Math 323: Graph Theory
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Solutions to the Midterm Examination
2013-March-02-Sat, 0900-1030 in room C304 of HR4

- ⋆ There are 6 problems. Each problem is worth 5 points. The maximum score is 30 points.
- ⋆ This is an open book exam.
- ⋆ Clearly state the results from the book that you use, referring to the page number.

(1) Prove that every set of six people contains at least three mutual acquaintances or three mutual strangers.

Solution. Let G be a graph with six vertices, one for each person, and an edge for each pair of people who are acquainted. Let x be a vertex. Since x has at most 5 neighbors, x has at least 3 neighbors or at least 3 nonneighbors. We may assume without loss of generality that x has at least 3 neighbors (if not, replace G with $\overline{G}$ (= the complement of G)). If any pair of these people are acquainted, then taking them together with x gives us a K_3 as an induced subgraph in G, or in other words, we have three mutual acquaintances. On the other hand, if no pair of the neighbors of x is acquainted, then the neighbors of x contain a set of three mutual strangers.

In purely graph theoretical terms, this problem says that any graph on six vertices has K_3 or $\overline{K}_3$ as an induced subgraph.

(2) Let G be a graph with $|G| \geq 2$. Show that G has two vertices of equal degree.

Solution. The degree of a vertex in a graph with n vertices is an element of $\{0, 1, \ldots, n-1\}$. These are n distinct values. Suppose that no two degrees are the same so that $0, 1, \ldots, n-1$ is the degree sequence of G. But a vertex of degree zero is isolated (not connected to any other vertex), and a vertex of degree $n-1$ is connected to every other vertex. These two conditions are contradictory.

(3) A saturated hydrocarbon C_kH_l is a molecule formed from k carbon atoms and l hydrogen atoms by adding bonds between atoms such that each carbon atom is in four bonds, each hydrogen atom is in one bond, and no sequence of bonds forms a cycle. Show that $l = 2k + 2$.

Solution. Let G be the graph made from the molecule, with vertices for the atoms and edges for the bonds. Thus $|G| = k + l$. The condition that there is no cycle means that G is a tree, so $|E(G)| = k + l - 1$. Now we use the fact that the sum of the degrees is equal to twice the number of edges to obtain $4k + l = 2(k + l - 1)$ and therefore $l = 2k + 2$.

(4) Let G be a weighted graph and let T be a minimum spanning tree. Let T' be any other spanning tree, not necessarily of minimum weight. Show that T' can be transformed into T by steps that exchange one edge of T' for one edge of T such that the edge set is always a spanning tree and the total weight never goes up.

Solution. The idea is simple: pick an edge e' in T' which is not in T and then carefully choose an edge e in T so that the weight of $T' - e' + e$ doesn't go up. After finitely many steps, this procedure will transform T' into T. Now let us give the details of the proof.

It suffices to find one step in the transformation from T' to T. For if that has been done, then the entire sequence of steps exists by induction on the number of edges in which the two trees differ.

Fix $e' \in E(T') - E(T)$. Then $T' - e' = U \sqcup V$ has two connected components. The path in T between the endpoints of e' must have an edge e from U to V. Since e is an edge of the path in T between the endpoints of e', the edge e belongs to the unique cycle in $T + e'$. Thus $T + e' - e$ is also a spanning tree. The proof will be complete if we show $w(T' - e' + e) \leq w(T')$.

Since T has minimum weight, $w(T) \leq w(T - e' + e)$, so we have $w(e) \leq w(e')$. Therefore $w(T' - e' + e) \leq w(T')$, as required.

(5) Show that if G' is obtained from a connected graph G by adