.<s0,s1,s2> = BraidGroup(4)
sage: B
Braid group on 4 strands
sage: s0*s1/s2/s1
s0*s1*s2^-1*s1^-1
sage: B((1, 2, -3, -2))
s0*s1*s2^-1*s1^-1
```

**LKB\_matrix**(*variables='x, y'*)

Return the Lawrence-Krammer-Bigelow representation matrix.

The matrix is expressed in the basis  $\{e_{i,j} \mid 1 \leq i < j \leq n\}$ , where the indices are ordered lexicographically. It is a matrix whose entries are in the ring of Laurent polynomials on the given variables. By default, the variables are 'x' and 'y'.

INPUT:

- *variables* – string (default: 'x, y'). A string containing the names of the variables, separated by a comma.

OUTPUT:

The matrix corresponding to the Lawrence-Krammer-Bigelow representation of the braid.

EXAMPLES:

```
sage: B = BraidGroup(3)
sage: b = B([1, 2, 1])
sage: b.LKB_matrix()
[      0 -x^4*y + x^3*y      -x^4*y]
[      0      -x^3*y      0]
[ -x^2*y  x^3*y - x^2*y      0]
sage: c = B([2, 1, 2])
sage: c.LKB_matrix()
[      0 -x^4*y + x^3*y      -x^4*y]
[      0      -x^3*y      0]
[ -x^2*y  x^3*y - x^2*y      0]
```

REFERENCES:

**alexander\_polynomial**(*var='t', normalized=True*)

Return the Alexander polynomial of the closure of the braid.

INPUT:

- *var* – string (default: 't'); the name of the variable in the entries of the matrix
- *normalized* – boolean (default: True); whether to return the normalized Alexander polynomial

OUTPUT:

The Alexander polynomial of the braid closure of the braid.

This is computed using the reduced Burau representation. The unnormalized Alexander polynomial is a Laurent polynomial, which is only well-defined up to multiplication by plus or minus times a power of  $t$ .

We normalize the polynomial by dividing by the largest power of  $t$  and then if the resulting constant coefficient is negative, we multiply by  $-1$ .

EXAMPLES:

We first construct the trefoil:

```
sage: B = BraidGroup(3)
sage: b = B([1, 2, 1, 2])
sage: b.alexander_polynomial(normalized=False)
1 - t + t^2
sage: b.alexander_polynomial()
t^-2 - t^-1 + 1
```

Next we construct the figure 8 knot:

```
sage: b = B([-1, 2, -1, 2])
sage: b.alexander_polynomial(normalized=False)
-t^-2 + 3*t^-1 - 1
sage: b.alexander_polynomial()
t^-2 - 3*t^-1 + 1
```

Our last example is the Kinoshita-Terasaka knot:

```
sage: B = BraidGroup(4)
sage: b = B([1, 1, 1, 3, 3, 2, -3, -1, -1, 2, -1, -3, -2])
sage: b.alexander_polynomial(normalized=False)
-t^-1
sage: b.alexander_polynomial()
1
```

REFERENCES:

- [Wikipedia article Alexander\\_polynomial](#)

**burau\_matrix** (*var='t', reduced=False*)

Return the Burau matrix of the braid.

INPUT:

- *var* – string (default: 't'); the name of the variable in the entries of the matrix
- *reduced* – boolean (default: False); whether to return the reduced or unreduced Burau representation

OUTPUT:

The Burau matrix of the braid. It is a matrix whose entries are Laurent polynomials in the variable *var*. If *reduced* is True, return the matrix for the reduced Burau representation instead.

EXAMPLES:

```
sage: B = BraidGroup(4)
sage: B.inject_variables()
Defining s0, s1, s2
sage: b = s0*s1/s2/s1
sage: b.burau_matrix()
[      1 - t      0      t - t^2      t^2]
```

```

[      1      0      0      0]
[      0      0      1      0]
[      0      t^-2 -t^-2 + t^-1  -t^-1 + 1]
sage: s2.burau_matrix('x')
[      1      0      0      0]
[      0      1      0      0]
[      0      0 1 - x      x]
[      0      0      1      0]
sage: s0.burau_matrix(reduced=True)
[-t  0  0]
[-t  1  0]
[-t  0  1]

```

## REFERENCES:

- [Wikipedia article Burau\\_representation](#)

**left\_normal\_form()**

Return the left normal form of the braid.

## OUTPUT:

A tuple of braid generators in the left normal form. The first element is a power of  $\Delta$ , and the rest are permutation braids.

## EXAMPLES:

```

sage: B = BraidGroup(4)
sage: b = B([1, 2, 3, -1, 2, -3])
sage: b.left_normal_form()
(s0^-1*s1^-1*s2^-1*s0^-1*s1^-1*s0^-1, s0*s1*s2*s1*s0, s0*s2*s1)
sage: c = B([1])
sage: c.left_normal_form()
(1, s0)

```

**permutation()**

Return the permutation induced by the braid in its strands.

## OUTPUT:

A permutation.

## EXAMPLES:

```

sage: B.<s0,s1,s2> = BraidGroup()
sage: b = s0*s1/s2/s1
sage: b.permutation()
[4, 1, 3, 2]
sage: b.permutation().cycle_string()
'(1,4,2)'

```

**plot** (*color='rainbow', orientation='bottom-top', gap=0.05, aspect\_ratio=1, axes=False, \*\*kwds*)

Plot the braid

The following options are available:

- *color* – (default: 'rainbow') the color of the strands. Possible values are:
  - 'rainbow', uses `rainbow()` according to the number of strands.
  - a valid color name for `bezier_path()` and `line()`. Used for all strands.
  - a list or a tuple of colors for each individual strand.

- **orientation** – (default: 'bottom-top') determines how the braid is printed. The possible values are:
  - 'bottom-top', the braid is printed from bottom to top
  - 'top-bottom', the braid is printed from top to bottom
  - 'left-right', the braid is printed from left to right
- **gap** – floating point number (default: 0.05). determines the size of the gap left when a strand goes under another.
- **aspect\_ratio** – floating point number (default: 1). The aspect ratio.
- **\*\*kwargs** – other keyword options that are passed to `bezier_path()` and `line()`.

**EXAMPLES:**

```

sage: B = BraidGroup(4, 's')
sage: b = B([1, 2, 3, 1, 2, 1])
sage: b.plot()
sage: b.plot(color=["red", "blue", "red", "blue"])

sage: B.<s,t> = BraidGroup(3)
sage: b = t^-1*s^2
sage: b.plot(orientation="left-right", color="red")

```

**plot3d**(color='rainbow')

Plots the braid in 3d.

The following option is available:

- **color** – (default: 'rainbow') the color of the strands. Possible values are:
  - 'rainbow', uses `rainbow()` according to the number of strands.
  - a valid color name for `bezier3d()`. Used for all strands.
  - a list or a tuple of colors for each individual strand.

**EXAMPLES:**

```

sage: B = BraidGroup(4, 's')
sage: b = B([1, 2, 3, 1, 2, 1])
sage: b.plot3d()
sage: b.plot3d(color="red")
sage: b.plot3d(color=["red", "blue", "red", "blue"])

```

**strands**()

Return the number of strands in the braid.

**EXAMPLES:**

```

sage: B = BraidGroup(4)
sage: b = B([1, 2, -1, 3, -2])
sage: b.strands()
4

```

**tropical\_coordinates**()

Return the tropical coordinates of `self` in the braid group  $B_n$ .

OUTPUT:

- a list of  $2n$  tropical integers

## EXAMPLES:

```
sage: B = BraidGroup(3)
sage: b = B([1])
sage: tc = b.tropical_coordinates(); tc
[1, 0, 0, 2, 0, 1]
sage: tc[0].parent()
Tropical semiring over Integer Ring

sage: b = B([-2, -2, -1, -1, 2, 2, 1, 1])
sage: b.tropical_coordinates()
[1, -19, -12, 9, 0, 13]
```

## REFERENCES:

`sage.groups.braid.BraidGroup` (*n=None*, *names='s'*)

Construct a Braid Group

## INPUT:

- *n* – integer or None (default). The number of strands. If not specified the *names* are counted and the group is assumed to have one more strand than generators.
- *names* – string or list/tuple/iterable of strings (default: 'x'). The generator names or name prefix.

## EXAMPLES:

```
sage: B.<a,b> = BraidGroup(); B
Braid group on 3 strands
sage: H = BraidGroup('a, b')
sage: B is H
True
sage: BraidGroup(3)
Braid group on 3 strands
```

The entry can be either a string with the names of the generators, or the number of generators and the prefix of the names to be given. The default prefix is 's'

```
sage: B=BraidGroup(3); B.generators()
(s0, s1)
sage: BraidGroup(3, 'g').generators()
(g0, g1)
```

Since the word problem for the braid groups is solvable, their Cayley graph can be locally obtained as follows (see [trac ticket #16059](#)):

```
sage: def ball(group, radius):
....:     ret = set()
....:     ret.add(group.one())
....:     for length in range(1, radius):
....:         for w in Words(alphabet=group.gens(), length=length):
....:             ret.add(prod(w))
....:     return ret
sage: B = BraidGroup(4)
sage: GB = B.cayley_graph(elements=ball(B, 4), generators=B.gens()); GB
Digraph on 31 vertices
```

Since the braid group has nontrivial relations, this graph contains less vertices than the one associated to the free group (which is a tree):

```
sage: F = FreeGroup(3)
sage: GF = F.cayley_graph(elements=ball(F, 4), generators=F.gens()); GF
```

Digraph on 40 vertices

TESTS:

```
sage: G1 = BraidGroup(3, 'a,b')
sage: G2 = BraidGroup('a,b')
sage: G3.<a,b> = BraidGroup()
sage: G1 is G2, G2 is G3
(True, True)
```

```
class sage.groups.braid.BraidGroup_class(names)
Bases: sage.groups.finitely_presented.FinitelyPresentedGroup
```

The braid group on  $n$  strands.

EXAMPLES:

```
sage: B1 = BraidGroup(5)
sage: B1
Braid group on 5 strands
sage: B2 = BraidGroup(3)
sage: B1==B2
False
sage: B2 is BraidGroup(3)
True
```

**Element**

alias of `Braid`

**an\_element()**

Return an element of the braid group.

This is used both for illustration and testing purposes.

EXAMPLES:

```
sage: B=BraidGroup(2)
sage: B.an_element()
s
```

**as\_permutation\_group()**

Return an isomorphic permutation group.

OUTPUT:

Raises a `ValueError` error since braid groups are infinite.

TESTS:

```
sage: B = BraidGroup(4, 'g')
sage: B.as_permutation_group()
Traceback (most recent call last):
...
ValueError: the group is infinite
```

**cardinality()**

Return the number of group elements.

OUTPUT:

Infinity.

TESTS:

```
sage: B1 = BraidGroup(5)
sage: B1.cardinality()
+Infinity
```

**mapping\_class\_action**(*F*)

Return the action of self in the free group *F* as mapping class group.

This action corresponds to the action of the braid over the punctured disk, whose fundamental group is the free group on as many generators as strands.

In Sage, this action is the result of multiplying a free group element with a braid. So you generally do not have to construct this action yourself.

OUTPUT:

A `MappingClassGroupAction`.

**EXAMPLES**

```
sage: B = BraidGroup(3)
sage: B.inject_variables()
Defining s0, s1
sage: F.<a,b,c> = FreeGroup(3)
sage: A = B.mapping_class_action(F)
sage: A(a,s0)
a*b*a^-1
sage: a * s0      # simpler notation
a*b*a^-1
```

**order**()

Return the number of group elements.

OUTPUT:

Infinity.

TESTS:

```
sage: B1 = BraidGroup(5)
sage: B1.cardinality()
+Infinity
```

**some\_elements**()

Return a list of some elements of the braid group.

This is used both for illustration and testing purposes.

**EXAMPLES:**

```
sage: B=BraidGroup(3)
sage: B.some_elements()
[s0, s0*s1, (s0*s1)^3]
```

**strands**()

Return the number of strands.

OUTPUT:

Integer.

EXAMPLES:



```

sage: B = BraidGroup(4)
sage: B.strands()
4

```

**class** `sage.groups.braid.MappingClassGroupAction` ( $G, M, is\_left=0$ )

Bases: `sage.categories.action.Action`

The action of the braid group the free group as the mapping class group of the punctured disk.

That is, this action is the action of the braid over the punctured disk, whose fundamental group is the free group on as many generators as strands.

This action is defined as follows:

$$x_j \cdot \sigma_i = \begin{cases} x_j \cdot x_{j+1} \cdot x_j^{-1} & \text{if } i = j \\ x_{j-1} & \text{if } i = j - 1, \\ x_j & \text{otherwise} \end{cases}$$

where  $\sigma_i$  are the generators of the braid group on  $n$  strands, and  $x_j$  the generators of the free group of rank  $n$ .

You should left multiplication of the free group element by the braid to compute the action. Alternatively, use the `mapping_class_action()` method of the braid group to construct this action.

EXAMPLES:

```

sage: B.<s0,s1,s2> = BraidGroup(4)
sage: F.<x0,x1,x2,x3> = FreeGroup(4)
sage: x0 * s1
x0
sage: x1 * s1
x1*x2*x1^-1
sage: x1^-1 * s1
x1*x2^-1*x1^-1

```

```

sage: A = B.mapping_class_action(F)
sage: A
Right action by Braid group on 4 strands on Free Group on generators {x0, x1, x2, x3}
sage: A(x0, s1)
x0
sage: A(x1, s1)
x1*x2*x1^-1
sage: A(x1^-1, s1)
x1*x2^-1*x1^-1

```



# INDEXED FREE GROUPS

Free groups and free abelian groups implemented using an indexed set of generators.

AUTHORS:

- Travis Scrimshaw (2013-10-16): Initial version

```
class sage.groups.indexed_free_group.IndexedFreeAbelianGroup(indices, prefix, category=None, **kws)
    Bases: sage.groups.indexed_free_group.IndexedGroup, sage.groups.group.AbelianGroup
```

An indexed free abelian group.

EXAMPLES:

```
sage: G = Groups().Commutative().free(index_set=ZZ)
sage: G
Free abelian group indexed by Integer Ring
sage: G = Groups().Commutative().free(index_set='abcde')
sage: G
Free abelian group indexed by {'a', 'b', 'c', 'd', 'e'}
```

```
class Element(F, x)
```

```
    Bases: sage.monoids.indexed_free_monoid.IndexedFreeAbelianMonoidElement,
           sage.groups.indexed_free_group.IndexedFreeGroup.Element
```

Create the element  $x$  of an indexed free abelian monoid  $F$ .

EXAMPLES:

```
sage: F = FreeAbelianMonoid(index_set=ZZ)
sage: x = F([(0, 1), (2, 2), (-1, 2)])
sage: y = F({0:1, 2:2, -1:2})
sage: z = F(reversed([(0, 1), (2, 2), (-1, 2)]))
sage: x == y and y == z
True
sage: TestSuite(x).run()
```

```
IndexedFreeAbelianGroup.gen(x)
```

The generator indexed by  $x$  of self.

EXAMPLES:

```
sage: G = Groups().Commutative().free(index_set=ZZ)
sage: G.gen(0)
F[0]
sage: G.gen(2)
F[2]
```

`IndexedFreeAbelianGroup.one()`

Return the identity element of self.

EXAMPLES:

```
sage: G = Groups().Commutative().free(index_set=ZZ)
```

```
sage: G.one()
```

```
1
```

**class** `sage.groups.indexed_free_group.IndexedFreeGroup` (*indices, prefix, category=None, \*\*kws*)

Bases: `sage.groups.indexed_free_group.IndexedGroup`, `sage.groups.group.Group`

An indexed free group.

EXAMPLES:

```
sage: G = Groups().free(index_set=ZZ)
```

```
sage: G
```

Free group indexed by Integer Ring

```
sage: G = Groups().free(index_set='abcde')
```

```
sage: G
```

Free group indexed by {'a', 'b', 'c', 'd', 'e'}

**class** `Element` (*F, x*)

Bases: `sage.monoids.indexed_free_monoid.IndexedFreeMonoidElement`

Create the element x of an indexed free abelian monoid F.

EXAMPLES:

```
sage: F = FreeMonoid(index_set=tuple('abcde'))
```

```
sage: x = F( [(1, 2), (0, 1), (3, 2), (0, 1)] )
```

```
sage: y = F( [(1, 2), (0, 1), [3, 2], [0, 1]] )
```

```
sage: z = F( reversed([(0, 1), (3, 2), (0, 1), (1, 2)]) )
```

```
sage: x == y and y == z
```

```
True
```

```
sage: TestSuite(x).run()
```

`length()`

Return the length of self.

EXAMPLES:

```
sage: G = Groups().free(index_set=ZZ)
```

```
sage: a,b,c,d,e = [G.gen(i) for i in range(5)]
```

```
sage: elt = a*c^-3*b^-2*a
```

```
sage: elt.length()
```

```
7
```

```
sage: len(elt)
```

```
7
```

```
sage: G = Groups().free(index_set=ZZ)
```

```
sage: a,b,c,d,e = [G.gen(i) for i in range(5)]
```

```
sage: elt = a*c^-3*b^-2*a
```

```
sage: elt.length()
```

```
7
```

```
sage: len(elt)
```

```
7
```

`to_word_list()`

Return self as a word represented as a list whose entries are the pairs (*i*, *s*) where *i* is the index and *s* is the sign.

## EXAMPLES:

```

sage: G = Groups().free(index_set=ZZ)
sage: a,b,c,d,e = [G.gen(i) for i in range(5)]
sage: x = a*b^2*e*a^-1
sage: x.to_word_list()
[(0, 1), (1, 1), (1, 1), (4, 1), (0, -1)]

```

`IndexedFreeGroup.gen(x)`

The generator indexed by `x` of `self`.

## EXAMPLES:

```

sage: G = Groups().free(index_set=ZZ)
sage: G.gen(0)
F[0]
sage: G.gen(2)
F[2]

```

`IndexedFreeGroup.one()`

Return the identity element of `self`.

## EXAMPLES:

```

sage: G = Groups().free(ZZ)
sage: G.one()
1

```

```

class sage.groups.indexed_free_group.IndexedGroup(indices, prefix, category=None,
                                                    names=None, **kwds)

```

Bases: `sage.monoids.indexed_free_monoid.IndexedMonoid`

Base class for free (abelian) groups whose generators are indexed by a set.

## TESTS:

We check finite properties:

```

sage: G = Groups().free(index_set=ZZ)
sage: G.is_finite()
False
sage: G = Groups().free(index_set='abc')
sage: G.is_finite()
False
sage: G = Groups().free(index_set=[])
sage: G.is_finite()
True

sage: G = Groups().Commutative().free(index_set=ZZ)
sage: G.is_finite()
False
sage: G = Groups().Commutative().free(index_set='abc')
sage: G.is_finite()
False
sage: G = Groups().Commutative().free(index_set=[])
sage: G.is_finite()
True

```

`gens()`

Return the group generators of `self`.

## EXAMPLES:

```
sage: G = Groups.free(index_set=ZZ)
sage: G.group_generators()
Lazy family (Generator map from Integer Ring to
Free group indexed by Integer Ring(i))_{i in Integer Ring}
sage: G = Groups().free(index_set='abcde')
sage: sorted(G.group_generators())
[F['a'], F['b'], F['c'], F['d'], F['e']]
```

**group\_generators()**

Return the group generators of `self`.

**EXAMPLES:**

```
sage: G = Groups.free(index_set=ZZ)
sage: G.group_generators()
Lazy family (Generator map from Integer Ring to
Free group indexed by Integer Ring(i))_{i in Integer Ring}
sage: G = Groups().free(index_set='abcde')
sage: sorted(G.group_generators())
[F['a'], F['b'], F['c'], F['d'], F['e']]
```

**order()**

Return the number of elements of `self`, which is  $\infty$  unless this is the trivial group.

**EXAMPLES:**

```
sage: G = Groups().free(index_set=ZZ)
sage: G.order()
+Infinity
sage: G = Groups().Commutative().free(index_set='abc')
sage: G.order()
+Infinity
sage: G = Groups().Commutative().free(index_set=[])
sage: G.order()
1
```

**rank()**

Return the rank of `self`.

This is the number of generators of `self`.

**EXAMPLES:**

```
sage: G = Groups().free(index_set=ZZ)
sage: G.rank()
+Infinity
sage: G = Groups().free(index_set='abc')
sage: G.rank()
3
sage: G = Groups().free(index_set=[])
sage: G.rank()
0

sage: G = Groups().Commutative().free(index_set=ZZ)
sage: G.rank()
+Infinity
sage: G = Groups().Commutative().free(index_set='abc')
sage: G.rank()
3
sage: G = Groups().Commutative().free(index_set=[])
```

```
sage: G.rank()  
0
```





# RIGHT-ANGLED ARTIN GROUPS

A *right-angled Artin group* (often abbreviated as RAAG) is a group which has a presentation whose only relations are commutators between generators. These are also known as graph groups, since they are (uniquely) encoded by (simple) graphs, or partially commutative groups.

AUTHORS:

- Travis Scrimshaw (2013-09-01): Initial version

**class** `sage.groups.raag.RightAngledArtinGroup` ( $G$ )

Bases: `sage.groups.finitely_presented.FinitelyPresentedGroup`

The right-angled Artin group defined by a graph  $G$ .

Let  $\Gamma = \{V(\Gamma), E(\Gamma)\}$  be a simple graph. A *right-angled Artin group* (commonly abbreviated as RAAG) is the group

$$A_\Gamma = \langle g_v : v \in V(\Gamma) \mid [g_u, g_v] \text{ if } \{u, v\} \notin E(\Gamma) \rangle.$$

These are sometimes known as graph groups or partially commutative groups. This RAAG's contains both free groups, given by the complete graphs, and free abelian groups, given by disjoint vertices.

**Warning:** This is the opposite convention of some papers.

Right-angled Artin groups contain many remarkable properties and have a very rich structure despite their simple presentation. Here are some known facts:

- The word problem is solvable.
- They are known to be rigid; that is for any finite simple graphs  $\Delta$  and  $\Gamma$ , we have  $A_\Delta \cong A_\Gamma$  if and only if  $\Delta \cong \Gamma$  [Droms1987].
- They embed as a finite index subgroup of a right-angled Coxeter group (which is the same definition as above except with the additional relations  $g_v^2 = 1$  for all  $v \in V(\Gamma)$ ).
- In [BB1997], it was shown they contain subgroups that satisfy the property  $FP_2$  but are not finitely presented by considering the kernel of  $\phi : A_\Gamma \rightarrow \mathbb{Z}$  by  $g_v \mapsto 1$  (i.e. words of exponent sum 0).
- $A_\Gamma$  has a finite  $K(\pi, 1)$  space.
- $A_\Gamma$  acts freely and cocompactly on a finite dimensional  $CAT(0)$  space, and so it is biautomatic.
- Given an Artin group  $B$  with generators  $s_i$ , then any subgroup generated by a collection of  $v_i = s_i^{k_i}$  where  $k_i \geq 2$  is a RAAG where  $[v_i, v_j] = 1$  if and only if  $[s_i, s_j] = 1$  [CP2001].

The normal forms for RAAG's in Sage are those described in [VW1994] and gathers commuting groups together.

EXAMPLES:

```
sage: Gamma = Graph(4)
sage: G = RightAngledArtinGroup(Gamma)
sage: a,b,c,d = G.gens()
sage: a*c*d^4*a^-3*b
v0^-2*v1*v2*v3^4

sage: Gamma = graphs.CompleteGraph(4)
sage: G = RightAngledArtinGroup(Gamma)
sage: a,b,c,d = G.gens()
sage: a*c*d^4*a^-3*b
v0*v2*v3^4*v0^-3*v1

sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G
Right-angled Artin group of Cycle graph
sage: a,b,c,d,e = G.gens()
sage: e^-1*c*b*e*b^-1*c^-4
v2^-3
```

## REFERENCES:

- [Wikipedia article Artin\\_group#Right-angled\\_Artin\\_groups](#)

**class Element** (*parent, lst*)

Bases: `sage.groups.finitely_presented.FinitelyPresentedGroupElement`

An element of a right-angled Artin group (RAAG).

Elements of RAAGs are modeled as lists of pairs  $[i, p]$  where  $i$  is the index of a vertex in the defining graph (with some fixed order of the vertices) and  $p$  is the power.

`RightAngledArtinGroup.as_permutation_group()`

Raise a `ValueError` error since right-angled Artin groups are infinite, so they have no isomorphic permutation group.

## EXAMPLES:

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.as_permutation_group()
Traceback (most recent call last):
...
ValueError: the group is infinite
```

`RightAngledArtinGroup.cardinality()`

Return the number of group elements.

## OUTPUT:

Infinity.

## EXAMPLES:

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.cardinality()
+Infinity
```

`RightAngledArtinGroup.gen(i)`

Return the  $i$ -th generator of `self`.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.gen(2)
v2
```

`RightAngledArtinGroup.gen()`

Return the generators of self.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.gens()
(v0, v1, v2, v3, v4)
sage: Gamma = Graph([('x', 'y'), ('y', 'zeta')])
sage: G = RightAngledArtinGroup(Gamma)
sage: G.gens()
(vx, vy, vzeta)
```

`RightAngledArtinGroup.graph()`

Return the defining graph of self.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.graph()
Cycle graph: Graph on 5 vertices
```

`RightAngledArtinGroup.ngens()`

Return the number of generators of self.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.ngens()
5
```

`RightAngledArtinGroup.one()`

Return the identity element 1.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.one()
1
```

`RightAngledArtinGroup.one_element()`

Return the identity element 1.

**EXAMPLES:**

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.one()
1
```

`RightAngledArtinGroup.order()`

Return the number of group elements.

OUTPUT:

Infinity.

EXAMPLES:

```
sage: Gamma = graphs.CycleGraph(5)
sage: G = RightAngledArtinGroup(Gamma)
sage: G.cardinality()
+Infinity
```

# MULTIPLICATIVE ABELIAN GROUPS

This module lets you compute with finitely generated Abelian groups of the form

$$G = \mathbf{Z}^r \oplus \mathbf{Z}_{k_1} \oplus \cdots \oplus \mathbf{Z}_{k_t}$$

It is customary to denote the infinite cyclic group  $\mathbf{Z}$  as having order 0, so the data defining the Abelian group can be written as an integer vector

$$\vec{k} = (0, \dots, 0, k_1, \dots, k_t)$$

where there are  $r$  zeroes and  $t$  non-zero values. To construct this Abelian group in Sage, you can either specify all entries of  $\vec{k}$  or only the non-zero entries together with the total number of generators:

```
sage: AbelianGroup([0,0,0,2,3])
Multiplicative Abelian group isomorphic to Z x Z x Z x C2 x C3
sage: AbelianGroup(5, [2,3])
Multiplicative Abelian group isomorphic to Z x Z x Z x C2 x C3
```

It is also legal to specify 1 as the order. The corresponding generator will be the neutral element, but it will still take up an index in the labelling of the generators:

```
sage: G = AbelianGroup([2,1,3], names='g')
sage: G.gens()
(g0, 1, g2)
```

Note that this presentation is not unique, for example  $\mathbf{Z}_6 = \mathbf{Z}_2 \times \mathbf{Z}_3$ . The orders of the generators  $\vec{k} = (0, \dots, 0, k_1, \dots, k_t)$  has previously been called invariants in Sage, even though they are not necessarily the (unique) invariant factors of the group. You should now use `gens_orders()` instead:

```
sage: J = AbelianGroup([2,0,3,2,4]); J
Multiplicative Abelian group isomorphic to C2 x Z x C3 x C2 x C4
sage: J.gens_orders()           # use this instead
(2, 0, 3, 2, 4)
sage: J.invariants()           # deprecated
(2, 0, 3, 2, 4)
sage: J.elementary_divisors()   # these are the "invariant factors"
(2, 2, 12, 0)
sage: for i in range(J.ngens()):
...     print i, J.gen(i), J.gen(i).order()   # or use this form
0 f0 2
1 f1 +Infinity
2 f2 3
3 f3 2
4 f4 4
```

Background on invariant factors and the Smith normal form (according to section 4.1 of [C1]): An abelian group is a group  $A$  for which there exists an exact sequence  $\mathbf{Z}^k \rightarrow \mathbf{Z}^\ell \rightarrow A \rightarrow 1$ , for some positive integers  $k, \ell$  with  $k \leq \ell$ . For example, a finite abelian group has a decomposition

$$A = \langle a_1 \rangle \times \cdots \times \langle a_\ell \rangle,$$

where  $\text{ord}(a_i) = p_i^{c_i}$ , for some primes  $p_i$  and some positive integers  $c_i$ ,  $i = 1, \dots, \ell$ . GAP calls the list (ordered by size) of the  $p_i^{c_i}$  the *abelian invariants*. In Sage they will be called *invariants*. In this situation,  $k = \ell$  and  $\phi : \mathbf{Z}^\ell \rightarrow A$  is the map  $\phi(x_1, \dots, x_\ell) = a_1^{x_1} \dots a_\ell^{x_\ell}$ , for  $(x_1, \dots, x_\ell) \in \mathbf{Z}^\ell$ . The matrix of relations  $M : \mathbf{Z}^k \rightarrow \mathbf{Z}^\ell$  is the matrix whose rows generate the kernel of  $\phi$  as a  $\mathbf{Z}$ -module. In other words,  $M = (M_{ij})$  is a  $\ell \times \ell$  diagonal matrix with  $M_{ii} = p_i^{c_i}$ . Consider now the subgroup  $B \subset A$  generated by  $b_1 = a_1^{f_{1,1}} \dots a_\ell^{f_{\ell,1}}$ , ...,  $b_m = a_1^{f_{1,m}} \dots a_\ell^{f_{\ell,m}}$ . The kernel of the map  $\phi_B : \mathbf{Z}^m \rightarrow B$  defined by  $\phi_B(y_1, \dots, y_m) = b_1^{y_1} \dots b_m^{y_m}$ , for  $(y_1, \dots, y_m) \in \mathbf{Z}^m$ , is the kernel of the matrix

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \vdots & & \ddots & \vdots \\ f_{\ell,1} & f_{\ell,2} & \cdots & f_{\ell,m} \end{pmatrix},$$

regarded as a map  $\mathbf{Z}^m \rightarrow (\mathbf{Z}/p_1^{c_1}\mathbf{Z}) \times \dots \times (\mathbf{Z}/p_\ell^{c_\ell}\mathbf{Z})$ . In particular,  $B \cong \mathbf{Z}^m / \ker(F)$ . If  $B = A$  then the Smith normal form (SNF) of a generator matrix of  $\ker(F)$  and the SNF of  $M$  are the same. The diagonal entries  $s_i$  of the SNF  $S = \text{diag}[s_1, s_2, s_3, \dots, s_r, 0, 0, \dots, 0]$ , are called *determinantal divisors* of  $F$ . where  $r$  is the rank. The {invariant factors} of  $A$  are:

$$s_1, s_2/s_1, s_3/s_2, \dots, s_r/s_{r-1}.$$

Sage supports multiplicative abelian groups on any prescribed finite number  $n \geq 0$  of generators. Use the `AbelianGroup()` function to create an abelian group, and the `gen()` and `gens()` methods to obtain the corresponding generators. You can print the generators as arbitrary strings using the optional `names` argument to the `AbelianGroup()` function.

#### EXAMPLE 1:

We create an abelian group in zero or more variables; the syntax `T(1)` creates the identity element even in the rank zero case:

```
sage: T = AbelianGroup(0, [])
sage: T
Trivial Abelian group
sage: T.gens()
()
sage: T(1)
1
```

#### EXAMPLE 2:

An Abelian group uses a multiplicative representation of elements, but the underlying representation is lists of integer exponents:

```
sage: F = AbelianGroup(5, [3, 4, 5, 5, 7], names = list("abcde"))
sage: F
Multiplicative Abelian group isomorphic to C3 x C4 x C5 x C5 x C7
sage: (a, b, c, d, e) = F.gens()
sage: a*b^2*d*e
a*b^2*d*e
sage: x = b^2*e*d*a^7
sage: x
a*b^2*d*e
sage: x.list()
[1, 2, 0, 1, 1]
```

## REFERENCES:

- [C1] H. Cohen Advanced topics in computational number theory, Springer, 2000.
- [C2] —, A course in computational algebraic number theory, Springer, 1996.
- [R] J. Rotman, An introduction to the theory of groups, 4th ed, Springer, 1995.

**Warning:** Many basic properties for infinite abelian groups are not implemented.

## AUTHORS:

- William Stein, David Joyner (2008-12): added (user requested) `is_cyclic`, fixed `elementary_divisors`.
- David Joyner (2006-03): (based on free abelian monoids by David Kohel)
- David Joyner (2006-05) several significant bug fixes
- David Joyner (2006-08) trivial changes to docs, added `random`, fixed bug in how invariants are recorded
- David Joyner (2006-10) added `dual_group` method
- David Joyner (2008-02) fixed serious bug in `word_problem`
- David Joyner (2008-03) fixed bug in trivial group case
- David Loeffler (2009-05) added `subgroups` method
- Volker Braun (2012-11) port to new Parent base. Use tuples for immutables. Rename invariants to `gens_orders`.

`sage.groups.abelian_gps.abelian_group.AbelianGroup` (*n*, *gens\_orders*=None, *names*='f')

Create the multiplicative abelian group in *n* generators with given orders of generators (which need not be prime powers).

## INPUT:

- ***n* – integer (optional).** If not specified, will be derived from `gens_orders`.
- ***gens\_orders* – a list of non-negative integers in the form**  $[a_0, a_1, \dots, a_{n-1}]$ , typically written in increasing order. This list is padded with zeros if it has length less than *n*. The orders of the commuting generators, with 0 denoting an infinite cyclic factor.
- ***names* – (optional)** names of generators

Alternatively, you can also give input in the form `AbelianGroup(gens_orders, names="f")`, where the `names` keyword argument must be explicitly named.

## OUTPUT:

Abelian group with generators and invariant type. The default name for generator `A.i` is `fi`, as in GAP.

## EXAMPLES:

```
sage: F = AbelianGroup(5, [5,5,7,8,9], names='abcde')
sage: F(1)
1
sage: (a, b, c, d, e) = F.gens()
sage: mul([a, b, a, c, b, d, c, d], F(1))
a^2*b^2*c^2*d^2
sage: d * b**2 * c**3
b^2*c^3*d
sage: F = AbelianGroup(3, [2]*3); F
Multiplicative Abelian group isomorphic to C2 x C2 x C2
sage: H = AbelianGroup([2,3], names="xy"); H
Multiplicative Abelian group isomorphic to C2 x C3
```

```
sage: AbelianGroup(5)
Multiplicative Abelian group isomorphic to  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ 
sage: AbelianGroup(5).order()
+Infinity
```

Notice that 0's are prepended if necessary:

```
sage: G = AbelianGroup(5, [2,3,4]); G
Multiplicative Abelian group isomorphic to  $\mathbb{Z} \times \mathbb{Z} \times C_2 \times C_3 \times C_4$ 
sage: G.gens_orders()
(0, 0, 2, 3, 4)
```

The invariant list must not be longer than the number of generators:

```
sage: AbelianGroup(2, [2,3,4])
Traceback (most recent call last):
...
ValueError: gens_orders (=(2, 3, 4)) must have length n (2)
```

```
class sage.groups.abelian_gps.abelian_group.AbelianGroup_class(generator_orders,
                                                                names)
Bases: sage.structure.unique_representation.UniqueRepresentation,
       sage.groups.group.AbelianGroup
```

The parent for Abelian groups with chosen generator orders.

**Warning:** You should use `AbelianGroup()` to construct Abelian groups and not instantiate this class directly.

INPUT:

- `generator_orders` – list of integers. The orders of the (commuting) generators. Zero denotes an infinite cyclic generator.
- `names` – names of the group generators (optional).

EXAMPLES:

```
sage: Z2xZ3 = AbelianGroup([2,3])
sage: Z6 = AbelianGroup([6])
sage: Z2xZ3 is Z2xZ3, Z6 is Z6
(True, True)
sage: Z2xZ3 is Z6
False
sage: Z2xZ3 == Z6
True

sage: F = AbelianGroup(5, [5,5,7,8,9], names = list("abcde")); F
Multiplicative Abelian group isomorphic to  $C_5 \times C_5 \times C_7 \times C_8 \times C_9$ 
sage: F = AbelianGroup(5, [2, 4, 12, 24, 120], names = list("abcde")); F
Multiplicative Abelian group isomorphic to  $C_2 \times C_4 \times C_{12} \times C_{24} \times C_{120}$ 
sage: F.elementary_divisors()
(2, 4, 12, 24, 120)

sage: F.category()
Category of groups
```

TESTS:



```

sage: AbelianGroup([]).gens_orders()
()
sage: AbelianGroup([1]).gens_orders()
(1,)
sage: AbelianGroup([1,1]).gens_orders()
(1, 1)
sage: AbelianGroup(0).gens_orders()
()

```

**Element**

alias of AbelianGroupElement

**dual\_group** (*names='X', base\_ring=None*)

Returns the dual group.

INPUT:

- *names* – string or list of strings. The generator names for the dual group.
- *base\_ring* – the base ring. If None (default), then a suitable cyclotomic field is picked automatically.

OUTPUT:

The `~sage.groups.abelian_gps.dual_abelian_group.DualAbelianGroup_class`

EXAMPLES:

```

sage: G = AbelianGroup([2])
sage: G.dual_group()
Dual of Abelian Group isomorphic to Z/2Z over Cyclotomic Field of order 2 and degree 1
sage: G.dual_group().gens()
(X,)
sage: G.dual_group(names='Z').gens()
(Z,)

sage: G.dual_group(base_ring=QQ)
Dual of Abelian Group isomorphic to Z/2Z over Rational Field

```

TESTS:

```

sage: H = AbelianGroup(1)
sage: H.dual_group()
Traceback (most recent call last):
...
ValueError: the group must be finite

```

**elementary\_divisors** ()

This returns the elementary divisors of the group, using Pari.

---

**Note:** Here is another algorithm for computing the elementary divisors  $d_1, d_2, d_3, \dots$ , of a finite abelian group (where  $d_1|d_2|d_3|\dots$  are composed of prime powers dividing the invariants of the group in a way described below). Just factor the invariants  $a_i$  that define the abelian group. Then the biggest  $d_i$  is the product of the maximum prime powers dividing some  $a_j$ . In other words, the largest  $d_i$  is the product of  $p^v$ , where  $v = \max(\text{ord}_p(a_j) \text{ for all } j)$ . Now divide out all those  $p^v$ 's into the list of invariants  $a_i$ , and get a new list of “smaller invariants”. Repeat the above procedure on these “smaller invariants” to compute  $d_{i-1}$ , and so on. (Thanks to Robert Miller for communicating this algorithm.)

---

OUTPUT:

A tuple of integers.

EXAMPLES:

```
sage: G = AbelianGroup(2, [2, 3])
sage: G.elementary_divisors()
(6,)
sage: G = AbelianGroup(1, [6])
sage: G.elementary_divisors()
(6,)
sage: G = AbelianGroup(2, [2, 6])
sage: G
Multiplicative Abelian group isomorphic to C2 x C6
sage: G.gens_orders()
(2, 6)
sage: G.elementary_divisors()
(2, 6)
sage: J = AbelianGroup([1, 3, 5, 12])
sage: J.elementary_divisors()
(3, 60)
sage: G = AbelianGroup(2, [0, 6])
sage: G.elementary_divisors()
(6, 0)
sage: AbelianGroup([3, 4, 5]).elementary_divisors()
(60,)
```

**exponent()**

Return the exponent of this abelian group.

EXAMPLES:

```
sage: G = AbelianGroup([2, 3, 7]); G
Multiplicative Abelian group isomorphic to C2 x C3 x C7
sage: G.exponent()
42
sage: G = AbelianGroup([2, 4, 6]); G
Multiplicative Abelian group isomorphic to C2 x C4 x C6
sage: G.exponent()
12
```

**gen(*i*=0)**

The *i*-th generator of the abelian group.

EXAMPLES:

```
sage: F = AbelianGroup(5, [], names='a')
sage: F.0
a0
sage: F.2
a2
sage: F.gens_orders()
(0, 0, 0, 0, 0)

sage: G = AbelianGroup([2, 1, 3])
sage: G.gens()
(f0, 1, f2)
```

**gens()**

Return the generators of the group.

OUTPUT:

A tuple of group elements. The generators according to the chosen `gens_orders()`.

EXAMPLES:

```
sage: F = AbelianGroup(5, [3, 2], names='abcde')
sage: F.gens()
(a, b, c, d, e)
sage: [ g.order() for g in F.gens() ]
[+Infinity, +Infinity, +Infinity, 3, 2]
```

**gens\_orders()**

Return the orders of the cyclic factors that this group has been defined with.

Use `elementary_divisors()` if you are looking for an invariant of the group.

OUTPUT:

A tuple of integers.

EXAMPLES:

```
sage: Z2xZ3 = AbelianGroup([2, 3])
sage: Z2xZ3.gens_orders()
(2, 3)
sage: Z2xZ3.elementary_divisors()
(6,)
```

```
sage: Z6 = AbelianGroup([6])
sage: Z6.gens_orders()
(6,)
sage: Z6.elementary_divisors()
(6,)
```

```
sage: Z2xZ3.is_isomorphic(Z6)
True
sage: Z2xZ3 is Z6
False
```

TESTS:

```
sage: F = AbelianGroup(3, [2], names='abc')
sage: map(type, F.gens_orders())
[<type 'sage.rings.integer.Integer'>,
 <type 'sage.rings.integer.Integer'>,
 <type 'sage.rings.integer.Integer'>]
```

**identity()**

Return the identity element of this group.

EXAMPLES:

```
sage: G = AbelianGroup([2, 2])
sage: e = G.identity()
sage: e
1
sage: g = G.gen(0)
sage: g*e
f0
sage: e*g
f0
```

**invariants()**

Return the orders of the cyclic factors that this group has been defined with.

For historical reasons this has been called invariants in Sage, even though they are not necessarily the invariant factors of the group. Use `gens_orders()` instead:

```
sage: J = AbelianGroup([2,0,3,2,4]); J
Multiplicative Abelian group isomorphic to C2 x Z x C3 x C2 x C4
sage: J.invariants()      # deprecated
(2, 0, 3, 2, 4)
sage: J.gens_orders()     # use this instead
(2, 0, 3, 2, 4)
sage: for i in range(J.ngens()):
...     print i, J.gen(i), J.gen(i).order()      # or use this
0 f0 2
1 f1 +Infinity
2 f2 3
3 f3 2
4 f4 4
```

Use `elementary_divisors()` if you are looking for an invariant of the group.

OUTPUT:

A tuple of integers. Zero means infinite cyclic factor.

EXAMPLES:

```
sage: J = AbelianGroup([2,3])
sage: J.invariants()
(2, 3)
sage: J.elementary_divisors()
(6,)
```

TESTS:

```
sage: F = AbelianGroup(3, [2], names='abc')
sage: map(type, F.gens_orders())
[<type 'sage.rings.integer.Integer'>,
 <type 'sage.rings.integer.Integer'>,
 <type 'sage.rings.integer.Integer'>]
```

**is\_commutative()**

Return True since this group is commutative.

EXAMPLES:

```
sage: G = AbelianGroup([2,3,9, 0])
sage: G.is_commutative()
True
sage: G.is_abelian()
True
```

**is\_cyclic()**

Return True if the group is a cyclic group.

EXAMPLES:

```
sage: J = AbelianGroup([2,3])
sage: J.gens_orders()
(2, 3)
sage: J.elementary_divisors()
(6,)
sage: J.is_cyclic()
```

```

True
sage: G = AbelianGroup([6])
sage: G.gens_orders()
(6,)
sage: G.is_cyclic()
True
sage: H = AbelianGroup([2,2])
sage: H.gens_orders()
(2, 2)
sage: H.is_cyclic()
False
sage: H = AbelianGroup([2,4])
sage: H.elementary_divisors()
(2, 4)
sage: H.is_cyclic()
False
sage: H.permutation_group().is_cyclic()
False
sage: T = AbelianGroup([])
sage: T.is_cyclic()
True
sage: T = AbelianGroup(1,[0]); T
Multiplicative Abelian group isomorphic to Z
sage: T.is_cyclic()
True
sage: B = AbelianGroup([3,4,5])
sage: B.is_cyclic()
True

```

**is\_isomorphic** (*left, right*)

Check whether *left* and *right* are isomorphic

INPUT:

- *right* – anything.

OUTPUT:

Boolean. Whether *left* and *right* are isomorphic as abelian groups.

EXAMPLES:

```

sage: G1 = AbelianGroup([2,3,4,5])
sage: G2 = AbelianGroup([2,3,4,5,1])
sage: G1.is_isomorphic(G2)
True
sage: G1 == G2      # syntactic sugar
True

```

**is\_subgroup** (*left, right*)

Test whether *left* is a subgroup of *right*.

EXAMPLES:

```

sage: G = AbelianGroup([2,3,4,5])
sage: G.is_subgroup(G)
True

sage: H = G.subgroup([G.1])
sage: H.is_subgroup(G)
True

```

```
sage: G.<a, b> = AbelianGroup(2)
sage: H.<c> = AbelianGroup(1)
sage: H < G
False
```

**is\_trivial()**

Return whether the group is trivial

A group is trivial if it has precisely one element.

EXAMPLES:

```
sage: AbelianGroup([2, 3]).is_trivial()
False
sage: AbelianGroup([1, 1]).is_trivial()
True
```

**list()**

Return tuple of all elements of this group.

EXAMPLES:

```
sage: G = AbelianGroup([2, 3], names = "ab")
sage: G.list()
(1, b, b^2, a, a*b, a*b^2)

sage: G = AbelianGroup([]); G
Trivial Abelian group
sage: G.list()
(1,)
```

**ngens()**

The number of free generators of the abelian group.

EXAMPLES:

```
sage: F = AbelianGroup(10000)
sage: F.ngens()
10000
```

**order()**

Return the order of this group.

EXAMPLES:

```
sage: G = AbelianGroup(2, [2, 3])
sage: G.order()
6
sage: G = AbelianGroup(3, [2, 3, 0])
sage: G.order()
+Infinity
```

**permutation\_group()**

Return the permutation group isomorphic to this abelian group.

If the invariants are  $q_1, \dots, q_n$  then the generators of the permutation will be of order  $q_1, \dots, q_n$ , respectively.

EXAMPLES:

```
sage: G = AbelianGroup(2, [2, 3]); G
Multiplicative Abelian group isomorphic to C2 x C3
sage: G.permutation_group()
Permutation Group with generators [(3, 4, 5), (1, 2)]
```

**random\_element()**

Return a random element of this group.

EXAMPLES:

```
sage: G = AbelianGroup([2, 3, 9])
sage: G.random_element()
f1^2
```

**subgroup(gensH, names='f')**

Create a subgroup of this group. The “big” group must be defined using “named” generators.

INPUT:

- **gensH** – list of elements which are products of the generators of the ambient abelian group  $G = \text{self}$

EXAMPLES:

```
sage: G.<a,b,c> = AbelianGroup(3, [2, 3, 4]); G
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: H = G.subgroup([a*b, a]); H
Multiplicative Abelian subgroup isomorphic to C2 x C3 generated by {a*b, a}
sage: H < G
True
sage: F = G.subgroup([a, b^2])
sage: F
Multiplicative Abelian subgroup isomorphic to C2 x C3 generated by {a, b^2}
sage: F.gens()
(a, b^2)
sage: F = AbelianGroup(5, [30, 64, 729], names = list("abcde"))
sage: a, b, c, d, e = F.gens()
sage: F.subgroup([a, b])
Multiplicative Abelian subgroup isomorphic to Z x Z generated by {a, b}
sage: F.subgroup([c, e])
Multiplicative Abelian subgroup isomorphic to C2 x C3 x C5 x C729 generated by {c, e}
```

**subgroup\_reduced(elts, verbose=False)**

Given a list of lists of integers (corresponding to elements of self), find a set of independent generators for the subgroup generated by these elements, and return the subgroup with these as generators, forgetting the original generators.

This is used by the `subgroups` routine.

An error will be raised if the elements given are not linearly independent over  $\mathbb{Q}\mathbb{Q}$ .

EXAMPLE:

```
sage: G = AbelianGroup([4, 4])
sage: G.subgroup([ G([1, 0]), G([1, 2]) ])
Multiplicative Abelian subgroup isomorphic to C2 x C4
generated by {f0, f0*f1^2}
sage: AbelianGroup([4, 4]).subgroup_reduced([ [1, 0], [1, 2] ])
Multiplicative Abelian subgroup isomorphic to C2 x C4
generated by {f1^2, f0}
```

**subgroups** (*check=False*)

Compute all the subgroups of this abelian group (which must be finite).

TODO: This is *many orders of magnitude* slower than Magma.

INPUT:

- check**: if True, performs the same computation in GAP and checks that the number of subgroups generated is the same. (I don't know how to convert GAP's output back into Sage, so we don't actually compare the subgroups).

ALGORITHM:

If the group is cyclic, the problem is easy. Otherwise, write it as a direct product  $A \times B$ , where  $B$  is cyclic. Compute the subgroups of  $A$  (by recursion).

Now, for every subgroup  $C$  of  $A \times B$ , let  $G$  be its *projection onto*  $A$  and  $H$  its *intersection with*  $B$ . Then there is a well-defined homomorphism  $f: G \rightarrow B/H$  that sends  $a$  in  $G$  to the class mod  $H$  of  $b$ , where  $(a,b)$  is any element of  $C$  lifting  $a$ ; and every subgroup  $C$  arises from a unique triple  $(G, H, f)$ .

EXAMPLES:

```
sage: AbelianGroup([2,3]).subgroups()
[Multiplicative Abelian subgroup isomorphic to C2 x C3 generated by {f0*f1^2},
 Multiplicative Abelian subgroup isomorphic to C2 generated by {f0},
 Multiplicative Abelian subgroup isomorphic to C3 generated by {f1},
 Trivial Abelian subgroup]
sage: len(AbelianGroup([2,4,8]).subgroups())
81
```

TESTS:

```
sage: AbelianGroup([]).subgroups()
[Trivial Abelian group]
```

**class** sage.groups.abelian\_gps.abelian\_group.**AbelianGroup\_subgroup** (*ambient, gens,*  
names='f')

Bases: sage.groups.abelian\_gps.abelian\_group.AbelianGroup\_class

Subgroup subclass of AbelianGroup\_class, so instance methods are inherited.

TODO:

- There should be a way to coerce an element of a subgroup into the ambient group.

**ambient\_group** ()

Return the ambient group related to self.

OUTPUT:

A multiplicative Abelian group.

EXAMPLES:

```
sage: G.<a,b,c> = AbelianGroup([2,3,4])
sage: H = G.subgroup([a, b^2])
sage: H.ambient_group() is G
True
```

**equals** (*left, right*)

Check whether *left* and *right* are the same (sub)group.

INPUT:



•right – anything.

OUTPUT:

Boolean. If `right` is a subgroup, test whether `left` and `right` are the same subset of the ambient group. If `right` is not a subgroup, test whether they are isomorphic groups, see `is_isomorphic()`.

EXAMPLES:

```
sage: G = AbelianGroup(3, [2,3,4], names="abc"); G
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: a,b,c = G.gens()
sage: F = G.subgroup([a,b^2]); F
Multiplicative Abelian subgroup isomorphic to C2 x C3 generated by {a, b^2}
sage: F<G
True

sage: A = AbelianGroup(1, [6])
sage: A.subgroup(list(A.gens())) == A
True

sage: G.<a,b> = AbelianGroup(2)
sage: A = G.subgroup([a])
sage: B = G.subgroup([b])
sage: A.equals(B)
False
sage: A == B           # same as A.equals(B)
False
sage: A.is_isomorphic(B)
True
```

**gen**(*n*)

Return the *n*th generator of this subgroup.

EXAMPLE:

```
sage: G.<a,b> = AbelianGroup(2)
sage: A = G.subgroup([a])
sage: A.gen(0)
a
```

**gens**()

Return the generators for this subgroup.

OUTPUT:

A tuple of group elements generating the subgroup.

EXAMPLES:

```
sage: G.<a,b> = AbelianGroup(2)
sage: A = G.subgroup([a])
sage: G.gens()
(a, b)
sage: A.gens()
(a,)
```

`sage.groups.abelian_gps.abelian_group.is_AbelianGroup(x)`

Return True if *x* is an Abelian group.

EXAMPLES:

```

sage: from sage.groups.abelian_gps.abelian_group import is_AbelianGroup
sage: F = AbelianGroup(5, [5, 5, 7, 8, 9], names = list("abcde")); F
Multiplicative Abelian group isomorphic to C5 x C5 x C7 x C8 x C9
sage: is_AbelianGroup(F)
True
sage: is_AbelianGroup(AbelianGroup(7, [3]*7))
True

```

`sage.groups.abelian_gps.abelian_group.word_problem(words, g, verbose=False)`

$G$  and  $H$  are abelian,  $g$  in  $G$ ,  $H$  is a subgroup of  $G$  generated by a list (words) of elements of  $G$ . If  $g$  is in  $H$ , return the expression for  $g$  as a word in the elements of (words).

The ‘word problem’ for a finite abelian group  $G$  boils down to the following matrix-vector analog of the Chinese remainder theorem.

Problem: Fix integers  $1 < n_1 \leq n_2 \leq \dots \leq n_k$  (indeed, these  $n_i$  will all be prime powers), fix a generating set  $g_i = (a_{i1}, \dots, a_{ik})$  (with  $a_{ij} \in \mathbb{Z}/n_j\mathbb{Z}$ ), for  $1 \leq i \leq \ell$ , for the group  $G$ , and let  $d = (d_1, \dots, d_k)$  be an element of the direct product  $\mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_k\mathbb{Z}$ . Find, if they exist, integers  $c_1, \dots, c_\ell$  such that  $c_1 g_1 + \dots + c_\ell g_\ell = d$ . In other words, solve the equation  $cA = d$  for  $c \in \mathbb{Z}^\ell$ , where  $A$  is the matrix whose rows are the  $g_i$ ’s. Of course, it suffices to restrict the  $c_i$ ’s to the range  $0 \leq c_i \leq N - 1$ , where  $N$  denotes the least common multiple of the integers  $n_1, \dots, n_k$ .

This function does not solve this directly, as perhaps it should. Rather (for both speed and as a model for a similar function valid for more general groups), it pushes it over to GAP, which has optimized (non-deterministic) algorithms for the word problem. Essentially, this function is a wrapper for the GAP function ‘Factorization’.

EXAMPLE:

```

sage: G.<a,b,c> = AbelianGroup(3, [2, 3, 4]); G
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: w = word_problem([a*b, a*c], b*c); w #random
[[a*b, 1], [a*c, 1]]
sage: prod([x^i for x, i in w]) == b*c
True
sage: w = word_problem([a*c, c], a); w #random
[[a*c, 1], [c, -1]]
sage: prod([x^i for x, i in w]) == a
True
sage: word_problem([a*c, c], a, verbose=True) #random
a = (a*c)^1*(c)^-1
[[a*c, 1], [c, -1]]

sage: A.<a,b,c,d,e> = AbelianGroup(5, [4, 5, 5, 7, 8])
sage: b1 = a^3*b*c*d^2*e^5
sage: b2 = a^2*b*c^2*d^3*e^3
sage: b3 = a^7*b^3*c^5*d^4*e^4
sage: b4 = a^3*b^2*c^2*d^3*e^5
sage: b5 = a^2*b^4*c^2*d^4*e^5
sage: w = word_problem([b1, b2, b3, b4, b5], e); w #random
[[a^3*b*c*d^2*e^5, 1], [a^2*b*c^2*d^3*e^3, 1], [a^3*b^3*d^4*e^4, 3], [a^2*b^4*c^2*d^4*e^5, 1]]
sage: prod([x^i for x, i in w]) == e
True
sage: word_problem([a, b, c, d, e], e)
[[e, 1]]
sage: word_problem([a, b, c, d, e], b)
[[b, 1]]

```

**Warning:**

1. Might have unpleasant effect when the word problem cannot be solved.
2. Uses permutation groups, so may be slow when group is large. The instance method `word_problem` of the class `AbelianGroupElement` is implemented differently (wrapping GAP's 'EpimorphismFrom-FreeGroup' and 'PreImagesRepresentative') and may be faster.



# MULTIPLICATIVE ABELIAN GROUPS WITH VALUES

Often, one ends up with a set that forms an Abelian group. It would be nice if one could return an Abelian group class to encapsulate the data. However, `AbelianGroup()` is an abstract Abelian group defined by generators and relations. This module implements `AbelianGroupWithValues` that allows the group elements to be decorated with values.

An example where this module is used is the unit group of a number field, see `sage.rings.number_field.unit_group`. The units form a finitely generated Abelian group. We can think of the elements either as abstract Abelian group elements or as particular numbers in the number field. The `AbelianGroupWithValues()` keeps track of these associated values.

**Warning:** Really, this requires a group homomorphism from the abstract Abelian group to the set of values. This is only checked if you pass the `check=True` option to `AbelianGroupWithValues()`.

## EXAMPLES:

Here is  $\mathbf{Z}_6$  with value  $-1$  assigned to the generator:

```
sage: Z6 = AbelianGroupWithValues([-1], [6], names='g')
sage: g = Z6.gen(0)
sage: g.value()
-1
sage: g*g
g^2
sage: (g*g).value()
1
sage: for i in range(7):
...     print i, g^i, (g^i).value()
0 1 1
1 g -1
2 g^2 1
3 g^3 -1
4 g^4 1
5 g^5 -1
6 1 1
```

The elements come with a coercion embedding into the `values_group()`, so you can use the group elements instead of the values:

```
sage: CF3.<zeta> = CyclotomicField(3)
sage: Z3.<g> = AbelianGroupWithValues([zeta], [3])
```

```
sage: Z3.values_group()
Cyclotomic Field of order 3 and degree 2
sage: g.value()
zeta
sage: CF3(g)
zeta
sage: g + zeta
2*zeta
sage: zeta + g
2*zeta
```

`sage.groups.abelian_gps.values.AbelianGroupWithValues` (*values*, *n*, *gens\_orders*=None, *names*='f', *check*=False, *values\_group*=None)

Construct an Abelian group with values associated to the generators.

INPUT:

- *values* – a list/tuple/iterable of values that you want to associate to the generators.
- *n* – integer (optional). If not specified, will be derived from *gens\_orders*.
- *gens\_orders* – a list of non-negative integers in the form  $[a_0, a_1, \dots, a_{n-1}]$ , typically written in increasing order. This list is padded with zeros if it has length less than *n*. The orders of the commuting generators, with 0 denoting an infinite cyclic factor.
- *names* – (optional) names of generators
- *values\_group* – a parent or None (default). The common parent of the values. This might be a group, but can also just contain the values. For example, if the values are units in a ring then the *values\_group* would be the whole ring. If None it will be derived from the values.

EXAMPLES:

```
sage: G = AbelianGroupWithValues([-1], [6])
sage: g = G.gen(0)
sage: for i in range(7):
...     print i, g^i, (g^i).value()
0 1 1
1 f -1
2 f^2 1
3 f^3 -1
4 f^4 1
5 f^5 -1
6 1 1
sage: G.values_group()
Integer Ring
```

The group elements come with a coercion embedding into the `values_group()`, so you can use them like their `value()`

```
sage: G.values_embedding()
Generic morphism:
  From: Multiplicative Abelian group isomorphic to C6
  To:   Integer Ring
sage: g.value()
-1
sage: 0 + g
-1
sage: 1 + 2*g
-1
```

**class** sage.groups.abelian\_gps.values.**AbelianGroupWithValuesElement** (*parent, exponents, value=None*)

Bases: sage.groups.abelian\_gps.abelian\_group\_element.AbelianGroupElement

An element of an Abelian group with values assigned to generators.

INPUT:

- *exponents* – tuple of integers. The exponent vector defining the group element.
- *parent* – the parent.
- *value* – the value assigned to the group element or None (default). In the latter case, the value is computed as needed.

EXAMPLES:

```
sage: F = AbelianGroupWithValues([1,-1], [2,4])
```

```
sage: a,b = F.gens()
```

```
sage: TestSuite(a*b).run()
```

**inverse()**

Return the inverse element.

EXAMPLE:

```
sage: G.<a,b> = AbelianGroupWithValues([2,-1], [0,4])
```

```
sage: a.inverse()
```

```
a^-1
```

```
sage: a.inverse().value()
```

```
1/2
```

```
sage: a.__invert__().value()
```

```
1/2
```

```
sage: (~a).value()
```

```
1/2
```

```
sage: (a*b).value()
```

```
-2
```

```
sage: (a*b).inverse().value()
```

```
-1/2
```

**value()**

Return the value of the group element.

OUTPUT:

The value according to the values for generators, see `gens_values()`.

EXAMPLES:

```
sage: G = AbelianGroupWithValues([5], 1)
```

```
sage: G.0.value()
```

```
5
```

**class** sage.groups.abelian\_gps.values.**AbelianGroupWithValuesEmbedding** (*domain, codomain*)

Bases: sage.categories.morphism.Morphism

The morphism embedding the Abelian group with values in its values group.

INPUT:

- *domain* – a `AbelianGroupWithValues_class`
- *codomain* – the values group (need not be in the category of groups, e.g. symbolic ring).

## EXAMPLES:

```
sage: Z4.<g> = AbelianGroupWithValues([I], [4])
sage: embedding = Z4.values_embedding(); embedding
Generic morphism:
  From: Multiplicative Abelian group isomorphic to C4
  To:   Symbolic Ring
sage: embedding(1)
1
sage: embedding(g)
I
sage: embedding(g^2)
-1
```

```
class sage.groups.abelian_gps.values.AbelianGroupWithValues_class(generator_orders,
                                                                    names, values,
                                                                    values_group)
```

Bases: `sage.groups.abelian_gps.abelian_group.AbelianGroup_class`

The class of an Abelian group with values associated to the generator.

## INPUT:

- `generator_orders` – tuple of integers. The orders of the generators.
- `names` – string or list of strings. The names for the generators.
- `values` – Tuple the same length as the number of generators. The values assigned to the generators.
- `values_group` – the common parent of the values.

## EXAMPLES:

```
sage: G.<a,b> = AbelianGroupWithValues([2,-1], [0,4])
sage: TestSuite(G).run()
```

**Element**

alias of `AbelianGroupWithValuesElement`

**gen** (*i=0*)

The *i*-th generator of the abelian group.

## INPUT:

- *i* – integer (default: 0). The index of the generator.

## OUTPUT:

A group element.

## EXAMPLES:

```
sage: F = AbelianGroupWithValues([1,2,3,4,5], 5, [], names='a')
sage: F.0
a0
sage: F.0.value()
1
sage: F.2
a2
sage: F.2.value()
3

sage: G = AbelianGroupWithValues([-1,0,1], [2,1,3])
sage: G.gens()
(f0, 1, f2)
```



**gens\_values()**

Return the values associated to the generators.

OUTPUT:

A tuple.

EXAMPLES:

```
sage: G = AbelianGroupWithValues([-1, 0, 1], [2, 1, 3])
sage: G.gens()
(f0, 1, f2)
sage: G.gens_values()
(-1, 0, 1)
```

**values\_embedding()**

Return the embedding of self in `values_group()`.

OUTPUT:

A morphism.

EXAMPLES:

```
sage: Z4 = AbelianGroupWithValues([I], [4])
sage: Z4.values_embedding()
Generic morphism:
  From: Multiplicative Abelian group isomorphic to C4
  To:   Symbolic Ring
```

**values\_group()**

The common parent of the values.

The values need to form a multiplicative group, but can be embedded in a larger structure. For example, if the values are units in a ring then the `values_group()` would be the whole ring.

OUTPUT:

The common parent of the values, containing the group generated by all values.

EXAMPLES:

```
sage: G = AbelianGroupWithValues([-1, 0, 1], [2, 1, 3])
sage: G.values_group()
Integer Ring

sage: Z4 = AbelianGroupWithValues([I], [4])
sage: Z4.values_group()
Symbolic Ring
```



# DUAL GROUPS OF FINITE MULTIPLICATIVE ABELIAN GROUPS

The basic idea is very simple. Let  $G$  be an abelian group and  $G^*$  its dual (i.e., the group of homomorphisms from  $G$  to  $\mathbb{C}^\times$ ). Let  $g_j, j = 1, \dots, n$ , denote generators of  $G$  - say  $g_j$  is of order  $m_j > 1$ . There are generators  $X_j, j = 1, \dots, n$ , of  $G^*$  for which  $X_j(g_j) = \exp(2\pi i/m_j)$  and  $X_i(g_j) = 1$  if  $i \neq j$ . These are used to construct  $G^*$ .

Sage supports multiplicative abelian groups on any prescribed finite number  $n > 0$  of generators. Use `AbelianGroup()` function to create an abelian group, the `dual_group()` method to create its dual, and then the `gen()` and `gens()` methods to obtain the corresponding generators. You can print the generators as arbitrary strings using the optional `names` argument to the `dual_group()` method.

EXAMPLES:

```
sage: F = AbelianGroup(5, [2, 5, 7, 8, 9], names='abcde')
sage: (a, b, c, d, e) = F.gens()
```

```
sage: Fd = F.dual_group(names='ABCDE')
sage: Fd.base_ring()
Cyclotomic Field of order 2520 and degree 576
sage: A,B,C,D,E = Fd.gens()
sage: A(a)
-1
sage: A(b), A(c), A(d), A(e)
(1, 1, 1, 1)
```

```
sage: Fd = F.dual_group(names='ABCDE', base_ring=CC)
sage: A,B,C,D,E = Fd.gens()
sage: A(a)      # abs tol 1e-8
-1.0000000000000000 + 0.0000000000000000*I
sage: A(b); A(c); A(d); A(e)
1.0000000000000000
1.0000000000000000
1.0000000000000000
1.0000000000000000
```

AUTHORS:

- David Joyner (2006-08) (based on `abelian_groups`)
- David Joyner (2006-10) modifications suggested by William Stein
- Volker Braun (2012-11) port to new Parent base. Use tuples for immutables. Default to cyclotomic base ring.

```
class sage.groups.abelian_gps.dual_abelian_group.DualAbelianGroup_class(G,
                                                                    names,
                                                                    base_ring)
```

Bases: `sage.structure.unique_representation.UniqueRepresentation`,  
`sage.groups.group.AbelianGroup`

Dual of abelian group.

EXAMPLES:

```
sage: F = AbelianGroup(5, [3, 5, 7, 8, 9], names="abcde")
sage: F.dual_group()
Dual of Abelian Group isomorphic to  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ 
over Cyclotomic Field of order 2520 and degree 576
sage: F = AbelianGroup(4, [15, 7, 8, 9], names="abcd")
sage: F.dual_group(base_ring=CC)
Dual of Abelian Group isomorphic to  $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ 
over Complex Field with 53 bits of precision
```

**Element**

alias of `DualAbelianGroupElement`

**base\_ring()**

Return the scalars over which the group is dualized.

EXAMPLES:

```
sage: F = AbelianGroup(3, [5, 64, 729], names=list("abc"))
sage: Fd = F.dual_group(base_ring=CC)
sage: Fd.base_ring()
Complex Field with 53 bits of precision
```

**gen** (*i=0*)

The *i*-th generator of the abelian group.

EXAMPLES:

```
sage: F = AbelianGroup(3, [1, 2, 3], names='a')
sage: Fd = F.dual_group(names="A")
sage: Fd.0
1
sage: Fd.1
A1
sage: Fd.gens_orders()
(1, 2, 3)
```

**gens** ()

Return the generators for the group.

OUTPUT:

A tuple of group elements generating the group.

EXAMPLES:

```
sage: F = AbelianGroup([7, 11]).dual_group()
sage: F.gens()
(x0, x1)
```

**gens\_orders** ()

The orders of the generators of the dual group.

OUTPUT:

A tuple of integers.

EXAMPLES:

```
sage: F = AbelianGroup([5]*1000)
sage: Fd = F.dual_group()
sage: invs = Fd.gens_orders(); len(invs)
1000
```

**group()**

Return the group that `self` is the dual of.

EXAMPLES:

```
sage: F = AbelianGroup(3, [5, 64, 729], names=list("abc"))
sage: Fd = F.dual_group(base_ring=CC)
sage: Fd.group() is F
True
```

**invariants()**

The invariants of the dual group.

You should use `gens_orders()` instead.

EXAMPLES:

```
sage: F = AbelianGroup([5]*1000)
sage: Fd = F.dual_group()
sage: invs = Fd.gens_orders(); len(invs)
1000
```

**is\_commutative()**

Return True since this group is commutative.

EXAMPLES:

```
sage: G = AbelianGroup([2, 3, 9])
sage: Gd = G.dual_group()
sage: Gd.is_commutative()
True
sage: Gd.is_abelian()
True
```

**list()**

Return tuple of all elements of this group.

EXAMPLES:

```
sage: G = AbelianGroup([2, 3], names="ab")
sage: Gd = G.dual_group(names="AB")
sage: Gd.list()
(1, B, B^2, A, A*B, A*B^2)
```

**ngens()**

The number of generators of the dual group.

EXAMPLES:

```
sage: F = AbelianGroup([7]*100)
sage: Fd = F.dual_group()
sage: Fd.ngens()
100
```

**order()**

Return the order of this group.

## EXAMPLES:

```
sage: G = AbelianGroup([2, 3, 9])
sage: Gd = G.dual_group()
sage: Gd.order()
54
```

**random\_element()**

Return a random element of this dual group.

## EXAMPLES:

```
sage: G = AbelianGroup([2, 3, 9])
sage: Gd = G.dual_group(base_ring=CC)
sage: Gd.random_element()
X1^2

sage: N = 43^2-1
sage: G = AbelianGroup([N], names="a")
sage: Gd = G.dual_group(names="A", base_ring=CC)
sage: a, = G.gens()
sage: A, = Gd.gens()
sage: x = a^(N/4); y = a^(N/3); z = a^(N/14)
sage: X = A*Gd.random_element(); X
A^615
sage: len([a for a in [x, y, z] if abs(X(a)-1)>10^(-8)])
2
```

`sage.groups.abelian_gps.dual_abelian_group.is_DualAbelianGroup(x)`

Return True if  $x$  is the dual group of an abelian group.

## EXAMPLES:

```
sage: from sage.groups.abelian_gps.dual_abelian_group import is_DualAbelianGroup
sage: F = AbelianGroup(5, [3, 5, 7, 8, 9], names=list("abcde"))
sage: Fd = F.dual_group()
sage: is_DualAbelianGroup(Fd)
True
sage: F = AbelianGroup(3, [1, 2, 3], names='a')
sage: Fd = F.dual_group()
sage: Fd.gens()
(1, X1, X2)
sage: F.gens()
(1, a1, a2)
```

# BASE CLASS FOR ABELIAN GROUP ELEMENTS

This is the base class for both `abelian_group_element` and `dual_abelian_group_element`.

As always, elements are immutable once constructed.

**class** `sage.groups.abelian_gps.element_base.AbelianGroupElementBase` (*parent*, *exponents*)

Bases: `sage.structure.element.MultiplicativeGroupElement`

Base class for abelian group elements

The group element is defined by a tuple whose  $i$ -th entry is an integer in the range from 0 (inclusively) to `G.gen(i).order()` (exclusively) if the  $i$ -th generator is of finite order, and an arbitrary integer if the  $i$ -th generator is of infinite order.

INPUT:

- `exponents` – 1 or a list/tuple/iterable of integers. The exponent vector (with respect to the parent generators) defining the group element.
- `parent` – Abelian group. The parent of the group element.

EXAMPLES:

```
sage: F = AbelianGroup(3, [7, 8, 9])
sage: Fd = F.dual_group(names="ABC")
sage: A, B, C = Fd.gens()
sage: A*B^-1 in Fd
True
```

**exponents()**

The exponents of the generators defining the group element.

OUTPUT:

A tuple of integers for an abelian group element. The integer can be arbitrary if the corresponding generator has infinite order. If the generator is of finite order, the integer is in the range from 0 (inclusive) to the order (exclusive).

EXAMPLES:

```
sage: F.<a,b,c,f> = AbelianGroup([7, 8, 9, 0])
sage: (a^3*b^2*c).exponents()
(3, 2, 1, 0)
sage: F([3, 2, 1, 0])
a^3*b^2*c
```

```
sage: (c^42).exponents()
(0, 0, 6, 0)
sage: (f^42).exponents()
(0, 0, 0, 42)
```

**inverse()**

Returns the inverse element.

EXAMPLE:

```
sage: G.<a,b> = AbelianGroup([0,5])
sage: a.inverse()
a^-1
sage: a.__invert__()
a^-1
sage: a^-1
a^-1
sage: ~a
a^-1
sage: (a*b).exponents()
(1, 1)
sage: (a*b).inverse().exponents()
(-1, 4)
```

**is\_trivial()**

Test whether self is the trivial group element 1.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: G.<a,b> = AbelianGroup([0,5])
sage: (a^5).is_trivial()
False
sage: (b^5).is_trivial()
True
```

**list()**

Return a copy of the exponent vector.

Use `exponents()` instead.

OUTPUT:

The underlying coordinates used to represent this element. If this is a word in an abelian group on  $n$  generators, then this is a list of nonnegative integers of length  $n$ .

EXAMPLES:

```
sage: F = AbelianGroup(5,[2, 3, 5, 7, 8], names="abcde")
sage: a,b,c,d,e = F.gens()
sage: Ad = F.dual_group(names="ABCDE")
sage: A,B,C,D,E = Ad.gens()
sage: (A*B*C^2*D^20*E^65).exponents()
(1, 1, 2, 6, 1)
sage: X = A*B*C^2*D^2*E^-6
sage: X.exponents()
(1, 1, 2, 2, 2)
```



**multiplicative\_order()**

Return the order of this element.

OUTPUT:

An integer or infinity.

EXAMPLES:

```
sage: F = AbelianGroup(3, [7, 8, 9])
sage: Fd = F.dual_group()
sage: A, B, C = Fd.gens()
sage: (B*C).order()
72

sage: F = AbelianGroup(3, [7, 8, 9]); F
Multiplicative Abelian group isomorphic to C7 x C8 x C9
sage: F.gens()[2].order()
9

sage: a, b, c = F.gens()
sage: (b*c).order()
72

sage: G = AbelianGroup(3, [7, 8, 9])
sage: type((G.0 * G.1).order())==Integer
True
```

**order()**

Return the order of this element.

OUTPUT:

An integer or infinity.

EXAMPLES:

```
sage: F = AbelianGroup(3, [7, 8, 9])
sage: Fd = F.dual_group()
sage: A, B, C = Fd.gens()
sage: (B*C).order()
72

sage: F = AbelianGroup(3, [7, 8, 9]); F
Multiplicative Abelian group isomorphic to C7 x C8 x C9
sage: F.gens()[2].order()
9

sage: a, b, c = F.gens()
sage: (b*c).order()
72

sage: G = AbelianGroup(3, [7, 8, 9])
sage: type((G.0 * G.1).order())==Integer
True
```



# ABELIAN GROUP ELEMENTS

## AUTHORS:

- David Joyner (2006-02); based on `free_abelian_monoid_element.py`, written by David Kohel.
- David Joyner (2006-05); bug fix in order
- David Joyner (2006-08); bug fix+new method in `pow` for negatives+fixed corresponding examples.
- David Joyner (2009-02): Fixed bug in order.
- Volker Braun (2012-11) port to new Parent base. Use tuples for immutables.

## EXAMPLES:

Recall an example from abelian groups:

```
sage: F = AbelianGroup(5, [4, 5, 5, 7, 8], names = list("abcde"))
sage: (a, b, c, d, e) = F.gens()
sage: x = a*b^2*e*d^20*e^12
sage: x
a*b^2*d^6*e^5
sage: x = a^10*b^12*c^13*d^20*e^12
sage: x
a^2*b^2*c^3*d^6*e^4
sage: y = a^13*b^19*c^23*d^27*e^72
sage: y
a*b^4*c^3*d^6
sage: x*y
a^3*b*c*d^5*e^4
sage: x.list()
[2, 2, 3, 6, 4]
```

```
class sage.groups.abelian_gps.abelian_group_element.AbelianGroupElement (parent,
                                                                           expo-
                                                                           nents)
```

Bases: `sage.groups.abelian_gps.element_base.AbelianGroupElementBase`

Elements of an `AbelianGroup`

## INPUT:

- `x` – list/tuple/iterable of integers (the element vector)
- `parent` – the parent `AbelianGroup`

## EXAMPLES:

```

sage: F = AbelianGroup(5, [3,4,5,8,7], 'abcde')
sage: a, b, c, d, e = F.gens()
sage: a^2 * b^3 * a^2 * b^-4
a*b^3
sage: b^-11
b
sage: a^-11
a
sage: a*b in F
True

```

**as\_permutation()**

Return the element of the permutation group  $G$  (isomorphic to the abelian group  $A$ ) associated to  $a$  in  $A$ .

EXAMPLES:

```

sage: G = AbelianGroup(3, [2,3,4], names="abc"); G
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: a,b,c=G.gens()
sage: Gp = G.permutation_group(); Gp
Permutation Group with generators [(6,7,8,9), (3,4,5), (1,2)]
sage: a.as_permutation()
(1,2)
sage: ap = a.as_permutation(); ap
(1,2)
sage: ap in Gp
True

```

**word\_problem(words)**

TODO - this needs a rewrite - see stuff in the matrix\_grp directory.

$G$  and  $H$  are abelian groups,  $g$  in  $G$ ,  $H$  is a subgroup of  $G$  generated by a list (words) of elements of  $G$ . If self is in  $H$ , return the expression for self as a word in the elements of (words).

This function does not solve the word problem in Sage. Rather it pushes it over to GAP, which has optimized (non-deterministic) algorithms for the word problem.

**Warning:** Don't use E (or other GAP-reserved letters) as a generator name.

EXAMPLE:

```

sage: G = AbelianGroup(2, [2,3], names="xy")
sage: x,y = G.gens()
sage: x.word_problem([x,y])
[[x, 1]]
sage: y.word_problem([x,y])
[[y, 1]]
sage: v = (y*x).word_problem([x,y]); v #random
[[x, 1], [y, 1]]
sage: prod([x^i for x,i in v]) == y*x
True

```

`sage.groups.abelian_gps.abelian_group_element.is_AbelianGroupElement(x)`

Return true if  $x$  is an abelian group element, i.e., an element of type `AbelianGroupElement`.

EXAMPLES: Though the integer 3 is in the integers, and the integers have an abelian group structure, 3 is not an `AbelianGroupElement`:

```
sage: from sage.groups.abelian_gps.abelian_group_element import is_AbelianGroupElement
sage: is_AbelianGroupElement(3)
False
sage: F = AbelianGroup(5, [3,4,5,8,7], 'abcde')
sage: is_AbelianGroupElement(F.0)
True
```



# ELEMENTS (CHARACTERS) OF THE DUAL GROUP OF A FINITE ABELIAN GROUP.

To obtain the dual group of a finite Abelian group, use the `dual_group()` method:

```
sage: F = AbelianGroup([2,3,5,7,8], names="abcde")
sage: F
Multiplicative Abelian group isomorphic to C2 x C3 x C5 x C7 x C8

sage: Fd = F.dual_group(names="ABCDE")
sage: Fd
Dual of Abelian Group isomorphic to Z/2Z x Z/3Z x Z/5Z x Z/7Z x Z/8Z
over Cyclotomic Field of order 840 and degree 192
```

The elements of the dual group can be evaluated on elements of the original group:

```
sage: a,b,c,d,e = F.gens()
sage: A,B,C,D,E = Fd.gens()
sage: A*B^2*D^7
A*B^2
sage: A(a)
-1
sage: B(b)
zeta840^140 - 1
sage: CC(_) # abs tol 1e-8
-0.4999999999999995 + 0.866025403784447*I
sage: A(a*b)
-1
sage: (A*B*C^2*D^20*E^65).exponents()
(1, 1, 2, 6, 1)
sage: B^(-1)
B^2
```

AUTHORS:

- David Joyner (2006-07); based on `abelian_group_element.py`.
- David Joyner (2006-10); modifications suggested by William Stein.
- Volker Braun (2012-11) port to new Parent base. Use tuples for immutables. Default to cyclotomic base ring.

**class** `sage.groups.abelian_gps.dual_abelian_group_element.DualAbelianGroupElement` (*parent, exponents*)

Bases: `sage.groups.abelian_gps.element_base.AbelianGroupElementBase`

Base class for abelian group elements

**word\_problem** (*words, display=True*)

This is a rather hackish method and is included for completeness.

The word problem for an instance of `DualAbelianGroup` as it can for an `AbelianGroup`. The reason why is that word problem for an instance of `AbelianGroup` simply calls GAP (which has abelian groups implemented) and invokes “EpimorphismFromFreeGroup” and “PreImagesRepresentative”. GAP does not have duals of abelian groups implemented. So, by using the same name for the generators, the method below converts the problem for the dual group to the corresponding problem on the group itself and uses GAP to solve that.

EXAMPLES:

```
sage: G = AbelianGroup(5, [3, 5, 5, 7, 8], names="abcde")
sage: Gd = G.dual_group(names="abcde")
sage: a, b, c, d, e = Gd.gens()
sage: u = a^3*b*c*d^2*e^5
sage: v = a^2*b*c^2*d^3*e^3
sage: w = a^7*b^3*c^5*d^4*e^4
sage: x = a^3*b^2*c^2*d^3*e^5
sage: y = a^2*b^4*c^2*d^4*e^5
sage: e.word_problem([u, v, w, x, y], display=False)
[[b^2*c^2*d^3*e^5, 245]]
```

The command `e.word_problem([u, v, w, x, y], display=True)` returns the same list but also prints  $e = (b^2 * c^2 * d^3 * e^5)^{245}$ .

`sage.groups.abelian_gps.dual_abelian_group_element.add_strings` (*x, z=0*)

This was in `sage.misc.misc` but commented out. Needed to add lists of strings in the `word_problem` method below.

Return the sum of the elements of *x*. If *x* is empty, return *z*.

INPUT:

- *x* – iterable
- *z* – the 0 that will be returned if *x* is empty.

OUTPUT:

The sum of the elements of *x*.

EXAMPLES:

```
sage: from sage.groups.abelian_gps.dual_abelian_group_element import add_strings
sage: add_strings([], z='empty')
'empty'
sage: add_strings(['a', 'b', 'c'])
'abc'
```

`sage.groups.abelian_gps.dual_abelian_group_element.is_DualAbelianGroupElement` (*x*)

Test whether *x* is a dual Abelian group element.

INPUT:

- *x* – anything.



OUTPUT:

Boolean.

EXAMPLES:

```
sage: from sage.groups.abelian_gps.dual_abelian_group import is_DualAbelianGroupElement
sage: F = AbelianGroup(5, [5, 5, 7, 8, 9], names = list("abcde")).dual_group()
sage: is_DualAbelianGroupElement(F)
False
sage: is_DualAbelianGroupElement(F.an_element())
True
```



# HOMOMORPHISMS OF ABELIAN GROUPS

TODO:

- there must be a homspace first
- there should be hom and Hom methods in abelian group

AUTHORS:

- David Joyner (2006-03-03): initial version

```
class sage.groups.abelian_gps.abelian_group_morphism.AbelianGroupMap(parent)
    Bases: sage.categories.morphism.Morphism
```

A set-theoretic map between AbelianGroups.

```
class sage.groups.abelian_gps.abelian_group_morphism.AbelianGroupMorphism(G,
                                                                              H,
                                                                              gens,
                                                                              imgss)
```

Bases: sage.categories.morphism.Morphism

Some python code for wrapping GAP's GroupHomomorphismByImages function for abelian groups. Returns "fail" if gens does not generate self or if the map does not extend to a group homomorphism, self - other.

EXAMPLES:

```
sage: G = AbelianGroup(3,[2,3,4],names="abc"); G
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: a,b,c = G.gens()
sage: H = AbelianGroup(2,[2,3],names="xy"); H
Multiplicative Abelian group isomorphic to C2 x C3
sage: x,y = H.gens()

sage: from sage.groups.abelian_gps.abelian_group_morphism import AbelianGroupMorphism
sage: phi = AbelianGroupMorphism(H,G,[x,y],[a,b])
```

AUTHORS:

- David Joyner (2006-02)

**image**(J)

Only works for finite groups.

J must be a subgroup of G. Computes the subgroup of H which is the image of J.

EXAMPLES:

```
sage: G = AbelianGroup(2, [2, 3], names="xy")
sage: x, y = G.gens()
sage: H = AbelianGroup(3, [2, 3, 4], names="abc")
sage: a, b, c = H.gens()
sage: phi = AbelianGroupMorphism(G, H, [x, y], [a, b])
```

**kernel()**

Only works for finite groups.

TODO: not done yet; returns a gap object but should return a Sage group.

EXAMPLES:

```
sage: H = AbelianGroup(3, [2, 3, 4], names="abc"); H
Multiplicative Abelian group isomorphic to C2 x C3 x C4
sage: a, b, c = H.gens()
sage: G = AbelianGroup(2, [2, 3], names="xy"); G
Multiplicative Abelian group isomorphic to C2 x C3
sage: x, y = G.gens()
sage: phi = AbelianGroupMorphism(G, H, [x, y], [a, b])
sage: phi.kernel()
'Group([ ])'
```

**class** sage.groups.abelian\_gps.abelian\_group\_morphism.**AbelianGroupMorphism\_id**(X)  
Bases: sage.groups.abelian\_gps.abelian\_group\_morphism.AbelianGroupMap

Return the identity homomorphism from X to itself.

EXAMPLES:

```
sage.groups.abelian_gps.abelian_group_morphism.is_AbelianGroupMorphism(f)
x.__init__(...) initializes x; see help(type(x)) for signature
```

# ADDITIVE ABELIAN GROUPS

Additive abelian groups are just modules over  $\mathbf{Z}$ . Hence the classes in this module derive from those in the module `sage.modules.fg_pid`. The only major differences are in the way elements are printed.

```
sage.groups.additive_abelian.additive_abelian_group.AdditiveAbelianGroup(invs,  
re-  
mem-  
ber_generators=True)
```

Construct a finitely-generated additive abelian group.

INPUTS:

- `invs` (list of integers): the invariants. These should all be greater than or equal to zero.
- `remember_generators` (boolean): whether or not to fix a set of generators (corresponding to the given invariants, which need not be in Smith form).

OUTPUT:

The abelian group  $\bigoplus_i \mathbf{Z}/n_i\mathbf{Z}$ , where  $n_i$  are the invariants.

EXAMPLE:

```
sage: AdditiveAbelianGroup([0, 2, 4])  
Additive abelian group isomorphic to  $\mathbf{Z}/2 + \mathbf{Z}/4 + \mathbf{Z}$ 
```

An example of the `remember_generators` switch:

```
sage: G = AdditiveAbelianGroup([0, 2, 3]); G  
Additive abelian group isomorphic to  $\mathbf{Z}/6 + \mathbf{Z}$   
sage: G.gens()  
((1, 0, 0), (0, 1, 0), (0, 0, 1))  
  
sage: H = AdditiveAbelianGroup([0, 2, 3], remember_generators = False); H  
Additive abelian group isomorphic to  $\mathbf{Z}/6 + \mathbf{Z}$   
sage: H.gens()  
((0, 1, 2), (1, 0, 0))
```

There are several ways to create elements of an additive abelian group. Realize that there are two sets of generators: the “obvious” ones composed of zeros and ones, one for each invariant given to construct the group, the other being a set of minimal generators. Which set is the default varies with the use of the `remember_generators` switch.

First with “obvious” generators. Note that a raw list will use the minimal generators and a vector (a module element) will use the generators that pair up naturally with the invariants. We create the same element repeatedly.

```
sage: H=AdditiveAbelianGroup([3,2,0], remember_generators=True)  
sage: H.gens()
```

```
((1, 0, 0), (0, 1, 0), (0, 0, 1))
sage: [H.0, H.1, H.2]
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]
sage: p=H.0+H.1+6*H.2; p
(1, 1, 6)

sage: H.smith_form_gens()
((2, 1, 0), (0, 0, 1))
sage: q=H([5,6]); q
(1, 1, 6)
sage: p==q
True

sage: r=H(vector([1,1,6])); r
(1, 1, 6)
sage: p==r
True

sage: s=H(p)
sage: p==s
True
```

Again, but now where the generators are the minimal set. Coercing a list or a vector works as before, but the default generators are different.

```
sage: G=AdditiveAbelianGroup([3,2,0], remember_generators=False)
sage: G.gens()
((2, 1, 0), (0, 0, 1))
sage: [G.0, G.1]
[(2, 1, 0), (0, 0, 1)]
sage: p=5*G.0+6*G.1; p
(1, 1, 6)

sage: H.smith_form_gens()
((2, 1, 0), (0, 0, 1))
sage: q=G([5,6]); q
(1, 1, 6)
sage: p==q
True

sage: r=G(vector([1,1,6])); r
(1, 1, 6)
sage: p==r
True

sage: s=H(p)
sage: p==s
True
```

**class** sage.groups.additive\_abelian.additive\_abelian\_group.**AdditiveAbelianGroupElement** (*parent*, *x*, *check=True*)

Bases: sage.modules.fg\_pid.fgp\_element.FGP\_Element

An element of an [AdditiveAbelianGroup\\_class](#).

---

```
class sage.groups.additive_abelian.additive_abelian_group.AdditiveAbelianGroup_class (cover,
                                                                                               re-
                                                                                               la-
                                                                                               tions)
```

```
Bases: sage.modules.fg_pid.fgp_module.FGP_Module_class,
sage.groups.old.AbelianGroup
```

An additive abelian group, implemented using the **Z**-module machinery.

INPUT:

- `cover` – the covering group as **Z**-module.
- `relations` – the relations as submodule of `cover`.

**Element**

alias of `AdditiveAbelianGroupElement`

**exponent()**

Return the exponent of this group (the smallest positive integer  $N$  such that  $Nx = 0$  for all  $x$  in the group). If there is no such integer, return 0.

EXAMPLES:

```
sage: AdditiveAbelianGroup([2, 4]).exponent()
4
sage: AdditiveAbelianGroup([0, 2, 4]).exponent()
0
sage: AdditiveAbelianGroup([]).exponent()
1
```

**is\_cyclic()**

Returns True if the group is cyclic.

EXAMPLES:

With no common factors between the orders of the generators, the group will be cyclic.

```
sage: G=AdditiveAbelianGroup([6, 7, 55])
sage: G.is_cyclic()
True
```

Repeating primes in the orders will create a non-cyclic group.

```
sage: G=AdditiveAbelianGroup([6, 15, 21, 33])
sage: G.is_cyclic()
False
```

A trivial group is trivially cyclic.

```
sage: T=AdditiveAbelianGroup([1])
sage: T.is_cyclic()
True
```

**is\_multiplicative()**

Return False since this is an additive group.

EXAMPLE:

```
sage: AdditiveAbelianGroup([0]).is_multiplicative()
False
```

**order()**

Return the order of this group (an integer or infinity)

EXAMPLES:

```
sage: AdditiveAbelianGroup([2, 4]).order()
8
sage: AdditiveAbelianGroup([0, 2, 4]).order()
+Infinity
sage: AdditiveAbelianGroup([]).order()
1
```

**short\_name()**

Return a name for the isomorphism class of this group.

EXAMPLE:

```
sage: AdditiveAbelianGroup([0, 2, 4]).short_name()
'Z/2 + Z/4 + Z'
sage: AdditiveAbelianGroup([0, 2, 3]).short_name()
'Z/6 + Z'
```

**class** sage.groups.additive\_abelian.additive\_abelian\_group.**AdditiveAbelianGroup\_fixed\_gens** (coverage, relations, generators)

Bases: sage.groups.additive\_abelian.additive\_abelian\_group.AdditiveAbelianGroup\_class

A variant which fixes a set of generators, which need not be in Smith form (or indeed independent).

**gens()**

Return the specified generators for self (as a tuple). Compare `self.smithform_gens()`.

EXAMPLE:

```
sage: G = AdditiveAbelianGroup([2, 3])
sage: G.gens()
((1, 0), (0, 1))
sage: G.smith_form_gens()
((1, 2),)
```

**identity()**

Return the identity (zero) element of this group.

EXAMPLE:

```
sage: G = AdditiveAbelianGroup([2, 3])
sage: G.identity()
(0, 0)
```

**permutation\_group()**

Return the permutation group attached to this group.

EXAMPLE:

```
sage: G = AdditiveAbelianGroup([2, 3])
sage: G.permutation_group()
Permutation Group with generators [(3, 4, 5), (1, 2)]
```

sage.groups.additive\_abelian.additive\_abelian\_group.**cover\_and\_relations\_from\_invariants** (invariants)

A utility function to construct modules required to initialize the super class.

Given a list of integers, this routine constructs the obvious pair of free modules such that the quotient of the two free modules over  $\mathbf{Z}$  is naturally isomorphic to the corresponding product of cyclic modules (and hence isomorphic to a direct sum of cyclic groups).

EXAMPLES:



```
sage: from sage.groups.additive_abelian.additive_abelian_group import cover_and_relations_from_i
sage: cr([0,2,3])
(Ambient free module of rank 3 over the principal ideal domain Integer Ring, Free module of degn
Echelon basis matrix:
[0 2 0]
[0 0 3])
```



# WRAPPER CLASS FOR ABELIAN GROUPS

This class is intended as a template for anything in Sage that needs the functionality of abelian groups. One can create an `AdditiveAbelianGroupWrapper` object from any given set of elements in some given parent, as long as an `_add_` method has been defined.

EXAMPLES:

We create a toy example based on the Mordell-Weil group of an elliptic curve over  $\mathbb{Q}$ :

```
sage: E = EllipticCurve('30a2')
sage: pts = [E(4,-7,1), E(7/4, -11/8, 1), E(3, -2, 1)]
sage: M = AdditiveAbelianGroupWrapper(pts[0].parent(), pts, [3, 2, 2])
sage: M
Additive abelian group isomorphic to  $\mathbb{Z}/2 + \mathbb{Z}/6$  embedded in Abelian group of
points on Elliptic Curve defined by  $y^2 + x*y + y = x^3 - 19*x + 26$  over
Rational Field
sage: M.gens()
((4 : -7 : 1), (7/4 : -11/8 : 1), (3 : -2 : 1))
sage: 3*M.0
(0 : 1 : 0)
sage: 30000000000000001 * M.0
(4 : -7 : 1)
sage: M == loads(dumps(M)) # known bug, see http://trac.sagemath.org/sage_trac/ticket/11599#comment
True
```

We check that ridiculous operations are being avoided:

```
sage: set_verbose(2, 'additive_abelian_wrapper.py')
sage: 300001 * M.0
verbose 1 (...: additive_abelian_wrapper.py, _discrete_exp) Calling discrete exp on (1, 0, 0)
(4 : -7 : 1)
sage: set_verbose(0, 'additive_abelian_wrapper.py')
```

TODO:

- Implement proper black-box discrete logarithm (using baby-step giant-step). The `discrete_exp` function can also potentially be speeded up substantially via caching.
- Think about subgroups and quotients, which probably won't work in the current implementation – some fiddly adjustments will be needed in order to be able to pass extra arguments to the subquotient's `init` method.

**class** sage.groups.additive\_abelian.additive\_abelian\_wrapper.**AdditiveAbelianGroupWrapper** (*universe*, *gens*, *invariants*)

Bases: sage.groups.additive\_abelian.additive\_abelian\_group.AdditiveAbelianGroup\_fixed\_gens

The parent of `AdditiveAbelianGroupWrapperElement`

**Element**

alias of `AdditiveAbelianGroupWrapperElement`

**generator\_orders()**

The orders of the generators with which this group was initialised. (Note that these are not necessarily a minimal set of generators.) Generators of infinite order are returned as 0. Compare `self.invariants()`, which returns the orders of a minimal set of generators.

EXAMPLE:

```
sage: V = Zmod(6)**2
sage: G = AdditiveAbelianGroupWrapper(V, [2*V.0, 3*V.1], [3, 2])
sage: G.generator_orders()
(3, 2)
sage: G.invariants()
(6,)
```

**universe()**

The ambient group in which this abelian group lives.

EXAMPLE:

```
sage: G = AdditiveAbelianGroupWrapper(QQbar, [sqrt(QQbar(2)), sqrt(QQbar(3))], [0, 0])
sage: G.universe()
Algebraic Field
```

**class** sage.groups.additive\_abelian.additive\_abelian\_wrapper.**AdditiveAbelianGroupWrapperElement**

Bases: sage.groups.additive\_abelian.additive\_abelian\_group.AdditiveAbelianGroupElement

An element of an `AdditiveAbelianGroupWrapper`.

**element()**

Return the underlying object that this element wraps.

EXAMPLE:

```
sage: T = EllipticCurve('65a').torsion_subgroup().gen(0)
sage: T; type(T)
(0 : 0 : 1)
<class 'sage.groups.additive_abelian.additive_abelian_wrapper.EllipticCurveTorsionSubgroup_wrapper'>
sage: T.element(); type(T.element())
(0 : 0 : 1)
<class 'sage.schemes.elliptic_curves.ell_point.EllipticCurvePoint_number_field'>
```

**class** sage.groups.additive\_abelian.additive\_abelian\_wrapper.**UnwrappingMorphism** (*domain*)

Bases: sage.categories.morphism.Morphism

The embedding into the ambient group. Used by the coercion framework.



# PERMUTATION GROUPS

A permutation group is a finite group  $G$  whose elements are permutations of a given finite set  $X$  (i.e., bijections  $X \rightarrow X$ ) and whose group operation is the composition of permutations. The number of elements of  $X$  is called the degree of  $G$ .

In Sage, a permutation is represented as either a string that defines a permutation using disjoint cycle notation, or a list of tuples, which represent disjoint cycles. That is:

```
(a, ..., b) (c, ..., d) ... (e, ..., f) <--> [(a, ..., b), (c, ..., d), ..., (e, ..., f)]  
() = identity <--> []
```

You can make the “named” permutation groups (see `permgp_named.py`) and use the following constructions:

- permutation group generated by elements,
- `direct_product_permgroups`, which takes a list of permutation groups and returns their direct product.

JOKE: Q: What’s hot, chunky, and acts on a polygon? A: Dihedral soup. Renteln, P. and Dundes, A. “Foolproof: A Sampling of Mathematical Folk Humor.” *Notices Amer. Math. Soc.* 52, 24-34, 2005.

AUTHORS:

- David Joyner (2005-10-14): first version
- David Joyner (2005-11-17)
- William Stein (2005-11-26): rewrite to better wrap Gap
- David Joyner (2005-12-21)
- William Stein and David Joyner (2006-01-04): added `conjugacy_class_representatives`
- David Joyner (2006-03): reorganization into subdirectory `perm_gps`; added `__contains__`, `has_element`; fixed `_cmp_`; added subgroup class+methods, PGL, PSL, PSp, PSU classes,
- David Joyner (2006-06): added PGU, functionality to `SymmetricGroup`, `AlternatingGroup`, `direct_product_permgroups`
- David Joyner (2006-08): added `degree`, `ramification_module_decomposition_modular_curve` and `ramification_module_decomposition_hurwitz_curve` methods to PSL(2,q), `MathieuGroup`, `is_isomorphic`
- Bobby Moretti (2006-10): Added `KleinFourGroup`, fixed bug in `DihedralGroup`
- David Joyner (2006-10): added `is_subgroup` (fixing a bug found by Kiran Kedlaya), `is_solvable`, `normalizer`, `is_normal_subgroup`, `Suzuki`
- David Kohel (2007-02): fixed `__contains__` to not enumerate group elements, following the convention for `__call__`
- David Harvey, Mike Hansen, Nick Alexander, William Stein (2007-02,03,04,05): Various patches

- Nathan Dunfield (2007-05): added orbits
- David Joyner (2007-06): added subgroup method (suggested by David Kohel), `composition_series`, `lower_central_series`, `upper_central_series`, `cayley_table`, `quotient_group`, `sylow_subgroup`, `is_cyclic`, `homology`, `homology_part`, `cohomology`, `cohomology_part`, `poincare_series`, `molien_series`, `is_simple`, `is_monomial`, `is_supersolvable`, `is_nilpotent`, `is_perfect`, `is_polycyclic`, `is_elementary_abelian`, `is_pgroup`, `gens_small`, `isomorphism_type_info_simple_group`. moved all the "named" groups to a new file.
- Nick Alexander (2007-07): move `is_isomorphic` to `isomorphism_to`, add `from_gap_list`
- William Stein (2007-07): put `is_isomorphic` back (and make it better)
- David Joyner (2007-08): fixed bugs in `composition_series`, `upper/lower_central_series`, `derived_series`,
- David Joyner (2008-06): modified `is_normal` (reported by W. J. Palenstijn), and added `normalizes`
- David Joyner (2008-08): Added example to docstring of `cohomology`.
- Simon King (2009-04): `__cmp__` methods for `PermutationGroup_generic` and `PermutationGroup_subgroup`
- Nicolas Borie (2009): Added `orbit`, `transversals`, `stabiliser` and `strong_generating_system` methods
- Christopher Swenson (2012): Added a special case to compute the order efficiently. (This patch Copyright 2012 Google Inc. All Rights Reserved. )
- Javier Lopez Pena (2013): Added conjugacy classes.

## REFERENCES:

- Cameron, P., *Permutation Groups*. New York: Cambridge University Press, 1999.
- Wielandt, H., *Finite Permutation Groups*. New York: Academic Press, 1964.
- Dixon, J. and Mortimer, B., *Permutation Groups*, Springer-Verlag, Berlin/New York, 1996.

---

**Note:** Though Suzuki groups are okay, Ree groups should *not* be wrapped as permutation groups - the construction is too slow - unless (for small values or the parameter) they are made using explicit generators.

---

```
sage.groups.perm_gps.permgroup.PermutationGroup(gens=None, gap_group=None, domain=None, canonicalize=True, category=None)
```

Return the permutation group associated to  $x$  (typically a list of generators).

## INPUT:

- `gens` - list of generators (default: None)
- `gap_group` - a gap permutation group (default: None)
- `canonicalize` - bool (default: True); if True, sort generators and remove duplicates

## OUTPUT:

- A permutation group.

## EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2,3), (4,5) ], [ (3,4) ]])
sage: G
Permutation Group with generators [(3,4), (1,2,3)(4,5)]
```

We can also make permutation groups from PARI groups:



```

sage: H = pari('x^4 - 2*x^3 - 2*x + 1').polgalois()
sage: G = PariGroup(H, 4); G
PARI group [8, -1, 3, "D(4)"] of degree 4
sage: H = PermutationGroup(G); H          # optional - database_gap
Transitive group number 3 of degree 4
sage: H.gens()                            # optional - database_gap
[(1,2,3,4), (1,3)]

```

We can also create permutation groups whose generators are Gap permutation objects:

```

sage: p = gap('(1,2)(3,7)(4,6)(5,8)'); p
(1,2)(3,7)(4,6)(5,8)
sage: PermutationGroup([p])
Permutation Group with generators [(1,2)(3,7)(4,6)(5,8)]

```

Permutation groups can work on any domain. In the following examples, the permutations are specified in list notation, according to the order of the elements of the domain:

```

sage: list(PermutationGroup([[ 'b', 'c', 'a' ]], domain=[ 'a', 'b', 'c' ]))
[(), ('a', 'b', 'c'), ('a', 'c', 'b')]
sage: list(PermutationGroup([[ 'b', 'c', 'a' ]], domain=[ 'b', 'c', 'a' ]))
[()]
sage: list(PermutationGroup([[ 'b', 'c', 'a' ]], domain=[ 'a', 'c', 'b' ]))
[(), ('a', 'b')]

```

There is an underlying gap object that implements each permutation group:

```

sage: G = PermutationGroup([(1,2,3,4)])
sage: G._gap_()
Group( [ (1,2,3,4) ] )
sage: gap(G)
Group( [ (1,2,3,4) ] )
sage: gap(G) is G._gap_()
True
sage: G = PermutationGroup([(1,2,3), (4,5)], [(3,4)])
sage: current_randstate().set_seed_gap()
sage: G._gap_().DerivedSeries()
[ Group( [ (3,4), (1,2,3)(4,5) ] ), Group( [ (1,5)(3,4), (1,5)(2,4), (1,5,3) ] ) ]

```

#### TESTS:

```

sage: r = Permutation("(1,7,9,3)(2,4,8,6)")
sage: f = Permutation("(1,3)(4,6)(7,9)")
sage: PermutationGroup([r,f]) #See Trac #12597
Permutation Group with generators [(1,3)(4,6)(7,9), (1,7,9,3)(2,4,8,6)]

```

```

sage: PermutationGroup(SymmetricGroup(5))
Traceback (most recent call last):
...
TypeError: gens must be a tuple, list, or GapElement

```

```

class sage.groups.perm_gps.permgroup.PermutationGroup_generic(gens=None,
                                                                gap_group=None,
                                                                canonicalize=True,
                                                                domain=None,
                                                                category=None)

```

Bases: `sage.groups.old.Group`

#### EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2,3), (4,5)], [(3,4)]])
sage: G
Permutation Group with generators [(3,4), (1,2,3)(4,5)]
sage: G.center()
Subgroup of (Permutation Group with generators [(3,4), (1,2,3)(4,5)]) generated by [()]
sage: G.group_id()          # optional - database_gap
[120, 34]
sage: n = G.order(); n
120
sage: G = PermutationGroup([[ (1,2,3), (4,5)], [(3,4)]])
sage: TestSuite(G).run()
```

**as\_finitely\_presented\_group** (*reduced=False*)

Return a finitely presented group isomorphic to self.

This method acts as wrapper for the GAP function `IsomorphismFpGroupByGenerators`, which yields an isomorphism from a given group to a finitely presented group.

INPUT:

- `reduced` – Default False, if True `FinitelyPresentedGroup.simplified` is called, attempting to simplify the presentation of the finitely presented group to be returned.

OUTPUT:

Finite presentation of self, obtained by taking the image of the isomorphism returned by the GAP function, `IsomorphismFpGroupByGenerators`.

ALGORITHM:

Uses GAP.

EXAMPLES:

```
sage: CyclicPermutationGroup(50).as_finitely_presented_group()
Finitely presented group < a | a^50 >
sage: DihedralGroup(4).as_finitely_presented_group()
Finitely presented group < a, b | b^2, a^4, (b*a)^2 >
sage: GeneralDihedralGroup([2,2]).as_finitely_presented_group()
Finitely presented group < a, b, c | a^2, b^2, c^2, (c*b)^2, (c*a)^2, (b*a)^2 >
```

GAP algorithm is not guaranteed to produce minimal or canonical presentation:

```
sage: G = PermutationGroup(['(1,2,3,4,5)', '(1,5)(2,4)'])
sage: G.is_isomorphic(DihedralGroup(5))
True
sage: K = G.as_finitely_presented_group(); K
Finitely presented group < a, b | b^2, (b*a)^2, b*a^-3*b*a^2 >
sage: K.as_permutation_group().is_isomorphic(DihedralGroup(5))
True
```

We can attempt to reduce the output presentation:

```
sage: PermutationGroup(['(1,2,3,4,5)', '(1,3,5,2,4)']).as_finitely_presented_group()
Finitely presented group < a, b | b^-2*a^-1, b*a^-2 >
sage: PermutationGroup(['(1,2,3,4,5)', '(1,3,5,2,4)']).as_finitely_presented_group(reduced=True)
Finitely presented group < a | a^5 >
```

TESTS:

```
sage: PermutationGroup([]).as_finitely_presented_group()
Finitely presented group < a | a >
```

```

sage: S = SymmetricGroup(6)
sage: perm_ls = [S.random_element() for i in range(3)]
sage: G = PermutationGroup(perm_ls)
sage: G.as_finitely_presented_group().as_permutation_group().is_isomorphic(G)
True

```

$D_9$  is the only non-Abelian group of order 18 with an automorphism group of order 54 [THOMAS-WOODS]:

```

sage: D = DihedralGroup(9).as_finitely_presented_group().gap()
sage: D.Order(), D.IsAbelian(), D.AutomorphismGroup().Order()
(18, false, 54)

```

$S_3$  is the only non-Abelian group of order 6 [THOMAS-WOODS]:

```

sage: S = SymmetricGroup(3).as_finitely_presented_group().gap()
sage: S.Order(), S.IsAbelian()
(6, false)

```

We can manually construct a permutation representation using GAP coset enumeration methods:

```

sage: D = GeneralDihedralGroup([3,3,4]).as_finitely_presented_group().gap()
sage: ctab = D.CosetTable(D.Subgroup([]))
sage: gen_ls = gap.List(ctab, gap.PermList)
sage: PermutationGroup(gen_ls).is_isomorphic(GeneralDihedralGroup([3,3,4]))
True
sage: A = AlternatingGroup(5).as_finitely_presented_group().gap()
sage: ctab = A.CosetTable(A.Subgroup([]))
sage: gen_ls = gap.List(ctab, gap.PermList)
sage: PermutationGroup(gen_ls).is_isomorphic(AlternatingGroup(5))
True

```

AUTHORS:

- Davis Shurbert (2013-06-21): initial version

**base** (*seed=None*)

Returns a (minimum) base of this permutation group. A base  $B$  of a permutation group is a subset of the domain of the group such that the only group element stabilizing all of  $B$  is the identity.

The argument *seed* is optional and must be a subset of the domain of *base*. When used, an attempt to create a base containing all or part of *seed* will be made.

EXAMPLES:

```

sage: G = PermutationGroup([(1,2,3),(6,7,8)])
sage: G.base()
[1, 6]
sage: G.base([2])
[2, 6]

sage: H = PermutationGroup([('a','b','c'),('a','y')])
sage: H.base()
['a', 'b', 'c']

sage: S = SymmetricGroup(13)
sage: S.base()
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]

sage: S = MathieuGroup(12)
sage: S.base()

```

```
[1, 2, 3, 4, 5]
sage: S.base([1,3,5,7,9,11]) # create a base for M12 with only odd integers
[1, 3, 5, 7, 9]
```

**blocks\_all** (*representatives=True*)

Returns the list of block systems of imprimitivity.

For more information on primitivity, see the [Wikipedia article on primitive group actions](#).

INPUT:

- *representative* (boolean) – whether to return all possible block systems of imprimitivity or only one of their representatives (the block can be obtained from its representative set  $S$  by computing the orbit of  $S$  under *self*).

This parameter is set to `True` by default (as it is GAP's default behaviour).

OUTPUT:

This method returns a description of *all* block systems. Hence, the output is a “list of lists of lists” or a “list of lists” depending on the value of *representatives*. A bit more clearly, output is :

- A list of length (#number of different block systems) of
  - block systems, each of them being defined as
    - \*If *representatives* = `True` : a list of representatives of each set of the block system
    - \*If *representatives* = `False` : a partition of the elements defining an imprimitivity block.

See Also:

- `is_primitive()`

EXAMPLE:

Picking an interesting group:

```
sage: g = graphs.DodecahedralGraph()
sage: g.is_vertex_transitive()
True
sage: ag = g.automorphism_group()
sage: ag.is_primitive()
False
```

Computing its blocks representatives:

```
sage: ag.blocks_all()
[[0, 15]]
```

Now the full blocks:

```
sage: ag.blocks_all(representatives = False)
[[[0, 15], [1, 16], [14, 19], [8, 17], [5, 10], [2, 12], [7, 18], [3, 13], [4, 9], [6, 11]]]
```

TESTS:

```
sage: g = PermutationGroup([("a", "b", "c", "d")])
sage: g.blocks_all()
[['a', 'c']]
sage: g.blocks_all(False)
[['a', 'c'], ['b', 'd']]
```

**cardinality()**

Return the number of elements of this group.

## EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2,3), (4,5) ], [ (1,2) ]])
sage: G.order()
12
sage: G = PermutationGroup([()])
sage: G.order()
1
sage: G = PermutationGroup([])
sage: G.order()
1
```

**center()**

Return the subgroup of elements that commute with every element of this group.

## EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2,3,4) ]])
sage: G.center()
Subgroup of (Permutation Group with generators [(1,2,3,4)]) generated by [(1,2,3,4)]
sage: G = PermutationGroup([[ (1,2,3,4) ], [ (1,2) ]])
sage: G.center()
Subgroup of (Permutation Group with generators [(1,2), (1,2,3,4)]) generated by [()]
```

**centralizer(g)**

Returns the centralizer of  $g$  in self.

## EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4) ]])
sage: g = G([ (1,3) ])
sage: G.centralizer(g)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(2,4), (1,3)]
sage: g = G([ (1,2,3,4) ])
sage: G.centralizer(g)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(1,2,3,4)]
sage: H = G.subgroup([G([ (1,2,3,4) ])])
sage: G.centralizer(H)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(1,2,3,4)]
```

**character(values)**

Returns a group character from values, where values is a list of the values of the character evaluated on the conjugacy classes.

## EXAMPLES:

```
sage: G = AlternatingGroup(4)
sage: n = len(G.conjugacy_classes_representatives())
sage: G.character([1]*n)
Character of Alternating group of order 4!/2 as a permutation group
```

**character\_table()**

Returns the matrix of values of the irreducible characters of a permutation group  $G$  at the conjugacy classes of  $G$ . The columns represent the conjugacy classes of  $G$  and the rows represent the different irreducible characters in the ordering given by GAP.

## EXAMPLES:

```

sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3) ]])
sage: G.order()
12
sage: G.character_table()
[      1      1      1      1]
[      1 -zeta3 - 1      zeta3      1]
[      1      zeta3 -zeta3 - 1      1]
[      3      0      0     -1]
sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3) ]])
sage: CT = gap(G).CharacterTable()

```

Type `print gap.eval("Display(%s) "%CT.name())` to display this nicely.

```

sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4) ]])
sage: G.order()
8
sage: G.character_table()
[ 1  1  1  1  1]
[ 1 -1 -1  1  1]
[ 1 -1  1 -1  1]
[ 1  1 -1 -1  1]
[ 2  0  0  0 -2]
sage: CT = gap(G).CharacterTable()

```

Again, type `print gap.eval("Display(%s) "%CT.name())` to display this nicely.

```

sage: SymmetricGroup(2).character_table()
[ 1 -1]
[ 1  1]
sage: SymmetricGroup(3).character_table()
[ 1 -1  1]
[ 2  0 -1]
[ 1  1  1]
sage: SymmetricGroup(5).character_table()
[ 1 -1  1  1 -1 -1  1]
[ 4 -2  0  1  1  0 -1]
[ 5 -1  1 -1 -1  1  0]
[ 6  0 -2  0  0  0  1]
[ 5  1  1 -1  1 -1  0]
[ 4  2  0  1 -1  0 -1]
[ 1  1  1  1  1  1  1]
sage: list(AlternatingGroup(6).character_table())
[(1, 1, 1, 1, 1, 1, 1), (5, 1, 2, -1, -1, 0, 0), (5, 1, -1, 2, -1, 0, 0), (8, 0, -1, -1, 0,

```

Suppose that you have a class function  $f(g)$  on  $G$  and you know the values  $v_1, \dots, v_n$  on the conjugacy class elements in `conjugacy_classes_representatives(G) =  $[g_1, \dots, g_n]$` . Since the irreducible characters  $\rho_1, \dots, \rho_n$  of  $G$  form an  $E$ -basis of the space of all class functions ( $E$  a “sufficiently large” cyclotomic field), such a class function is a linear combination of these basis elements,  $f = c_1\rho_1 + \dots + c_n\rho_n$ . To find the coefficients  $c_i$ , you simply solve the linear system `character_table_values(G)  $[v_1, \dots, v_n]$  =  $[c_1, \dots, c_n]$` , where  $[v_1, \dots, v_n] = \text{character\_table\_values}(G)^{(-1)}[c_1, \dots, c_n]$ .

AUTHORS:

- David Joyner and William Stein (2006-01-04)

#### **cohomology** ( $n, p=0$ )

Computes the group cohomology  $H^n(G, F)$ , where  $F = \mathbf{Z}$  if  $p = 0$  and  $F = \mathbf{Z}/p\mathbf{Z}$  if  $p > 0$  is a prime. Wraps HAP’s `GroupHomology` function, written by Graham Ellis.

REQUIRES: GAP package HAP (in gap\_packages-\*.spkg).

EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: G.cohomology(1,2) # optional - gap_packages
Multiplicative Abelian group isomorphic to C2
sage: G = SymmetricGroup(3)
sage: G.cohomology(5) # optional - gap_packages
Trivial Abelian group
sage: G.cohomology(5,2) # optional - gap_packages
Multiplicative Abelian group isomorphic to C2
sage: G.homology(5,3) # optional - gap_packages
Trivial Abelian group
sage: G.homology(5,4) # optional - gap_packages
Traceback (most recent call last):
...
ValueError: p must be 0 or prime
```

This computes  $H^4(S_3, \mathbf{Z})$  and  $H^4(S_3, \mathbf{Z}/2\mathbf{Z})$ , respectively.

AUTHORS:

- David Joyner and Graham Ellis

REFERENCES:

- G. Ellis, ‘Computing group resolutions’, J. Symbolic Computation. Vol.38, (2004)1077-1118 (Available at <http://hamilton.nuigalway.ie/>).
- D. Joyner, ‘A primer on computational group homology and cohomology’, <http://front.math.ucdavis.edu/0706.0549>.

**cohomology\_part** ( $n, p=0$ )

Computes the  $p$ -part of the group cohomology  $H^n(G, F)$ , where  $F = \mathbf{Z}$  if  $p = 0$  and  $F = \mathbf{Z}/p\mathbf{Z}$  if  $p > 0$  is a prime. Wraps HAP’s Homology function, written by Graham Ellis, applied to the  $p$ -Sylow subgroup of  $G$ .

REQUIRES: GAP package HAP (in gap\_packages-\*.spkg).

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.cohomology_part(7,2) # optional - gap_packages
Multiplicative Abelian group isomorphic to C2 x C2 x C2
sage: G = SymmetricGroup(3)
sage: G.cohomology_part(2,3) # optional - gap_packages
Multiplicative Abelian group isomorphic to C3
```

AUTHORS:

- David Joyner and Graham Ellis

**commutator** ( $other=None$ )

Returns the commutator subgroup of a group, or of a pair of groups.

INPUT:

- other - default: None - a permutation group.

OUTPUT:

Let  $G$  denote `self`. If `other` is `None` then this method returns the subgroup of  $G$  generated by the set of commutators,

$$\{[g_1, g_2] | g_1, g_2 \in G\} = \{g_1^{-1}g_2^{-1}g_1g_2 | g_1, g_2 \in G\}$$

Let  $H$  denote `other`, in the case that it is not `None`. Then this method returns the group generated by the set of commutators,

$$\{[g, h] | g \in G, h \in H\} = \{g^{-1}h^{-1}gh | g \in G, h \in H\}$$

The two groups need only be permutation groups, there is no notion of requiring them to explicitly be subgroups of some other group.

---

**Note:** For the identical statement, the generators of the returned group can vary from one execution to the next.

---

#### EXAMPLES:

```
sage: G = DiCyclicGroup(4)
sage: G.commutator()
Permutation Group with generators [(1, 3, 5, 7) (2, 4, 6, 8) (9, 11, 13, 15) (10, 12, 14, 16)]
```

```
sage: G = SymmetricGroup(5)
sage: H = CyclicPermutationGroup(5)
sage: C = G.commutator(H)
sage: C.is_isomorphic(AlternatingGroup(5))
True
```

An abelian group will have a trivial commutator.

```
sage: G = CyclicPermutationGroup(10)
sage: G.commutator()
Permutation Group with generators [()]
```

The quotient of a group by its commutator is always abelian.

```
sage: G = DihedralGroup(20)
sage: C = G.commutator()
sage: Q = G.quotient(C)
sage: Q.is_abelian()
True
```

When forming commutators from two groups, the order of the groups does not matter.

```
sage: D = DihedralGroup(3)
sage: S = SymmetricGroup(2)
sage: C1 = D.commutator(S); C1
Permutation Group with generators [(1, 2, 3)]
sage: C2 = S.commutator(D); C2
Permutation Group with generators [(1, 3, 2)]
sage: C1 == C2
True
```

This method calls two different functions in GAP, so this tests that their results are consistent. The commutator groups may have different generators, but the groups are equal.

```
sage: G = DiCyclicGroup(3)
sage: C = G.commutator(); C
Permutation Group with generators [(5, 7, 6)]
sage: CC = G.commutator(G); CC
```



```

Permutation Group with generators [(5,6,7)]
sage: C == CC
True

```

The second group is checked.

```

sage: G = SymmetricGroup(2)
sage: G.commutator('junk')
Traceback (most recent call last):
...
TypeError: junk is not a permutation group

```

### **composition\_series()**

Return the composition series of this group as a list of permutation groups.

EXAMPLES:

These computations use pseudo-random numbers, so we set the seed for reproducible testing.

```

sage: set_random_seed(0)
sage: G = PermutationGroup([[ (1,2,3), (4,5) ], [ (3,4) ]])
sage: G.composition_series() # random output
[Permutation Group with generators [(1,2,3)(4,5), (3,4)], Permutation Group with generators
sage: G = PermutationGroup([[ (1,2,3), (4,5) ], [ (1,2) ]])
sage: CS = G.composition_series()
sage: CS[3]
Subgroup of (Permutation Group with generators [(1,2), (1,2,3)(4,5)]) generated by [()]

```

### **conjugacy\_class(g)**

Return the conjugacy class of  $g$  inside the group `self`.

INPUT:

- $g$  – an element of the permutation group `self`

OUTPUT:

The conjugacy class of  $g$  in the group `self`. If `self` is the group denoted by  $G$ , this method computes the set  $\{x^{-1}gx \mid x \in G\}$

EXAMPLES:

```

sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: G.conjugacy_class(g)
Conjugacy class of (1,2,3,4) in Symmetric group of order 4! as a permutation group

```

### **conjugacy\_classes()**

Return a list with all the conjugacy classes of `self`.

EXAMPLES:

```

sage: G = DihedralGroup(3)
sage: G.conjugacy_classes()
[Conjugacy class of () in Dihedral group of order 6 as a permutation group,
Conjugacy class of (2,3) in Dihedral group of order 6 as a permutation group,
Conjugacy class of (1,2,3) in Dihedral group of order 6 as a permutation group]

```

### **conjugacy\_classes\_representatives()**

Returns a complete list of representatives of conjugacy classes in a permutation group  $G$ . The ordering is that given by GAP.

EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4) ]])
sage: cl = G.conjugacy_classes_representatives(); cl
[ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3)(2,4) ]
sage: cl[3] in G
True

sage: G = SymmetricGroup(5)
sage: G.conjugacy_classes_representatives()
[ (), (1,2), (1,2)(3,4), (1,2,3), (1,2,3)(4,5), (1,2,3,4), (1,2,3,4,5) ]

sage: S = SymmetricGroup(['a','b','c'])
sage: S.conjugacy_classes_representatives()
[ (), ('a','b'), ('a','b','c') ]
```

AUTHORS:

•David Joyner and William Stein (2006-01-04)

**conjugacy\_classes\_subgroups()**

Returns a complete list of representatives of conjugacy classes of subgroups in a permutation group  $G$ . The ordering is that given by GAP.

EXAMPLES:

```
sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4) ]])
sage: cl = G.conjugacy_classes_subgroups()
sage: cl
[Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [()],
sage: G = SymmetricGroup(3)
sage: G.conjugacy_classes_subgroups()
[Subgroup of (Symmetric group of order 3! as a permutation group) generated by [()], Subgroup
```

AUTHORS:

•David Joyner (2006-10)

**conjugate(g)**

Returns the group formed by conjugating `self` with `g`.

INPUT:

•`g` - a permutation group element, or an object that converts to a permutation group element, such as a list of integers or a string of cycles.

OUTPUT:

If `self` is the group denoted by  $H$ , then this method computes the group

$$g^{-1}Hg = \{g^{-1}hg | h \in H\}$$

which is the group  $H$  conjugated by  $g$ .

There are no restrictions on `self` and `g` belonging to a common permutation group, and correspondingly, there is no relationship (such as a common parent) between `self` and the output group.

EXAMPLES:

```
sage: G = DihedralGroup(6)
sage: a = PermutationGroupElement("(1,2,3,4)")
sage: G.conjugate(a)
Permutation Group with generators [(1,4)(2,6)(3,5), (1,5,6,2,3,4)]
```

The element performing the conjugation can be specified in several ways.

```
sage: G = DihedralGroup(6)
sage: strng = "(1,2,3,4)"
sage: G.conjugate(strng)
Permutation Group with generators [(1,4)(2,6)(3,5), (1,5,6,2,3,4)]
sage: G = DihedralGroup(6)
sage: lst = [2,3,4,1]
sage: G.conjugate(lst)
Permutation Group with generators [(1,4)(2,6)(3,5), (1,5,6,2,3,4)]
sage: G = DihedralGroup(6)
sage: cycles = [(1,2,3,4)]
sage: G.conjugate(cycles)
Permutation Group with generators [(1,4)(2,6)(3,5), (1,5,6,2,3,4)]
```

Conjugation is a group automorphism, so conjugate groups will be isomorphic.

```
sage: G = DiCyclicGroup(6)
sage: G.degree()
11
sage: cycle = [i+1 for i in range(1,11)] + [1]
sage: C = G.conjugate(cycle)
sage: G.is_isomorphic(C)
True
```

The conjugating element may be from a symmetric group with larger degree than the group being conjugated.

```
sage: G = AlternatingGroup(5)
sage: G.degree()
5
sage: g = "(1,3)(5,6,7)"
sage: H = G.conjugate(g); H
Permutation Group with generators [(1,4,6,3,2), (1,4,6)]
sage: H.degree()
6
```

The conjugating element is checked.

```
sage: G = SymmetricGroup(3)
sage: G.conjugate("junk")
Traceback (most recent call last):
...
TypeError: junk does not convert to a permutation group element
```

#### construction()

EXAMPLES:

```
sage: P1 = PermutationGroup([(1,2)])
sage: P1.construction()
(PermutationGroupFunctor[(1,2)], Permutation Group with generators [])

sage: PermutationGroup([]).construction() is None
True
```

This allows us to perform computations like the following:

```
sage: P1 = PermutationGroup([(1,2)]); p1 = P1.gen()
sage: P2 = PermutationGroup([(1,3)]); p2 = P2.gen()
sage: p = p1*p2; p
(1,2,3)
```

```
sage: p.parent()
Permutation Group with generators [(1,2), (1,3)]
sage: p.parent().domain()
{1, 2, 3}
```

Note that this will merge permutation groups with different domains:

```
sage: g1 = PermutationGroupElement([(1,2), (3,4,5)])
sage: g2 = PermutationGroup([( 'a', 'b' )], domain=[ 'a', 'b' ]).gens()[0]
sage: g2
( 'a', 'b' )
sage: p = g1*g2; p
(1,2) (3,4,5) ( 'a', 'b' )
```

**cosets** (*S*, *side*=*'right'*)

Returns a list of the cosets of *S* in *self*.

INPUT:

- *S* - a subgroup of *self*. An error is raised if *S* is not a subgroup.
- *side* - default: *'right'* - determines if right cosets or left cosets are returned. *side* refers to where the representative is placed in the products forming the cosets and thus allowable values are only *'right'* and *'left'*.

OUTPUT:

A list of lists. Each inner list is a coset of the subgroup in the group. The first element of each coset is the smallest element (based on the ordering of the elements of *self*) of all the group elements that have not yet appeared in a previous coset. The elements of each coset are in the same order as the subgroup elements used to build the coset's elements.

As a consequence, the subgroup itself is the first coset, and its first element is the identity element. For each coset, the first element listed is the element used as a representative to build the coset. These representatives form an increasing sequence across the list of cosets, and within a coset the representative is the smallest element of its coset (both orderings are based on of the ordering of elements of *self*).

In the case of a normal subgroup, left and right cosets should appear in the same order as part of the outer list. However, the list of the elements of a particular coset may be in a different order for the right coset versus the order in the left coset. So, if you check to see if a subgroup is normal, it is necessary to sort each individual coset first (but not the list of cosets, due to the ordering of the representatives). See below for examples of this.

---

**Note:** This is a naive implementation intended for instructional purposes, and hence is slow for larger groups. Sage and GAP provide more sophisticated functions for working quickly with cosets of larger groups.

---

EXAMPLES:

The default is to build right cosets. This example works with the symmetry group of an 8-gon and a normal subgroup. Notice that a straight check on the equality of the output is not sufficient to check normality, while sorting the individual cosets is sufficient to then simply test equality of the list of lists. Study the second coset in each list to understand the need for sorting the elements of the cosets.

```
sage: G = DihedralGroup(8)
sage: quarter_turn = G.list()[5]; quarter_turn
(1,3,5,7) (2,4,6,8)
sage: S = G.subgroup([quarter_turn])
sage: rc = G.cosets(S); rc
```

```

[[(), (1,3,5,7)(2,4,6,8), (1,5)(2,6)(3,7)(4,8), (1,7,5,3)(2,8,6,4)],
 [(2,8)(3,7)(4,6), (1,7)(2,6)(3,5), (1,5)(2,4)(6,8), (1,3)(4,8)(5,7)],
 [(1,2)(3,8)(4,7)(5,6), (1,8)(2,7)(3,6)(4,5), (1,6)(2,5)(3,4)(7,8), (1,4)(2,3)(5,8)(6,7)],
 [(1,2,3,4,5,6,7,8), (1,4,7,2,5,8,3,6), (1,6,3,8,5,2,7,4), (1,8,7,6,5,4,3,2)]]
sage: lc = G.cosets(S, side='left'); lc
[[(), (1,3,5,7)(2,4,6,8), (1,5)(2,6)(3,7)(4,8), (1,7,5,3)(2,8,6,4)],
 [(2,8)(3,7)(4,6), (1,3)(4,8)(5,7), (1,5)(2,4)(6,8), (1,7)(2,6)(3,5)],
 [(1,2)(3,8)(4,7)(5,6), (1,4)(2,3)(5,8)(6,7), (1,6)(2,5)(3,4)(7,8), (1,8)(2,7)(3,6)(4,5)],
 [(1,2,3,4,5,6,7,8), (1,4,7,2,5,8,3,6), (1,6,3,8,5,2,7,4), (1,8,7,6,5,4,3,2)]]

sage: S.is_normal(G)
True
sage: rc == lc
False
sage: rc_sorted = [sorted(c) for c in rc]
sage: lc_sorted = [sorted(c) for c in lc]
sage: rc_sorted == lc_sorted
True

```

An example with the symmetry group of a regular tetrahedron and a subgroup that is not normal. Thus, the right and left cosets are different (and so are the representatives). With each individual coset sorted, a naive test of normality is possible.

```

sage: A = AlternatingGroup(4)
sage: face_turn = A.list()[4]; face_turn
(1,2,3)
sage: stabilizer = A.subgroup([face_turn])
sage: rc = A.cosets(stabilizer, side='right'); rc
[[(), (1,2,3), (1,3,2)],
 [(2,3,4), (1,3)(2,4), (1,4,2)],
 [(2,4,3), (1,4,3), (1,2)(3,4)],
 [(1,2,4), (1,4)(2,3), (1,3,4)]]
sage: lc = A.cosets(stabilizer, side='left'); lc
[[(), (1,2,3), (1,3,2)],
 [(2,3,4), (1,2)(3,4), (1,3,4)],
 [(2,4,3), (1,2,4), (1,3)(2,4)],
 [(1,4,2), (1,4,3), (1,4)(2,3)]]

sage: stabilizer.is_normal(A)
False
sage: rc_sorted = [sorted(c) for c in rc]
sage: lc_sorted = [sorted(c) for c in lc]
sage: rc_sorted == lc_sorted
False

```

#### TESTS:

The keyword `side` is checked for the two possible values.

```

sage: G = SymmetricGroup(3)
sage: S = G.subgroup([G("(1,2)")])
sage: G.cosets(S, side='junk')
Traceback (most recent call last):
...
ValueError: side should be 'right' or 'left', not junk

```

The subgroup argument is checked to see if it is a permutation group. Even a legitimate GAP object can be rejected.

```
sage: G=SymmetricGroup(3)
sage: G.cosets(gap(3))
Traceback (most recent call last):
...
TypeError: 3 is not a permutation group
```

The subgroup is verified as a subgroup of self.

```
sage: A = AlternatingGroup(3)
sage: G = SymmetricGroup(3)
sage: S = G.subgroup([G("(1,2)")])
sage: A.cosets(S)
Traceback (most recent call last):
...
ValueError: Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(
```

AUTHOR:

- Rob Beezer (2011-01-31)

### **degree()**

Returns the degree of this permutation group.

EXAMPLES:

```
sage: S = SymmetricGroup(['a','b','c'])
sage: S.degree()
3
sage: G = PermutationGroup([(1,3),(4,5)])
sage: G.degree()
5
```

Note that you can explicitly specify the domain to get a permutation group of smaller degree:

```
sage: G = PermutationGroup([(1,3),(4,5)], domain=[1,3,4,5])
sage: G.degree()
4
```

### **derived\_series()**

Return the derived series of this group as a list of permutation groups.

EXAMPLES:

These computations use pseudo-random numbers, so we set the seed for reproducible testing.

```
sage: set_random_seed(0)
sage: G = PermutationGroup([(1,2,3),(4,5)], [(3,4)])
sage: G.derived_series() # random output
[Permutation Group with generators [(1,2,3)(4,5), (3,4)], Permutation Group with generators
```

### **direct\_product(other, maps=True)**

Wraps GAP's DirectProduct, Embedding, and Projection.

Sage calls GAP's DirectProduct, which chooses an efficient representation for the direct product. The direct product of permutation groups will be a permutation group again. For a direct product  $D$ , the GAP operation `Embedding(D, i)` returns the homomorphism embedding the  $i$ -th factor into  $D$ . The GAP operation `Projection(D, i)` gives the projection of  $D$  onto the  $i$ -th factor. This method returns a 5-tuple: a permutation group and 4 morphisms.

INPUT:

- self, other - permutation groups

## OUTPUT:

- `D` - a direct product of the inputs, returned as a permutation group as well
- `iota1` - an embedding of `self` into `D`
- `iota2` - an embedding of `other` into `D`
- `pr1` - the projection of `D` onto `self` (giving a splitting `1 - other - D - self - 1`)
- `pr2` - the projection of `D` onto `other` (giving a splitting `1 - self - D - other - 1`)

## EXAMPLES:

```

sage: G = CyclicPermutationGroup(4)
sage: D = G.direct_product(G, False)
sage: D
Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
sage: D, iota1, iota2, pr1, pr2 = G.direct_product(G)
sage: D; iota1; iota2; pr1; pr2
Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
Permutation group morphism:
  From: Cyclic group of order 4 as a permutation group
  To:   Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
  Defn: Embedding( Group( [ (1, 2, 3, 4), (5, 6, 7, 8) ] ), 1 )
Permutation group morphism:
  From: Cyclic group of order 4 as a permutation group
  To:   Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
  Defn: Embedding( Group( [ (1, 2, 3, 4), (5, 6, 7, 8) ] ), 2 )
Permutation group morphism:
  From: Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
  To:   Cyclic group of order 4 as a permutation group
  Defn: Projection( Group( [ (1, 2, 3, 4), (5, 6, 7, 8) ] ), 1 )
Permutation group morphism:
  From: Permutation Group with generators [(5, 6, 7, 8), (1, 2, 3, 4)]
  To:   Cyclic group of order 4 as a permutation group
  Defn: Projection( Group( [ (1, 2, 3, 4), (5, 6, 7, 8) ] ), 2 )
sage: g=D([(1,3),(2,4)]); g
(1, 3) (2, 4)
sage: d=D([(1,4,3,2),(5,7),(6,8)]); d
(1, 4, 3, 2) (5, 7) (6, 8)
sage: iota1(g); iota2(g); pr1(d); pr2(d)
(1, 3) (2, 4)
(5, 7) (6, 8)
(1, 4, 3, 2)
(1, 3) (2, 4)

```

**domain()**

Returns the underlying set that this permutation group acts on.

## EXAMPLES:

```

sage: P = PermutationGroup([(1,2),(3,5)])
sage: P.domain()
{1, 2, 3, 4, 5}
sage: S = SymmetricGroup(['a', 'b', 'c'])
sage: S.domain()
{'a', 'b', 'c'}

```

**exponent()**

Computes the exponent of the group. The exponent  $e$  of a group  $G$  is the LCM of the orders of its elements, that is,  $e$  is the smallest integer such that  $g^e = 1$  for all  $g \in G$ .

EXAMPLES:

```
sage: G = AlternatingGroup(4)
sage: G.exponent()
6
```

**fitting\_subgroup()**

Returns the Fitting subgroup of `self`. The Fitting subgroup of a group  $G$  is the largest nilpotent normal subgroup of  $G$ .

EXAMPLES:

```
sage: G=PermutationGroup([[ (1,2,3,4) ], [ (2,4) ]])
sage: G.fitting_subgroup()
Subgroup of (Permutation Group with generators [ (2,4), (1,2,3,4) ]) generated by [ (2,4), (1,2,3,4) ]
sage: G=PermutationGroup([[ (1,2,3,4) ], [ (1,2) ]])
sage: G.fitting_subgroup()
Subgroup of (Permutation Group with generators [ (1,2), (1,2,3,4) ]) generated by [ (1,2) (3,4), (1,2,3,4) ]
```

**fixed\_points()**

Returns the list of points fixed by `self`, i.e., the subset of `.domain()` not moved by any element of `self`.

EXAMPLES:

```
sage: G=PermutationGroup([ (1,2,3) ])
sage: G.fixed_points()
[]
sage: G=PermutationGroup([ (1,2,3), (5,6) ])
sage: G.fixed_points()
[4]
sage: G=PermutationGroup([ [ (1,4,7) ], [ (4,3), (6,7) ] ])
sage: G.fixed_points()
[2, 5]
```

**frattini\_subgroup()**

Returns the Frattini subgroup of `self`. The Frattini subgroup of a group  $G$  is the intersection of all maximal subgroups of  $G$ .

EXAMPLES:

```
sage: G=PermutationGroup([ [ (1,2,3,4) ], [ (2,4) ] ])
sage: G.frattini_subgroup()
Subgroup of (Permutation Group with generators [ (2,4), (1,2,3,4) ]) generated by [ (1,3) (2,4) ]
sage: G=SymmetricGroup(4)
sage: G.frattini_subgroup()
Subgroup of (Symmetric group of order 4! as a permutation group) generated by [ () ]
```

**gen(*i=None*)**

Returns the  $i$ -th generator of `self`; that is, the  $i$ -th element of the list `self.gens()`.

The argument  $i$  may be omitted if there is only one generator (but this will raise an error otherwise).

EXAMPLES:

We explicitly construct the alternating group on four elements:

```
sage: A4 = PermutationGroup([ [ (1,2,3) ], [ (2,3,4) ] ]); A4
Permutation Group with generators [ (2,3,4), (1,2,3) ]
sage: A4.gens()
[ (2,3,4), (1,2,3) ]
sage: A4.gen(0)
```



```

(2, 3, 4)
sage: A4.gen(1)
(1, 2, 3)
sage: A4.gens()[0]; A4.gens()[1]
(2, 3, 4)
(1, 2, 3)

sage: P1 = PermutationGroup([[ (1, 2) ]]); P1.gen()
(1, 2)

```

**gens()**

Return tuple of generators of this group. These need not be minimal, as they are the generators used in defining this group.

**EXAMPLES:**

```

sage: G = PermutationGroup([[ (1, 2, 3) ], [ (1, 2) ]])
sage: G.gens()
[ (1, 2), (1, 2, 3) ]

```

Note that the generators need not be minimal, though duplicates are removed:

```

sage: G = PermutationGroup([[ (1, 2) ], [ (1, 3) ], [ (2, 3) ], [ (1, 2) ]])
sage: G.gens()
[ (2, 3), (1, 2), (1, 3) ]

```

We can use index notation to access the generators returned by `self.gens`:

```

sage: G = PermutationGroup([[ (1, 2, 3, 4), (5, 6) ], [ (1, 2) ]])
sage: g = G.gens()
sage: g[0]
(1, 2)
sage: g[1]
(1, 2, 3, 4) (5, 6)

```

**TESTS:**

We make sure that the trivial group gets handled correctly:

```

sage: SymmetricGroup(1).gens()
[()]

```

**gens\_small()**

For this group, returns a generating set which has few elements. As neither irredundancy nor minimal length is proven, it is fast.

**EXAMPLES:**

```

sage: R = "(25, 27, 32, 30) (26, 29, 31, 28) ( 3, 38, 43, 19) ( 5, 36, 45, 21) ( 8, 33, 48, 24) " ## R = right
sage: U = "( 1, 3, 8, 6) ( 2, 5, 7, 4) ( 9, 33, 25, 17) (10, 34, 26, 18) (11, 35, 27, 19) " ## U = top
sage: L = "( 9, 11, 16, 14) (10, 13, 15, 12) ( 1, 17, 41, 40) ( 4, 20, 44, 37) ( 6, 22, 46, 35) " ## L = left
sage: F = "(17, 19, 24, 22) (18, 21, 23, 20) ( 6, 25, 43, 16) ( 7, 28, 42, 13) ( 8, 30, 41, 11) " ## F = front
sage: B = "(33, 35, 40, 38) (34, 37, 39, 36) ( 3, 9, 46, 32) ( 2, 12, 47, 29) ( 1, 14, 48, 27) " ## B = back or
sage: D = "(41, 43, 48, 46) (42, 45, 47, 44) (14, 22, 30, 38) (15, 23, 31, 39) (16, 24, 32, 40) " ## D = down or
sage: G = PermutationGroup([R, L, U, F, B, D])
sage: len(G.gens_small())
2

```

The output may be unpredictable, due to the use of randomized algorithms in GAP. Note that both the following answers are equally valid.

```
sage: G = PermutationGroup([(['a','b']), [(['b', 'c']), [(['a', 'c')]]])
sage: G.gens_small() # random
[(['b','c'), (['a','c','b'])] ## (on 64-bit Linux)
[(['a','b'), (['a','c','b'])] ## (on Solaris)
sage: len(G.gens_small()) == 2
True
```

**group\_id()**

Return the ID code of this group, which is a list of two integers. Requires “optional” database\_gap-4.4.x package.

EXAMPLES:

```
sage: G = PermutationGroup([[(1,2,3), (4,5)], [(1,2)]]
sage: G.group_id() # optional - database_gap
[12, 4]
```

**group\_primitive\_id()**

Return the index of this group in the GAP database of primitive groups.

Requires “optional” database\_gap-4.4.x package.

OUTPUT:

A positive integer, following GAP’s conventions. A `ValueError` is raised if the group is not primitive.

EXAMPLES:

```
sage: G = PermutationGroup([[(1,2,3,4,5)], [(1,5), (2,4)]]
sage: G.group_primitive_id() # optional - database_gap
2
sage: G.degree()
5
```

From the information of the degree and the identification number, you can recover the isomorphism class of your group in the GAP database:

```
sage: H = PrimitiveGroup(5,2) # optional - database_gap
sage: G == H # optional - database_gap
False
sage: G.is_isomorphic(H) # optional - database_gap
True
```

**has\_element(item)**

Returns boolean value of `item` in `self` - however *ignores* parentage.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
sage: gens = G.gens()
sage: H = DihedralGroup(4)
sage: g = G([(1,2,3,4)]); g
(1,2,3,4)
sage: G.has_element(g)
True
sage: h = H([(1,2), (3,4)]); h
(1,2)(3,4)
sage: G.has_element(h)
False
```

**holomorph()**

The holomorph of a group as a permutation group.

The holomorph of a group  $G$  is the semidirect product  $G \rtimes_{id} \text{Aut}(G)$ , where  $id$  is the identity function on  $\text{Aut}(G)$ , the automorphism group of  $G$ .

See [Wikipedia article Holomorph \(mathematics\)](#)

OUTPUT:

Returns the holomorph of a given group as permutation group via a wrapping of GAP's semidirect product function.

EXAMPLES:

Thomas and Wood's 'Group Tables' (Shiva Publishing, 1980) tells us that the holomorph of  $C_5$  is the unique group of order 20 with a trivial center.

```
sage: C5 = CyclicPermutationGroup(5)
sage: A = C5.holomorph()
sage: A.order()
20
sage: A.is_abelian()
False
sage: A.center()
Subgroup of (Permutation Group with generators
[(5,6,7,8,9), (1,2,4,3)(6,7,9,8)]) generated by [()]
sage: A
Permutation Group with generators [(5,6,7,8,9), (1,2,4,3)(6,7,9,8)]
```

Noting that the automorphism group of  $D_4$  is itself  $D_4$ , it can easily be shown that the holomorph is indeed an internal semidirect product of these two groups.

```
sage: D4 = DihedralGroup(4)
sage: H = D4.holomorph()
sage: H.gens()
[(3,8)(4,7), (2,3,5,8), (2,5)(3,8), (1,4,6,7)(2,3,5,8), (1,8)(2,7)(3,6)(4,5)]
sage: G = H.subgroup([H.gens()[0], H.gens()[1], H.gens()[2]])
sage: N = H.subgroup([H.gens()[3], H.gens()[4]])
sage: N.is_normal(H)
True
sage: G.is_isomorphic(D4)
True
sage: N.is_isomorphic(D4)
True
sage: G.intersection(N)
Permutation Group with generators [()]
sage: L = [H(x)*H(y) for x in G for y in N]; L.sort()
sage: L1 = H.list(); L1.sort()
sage: L == L1
True
```

Author:

•Kevin Halasz (2012-08-14)

**homology** ( $n, p=0$ )

Computes the group homology  $H_n(G, F)$ , where  $F = \mathbf{Z}$  if  $p = 0$  and  $F = \mathbf{Z}/p\mathbf{Z}$  if  $p > 0$  is a prime. Wraps HAP's GroupHomology function, written by Graham Ellis.

REQUIRES: GAP package HAP (in gap\_packages-\*.spkg).

AUTHORS:

•David Joyner and Graham Ellis

The example below computes  $H_7(S_5, \mathbf{Z})$ ,  $H_7(S_5, \mathbf{Z}/2\mathbf{Z})$ ,  $H_7(S_5, \mathbf{Z}/3\mathbf{Z})$ , and  $H_7(S_5, \mathbf{Z}/5\mathbf{Z})$ , respectively. To compute the 2-part of  $H_7(S_5, \mathbf{Z})$ , use the `homology_part` function.

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.homology(7)                                     # optional - gap_packages
Multiplicative Abelian group isomorphic to C2 x C2 x C4 x C3 x C5
sage: G.homology(7, 2)                                  # optional - gap_packages
Multiplicative Abelian group isomorphic to C2 x C2 x C2 x C2 x C2
sage: G.homology(7, 3)                                  # optional - gap_packages
Multiplicative Abelian group isomorphic to C3
sage: G.homology(7, 5)                                  # optional - gap_packages
Multiplicative Abelian group isomorphic to C5
```

REFERENCES:

- G. Ellis, “Computing group resolutions”, J. Symbolic Computation. Vol.38, (2004)1077-1118 (Available at <http://hamilton.nuigalway.ie/>).
- D. Joyner, “A primer on computational group homology and cohomology”, <http://front.math.ucdavis.edu/0706.0549>

**homology\_part** (*n, p=0*)

Computes the  $p$ -part of the group homology  $H_n(G, F)$ , where  $F = \mathbf{Z}$  if  $p = 0$  and  $F = \mathbf{Z}/p\mathbf{Z}$  if  $p > 0$  is a prime. Wraps HAP’s Homology function, written by Graham Ellis, applied to the  $p$ -Sylow subgroup of  $G$ .

REQUIRES: GAP package HAP (in `gap_packages-*.spkg`).

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.homology_part(7, 2)                             # optional - gap_packages
Multiplicative Abelian group isomorphic to C2 x C2 x C2 x C2 x C4
```

AUTHORS:

•David Joyner and Graham Ellis

**id** ()

(Same as `self.group_id()`.) Return the ID code of this group, which is a list of two integers. Requires “optional” `database_gap-4.4.x` package.

EXAMPLES:

```
sage: G = PermutationGroup([[ (1, 2, 3), (4, 5) ], [ (1, 2) ]])
sage: G.group_id()                                     # optional - database_gap
[12, 4]
```

**identity** ()

Return the identity element of this group.

EXAMPLES:

```
sage: G = PermutationGroup([[ (1, 2, 3), (4, 5) ]])
sage: e = G.identity()
sage: e
()
sage: g = G.gen(0)
sage: g*e
```

```

(1,2,3) (4,5)
sage: e*g
(1,2,3) (4,5)

sage: S = SymmetricGroup(['a','b','c'])
sage: S.identity()
()
```

**intersection(*other*)**

Returns the permutation group that is the intersection of *self* and *other*.

INPUT:

- *other* - a permutation group.

OUTPUT:

A permutation group that is the set-theoretic intersection of *self* with *other*. The groups are viewed as subgroups of a symmetric group big enough to contain both group's symbol sets. So there is no strict notion of the two groups being subgroups of a common parent.

EXAMPLES:

```

sage: H = DihedralGroup(4)

sage: K = CyclicPermutationGroup(4)
sage: H.intersection(K)
Permutation Group with generators [(1,2,3,4)]

sage: L = DihedralGroup(5)
sage: H.intersection(L)
Permutation Group with generators [(1,4)(2,3)]

sage: M = PermutationGroup(["()"])
sage: H.intersection(M)
Permutation Group with generators [()]
```

Some basic properties.

```

sage: H = DihedralGroup(4)
sage: L = DihedralGroup(5)
sage: H.intersection(L) == L.intersection(H)
True
sage: H.intersection(H) == H
True
```

The group *other* is verified as such.

```

sage: H = DihedralGroup(4)
sage: H.intersection('junk')
Traceback (most recent call last):
...
TypeError: junk is not a permutation group
```

**irreducible\_characters()**

Returns a list of the irreducible characters of *self*.

EXAMPLES:

```

sage: irr = SymmetricGroup(3).irreducible_characters()
sage: [x.values() for x in irr]
```

```
[ [1, -1, 1], [2, 0, -1], [1, 1, 1] ]
```

**is\_abelian()**

Return True if this group is abelian.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_abelian()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_abelian()
True
```

**is\_commutative()**

Return True if this group is commutative.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_commutative()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_commutative()
True
```

**is\_cyclic()**

Return True if this group is cyclic.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_cyclic()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_cyclic()
True
```

**is\_elementary\_abelian()**

Return True if this group is elementary abelian. An elementary abelian group is a finite abelian group, where every nontrivial element has order  $p$ , where  $p$  is a prime.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_elementary_abelian()
False
sage: G = PermutationGroup(['(1,2,3)', '(4,5,6)'])
sage: G.is_elementary_abelian()
True
```

**is\_isomorphic(right)**

Return True if the groups are isomorphic.

INPUT:

- self - this group
- right - a permutation group

OUTPUT:

•boolean; True if self and right are isomorphic groups; False otherwise.

#### EXAMPLES:

```
sage: v = ['(1,2,3)(4,5)', '(1,2,3,4,5)']
sage: G = PermutationGroup(v)
sage: H = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_isomorphic(H)
False
sage: G.is_isomorphic(G)
True
sage: G.is_isomorphic(PermutationGroup(list(reversed(v))))
True
```

#### **is\_monomial()**

Returns True if the group is monomial. A finite group is monomial if every irreducible complex character is induced from a linear character of a subgroup.

#### EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_monomial()
True
```

#### **is\_nilpotent()**

Return True if this group is nilpotent.

#### EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_nilpotent()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_nilpotent()
True
```

#### **is\_normal(other)**

Return True if this group is a normal subgroup of other.

#### EXAMPLES:

```
sage: AlternatingGroup(4).is_normal(SymmetricGroup(4))
True
sage: H = PermutationGroup(['(1,2,3)(4,5)'])
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: H.is_normal(G)
False
```

#### **is\_perfect()**

Return True if this group is perfect. A group is perfect if it equals its derived subgroup.

#### EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_perfect()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_perfect()
False
```

#### **is\_pgroup()**

Returns True if this group is a  $p$ -group. A finite group is a  $p$ -group if its order is of the form  $p^n$  for a

prime integer  $p$  and a nonnegative integer  $n$ .

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3,4,5)'])
sage: G.is_pgroup()
True
```

**is\_polycyclic()**

Return True if this group is polycyclic. A group is polycyclic if it has a subnormal series with cyclic factors. (For finite groups, this is the same as if the group is solvable - see `is_solvable`.)

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: G.is_polycyclic()
False
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_polycyclic()
True
```

**is\_primitive** (*domain=None*)

Returns True if `self` acts primitively on `domain`. A group  $G$  acts primitively on a set  $S$  if

1.  $G$  acts transitively on  $S$  and
2. the action induces no non-trivial block system on  $S$ .

INPUT:

- `domain` (optional)

See Also:

- `blocks_all()`

EXAMPLES:

By default, test for primitivity of `self` on its domain:

```
sage: G = PermutationGroup([(1,2,3,4)], [(1,2)]])
sage: G.is_primitive()
True
sage: G = PermutationGroup([(1,2,3,4)], [(2,4)]])
sage: G.is_primitive()
False
```

You can specify a domain on which to test primitivity:

```
sage: G = PermutationGroup([(1,2,3,4)], [(2,4)]])
sage: G.is_primitive([1..4])
False
sage: G.is_primitive([1,2,3])
True
sage: G = PermutationGroup([(3,4,5,6)], [(3,4)]]) #S_4 on [3..6]
sage: G.is_primitive(G.non_fixed_points())
True
```

**is\_regular** (*domain=None*)

Returns True if `self` acts regularly on `domain`. A group  $G$  acts regularly on a set  $S$  if

1.  $G$  acts transitively on  $S$  and



2.  $G$  acts semi-regularly on  $S$ .

EXAMPLES:

```
sage: G = PermutationGroup([[ (1, 2, 3, 4) ]])
sage: G.is_regular()
True
sage: G = PermutationGroup([[ (1, 2, 3, 4) ], [ (5, 6) ]])
sage: G.is_regular()
False
```

You can pass in a domain on which to test regularity:

```
sage: G = PermutationGroup([[ (1, 2, 3, 4) ], [ (5, 6) ]])
sage: G.is_regular([1..4])
True
sage: G.is_regular(G.non_fixed_points())
False
```

**is\_semi\_regular** (*domain=None*)

Returns True if self acts semi-regularly on domain. A group  $G$  acts semi-regularly on a set  $S$  if the point stabilizers of  $S$  in  $G$  are trivial.

domain is optional and may take several forms. See examples.

EXAMPLES:

```
sage: G = PermutationGroup([[ (1, 2, 3, 4) ]])
sage: G.is_semi_regular()
True
sage: G = PermutationGroup([[ (1, 2, 3, 4) ], [ (5, 6) ]])
sage: G.is_semi_regular()
False
```

You can pass in a domain to test semi-regularity:

```
sage: G = PermutationGroup([[ (1, 2, 3, 4) ], [ (5, 6) ]])
sage: G.is_semi_regular([1..4])
True
sage: G.is_semi_regular(G.non_fixed_points())
False
```

**is\_simple** ()

Returns True if the group is simple. A group is simple if it has no proper normal subgroups.

EXAMPLES:

```
sage: G = PermutationGroup([' (1, 2, 3) (4, 5) '])
sage: G.is_simple()
False
```

**is\_solvable** ()

Returns True if the group is solvable.

EXAMPLES:

```
sage: G = PermutationGroup([' (1, 2, 3) (4, 5) '])
sage: G.is_solvable()
True
```

**is\_subgroup** (*other*)

Returns True if self is a subgroup of other.

EXAMPLES:

```
sage: G = AlternatingGroup(5)
sage: H = SymmetricGroup(5)
sage: G.is_subgroup(H)
True
```

**is\_supersolvable()**

Returns True if the group is supersolvable. A finite group is supersolvable if it has a normal series with cyclic factors.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.is_supersolvable()
True
```

**is\_transitive(domain=None)**

Returns True if self acts transitively on domain. A group  $G$  acts transitively on set  $S$  if for all  $x, y \in S$  there is some  $g \in G$  such that  $x^g = y$ .

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.is_transitive()
True
sage: G = PermutationGroup(['(1,2)(3,4)(5,6)'])
sage: G.is_transitive()
False

sage: G = PermutationGroup([(1,2,3,4,5)], [(1,2)]]) #S_5 on [1..5]
sage: G.is_transitive([1,4,5])
True
sage: G.is_transitive([2..6])
False
sage: G.is_transitive(G.non_fixed_points())
True
sage: H = PermutationGroup([(1,2,3)], [(4,5,6)]])
sage: H.is_transitive(H.non_fixed_points())
False
```

Note that this differs from the definition in GAP, where `IsTransitive` returns whether the group is transitive on the set of points moved by the group.

```
sage: G = PermutationGroup([(2,3)])
sage: G.is_transitive()
False
sage: gap(G).IsTransitive()
true
```

**isomorphism\_to(right)**

Return an isomorphism from self to right if the groups are isomorphic, otherwise None.

INPUT:

- self - this group
- right - a permutation group

OUTPUT:

- None or a morphism of permutation groups.

## EXAMPLES:

```

sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: H = PermutationGroup(['(1,2,3)(4,5)'])
sage: G.isomorphism_to(H) is None
True
sage: G = PermutationGroup([(1,2,3), (2,3)])
sage: H = PermutationGroup([(1,2,4), (1,4)])
sage: G.isomorphism_to(H) # not tested, see below
Permutation group morphism:
  From: Permutation Group with generators [(2,3), (1,2,3)]
  To:   Permutation Group with generators [(1,2,4), (1,4)]
  Defn: [(2,3), (1,2,3)] -> [(2,4), (1,2,4)]

```

## TESTS:

Partial check that the output makes some sense:

```

sage: G.isomorphism_to(H)
Permutation group morphism:
  From: Permutation Group with generators [(2,3), (1,2,3)]
  To:   Permutation Group with generators [(1,2,4), (1,4)]
  Defn: [(2,3), (1,2,3)] -> [...]

```

**isomorphism\_type\_info\_simple\_group()**

If the group is simple, then this returns the name of the group.

## EXAMPLES:

```

sage: G = CyclicPermutationGroup(5)
sage: G.isomorphism_type_info_simple_group()
rec(
  name := "Z(5)",
  parameter := 5,
  series := "Z" )

```

TESTS: This shows that the issue at trac ticket 7360 is fixed:

```

sage: G = KleinFourGroup()
sage: G.is_simple()
False
sage: G.isomorphism_type_info_simple_group()
Traceback (most recent call last):
...
TypeError: Group must be simple.

```

**largest\_moved\_point()**

Return the largest point moved by a permutation in this group.

## EXAMPLES:

```

sage: G = PermutationGroup([(1,2), (3,4)], [(1,2,3,4)])
sage: G.largest_moved_point()
4
sage: G = PermutationGroup([(1,2), (3,4)], [(1,2,3,4,10)])
sage: G.largest_moved_point()
10

sage: G = PermutationGroup([(a', b', c'), (d', e')])
sage: G.largest_moved_point()
'e'

```

**Warning:** The name of this function is not good; this function should be deprecated in term of degree:

```
sage: P = PermutationGroup([[1,2,3,4]])
sage: P.largest_moved_point()
4
sage: P.cardinality()
1
```

**list()**

Return list of all elements of this group.

EXAMPLES:

```
sage: G = PermutationGroup([[1,2,3,4]], [(1,2)])
sage: G.list()
[(), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3), (1,2,3,4), (1,2,4,3)]

sage: G = PermutationGroup([(('a','b')]], domain=('a', 'b')); G
Permutation Group with generators [(('a','b')]]
sage: G.list()
[(), (('a','b'))]
```

**lower\_central\_series()**

Return the lower central series of this group as a list of permutation groups.

EXAMPLES:

These computations use pseudo-random numbers, so we set the seed for reproducible testing.

```
sage: set_random_seed(0)
sage: G = PermutationGroup([(1,2,3), (4,5)], [(3,4)])
sage: G.lower_central_series() # random output
[Permutation Group with generators [(1,2,3)(4,5), (3,4)], Permutation Group with generators
```

**molien\_series()**

Returns the Molien series of a transitive permutation group. The function

$$M(x) = (1/|G|) \sum_{g \in G} \det(1 - x * g)^{-1}$$

is sometimes called the “Molien series” of  $G$ . GAP’s `MolienSeries` is associated to a character of a group  $G$ . How are these related? A group  $G$ , given as a permutation group on  $n$  points, has a “natural” representation of dimension  $n$ , given by permutation matrices. The Molien series of  $G$  is the one associated to that permutation representation of  $G$  using the above formula. Character values then count fixed points of the corresponding permutations.

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.molien_series()
1/(-x^15 + x^14 + x^13 - x^10 - x^9 - x^8 + x^7 + x^6 + x^5 - x^2 - x + 1)
sage: G = SymmetricGroup(3)
sage: G.molien_series()
1/(-x^6 + x^5 + x^4 - x^2 - x + 1)
```

**non\_fixed\_points()**

Returns the list of points not fixed by `self`, i.e., the subset of `self.domain()` moved by some element of `self`.

EXAMPLES:

```

sage: G = PermutationGroup([[ (3,4,5) ], [ (7,10) ]])
sage: G.non_fixed_points()
[3, 4, 5, 7, 10]
sage: G = PermutationGroup([[ (2,3,6) ], [ (9,) ]]) # note: 9 is fixed
sage: G.non_fixed_points()
[2, 3, 6]

```

**normal\_subgroups()**

Return the normal subgroups of this group as a (sorted in increasing order) list of permutation groups.

The normal subgroups of  $H = PSL(2,7) \times PSL(2,7)$  are 1, two copies of  $PSL(2,7)$  and  $H$  itself, as the following example shows.

**EXAMPLES:**

```

sage: G = PSL(2,7)
sage: D = G.direct_product(G)
sage: H = D[0]
sage: NH = H.normal_subgroups()
sage: len(NH)
4
sage: NH[1].is_isomorphic(G)
True
sage: NH[2].is_isomorphic(G)
True

```

**normalizer(g)**

Returns the normalizer of  $g$  in self.

**EXAMPLES:**

```

sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4) ]])
sage: g = G([ (1,3) ])
sage: G.normalizer(g)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(2,4),
sage: g = G([ (1,2,3,4) ])
sage: G.normalizer(g)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(2,4),
sage: H = G.subgroup([G([ (1,2,3,4) ])])
sage: G.normalizer(H)
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4)]) generated by [(2,4),

```

**normalizes(other)**

Returns True if the group *other* is normalized by self. Wraps GAP's IsNormal function.

A group  $G$  normalizes a group  $U$  if and only if for every  $g \in G$  and  $u \in U$  the element  $u^g$  is a member of  $U$ . Note that  $U$  need not be a subgroup of  $G$ .

**EXAMPLES:**

```

sage: G = PermutationGroup([' (1,2,3) (4,5) '])
sage: H = PermutationGroup([' (1,2,3) (4,5) ', ' (1,2,3,4,5) '])
sage: H.normalizes(G)
False
sage: G = SymmetricGroup(3)
sage: H = PermutationGroup([ (4,5,6) ])
sage: G.normalizes(H)
True
sage: H.normalizes(G)
True

```

In the last example,  $G$  and  $H$  are disjoint, so each normalizes the other.

**orbit** (*point*, *action*='OnPoints')

Return the orbit of a point under a group action.

INPUT:

- *point* – can be a point or any of the list above, depending on the action to be considered.
- *action* – string. if *point* is an element from the domain, a tuple of elements of the domain, a tuple of tuples [...], this variable describes how the group is acting.

The actions currently available through this method are "OnPoints", "OnTuples", "OnSets", "OnPairs", "OnSetsSets", "OnSetsDisjointSets", "OnSetsTuples", "OnTuplesSets", "OnTuplesTuples". They are taken from GAP's list of group actions, see `gap.help('Group Actions')`.

It is set to "OnPoints" by default. See below for examples.

OUTPUT:

The orbit of *point* as a tuple. Each entry is an image under the action of the permutation group, if necessary converted to the corresponding container. That is, if *action*='OnSets' then each entry will be a set even if *point* was given by a list/tuple/iterable.

EXAMPLES:

```
sage: G = PermutationGroup([ [(3,4)], [(1,3)] ])
sage: G.orbit(3)
(3, 4, 1)
sage: G = PermutationGroup([ [(1,2), (3,4)], [(1,2,3,4,10)] ])
sage: G.orbit(3)
(3, 4, 10, 1, 2)
sage: G = PermutationGroup([ [('c','d')], [('a','c')] ])
sage: G.orbit('a')
('a', 'c', 'd')
```

Action of  $S_3$  on sets:

```
sage: S3 = groups.permutation.Symmetric(3)
sage: S3.orbit((1,2), action = "OnSets")
({1, 2}, {2, 3}, {1, 3})
```

On tuples:

```
sage: S3.orbit((1,2), action = "OnTuples")
((1, 2), (2, 3), (2, 1), (3, 1), (1, 3), (3, 2))
```

Action of  $S_4$  on sets of disjoint sets:

```
sage: S4 = groups.permutation.Symmetric(4)
sage: S4.orbit(((1,2), (3,4)), action = "OnSetsDisjointSets")
({{1, 2}, {3, 4}}, {{2, 3}, {1, 4}}, {{1, 3}, {2, 4}})
```

Action of  $S_4$  (on a nonstandard domain) on tuples of sets:

```
sage: S4 = PermutationGroup([ [('c','d')], [('a','c')], [('a','b')] ])
sage: S4.orbit(((('a','c'), ('b','d'))), "OnTuplesSets")
(((('a', 'c'), ('b', 'd')),
  (('a', 'd'), ('c', 'b')),
  (('c', 'b'), ('a', 'd')),
  (('b', 'd'), ('a', 'c')),
  (('c', 'd'), ('a', 'b')),
  (('a', 'b'), ('c', 'd'))))
```

Action of  $S_4$  (on a very nonstandard domain) on tuples of sets:

```
sage: S4 = PermutationGroup([ [(11, (12, 13)), 'd'],
...                             [(12, (12, 11)), (11, (12, 13))], [(12, (12, 11)), 'b'] ])
sage: S4.orbit([(11, (12, 13)), (12, (12, 11)), ('b', 'd')], "OnTuplesSets")
([(11, (12, 13)), (12, (12, 11)), {'b', 'd'}],
 [{'d', (12, (12, 11))}, {(11, (12, 13)), 'b'}],
 [{(11, (12, 13)), 'b'}, {'d', (12, (12, 11))}],
 [{(11, (12, 13)), 'd'}, {'b', (12, (12, 11))}],
 [{'b', 'd'}, {(11, (12, 13)), (12, (12, 11))}],
 [{'b', (12, (12, 11))}, {(11, (12, 13)), 'd'}])
```

### `orbits()`

Returns the orbits of the elements of the domain under the default group action.

EXAMPLES:

```
sage: G = PermutationGroup([ [(3, 4)], [(1, 3)] ])
sage: G.orbits()
[[1, 3, 4], [2]]
sage: G = PermutationGroup([ [(1, 2), (3, 4)], [(1, 2, 3, 4, 10)] ])
sage: G.orbits()
[[1, 2, 3, 4, 10], [5], [6], [7], [8], [9]]

sage: G = PermutationGroup([ [('c', 'd')], [('a', 'c')], [('b', )]])
sage: G.orbits()
[['a', 'c', 'd'], ['b']]
```

The answer is cached:

```
sage: G.orbits() is G.orbits()
True
```

AUTHORS:

•Nathan Dunfield

### `order()`

Return the number of elements of this group.

EXAMPLES:

```
sage: G = PermutationGroup([ [(1, 2, 3), (4, 5)], [(1, 2)] ])
sage: G.order()
12
sage: G = PermutationGroup([ () ])
sage: G.order()
1
sage: G = PermutationGroup([ ])
sage: G.order()
1
```

### `poincare_series(p=2, n=10)`

Returns the Poincare series of  $G \bmod p$  ( $p \geq 2$  must be a prime), for  $n$  large. In other words, if you input a finite group  $G$ , a prime  $p$ , and a positive integer  $n$ , it returns a quotient of polynomials  $f(x) = P(x)/Q(x)$  whose coefficient of  $x^k$  equals the rank of the vector space  $H_k(G, \mathbf{Z}/p\mathbf{Z})$ , for all  $k$  in the range  $1 \leq k \leq n$ .

REQUIRES: GAP package HAP (in gap\_packages-\*.spkg).

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: G.poincare_series(2,10)                # optional - gap_packages
(x^2 + 1)/(x^4 - x^3 - x + 1)
sage: G = SymmetricGroup(3)
sage: G.poincare_series(2,10)                # optional - gap_packages
1/(-x + 1)
```

AUTHORS:

•David Joyner and Graham Ellis

**quotient** (*N*)

Returns the quotient of this permutation group by the normal subgroup *N*, as a permutation group.

Wraps the GAP operator “/”.

EXAMPLES:

```
sage: G = PermutationGroup([(1,2,3), (2,3)])
sage: N = PermutationGroup([(1,2,3)])
sage: G.quotient(N)
Permutation Group with generators [(1,2)]
sage: G.quotient(G)
Permutation Group with generators [()]
```

**random\_element** ()

Return a random element of this group.

EXAMPLES:

```
sage: G = PermutationGroup([(1,2,3), (4,5)], [(1,2)])
sage: a = G.random_element()
sage: a in G
True
sage: a.parent() is G
True
sage: a^6
()
```

**semidirect\_product** (*N*, *mapping*, *check=True*)

The semidirect product of *self* with *N*.

INPUT:

- N* - A group which is acted on by *self* and naturally embeds as a normal subgroup of the returned semidirect product.
- mapping* - A pair of lists that together define a homomorphism,  $\phi : \text{self} \rightarrow \text{Aut}(N)$ , by giving, in the second list, the images of the generators of *self* in the order given in the first list.
- check* - A boolean that, if set to False, will skip the initial tests which are made on *mapping*. This may be beneficial for large *N*, since in such cases the injectivity test can be expensive. Set to True by default.

OUTPUT:

The semidirect product of *self* and *N* defined by the action of *self* on *N* given in *mapping* (note that a homomorphism from *A* to the automorphism group of *B* is equivalent to an action of *A* on the *B*’s underlying set). The semidirect product of two groups, *H* and *N*, is a construct similar to the direct product in so far as the elements are the Cartesian product of the elements of *H* and the elements of *N*.



The operation, however, is built upon an action of  $H$  on  $N$ , and is defined as such:

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1^{h_2} n_2)$$

This function is a wrapper for GAP's `SemidirectProduct` command. The permutation group returned is built upon a permutation representation of the semidirect product of `self` and `N` on a set of size  $|N|$ . The generators of `N` are given as their right regular representations, while the generators of `self` are defined by the underlying action of `self` on `N`. It should be noted that the defining action is not always faithful, and in this case the inputted representations of the generators of `self` are placed on additional letters and adjoined to the output's generators of `self`.

EXAMPLES:

Perhaps the most common example of a semidirect product comes from the family of dihedral groups. Each dihedral group is the semidirect product of  $C_2$  with  $C_n$ , where, by convention,  $3 \leq n$ . In this case, the nontrivial element of  $C_2$  acts on  $C_n$  so as to send each element to its inverse.

```
sage: C2 = CyclicPermutationGroup(2)
sage: C8 = CyclicPermutationGroup(8)
sage: alpha = PermutationGroupMorphism_im_gens(C8, C8, [(1, 8, 7, 6, 5, 4, 3, 2)])
sage: S = C2.semirect_product(C8, [(1, 2)], [alpha])
sage: S == DihedralGroup(8)
False
sage: S.is_isomorphic(DihedralGroup(8))
True
sage: S.gens()
[(3, 4, 5, 6, 7, 8, 9, 10), (1, 2) (4, 10) (5, 9) (6, 8)]
```

A more complicated example can be drawn from [\[THOMAS-WOODS\]](#). It is there given that a semidirect product of  $D_4$  and  $C_3$  is isomorphic to one of  $C_2$  and the dicyclic group of order 12. This nonabelian group of order 24 has very similar structure to the dicyclic and dihedral groups of order 24, the three being the only groups of order 24 with a two-element center and 9 conjugacy classes.

```
sage: D4 = DihedralGroup(4)
sage: C3 = CyclicPermutationGroup(3)
sage: alpha1 = PermutationGroupMorphism_im_gens(C3, C3, [(1, 3, 2)])
sage: alpha2 = PermutationGroupMorphism_im_gens(C3, C3, [(1, 2, 3)])
sage: S1 = D4.semirect_product(C3, [(1, 2, 3, 4), (1, 3)], [alpha1, alpha2])
sage: C2 = CyclicPermutationGroup(2)
sage: Q = DiCyclicGroup(3)
sage: a = Q.gens()[0]; b = Q.gens()[1].inverse()
sage: alpha = PermutationGroupMorphism_im_gens(Q, Q, [a, b])
sage: S2 = C2.semirect_product(Q, [(1, 2)], [alpha])
sage: S1.is_isomorphic(S2)
True
sage: S1.is_isomorphic(DihedralGroup(12))
False
sage: S1.is_isomorphic(DiCyclicGroup(6))
False
sage: S1.center()
Subgroup of (Permutation Group with generators
[(5, 6, 7), (1, 2, 3, 4) (6, 7), (1, 3)]) generated by [(1, 3) (2, 4)]
sage: len(S1.conjugacy_classes_representatives())
9
```

If your normal subgroup is large, and you are confident that your inputs will successfully create a semidirect product, then it is beneficial, for the sake of time efficiency, to set the `check` parameter to `False`.

```
sage: C2 = CyclicPermutationGroup(2)
sage: C2000 = CyclicPermutationGroup(500)
```

```
sage: alpha = PermutationGroupMorphism(C2000,C2000,[C2000.gen().inverse()])
sage: S = C2.semidirect_product(C2000,[[ (1,2)], [alpha]], check=False)
```

**TESTS:**

```
sage: C3 = CyclicPermutationGroup(3)
sage: D4 = DihedralGroup(4)
sage: alpha = PermutationGroupMorphism(C3,C3,[C3("(1,3,2)")] )
sage: alpha1 = PermutationGroupMorphism(C3,C3,[C3("(1,2,3)")] )

sage: s = D4.semidirect_product('junk', [[(1,2,3,4),(1,2)], [alpha, alpha1]])
Traceback (most recent call last):
...
TypeError: junk is not a permutation group

sage: s = D4.semidirect_product(C3, [[(1,2,3,4),(1,2)], [alpha, alpha1]])
Traceback (most recent call last):
...
ValueError: the generator list must generate the calling group, [(1, 2, 3, 4), (1, 2)]
does not generate Dihedral group of order 8 as a permutation group

sage: s = D4.semidirect_product(C3, [[(1,2,3,4),(1,3)], [alpha]])
Traceback (most recent call last):
...
ValueError: the list of generators and the list of morphisms must be of equal length

sage: alpha2 = PermutationGroupMorphism(C3, D4, [D4("()")])
sage: s = D4.semidirect_product(C3, [[(1,2,3,4),(1,3)], [alpha, alpha2]])
Traceback (most recent call last):
...
ValueError: an element of the automorphism list is not an endomorphism (and is therefore not an

sage: alpha3 = PermutationGroupMorphism(C3,C3,[C3("()")])
sage: s = D4.semidirect_product(C3, [[(1,2,3,4),(1,3)], [alpha, alpha3]])
Traceback (most recent call last):
...
ValueError: an element of the automorphism list is not an injection (and is therefore not an
```

**REFERENCES:****AUTHOR:**

•Kevin Halasz (2012-8-12)

**smallest\_moved\_point()**

Return the smallest point moved by a permutation in this group.

**EXAMPLES:**

```
sage: G = PermutationGroup([[ (3,4)], [ (2,3,4) ]])
sage: G.smallest_moved_point()
2
sage: G = PermutationGroup([[ (1,2), (3,4) ], [ (1,2,3,4,10) ]])
sage: G.smallest_moved_point()
1
```

Note that this function uses the ordering from the domain:

```
sage: S = SymmetricGroup(['a', 'b', 'c'])
sage: S.smallest_moved_point()
'a'
```

**socle()**

Returns the socle of *self*. The socle of a group *G* is the subgroup generated by all minimal normal subgroups.

**EXAMPLES:**

```
sage: G=SymmetricGroup(4)
```

```
sage: G.socle()
```

Subgroup of (Symmetric group of order 4! as a permutation group) generated by [(1,2)(3,4), (1,2,3,4)]

```
sage: G.socle().socle()
```

Subgroup of (Subgroup of (Symmetric group of order 4! as a permutation group) generated by [(1,2)(3,4), (1,2,3,4)]

**solvable\_radical()**

Returns the solvable radical of *self*. The solvable radical (or just radical) of a group *G* is the largest solvable normal subgroup of *G*.

**EXAMPLES:**

```
sage: G=SymmetricGroup(4)
```

```
sage: G.solvable_radical()
```

Subgroup of (Symmetric group of order 4! as a permutation group) generated by [(1,2), (1,2,3,4)]

```
sage: G=SymmetricGroup(5)
```

```
sage: G.solvable_radical()
```

Subgroup of (Symmetric group of order 5! as a permutation group) generated by [(1,2,3,4,5)]

**stabilizer(*point*)**

Return the subgroup of *self* which stabilize the given position. *self* and its stabilizers must have same degree.

**EXAMPLES:**

```
sage: G = PermutationGroup([ [(3,4)], [(1,3)] ])
sage: G.stabilizer(1)
```

```
Subgroup of (Permutation Group with generators [(3,4), (1,3)] generated by [(3,4)]
```

```
sage: G.stabilizer(3)
```

```
Subgroup of (Permutation Group with generators [(3,4), (1,3)] generated by [(1,4)]
```

```
sage: G = PermutationGroup([ [(1,2), (3,4)], [(1,2,3,4,10)] ])
sage: G.stabilizer(10)
```

```
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4,10)] generated by [(2,3,4,10)]
```

```
sage: G.stabilizer(1)
```

```
Subgroup of (Permutation Group with generators [(1,2)(3,4), (1,2,3,4,10)] generated by [(2,3,4,10)]
```

```
sage: G = PermutationGroup([ [(2,3,4)], [(6,7)] ])
sage: G.stabilizer(1)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7), (2,3,4)]
```

```
sage: G.stabilizer(2)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7)]
```

```
sage: G.stabilizer(3)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7)]
```

```
sage: G.stabilizer(4)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7)]
```

```
sage: G.stabilizer(5)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7), (2,3,4)]
```

```
sage: G.stabilizer(6)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(2,3,4)]
```

```
sage: G.stabilizer(7)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(2,3,4)]
```

```
sage: G.stabilizer(8)
```

```
Subgroup of (Permutation Group with generators [(6,7), (2,3,4)] generated by [(6,7), (2,3,4)]
```

```

sage: G = PermutationGroup([ ['c','d'], ['a','c'] ], domain='abcd')
sage: G.stabilizer('a')
Subgroup of (Permutation Group with generators [('c','d'), ('a','c')]) generated by [('c','d')]
sage: G.stabilizer('b')
Subgroup of (Permutation Group with generators [('c','d'), ('a','c')]) generated by [('c','d')]
sage: G.stabilizer('c')
Subgroup of (Permutation Group with generators [('c','d'), ('a','c')]) generated by [('a','c')]
sage: G.stabilizer('d')
Subgroup of (Permutation Group with generators [('c','d'), ('a','c')]) generated by [('a','c')]

```

#### **strong\_generating\_system**(*base\_of\_group=None*)

Return a Strong Generating System of *self* according the given base for the right action of *self* on itself.

*base\_of\_group* is a list of the positions on which *self* acts, in any order. The algorithm returns a list of transversals and each transversal is a list of permutations. By default, *base\_of\_group* is  $[1, 2, 3, \dots, d]$  where  $d$  is the degree of the group.

For *base\_of\_group* =  $[pos_1, pos_2, \dots, pos_d]$  let  $G_i$  be the subgroup of  $G = \text{self}$  which stabilizes  $pos_1, pos_2, \dots, pos_i$ , so

$$G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n = \{e\}$$

Then the algorithm returns  $[G_i.\text{transversals}(pos_{i+1})]_{1 \leq i \leq n}$

INPUT:

- *base\_of\_group* (optional) – default:  $[1, 2, 3, \dots, d]$  – a list containing the integers  $1, 2, \dots, d$  in any order ( $d$  is the degree of *self*)

OUTPUT:

- A list of lists of permutations from the group, which form a strong generating system.

TESTS:

```

sage: G = SymmetricGroup(10)
sage: H = PermutationGroup([G.random_element() for i in range(randrange(1,3,1))])
sage: prod(map(lambda x : len(x), H.strong_generating_system()),1) == H.cardinality()
True

```

EXAMPLES:

```

sage: G = PermutationGroup([[(7,8)],[(3,4)],[(4,5)]]])
sage: G.strong_generating_system()
[[()], [(3,4,5), (3,5)], [(4,5)], [()], [(7,8)], [()]]
sage: G = PermutationGroup([[(1,2,3,4)],[(1,2)]]])
sage: G.strong_generating_system()
[[()], (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)], [(2,3,4), (2,4,3)], [(3,4)], [()]]
sage: G = PermutationGroup([[(1,2,3)],[(4,5,7)],[(1,4,6)]]])
sage: G.strong_generating_system()
[[()], (1,2,3), (1,4,6), (1,3,2), (1,5,7,4,6), (1,6,4), (1,7,5,4,6)], [(2,6,3), (2,5,7,6,4)], [(3,4,5)]]
sage: G = PermutationGroup([[(1,2,3)],[(2,3,4)],[(3,4,5)]]])
sage: G.strong_generating_system([5,4,3,2,1])
[[()], (1,5,3,4,2), (1,5,4,3,2), (1,5)(2,3), (1,5,2)], [()], (1,3)(2,4), (1,2)(3,4), (1,4)(2,3)]
sage: G = PermutationGroup([[(3,4)]]])
sage: G.strong_generating_system()
[[()], [()], [()], (3,4)], [()]]
sage: G.strong_generating_system(base_of_group=[3,1,2,4])
[[()], (3,4)], [()], [()], [()]]
sage: G = TransitiveGroup(12,17) # optional

```

```
sage: G.strong_generating_system() # optional
[[(), (1, 4, 11, 2) (3, 6, 5, 8) (7, 10, 9, 12), (1, 8, 3, 2) (4, 11, 10, 9) (5, 12, 7, 6), (1, 7) (2, 8) (3, 9) (4, 10) (
```

**structure\_description** (*latex=False*)

Return a string that tries to describe the structure of self.

This methods wraps GAP's StructureDescription method.

Requires the *optional* database\_gap package.

For full details, including the form of the returned string and the algorithm to build it, see [GAP's documentation](#).

INPUT:

- *latex* – a boolean (default: False). If True return a LaTeX formatted string.

OUTPUT:

- string

**Warning:** From GAP's documentation: The string returned by StructureDescription is **not** an isomorphism invariant: non-isomorphic groups can have the same string value, and two isomorphic groups in different representations can produce different strings.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(6)
sage: G.structure_description() # optional - database_gap
'C6'
sage: G.structure_description(latex=True) # optional - database_gap
'C_{6}'
sage: G2 = G.direct_product(G, maps=False)
sage: LatexExpr(G2.structure_description(latex=True)) # optional - database_gap
C_{6} \times C_{6}
```

This method is mainly intended for small groups or groups with few normal subgroups. Even then there are some surprises:

```
sage: D3 = DihedralGroup(3)
sage: D3.structure_description() # optional - database_gap
'S3'
```

We use the Sage notation for the degree of dihedral groups:

```
sage: D4 = DihedralGroup(4)
sage: D4.structure_description() # optional - database_gap
'D4'
```

**subgroup** (*gens=None, gap\_group=None, domain=None, category=None, canonicalize=True, check=True*)

Wraps the `PermutationGroup_subgroup` constructor. The argument *gens* is a list of elements of self.

EXAMPLES:

```
sage: G = PermutationGroup([(1, 2, 3), (3, 4, 5)])
sage: g = G((1, 2, 3))
sage: G.subgroup([g])
Subgroup of (Permutation Group with generators [(3, 4, 5), (1, 2, 3)]) generated by [(1, 2, 3)]
```

**subgroups()**

Returns a list of all the subgroups of `self`.

OUTPUT:

Each possible subgroup of `self` is contained once in the returned list. The list is in order, according to the size of the subgroups, from the trivial subgroup with one element on through up to the whole group. Conjugacy classes of subgroups are contiguous in the list.

**Warning:** For even relatively small groups this method can take a very long time to execute, or create vast amounts of output. Likely both. Its purpose is instructional, as it can be useful for studying small groups. The 156 subgroups of the full symmetric group on 5 symbols of order 120,  $S_5$ , can be computed in about a minute on commodity hardware in 2011. The 64 subgroups of the cyclic group of order  $30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$  takes about twice as long. For faster results, which still exhibit the structure of the possible subgroups, use `conjugacy_classes_subgroups()`.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
```

```
sage: G.subgroups()
```

```
[Subgroup of (Symmetric group of order 3! as a permutation group) generated by [()],
 Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(2,3)],
 Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(1,2)],
 Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(1,3)],
 Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(1,2,3)],
 Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(2,3), (1,2,3)]]
```

```
sage: G = CyclicPermutationGroup(14)
```

```
sage: G.subgroups()
```

```
[Subgroup of (Cyclic group of order 14 as a permutation group) generated by [()],
 Subgroup of (Cyclic group of order 14 as a permutation group) generated by [(1,8)(2,9)(3,10)(4,11)(5,12)(6,13)],
 Subgroup of (Cyclic group of order 14 as a permutation group) generated by [(1,3,5,7,9,11,13)],
 Subgroup of (Cyclic group of order 14 as a permutation group) generated by [(1,2,3,4,5,6,7,8)]]
```

AUTHOR:

•Rob Beezer (2011-01-24)

**syLOW\_subgroup(p)**

Returns a Sylow  $p$ -subgroup of the finite group  $G$ , where  $p$  is a prime. This is a  $p$ -subgroup of  $G$  whose index in  $G$  is coprime to  $p$ . Wraps the GAP function `SylowSubgroup`.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)', '(2,3)'])
```

```
sage: G.sylow_subgroup(2)
```

```
Subgroup of (Permutation Group with generators [(2,3), (1,2,3)]) generated by [(2,3)]
```

```
sage: G.sylow_subgroup(5)
```

```
Subgroup of (Permutation Group with generators [(2,3), (1,2,3)]) generated by [()]
```

TESTS:

Implementation details should not prevent us from computing large subgroups (trac #5491):

```
sage: PSL(10,2).syLOW_subgroup(7)
```

```
Subgroup of...
```

**transversals(point)**

If  $G$  is a permutation group acting on the set  $X = \{1, 2, \dots, n\}$  and  $H$  is the stabilizer subgroup of

<integer>, a right (respectively left) transversal is a set containing exactly one element from each right (respectively left) coset of  $H$ . This method returns a right transversal of `self` by the stabilizer of `self` on <integer> position.

EXAMPLES:

```
sage: G = PermutationGroup([ [(3,4)], [(1,3)] ])
sage: G.transversals(1)
[(), (1,3,4), (1,4,3)]
sage: G = PermutationGroup([ [(1,2), (3,4)], [(1,2,3,4,10)] ])
sage: G.transversals(1)
[(), (1,2)(3,4), (1,3,2,10,4), (1,4,2,10,3), (1,10,4,3,2)]

sage: G = PermutationGroup([ [('c','d')], [('a','c')] ])
sage: G.transversals('a')
[(), ('a','c','d'), ('a','d','c')]
```

**trivial\_character()**

Returns the trivial character of `self`.

EXAMPLES:

```
sage: SymmetricGroup(3).trivial_character()
Character of Symmetric group of order 3! as a permutation group
```

**upper\_central\_series()**

Return the upper central series of this group as a list of permutation groups.

EXAMPLES:

These computations use pseudo-random numbers, so we set the seed for reproducible testing:

```
sage: G = PermutationGroup([ [(1,2,3), (4,5)], [(3,4)] ])
sage: G.upper_central_series()
[Subgroup of (Permutation Group with generators [(3,4), (1,2,3)(4,5)]) generated by [()]]
```

```
class sage.groups.perm_gps.permgroup.PermutationGroup_subgroup(ambient,
                                                                gens=None,
                                                                gap_group=None,
                                                                domain=None,
                                                                category=None,
                                                                canonicalize=True,
                                                                check=True)
```

Bases: `sage.groups.perm_gps.permgroup.PermutationGroup_generic`

Subgroup subclass of `PermutationGroup_generic`, so instance methods are inherited.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
sage: gens = G.gens()
sage: H = DihedralGroup(4)
sage: H.subgroup(gens)
Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4)]
sage: K = H.subgroup(gens)
sage: K.list()
[(), (1,2,3,4), (1,3)(2,4), (1,4,3,2)]
sage: K.ambient_group()
Dihedral group of order 8 as a permutation group
sage: K.gens()
[(1,2,3,4)]
```

**ambient\_group()**

Return the ambient group related to self.

EXAMPLES:

An example involving the dihedral group on four elements,  $D_8$ :

```
sage: G = DihedralGroup(4)
sage: H = CyclicPermutationGroup(4)
sage: gens = H.gens()
sage: S = PermutationGroup_subgroup(G, list(gens))
sage: S.ambient_group()
Dihedral group of order 8 as a permutation group
sage: S.ambient_group() == G
True
```

**is\_normal(other=None)**

Return True if this group is a normal subgroup of other. If other is not specified, then it is assumed to be the ambient group.

EXAMPLES:

```
sage: S = SymmetricGroup(['a', 'b', 'c'])
sage: H = S.subgroup(['a', 'b', 'c']); H
Subgroup of (Symmetric group of order 3! as a permutation group) generated by ['a', 'b', 'c']
sage: H.is_normal()
True
```

**sage.groups.perm\_gps.permgroup.direct\_product\_permgroups(P)**

Takes the direct product of the permutation groups listed in P.

EXAMPLES:

```
sage: G1 = AlternatingGroup([1, 2, 4, 5])
sage: G2 = AlternatingGroup([3, 4, 6, 7])
sage: D = direct_product_permgroups([G1, G2, G1])
sage: D.order()
1728
sage: D = direct_product_permgroups([G1])
sage: D==G1
True
sage: direct_product_permgroups([])
Symmetric group of order 0! as a permutation group
```

**sage.groups.perm\_gps.permgroup.from\_gap\_list(G, src)**

Convert a string giving a list of GAP permutations into a list of elements of G.

EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup import from_gap_list
sage: G = PermutationGroup([(1, 2, 3), (4, 5)], [(3, 4)])
sage: L = from_gap_list(G, "[ (1, 2, 3) (4, 5), (3, 4) ]"); L
[(1, 2, 3) (4, 5), (3, 4)]
sage: L[0].parent() is G
True
sage: L[1].parent() is G
True
```

**sage.groups.perm\_gps.permgroup.hap\_decorator(f)**

A decorator for permutation group methods that require HAP. It checks to see that HAP is installed as well as checks that the argument p is either 0 or prime.



## EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup import hap_decorator
sage: def foo(self, n, p=0): print "Done"
sage: foo = hap_decorator(foo)
sage: foo(None, 3)      #optional - gap_packages
Done
sage: foo(None, 3, 0) # optional - gap_packages
Done
sage: foo(None, 3, 5) # optional - gap_packages
Done
sage: foo(None, 3, 4) #optional - gap_packages
Traceback (most recent call last):
...
ValueError: p must be 0 or prime
```

`sage.groups.perm_gps.permgroup.load_hap()`

Load the GAP hap package into the default GAP interpreter interface. If this fails, try one more time to load it.

## EXAMPLES:

```
sage: sage.groups.perm_gps.permgroup.load_hap() # optional - gap_packages
```



# “NAMED” PERMUTATION GROUPS (SUCH AS THE SYMMETRIC GROUP, S\_N)

You can construct the following permutation groups:

- **SymmetricGroup**,  $S_n$  of order  $n!$  (n can also be a list  $X$  of distinct positive integers, in which case it returns  $\$S\_X\$$ )
- **AlternatingGroup**,  $A_n$  or order  $n!/2$  (n can also be a list  $X$  of distinct positive integers, in which case it returns  $\$A\_X\$$ )
- **DihedralGroup**,  $D_n$  of order  $2n$
- **GeneralDihedralGroup**,  $Dih(G)$ , where  $G$  is an abelian group
- **CyclicPermutationGroup**,  $C_n$  of order  $n$
- **DiCyclicGroup**, nonabelian groups of order  $4m$  with a unique element of order 2
- **TransitiveGroup**,  $n^{th}$  transitive group of degree  $d$  from the GAP tables of transitive groups (requires the “optional” package database\_gap)
- **TransitiveGroups(d)**, **TransitiveGroups()**, set of all of the above
- **PrimitiveGroup**,  $n^{th}$  primitive group of degree  $d$  from the GAP tables of primitive groups (requires the “optional” package database\_gap)
- **PrimitiveGroups(d)**, **PrimitiveGroups()**, set of all of the above
- **MathieuGroup(degree)**, Mathieu group of degree 9, 10, 11, 12, 21, 22, 23, or 24.
- **KleinFourGroup**, subgroup of  $S_4$  of order 4 which is not  $C_2 \times C_2$
- **QuaternionGroup**, non-abelian group of order 8,  $\{\pm 1, \pm I, \pm J, \pm K\}$
- **SplitMetacyclicGroup**, nonabelian groups of order  $p^m$  with cyclic subgroups of index  $p$
- **SemidihedralGroup**, nonabelian 2-groups with cyclic subgroups of index 2
- **PGL(n,q)**, projective general linear group of  $n \times n$  matrices over the finite field  $GF(q)$
- **PSL(n,q)**, projective special linear group of  $n \times n$  matrices over the finite field  $GF(q)$
- **PSp(2n,q)**, projective symplectic linear group of  $2n \times 2n$  matrices over the finite field  $GF(q)$
- **PSU(n,q)**, projective special unitary group of  $n \times n$  matrices having coefficients in the finite field  $GF(q^2)$  that respect a fixed nondegenerate sesquilinear form, of determinant 1.

- **PGU( $n, q$ )**, **projective general unitary group of  $n \times n$  matrices having** coefficients in the finite field  $\text{GF}(q^2)$  that respect a fixed nondegenerate sesquilinear form, modulo the centre.
- **SuzukiGroup( $q$ )**, Suzuki group over  $\text{GF}(q)$ ,  ${}^2B_2(2^{2k+1}) = Sz(2^{2k+1})$ .

**AUTHOR:**

- David Joyner (2007-06): split from permgp.py (suggested by Nick Alexander)

**REFERENCES:** Cameron, P., *Permutation Groups*. New York: Cambridge University Press, 1999. Wielandt, H., *Finite Permutation Groups*. New York: Academic Press, 1964. Dixon, J. and Mortimer, B., *Permutation Groups*, Springer-Verlag, Berlin/New York, 1996.

**NOTE:** Though Suzuki groups are okay, Ree groups should *not* be wrapped as permutation groups - the construction is too slow - unless (for small values or the parameter) they are made using explicit generators.

**class** sage.groups.perm\_gps.permgroup\_named.**AlternatingGroup**(domain=None)  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_symalt

The alternating group of order  $n!/2$ , as a permutation group.

**INPUT:**

$n$  – a positive integer, or list or tuple thereof

---

**Note:** This group is also available via `groups.permutation.Alternating()`.

---

**EXAMPLES:**

```
sage: G = AlternatingGroup(6)
sage: G.order()
360
sage: G
Alternating group of order 6!/2 as a permutation group
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run() # long time

sage: G = AlternatingGroup([1, 2, 4, 5])
sage: G
Alternating group of order 4!/2 as a permutation group
sage: G.domain()
{1, 2, 4, 5}
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run()
```

**TESTS:**

```
sage: groups.permutation.Alternating(6)
Alternating group of order 6!/2 as a permutation group
```

**class** sage.groups.perm\_gps.permgroup\_named.**CyclicPermutationGroup**( $n$ )  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

A cyclic group of order  $n$ , as a permutation group.

**INPUT:**  $n$  – a positive integer

---

**Note:** This group is also available via `groups.permutation.Cyclic()`.

---

**EXAMPLES:**

```

sage: G = CyclicPermutationGroup(8)
sage: G.order()
8
sage: G
Cyclic group of order 8 as a permutation group
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run()
sage: C = CyclicPermutationGroup(10)
sage: C.is_abelian()
True
sage: C = CyclicPermutationGroup(10)
sage: C.as_AbelianGroup()
Multiplicative Abelian group isomorphic to C2 x C5

```

**TESTS:**

```

sage: groups.permutation.Cyclic(6)
Cyclic group of order 6 as a permutation group

```

**as\_AbelianGroup()**

Returns the corresponding Abelian Group instance.

**EXAMPLES:**

```

sage: C = CyclicPermutationGroup(8)
sage: C.as_AbelianGroup()
Multiplicative Abelian group isomorphic to C8

```

**is\_abelian()**

Return True if this group is abelian.

**EXAMPLES:**

```

sage: C = CyclicPermutationGroup(8)
sage: C.is_abelian()
True

```

**is\_commutative()**

Return True if this group is commutative.

**EXAMPLES:**

```

sage: C = CyclicPermutationGroup(8)
sage: C.is_commutative()
True

```

**class** sage.groups.perm\_gps.permgroup\_named.**DiCyclicGroup**(*n*)

Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The dicyclic group of order  $4n$ , for  $n \geq 2$ .

**INPUT:**

- *n* – a positive integer, two or greater

**OUTPUT:**

This is a nonabelian group similar in some respects to the dihedral group of the same order, but with far fewer

elements of order 2 (it has just one). The permutation representation constructed here is based on the presentation

$$\langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$$

For  $n = 2$  this is the group of quaternions  $(\pm 1, \pm I, \pm J, \pm K)$ , which is the nonabelian group of order 8 that is not the dihedral group  $D_4$ , the symmetries of a square. For  $n = 3$  this is the nonabelian group of order 12 that is not the dihedral group  $D_6$  nor the alternating group  $A_4$ . This group of order 12 is also the semi-direct product of  $C_2$  by  $C_4$ ,  $C_3 \rtimes C_4$ . [CONRAD2009]

When the order of the group is a power of 2 it is known as a “generalized quaternion group.”

#### IMPLEMENTATION:

The presentation above means every element can be written as  $a^i x^j$  with  $0 \leq i < 2n$ ,  $j = 0, 1$ . We code  $a^i$  as the symbol  $i + 1$  and code  $a^i x$  as the symbol  $2n + i + 1$ . The two generators are then represented using a left regular representation.

---

**Note:** This group is also available via `groups.permutation.DiCyclic()`.

---

#### EXAMPLES:

A dicyclic group of order 384, with a large power of 2 as a divisor:

```
sage: n = 3*2^5
sage: G = DiCyclicGroup(n)
sage: G.order()
384
sage: a = G.gen(0)
sage: x = G.gen(1)
sage: a^(2*n)
()
sage: a^n==x^2
True
sage: x^-1*a*x==a^-1
True
```

A large generalized quaternion group (order is a power of 2):

```
sage: n = 2^10
sage: G=DiCyclicGroup(n)
sage: G.order()
4096
sage: a = G.gen(0)
sage: x = G.gen(1)
sage: a^(2*n)
()
sage: a^n==x^2
True
sage: x^-1*a*x==a^-1
True
```

Just like the dihedral group, the dicyclic group has an element whose order is half the order of the group. Unlike the dihedral group, the dicyclic group has only one element of order 2. Like the dihedral groups of even order, the center of the dicyclic group is a subgroup of order 2 (thus has the unique element of order 2 as its non-identity element).

```
sage: G=DiCyclicGroup(3*5*4)
sage: G.order()
240
sage: two = [g for g in G if g.order()==2]; two
```

```
[ (1,5) (2,6) (3,7) (4,8) (9,13) (10,14) (11,15) (12,16) ]
sage: G.center().order()
2
```

For small orders, we check this is really a group we do not have in Sage otherwise.

```
sage: G = DiCyclicGroup(2)
sage: H = DihedralGroup(4)
sage: G.is_isomorphic(H)
False
sage: G = DiCyclicGroup(3)
sage: H = DihedralGroup(6)
sage: K = AlternatingGroup(6)
sage: G.is_isomorphic(H) or G.is_isomorphic(K)
False
```

#### TESTS:

```
sage: groups.permutation.DiCyclic(6)
Dicyclic group of order 24 as a permutation group
```

#### REFERENCES:

#### AUTHOR:

- Rob Beezer (2009-10-18)

#### **is\_abelian()**

Return True if this group is abelian.

##### EXAMPLES:

```
sage: D = DiCyclicGroup(12)
sage: D.is_abelian()
False
```

#### **is\_commutative()**

Return True if this group is commutative.

##### EXAMPLES:

```
sage: D = DiCyclicGroup(12)
sage: D.is_commutative()
False
```

**class** sage.groups.perm\_gps.permgroup\_named.**DihedralGroup**(*n*)  
 Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The Dihedral group of order  $2n$  for any integer  $n \geq 1$ .

**INPUT:** *n* – a positive integer

**OUTPUT:** – the dihedral group of order  $2*n$ , as a permutation group

---

**Note:** This group is also available via `groups.permutation.Dihedral()`.

---

#### EXAMPLES:

```
sage: DihedralGroup(1)
Dihedral group of order 2 as a permutation group

sage: DihedralGroup(2)
Dihedral group of order 4 as a permutation group
sage: DihedralGroup(2).gens()
[(3,4), (1,2)]

sage: DihedralGroup(5).gens()
[(1,2,3,4,5), (1,5)(2,4)]
sage: list(DihedralGroup(5))
[(), (2,5)(3,4), (1,2)(3,5), (1,2,3,4,5), (1,3)(4,5), (1,3,5,2,4), (1,4)(2,3), (1,4,2,5,3), (1,5)(3,4)]

sage: G = DihedralGroup(6)
sage: G.order()
12
sage: G = DihedralGroup(5)
sage: G.order()
10
sage: G
Dihedral group of order 10 as a permutation group
sage: G.gens()
[(1,2,3,4,5), (1,5)(2,4)]

sage: DihedralGroup(0)
Traceback (most recent call last):
...
ValueError: n must be positive
```

**TESTS:**

```
sage: TestSuite(G).run()
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run()

sage: groups.permutation.Dihedral(6)
Dihedral group of order 12 as a permutation group
```

**class** sage.groups.perm\_gps.permgroup\_named.**GeneralDihedralGroup**(*factors*)  
Bases: sage.groups.perm\_gps.permgroup.PermutationGroup\_generic

The Generalized Dihedral Group generated by the abelian group with direct factors in the input list.

**INPUT:**

- *factors* - a list of the sizes of the cyclic factors of the abelian group being dihedralized (this will be sorted once entered)

**OUTPUT:**

For a given abelian group (noting that each finite abelian group can be represented as the direct product of cyclic groups), the General Dihedral Group it generates is simply the semi-direct product of the given group with  $C_2$ , where the nonidentity element of  $C_2$  acts on the abelian group by turning each element into its inverse. In this implementation, each input abelian group will be standardized so as to act on a minimal amount of letters. This will be done by breaking the direct factors into products of  $p$ -groups, before this new set of factors is ordered from smallest to largest for complete standardization. Note that the generalized dihedral group corresponding to a cyclic group,  $C_n$ , is simply the dihedral group  $D_n$ .

**EXAMPLES:**



As is noted in [1],  $Dih(C_3 \times C_3)$  has the presentation

$$\langle a, b, c \mid a^3, b^3, c^2, ab = ba, ac = ca^{-1}, bc = cb^{-1} \rangle$$

Note also the fact, verified by <sup>1</sup>, that the dihedralization of  $C_3 \times C_3$  is the only nonabelian group of order 18 with no element of order 6.

```
sage: G = GeneralDihedralGroup([3,3])
sage: G
Generalized dihedral group generated by C3 x C3
sage: G.order()
18
sage: G.gens()
[(4,5,6), (2,3)(5,6), (1,2,3)]
sage: a = G.gens()[2]; b = G.gens()[0]; c = G.gens()[1]
sage: a.order() == 3, b.order() == 3, c.order() == 2
(True, True, True)
sage: a*b == b*a, a*c == c*a.inverse(), b*c == c*b.inverse()
(True, True, True)
sage: G.subgroup([a,b,c]) == G
True
sage: G.is_abelian()
False
sage: all([x.order() != 6 for x in G])
True
```

If all of the direct factors are  $C_2$ , then the action turning each element into its inverse is trivial, and the semi-direct product becomes a direct product.

```
sage: G = GeneralDihedralGroup([2,2,2])
sage: G.order()
16
sage: G.gens()
[(7,8), (5,6), (3,4), (1,2)]
sage: G.is_abelian()
True
sage: H = KleinFourGroup()
sage: G.is_isomorphic(H.direct_product(H)[0])
True
```

If two nonidentical input lists generate isomorphic abelian groups, then they will generate identical groups (with each direct factor broken up into its prime factors), but they will still have distinct descriptions. Note that If  $\gcd(n, m) = 1$ , then  $C_n \times C_m \cong C_{nm}$ , while the general dihedral groups generated by isomorphic abelian groups should be themselves isomorphic.

```
sage: G = GeneralDihedralGroup([6,34,46,14])
sage: H = GeneralDihedralGroup([7,17,3,46,2,2,2])
sage: G == H, G.gens() == H.gens()
(True, True)
sage: [x.order() for x in G.gens()]
[23, 17, 7, 2, 3, 2, 2, 2, 2]
sage: G
Generalized dihedral group generated by C6 x C34 x C46 x C14
sage: H
Generalized dihedral group generated by C7 x C17 x C3 x C46 x C2 x C2 x C2
```

A cyclic input yields a Classical Dihedral Group.

<sup>1</sup> A.D. Thomas and G.V. Wood, Group Tables (Exeter: Shiva Publishing, 1980)

```
sage: G = GeneralDihedralGroup([6])
sage: D = DihedralGroup(6)
sage: G.is_isomorphic(D)
True
```

A Generalized Dihedral Group will always have size twice the underlying group, be solvable (as it has an abelian subgroup with index 2), and, unless the underlying group is of the form  $C_2^n$ , be nonabelian (by the structure theorem of finite abelian groups and the fact that a semi-direct product is a direct product only when the underlying action is trivial).

```
sage: G = GeneralDihedralGroup([6, 18, 33, 60])
sage: (6*18*33*60)*2
427680
sage: G.order()
427680
sage: G.is_solvable()
True
sage: G.is_abelian()
False
```

#### TESTS:

```
sage: G = GeneralDihedralGroup("foobar")
Traceback (most recent call last):
...
TypeError: factors of abelian group must be a list, not foobar

sage: GeneralDihedralGroup([])
Traceback (most recent call last):
...
ValueError: there must be at least one direct factor in the abelian group being dihedralized

sage: GeneralDihedralGroup([3, 1.5])
Traceback (most recent call last):
...
TypeError: the input list must consist of Integers

sage: GeneralDihedralGroup([4, -8])
Traceback (most recent call last):
...
ValueError: all direct factors must be greater than 1
```

#### REFERENCES:

##### AUTHOR:

•Kevin Halasz (2012-7-12)

**class** sage.groups.perm\_gps.permgroup\_named.**KleinFourGroup**  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The Klein 4 Group, which has order 4 and exponent 2, viewed as a subgroup of  $S_4$ .

**OUTPUT:** – the Klein 4 group of order 4, as a permutation group of degree 4.

---

**Note:** This group is also available via `groups.permutation.KleinFour()`.

---

#### EXAMPLES:

```
sage: G = KleinFourGroup(); G
The Klein 4 group of order 4, as a permutation group
sage: list(G)
[(), (3,4), (1,2), (1,2)(3,4)]
```

**TESTS:**

```
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run()

sage: groups.permutation.KleinFour()
The Klein 4 group of order 4, as a permutation group
```

**AUTHOR:** – Bobby Moretti (2006-10)

```
class sage.groups.perm_gps.permgroup_named.MathieuGroup(n)
Bases: sage.groups.perm_gps.permgroup_named.PermutationGroup_unique
```

The Mathieu group of degree  $n$ .

**INPUT:**  $n$  – a positive integer in  $\{9, 10, 11, 12, 21, 22, 23, 24\}$ .

**OUTPUT:** – the Mathieu group of degree  $n$ , as a permutation group

---

**Note:** This group is also available via `groups.permutation.Mathieu()`.

---

**EXAMPLES:**

```
sage: G = MathieuGroup(12)
sage: G
Mathieu group of degree 12 and order 95040 as a permutation group
```

**TESTS:**

```
sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run(skip=["_test_enumerated_set_contains", "_test_enumerated_set_iter_list"])

sage: groups.permutation.Mathieu(9)
Mathieu group of degree 9 and order 72 as a permutation group
```

Note: this is a fairly big group, so we skip some tests that currently require to list all the elements of the group, because there is no proper iterator yet.

```
class sage.groups.perm_gps.permgroup_named.PGL(n, q, name='a')
Bases: sage.groups.perm_gps.permgroup_named.PermutationGroup_plg
```

The projective general linear groups over  $\text{GF}(q)$ .

**INPUT:**  $n$  – positive integer; the degree  $q$  – prime power; the size of the ground field name – (default: ‘a’) variable name of indeterminate of finite field  $\text{GF}(q)$

**OUTPUT:**  $\text{PGL}(n, q)$

---

**Note:** This group is also available via `groups.permutation.PGL()`.

---

**EXAMPLES:**

```
sage: G = PGL(2, 3); G
Permutation Group with generators [(3, 4), (1, 2, 4)]
sage: print G
The projective general linear group of degree 2 over Finite Field of size 3
sage: G.base_ring()
Finite Field of size 3
sage: G.order()
24

sage: G = PGL(2, 9, 'b'); G
Permutation Group with generators [(3, 10, 9, 8, 4, 7, 6, 5), (1, 2, 4) (5, 6, 8) (7, 9, 10)]
sage: G.base_ring()
Finite Field in b of size 3^2

sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run() # long time
```

**TESTS:**

```
sage: groups.permutation.PGL(2, 3)
Permutation Group with generators [(3, 4), (1, 2, 4)]
```

**class** sage.groups.perm\_gps.permgroup\_named.**PGU**(*n, q, name='a'*)  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_pug

The projective general unitary groups over GF(*q*).

**INPUT:** *n* – positive integer; the degree *q* – prime power; the size of the ground field name – (default: 'a')  
variable name of indeterminate of finite field GF(*q*)

**OUTPUT:** PGU(*n, q*)

---

**Note:** This group is also available via groups.permutation.PGU().

---

**EXAMPLES:**

```
sage: PGU(2, 3)
The projective general unitary group of degree 2 over Finite Field of size 3

sage: G = PGU(2, 8, name='alpha'); G
The projective general unitary group of degree 2 over Finite Field in alpha of size 2^3
sage: G.base_ring()
Finite Field in alpha of size 2^3
```

**TESTS:**

```
sage: groups.permutation.PGU(2, 3)
The projective general unitary group of degree 2 over Finite Field of size 3
```

**class** sage.groups.perm\_gps.permgroup\_named.**PSL**(*n, q, name='a'*)  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_plg

The projective special linear groups over GF(*q*).

**INPUT:**

- *n* – positive integer; the degree
- *q* – either a prime power (the size of the ground field) or a finite field

•name – (default: ‘a’) variable name of indeterminate of finite field GF(q)

OUTPUT:

the group PSL(n,q)

---

**Note:** This group is also available via `groups.permutation.PSL()`.

---

EXAMPLES:

```
sage: G = PSL(2,3); G
Permutation Group with generators [(2,3,4), (1,2)(3,4)]
sage: G.order()
12
sage: G.base_ring()
Finite Field of size 3
sage: print G
The projective special linear group of degree 2 over Finite Field of size 3
```

We create two groups over nontrivial finite fields:

```
sage: G = PSL(2, 4, 'b'); G
Permutation Group with generators [(3,4,5), (1,2,3)]
sage: G.base_ring()
Finite Field in b of size 2^2
sage: G = PSL(2, 8); G
Permutation Group with generators [(3,8,6,4,9,7,5), (1,2,3)(4,7,5)(6,9,8)]
sage: G.base_ring()
Finite Field in a of size 2^3

sage: G.category()
Category of finite permutation groups
sage: TestSuite(G).run() # long time
```

TESTS:

```
sage: groups.permutation.PSL(2, 3)
Permutation Group with generators [(2,3,4), (1,2)(3,4)]
```

Check that [trac ticket #7424](#) is handled:

```
sage: PSL(2, GF(7, 'x'))
Permutation Group with generators [(3,7,5)(4,8,6), (1,2,6)(3,4,8)]
sage: PSL(2, GF(3))
Permutation Group with generators [(2,3,4), (1,2)(3,4)]
sage: PSL(2, QQ)
Traceback (most recent call last):
...
ValueError: q must be a prime power or a finite field
```

**ramification\_module\_decomposition\_hurwitz\_curve()**

Helps compute the decomposition of the ramification module for the Hurwitz curves  $X$  (over CC say) with automorphism group  $G = \text{PSL}(2,q)$ ,  $q$  a “Hurwitz prime” (ie,  $p$  is  $\pm 1 \pmod{7}$ ). Using this computation and Borne’s formula helps determine the  $G$ -module structure of the RR spaces of equivariant divisors can be determined explicitly.

The output is a list of integer multiplicities:  $[m_1, \dots, m_n]$ , where  $n$  is the number of conj classes of  $G = \text{PSL}(2,p)$  and  $m_i$  is the multiplicity of  $\pi_i$  in the ramification module of a Hurwitz curve with automorphism group  $G$ . Here  $\text{IrrRepns}(G) = [\pi_1, \dots, \pi_n]$  (in the order listed in the output of `self.character_table()`).

**REFERENCE:** David Joyner, Amy Ksir, Roger Vogeler, “Group representations on Riemann-Roch spaces of some Hurwitz curves,” preprint, 2006.

EXAMPLES:

```
sage: G = PSL(2,13)
sage: G.ramification_module_decomposition_hurwitz_curve() # random, optional - database_gap
[0, 7, 7, 12, 12, 12, 13, 15, 14]
```

This means, for example, that the trivial representation does not occur in the ramification module of a Hurwitz curve with automorphism group  $\text{PSL}(2,13)$ , since the trivial representation is listed first and that entry has multiplicity 0. The “randomness” is due to the fact that GAP randomly orders the conjugacy classes of the same order in the list of all conjugacy classes. Similarly, there is some randomness to the ordering of the characters.

If you try to use this function on a group  $\text{PSL}(2,q)$  where  $q$  is not a (smallish) “Hurwitz prime”, an error message will be printed.

**ramification\_module\_decomposition\_modular\_curve()**

Helps compute the decomposition of the ramification module for the modular curve  $X(p)$  (over  $\mathbb{C}$  say) with automorphism group  $G = \text{PSL}(2,q)$ ,  $q$  a prime  $> 5$ . Using this computation and Borne’s formula helps determine the  $G$ -module structure of the RR spaces of equivariant divisors can be determined explicitly.

The output is a list of integer multiplicities:  $[m_1, \dots, m_n]$ , where  $n$  is the number of conj classes of  $G = \text{PSL}(2,p)$  and  $m_i$  is the multiplicity of  $\pi_i$  in the ramification module of a modular curve with automorphism group  $G$ . Here  $\text{IrrRepns}(G) = [\pi_1, \dots, \pi_n]$  (in the order listed in the output of `self.character_table()`).

**REFERENCE:** D. Joyner and A. Ksir, ‘Modular representations on some Riemann-Roch spaces of modular curves  $X(N)$ ’, Computational Aspects of Algebraic Curves, (Editor: T. Shaska) Lecture Notes in Computing, WorldScientific, 2005.)

EXAMPLES:

```
sage: G = PSL(2,7)
sage: G.ramification_module_decomposition_modular_curve() # random, optional - database_gap
[0, 4, 3, 6, 7, 8]
```

This means, for example, that the trivial representation does not occur in the ramification module of  $X(7)$ , since the trivial representation is listed first and that entry has multiplicity 0. The “randomness” is due to the fact that GAP randomly orders the conjugacy classes of the same order in the list of all conjugacy classes. Similarly, there is some randomness to the ordering of the characters.

```
sage.groups.perm_gps.permgroup_named.PSP
alias of PSp
```

```
class sage.groups.perm_gps.permgroup_named.PSU(n, q, name='a')
Bases: sage.groups.perm_gps.permgroup_named.PermutationGroup_pug
```

The projective special unitary groups over  $\text{GF}(q)$ .

**INPUT:**  $n$  – positive integer; the degree  $q$  – prime power; the size of the ground field name – (default: ‘a’) variable name of indeterminate of finite field  $\text{GF}(q)$

**OUTPUT:**  $\text{PSU}(n,q)$

---

**Note:** This group is also available via `groups.permutation.PSU()`.

---

EXAMPLES:

**sage:** `PSU(2, 3)`

The projective special unitary group of degree 2 over Finite Field of size 3

**sage:** `G = PSU(2, 8, name='alpha'); G`

The projective special unitary group of degree 2 over Finite Field in alpha of size  $2^3$

**sage:** `G.base_ring()`

Finite Field in alpha of size  $2^3$

TESTS:

**sage:** `groups.permutation.PSU(2, 3)`

The projective special unitary group of degree 2 over Finite Field of size 3

**class** `sage.groups.perm_gps.permgroup_named.PSp(n, q, name='a')`

Bases: `sage.groups.perm_gps.permgroup_named.PermutationGroup_plg`

The projective symplectic linear groups over  $\text{GF}(q)$ .

**INPUT:**  $n$  – positive integer; the degree  $q$  – prime power; the size of the ground field name – (default: 'a') variable name of indeterminate of finite field  $\text{GF}(q)$

**OUTPUT:**  $\text{PSp}(n, q)$

---

**Note:** This group is also available via `groups.permutation.PSp()`.

---

EXAMPLES:

**sage:** `G = PSp(2, 3); G`

Permutation Group with generators [(2, 3, 4), (1, 2)(3, 4)]

**sage:** `G.order()`

12

**sage:** `G = PSp(4, 3); G`

Permutation Group with generators [(3, 4)(6, 7)(9, 10)(12, 13)(17, 20)(18, 21)(19, 22)(23, 32)(24, 33)(25,

**sage:** `G.order()`

25920

**sage:** `print G`

The projective symplectic linear group of degree 4 over Finite Field of size 3

**sage:** `G.base_ring()`

Finite Field of size 3

**sage:** `G = PSp(2, 8, name='alpha'); G`

Permutation Group with generators [(3, 8, 6, 4, 9, 7, 5), (1, 2, 3)(4, 7, 5)(6, 9, 8)]

**sage:** `G.base_ring()`

Finite Field in alpha of size  $2^3$

TESTS:

**sage:** `groups.permutation.PSp(2, 3)`

Permutation Group with generators [(2, 3, 4), (1, 2)(3, 4)]

**class** `sage.groups.perm_gps.permgroup_named.PermutationGroup_plg(gens=None, gap_group=None, canonical-ize=True, do-main=None, category=None)`

Bases: `sage.groups.perm_gps.permgroup_named.PermutationGroup_unique`

**INPUT:**

- gens - list of generators (default: None)
- gap\_group - a gap permutation group (default: None)
- canonicalize - bool (default: True); if True, sort generators and remove duplicates

OUTPUT:

- A permutation group.

EXAMPLES:

We explicitly construct the alternating group on four elements:

```
sage: A4 = PermutationGroup([[ (1,2,3) ], [ (2,3,4) ]]); A4
Permutation Group with generators [ (2,3,4), (1,2,3) ]
sage: A4.__init__([[ (1,2,3) ], [ (2,3,4) ]]); A4
Permutation Group with generators [ (2,3,4), (1,2,3) ]
sage: A4.center()
Subgroup of (Permutation Group with generators [ (2,3,4), (1,2,3) ]) generated by [()]
sage: A4.category()
Category of finite permutation groups
sage: TestSuite(A4).run()
```

TESTS:

```
sage: TestSuite(PermutationGroup([[]])).run()
sage: TestSuite(PermutationGroup([])).run()
sage: TestSuite(PermutationGroup([(0,1)])).run()
```

**base\_ring()**

EXAMPLES:

```
sage: G = PGL(2,3)
sage: G.base_ring()
Finite Field of size 3

sage: G = PSL(2,3)
sage: G.base_ring()
Finite Field of size 3
```

**matrix\_degree()**

EXAMPLES:

```
sage: G = PSL(2,3)
sage: G.matrix_degree()
2
```

```
class sage.groups.perm_gps.permgroup_named.PermutationGroup_pug(gens=None,
                                                                gap_group=None,
                                                                canonical-
                                                                ize=True,      do-
                                                                main=None,
                                                                category=None)
```

Bases: `sage.groups.perm_gps.permgroup_named.PermutationGroup_plg`

INPUT:

- gens - list of generators (default: None)
- gap\_group - a gap permutation group (default: None)
- canonicalize - bool (default: True); if True, sort generators and remove duplicates



OUTPUT:

•A permutation group.

EXAMPLES:

We explicitly construct the alternating group on four elements:

```
sage: A4 = PermutationGroup([[ (1,2,3)], [(2,3,4)]]); A4
Permutation Group with generators [(2,3,4), (1,2,3)]
sage: A4.__init__([[ (1,2,3)], [(2,3,4)]]); A4
Permutation Group with generators [(2,3,4), (1,2,3)]
sage: A4.center()
Subgroup of (Permutation Group with generators [(2,3,4), (1,2,3)]) generated by [()]
sage: A4.category()
Category of finite permutation groups
sage: TestSuite(A4).run()
```

TESTS:

```
sage: TestSuite(PermutationGroup([[]])).run()
sage: TestSuite(PermutationGroup([])).run()
sage: TestSuite(PermutationGroup([(0,1)])).run()
```

**field\_of\_definition()**

EXAMPLES:

```
sage: PSU(2,3).field_of_definition()
Finite Field in a of size 3^2
```

```
class sage.groups.perm_gps.permgroup_named.PermutationGroup_symalt (gens=None,
                                                                    gap_group=None,
                                                                    canonical-
                                                                    ize=True, do-
                                                                    main=None,
                                                                    cate-
                                                                    gory=None)
```

Bases: `sage.groups.perm_gps.permgroup_named.PermutationGroup_unique`

This is a class used to factor out some of the commonality in the SymmetricGroup and AlternatingGroup classes.

**set** (\*args, \*\*kws)

Deprecated: Use `domain()` instead. See [trac ticket #10335](#) for details.

```
class sage.groups.perm_gps.permgroup_named.PermutationGroup_unique (gens=None,
                                                                    gap_group=None,
                                                                    canonical-
                                                                    ize=True, do-
                                                                    main=None,
                                                                    cate-
                                                                    gory=None)
```

Bases: `sage.structure.unique_representation.CachedRepresentation`,  
`sage.groups.perm_gps.permgroup.PermutationGroup_generic`

---

**Todo**

Fix the broken hash.

```
sage: G = SymmetricGroup(6)
sage: G3 = G.subgroup([G((1,2,3,4,5,6)), G((1,2))])
```

```
sage: hash(G) == hash(G3) # todo: Should be True!
False
```

---

**class** `sage.groups.perm_gps.permgroup_named.PrimitiveGroup(d, n)`  
Bases: `sage.groups.perm_gps.permgroup_named.PermutationGroup_unique`

The primitive group from the GAP tables of primitive groups.

INPUT:

- *d* – non-negative integer. the degree of the group.
- *n* – positive integer. the index of the group in the GAP database, starting at 1

OUTPUT:

The *n*-th primitive group of degree *d*.

EXAMPLES:

```
sage: PrimitiveGroup(0,1)
Trivial group
sage: PrimitiveGroup(1,1)
Trivial group
sage: G = PrimitiveGroup(5, 2); G # optional - database_gap
D(2*5)
sage: G.gens() # optional - database_gap
[(2,4)(3,5), (1,2,3,5,4)]
sage: G.category() # optional - database_gap
Category of finite permutation groups
```

**Warning:** this follows GAP’s naming convention of indexing the primitive groups starting from 1:

```
sage: PrimitiveGroup(5,0) # optional - database_gap
Traceback (most recent call last):
...
ValueError: Index n must be in {1,...,5}
```

Only primitive groups of “small” degree are available in GAP’s database:

```
sage: PrimitiveGroup(2500,1) # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the primitive groups of degree less
than 2500 are available in GAP’s database
```

**group\_primitive\_id()**

Return the index of this group in the GAP database of primitive groups.

Requires “optional” `database_gap` package.

OUTPUT:

A positive integer, following GAP’s conventions.

EXAMPLES:

```
sage: G = PrimitiveGroup(5,2); G.group_primitive_id() # optional - database_gap
2
```

`sage.groups.perm_gps.permgroup_named.PrimitiveGroups` ( $d=None$ )

Return the set of all primitive groups of a given degree  $d$

INPUT:

- $d$  – an integer (optional)

OUTPUT:

The set of all primitive groups of a given degree  $d$  up to isomorphisms using GAP. If  $d$  is not specified, it returns the set of all primitive groups up to isomorphisms stored in GAP.

**Attention:** `PrimitiveGroups` requires the optional GAP database package. Please install it with `install_package('database_gap')`.

EXAMPLES:

```
sage: PrimitiveGroups(3)
Primitive Groups of degree 3
sage: PrimitiveGroups(7)
Primitive Groups of degree 7
sage: PrimitiveGroups(8)
Primitive Groups of degree 8
sage: PrimitiveGroups()
Primitive Groups
```

The database currently only contains primitive groups up to degree 2499:

```
sage: PrimitiveGroups(2500).cardinality() # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the primitive groups of degree less
than 2500 are available in GAP's database
```

TODO:

This enumeration helper could be extended based on `PrimitiveGroupsIterator` in GAP. This method allows to enumerate groups with specified properties such as transitivity, solvability, ..., without creating all groups.

**class** `sage.groups.perm_gps.permgroup_named.PrimitiveGroupsAll`

Bases: `sage.sets.disjoint_union_enumerated_sets.DisjointUnionEnumeratedSets`

The infinite set of all primitive groups up to isomorphisms.

EXAMPLES:

```
sage: L = PrimitiveGroups(); L
Primitive Groups
sage: L.category()
Category of infinite enumerated sets
sage: L.cardinality()
+Infinity

sage: p = L.__iter__() # optional - database_gap
sage: (p.next(), p.next(), p.next(), p.next(), # optional - database_gap
...     p.next(), p.next(), p.next(), p.next())
(Trivial group, Trivial group, S(2), A(3), S(3), A(4), S(4), C(5))
```

TESTS:

```
sage: TestSuite(PrimitiveGroups()).run() # optional - database_gap # long time
```

```
class sage.groups.perm_gps.permgroup_named.PrimitiveGroupsOfDegree(n)
    Bases: sage.structure.unique_representation.CachedRepresentation,
           sage.structure.parent.Parent
```

The set of all primitive groups of a given degree up to isomorphisms.

EXAMPLES:

```
sage: S = PrimitiveGroups(5); S # optional - database_gap
Primitive Groups of degree 5
sage: S.list() # optional - database_gap
[C(5), D(2*5), AGL(1, 5), A(5), S(5)]
sage: S.an_element() # optional - database_gap
C(5)
```

We write the cardinality of all primitive groups of degree 5:

```
sage: for G in PrimitiveGroups(5): # optional - database_gap
...     print G.cardinality()
5
10
20
60
120
```

TESTS:

```
sage: TestSuite(PrimitiveGroups(3)).run() # optional - database_gap
```

**cardinality()**

Return the cardinality of self.

OUTPUT:

An integer. The number of primitive groups of a given degree up to isomorphism.

EXAMPLES:

```
sage: PrimitiveGroups(0).cardinality() # optional - database_gap
1
sage: PrimitiveGroups(2).cardinality() # optional - database_gap
1
sage: PrimitiveGroups(7).cardinality() # optional - database_gap
7
sage: PrimitiveGroups(12).cardinality() # optional - database_gap
6
sage: [PrimitiveGroups(i).cardinality() for i in range(11)] # optional - database_gap
[1, 1, 1, 2, 2, 5, 4, 7, 7, 11, 9]
```

The database\_gap contains all primitive groups up to degree 2499:

```
sage: PrimitiveGroups(2500).cardinality() # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the primitive groups of degree less than
2500 are available in GAP's database
```

TESTS:

```

sage: type(PrimitiveGroups(12).cardinality())           # optional - database_gap
<type 'sage.rings.integer.Integer'>
sage: type(PrimitiveGroups(0).cardinality())
<type 'sage.rings.integer.Integer'>

```

**class** sage.groups.perm\_gps.permgroup\_named.**QuaternionGroup**  
 Bases: sage.groups.perm\_gps.permgroup\_named.DiCyclicGroup

The quaternion group of order 8.

**OUTPUT:** The quaternion group of order 8, as a permutation group. See the DiCyclicGroup class for a generalization of this construction.

---

**Note:** This group is also available via groups.permutation.Quaternion().

---

#### EXAMPLES:

The quaternion group is one of two non-abelian groups of order 8, the other being the dihedral group  $D_4$ . One way to describe this group is with three generators,  $I, J, K$ , so the whole group is then given as the set  $\{\pm 1, \pm I, \pm J, \pm K\}$  with relations such as  $I^2 = J^2 = K^2 = -1$ ,  $IJ = K$  and  $JI = -K$ .

The examples below illustrate how to use this group in a similar manner, by testing some of these relations. The representation used here is the left-regular representation.

```

sage: Q = QuaternionGroup()
sage: I = Q.gen(0)
sage: J = Q.gen(1)
sage: K = I*J
sage: [I, J, K]
[(1, 2, 3, 4) (5, 6, 7, 8), (1, 5, 3, 7) (2, 8, 4, 6), (1, 8, 3, 6) (2, 7, 4, 5)]
sage: neg_one = I^2; neg_one
(1, 3) (2, 4) (5, 7) (6, 8)
sage: J^2 == neg_one and K^2 == neg_one
True
sage: J*I == neg_one*K
True
sage: Q.center().order() == 2
True
sage: neg_one in Q.center()
True

```

#### TESTS:

```

sage: groups.permutation.Quaternion()
Quaternion group of order 8 as a permutation group

```

**AUTHOR:** – Rob Beezer (2009-10-09)

**class** sage.groups.perm\_gps.permgroup\_named.**SemidihedralGroup**( $m$ )  
 Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The semidihedral group of order  $2^m$ .

INPUT:

- $m$  - a positive integer; the power of 2 that is the group's order

OUTPUT:

The semidihedral group of order  $2^m$ . These groups can be thought of as a semidirect product of  $C_{2^{m-1}}$  with  $C_2$ , where the nontrivial element of  $C_2$  is sent to the element of the automorphism group of  $C_{2^{m-1}}$  that sends elements to their  $-1 + 2^{m-2}$  th power. Thus, the group has the presentation:

$$\langle x, y \mid x^{2^{m-1}}, y^2, y^{-1}xy = x^{-1+2^{m-2}} \rangle$$

This family is notable because it is made up of non-abelian 2-groups that all contain cyclic subgroups of index 2. It is one of only four such families.

#### EXAMPLES:

In [GORENSTEIN1980] it is shown that the semidihedral groups have center of order 2. It is also shown that they have a Frattini subgroup equal to their commutator, which is a cyclic subgroup of order  $2^{m-2}$ .

```
sage: G = SemidihedralGroup(12)
sage: G.order() == 2^12
True
sage: G.commutator() == G.frattini_subgroup()
True
sage: G.commutator().order() == 2^10
True
sage: G.commutator().is_cyclic()
True
sage: G.center().order()
2

sage: G = SemidihedralGroup(4)
sage: len([H for H in G.subgroups() if H.is_cyclic() and H.order() == 8])
1
sage: G.gens()
[(2, 4) (3, 7) (6, 8), (1, 2, 3, 4, 5, 6, 7, 8)]
sage: x = G.gens()[1]; y = G.gens()[0]
sage: x.order() == 2^3; y.order() == 2
True
True
sage: y*x*y == x^(-1+2^2)
True
```

#### TESTS:

```
sage: G = SemidihedralGroup(4.4)
Traceback (most recent call last):
...
TypeError: m must be an integer, not 4.400000000000000

sage: G = SemidihedralGroup(-5)
Traceback (most recent call last):
...
ValueError: the exponent must be greater than 3, not -5
```

#### REFERENCES:

#### AUTHOR:

•Kevin Halasz (2012-8-7)

**class** sage.groups.perm\_gps.permgroup\_named.**SplitMetacyclicGroup**( $p, m$ )  
 Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The split metacyclic group of order  $p^m$ .

#### INPUT:

- $p$  - a prime number that is the prime underlying this  $p$ -group
- $m$  - a positive integer such that the order of this group is the  $p^m$ . Be aware that, for even  $p$ ,  $m$  must be greater than 3, while for odd  $p$ ,  $m$  must be greater than 2.

OUTPUT:

The split metacyclic group of order  $p^m$ . This family of groups has presentation

$$\langle x, y \mid x^{p^{m-1}}, y^p, y^{-1}xy = x^{1+p^{m-2}} \rangle$$

This family is notable because, for odd  $p$ , these are the only  $p$ -groups with a cyclic subgroup of index  $p$ , a result proven in [GORENSTEIN]. It is also shown in [GORENSTEIN] that this is one of four families containing nonabelian 2-groups with a cyclic subgroup of index 2 (with the others being the dicyclic groups, the dihedral groups, and the semidihedral groups).

EXAMPLES:

Using the last relation in the group's presentation, one can see that the elements of the form  $y^i x$ ,  $0 \leq i \leq p-1$  all have order  $p^{m-1}$ , as it can be shown that their  $p$ th powers are all  $x^{p^{m-2}+p}$ , an element with order  $p^{m-2}$ . Manipulation of the same relation shows that none of these elements are powers of any other. Thus, there are  $p$  cyclic maximal subgroups in each split metacyclic group. It is also proven in [GORENSTEIN] that this family has commutator subgroup of order  $p$ , and the Frattini subgroup is equal to the center, with this group being cyclic of order  $p^{m-2}$ . These characteristics are necessary to identify these groups in the case that  $p = 2$ , although the possession of a cyclic maximal subgroup in a non-abelian  $p$ -group is enough for odd  $p$  given the group's order.

```
sage: G = SplitMetacyclicGroup(2, 8)
sage: G.order() == 2**8
True
sage: G.is_abelian()
False
sage: len([H for H in G.subgroups() if H.order() == 2^7 and H.is_cyclic()])
2
sage: G.commutator().order()
2
sage: G.frattini_subgroup() == G.center()
True
sage: G.center().order() == 2^6
True
sage: G.center().is_cyclic()
True

sage: G = SplitMetacyclicGroup(3, 3)
sage: len([H for H in G.subgroups() if H.order() == 3^2 and H.is_cyclic()])
3
sage: G.commutator().order()
3
sage: G.frattini_subgroup() == G.center()
True
sage: G.center().order()
3
```

TESTS:

```
sage: G = SplitMetacyclicGroup(3, 2.5)
Traceback (most recent call last):
...
TypeError: both p and m must be integers

sage: G = SplitMetacyclicGroup(4, 3)
Traceback (most recent call last):
```

```
...
ValueError: p must be prime, 4 is not prime

sage: G = SplitMetacyclicGroup(2,2)
Traceback (most recent call last):
...
ValueError: if prime is 2, the exponent must be greater than 3, not 2

sage: G = SplitMetacyclicGroup(11,2)
Traceback (most recent call last):
...
ValueError: if prime is odd, the exponent must be greater than 2, not 2
```

## REFERENCES:

## AUTHOR:

- Kevin Halasz (2012-8-7)

**class** sage.groups.perm\_gps.permgroup\_named.**SuzukiGroup**(*q, name='a'*)  
Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The Suzuki group over  $\text{GF}(q)$ ,  ${}^2B_2(2^{2k+1}) = Sz(2^{2k+1})$ .

A wrapper for the GAP function SuzukiGroup.

## INPUT:

**q –  $2^n$ , an odd power of 2; the size of the ground field.** (Strictly speaking,  $n$  should be greater than 1, or else this group is not simple.)

**name – (default: 'a') variable name of indeterminate of finite field  $\text{GF}(q)$**

## OUTPUT:

- A Suzuki group.

---

**Note:** This group is also available via `groups.permutation.Suzuki()`.

---

## EXAMPLES:

```
sage: SuzukiGroup(8)
Permutation Group with generators [(1,2) (3,10) (4,42) (5,18) (6,50) (7,26) (8,58) (9,34) (12,28) (13,45)
(1,28,10,44) (3,50,11,42) (4,43,53,64) (5,9,39,52) (6,36,63,13) (7,51,60,57) (8,33,37,16) (12,24,55,29)]
sage: print SuzukiGroup(8)
The Suzuki group over Finite Field in a of size 2^3

sage: G = SuzukiGroup(32, name='alpha')
sage: G.order()
32537600
sage: G.order().factor()
2^10 * 5^2 * 31 * 41
sage: G.base_ring()
Finite Field in alpha of size 2^5
```

## TESTS:

```
sage: groups.permutation.Suzuki(8)
Permutation Group with generators [(1,2) (3,10) (4,42) (5,18) (6,50) (7,26) (8,58) (9,34) (12,28) (13,45)
(1,28,10,44) (3,50,11,42) (4,43,53,64) (5,9,39,52) (6,36,63,13) (7,51,60,57) (8,33,37,16) (12,24,55,29)]
```



## REFERENCES:

- [http://en.wikipedia.org/wiki/Group\\_of\\_Lie\\_type#Suzuki-Ree\\_groups](http://en.wikipedia.org/wiki/Group_of_Lie_type#Suzuki-Ree_groups)

**base\_ring()**

## EXAMPLES:

```
sage: G = SuzukiGroup(32, name='alpha')
sage: G.base_ring()
Finite Field in alpha of size 2^5
```

**class** sage.groups.perm\_gps.permgroup\_named.**SymmetricGroup**(domain=None)  
 Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_symalt

The full symmetric group of order  $n!$ , as a permutation group.

If  $n$  is a list or tuple of positive integers then it returns the symmetric group of the associated set.

## INPUT:

- $n$  – a positive integer, or list or tuple thereof

---

**Note:** This group is also available via `groups.permutation.Symmetric()`.

---

## EXAMPLES:

```
sage: G = SymmetricGroup(8)
sage: G.order()
40320
sage: G
Symmetric group of order 8! as a permutation group
sage: G.degree()
8
sage: S8 = SymmetricGroup(8)
sage: G = SymmetricGroup([1,2,4,5])
sage: G
Symmetric group of order 4! as a permutation group
sage: G.domain()
{1, 2, 4, 5}
sage: G = SymmetricGroup(4)
sage: G
Symmetric group of order 4! as a permutation group
sage: G.domain()
{1, 2, 3, 4}
sage: G.category()
Join of Category of finite permutation groups and Category of finite weyl groups
sage: TestSuite(G).run()
```

## TESTS:

```
sage: TestSuite(SymmetricGroup(0)).run()

sage: groups.permutation.Symmetric(4)
Symmetric group of order 4! as a permutation group
```

**algebra**(base\_ring)

Return the symmetric group algebra associated to self.

## EXAMPLES:

```
sage: S4 = SymmetricGroup(4)
sage: S4.algebra(QQ)
Symmetric group algebra of order 4 over Rational Field
```

**cartan\_type()**

Return the Cartan type of self

The symmetric group  $S_n$  is a Coxeter group of type  $A_{n-1}$ .

EXAMPLES:

```
sage: A = SymmetricGroup([2, 3, 7]); A.cartan_type()
['A', 2]
```

**index\_set()**

Indexing sets of descent of the symmetric group.

EXAMPLES:

```
sage: S8 = SymmetricGroup(8)
sage: S8.index_set()
[1, 2, 3, 4, 5, 6, 7]

sage: S = SymmetricGroup([3, 1, 4, 5])
sage: S.index_set()
[3, 1, 4]
```

**major\_index(parameter=None)**

Returns the *major index generating polynomial* of self, which is a gadget counting the elements of self by major index.

INPUT:

- parameter - an element of a ring. The result is more explicit with a formal variable. (default: element q of Univariate Polynomial Ring in q over Integer Ring)

$$P(q) = \sum_{g \in S_n} q^{\text{major index}(g)}$$

EXAMPLES:

```
sage: S4 = SymmetricGroup(4)
sage: S4.major_index()
q^6 + 3*q^5 + 5*q^4 + 6*q^3 + 5*q^2 + 3*q + 1
sage: K.<t> = QQ[]
sage: S4.major_index(t)
t^6 + 3*t^5 + 5*t^4 + 6*t^3 + 5*t^2 + 3*t + 1
```

**simple\_reflection(i)**

For  $i$  in the index set of self, this returns the elementary transposition  $s_i = (i, i + 1)$ .

EXAMPLES:

```
sage: A = SymmetricGroup(5)
sage: A.simple_reflection(3)
(3, 4)

sage: A = SymmetricGroup([2, 3, 7])
sage: A.simple_reflections()
Finite family {2: (2, 3), 3: (3, 7)}
```

**young\_subgroup** (*comp*)

Return the Young subgroup associated with the composition *comp*.

## EXAMPLES:

```
sage: S = SymmetricGroup(8)
sage: c = Composition([2,2,2,2])
sage: S.young_subgroup(c)
Subgroup of (Symmetric group of order 8! as a permutation group) generated by [(7,8), (5,6), (4,5)]

sage: S = SymmetricGroup(['a','b','c'])
sage: S.young_subgroup([2,1])
Subgroup of (Symmetric group of order 3! as a permutation group) generated by [('a','b')]

sage: Y = S.young_subgroup([2,2,2,2])
Traceback (most recent call last):
...
ValueError: The composition is not of expected size
```

**class** sage.groups.perm\_gps.permgroup\_named.**TransitiveGroup** (*d, n*)

Bases: sage.groups.perm\_gps.permgroup\_named.PermutationGroup\_unique

The transitive group from the GAP tables of transitive groups.

**INPUT:** *d* – non-negative integer; the degree *n* – positive integer; the index of the group in the GAP database, starting at 1

**OUTPUT:** the *n*-th transitive group of degree *d*

---

**Note:** This group is also available via `groups.permutation.Transitive()`.

---

## EXAMPLES:

```
sage: TransitiveGroup(0,1)
Transitive group number 1 of degree 0
sage: TransitiveGroup(1,1)
Transitive group number 1 of degree 1
sage: G = TransitiveGroup(5, 2); G          # optional - database_gap
Transitive group number 2 of degree 5
sage: G.gens()                             # optional - database_gap
[(1,2,3,4,5), (1,4)(2,3)]

sage: G.category()                         # optional - database_gap
Category of finite permutation groups
```

**Warning:** this follows GAP's naming convention of indexing the transitive groups starting from 1:

```
sage: TransitiveGroup(5,0)                  # optional - database_gap
Traceback (most recent call last):
...
ValueError: Index n must be in {1,...,5}
```

**Warning:** only transitive groups of “small” degree are available in GAP’s database:

```
sage: TransitiveGroup(31,1) # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the transitive groups of order less than 30 are available in GAP’s da
```

TESTS:

```
sage: groups.permutation.Transitive(1, 1)
Transitive group number 1 of degree 1

sage: TestSuite(TransitiveGroup(0,1)).run()
sage: TestSuite(TransitiveGroup(1,1)).run()
sage: TestSuite(TransitiveGroup(5,2)).run() # optional - database_gap

sage: TransitiveGroup(1,5) # optional - database_gap
Traceback (most recent call last):
...
ValueError: Index n must be in {1,...,1}
```

sage.groups.perm\_gps.permgroup\_named.**TransitiveGroups** (*d=None*)

INPUT:

- *d* – an integer (optional)

Returns the set of all transitive groups of a given degree *d* up to isomorphisms. If *d* is not specified, it returns the set of all transitive groups up to isomorphisms.

Warning: TransitiveGroups requires the optional GAP database package. Please install it with `sage -i database_gap`.

EXAMPLES:

```
sage: TransitiveGroups(3)
Transitive Groups of degree 3
sage: TransitiveGroups(7)
Transitive Groups of degree 7
sage: TransitiveGroups(8)
Transitive Groups of degree 8

sage: TransitiveGroups()
Transitive Groups
```

**Warning:** in practice, the database currently only contains transitive groups up to degree 30:

```
sage: TransitiveGroups(31).cardinality() # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the transitive groups of order less than 30 are available in GAP’s da
```

**class** sage.groups.perm\_gps.permgroup\_named.**TransitiveGroupsAll**

Bases: sage.sets.disjoint\_union\_enumerated\_sets.DisjointUnionEnumeratedSets

The infinite set of all transitive groups up to isomorphisms.

EXAMPLES:

```

sage: L = TransitiveGroups(); L
Transitive Groups
sage: L.category()
Category of infinite enumerated sets
sage: L.cardinality()
+Infinity

sage: p = L.__iter__() # optional - database_gap
sage: (p.next(), p.next(), p.next(), p.next(), p.next(), p.next(), p.next(), p.next()) # optional
(Transitive group number 1 of degree 0, Transitive group number 1 of degree 1, Transitive group

```

**TESTS:**

```

sage: TestSuite(TransitiveGroups()).run() # optional - database_gap # long time

```

```

class sage.groups.perm_gps.permgroup_named.TransitiveGroupsOfDegree(n)
    Bases: sage.structure.unique_representation.CachedRepresentation,
           sage.structure.parent.Parent

```

The set of all transitive groups of a given (small) degree up to isomorphisms.

**EXAMPLES:**

```

sage: S = TransitiveGroups(4); S # optional - database_gap
Transitive Groups of degree 4
sage: list(S) # optional - database_gap
[Transitive group number 1 of degree 4, Transitive group number 2 of degree 4, Transitive group

sage: TransitiveGroups(5).an_element() # optional - database_gap
Transitive group number 1 of degree 5

```

We write the cardinality of all transitive groups of degree 5:

```

sage: for G in TransitiveGroups(5): # optional - database_gap
...     print G.cardinality()
5
10
20
60
120

```

**TESTS:**

```

sage: TestSuite(TransitiveGroups(3)).run() # optional - database_gap

```

**cardinality()**

Returns the cardinality of self, that is the number of transitive groups of a given degree.

**EXAMPLES:**

```

sage: TransitiveGroups(0).cardinality() # optional - database_gap
1
sage: TransitiveGroups(2).cardinality() # optional - database_gap
1
sage: TransitiveGroups(7).cardinality() # optional - database_gap
7
sage: TransitiveGroups(12).cardinality() # optional - database_gap
301
sage: [TransitiveGroups(i).cardinality() for i in range(11)] # optional - database_gap
[1, 1, 1, 2, 5, 5, 16, 7, 50, 34, 45]

```

**Warning:** The database\_gap contains all transitive groups up to degree 30:

```
sage: TransitiveGroups(31).cardinality()           # optional - database_gap
Traceback (most recent call last):
...
NotImplementedError: Only the transitive groups of order less than 30 are available in GAP
```

TESTS:

```
sage: type(TransitiveGroups(12).cardinality())    # optional - database_gap
<type 'sage.rings.integer.Integer'>
sage: type(TransitiveGroups(0).cardinality())
<type 'sage.rings.integer.Integer'>
```

# PERMUTATION GROUP ELEMENTS

Permutation group elements

AUTHORS:

- David Joyner (2006-02)
- David Joyner (2006-03): word problem method and reorganization
- Robert Bradshaw (2007-11): convert to Cython

EXAMPLES: The Rubik's cube group:

```
sage: f= [(17,19,24,22), (18,21,23,20), (6,25,43,16), (7,28,42,13), (8,30,41,11)]
sage: b=[ (33,35,40,38), (34,37,39,36), ( 3, 9,46,32), ( 2,12,47,29), ( 1,14,48,27)]
sage: l=[ ( 9,11,16,14), (10,13,15,12), ( 1,17,41,40), ( 4,20,44,37), ( 6,22,46,35)]
sage: r=[ (25,27,32,30), (26,29,31,28), ( 3,38,43,19), ( 5,36,45,21), ( 8,33,48,24)]
sage: u=[ ( 1, 3, 8, 6), ( 2, 5, 7, 4), ( 9,33,25,17), (10,34,26,18), (11,35,27,19)]
sage: d=[ (41,43,48,46), (42,45,47,44), (14,22,30,38), (15,23,31,39), (16,24,32,40)]
sage: cube = PermutationGroup([f,b,l,r,u,d])
sage: F=cube.gens()[0]
sage: B=cube.gens()[1]
sage: L=cube.gens()[2]
sage: R=cube.gens()[3]
sage: U=cube.gens()[4]
sage: D=cube.gens()[5]
sage: cube.order()
43252003274489856000
sage: F.order()
4
```

The interested user may wish to explore the following commands: `move = cube.random_element()` and `time word_problem([F,B,L,R,U,D], move, False)`. This typically takes about 5 minutes (on a 2 Ghz machine) and outputs a word ('solving' the cube in the position `move`) with about 60 terms or so.

OTHER EXAMPLES: We create element of a permutation group of large degree.

```
sage: G = SymmetricGroup(30)
sage: s = G(srange(30,0,-1)); s
(1,30) (2,29) (3,28) (4,27) (5,26) (6,25) (7,24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16)
```

```
class sage.groups.perm_gps.permgroup_element.PermutationGroupElement
    Bases: sage.structure.element.MultiplicativeGroupElement
```

An element of a permutation group.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: G
Permutation Group with generators [(1,2,3)(4,5)]
sage: g = G.random_element()
sage: g in G
True
sage: g = G.gen(0); g
(1,2,3)(4,5)
sage: print g
(1,2,3)(4,5)
sage: g*g
(1,3,2)
sage: g**(-1)
(1,3,2)(4,5)
sage: g**2
(1,3,2)
sage: G = PermutationGroup([(1,2,3)])
sage: g = G.gen(0); g
(1,2,3)
sage: g.order()
3
```

This example illustrates how permutations act on multivariate polynomials.

```
sage: R = PolynomialRing(RationalField(), 5, ["x","y","z","u","v"])
sage: x, y, z, u, v = R.gens()
sage: f = x**2 - y**2 + 3*z**2
sage: G = PermutationGroup(['(1,2,3)(4,5)', '(1,2,3,4,5)'])
sage: sigma = G.gen(0)
sage: f * sigma
3*x^2 + y^2 - z^2
```

#### **conjugacy\_class()**

Return the conjugacy class of self.

EXAMPLES:

```
sage: D = DihedralGroup(5)
sage: g = D((1,3,5,2,4))
sage: g.conjugacy_class()
Conjugacy class of (1,3,5,2,4) in Dihedral group of order 10 as a permutation group
```

#### **cycle\_string(*singletons=False*)**

Return string representation of this permutation.

EXAMPLES:

```
sage: g = PermutationGroupElement([(1,2,3), (4,5)])
sage: g.cycle_string()
'(1,2,3)(4,5) '

sage: g = PermutationGroupElement([3,2,1])
sage: g.cycle_string(singletons=True)
'(1,3)(2) '
```

#### **cycle\_tuples(*singletons=False*)**

Return self as a list of disjoint cycles, represented as tuples rather than permutation group elements.

INPUT:



- `singletons` - boolean (default: False) whether or not consider the cycle that correspond to fixed point

**EXAMPLES:**

```
sage: p = PermutationGroupElement('(2,6)(4,5,1)')
sage: p.cycle_tuples()
[(1, 4, 5), (2, 6)]
sage: p.cycle_tuples(singletons=True)
[(1, 4, 5), (2, 6), (3,)]
```

**EXAMPLES:**

```
sage: S = SymmetricGroup(4)
sage: S.gen(0).cycle_tuples()
[(1, 2, 3, 4)]

sage: S = SymmetricGroup(['a','b','c','d'])
sage: S.gen(0).cycle_tuples()
[('a', 'b', 'c', 'd')]
sage: S([('a', 'b'), ('c', 'd')]).cycle_tuples()
[('a', 'b'), ('c', 'd')]
```

**`cycles()`**

Return self as a list of disjoint cycles.

**EXAMPLES:**

```
sage: G = PermutationGroup(['(1,2,3)(4,5,6,7)'])
sage: g = G.0
sage: g.cycles()
[(1,2,3), (4,5,6,7)]
sage: a, b = g.cycles()
sage: a(1), b(1)
(2, 1)
```

**`dict()`**

Returns a dictionary associating each element of the domain with its image.

**EXAMPLES:**

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4)); g
(1,2,3,4)
sage: v = g.dict(); v
{1: 2, 2: 3, 3: 4, 4: 1}
sage: type(v[1])
<type 'int'>
sage: x = G([2,1]); x
(1,2)
sage: x.dict()
{1: 2, 2: 1, 3: 3, 4: 4}
```

**`domain()`**

Returns the domain of self.

**EXAMPLES:**

```
sage: G = SymmetricGroup(4)
sage: x = G([2,1,4,3]); x
(1,2)(3,4)
sage: v = x.domain(); v
```

```
[2, 1, 4, 3]
sage: type(v[0])
<type 'int'>
sage: x = G([2,1]); x
(1,2)
sage: x.domain()
[2, 1, 3, 4]
```

**TESTS:**

```
sage: S = SymmetricGroup(0)
sage: x = S.one(); x
()
sage: x.domain()
[]
```

**has\_descent** (*i*, *side*='right', *positive*=False)

INPUT:

- *i*: an element of the index set
- *side*: “left” or “right” (default: “right”)
- *positive*: a boolean (default: False)

Returns whether *self* has a left (resp. right) descent at position *i*. If *positive* is True, then test for a non descent instead.

Beware that, since permutations are acting on the right, the meaning of descents is the reverse of the usual convention. Hence, *self* has a left descent at position *i* if *self*(*i*) > *self*(*i*+1).

**EXAMPLES:**

```
sage: S = SymmetricGroup([1,2,3])
sage: S.one().has_descent(1)
False
sage: S.one().has_descent(2)
False
sage: s = S.simple_reflections()
sage: x = s[1]*s[2]
sage: x.has_descent(1, side = "right")
False
sage: x.has_descent(2, side = "right")
True
sage: x.has_descent(1, side = "left")
True
sage: x.has_descent(2, side = "left")
False
sage: S._test_has_descent()
```

The symmetric group acting on a set not of the form  $(1, \dots, n)$  is also supported:

```
sage: S = SymmetricGroup([2,4,1])
sage: s = S.simple_reflections()
sage: x = s[2]*s[4]
sage: x.has_descent(4)
True
sage: S._test_has_descent()
```

**inverse** ()

Returns the inverse permutation.

OUTPUT:

For an element of a permutation group, this method returns the inverse element, which is both the inverse function and the inverse as an element of a group.

EXAMPLES:

```
sage: s = PermutationGroupElement("(1,2,3)(4,5)")
sage: s.inverse()
(1,3,2)(4,5)

sage: A = AlternatingGroup(4)
sage: t = A("(1,2,3)")
sage: t.inverse()
(1,3,2)
```

There are several ways (syntactically) to get an inverse of a permutation group element.

```
sage: s = PermutationGroupElement("(1,2,3,4)(6,7,8)")
sage: s.inverse() == s^-1
True
sage: s.inverse() == ~s
True
```

**list** (\*args, \*\*kws)

Deprecated: Use `domain()` instead. See [trac ticket #14319](#) for details.

**matrix** ()

Returns deg x deg permutation matrix associated to the permutation self

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: g = G.gen(0)
sage: g.matrix()
[0 1 0 0 0]
[0 0 1 0 0]
[1 0 0 0 0]
[0 0 0 0 1]
[0 0 0 1 0]
```

**orbit** (n, sorted=True)

Returns the orbit of the integer  $n$  under this group element, as a sorted list.

EXAMPLES:

```
sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: g = G.gen(0)
sage: g.orbit(4)
[4, 5]
sage: g.orbit(3)
[1, 2, 3]
sage: g.orbit(10)
[10]

sage: s = SymmetricGroup(['a', 'b']).gen(0); s
('a', 'b')
sage: s.orbit('a')
['a', 'b']
```

**order** ()

Return the order of this group element, which is the smallest positive integer  $n$  for which  $g^n = 1$ .

EXAMPLES:

```
sage: s = PermutationGroupElement('(1,2)(3,5,6)')
sage: s.order()
6
```

TESTS:

```
sage: prod(primes(150))
1492182350939279320058875736615841068547583863326864530410
sage: L = [tuple(range(sum(primes(p))+1, sum(primes(p))+1+p)) for p in primes(150)]
sage: PermutationGroupElement(L).order()
1492182350939279320058875736615841068547583863326864530410
```

**sign()**

Returns the sign of self, which is  $(-1)^s$ , where  $s$  is the number of swaps.

EXAMPLES:

```
sage: s = PermutationGroupElement('(1,2)(3,5,6)')
sage: s.sign()
-1
```

ALGORITHM: Only even cycles contribute to the sign, thus

$$\text{sign}(\sigma) = (-1)^{\sum_c \text{len}(c)-1}$$

where the sum is over cycles in self.

**tuple()**

Return tuple of images of the domain under self.

EXAMPLES:

```
sage: G = SymmetricGroup(5)
sage: s = G([2,1,5,3,4])
sage: s.tuple()
(2, 1, 5, 3, 4)

sage: S = SymmetricGroup(['a', 'b'])
sage: S.gen().tuple()
('b', 'a')
```

**word\_problem(words, display=True)**

$G$  and  $H$  are permutation groups,  $g$  in  $G$ ,  $H$  is a subgroup of  $G$  generated by a list (words) of elements of  $G$ . If  $g$  is in  $H$ , return the expression for  $g$  as a word in the elements of (words).

This function does not solve the word problem in Sage. Rather it pushes it over to GAP, which has optimized algorithms for the word problem. Essentially, this function is a wrapper for the GAP functions “EpimorphismFromFreeGroup” and “PreImagesRepresentative”.

EXAMPLE:

```
sage: G = PermutationGroup([[ (1,2,3), (4,5) ], [ (3,4) ]], canonicalize=False)
sage: g1, g2 = G.gens()
sage: h = g1^2*g2*g1
sage: h.word_problem([g1,g2], False)
('x1^2*x2^-1*x1', '(1,2,3)(4,5)^2*(3,4)^-1*(1,2,3)(4,5)')
sage: h.word_problem([g1,g2])
x1^2*x2^-1*x1
[[ ('(1,2,3)(4,5)', 2), ['(3,4)', -1], ['(1,2,3)(4,5)', 1]]
('x1^2*x2^-1*x1', '(1,2,3)(4,5)^2*(3,4)^-1*(1,2,3)(4,5)')
```

```
sage.groups.perm_gps.permgroup_element.is_PermutationGroupElement(x)
```

Returns True if  $x$  is a `PermutationGroupElement`.

EXAMPLES:

```
sage: p = PermutationGroupElement([(1,2),(3,4,5)])
sage: from sage.groups.perm_gps.permgroup_element import is_PermutationGroupElement
sage: is_PermutationGroupElement(p)
True
```

```
sage.groups.perm_gps.permgroup_element.make_permgroup_element(G,x)
```

Returns a `PermutationGroupElement` given the permutation group  $G$  and the permutation  $x$  in list notation.

This function is used when unpickling old (pre-domain) versions of permutation groups and their elements. This now does a bit of processing and calls `make_permgroup_element_v2()` which is used in unpickling the current `PermutationGroupElements`.

EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup_element import make_permgroup_element
sage: S = SymmetricGroup(3)
sage: make_permgroup_element(S, [1,3,2])
(2,3)
```

```
sage.groups.perm_gps.permgroup_element.make_permgroup_element_v2(G, x, domain)
```

Returns a `PermutationGroupElement` given the permutation group  $G$ , the permutation  $x$  in list notation, and the domain `domain` of the permutation group.

This function is used when unpickling permutation groups and their elements.

EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup_element import make_permgroup_element_v2
sage: S = SymmetricGroup(3)
sage: make_permgroup_element_v2(S, [1,3,2], S.domain())
(2,3)
```

```
sage.groups.perm_gps.permgroup_element.standardize_generator(g, convert_dict=None)
```

Standardizes the input for permutation group elements to a list of tuples. This was factored out of the `PermutationGroupElement.__init__` since `PermutationGroup_generic.__init__` needs to do the same computation in order to compute the domain of a group when it's not explicitly specified.

INPUT:

- $g$  - a list, tuple, string, `GapElement`, `PermutationGroupElement`, `Permutation`
- `convert_dict` - (optional) a dictionary used to convert the points to a number compatible with GAP.

OUTPUT:

The permutation in as a list of cycles.

EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup_element import standardize_generator
sage: standardize_generator('(1,2)')
[(1, 2)]
sage: p = PermutationGroupElement([(1,2)])
sage: standardize_generator(p)
[(1, 2)]
sage: standardize_generator(p._gap_())
```

```
[ (1, 2) ]
sage: standardize_generator((1,2))
[ (1, 2) ]
sage: standardize_generator([(1,2)])
[ (1, 2) ]
sage: standardize_generator(Permutation([2,1,3]))
[ (1, 2), (3,) ]

sage: d = {'a': 1, 'b': 2}
sage: p = SymmetricGroup(['a', 'b']).gen(0); p
('a','b')
sage: standardize_generator(p, convert_dict=d)
[ (1, 2) ]
sage: standardize_generator(p._gap_(), convert_dict=d)
[ (1, 2) ]
sage: standardize_generator(('a','b'), convert_dict=d)
[ (1, 2) ]
sage: standardize_generator([('a','b')], convert_dict=d)
[ (1, 2) ]
```

`sage.groups.perm_gps.permgroup_element.string_to_tuples(g)`

EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup_element import string_to_tuples
sage: string_to_tuples('(1,2,3)')
[ (1, 2, 3) ]
sage: string_to_tuples('(1,2,3)(4,5)')
[ (1, 2, 3), (4, 5) ]
sage: string_to_tuples('(1,2, 3)(4,5)')
[ (1, 2, 3), (4, 5) ]
sage: string_to_tuples('(1,2)(3)')
[ (1, 2), (3,) ]
```

# PERMUTATION GROUP HOMOMORPHISMS

AUTHORS:

- David Joyner (2006-03-21): first version
- David Joyner (2008-06): fixed kernel and image to return a group, instead of a string.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
sage: H = DihedralGroup(4)
sage: g = G([(1,2,3,4)])
sage: phi = PermutationGroupMorphism_im_gens(G, H, map(H, G.gens()))
sage: phi.image(G)
Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4)]
sage: phi.kernel()
Subgroup of (Cyclic group of order 4 as a permutation group) generated by [()]
sage: phi.image(g)
(1,2,3,4)
sage: phi(g)
(1,2,3,4)
sage: phi.codomain()
Dihedral group of order 8 as a permutation group
sage: phi.codomain()
Dihedral group of order 8 as a permutation group
sage: phi.domain()
Cyclic group of order 4 as a permutation group
```

```
class sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism
    Bases: sage.categories.morphism.Morphism
```

A set-theoretic map between PermutationGroups.

**image**(*J*)

*J* must be a subgroup of *G*. Computes the subgroup of *H* which is the image of *J*.

EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
sage: H = DihedralGroup(4)
sage: g = G([(1,2,3,4)])
sage: phi = PermutationGroupMorphism_im_gens(G, H, map(H, G.gens()))
sage: phi.image(G)
Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4)]
```

```

sage: phi.image(g)
(1, 2, 3, 4)

sage: G = PSL(2, 7)
sage: D = G.direct_product(G)
sage: H = D[0]
sage: pr1 = D[3]
sage: pr1.image(G)
Subgroup of (The projective special linear group of degree 2 over Finite Field of size 7) ge
sage: G.is_isomorphic(pr1.image(G))
True

```

**kernel()**

Returns the kernel of this homomorphism as a permutation group.

**EXAMPLES:**

```

sage: G = CyclicPermutationGroup(4)
sage: H = DihedralGroup(4)
sage: g = G([(1, 2, 3, 4)])
sage: phi = PermutationGroupMorphism_im_gens(G, H, [1])
sage: phi.kernel()
Subgroup of (Cyclic group of order 4 as a permutation group) generated by [(1, 2, 3, 4)]

sage: G = PSL(2, 7)
sage: D = G.direct_product(G)
sage: H = D[0]
sage: pr1 = D[3]
sage: G.is_isomorphic(pr1.kernel())
True

```

```

class sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism_from_gap(G,
                                                                                   H,
                                                                                   gap_hom)

```

Bases: `sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism`

This is a Python trick to allow Sage programmers to create a group homomorphism using GAP using very general constructions. An example of its usage is in the `direct_product` instance method of the `PermutationGroup_generic` class in `permgroup.py`.

Basic syntax:

`PermutationGroupMorphism_from_gap(domain_group, range_group, phi:=gap_hom_command, 'phi')` And don't forget the line: `from sage.groups.perm_gps.permgroup_morphism import PermutationGroupMorphism_from_gap` in your program.

**EXAMPLES:**

```

sage: from sage.groups.perm_gps.permgroup_morphism import PermutationGroupMorphism_from_gap
sage: G = PermutationGroup([(1, 2), (3, 4)], [(1, 2, 3, 4)])
sage: H = G.subgroup([G([(1, 2, 3, 4))])
sage: PermutationGroupMorphism_from_gap(H, G, gap.Identity)
Permutation group morphism:
From: Subgroup of (Permutation Group with generators [(1, 2) (3, 4), (1, 2, 3, 4)]) generated by [(1, 2, 3, 4)]
To: Permutation Group with generators [(1, 2) (3, 4), (1, 2, 3, 4)]
Defn: Identity

```

```

class sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism_id
Bases: sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism

```



## INPUT:

There can be one or two arguments of this init method. If it is one argument, it must be a hom space. If it is two arguments, it must be two parent structures that will be domain and codomain of the map-to-be-created.

## TESTS:

```
sage: from sage.categories.map import Map
```

## Using a hom space:

```
sage: Map(Hom(QQ, ZZ, Rings()))
```

Generic map:

```
From: Rational Field
To: Integer Ring
```

## Using domain and codomain:

```
sage: Map(QQ['x'], SymmetricGroup(6))
```

Generic map:

```
From: Univariate Polynomial Ring in x over Rational Field
To: Symmetric group of order 6! as a permutation group
```

```
class sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism_im_gens(G,
                                                                                      H,
                                                                                      gens=None)
```

Bases: `sage.groups.perm_gps.permgroup_morphism.PermutationGroupMorphism`

Some python code for wrapping GAP's GroupHomomorphismByImages function but only for permutation groups. Can be expensive if G is large. Returns "fail" if gens does not generate self or if the map does not extend to a group homomorphism, self - other.

## EXAMPLES:

```
sage: G = CyclicPermutationGroup(4)
```

```
sage: H = DihedralGroup(4)
```

```
sage: phi = PermutationGroupMorphism_im_gens(G, H, map(H, G.gens())); phi
Permutation group morphism:
```

```
From: Cyclic group of order 4 as a permutation group
To: Dihedral group of order 8 as a permutation group
Defn: [(1,2,3,4)] -> [(1,2,3,4)]
```

```
sage: g = G([(1,3),(2,4)]); g
```

```
(1,3)(2,4)
```

```
sage: phi(g)
```

```
(1,3)(2,4)
```

```
sage: images = ((4,3,2,1),)
```

```
sage: phi = PermutationGroupMorphism_im_gens(G, G, images)
```

```
sage: g = G([(1,2,3,4)]); g
```

```
(1,2,3,4)
```

```
sage: phi(g)
```

```
(1,4,3,2)
```

## AUTHORS:

•David Joyner (2006-02)

```
sage.groups.perm_gps.permgroup_morphism.is_PermutationGroupMorphism(f)
```

Returns True if the argument f is a PermutationGroupMorphism.

## EXAMPLES:

```
sage: from sage.groups.perm_gps.permgroup_morphism import is_PermutationGroupMorphism
```

```
sage: G = CyclicPermutationGroup(4)
```

```
sage: H = DihedralGroup(4)
sage: phi = PermutationGroupMorphism_im_gens(G, H, map(H, G.gens()))
sage: is_PermutationGroupMorphism(phi)
True
```

# RUBIK'S CUBE GROUP FUNCTIONS

---

**Note:** “Rubiks cube” is trademarked. We shall omit the trademark symbol below for simplicity.

---

## NOTATION:

$B$  denotes a clockwise quarter turn of the back face,  $D$  denotes a clockwise quarter turn of the down face, and similarly for  $F$  (front),  $L$  (left),  $R$  (right), and  $U$  (up). Products of moves are read right to left, so for example,  $R \cdot U$  means move  $U$  first and then  $R$ .

See `CubeGroup.parse()` for all possible input notations.

The “Singmaster notation”:

- moves:  $U, D, R, L, F, B$  as in the diagram below,
- corners:  $xyz$  means the facet is on face  $x$  (in  $R, F, L, U, D, B$ ) and the clockwise rotation of the corner sends  $x - y - z$
- edges:  $xy$  means the facet is on face  $x$  and a flip of the edge sends  $x - y$ .

**sage:** `rubik = CubeGroup()`

**sage:** `rubik.display2d("")`

```

+-----+
|  1    2    3 |
|  4  top  5 |
|  6    7    8 |
+-----+
+-----+-----+-----+-----+
|  9 10 11 | 17 18 19 | 25 26 27 | 33 34 35 |
| 12 left 13 | 20 front 21 | 28 right 29 | 36 rear 37 |
| 14 15 16 | 22 23 24 | 30 31 32 | 38 39 40 |
+-----+-----+-----+-----+
| 41 42 43 |
| 44 bottom 45 |
| 46 47 48 |
+-----+
```

## AUTHORS:

- David Joyner (2006-10-21): first version
- David Joyner (2007-05): changed faces, added legal and solve
- David Joyner(2007-06): added plotting functions
- David Joyner (2007, 2008): colors corrected, “solve” rewritten (again), typos fixed.
- Robert Miller (2007, 2008): editing, cleaned up display2d

- Robert Bradshaw (2007, 2008): RubiksCube object, 3d plotting.
- David Joyner (2007-09): rewrote docstring for CubeGroup’s “solve”.
- Robert Bradshaw (2007-09): Versatile parse function for all input types.
- Robert Bradshaw (2007-11): Cleanup.

## REFERENCES:

- Cameron, P., Permutation Groups. New York: Cambridge University Press, 1999.
- Wielandt, H., Finite Permutation Groups. New York: Academic Press, 1964.
- Dixon, J. and Mortimer, B., Permutation Groups, Springer-Verlag, Berlin/New York, 1996.
- Joyner, D., Adventures in Group Theory, Johns Hopkins Univ Press, 2002.

**class** `sage.groups.perm_gps.cubegroup.CubeGroup`  
 Bases: `sage.groups.perm_gps.permgroup.PermutationGroup_generic`  
 A python class to help compute Rubik’s cube group actions.

---

**Note:** This group is also available via `groups.permutation.RubiksCube()`.

---

## EXAMPLES:

If  $G$  denotes the cube group then it may be regarded as a subgroup of `SymmetricGroup(48)`, where the 48 facets are labeled as follows.

```
sage: rubik = CubeGroup()
sage: rubik.display2d("")
```

```

      +-----+
      |  1    2    3 |
      |  4  top  5 |
      |  6    7    8 |
+-----+-----+-----+-----+
|  9  10  11 | 17  18  19 | 25  26  27 | 33  34  35 |
| 12 left 13 | 20  front 21 | 28  right 29 | 36  rear 37 |
| 14  15  16 | 22  23  24 | 30   31  32 | 38  39  40 |
+-----+-----+-----+-----+
      | 41    42    43 |
      | 44 bottom 45 |
      | 46    47    48 |
      +-----+

```

```
sage: rubik
```

The Rubik’s cube group with generators  $R, L, F, B, U, D$  in `SymmetricGroup(48)`.

## TESTS::

```
sage: groups.permutation.RubiksCube()
The Rubik’s cube group with generators R,L,F,B,U,D in SymmetricGroup(48).
```

**B()**

Return the generator  $B$  in Singmaster notation.

## EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.B()
(1, 14, 48, 27) (2, 12, 47, 29) (3, 9, 46, 32) (33, 35, 40, 38) (34, 37, 39, 36)
```

**D()**Return the generator  $D$  in Singmaster notation.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.D()
(14, 22, 30, 38) (15, 23, 31, 39) (16, 24, 32, 40) (41, 43, 48, 46) (42, 45, 47, 44)
```

**F()**Return the generator  $F$  in Singmaster notation.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.F()
(6, 25, 43, 16) (7, 28, 42, 13) (8, 30, 41, 11) (17, 19, 24, 22) (18, 21, 23, 20)
```

**L()**Return the generator  $L$  in Singmaster notation.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.L()
(1, 17, 41, 40) (4, 20, 44, 37) (6, 22, 46, 35) (9, 11, 16, 14) (10, 13, 15, 12)
```

**R()**Return the generator  $R$  in Singmaster notation.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.R()
(3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24) (25, 27, 32, 30) (26, 29, 31, 28)
```

**U()**Return the generator  $U$  in Singmaster notation.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.U()
(1, 3, 8, 6) (2, 5, 7, 4) (9, 33, 25, 17) (10, 34, 26, 18) (11, 35, 27, 19)
```

**display2d(mv)**

Print the 2d representation of self.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.display2d("R")
+-----+
| 1      2      38 |
| 4      top    36 |
| 6      7      33 |
+-----+
+-----+-----+-----+-----+
| 9 10 11 | 17 18 3 | 27 29 32 | 48 34 35 |
| 12 left 13 | 20 front 5 | 26 right 31 | 45 rear 37 |
| 14 15 16 | 22 23 8 | 25 28 30 | 43 39 40 |
+-----+-----+-----+-----+
| 41 42 19 |
| 44 bottom 21 |
```

```
| 46  47  24 |
+-----+
```

**faces** (*mv*)

Return the dictionary of faces created by the effect of the move *mv*, which is a string of the form  $X^a Y^b \dots$ , where  $X, Y, \dots$  are in  $\{R, L, F, B, U, D\}$  and  $a, b, \dots$  are integers. We call this ordering of the faces the “BDFLRU, L2R, T2B ordering”.

EXAMPLES:

```
sage: rubik = CubeGroup()
```

Here is the dictionary of the solved state:

```
sage: sorted(rubik.faces("").items())
[('back', [[33, 34, 35], [36, 0, 37], [38, 39, 40]]),
 ('down', [[41, 42, 43], [44, 0, 45], [46, 47, 48]]),
 ('front', [[17, 18, 19], [20, 0, 21], [22, 23, 24]]),
 ('left', [[9, 10, 11], [12, 0, 13], [14, 15, 16]]),
 ('right', [[25, 26, 27], [28, 0, 29], [30, 31, 32]]),
 ('up', [[1, 2, 3], [4, 0, 5], [6, 7, 8]])]
```

Now the dictionary of the state obtained after making the move *R* followed by *L*:

```
sage: sorted(rubik.faces("R*U").items())
[('back', [[48, 26, 27], [45, 0, 37], [43, 39, 40]]),
 ('down', [[41, 42, 11], [44, 0, 21], [46, 47, 24]]),
 ('front', [[9, 10, 8], [20, 0, 7], [22, 23, 6]]),
 ('left', [[33, 34, 35], [12, 0, 13], [14, 15, 16]]),
 ('right', [[19, 29, 32], [18, 0, 31], [17, 28, 30]]),
 ('up', [[3, 5, 38], [2, 0, 36], [1, 4, 25]])]
```

**facets** (*g=None*)

Return the set of facets on which the group acts. This function is a “constant”.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.facets() == range(1,49)
True
```

**gen\_names** ()

Return the names of the generators.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.gen_names()
['B', 'D', 'F', 'L', 'R', 'U']
```

**group** ()

This is deprecated in trac:11360. Use the `CubeGroup` instead.

EXAMPLES:

```
sage: CubeGroup().group()
doctest:...: DeprecationWarning: group() is deprecated. Use the CubeGroup instead.
See http://trac.sagemath.org/11360 for details.
The Rubik's cube group with generators R,L,F,B,U,D in SymmetricGroup(48).
```

**legal** (*state, mode='quiet'*)

Return 1 (true) if the dictionary `state` (in the same format as returned by the `faces` method) represents a legal position (or state) of the Rubik's cube or 0 (false) otherwise.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: r0 = rubik.faces("")
sage: r1 = {'back': [[33, 34, 35], [36, 0, 37], [38, 39, 40]], 'down': [[41, 42, 43], [44, 0, 45], [46, 47, 48]]}
sage: rubik.legal(r0)
1
sage: rubik.legal(r0, "verbose")
(1, ())
sage: rubik.legal(r1)
0
```

**move** (*mv*)

Return the group element and the reordered list of facets, as moved by the list `mv` (read left-to-right)

INPUT:

•`mv` – A string of the form `Xa*Yb*...`, where `X, Y, ...` are in `R, L, F, B, U, D` and `a, b, ...` are integers.

EXAMPLES:

```
sage: rubik = CubeGroup()
sage: rubik.move("")[0]
()
sage: rubik.move("R")[0]
(3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24) (25, 27, 32, 30) (26, 29, 31, 28)
sage: rubik.R()
(3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24) (25, 27, 32, 30) (26, 29, 31, 28)
```

**parse** (*mv*, *check=True*)

This function allows one to create the permutation group element from a variety of formats.

INPUT:

•`mv` – Can one of the following:

- `list` - list of facets (as returned by `self.facets()`)
- `dict` - list of faces (as returned by `self.faces()`)
- `str` - either cycle notation (passed to GAP) or a product of generators or Singmaster notation
- `perm_group element` - returned as an element of `self`

•`check` – check if the input is valid

EXAMPLES:

```
sage: C = CubeGroup()
sage: C.parse(range(1, 49))
()
sage: g = C.parse("L"); g
(1, 17, 41, 40) (4, 20, 44, 37) (6, 22, 46, 35) (9, 11, 16, 14) (10, 13, 15, 12)
sage: C.parse(str(g)) == g
True
sage: facets = C.facets(g); facets
[17, 2, 3, 20, 5, 22, 7, 8, 11, 13, 16, 10, 15, 9, 12, 14, 41, 18, 19, 44, 21, 46, 23, 24, 2]
sage: C.parse(facets)
(1, 17, 41, 40) (4, 20, 44, 37) (6, 22, 46, 35) (9, 11, 16, 14) (10, 13, 15, 12)
sage: C.parse(facets) == g
True
```

```

sage: faces = C.faces("L"); faces
{'right': [[25, 26, 27], [28, 0, 29], [30, 31, 32]], 'up': [[17, 2, 3], [20, 0, 5], [22, 7,
sage: C.parse(faces) == C.parse("L")
True
sage: C.parse("L' R2") == C.parse("L^(-1)*R^2")
True
sage: C.parse("L' R2")
(1, 40, 41, 17) (3, 43) (4, 37, 44, 20) (5, 45) (6, 35, 46, 22) (8, 48) (9, 14, 16, 11) (10, 12, 15, 13) (19, 38) (21, 36)
sage: C.parse("L^4")
()
sage: C.parse("L^(-1)*R")
(1, 40, 41, 17) (3, 38, 43, 19) (4, 37, 44, 20) (5, 36, 45, 21) (6, 35, 46, 22) (8, 33, 48, 24) (9, 14, 16, 11) (10, 12, 15, 13)

```

#### **plot3d\_cube** (mv, title=True)

Displays  $F, U, R$  faces of the cube after the given move mv. Mostly included for the purpose of drawing pictures and checking moves.

INPUT:

- mv – A string in the Singmaster notation
- title – (Default: True) Display the title information

The first one below is “superflip+4 spot” (in 26q\* moves) and the second one is the superflip (in 20f\* moves). Type show(P) to view them.

EXAMPLES:

```

sage: rubik = CubeGroup()
sage: P = rubik.plot3d_cube("U^2*F*U^2*L*R^(-1)*F^2*U*F^3*B^3*R*L*U^2*R*D^3*U*L^3*R*D*R^3*L^3")
sage: P = rubik.plot3d_cube("R*L*D^2*B^3*L^2*F^2*R^2*U^3*D*R^3*D^2*F^3*B^3*D^3*F^2*D^3*R^2*U^3")

```

#### **plot\_cube** (mv, title=True, colors=[(1, 0.63, 1), (1, 1, 0), (1, 0, 0), (0, 1, 0), (1, 0.6, 0.3), (0, 0, 1)])

Input the move mv, as a string in the Singmaster notation, and output the 2D plot of the cube in that state.

Type P.show() to display any of the plots below.

EXAMPLES:

```

sage: rubik = CubeGroup()
sage: P = rubik.plot_cube("R^2*U^2*R^2*U^2*R^2*U^2", title = False)
sage: # (R^2*U^2)^3 permutes 2 pairs of edges (uf,ub) (fr,br)
sage: P = rubik.plot_cube("R*L*D^2*B^3*L^2*F^2*R^2*U^3*D*R^3*D^2*F^3*B^3*D^3*F^2*D^3*R^2*U^3")
sage: # the superflip (in 20f* moves)
sage: P = rubik.plot_cube("U^2*F*U^2*L*R^(-1)*F^2*U*F^3*B^3*R*L*U^2*R*D^3*U*L^3*R*D*R^3*L^3")
sage: # "superflip+4 spot" (in 26q* moves)

```

#### **repr2d** (mv)

Displays a 2D map of the Rubik’s cube after the move mv has been made. Nothing is returned.

EXAMPLES:

```

sage: rubik = CubeGroup()
sage: print rubik.repr2d("")
+-----+
| 1      2      3 |
| 4      top    5 |
| 6      7      8 |
+-----+
| 9 10 11 | 17 18 19 | 25 26 27 | 33 34 35 |
| 12 left 13 | 20 front 21 | 28 right 29 | 36 rear 37 |
| 14 15 16 | 22 23 24 | 30 31 32 | 38 39 40 |

```



```

+-----+-----+-----+-----+
| 41  42  43 |
| 44 bottom 45 |
| 46  47  48 |
+-----+

sage: print rubik.repr2d("R")
+-----+
| 1    2   38 |
| 4    top 36 |
| 6    7   33 |
+-----+
+-----+-----+-----+-----+
| 9 10 11 | 17 18 3 | 27 29 32 | 48 34 35 |
| 12 left 13 | 20 front 5 | 26 right 31 | 45 rear 37 |
| 14 15 16 | 22 23 8 | 25 28 30 | 43 39 40 |
+-----+-----+-----+-----+
| 41  42  19 |
| 44 bottom 21 |
| 46  47  24 |
+-----+

```

You can see the right face has been rotated but not the left face.

**solve** (*state*, *algorithm*='default')

Solves the cube in the *state*, given as a dictionary as in `legal`. See the `solve` method of the `RubiksCube` class for more details.

This may use GAP's `EpimorphismFromFreeGroup` and `PreImagesRepresentative` as explained below, if 'gap' is passed in as the algorithm.

This algorithm

1. constructs the free group on 6 generators then computes a reasonable set of relations which they satisfy
2. computes a homomorphism from the cube group to this free group quotient
3. takes the cube position, regarded as a group element, and maps it over to the free group quotient
4. using those relations and tricks from combinatorial group theory (stabilizer chains), solves the "word problem" for that element.
5. uses python string parsing to rewrite that in cube notation.

The Rubik's cube group has about  $4.3 \times 10^{19}$  elements, so this process is time-consuming. See <http://www.gap-system.org/Doc/Examples/rubik.html> for an interesting discussion of some GAP code analyzing the Rubik's cube.

EXAMPLES:

```

sage: rubik = CubeGroup()
sage: state = rubik.faces("R")
sage: rubik.solve(state)
'R'
sage: state = rubik.faces("R*U")
sage: rubik.solve(state, algorithm='gap')           # long time
'R*U'

```

You can also check this another (but similar) way using the `word_problem` method (eg, `G = rubik.group()`; `g = G("(3,38,43,19)(5,36,45,21)(8,33,48,24)(25,27,32,30)(26,29,31,28)")`; `g.word_problem([b,d,f,l,r,u])`), though the output will be less intuitive).

```
class sage.groups.perm_gps.cubegroup.RubiksCube (state=None, history=[], colors=[(1, 0.63,
1), (1, 1, 0), (1, 0, 0), (0, 1, 0), (1, 0.6, 0.3),
(0, 0, 1)])
```

Bases: `sage.structure.sage_object.SageObject`

The Rubik's cube (in a given state).

EXAMPLES:

```
sage: C = RubiksCube().move("R U R'")
```

```
sage: C.show3d()
```

```
sage: C = RubiksCube("R*L"); C
```

```
+-----+
| 17      2    38 |
| 20      top  36 |
| 22      7    33 |
+-----+
+-----+-----+-----+
| 11  13  16 | 41   18   3 | 27   29  32 | 48   34   6 |
| 10 left 15 | 44  front  5 | 26 right 31 | 45 rear  4 |
|  9  12  14 | 46   23   8 | 25   28  30 | 43   39   1 |
+-----+-----+-----+
| 40      42   19 |
| 37 bottom 21 |
| 35      47   24 |
+-----+
```

```
sage: C.show()
```

```
sage: C.solve(algorithm='gap') # long time
'L R'
```

```
sage: C == RubiksCube("L*R")
```

```
True
```

**cubie** (*size, gap, x, y, z, colors, stickers=True*)

Return the cubie at  $(x, y, z)$ .

INPUT:

- *size* – The size of the cubie
- *gap* – The gap between cubies
- *x, y, z* – The position of the cubie
- *colors* – The list of colors
- *stickers* – (Default True) Boolean to display stickers

EXAMPLES:

```
sage: C = RubiksCube("R*U")
```

```
sage: C.cubie(0.15, 0.025, 0, 0, 0, C.colors*3)
```

**facets** ()

Return the facets of `self`.

EXAMPLES:

```
sage: C = RubiksCube("R*U")
```

```
sage: C.facets()
```

```
[3, 5, 38, 2, 36, 1, 4, 25, 33, 34, 35, 12, 13, 14, 15, 16, 9, 10,
8, 20, 7, 22, 23, 6, 19, 29, 32, 18, 31, 17, 28, 30, 48, 26, 27,
45, 37, 43, 39, 40, 41, 42, 11, 44, 21, 46, 47, 24]
```

**move**(*g*)Move the Rubik's cube by *g*.

EXAMPLES:

```
sage: RubiksCube().move("R*U") == RubiksCube("R*U")
True
```

**plot**()

Return a plot of self.

EXAMPLES:

```
sage: C = RubiksCube("R*U")
sage: C.plot()
```

**plot3d**(*stickers=True*)

Return a 3D plot of self.

EXAMPLES:

```
sage: C = RubiksCube("R*U")
sage: C.plot3d()
```

**scramble**(*moves=30*)

Scramble the Rubik's cube.

EXAMPLES:

```
sage: C = RubiksCube()
sage: C.scramble() # random
```

```

+-----+
| 38  29  35 |
| 20  top  42 |
| 11  44  30 |
+-----+
+-----+-----+-----+-----+
| 48 13 17 | 6  15  24 | 43  23  9 | 1  36 32 |
| 4 left 18 | 7  front 37 | 12 right 26 | 5 rear 10 |
| 33 31 40 | 14  28  8 | 25  47 16 | 22  2  3 |
+-----+-----+-----+-----+
| 46  21  19 |
| 45 bottom 39 |
| 27  34  41 |
+-----+
```

**show**()

Show a plot of self.

EXAMPLES:

```
sage: C = RubiksCube("R*U")
sage: C.show()
```

**show3d**()

Show a 3D plot of self.

EXAMPLES:

```
sage: C = RubiksCube("R*U")
sage: C.show3d()
```

**solve**(*algorithm='hybrid', timeout=15*)

Solve the Rubik's cube.

INPUT:

- algorithm – must be one of the following:
  - hybrid - try kociemba for timeout seconds, then dietz
  - kociemba - Use Dik T. Winter's program (reasonable speed, few moves)
  - dietz - Use Eric Dietz's cubex program (fast but lots of moves)
  - optimal - Use Michael Reid's optimal program (may take a long time)
  - gap - Use GAP word solution (can be slow)

EXAMPLES:

```
sage: C = RubiksCube("R U F L B D")
sage: C.solve()
'R U F L B D'
```

Dietz's program is much faster, but may give highly non-optimal solutions:

```
sage: s = C.solve('dietz'); s
"U' L' L' U L U' L U D L L D' L' D L' D' L D L' U' L D' L' U L' B' U' L' U B L D L D' U' L'
sage: C2 = RubiksCube(s)
sage: C == C2
True
```

**undo()**

Undo the last move of the Rubik's cube.

EXAMPLES:

```
sage: C = RubiksCube()
sage: D = C.move("R*U")
sage: D.undo() == C
True
```

`sage.groups.perm_gps.cubegroup.color_of_square` (*facet*, *colors*=['purple', 'yellow', 'red', 'green', 'orange', 'blue'])

Return the color the facet has in the solved state.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import *
sage: color_of_square(41)
'blue'
```

`sage.groups.perm_gps.cubegroup.create_poly` (*face*, *color*)

Create the polygon given by *face* with color *color*.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import create_poly, red
sage: create_poly('ur', red)
```

`sage.groups.perm_gps.cubegroup.cubie_centers` (*label*)

Return the cubie center list element given by *label*.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import cubie_centers
sage: cubie_centers(3)
[0, 2, 2]
```

`sage.groups.perm_gps.cubegroup.cubie_colors(label, state0)`

Return the color of the cubie given by `label` at `state0`.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import cubie_colors
sage: G = CubeGroup()
sage: g = G.parse("R*U")
sage: cubie_colors(3, G.facets(g))
[(1, 1, 1), (1, 0.63, 1), (1, 0.6, 0.3)]
```

`sage.groups.perm_gps.cubegroup.cubie_faces()`

This provides a map from the 6 faces of the 27 cubies to the 48 facets of the larger cube.

-1,-1,-1 is left, top, front

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import cubie_faces
sage: sorted(cubie_faces().items())
[((-1, -1, -1), [6, 17, 11, 0, 0, 0]),
 ((-1, -1, 0), [4, 0, 10, 0, 0, 0]),
 ((-1, -1, 1), [1, 0, 9, 0, 35, 0]),
 ((-1, 0, -1), [0, 20, 13, 0, 0, 0]),
 ((-1, 0, 0), [0, 0, -5, 0, 0, 0]),
 ((-1, 0, 1), [0, 0, 12, 0, 37, 0]),
 ((-1, 1, -1), [0, 22, 16, 41, 0, 0]),
 ((-1, 1, 0), [0, 0, 15, 44, 0, 0]),
 ((-1, 1, 1), [0, 0, 14, 46, 40, 0]),
 ((0, -1, -1), [7, 18, 0, 0, 0, 0]),
 ((0, -1, 0), [-6, 0, 0, 0, 0, 0]),
 ((0, -1, 1), [2, 0, 0, 0, 34, 0]),
 ((0, 0, -1), [0, -4, 0, 0, 0, 0]),
 ((0, 0, 0), [0, 0, 0, 0, 0, 0]),
 ((0, 0, 1), [0, 0, 0, 0, -2, 0]),
 ((0, 1, -1), [0, 23, 0, 42, 0, 0]),
 ((0, 1, 0), [0, 0, 0, -1, 0, 0]),
 ((0, 1, 1), [0, 0, 0, 47, 39, 0]),
 ((1, -1, -1), [8, 19, 0, 0, 0, 25]),
 ((1, -1, 0), [5, 0, 0, 0, 0, 26]),
 ((1, -1, 1), [3, 0, 0, 0, 33, 27]),
 ((1, 0, -1), [0, 21, 0, 0, 0, 28]),
 ((1, 0, 0), [0, 0, 0, 0, 0, -3]),
 ((1, 0, 1), [0, 0, 0, 0, 36, 29]),
 ((1, 1, -1), [0, 24, 0, 43, 0, 30]),
 ((1, 1, 0), [0, 0, 0, 45, 0, 31]),
 ((1, 1, 1), [0, 0, 0, 48, 38, 32])]
```

`sage.groups.perm_gps.cubegroup.index2singmaster(facet)`

Translate index used (eg, 43) to Singmaster facet notation (eg, fdr).

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import *
sage: index2singmaster(41)
'dlf'
```

`sage.groups.perm_gps.cubegroup.inv_list(lst)`

Input a list of ints  $1, \dots, m$  (in any order), outputs inverse perm.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import inv_list
sage: L = [2, 3, 1]
sage: inv_list(L)
[3, 1, 2]
```

`sage.groups.perm_gps.cubegroup.plot3d_cubie(cnt, clrs)`  
Plot the front, up and right face of a cubie centered at `cnt` and `rgbcolors` given by `clrs` (in the order FUR).  
Type `P.show()` to view.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import *
sage: clrF = blue; clrU = red; clrR = green
sage: P = plot3d_cubie([1/2, 1/2, 1/2], [clrF, clrU, clrR])
```

`sage.groups.perm_gps.cubegroup.polygon_plot3d(points, tilt=30, turn=30, **kwargs)`  
Plot a polygon viewed from an angle determined by `tilt`, `turn`, and vertices `points`.

**Warning:** The ordering of the points is important to get “correct” and if you add several of these plots together, the one added first is also drawn first (ie, addition of Graphics objects is not commutative).

The following example produced a green-colored square with vertices at the points indicated.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import polygon_plot3d, green
sage: P = polygon_plot3d([[1, 3, 1], [2, 3, 1], [2, 3, 2], [1, 3, 2], [1, 3, 1]], rgbcolor=green)
```

`sage.groups.perm_gps.cubegroup.rotation_list(tilt, turn)`  
Return a list  $[\sin(\theta), \sin(\phi), \cos(\theta), \cos(\phi)]$  of rotations where  $\theta$  is `tilt` and  $\phi$  is `turn`.

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import rotation_list
sage: rotation_list(30, 45)
[0.5, 0.707106781187, 0.866025403784, 0.707106781187]
```

`sage.groups.perm_gps.cubegroup.xproj(x, y, z, r)`  
Return the  $x$ -projection of  $(x, y, z)$  rotated by  $r$ .

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import rotation_list, xproj
sage: rot = rotation_list(30, 45)
sage: xproj(1, 2, 3, rot)
0.612372435696
```

`sage.groups.perm_gps.cubegroup.yproj(x, y, z, r)`  
Return the  $y$ -projection of  $(x, y, z)$  rotated by  $r$ .

EXAMPLES:

```
sage: from sage.groups.perm_gps.cubegroup import rotation_list, yproj
sage: rot = rotation_list(30, 45)
sage: yproj(1, 2, 3, rot)
1.37849741698
```

# BASE CLASSES FOR MATRIX GROUPS

Loading, saving, ... works:

```
sage: G = GL(2, 5); G
General Linear Group of degree 2 over Finite Field of size 5
sage: TestSuite(G).run()

sage: g = G.1; g
[4 1]
[4 0]
sage: TestSuite(g).run()
```

We test that [trac ticket #9437](#) is fixed:

```
sage: len(list(SL(2, Zmod(4))))
48
```

AUTHORS:

- William Stein: initial version
- David Joyner (2006-03-15): degree, base\_ring, \_contains\_, list, random, order methods; examples
- William Stein (2006-12): rewrite
- David Joyner (2007-12): Added invariant\_generators (with Martin Albrecht and Simon King)
- David Joyner (2008-08): Added module\_composition\_factors (interface to GAP's MeatAxe implementation) and as\_permutation\_group (returns isomorphic PermutationGroup).
- Simon King (2010-05): Improve invariant\_generators by using GAP for the construction of the Reynolds operator in Singular.

```
class sage.groups.matrix_gps.matrix_group.MatrixGroup_base
    Bases: sage.groups.group.Group
```

Base class for all matrix groups.

This base class just holds the base ring, but not the degree. So it can be a base for affine groups where the natural matrix is larger than the degree of the affine group. Makes no assumption about the group except that its elements have a `matrix()` method.

```
as_matrix_group()
```

Return a new matrix group from the generators.

This will throw away any extra structure (encoded in a derived class) that a group of special matrices has.

EXAMPLES:

```

sage: G = SU(4,GF(5))
sage: G.as_matrix_group()
Matrix group over Finite Field in a of size 5^2 with 2 generators (
[      a      0      0      0] [      1      0 4*a + 3      0]
[      0 2*a + 3      0      0] [      1      0      0      0]
[      0      0 4*a + 1      0] [      0 2*a + 4      0      1]
[      0      0      0      3*a], [      0 3*a + 1      0      0]
)

sage: G = GO(3,GF(5))
sage: G.as_matrix_group()
Matrix group over Finite Field of size 5 with 2 generators (
[2 0 0] [0 1 0]
[0 3 0] [1 4 4]
[0 0 1], [0 2 1]
)

```

**base\_field()**

Deprecated alias of `base_ring()`

**EXAMPLES:**

```

sage: G = SU(3,GF(5))
sage: G.base_field()
doctest:...: DeprecationWarning: Use base_ring() instead.
See http://trac.sagemath.org/14014 for details.
Finite Field in a of size 5^2

```

**field\_of\_definition(\*\*kws)**

Return a field that contains all the matrices in this matrix group.

**EXAMPLES:**

```

sage: G = SU(3,GF(5))
sage: G.base_ring()
Finite Field in a of size 5^2
sage: G.field_of_definition()
doctest:...: DeprecationWarning: Use base_ring() instead.
See http://trac.sagemath.org/14014 for details.
Finite Field in a of size 5^2
sage: G = GO(4,GF(7),1)
sage: G.field_of_definition()
Finite Field of size 7
sage: G.base_ring()
Finite Field of size 7

```

```

class sage.groups.matrix_gps.matrix_group.MatrixGroup_gap (degree,          base_ring,
                                                            libgap_group,          am-
                                                            bient=None,            cate-
                                                            gory=None)

```

Bases: `sage.groups.libgap_mixin.GroupMixinLibGAP`, `sage.groups.matrix_gps.matrix_group.MatrixGroup`, `sage.groups.libgap_wrapper.ParentLibGAP`

Base class for matrix groups that implements GAP interface.

**INPUT:**

- `degree` – integer. The degree (matrix size) of the matrix group.
- `base_ring` – ring. The base ring of the matrices.



- `libgap_group` – the defining libgap group.
- `ambient` – A derived class of `ParentLibGAP` or `None` (default). The ambient class if `libgap_group` has been defined as a subgroup.

**TESTS:**

```
sage: from sage.groups.matrix_gps.matrix_group import MatrixGroup_gap
sage: MatrixGroup_gap(2, ZZ, libgap.eval('GL(2, Integers)'))
Matrix group over Integer Ring with 3 generators (
[0 1]  [-1  0]  [1 1]
[1 0], [ 0  1], [0 1]
)
```

**Element**

alias of `MatrixGroupElement_gap`

**list()**

List all elements of this group.

This method overrides the matrix group enumerator in GAP which is very slow, see <http://tracker.gap-system.org/issues/369>.

**OUTPUT:**

A tuple containing all group elements in a random but fixed order.

**EXAMPLES:**

```
sage: F = GF(3)
sage: gens = [matrix(F, 2, [1, 0, -1, 1]), matrix(F, 2, [1, 1, 0, 1])]
sage: G = MatrixGroup(gens)
sage: G.cardinality()
24
sage: v = G.list()
sage: len(v)
24
sage: v[:5]
(
[1 0]  [2 0]  [0 1]  [0 2]  [1 2]
[0 1], [0 2], [2 0], [1 0], [2 2]
)
sage: all(g in G for g in G.list())
True
```

An example over a ring (see trac 5241):

```
sage: M1 = matrix(ZZ, 2, [[-1, 0], [0, 1]])
sage: M2 = matrix(ZZ, 2, [[1, 0], [0, -1]])
sage: M3 = matrix(ZZ, 2, [[-1, 0], [0, -1]])
sage: MG = MatrixGroup([M1, M2, M3])
sage: MG.list()
(
[1 0]  [ 1  0]  [-1  0]  [-1  0]
[0 1], [ 0 -1], [ 0  1], [ 0 -1]
)
sage: MG.list()[1]
[ 1  0]
[ 0 -1]
sage: MG.list()[1].parent()
Matrix group over Integer Ring with 3 generators (
[-1  0]  [ 1  0]  [-1  0]
```

```
[ 0  1], [ 0 -1], [ 0 -1]
)
```

An example over a field (see trac 10515):

```
sage: gens = [matrix(QQ, 2, [1, 0, 0, 1])]
sage: MatrixGroup(gens).list()
(
[1 0]
[0 1]
)
```

Another example over a ring (see trac 9437):

```
sage: len(SL(2, Zmod(4)).list())
48
```

An error is raised if the group is not finite:

```
sage: GL(2, ZZ).list()
Traceback (most recent call last):
...
NotImplementedError: group must be finite
```

```
class sage.groups.matrix_gps.matrix_group.MatrixGroup_generic(degree, base_ring,
                                                                category=None)
    Bases: sage.groups.matrix_gps.matrix_group.MatrixGroup_base
```

Base class for matrix groups over generic base rings

You should not use this class directly. Instead, use one of the more specialized derived classes.

INPUT:

- *degree* – integer. The degree (matrix size) of the matrix group.
- *base\_ring* – ring. The base ring of the matrices.

TESTS:

```
sage: G = GL(2, QQ)
sage: from sage.groups.matrix_gps.matrix_group import MatrixGroup_generic
sage: isinstance(G, MatrixGroup_generic)
True
```

**Element**

alias of `MatrixGroupElement_generic`

**degree()**

Return the degree of this matrix group.

OUTPUT:

Integer. The size (number of rows equals number of columns) of the matrices.

EXAMPLES:

```
sage: SU(5, 5).degree()
5
```

**hom(*x*)**

Return the group homomorphism defined by *x*

INPUT:

•x – a list/tuple/iterable of matrix group elements.

OUTPUT:

The group homomorphism defined by x.

EXAMPLES:

```
sage: G = MatrixGroup([matrix(GF(5), [[1,3],[0,1]])])
sage: H = MatrixGroup([matrix(GF(5), [[1,2],[0,1]])])
sage: G.hom([H.gen(0)])
Homomorphism : Matrix group over Finite Field of size 5 with 1 generators (
[1 3]
[0 1]
) --> Matrix group over Finite Field of size 5 with 1 generators (
[1 2]
[0 1]
)
```

**matrix\_space()**

Return the matrix space corresponding to this matrix group.

This is a matrix space over the field of definition of this matrix group.

EXAMPLES:

```
sage: F = GF(5); MS = MatrixSpace(F,2,2)
sage: G = MatrixGroup([MS(1), MS([1,2,3,4])])
sage: G.matrix_space()
Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 5
sage: G.matrix_space() is MS
True
```

`sage.groups.matrix_gps.matrix_group.is_MatrixGroup(x)`

Test whether x is a matrix group.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.matrix_group import is_MatrixGroup
sage: is_MatrixGroup(MatrixSpace(QQ,3))
False
sage: is_MatrixGroup(Mat(QQ,3))
False
sage: is_MatrixGroup(GL(2,ZZ))
True
sage: is_MatrixGroup(MatrixGroup([matrix(2,[1,1,0,1])]))
True
```



# MATRIX GROUP ELEMENTS

EXAMPLES:

```
sage: F = GF(3); MS = MatrixSpace(F,2,2)
sage: gens = [MS([[1,0],[0,1]]),MS([[1,1],[0,1]])]
sage: G = MatrixGroup(gens); G
Matrix group over Finite Field of size 3 with 2 generators (
[1 0]  [1 1]
[0 1], [0 1]
)
sage: g = G([[1,1],[0,1]])
sage: h = G([[1,2],[0,1]])
sage: g*h
[1 0]
[0 1]
```

You cannot add two matrices, since this is not a group operation. You can coerce matrices back to the matrix space and add them there:

```
sage: g + h
Traceback (most recent call last):
...
TypeError: unsupported operand type(s) for +:
'FinitelyGeneratedMatrixGroup_gap_with_category.element_class' and
'FinitelyGeneratedMatrixGroup_gap_with_category.element_class'

sage: g.matrix() + h.matrix()
[2 0]
[0 2]
```

Similarly, you cannot multiply group elements by scalars but you can do it with the underlying matrices:

```
sage: 2*g
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for '*': 'Integer Ring' and 'Matrix group over Finite Field
[1 0]  [1 1]
[0 1], [0 1]
)'
```

AUTHORS:

- David Joyner (2006-05): initial version David Joyner
- David Joyner (2006-05): various modifications to address William Stein's TODO's.

- William Stein (2006-12-09): many revisions.
- Volker Braun (2013-1) port to new Parent, libGAP.

**class** `sage.groups.matrix_gps.group_element.MatrixGroupElement_base`  
Bases: `sage.structure.element.MultiplicativeGroupElement`

Base class for elements of matrix groups.

You should use one of the two subclasses:

- `MatrixGroupElement_sage` implements the group multiplication using Sage matrices.
- `MatrixGroupElement_gap` implements the group multiplication using libGAP matrices.

The base class only assumes that derived classes implement `matrix()`.

EXAMPLES:

```
sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
      sage: gens = [MS([[1, 0], [0, 1]]), MS([[1, 1], [0, 1]])]
sage: G = MatrixGroup(gens)
sage: g = G.random_element()
sage: type(g)
<class 'sage.groups.matrix_gps.group_element.FinitelyGeneratedMatrixGroup_gap_with_category.element'>
```

**list()**

Return list representation of this matrix.

EXAMPLES:

```
sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[1, 0], [0, 1]]), MS([[1, 1], [0, 1]])]
sage: G = MatrixGroup(gens)
sage: g = G.0
sage: g
[1 0]
[0 1]
sage: g.list()
[[1, 0], [0, 1]]
```

**class** `sage.groups.matrix_gps.group_element.MatrixGroupElement_gap` (*parent*, *M*,  
*check=True*,  
*convert=True*)  
Bases: `sage.groups.libgap_mixin.GroupElementMixinLibGAP`,  
`sage.groups.matrix_gps.group_element.MatrixGroupElement_base`,  
`sage.groups.libgap_wrapper.ElementLibGAP`

Element of a matrix group over a generic ring.

The group elements are implemented as Sage matrices.

INPUT:

- *M* – a matrix.
- *parent* – the parent.
- *check* – bool (default: True). If true does some type checking.
- *convert* – bool (default: True). If true convert *M* to the right matrix space.

TESTS:

```

sage: MS = MatrixSpace(GF(3), 2, 2)
sage: G = MatrixGroup(MS([[1, 0], [0, 1]]), MS([[1, 1], [0, 1]]))
sage: G.gen(0)
[1 0]
[0 1]
sage: g = G.random_element()
sage: TestSuite(g).run()

```

**conjugacy\_class()**

Return the conjugacy class of self.

OUTPUT:

The conjugacy class of  $g$  in the group self. If self is the group denoted by  $G$ , this method computes the set  $\{x^{-1}gx \mid x \in G\}$ .

EXAMPLES:

```

sage: G = SL(2, QQ)
sage: g = G([[1, 1], [0, 1]])
sage: g.conjugacy_class()
Conjugacy class of [1 1]
[0 1] in Special Linear Group of degree 2 over Rational Field

```

**matrix()**

Obtain the usual matrix (as an element of a matrix space) associated to this matrix group element.

EXAMPLES:

```

sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[1, 0], [0, 1]]), MS([[1, 1], [0, 1]])]
sage: G = MatrixGroup(gens)
sage: G.gen(0).matrix()
[1 0]
[0 1]
sage: _.parent()
Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 3

```

```

class sage.groups.matrix_gps.group_element.MatrixGroupElement_generic(parent,
                                                                           M,
                                                                           check=True,
                                                                           con-
                                                                           vert=True)

```

Bases: `sage.groups.matrix_gps.group_element.MatrixGroupElement_base`

Element of a matrix group over a generic ring.

The group elements are implemented as Sage matrices.

INPUT:

- $M$  – a matrix.
- parent – the parent.
- **check** – bool (default: **True**). If true does some type checking.
- **convert** – bool (default: **True**). If true convert  $M$  to the right matrix space.

TESTS:

```

sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[1, 0], [0, 1]]), MS([[1, 1], [0, 1]])]

```

```
sage: G = MatrixGroup(gens)
sage: g = G.random_element()
sage: TestSuite(g).run()
```

**conjugacy\_class()**

Return the conjugacy class of `self`.

OUTPUT:

The conjugacy class of  $g$  in the group `self`. If `self` is the group denoted by  $G$ , this method computes the set  $\{x^{-1}gx \mid x \in G\}$ .

EXAMPLES:

```
sage: G = SL(2, GF(2))
sage: g = G.gens()[0]
sage: g.conjugacy_class()
Conjugacy class of [1 1]
[0 1] in Special Linear Group of degree 2 over Finite Field of size 2
```

**inverse()**

Return the inverse group element

OUTPUT:

A matrix group element.

EXAMPLES:

```
sage: G = GL(2, 3)
sage: g = G([1, 2, 1, 0]); g
[1 2]
[1 0]
sage: g.__invert__()
[0 1]
[2 1]
sage: g * (~g)
[1 0]
[0 1]
```

**matrix()**

Obtain the usual matrix (as an element of a matrix space) associated to this matrix group element.

One reason to compute the associated matrix is that matrices support a huge range of functionality.

EXAMPLES:

```
sage: k = GF(7); G = MatrixGroup([matrix(k, 2, [1, 1, 0, 1]), matrix(k, 2, [1, 0, 0, 2])])
sage: g = G.0
sage: g.matrix()
[1 1]
[0 1]
sage: parent(g.matrix())
Full MatrixSpace of 2 by 2 dense matrices over Finite Field of size 7
```

Matrices have extra functionality that matrix group elements do not have:

```
sage: g.matrix().charpoly('t')
t^2 + 5*t + 1
```

```
sage.groups.matrix_gps.group_element.is_MatrixGroupElement(x)
Test whether x is a matrix group element
```



INPUT:

•x – anything.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.group_element import is_MatrixGroupElement
sage: is_MatrixGroupElement('helloooo')
False

sage: G = GL(2,3)
sage: is_MatrixGroupElement(G.an_element())
True
```



# FINITELY GENERATED MATRIX GROUPS

This class is designed for computing with matrix groups defined by a finite set of generating matrices.

EXAMPLES:

```
sage: F = GF(3)
sage: gens = [matrix(F, 2, [1, 0, -1, 1]), matrix(F, 2, [1, 1, 0, 1])]
sage: G = MatrixGroup(gens)
sage: G.conjugacy_class_representatives()
(
[1 0] [0 2] [0 1] [2 0] [0 2] [0 1] [0 2]
[0 1], [1 1], [2 1], [0 2], [1 2], [2 2], [1 0]
)
```

The finitely generated matrix groups can also be constructed as subgroups of matrix groups:

```
sage: SL2Z = SL(2, ZZ)
sage: S, T = SL2Z.gens()
sage: SL2Z.subgroup([T^2])
Matrix group over Integer Ring with 1 generators (
[1 2]
[0 1]
)
```

AUTHORS:

- William Stein: initial version
- David Joyner (2006-03-15): degree, base\_ring, \_contains\_, list, random, order methods; examples
- William Stein (2006-12): rewrite
- David Joyner (2007-12): Added invariant\_generators (with Martin Albrecht and Simon King)
- David Joyner (2008-08): Added module\_composition\_factors (interface to GAP's MeatAxe implementation) and as\_permutation\_group (returns isomorphic PermutationGroup).
- Simon King (2010-05): Improve invariant\_generators by using GAP for the construction of the Reynolds operator in Singular.
- Volker Braun (2013-1) port to new Parent, libGAP.

```
class sage.groups.matrix_gps.finitely_generated.FinitelyGeneratedMatrixGroup_gap (degree,
                                                                                   base_ring,
                                                                                   lib-
                                                                                   gap_group,
                                                                                   am-
                                                                                   bi-
                                                                                   ent=None,
                                                                                   cat-
                                                                                   e-
                                                                                   gory=None)
```

Bases: `sage.groups.matrix_gps.matrix_group.MatrixGroup_gap`

Matrix group generated by a finite number of matrices.

EXAMPLES:

```
sage: m1 = matrix(GF(11), [[1,2],[3,4]])
sage: m2 = matrix(GF(11), [[1,3],[10,0]])
sage: G = MatrixGroup(m1, m2); G
Matrix group over Finite Field of size 11 with 2 generators (
[1 2] [ 1 3]
[3 4], [10 0]
)
sage: type(G)
<class 'sage.groups.matrix_gps.finitely_generated.FinitelyGeneratedMatrixGroup_gap_with_category'
sage: TestSuite(G).run()
```

**as\_permutation\_group** (*algorithm=None*)

Return a permutation group representation for the group.

In most cases occurring in practice, this is a permutation group of minimal degree (the degree begin determined from orbits under the group action). When these orbits are hard to compute, the procedure can be time-consuming and the degree may not be minimal.

INPUT:

- `algorithm=None` or `'smaller'`. In the latter case, try harder to find a permutation representation of small degree.

OUTPUT:

A permutation group isomorphic to self. The `algorithm='smaller'` option tries to return an isomorphic group of low degree, but is not guaranteed to find the smallest one.

EXAMPLES:

```
sage: MS = MatrixSpace(GF(2), 5, 5)
sage: A = MS([[0,0,0,0,1],[0,0,0,1,0],[0,0,1,0,0],[0,1,0,0,0],[1,0,0,0,0]])
sage: G = MatrixGroup([A])
sage: G.as_permutation_group()
Permutation Group with generators [(1,2)]
sage: MS = MatrixSpace(GF(7), 12, 12)
sage: GG = gap("ImfMatrixGroup( 12, 3 )")
sage: GG.GeneratorsOfGroup().Length()
3
sage: g1 = MS(eval(str(GG.GeneratorsOfGroup()[1]).replace("\n", "")))
sage: g2 = MS(eval(str(GG.GeneratorsOfGroup()[2]).replace("\n", "")))
sage: g3 = MS(eval(str(GG.GeneratorsOfGroup()[3]).replace("\n", "")))
sage: G = MatrixGroup([g1, g2, g3])
sage: G.cardinality()
21499084800
```

```

sage: set_random_seed(0); current_randstate().set_seed_gap()
sage: P = G.as_permutation_group()
sage: P.cardinality()
21499084800
sage: P.degree() # random output
144
sage: set_random_seed(3); current_randstate().set_seed_gap()
sage: Psmaller = G.as_permutation_group(algorithm="smaller")
sage: Psmaller.cardinality()
21499084800
sage: Psmaller.degree() # random output
108

```

In this case, the “smaller” option returned an isomorphic group of lower degree. The above example used GAP’s library of irreducible maximal finite (“imf”) integer matrix groups to construct the MatrixGroup  $G$  over  $\text{GF}(7)$ . The section “Irreducible Maximal Finite Integral Matrix Groups” in the GAP reference manual has more details.

### **invariant\_generators()**

Return invariant ring generators.

Computes generators for the polynomial ring  $F[x_1, \dots, x_n]^G$ , where  $G$  in  $GL(n, F)$  is a finite matrix group.

In the “good characteristic” case the polynomials returned form a minimal generating set for the algebra of  $G$ -invariant polynomials. In the “bad” case, the polynomials returned are primary and secondary invariants, forming a not necessarily minimal generating set for the algebra of  $G$ -invariant polynomials.

### **ALGORITHM:**

Wraps Singular’s `invariant_algebra_reynolds` and `invariant_ring` in `finvar.lib`.

### **EXAMPLES:**

```

sage: F = GF(7); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[0, 1], [-1, 0]]), MS([[1, 1], [2, 3]])]
sage: G = MatrixGroup(gens)
sage: G.invariant_generators()
[x1^7*x2 - x1*x2^7, x1^12 - 2*x1^9*x2^3 - x1^6*x2^6 + 2*x1^3*x2^9 + x2^12, x1^18 + 2*x1^15*x2^3]
sage: q = 4; a = 2
sage: MS = MatrixSpace(QQ, 2, 2)
sage: gen1 = [[1/a, (q-1)/a], [1/a, -1/a]]; gen2 = [[1, 0], [0, -1]]; gen3 = [[-1, 0], [0, 1]]
sage: G = MatrixGroup([MS(gen1), MS(gen2), MS(gen3)])
sage: G.cardinality()
12
sage: G.invariant_generators()
[x1^2 + 3*x2^2, x1^6 + 15*x1^4*x2^2 + 15*x1^2*x2^4 + 33*x2^6]
sage: F = GF(5); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[1, 2], [-1, 1]]), MS([[1, 1], [-1, 1]])]
sage: G = MatrixGroup(gens)
sage: G.invariant_generators() # long time (67s on sage.math, 2012)
[x1^20 + x1^16*x2^4 + x1^12*x2^8 + x1^8*x2^12 + x1^4*x2^16 + x2^20, x1^20*x2^4 + x1^16*x2^8]
sage: F=CyclotomicField(8)
sage: z=F.gen()
sage: a=z+1/z
sage: b=z^2
sage: MS=MatrixSpace(F, 2, 2)
sage: g1=MS([[1/a, 1/a], [1/a, -1/a]])
sage: g2=MS([[1, 0], [0, b]])
sage: g3=MS([[b, 0], [0, 1]])

```

```

sage: G=MatrixGroup([g1,g2,g3])
sage: G.invariant_generators() # long time (12s on sage.math, 2011)
[x1^8 + 14*x1^4*x2^4 + x2^8,
 x1^24 + 10626/1025*x1^20*x2^4 + 735471/1025*x1^16*x2^8 + 2704156/1025*x1^12*x2^12 + 735471/1025*x1^8*x2^16 + x2^24]

```

## AUTHORS:

- David Joyner, Simon King and Martin Albrecht.

## REFERENCES:

- Singular reference manual
- B. Sturmfels, “Algorithms in invariant theory”, Springer-Verlag, 1993.
- S. King, “Minimal Generating Sets of non-modular invariant rings of finite groups”, [Arxiv math/0703035](https://arxiv.org/abs/math/0703035).

**module\_composition\_factors** (*algorithm=None*)

Return a list of triples consisting of [base field, dimension, irreducibility], for each of the Meataxe composition factors modules. The `algorithm="verbose"` option returns more information, but in Meataxe notation.

## EXAMPLES:

```

sage: F=GF(3); MS=MatrixSpace(F,4,4)
sage: M=MS(0)
sage: M[0,1]=1; M[1,2]=1; M[2,3]=1; M[3,0]=1
sage: G = MatrixGroup(M)
sage: G.module_composition_factors()
[(Finite Field of size 3, 1, True),
 (Finite Field of size 3, 1, True),
 (Finite Field of size 3, 2, True)]
sage: F = GF(7); MS = MatrixSpace(F,2,2)
sage: gens = [MS([[0,1],[-1,0]]),MS([[1,1],[2,3]])]
sage: G = MatrixGroup(gens)
sage: G.module_composition_factors()
[(Finite Field of size 7, 2, True)]

```

Type `G.module_composition_factors(algorithm='verbose')` to get a more verbose version.

For more on MeatAxe notation, see <http://www.gap-system.org/Manuals/doc/ref/chap69.html>

**class** `sage.groups.matrix_gps.finitely_generated.FinitelyGeneratedMatrixGroup_generic` (*degree,*  
*base\_ring,*  
*gen-*  
*er-*  
*a-*  
*tor\_matrices,*  
*cat-*  
*e-*  
*gory=None*)

Bases: `sage.groups.matrix_gps.matrix_group.MatrixGroup_generic`

Matrix group generated by a finite number of matrices.

## EXAMPLES:

```

sage: m1 = matrix(SR, [[1,2],[3,4]])
sage: m2 = matrix(SR, [[1,3],[-1,0]])
sage: G = MatrixGroup(m1, m2)

```

```

sage: TestSuite(G).run()
sage: type(G)
<class 'sage.groups.matrix_gps.finitely_generated.FinitelyGeneratedMatrixGroup_generic_with_cate

sage: from sage.groups.matrix_gps.finitely_generated import .....: FinitelyGenera
sage: G = FinitelyGeneratedMatrixGroup_generic(2, QQ, [matrix(QQ, [[1,2],[3,4]])])
sage: G.gens()
(
[1 2]
[3 4]
)

```

**gen(*i*)**

Return the *i*-th generator

OUTPUT:

The *i*-th generator of the group.

EXAMPLES:

```

sage: H = GL(2, GF(3))
sage: h1, h2 = H([[1,0],[2,1]]), H([[1,1],[0,1]])
sage: G = H.subgroup([h1, h2])
sage: G.gen(0)
[1 0]
[2 1]
sage: G.gen(0).matrix() == h1.matrix()
True

```

**gens()**

Return the generators of the matrix group.

EXAMPLES:

```

sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[1,0],[0,1]]), MS([[1,1],[0,1]])]
sage: G = MatrixGroup(gens)
sage: gens[0] in G
True
sage: gens = G.gens()
sage: gens[0] in G
True
sage: gens = [MS([[1,0],[0,1]]), MS([[1,1],[0,1]])]

sage: F = GF(5); MS = MatrixSpace(F, 2, 2)
sage: G = MatrixGroup([MS(1), MS([1,2,3,4])])
sage: G
Matrix group over Finite Field of size 5 with 2 generators (
[1 0] [1 2]
[0 1], [3 4]
)
sage: G.gens()
(
[1 0] [1 2]
[0 1], [3 4]
)

```

**ngens()**

Return the number of generators

OUTPUT:

An integer. The number of generators.

EXAMPLES:

```
sage: H = GL(2, GF(3))
sage: h1, h2 = H([[1,0],[2,1]]), H([[1,1],[0,1]])
sage: G = H.subgroup([h1, h2])
sage: G.ngens()
2
```

```
sage.groups.matrix_gps.finitely_generated.MatrixGroup(*gens, **kws)
```

Return the matrix group with given generators.

INPUT:

- `*gens` – matrices, or a single list/tuple/iterable of matrices, or a matrix group.
- `check` – boolean keyword argument (optional, default: `True`). Whether to check that each matrix is invertible.

EXAMPLES:

```
sage: F = GF(5)
sage: gens = [matrix(F,2,[1,2,-1,1]), matrix(F,2,[1,1,0,1])]
sage: G = MatrixGroup(gens); G
Matrix group over Finite Field of size 5 with 2 generators (
[1 2] [1 1]
[4 1], [0 1]
)
```

In the second example, the generators are a matrix over  $\mathbb{Z}$ , a matrix over a finite field, and the integer 2. Sage determines that they both canonically map to matrices over the finite field, so creates that matrix group there:

```
sage: gens = [matrix(2,[1,2,-1,1]), matrix(GF(7),2,[1,1,0,1]), 2]
sage: G = MatrixGroup(gens); G
Matrix group over Finite Field of size 7 with 3 generators (
[1 2] [1 1] [2 0]
[6 1], [0 1], [0 2]
)
```

Each generator must be invertible:

```
sage: G = MatrixGroup([matrix(ZZ,2,[1,2,3,4])])
Traceback (most recent call last):
...
ValueError: each generator must be an invertible matrix

sage: F = GF(5); MS = MatrixSpace(F,2,2)
sage: MatrixGroup([MS.0])
Traceback (most recent call last):
...
ValueError: each generator must be an invertible matrix
sage: MatrixGroup([MS.0], check=False) # works formally but is mathematical nonsense
Matrix group over Finite Field of size 5 with 1 generators (
[1 0]
[0 0]
)
```

Some groups are not supported, or do not have much functionality implemented:



```

sage: G = SL(0, QQ)
Traceback (most recent call last):
...
ValueError: the degree must be at least 1

sage: SL2C = SL(2, CC); SL2C
Special Linear Group of degree 2 over Complex Field with 53 bits of precision
sage: SL2C.gens()
Traceback (most recent call last):
...
AttributeError: 'LinearMatrixGroup_generic_with_category' object has no attribute 'gens'

```

```
sage.groups.matrix_gps.finitely_generated.QuaternionMatrixGroupGF3()
```

The quaternion group as a set of  $2 \times 2$  matrices over  $GF(3)$ .

OUTPUT:

A matrix group consisting of  $2 \times 2$  matrices with elements from the finite field of order 3. The group is the quaternion group, the nonabelian group of order 8 that is not isomorphic to the group of symmetries of a square (the dihedral group  $D_4$ ).

---

**Note:** This group is most easily available via `groups.matrix.QuaternionGF3()`.

---

EXAMPLES:

The generators are the matrix representations of the elements commonly called  $I$  and  $J$ , while  $K$  is the product of  $I$  and  $J$ .

```

sage: from sage.groups.matrix_gps.finitely_generated import QuaternionMatrixGroupGF3
sage: Q = QuaternionMatrixGroupGF3()
sage: Q.order()
8
sage: aye = Q.gens()[0]; aye
[1 1]
[1 2]
sage: jay = Q.gens()[1]; jay
[2 1]
[1 1]
sage: kay = aye*jay; kay
[0 2]
[1 0]

```

TESTS:

```

sage: groups.matrix.QuaternionGF3()
Matrix group over Finite Field of size 3 with 2 generators (
[1 1] [2 1]
[1 2], [1 1]
)

sage: Q = QuaternionMatrixGroupGF3()
sage: QP = Q.as_permutation_group()
sage: QP.is_isomorphic(QuaternionGroup())
True
sage: H = DihedralGroup(4)
sage: H.order()
8
sage: QP.is_abelian(), H.is_abelian()

```

```
(False, False)
sage: QP.is_isomorphic(H)
False
```

`sage.groups.matrix_gps.finitely_generated.normalize_square_matrices(matrices)`  
Find a common space for all matrices.

OUTPUT:

A list of matrices, all elements of the same matrix space.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.finitely_generated import normalize_square_matrices
sage: m1 = [[1, 2], [3, 4]]
sage: m2 = [2, 3, 4, 5]
sage: m3 = matrix(QQ, [[1/2, 1/3], [1/4, 1/5]])
sage: m4 = MatrixGroup(m3).gen(0)
sage: normalize_square_matrices([m1, m2, m3, m4])
[
[1 2]  [2 3]  [1/2 1/3]  [1/2 1/3]
[3 4], [4 5], [1/4 1/5], [1/4 1/5]
]
```

# HOMOMORPHISMS BETWEEN MATRIX GROUPS

## AUTHORS:

- David Joyner and William Stein (2006-03): initial version
- David Joyner (2006-05): examples
- Simon King (2011-01): cleaning and improving code
- Volker Braun (2013-1) port to new Parent, libGAP.

```
class sage.groups.matrix_gps.morphism.MatrixGroupMap(parent)
    Bases: sage.categories.morphism.Morphism
```

Set-theoretic map between matrix groups.

### EXAMPLES:

```
sage: from sage.groups.matrix_gps.morphism import MatrixGroupMap
sage: MatrixGroupMap(ZZ.Hom(ZZ)) # mathematical nonsense
MatrixGroup endomorphism of Integer Ring
```

```
class sage.groups.matrix_gps.morphism.MatrixGroupMorphism(parent)
    Bases: sage.groups.matrix_gps.morphism.MatrixGroupMap
```

Set-theoretic map between matrix groups.

### EXAMPLES:

```
sage: from sage.groups.matrix_gps.morphism import MatrixGroupMap
sage: MatrixGroupMap(ZZ.Hom(ZZ)) # mathematical nonsense
MatrixGroup endomorphism of Integer Ring
```

```
class sage.groups.matrix_gps.morphism.MatrixGroupMorphism_im_gens(homset, imgsH,
                                                                    check=True)
```

Bases: sage.groups.matrix\_gps.morphism.MatrixGroupMorphism

Group morphism specified by images of generators.

Some Python code for wrapping GAP's GroupHomomorphismByImages function but only for matrix groups.  
Can be expensive if G is large.

### EXAMPLES:

```
sage: F = GF(5); MS = MatrixSpace(F, 2, 2)
sage: G = MatrixGroup([MS([1, 1, 0, 1])])
sage: H = MatrixGroup([MS([1, 0, 1, 1])])
```

```
sage: phi = G.hom(H.gens())
sage: phi
Homomorphism : Matrix group over Finite Field of size 5 with 1 generators (
[1 1]
[0 1]
) --> Matrix group over Finite Field of size 5 with 1 generators (
[1 0]
[1 1]
)
sage: phi(MS([1,1,0,1]))
[1 0]
[1 1]
sage: F = GF(7); MS = MatrixSpace(F,2,2)
sage: F.multiplicative_generator()
3
sage: G = MatrixGroup([MS([3,0,0,1])])
sage: a = G.gens()[0]^2
sage: phi = G.hom([a])
```

**gap()**

Return the underlying LibGAP group homomorphism

OUTPUT:

A LibGAP element.

EXAMPLES:

```
sage: F = GF(5); MS = MatrixSpace(F,2,2)
sage: G = MatrixGroup([MS([1,1,0,1])])
sage: H = MatrixGroup([MS([1,0,1,1])])
sage: phi = G.hom(H.gens())
sage: phi.gap()
CompositionMapping( [ (6,7,8,10,9) (11,13,14,12,15) (16,19,20,18,17) (21,25,22,24,23) ]
-> [ [ [ Z(5)^0, 0*Z(5) ], [ Z(5)^0, Z(5)^0 ] ] ], <action isomorphism> )
sage: type(_)
<type 'sage.libs.gap.element.GapElement'>
```

**image(J, \*args, \*\*kws)**

The image of an element or a subgroup.

INPUT:

J – a subgroup or an element of the domain of self.

OUTPUT:

The image of J under self

NOTE:

pushforward is the method that is used when a map is called on anything that is not an element of its domain. For historical reasons, we keep the alias `image()` for this method.

EXAMPLES:

```
sage: F = GF(7); MS = MatrixSpace(F,2,2)
sage: F.multiplicative_generator()
3
sage: G = MatrixGroup([MS([3,0,0,1])])
sage: a = G.gens()[0]^2
sage: phi = G.hom([a])
```

```

sage: phi.image(G.gens()[0]) # indirect doctest
[2 0]
[0 1]
sage: H = MatrixGroup([MS(a.list())])
sage: H
Matrix group over Finite Field of size 7 with 1 generators (
[2 0]
[0 1]
)

```

The following tests against trac ticket #10659:

```

sage: phi(H) # indirect doctest
Matrix group over Finite Field of size 7 with 1 generators (
[4 0]
[0 1]
)

```

### **kernel()**

Return the kernel of `self`, i.e., a matrix group.

EXAMPLES:

```

sage: F = GF(7); MS = MatrixSpace(F, 2, 2)
sage: F.multiplicative_generator()
3
sage: G = MatrixGroup([MS([3, 0, 0, 1])])
sage: a = G.gens()[0]^2
sage: phi = G.hom([a])
sage: phi.kernel()
Matrix group over Finite Field of size 7 with 1 generators (
[6 0]
[0 1]
)

```

### **pushforward(J, \*args, \*\*kws)**

The image of an element or a subgroup.

INPUT:

`J` – a subgroup or an element of the domain of `self`.

OUTPUT:

The image of `J` under `self`

NOTE:

`pushforward` is the method that is used when a `map` is called on anything that is not an element of its domain. For historical reasons, we keep the alias `image()` for this method.

EXAMPLES:

```

sage: F = GF(7); MS = MatrixSpace(F, 2, 2)
sage: F.multiplicative_generator()
3
sage: G = MatrixGroup([MS([3, 0, 0, 1])])
sage: a = G.gens()[0]^2
sage: phi = G.hom([a])
sage: phi.image(G.gens()[0]) # indirect doctest
[2 0]
[0 1]

```

```
sage: H = MatrixGroup([MS(a.list())])
sage: H
Matrix group over Finite Field of size 7 with 1 generators (
[2 0]
[0 1]
)
```

The following tests against trac ticket #10659:

```
sage: phi(H)      # indirect doctest
Matrix group over Finite Field of size 7 with 1 generators (
[4 0]
[0 1]
)
```

`sage.groups.matrix_gps.morphism.to_libgap(x)`  
Helper to convert `x` to a LibGAP matrix or matrix group element.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.morphism import to_libgap
sage: to_libgap(GL(2,3).gen(0))
[ [ Z(3), 0*Z(3) ], [ 0*Z(3), Z(3)^0 ] ]
sage: to_libgap(matrix(QQ, [[1,2],[3,4]]))
[ [ 1, 2 ], [ 3, 4 ] ]
```

# MATRIX GROUP HOMSETS

AUTHORS:

- William Stein (2006-05-07): initial version
- Volker Braun (2013-1) port to new Parent, libGAP

**class** `sage.groups.matrix_gps.homset.MatrixGroupHomset` ( $G, H$ )

Bases: `sage.groups.group_homset.GroupHomset_generic`

Return the homset of two matrix groups.

INPUT:

•  $G$  – a matrix group.

•  $H$  – a matrix group.

OUTPUT:

The homset of two matrix groups.

EXAMPLES:

```
sage: F = GF(5)
sage: gens = [matrix(F, 2, [1, 2, -1, 1]), matrix(F, 2, [1, 1, 0, 1])]
sage: G = MatrixGroup(gens)
sage: from sage.groups.matrix_gps.homset import MatrixGroupHomset
sage: MatrixGroupHomset(G, G)
Set of Homomorphisms from
Matrix group over Finite Field of size 5 with 2 generators (
[1 2] [1 1]
[4 1], [0 1]
) to Matrix group over Finite Field of size 5 with 2 generators (
[1 2] [1 1]
[4 1], [0 1]
)
```

`sage.groups.matrix_gps.homset.is_MatrixGroupHomset` ( $x$ )

Test whether  $x$  is a homset.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.homset import is_MatrixGroupHomset
sage: is_MatrixGroupHomset(4)
False

sage: F = GF(5)
sage: gens = [matrix(F, 2, [1, 2, -1, 1]), matrix(F, 2, [1, 1, 0, 1])]
```

```
sage: G = MatrixGroup(gens)
sage: from sage.groups.matrix_gps.homset import MatrixGroupHomset
sage: M = MatrixGroupHomset(G, G)
sage: is_MatrixGroupHomset(M)
True
```



# COXETER GROUPS AS MATRIX GROUPS

This implements a general Coxeter group as a matrix group by using the reflection representation.

AUTHORS:

- Travis Scrimshaw (2013-08-28): Initial version

```
class sage.groups.matrix_gps.coxeter_group.CoxeterMatrixGroup (coxeter_matrix,
                                                                base_ring,          in-
                                                                dex_set)
Bases: sage.groups.matrix_gps.finitely_generated.FinitelyGeneratedMatrixGroup_generic,
sage.structure.unique_representation.UniqueRepresentation
```

A Coxeter group represented as a matrix group.

Let  $(W, S)$  be a Coxeter system and we construct a vector space  $V$  over  $\mathbf{R}$  with a basis of  $\{\alpha_s\}_{s \in S}$  and inner product

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right)$$

where we have  $B(\alpha_s, \alpha_t) = -1$  if  $m_{st} = \infty$ . Next we define a representation  $\sigma_s : V \rightarrow V$  by

$$\sigma_s \lambda = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

This representation is faithful so we can represent the Coxeter group  $W$  by the set of matrices  $\sigma_s$  acting on  $V$ .

INPUT:

- `data` – a Coxeter matrix or graph or a Cartan type
- `base_ring` – (default: the universal cyclotomic field) the base ring which contains all values  $\cos(\pi/m_{ij})$  where  $(m_{ij})_{ij}$  is the Coxeter matrix
- `index_set` – (optional) an indexing set for the generators

For more on creating Coxeter groups, see `CoxeterGroup()`.

---

## Todo

Currently the label  $\infty$  is implemented as  $-1$  in the Coxeter matrix.

---

## EXAMPLES:

We can create Coxeter groups from Coxeter matrices:

```

sage: W = CoxeterGroup([[1, 6, 3], [6, 1, 10], [3, 10, 1]])
sage: W
Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[ 1  6  3]
[ 6  1 10]
[ 3 10  1]
sage: W.gens()
(
[
[          -1 -E(12)^7 + E(12)^11          1]
[          0          1          0]
[          0          0          1],

[
[          1          0          0]
[-E(12)^7 + E(12)^11      -1      E(20) - E(20)^9]
[          0          0          1],

[
[          1          0          0]
[          0          1          0]
[          1 E(20) - E(20)^9      -1]
]
)
sage: m = matrix([[1, 3, 3, 3], [3, 1, 3, 2], [3, 3, 1, 2], [3, 2, 2, 1]])
sage: W = CoxeterGroup(m)
sage: W.gens()
(
[-1  1  1  1] [ 1  0  0  0] [ 1  0  0  0] [ 1  0  0  0]
[ 0  1  0  0] [ 1 -1  1  0] [ 0  1  0  0] [ 0  1  0  0]
[ 0  0  1  0] [ 0  0  1  0] [ 1  1 -1  0] [ 0  0  1  0]
[ 0  0  0  1], [ 0  0  0  1], [ 0  0  0  1], [ 1  0  0 -1]
)
sage: a,b,c,d = W.gens()
sage: (a*b*c)^3
[ 5  1 -5  7]
[ 5  0 -4  5]
[ 4  1 -4  4]
[ 0  0  0  1]
sage: (a*b)^3
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: b*d == d*b
True
sage: a*c*a == c*a*c
True

```

We can create the matrix representation over different base rings and with different index sets. Note that the base ring must contain all  $2 * \cos(\pi/m_{ij})$  where  $(m_{ij})_{ij}$  is the Coxeter matrix:

```

sage: W = CoxeterGroup(m, base_ring=RR, index_set=['a', 'b', 'c', 'd'])
sage: W.base_ring()
Real Field with 53 bits of precision
sage: W.index_set()
('a', 'b', 'c', 'd')

sage: CoxeterGroup(m, base_ring=ZZ)
Coxeter group over Integer Ring with Coxeter matrix:
[1 3 3 3]
[3 1 3 2]

```

```
[3 3 1 2]
[3 2 2 1]
sage: CoxeterGroup([[1,4],[4,1]], base_ring=QQ)
Traceback (most recent call last):
...
TypeError: unable to convert sqrt(2) to a rational
```

Using the well-known conversion between Coxeter matrices and Coxeter graphs, we can input a Coxeter graph. Following the standard convention, edges with no label (i.e. labelled by None) are treated as 3:

```
sage: G = Graph([(0,3,None), (1,3,15), (2,3,7), (0,1,3)])
sage: W = CoxeterGroup(G); W
Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[ 1  3  2  3]
[ 3  1  2 15]
[ 2  2  1  7]
[ 3 15  7  1]
sage: G2 = W.coxeter_graph()
sage: CoxeterGroup(G2) is W
True
```

Because there currently is no class for  $\mathbb{Z} \cup \{\infty\}$ , labels of  $\infty$  are given by  $-1$  in the Coxeter matrix:

```
sage: G = Graph([(0,1,None), (1,2,4), (0,2,oo)])
sage: W = CoxeterGroup(G)
sage: W.coxeter_matrix()
[ 1  3 -1]
[ 3  1  4]
[-1  4  1]
```

We can also create Coxeter groups from Cartan types using the implementation keyword:

```
sage: W = CoxeterGroup(['D',5], implementation="reflection")
sage: W
Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[1 3 2 2 2]
[3 1 3 2 2]
[2 3 1 3 3]
[2 2 3 1 2]
[2 2 3 2 1]
sage: W = CoxeterGroup(['H',3], implementation="reflection")
sage: W
Coxeter group over Universal Cyclotomic Field with Coxeter matrix:
[1 3 2]
[3 1 5]
[2 5 1]
```

```
class Element (parent, M, check=True, convert=True)
    Bases: sage.groups.matrix_gps.group_element.MatrixGroupElement_generic
    A Coxeter group element.
```

```
canonical_matrix()
```

Return the matrix of self in the canonical faithful representation, which is self as a matrix.

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], implementation="reflection")
sage: a,b,c = W.gens()
sage: elt = a*b*c
```

```

sage: elt.canonical_matrix()
[ 0  0 -1]
[ 1  0 -1]
[ 0  1 -1]

```

**has\_right\_descent**(*i*)

Return whether *i* is a right descent of *self*.

A Coxeter system  $(W, S)$  has a root system defined as  $\{w(\alpha_s)\}_{w \in W}$  and we define the positive (resp. negative) roots  $\alpha = \sum_{s \in S} c_s \alpha_s$  by all  $c_s \geq 0$  (resp.  $c_s \leq 0$ ). In particular, we note that if  $\ell(ws) > \ell(w)$  then  $w(\alpha_s) > 0$  and if  $\ell(ws) < \ell(w)$  then  $w(\alpha_s) < 0$ . Thus  $i \in I$  is a right descent if  $w(\alpha_{s_i}) < 0$  or equivalently if the matrix representing  $w$  has all entries of the  $i$ -th column being non-positive.

INPUT:

- *i* – an element in the index set

EXAMPLES:

```

sage: W = CoxeterGroup(['A', 3], implementation="reflection")
sage: a, b, c = W.gens()
sage: elt = b*a*c
sage: map(lambda i: elt.has_right_descent(i), [1, 2, 3])
[True, False, True]

```

CoxeterMatrixGroup.**canonical\_representation**()

Return the canonical faithful representation of *self*, which is *self*.

EXAMPLES:

```

sage: W = CoxeterGroup([[1, 3], [3, 1]])
sage: W.canonical_representation() is W
True

```

CoxeterMatrixGroup.**coxeter\_graph**()

Return the Coxeter graph of *self*.

EXAMPLES:

```

sage: W = CoxeterGroup(['H', 3], implementation="reflection")
sage: G = W.coxeter_graph(); G
Graph on 3 vertices
sage: G.edges()
[(1, 2, None), (2, 3, 5)]
sage: CoxeterGroup(G) is W
True
sage: G = Graph([(0, 1, 3), (1, 2, oo)])
sage: W = CoxeterGroup(G)
sage: W.coxeter_graph() == G
True
sage: CoxeterGroup(W.coxeter_graph()) is W
True

```

CoxeterMatrixGroup.**coxeter\_matrix**()

Return the Coxeter matrix of *self*.

EXAMPLES:

```

sage: W = CoxeterGroup([[1, 3], [3, 1]])
sage: W.coxeter_matrix()
[1 3]
[3 1]
sage: W = CoxeterGroup(['H', 3])

```

```

sage: W.coxeter_matrix()
[1 3 2]
[3 1 5]
[2 5 1]

```

`CoxeterMatrixGroup.index_set()`

Return the index set of self.

EXAMPLES:

```

sage: W = CoxeterGroup([[1,3],[3,1]])
sage: W.index_set()
(0, 1)
sage: W = CoxeterGroup([[1,3],[3,1]], index_set=['x', 'y'])
sage: W.index_set()
('x', 'y')
sage: W = CoxeterGroup(['H',3])
sage: W.index_set()
(1, 2, 3)

```

`CoxeterMatrixGroup.simple_reflection(i)`

Return the simple reflection  $s_i$ .

INPUT:

- $i$  – an element from the index set

EXAMPLES:

```

sage: W = CoxeterGroup(['A',3], implementation="reflection")
sage: W.simple_reflection(1)
[-1  1  0]
[ 0  1  0]
[ 0  0  1]
sage: W.simple_reflection(2)
[ 1  0  0]
[ 1 -1  1]
[ 0  0  1]
sage: W.simple_reflection(3)
[ 1  0  0]
[ 0  1  0]
[ 0  1 -1]

```



# LINEAR GROUPS

## EXAMPLES:

```
sage: GL(4,QQ)
General Linear Group of degree 4 over Rational Field
sage: GL(1,ZZ)
General Linear Group of degree 1 over Integer Ring
sage: GL(100,RR)
General Linear Group of degree 100 over Real Field with 53 bits of precision
sage: GL(3,GF(49,'a'))
General Linear Group of degree 3 over Finite Field in a of size 7^2

sage: SL(2, ZZ)
Special Linear Group of degree 2 over Integer Ring
sage: G = SL(2,GF(3)); G
Special Linear Group of degree 2 over Finite Field of size 3
sage: G.is_finite()
True
sage: G.conjugacy_class_representatives()
(
 [1 0]  [0 2]  [0 1]  [2 0]  [0 2]  [0 1]  [0 2]
 [0 1], [1 1], [2 1], [0 2], [1 2], [2 2], [1 0]
)
sage: G = SL(6,GF(5))
sage: G.gens()
(
 [2 0 0 0 0 0]  [4 0 0 0 0 1]
 [0 3 0 0 0 0]  [4 0 0 0 0 0]
 [0 0 1 0 0 0]  [0 4 0 0 0 0]
 [0 0 0 1 0 0]  [0 0 4 0 0 0]
 [0 0 0 0 1 0]  [0 0 0 4 0 0]
 [0 0 0 0 0 1], [0 0 0 0 4 0]
)
```

## AUTHORS:

- William Stein: initial version
- David Joyner: degree, base\_ring, random, order methods; examples
- David Joyner (2006-05): added center, more examples, renamed random attributes, bug fixes.
- William Stein (2006-12): total rewrite
- Volker Braun (2013-1) port to new Parent, libGAP, extreme refactoring.

## REFERENCES:

- [KL] Peter Kleidman and Martin Liebeck. The subgroup structure of the finite classical groups. Cambridge University Press, 1990.
- [C] R. W. Carter. Simple groups of Lie type, volume 28 of Pure and Applied Mathematics. John Wiley and Sons, 1972.

```
sage.groups.matrix_gps.linear.GL(n, R, var='a')
```

Return the general linear group.

The general linear group  $GL(d, R)$  consists of all  $dimesd$  matrices that are invertible over the ring  $R$ .

---

**Note:** This group is also available via `groups.matrix.GL()`.

---

INPUT:

- `n` – a positive integer.
- `R` – ring or an integer. If an integer is specified, the corresponding finite field is used.
- `var` – variable used to represent generator of the finite field, if needed.

EXAMPLES:

```
sage: G = GL(6, GF(5))
sage: G.order()
110644754220000000000000000
sage: G.base_ring()
Finite Field of size 5
sage: G.category()
Category of finite groups
sage: TestSuite(G).run()
```

```
sage: G = GL(6, QQ)
sage: G.category()
Category of groups
sage: TestSuite(G).run()
```

Here is the Cayley graph of (relatively small) finite General Linear Group:

```
sage: g = GL(2, 3)
sage: d = g.cayley_graph(); d
Digraph on 48 vertices
sage: d.show(color_by_label=True, vertex_size=0.03, vertex_labels=False)
sage: d.show3d(color_by_label=True)
```

```
sage: F = GF(3); MS = MatrixSpace(F, 2, 2)
sage: gens = [MS([[2, 0], [0, 1]]), MS([[2, 1], [2, 0]])]
sage: G = MatrixGroup(gens)
sage: G.order()
48
sage: G.cardinality()
48
sage: H = GL(2, F)
sage: H.order()
48
sage: H == G
True
sage: H.gens() == G.gens()
True
sage: H.as_matrix_group() == H
True
```



```
sage: H.gens()
(
 [2 0]  [2 1]
 [0 1], [2 0]
)
```

**TESTS:**

```
sage: groups.matrix.GL(2, 3)
General Linear Group of degree 2 over Finite Field of size 3
```

```
class sage.groups.matrix_gps.linear.LinearMatrixGroup_gap(degree, base_ring, special,
                                                           sage_name, latex_string,
                                                           gap_command_string)
Bases: sage.groups.matrix_gps.named_group.NamedMatrixGroup_gap,
       sage.groups.matrix_gps.linear.LinearMatrixGroup_generic
Base class for “named” matrix groups using LibGAP
```

**INPUT:**

- degree – integer. The degree (number of rows/columns of matrices).
- base\_ring – rinrg. The base ring of the matrices.
- special – boolean. Whether the matrix group is special, that is, elements have determinant one.
- latex\_string – string. The latex representation.
- gap\_command\_string – string. The GAP command to construct the matrix group.

**EXAMPLES:**

```
sage: G = GL(2, GF(3))
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_gap
sage: isinstance(G, NamedMatrixGroup_gap)
True
```

```
class sage.groups.matrix_gps.linear.LinearMatrixGroup_generic(degree, base_ring,
                                                                special, sage_name,
                                                                latex_string)
Bases: sage.groups.matrix_gps.named_group.NamedMatrixGroup_generic
Base class for “named” matrix groups
```

**INPUT:**

- degree – integer. The degree (number of rows/columns of matrices).
- base\_ring – rinrg. The base ring of the matrices.
- special – boolean. Whether the matrix group is special, that is, elements have determinant one.
- latex\_string – string. The latex representation.

**EXAMPLES:**

```
sage: G = GL(2, QQ)
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_generic
sage: isinstance(G, NamedMatrixGroup_generic)
True
```

```
sage.groups.matrix_gps.linear.SL(n, R, var='a')
```

Return the special linear group.

The special linear group  $GL(d, R)$  consists of all  $d \times d$  matrices that are invertible over the ring  $R$  with determinant one.

---

**Note:** This group is also available via `groups.matrix.SL()`.

---

INPUT:

- `n` – a positive integer.
- `R` – ring or an integer. If an integer is specified, the corresponding finite field is used.
- `var` – variable used to represent generator of the finite field, if needed.

EXAMPLES:

```
sage: SL(3, GF(2))
Special Linear Group of degree 3 over Finite Field of size 2
sage: G = SL(15, GF(7)); G
Special Linear Group of degree 15 over Finite Field of size 7
sage: G.category()
Category of finite groups
sage: G.order()
19567125956981469620152190624295863411240180071820494789160673696387130667378823633935199663
sage: len(G.gens())
2
sage: G = SL(2, ZZ); G
Special Linear Group of degree 2 over Integer Ring
sage: G.gens()
(
 [ 0  1]  [1  1]
 [-1  0], [0  1]
)
```

Next we compute generators for  $SL_3(\mathbb{Z})$

```
sage: G = SL(3, ZZ); G
Special Linear Group of degree 3 over Integer Ring
sage: G.gens()
(
 [0 1 0]  [ 0  1  0]  [1  1  0]
 [0 0 1]  [-1  0  0]  [0  1  0]
 [1 0 0], [ 0  0  1], [0 0 1]
)
sage: TestSuite(G).run()
```

TESTS:

```
sage: groups.matrix.SL(2, 3)
Special Linear Group of degree 2 over Finite Field of size 3
```

# ORTHOGONAL LINEAR GROUPS

The general orthogonal group  $GO(n, R)$  consists of all  $n \times n$  matrices over the ring  $R$  preserving an  $n$ -ary positive definite quadratic form. In cases where there are multiple non-isomorphic quadratic forms, additional data needs to be specified to disambiguate. The special orthogonal group is the normal subgroup of matrices of determinant one.

In characteristics different from 2, a quadratic form is equivalent to a bilinear symmetric form. Furthermore, over the real numbers a positive definite quadratic form is equivalent to the diagonal quadratic form, equivalent to the bilinear symmetric form defined by the identity matrix. Hence, the orthogonal group  $GO(n, \mathbf{R})$  is the group of orthogonal matrices in the usual sense.

In the case of a finite field and if the degree  $n$  is even, then there are two inequivalent quadratic forms and a third parameter  $e$  must be specified to disambiguate these two possibilities. The index of  $SO(e, d, q)$  in  $GO(e, d, q)$  is 2 if  $q$  is odd, but  $SO(e, d, q) = GO(e, d, q)$  if  $q$  is even.)

**Warning:** GAP and Sage use different notations:

- GAP notation: The optional  $e$  comes first, that is,  $GO([e, ] d, q), SO([e, ] d, q)$ .
- Sage notation: The optional  $e$  comes last, the standard Python convention:  $GO(d, GF(q), e=0), SO(d, GF(q), e=0)$ .

## EXAMPLES:

```
sage: GO(3, 7)
General Orthogonal Group of degree 3 over Finite Field of size 7

sage: G = SO( 4, GF(7), 1); G
Special Orthogonal Group of degree 4 and form parameter 1 over Finite Field of size 7
sage: G.random_element()      # random
[4 3 5 2]
[6 6 4 0]
[0 4 6 0]
[4 4 5 1]
```

## TESTS:

```
sage: G = GO(3, GF(5))
sage: latex(G)
\text{GO}_{3}(\text{\boldsymbol{F}}_{5})
sage: G = SO(3, GF(5))
sage: latex(G)
\text{SO}_{3}(\text{\boldsymbol{F}}_{5})
sage: G = SO(4, GF(5), 1)
sage: latex(G)
\text{SO}_{4}(\text{\boldsymbol{F}}_{5}, +)
```

## AUTHORS:

- David Joyner (2006-03): initial version
- David Joyner (2006-05): added examples, `_latex_`, `__str__`, `gens`, `as_matrix_group`
- William Stein (2006-12-09): rewrite
- Volker Braun (2013-1) port to new Parent, libGAP, extreme refactoring.

```
sage.groups.matrix_gps.orthogonal.GO(n, R, e=0, var='a')
```

Return the general orthogonal group.

The general orthogonal group  $GO(n, R)$  consists of all  $n \times n$  matrices over the ring  $R$  preserving an  $n$ -ary positive definite quadratic form. In cases where there are multiple non-isomorphic quadratic forms, additional data needs to be specified to disambiguate.

In the case of a finite field and if the degree  $n$  is even, then there are two inequivalent quadratic forms and a third parameter  $e$  must be specified to disambiguate these two possibilities.

---

**Note:** This group is also available via `groups.matrix.GO()`.

---

## INPUT:

- $n$  – integer. The degree.
- $R$  – ring or an integer. If an integer is specified, the corresponding finite field is used.
- $e$  –  $+1$  or  $-1$ , and ignored by default. Only relevant for finite fields and if the degree is even. A parameter that distinguishes inequivalent invariant forms.

## OUTPUT:

The general orthogonal group of given degree, base ring, and choice of invariant form.

## EXAMPLES:

```
sage: GO(3, GF(7)) General Orthogonal Group of degree 3 over Finite Field of size 7
sage: GO(3, GF(7)).order() 672
sage: GO(3, GF(7)).gens() ([3 0 0] [0 1 0] [0 5 0] [1 6 6] [0 0 1], [0 2 1])
```

## TESTS:

```
sage: groups.matrix.GO(2, 3, e=-1)
General Orthogonal Group of degree 2 and form parameter -1 over Finite Field of size 3
```

```
class sage.groups.matrix_gps.orthogonal.OrthogonalMatrixGroup_gap(degree,
                                                                    base_ring,
                                                                    special,
                                                                    sage_name,
                                                                    latex_string,
                                                                    gap_command_string)
Bases: sage.groups.matrix_gps.orthogonal.OrthogonalMatrixGroup_generic,
       sage.groups.matrix_gps.named_group.NamedMatrixGroup_gap
```

Base class for “named” matrix groups using LibGAP

## INPUT:

- `degree` – integer. The degree (number of rows/columns of matrices).
- `base_ring` – ring. The base ring of the matrices.

- `special` – boolean. Whether the matrix group is special, that is, elements have determinant one.
- `latex_string` – string. The latex representation.
- `gap_command_string` – string. The GAP command to construct the matrix group.

**EXAMPLES:**

```
sage: G = GL(2, GF(3))
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_gap
sage: isinstance(G, NamedMatrixGroup_gap)
True
```

**invariant\_bilinear\_form()**

Return the symmetric bilinear form preserved by the orthogonal group.

**OUTPUT:**

A matrix  $M$  such that, for every group element  $g$ , the identity  $gm g^T = m$  holds. In characteristic different from two, this uniquely determines the orthogonal group.

**EXAMPLES:**

```
sage: G = GO(4, GF(7), -1)
sage: G.invariant_bilinear_form()
[0 1 0 0]
[1 0 0 0]
[0 0 2 0]
[0 0 0 2]
```

```
sage: G = GO(4, GF(7), +1)
sage: G.invariant_bilinear_form()
[0 1 0 0]
[1 0 0 0]
[0 0 6 0]
[0 0 0 2]
```

```
sage: G = GO(4, QQ)
sage: G.invariant_bilinear_form()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: G = SO(4, GF(7), -1)
sage: G.invariant_bilinear_form()
[0 1 0 0]
[1 0 0 0]
[0 0 2 0]
[0 0 0 2]
```

**invariant\_quadratic\_form()**

Return the quadratic form preserved by the orthogonal group.

**OUTPUT:**

The matrix  $Q$  defining “orthogonal” as follows. The matrix determines a quadratic form  $q$  on the natural vector space  $V$ , on which  $G$  acts, by  $q(v) = vQv^t$ . A matrix  $M$  is an element of the orthogonal group if  $q(v) = q(vM)$  for all  $v \in V$ .

**EXAMPLES:**

```
sage: G = GO(4, GF(7), -1)
sage: G.invariant_quadratic_form()
[0 1 0 0]
[0 0 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: G = GO(4, GF(7), +1)
sage: G.invariant_quadratic_form()
[0 1 0 0]
[0 0 0 0]
[0 0 3 0]
[0 0 0 1]
```

```
sage: G = GO(4, QQ)
sage: G.invariant_quadratic_form()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: G = SO(4, GF(7), -1)
sage: G.invariant_quadratic_form()
[0 1 0 0]
[0 0 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
class sage.groups.matrix_gps.orthogonal.OrthogonalMatrixGroup_generic(degree,
                                                                           base_ring,
                                                                           special,
                                                                           sage_name,
                                                                           la-
                                                                           tex_string)
```

Bases: `sage.groups.matrix_gps.named_group.NamedMatrixGroup_generic`

Base class for “named” matrix groups

INPUT:

- `degree` – integer. The degree (number of rows/columns of matrices).
- `base_ring` – ring. The base ring of the matrices.
- `special` – boolean. Whether the matrix group is special, that is, elements have determinant one.
- `latex_string` – string. The latex representation.

EXAMPLES:

```
sage: G = GL(2, QQ)
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_generic
sage: isinstance(G, NamedMatrixGroup_generic)
True
```

**`invariant_bilinear_form()`**

Return the symmetric bilinear form preserved by the orthogonal group.

OUTPUT:

A matrix.

## EXAMPLES:

```
sage: GO(2,3,+1).invariant_bilinear_form()
[0 1]
[1 0]
sage: GO(2,3,-1).invariant_bilinear_form()
[2 1]
[1 1]
```

**invariant\_quadratic\_form()**

Return the quadratic form preserved by the orthogonal group.

## OUTPUT:

A matrix.

## EXAMPLES:

```
sage: GO(2,3,+1).invariant_quadratic_form()
[0 1]
[0 0]
sage: GO(2,3,-1).invariant_quadratic_form()
[1 1]
[0 2]
```

```
sage.groups.matrix_gps.orthogonal.SO(n, R, e=None, var='a')
```

Return the special orthogonal group.

The special orthogonal group  $GO(n, R)$  consists of all  $n \times n$  matrices with determinant one over the ring  $R$  preserving an  $n$ -ary positive definite quadratic form. In cases where there are multiple non-isomorphic quadratic forms, additional data needs to be specified to disambiguate.

---

**Note:** This group is also available via `groups.matrix.SO()`.

---

## INPUT:

- $n$  – integer. The degree.
- $R$  – ring or an integer. If an integer is specified, the corresponding finite field is used.
- $e$  –  $+1$  or  $-1$ , and ignored by default. Only relevant for finite fields and if the degree is even. A parameter that distinguishes inequivalent invariant forms.

## OUTPUT:

The special orthogonal group of given degree, base ring, and choice of invariant form.

## EXAMPLES:

```
sage: G = SO(3,GF(5))
sage: G
Special Orthogonal Group of degree 3 over Finite Field of size 5

sage: G = SO(3,GF(5))
sage: G.gens()
(
[2 0 0] [3 2 3] [1 4 4]
[0 3 0] [0 2 0] [4 0 0]
[0 0 1], [0 3 1], [2 0 4]
)
sage: G = SO(3,GF(5))
sage: G.as_matrix_group()
```

```
Matrix group over Finite Field of size 5 with 3 generators (  
[2 0 0] [3 2 3] [1 4 4]  
[0 3 0] [0 2 0] [4 0 0]  
[0 0 1], [0 3 1], [2 0 4]  
)
```

**TESTS:**

```
sage: groups.matrix.SO(2, 3, e=1)
```

Special Orthogonal Group of degree 2 and form parameter 1 over Finite Field of size 3

```
sage.groups.matrix_gps.orthogonal.normalize_args_e(degree, ring, e)
```

Normalize the arguments that relate the choice of quadratic form for special orthogonal groups over finite fields.

**INPUT:**

- *degree* – integer. The degree of the affine group, that is, the dimension of the affine space the group is acting on.
- *ring* – a ring. The base ring of the affine space.
- *e* – integer, one of +1, 0, -1. Only relevant for finite fields and if the degree is even. A parameter that distinguishes inequivalent invariant forms.

**OUTPUT:**

The integer *e* with values required by GAP.

**TESTS:**

```
sage: from sage.groups.matrix_gps.orthogonal import normalize_args_e
```

```
sage: normalize_args_e(2, GF(3), +1)
```

```
1
```

```
sage: normalize_args_e(3, GF(3), 0)
```

```
0
```

```
sage: normalize_args_e(3, GF(3), +1)
```

```
0
```

```
sage: normalize_args_e(2, GF(3), 0)
```

```
Traceback (most recent call last):
```

```
...
```

```
ValueError: must have e=-1 or e=1 for even degree
```



# SYMPLECTIC LINEAR GROUPS

## EXAMPLES:

```
sage: G = Sp(4, GF(7)); G
Symplectic Group of degree 4 over Finite Field of size 7
sage: g = prod(G.gens()); g
[3 0 3 0]
[1 0 0 0]
[0 1 0 1]
[0 2 0 0]
sage: m = g.matrix()
sage: m * G.invariant_form() * m.transpose() == G.invariant_form()
True
sage: G.order()
276595200
```

## AUTHORS:

- David Joyner (2006-03): initial version, modified from `special_linear` (by W. Stein)
- Volker Braun (2013-1) port to new Parent, libGAP, extreme refactoring.

`sage.groups.matrix_gps.symplectic.Sp(n, R, var='a')`

Return the symplectic group.

The special linear group  $GL(d, R)$  consists of all  $d \times d$  matrices that are invertible over the ring  $R$  with determinant one.

---

**Note:** This group is also available via `groups.matrix.Sp()`.

---

## INPUT:

- `n` – a positive integer.
- `R` – ring or an integer. If an integer is specified, the corresponding finite field is used.
- `var` – variable used to represent generator of the finite field, if needed.

## EXAMPLES:

```
sage: Sp(4, 5)
Symplectic Group of degree 4 over Finite Field of size 5

sage: Sp(4, IntegerModRing(15))
Symplectic Group of degree 4 over Ring of integers modulo 15

sage: Sp(3, GF(7))
```

```
Traceback (most recent call last):
...
ValueError: the degree must be even
```

**TESTS:**

```
sage: groups.matrix.Sp(2, 3)
Symplectic Group of degree 2 over Finite Field of size 3
```

```
sage: G = Sp(4, 5)
sage: TestSuite(G).run()
```

```
class sage.groups.matrix_gps.symplectic.SymplecticMatrixGroup_gap(degree,
                                                                    base_ring,
                                                                    special,
                                                                    sage_name,
                                                                    latex_string,
                                                                    gap_command_string)
```

```
Bases:      sage.groups.matrix_gps.symplectic.SymplecticMatrixGroup_generic,
sage.groups.matrix_gps.named_group.NamedMatrixGroup_gap
```

Symplectic group in GAP

**EXAMPLES:**

```
sage: Sp(2, 4)
Symplectic Group of degree 2 over Finite Field in a of size 2^2
```

```
sage: latex(Sp(4, 5))
\text{Sp}_{4}(\text{Bold{F}}_{5})
```

**invariant\_form()**

Return the quadratic form preserved by the orthogonal group.

**OUTPUT:**

A matrix.

**EXAMPLES:**

```
sage: Sp(4, GF(3)).invariant_form()
[0 0 0 1]
[0 0 1 0]
[0 2 0 0]
[2 0 0 0]
```

```
class sage.groups.matrix_gps.symplectic.SymplecticMatrixGroup_generic(degree,
                                                                    base_ring,
                                                                    special,
                                                                    sage_name,
                                                                    la-
                                                                    tex_string)
```

```
Bases: sage.groups.matrix_gps.named_group.NamedMatrixGroup_generic
```

Base class for “named” matrix groups

**INPUT:**

- degree – integer. The degree (number of rows/columns of matrices).
- base\_ring – rinrg. The base ring of the matrices.

- `special` – boolean. Whether the matrix group is special, that is, elements have determinant one.
- `latex_string` – string. The latex representation.

EXAMPLES:

```
sage: G = GL(2, QQ)
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_generic
sage: isinstance(G, NamedMatrixGroup_generic)
True
```

**`invariant_form()`**

Return the quadratic form preserved by the orthogonal group.

OUTPUT:

A matrix.

EXAMPLES:

```
sage: Sp(4, QQ).invariant_form()
[0 0 0 1]
[0 0 1 0]
[0 1 0 0]
[1 0 0 0]
```



# UNITARY GROUPS $GU(N, Q)$ AND $SU(N, Q)$

These are  $n \times n$  unitary matrices with entries in  $GF(q^2)$ .

EXAMPLES:

```
sage: G = SU(3, 5)
sage: G.order()
378000
sage: G
Special Unitary Group of degree 3 over Finite Field in a of size 5^2
sage: G.gens()
(
[      a      0      0] [4*a  4  1]
[      0 2*a + 2      0] [ 4  4  0]
[      0      0 3*a], [ 1  0  0]
)
sage: G.base_ring()
Finite Field in a of size 5^2
```

AUTHORS:

- David Joyner (2006-03): initial version, modified from special\_linear (by W. Stein)
- David Joyner (2006-05): minor additions (examples, \_latex\_, \_\_str\_\_, gens)
- William Stein (2006-12): rewrite
- Volker Braun (2013-1) port to new Parent, libGAP, extreme refactoring.

```
sage.groups.matrix_gps.unitary.GU(n, R, var='a')
```

Return the general unitary group.

The general unitary group  $GU(d, R)$  consists of all  $d \times d$  matrices that preserve a nondegenerate sesquilinear form over the ring  $R$ .

---

**Note:** For a finite field the matrices that preserve a sesquilinear form over  $F_q$  live over  $F_{q^2}$ . So  $GU(n, q)$  for integer  $q$  constructs the matrix group over the base ring  $GF(q^2)$ .

---

---

**Note:** This group is also available via `groups.matrix.GU()`.

---

INPUT:

- $n$  – a positive integer.
- $R$  – ring or an integer. If an integer is specified, the corresponding finite field is used.
- $\text{var}$  – variable used to represent generator of the finite field, if needed.

OUTPUT:

Return the general unitary group.

EXAMPLES:

```
sage: G = GU(3, 7); G
General Unitary Group of degree 3 over Finite Field in a of size 7^2
sage: G.gens()
(
 [ a    0    0] [6*a    6    1]
 [ 0    1    0] [ 6    6    0]
 [ 0    0 5*a] [ 1    0    0]
)
sage: GU(2, QQ)
General Unitary Group of degree 2 over Rational Field

sage: G = GU(3, 5, var='beta')
sage: G.base_ring()
Finite Field in beta of size 5^2
sage: G.gens()
(
 [ beta    0    0] [4*beta    4    1]
 [ 0    1    0] [ 4    4    0]
 [ 0    0 3*beta] [ 1    0    0]
)
```

TESTS:

```
sage: groups.matrix.GU(2, 3)
General Unitary Group of degree 2 over Finite Field in a of size 3^2
```

`sage.groups.matrix_gps.unitary.SU( $n, R, \text{var}='a'$ )`

The special unitary group  $SU(d, R)$  consists of all  $d \times d$  matrices that preserve a nondegenerate sesquilinear form over the ring  $R$  and have determinant one.

---

**Note:** For a finite field the matrices that preserve a sesquilinear form over  $F_q$  live over  $F_{q^2}$ . So  $SU(n, q)$  for integer  $q$  constructs the matrix group over the base ring  $\text{GF}(q^2)$ .

---

---

**Note:** This group is also available via `groups.matrix.SU()`.

---

INPUT:

- $n$  – a positive integer.
- $R$  – ring or an integer. If an integer is specified, the corresponding finite field is used.
- $\text{var}$  – variable used to represent generator of the finite field, if needed.

OUTPUT:

Return the special unitary group.

EXAMPLES:

```

sage: SU(3, 5)
Special Unitary Group of degree 3 over Finite Field in a of size 5^2
sage: SU(3, GF(5))
Special Unitary Group of degree 3 over Finite Field in a of size 5^2
sage: SU(3, QQ)
Special Unitary Group of degree 3 over Rational Field

```

## TESTS:

```

sage: groups.matrix.SU(2, 3)
Special Unitary Group of degree 2 over Finite Field in a of size 3^2

```

```

class sage.groups.matrix_gps.unitary.UnitaryMatrixGroup_gap(degree,      base_ring,
                                                             special,      sage_name,
                                                             latex_string,
                                                             gap_command_string)
Bases:      sage.groups.matrix_gps.unitary.UnitaryMatrixGroup_generic,
sage.groups.matrix_gps.named_group.NamedMatrixGroup_gap

```

Base class for “named” matrix groups using LibGAP

## INPUT:

- degree – integer. The degree (number of rows/columns of matrices).
- base\_ring – ring. The base ring of the matrices.
- special – boolean. Whether the matrix group is special, that is, elements have determinant one.
- latex\_string – string. The latex representation.
- gap\_command\_string – string. The GAP command to construct the matrix group.

## EXAMPLES:

```

sage: G = GL(2, GF(3))
sage: from sage.groups.matrix_gps.named_group import NamedMatrixGroup_gap
sage: isinstance(G, NamedMatrixGroup_gap)
True

```

```

class sage.groups.matrix_gps.unitary.UnitaryMatrixGroup_generic(degree, base_ring,
                                                                  special,
                                                                  sage_name,
                                                                  latex_string)

```

Bases: sage.groups.matrix\_gps.named\_group.**NamedMatrixGroup\_generic**

General Unitary Group over arbitrary rings.

## EXAMPLES:

```

sage: G = GU(3, GF(7)); G
General Unitary Group of degree 3 over Finite Field in a of size 7^2
sage: latex(G)
\text{GU}_{3}(\text{Bold{F}}_{7^2})

sage: G = SU(3, GF(5)); G
Special Unitary Group of degree 3 over Finite Field in a of size 5^2
sage: latex(G)
\text{SU}_{3}(\text{Bold{F}}_{5^2})

```

```

sage.groups.matrix_gps.unitary.finite_field_sqrt(ring)
Helper function.

```

INPUT:

A ring.

OUTPUT:

Integer  $q$  such that `ring` is the finite field with  $q^2$  elements.

EXAMPLES:

```
sage: from sage.groups.matrix_gps.unitary import finite_field_sqrt
```

```
sage: finite_field_sqrt(GF(4, 'a'))
```

```
2
```



# AFFINE GROUPS

AUTHORS:

- Volker Braun: initial version

```
class sage.groups.affine_gps.affine_group.AffineGroup(degree, ring)
    Bases: sage.structure.unique_representation.UniqueRepresentation,
           sage.groups.group.Group
```

An affine group.

The affine group  $\text{Aff}(A)$  (or general affine group) of an affine space  $A$  is the group of all invertible affine transformations from the space into itself.

If we let  $A_V$  be the affine space of a vector space  $V$  (essentially, forgetting what is the origin) then the affine group  $\text{Aff}(A_V)$  is the group generated by the general linear group  $GL(V)$  together with the translations. Recall that the group of translations acting on  $A_V$  is just  $V$  itself. The general linear and translation subgroups do not quite commute, and in fact generate the semidirect product

$$\text{Aff}(A_V) = GL(V) \ltimes V.$$

As such, the group elements can be represented by pairs  $(A, b)$  of a matrix and a vector. This pair then represents the transformation

$$x \mapsto Ax + b.$$

We can also represent affine transformations as linear transformations by considering  $\dim(V) + 1$  dimensional space. We take the affine transformation  $(A, b)$  to

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

and lifting  $x = (x_1, \dots, x_n)$  to  $(x_1, \dots, x_n, 1)$ . Here the  $(n + 1)$ -th component is always 1, so the linear representations acts on the affine hyperplane  $x_{n+1} = 1$  as affine transformations which can be seen directly from the matrix multiplication.

INPUT:

Something that defines an affine space. For example

- An affine space itself:
  - $A$  – affine space
- A vector space:
  - $V$  – a vector space
- Degree and base ring:

- degree – An integer. The degree of the affine group, that is, the dimension of the affine space the group is acting on.
- ring – A ring or an integer. The base ring of the affine space. If an integer is given, it must be a prime power and the corresponding finite field is constructed.
- var – (Default: 'a') Keyword argument to specify the finite field generator name in the case where ring is a prime power.

**EXAMPLES:**

```
sage: F = AffineGroup(3, QQ); F
Affine Group of degree 3 over Rational Field
sage: F(matrix(QQ, [[1,2,3],[4,5,6],[7,8,0]]), vector(QQ, [10,11,12]))
[1 2 3] [10]
x |-> [4 5 6] x + [11]
[7 8 0] [12]
sage: F([[1,2,3],[4,5,6],[7,8,0]], [10,11,12])
[1 2 3] [10]
x |-> [4 5 6] x + [11]
[7 8 0] [12]
sage: F([1,2,3,4,5,6,7,8,0], [10,11,12])
[1 2 3] [10]
x |-> [4 5 6] x + [11]
[7 8 0] [12]
```

Instead of specifying the complete matrix/vector information, you can also create special group elements:

```
sage: F.linear([1,2,3,4,5,6,7,8,0])
[1 2 3] [0]
x |-> [4 5 6] x + [0]
[7 8 0] [0]
sage: F.translation([1,2,3])
[1 0 0] [1]
x |-> [0 1 0] x + [2]
[0 0 1] [3]
```

Some additional ways to create affine groups:

```
sage: A = AffineSpace(2, GF(4, 'a')); A
Affine Space of dimension 2 over Finite Field in a of size 2^2
sage: G = AffineGroup(A); G
Affine Group of degree 2 over Finite Field in a of size 2^2
sage: G is AffineGroup(2,4) # shorthand
True

sage: V = ZZ^3; V
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: AffineGroup(V)
Affine Group of degree 3 over Integer Ring
```

**REFERENCES:**

- [Wikipedia article Affine\\_group](#)

**Element**

alias of `AffineGroupElement`

**degree()**

Return the dimension of the affine space.

**OUTPUT:**

An integer.

EXAMPLES:

```
sage: G = AffineGroup(6, GF(5))
sage: g = G.an_element()
sage: G.degree()
6
sage: G.degree() == g.A().nrows() == g.A().ncols() == g.b().degree()
True
```

**linear(A)**

Construct the general linear transformation by A.

INPUT:

- A – anything that determines a matrix

OUTPUT:

The affine group element  $x \mapsto Ax$ .

EXAMPLES:

```
sage: G = AffineGroup(3, GF(5))
sage: G.linear([1, 2, 3, 4, 5, 6, 7, 8, 0])
      [1 2 3]      [0]
x |-> [4 0 1] x + [0]
      [2 3 0]      [0]
```

**linear\_space()**

Return the space of the affine transformations represented as linear transformations.

We can represent affine transformations  $Ax + b$  as linear transformations by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

and lifting  $x = (x_1, \dots, x_n)$  to  $(x_1, \dots, x_n, 1)$ .

See Also:

- `sage.groups.affine_gps.group_element.AffineGroupElement.matrix()`

EXAMPLES:

```
sage: G = AffineGroup(3, GF(5))
sage: G.linear_space()
Full MatrixSpace of 4 by 4 dense matrices over Finite Field of size 5
```

**matrix\_space()**

Return the space of matrices representing the general linear transformations.

OUTPUT:

The parent of the matrices  $A$  defining the affine group element  $Ax + b$ .

EXAMPLES:

```
sage: G = AffineGroup(3, GF(5))
sage: G.matrix_space()
Full MatrixSpace of 3 by 3 dense matrices over Finite Field of size 5
```

**random\_element()**

Return a random element of this group.

EXAMPLES:

```
sage: G = AffineGroup(4, GF(3))
sage: G.random_element() # random
      [2 0 1 2]      [1]
      [2 1 1 2]      [2]
x |-> [1 0 2 2] x + [2]
      [1 1 1 1]      [2]
sage: G.random_element() in G
True
```

**reflection(v)**

Construct the Householder reflection.

A Householder reflection (transformation) is the affine transformation corresponding to an elementary reflection at the hyperplane perpendicular to  $v$ .

INPUT:

- $v$  – a vector, or something that determines a vector.

OUTPUT:

The affine group element that is just the Householder transformation (a.k.a. Householder reflection, elementary reflection) at the hyperplane perpendicular to  $v$ .

EXAMPLES:

```
sage: G = AffineGroup(3, QQ)
sage: G.reflection([1,0,0])
      [-1  0  0]      [0]
x |-> [ 0  1  0] x + [0]
      [ 0  0  1]      [0]
sage: G.reflection([3,4,-5])
      [ 16/25 -12/25  3/5]      [0]
x |-> [-12/25  9/25  4/5] x + [0]
      [  3/5  4/5  0]      [0]
```

**translation(b)**

Construct the translation by  $b$ .

INPUT:

- $b$  – anything that determines a vector

OUTPUT:

The affine group element  $x \mapsto x + b$ .

EXAMPLES:

```
sage: G = AffineGroup(3, GF(5))
sage: G.translation([1,4,8])
      [1 0 0]      [1]
x |-> [0 1 0] x + [4]
      [0 0 1]      [3]
```

**vector\_space()**

Return the vector space of the underlying affine space.

EXAMPLES:

```
sage: G = AffineGroup(3, GF(5))
sage: G.vector_space()
Vector space of dimension 3 over Finite Field of size 5
```



# EUCLIDEAN GROUPS

AUTHORS:

- Volker Braun: initial version

**class** `sage.groups.affine_gps.euclidean_group.EuclideanGroup` (*degree, ring*)  
Bases: `sage.groups.affine_gps.affine_group.AffineGroup`

A Euclidean group.

The Euclidean group  $E(A)$  (or general affine group) of an affine space  $A$  is the group of all invertible affine transformations from the space into itself preserving the Euclidean metric.

If we let  $A_V$  be the affine space of a vector space  $V$  (essentially, forgetting what is the origin) then the Euclidean group  $E(A_V)$  is the group generated by the general linear group  $SO(V)$  together with the translations. Recall that the group of translations acting on  $A_V$  is just  $V$  itself. The general linear and translation subgroups do not quite commute, and in fact generate the semidirect product

$$E(A_V) = SO(V) \ltimes V.$$

As such, the group elements can be represented by pairs  $(A, b)$  of a matrix and a vector. This pair then represents the transformation

$$x \mapsto Ax + b.$$

We can also represent this as a linear transformation in  $\dim(V) + 1$  dimensional space as

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

and lifting  $x = (x_1, \dots, x_n)$  to  $(x_1, \dots, x_n, 1)$ .

**See Also:**

- `AffineGroup`

INPUT:

Something that defines an affine space. For example

- An affine space itself:
  - $A$  – affine space
- A vector space:
  - $V$  – a vector space
- Degree and base ring:

- degree – An integer. The degree of the affine group, that is, the dimension of the affine space the group is acting on.
- ring – A ring or an integer. The base ring of the affine space. If an integer is given, it must be a prime power and the corresponding finite field is constructed.
- var – (Default: 'a') Keyword argument to specify the finite field generator name in the case where ring is a prime power.

**EXAMPLES:**

```
sage: E3 = EuclideanGroup(3, QQ); E3
Euclidean Group of degree 3 over Rational Field
sage: E3(matrix(QQ, [(6/7, -2/7, 3/7), (-2/7, 3/7, 6/7), (3/7, 6/7, -2/7)]), vector(QQ, [10,11,12])
[ 6/7 -2/7 3/7] [10]
x |-> [-2/7 3/7 6/7] x + [11]
[ 3/7 6/7 -2/7] [12]
sage: E3([[6/7, -2/7, 3/7], [-2/7, 3/7, 6/7], [3/7, 6/7, -2/7]], [10,11,12])
[ 6/7 -2/7 3/7] [10]
x |-> [-2/7 3/7 6/7] x + [11]
[ 3/7 6/7 -2/7] [12]
sage: E3([6/7, -2/7, 3/7, -2/7, 3/7, 6/7, 3/7, 6/7, -2/7], [10,11,12])
[ 6/7 -2/7 3/7] [10]
x |-> [-2/7 3/7 6/7] x + [11]
[ 3/7 6/7 -2/7] [12]
```

Instead of specifying the complete matrix/vector information, you can also create special group elements:

```
sage: E3.linear([6/7, -2/7, 3/7, -2/7, 3/7, 6/7, 3/7, 6/7, -2/7])
[ 6/7 -2/7 3/7] [0]
x |-> [-2/7 3/7 6/7] x + [0]
[ 3/7 6/7 -2/7] [0]
sage: E3.reflection([4,5,6])
[ 45/77 -40/77 -48/77] [0]
x |-> [-40/77 27/77 -60/77] x + [0]
[-48/77 -60/77 5/77] [0]
sage: E3.translation([1,2,3])
[1 0 0] [1]
x |-> [0 1 0] x + [2]
[0 0 1] [3]
```

Some additional ways to create Euclidean groups:

```
sage: A = AffineSpace(2, GF(4, 'a')); A
Affine Space of dimension 2 over Finite Field in a of size 2^2
sage: G = EuclideanGroup(A); G
Euclidean Group of degree 2 over Finite Field in a of size 2^2
sage: G is EuclideanGroup(2,4) # shorthand
True

sage: V = ZZ^3; V
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: EuclideanGroup(V)
Euclidean Group of degree 3 over Integer Ring

sage: EuclideanGroup(2, QQ)
Euclidean Group of degree 2 over Rational Field
```

**TESTS:**



```

sage: E6 = EuclideanGroup(6, QQ)
sage: E6 is E6
True
sage: V = QQ^6
sage: E6 is EuclideanGroup(V)
True
sage: G = EuclideanGroup(2, GF(5)); G
Euclidean Group of degree 2 over Finite Field of size 5
sage: TestSuite(G).run()

```

## REFERENCES:

- [Wikipedia article Euclidean\\_group](#)

**random\_element()**

Return a random element of this group.

## EXAMPLES:

```

sage: G = EuclideanGroup(4, GF(3))
sage: G.random_element() # random
[2 1 2 1] [1]
[1 2 2 1] [0]
x |-> [2 2 2 2] x + [1]
[1 1 2 2] [2]
sage: G.random_element() in G
True

```

## TESTS:

```

sage: G.random_element().A().is_unitary()
True

```



## ELEMENTS OF AFFINE GROUPS

The class in this module is used to represent the elements of `AffineGroup()` and its subgroups.

EXAMPLES:

```
sage: F = AffineGroup(3, QQ)
sage: F([1,2,3,4,5,6,7,8,0], [10,11,12])
      [1 2 3]      [10]
x |-> [4 5 6] x + [11]
      [7 8 0]      [12]

sage: G = AffineGroup(2, ZZ)
sage: g = G([1,1],[0,1], [1,0])
sage: h = G([1,2],[0,1], [0,1])
sage: g*h
      [1 3]      [2]
x |-> [0 1] x + [1]
sage: h*g
      [1 3]      [1]
x |-> [0 1] x + [1]
sage: g*h != h*g
True
```

AUTHORS:

- Volker Braun

```
class sage.groups.affine_gps.group_element.AffineGroupElement (parent, A, b=0,
                                                                convert=True,
                                                                check=True)
```

Bases: `sage.groups.matrix_gps.group_element.MatrixGroupElement_base`

An affine group element.

INPUT:

- **A** – an invertible matrix, or something defining a matrix if `convert==True`.
- **b** – a vector, or something defining a vector if `convert==True` (default: 0, defining the zero vector).
- **parent** – the parent affine group.
- **convert** - bool (default: `True`). Whether to convert **A** into the correct matrix space and **b** into the correct vector space.
- **check** - bool (default: `True`). Whether to do some checks or just accept the input as valid.

As a special case,  $A$  can be a matrix obtained from `matrix()`, that is, one row and one column larger. In that case, the group element defining that matrix is reconstructed.

OUTPUT:

The affine group element  $x \mapsto Ax + b$

EXAMPLES:

```
sage: G = AffineGroup(2, GF(3))
sage: g = G.random_element()
sage: type(g)
<class 'sage.groups.affine_gps.group_element.AffineGroup_with_category.element_class'>
sage: G(g.matrix()) == g
True
sage: G(2)
      [2 0]      [0]
x |-> [0 2] x + [0]
```

**A()**

Return the general linear part of an affine group element.

OUTPUT:

The matrix  $A$  of the affine group element  $Ax + b$ .

EXAMPLES:

```
sage: G = AffineGroup(3, QQ)
sage: g = G([1, 2, 3, 4, 5, 6, 7, 8, 0], [10, 11, 12])
sage: g.A()
[1 2 3]
[4 5 6]
[7 8 0]
```

**b()**

Return the translation part of an affine group element.

OUTPUT:

The vector  $b$  of the affine group element  $Ax + b$ .

EXAMPLES:

```
sage: G = AffineGroup(3, QQ)
sage: g = G([1, 2, 3, 4, 5, 6, 7, 8, 0], [10, 11, 12])
sage: g.b()
(10, 11, 12)
```

**inverse()**

Return the inverse group element.

OUTPUT:

Another affine group element.

EXAMPLES:

```
sage: G = AffineGroup(2, GF(3))
sage: g = G([1, 2, 3, 4], [5, 6])
sage: g
      [1 2]      [2]
x |-> [0 1] x + [0]
sage: ~g
```

```

      [1 1]      [1]
x |-> [0 1] x + [0]
sage: g * g.inverse()
      [1 0]      [0]
x |-> [0 1] x + [0]
sage: g * g.inverse() == g.inverse() * g == G(1)
True

```

**matrix()**

Return the standard matrix representation of `self`.

**See Also:**

• `AffineGroup.linear_space()`

**EXAMPLES:**

```

sage: G = AffineGroup(3, GF(7))
sage: g = G([1,2,3,4,5,6,7,8,0], [10,11,12])
sage: g
      [1 2 3]      [3]
x |-> [4 5 6] x + [4]
      [0 1 0]      [5]
sage: g.matrix()
[1 2 3|3]
[4 5 6|4]
[0 1 0|5]
[-----+--]
[0 0 0|1]
sage: parent(g.matrix())
Full MatrixSpace of 4 by 4 dense matrices over Finite Field of size 7
sage: g.matrix() == matrix(g)
True

```

Composition of affine group elements equals multiplication of the matrices:

```

sage: g1 = G.random_element()
sage: g2 = G.random_element()
sage: g1.matrix() * g2.matrix() == (g1*g2).matrix()
True

```



# MISCELLANEOUS GROUPS

This is a collection of groups that may not fit into some of the other infinite families described elsewhere.





# SEMIMONOMIAL TRANSFORMATION GROUP

The semimonomial transformation group of degree  $n$  over a ring  $R$  is the semidirect product of the monomial transformation group of degree  $n$  (also known as the complete monomial group over the group of units  $R^\times$  of  $R$ ) and the group of ring automorphisms.

The multiplication of two elements  $(\phi, \pi, \alpha)(\psi, \sigma, \beta)$  with

- $\phi, \psi \in R^{\times n}$
- $\pi, \sigma \in S_n$  (with the multiplication  $\pi\sigma$  done from left to right (like in GAP) – that is,  $(\pi\sigma)(i) = \sigma(\pi(i))$  for all  $i$ .)
- $\alpha, \beta \in \text{Aut}(R)$

is defined by

$$(\phi, \pi, \alpha)(\psi, \sigma, \beta) = (\phi \cdot \psi^{\pi, \alpha}, \pi\sigma, \alpha \circ \beta)$$

where  $\psi^{\pi, \alpha} = (\alpha(\psi_{\pi(1)-1}), \dots, \alpha(\psi_{\pi(n)-1}))$  and the multiplication of vectors is defined elementwisely. (The indexing of vectors is 0-based here, so  $\psi = (\psi_0, \psi_1, \dots, \psi_{n-1})$ .)

---

## Todo

Up to now, this group is only implemented for finite fields because of the limited support of automorphisms for arbitrary rings.

---

## AUTHORS:

- Thomas Feulner (2012-11-15): initial version

## EXAMPLES:

```
sage: S = SemimonomialTransformationGroup(GF(4, 'a'), 4)
sage: G = S.gens()
sage: G[0]*G[1]
((a, 1, 1, 1); (1,2,3,4), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a)
```

## TESTS:

```
sage: TestSuite(S).run()
sage: TestSuite(S.an_element()).run()
```

**class** sage.groups.semimonomial\_transformations.semimonomial\_transformation\_group.Semimonomial

Bases: sage.categories.action.Action

The action of `SemimonomialTransformationGroup` on matrices over the same ring whose number of columns is equal to the degree. See `SemimonomialActionVec` for the definition of the action on the row vectors of such a matrix.

**class** sage.groups.semimonomial\_transformations.semimonomial\_transformation\_group.Semimonomial

Bases: sage.categories.action.Action

The natural action of the semimonomial group on vectors.

The action is defined by:  $(\phi, \pi, \alpha) * (v_0, \dots, v_{n-1}) := (\alpha(v_{\pi(1)-1}) \cdot \phi_0^{-1}, \dots, \alpha(v_{\pi(n)-1}) \cdot \phi_{n-1}^{-1})$ . (The indexing of vectors is 0-based here, so  $\psi = (\psi_0, \psi_1, \dots, \psi_{n-1})$ .)

**class** sage.groups.semimonomial\_transformations.semimonomial\_transformation\_group.Semimonomial

Bases: sage.groups.group.FiniteGroup, sage.structure.unique\_representation.UniqueRepresent

A semimonomial transformation group over a ring.

The semimonomial transformation group of degree  $n$  over a ring  $R$  is the semidirect product of the monomial transformation group of degree  $n$  (also known as the complete monomial group over the group of units  $R^\times$  of  $R$ ) and the group of ring automorphisms.

The multiplication of two elements  $(\phi, \pi, \alpha)(\psi, \sigma, \beta)$  with

- $\phi, \psi \in R^{\times n}$
- $\pi, \sigma \in S_n$  (with the multiplication  $\pi\sigma$  done from left to right (like in GAP) – that is,  $(\pi\sigma)(i) = \sigma(\pi(i))$  for all  $i$ .)
- $\alpha, \beta \in \text{Aut}(R)$

is defined by

$$(\phi, \pi, \alpha)(\psi, \sigma, \beta) = (\phi \cdot \psi^{\pi, \alpha}, \pi\sigma, \alpha \circ \beta)$$

where  $\psi^{\pi, \alpha} = (\alpha(\psi_{\pi(1)-1}), \dots, \alpha(\psi_{\pi(n)-1}))$  and the multiplication of vectors is defined elementwisely. (The indexing of vectors is 0-based here, so  $\psi = (\psi_0, \psi_1, \dots, \psi_{n-1})$ .)

---

### Todo

Up to now, this group is only implemented for finite fields because of the limited support of automorphisms for arbitrary rings.

---

### EXAMPLES:

```
sage: F.<a> = GF(9)
sage: S = SemimonomialTransformationGroup(F, 4)
sage: g = S(v = [2, a, 1, 2])
sage: h = S(perm = Permutation('(1,2,3,4)'), autom=F.hom([a**3]))
sage: g*h
((2, a, 1, 2); (1,2,3,4), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> 2*a + 1)
sage: h*g
((2*a + 1, 1, 2, 2); (1,2,3,4), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> a^3)
sage: S(g)
((2, a, 1, 2); (), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> a)
```

```
sage: S(1)
((1, 1, 1, 1); (), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> a)
```

**Element**

alias of SemimonomialTransformation

**base\_ring()**

Returns the underlying ring of self.

EXAMPLES:

```
sage: F.<a> = GF(4)
sage: SemimonomialTransformationGroup(F, 3).base_ring() is F
True
```

**degree()**

Returns the degree of self.

EXAMPLES:

```
sage: F.<a> = GF(4)
sage: SemimonomialTransformationGroup(F, 3).degree()
3
```

**gens()**

Return a tuple of generators of self.

EXAMPLES:

```
sage: F.<a> = GF(4)
sage: SemimonomialTransformationGroup(F, 3).gens()
[((a, 1, 1); ()), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a), ((1, 1, 1); (1,2,3), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a), ((1, 1, 1); (1,2), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a), ((1, 1, 1); ()), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a + 1)]
```

**order()**

Returns the number of elements of self.

EXAMPLES:

```
sage: F.<a> = GF(4)
sage: SemimonomialTransformationGroup(F, 5).order() == (4-1)**5 * factorial(5) * 2
True
```



# ELEMENTS OF A SEMIMONOMIAL TRANSFORMATION GROUP.

Elements of a semimonomial transformation group.

The semimonomial transformation group of degree  $n$  over a ring  $R$  is the semidirect product of the monomial transformation group of degree  $n$  (also known as the complete monomial group over the group of units  $R^\times$  of  $R$ ) and the group of ring automorphisms.

The multiplication of two elements  $(\phi, \pi, \alpha)(\psi, \sigma, \beta)$  with

- $\phi, \psi \in R^\times^n$
- $\pi, \sigma \in S_n$  (with the multiplication  $\pi\sigma$  done from left to right (like in GAP) – that is,  $(\pi\sigma)(i) = \sigma(\pi(i))$  for all  $i$ .)
- $\alpha, \beta \in \text{Aut}(R)$

is defined by

$$(\phi, \pi, \alpha)(\psi, \sigma, \beta) = (\phi \cdot \psi^{\pi, \alpha}, \pi\sigma, \alpha \circ \beta)$$

with  $\psi^{\pi, \alpha} = (\alpha(\psi_{\pi(1)-1}), \dots, \alpha(\psi_{\pi(n)-1}))$  and an elementwisely defined multiplication of vectors. (The indexing of vectors is 0-based here, so  $\psi = (\psi_0, \psi_1, \dots, \psi_{n-1})$ .)

The parent is `SemimonomialTransformationGroup`.

AUTHORS:

- Thomas Feulner (2012-11-15): initial version
- **Thomas Feulner (2013-12-27): [trac ticket #15576](#) dissolve dependency on `Permutations().global_options()['mul']`**

EXAMPLES:

```
sage: S = SemimonomialTransformationGroup(GF(4, 'a'), 4)
sage: G = S.gens()
sage: G[0]*G[1]
((a, 1, 1, 1); (1,2,3,4), Ring endomorphism of Finite Field in a of size 2^2
Defn: a |--> a)
```

TESTS:

```
sage: TestSuite(G[0]).run()
```

**class** sage.groups.semimonomial\_transformations.semimonomial\_transformation.SemimonomialTransformationGroup  
 Bases: sage.structure.element.MultiplicativeGroupElement

An element in the semimonomial group over a ring  $R$ . See [SemimonomialTransformationGroup](#) for the details on the multiplication of two elements.

The `init` method should never be called directly. Use the call via the parent [SemimonomialTransformationGroup](#). instead.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: S = SemimonomialTransformationGroup(F, 4)
sage: g = S(v = [2, a, 1, 2])
sage: h = S(perm = Permutation('(1,2,3,4)'), autom=F.hom([a**3]))
sage: g*h
((2, a, 1, 2); (1,2,3,4), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> 2*a + 1)
sage: h*g
((2*a + 1, 1, 2, 2); (1,2,3,4), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> 2*a + 1)
sage: S(g)
((2, a, 1, 2); (), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> a)
sage: S(1) # the one element in the group
((1, 1, 1, 1); (), Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> a)
```

**get\_autom()**

Returns the component corresponding to  $Aut(R)$  of self.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: SemimonomialTransformationGroup(F, 4).an_element().get_autom()
Ring endomorphism of Finite Field in a of size 3^2 Defn: a |--> 2*a + 1
```

**get\_perm()**

Returns the component corresponding to  $S_n$  of self.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: SemimonomialTransformationGroup(F, 4).an_element().get_perm()
[4, 1, 2, 3]
```

**get\_v()**

Returns the component corresponding to  $R^{imes^n}$  of self.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: SemimonomialTransformationGroup(F, 4).an_element().get_v()
(a, 1, 1, 1)
```

**get\_v\_inverse()**

Returns the (elementwise) inverse of the component corresponding to  $R^{imes^n}$  of self.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: SemimonomialTransformationGroup(F, 4).an_element().get_v_inverse()
(a + 2, 1, 1, 1)
```

**invert\_v()**

Elementwisely inverts all entries of self which correspond to the component  $R^{imes^n}$ .

The other components of `self` keep unchanged.

EXAMPLES:

```
sage: F.<a> = GF(9)
sage: x = copy(SemimonomialTransformationGroup(F, 4).an_element())
sage: x.invert_v();
sage: x.get_v() == SemimonomialTransformationGroup(F, 4).an_element().get_v_inverse()
True
```





# CONJUGACY CLASSES OF GROUPS

This module implements a wrapper of GAP's `ConjugacyClass` function.

There are two main classes, `ConjugacyClass` and `ConjugacyClassGAP`. All generic methods should go into `ConjugacyClass`, whereas `ConjugacyClassGAP` should only contain wrappers for GAP functions. `ConjugacyClass` contains some fallback methods in case some group cannot be defined as a GAP object.

---

## Todo

- Implement a non-naive fallback method for computing all the elements of the conjugacy class when the group is not defined in GAP, as the one in Butler's paper.
  - Define a sage method for gap matrices so that groups of matrices can use the quicker GAP algorithm rather than the naive one.
- 

## EXAMPLES:

Conjugacy classes for groups of permutations:

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: G.conjugacy_class(g)
Conjugacy class of (1,2,3,4) in Symmetric group of order 4! as a permutation group
```

Conjugacy classes for groups of matrices:

```
sage: F = GF(5)
sage: gens = [matrix(F,2,[1,2,-1,1]), matrix(F,2,[1,1,0,1])]
sage: H = MatrixGroup(gens)
sage: h = H(matrix(F,2,[1,2,-1,1]))
sage: H.conjugacy_class(h)
Conjugacy class of [1 2]
[4 1] in Matrix group over Finite Field of size 5 with 2 generators (
[1 2]  [1 1]
[4 1], [0 1]
)
```

## TESTS:

```
sage: G = SymmetricGroup(3)
sage: g = G((1,2,3))
sage: C = ConjugacyClass(G,g)
sage: TestSuite(C).run()
```

**class** `sage.groups.conjugacy_classes.ConjugacyClass` (*group*, *element*)  
Bases: `sage.structure.parent.Parent`

Generic conjugacy classes for elements in a group.

This is the default fall-back implementation to be used whenever GAP cannot handle the group.

EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: ConjugacyClass(G,g)
Conjugacy class of (1,2,3,4) in Symmetric group of order 4! as a
permutation group
```

**is\_rational()**

Checks if `self` is rational (closed for powers).

EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: c = ConjugacyClass(G,g)
sage: c.is_rational()
False
```

**is\_real()**

Checks if `self` is real (closed for inverses).

EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: c = ConjugacyClass(G,g)
sage: c.is_real()
True
```

**list()**

Return a list with all the elements of `self`.

EXAMPLES:

Groups of permutations:

```
sage: G = SymmetricGroup(3)
sage: g = G((1,2,3))
sage: c = ConjugacyClass(G,g)
sage: L = c.list()
sage: Set(L) == Set([G((1,3,2)), G((1,2,3))])
True
```

**representative()**

Return a representative of `self`.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: g = G((1,2,3))
sage: C = ConjugacyClass(G,g)
sage: C.representative()
(1,2,3)
```

**set()**

Naive algorithm to give a set with all the elements of the conjugacy class.

---

**Todo**

Implement a non-naive algorithm, cf. for instance G. Butler: “An Inductive Schema for Computing Conjugacy Classes in Permutation Groups”, Math. of Comp. Vol. 62, No. 205 (1994)

---

**EXAMPLES:**

Groups of permutations:

```

sage: G = SymmetricGroup(3)
sage: g = G((1,2))
sage: C = ConjugacyClass(G,g)
sage: S = [(2,3), (1,2), (1,3)]
sage: C.set() == Set(G(x) for x in S)
True

```

Groups of matrices over finite fields:

```

sage: F = GF(5)
sage: gens = [matrix(F,2,[1,2,-1,1]), matrix(F,2,[1,1,0,1])]
sage: H = MatrixGroup(gens)
sage: h = H(matrix(F,2,[1,2,-1,1]))
sage: C = ConjugacyClass(H,h)
sage: S = [[[3, 2], [2, 4]], [[0, 1], [2, 2]], [[3, 4], [1, 4]], \
[[0, 3], [4, 2]], [[1, 2], [4, 1]], [[2, 1], [2, 0]], \
[[4, 1], [4, 3]], [[4, 4], [1, 3]], [[2, 4], [3, 0]], \
[[1, 4], [2, 1]], [[3, 3], [3, 4]], [[2, 3], [4, 0]], \
[[0, 2], [1, 2]], [[1, 3], [1, 1]], [[4, 3], [3, 3]], \
[[4, 2], [2, 3]], [[0, 4], [3, 2]], [[1, 1], [3, 1]], \
[[2, 2], [1, 0]], [[3, 1], [4, 4]]]
sage: C.set() == Set(H(x) for x in S)
True

```

**class** sage.groups.conjugacy\_classes.**ConjugacyClassGAP**(group, element)

Bases: sage.groups.conjugacy\_classes.ConjugacyClass

Class for a conjugacy class for groups defined over GAP. Intended for wrapping GAP methods on conjugacy classes.

**INPUT:**

- group – the group in which the conjugacy class is taken
- element – the element generating the conjugacy class

**EXAMPLES:**

```

sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: ConjugacyClassGAP(G,g)
Conjugacy class of (1,2,3,4) in Symmetric group of order 4! as a
permutation group

```

**set()**

Return a Sage Set with all the elements of the conjugacy class.

By default attempts to use GAP construction of the conjugacy class. If GAP method is not implemented for the given group, and the group is finite, falls back to a naive algorithm.

**Warning:** The naive algorithm can be really slow and memory intensive.

EXAMPLES:

Groups of permutations:

```
sage: G = SymmetricGroup(4)
sage: g = G((1,2,3,4))
sage: C = ConjugacyClassGAP(G,g)
sage: S = [(1,3,2,4), (1,4,3,2), (1,3,4,2), (1,2,3,4), (1,4,2,3), (1,2,4,3)]
sage: C.set() == Set(G(x) for x in S)
True
```

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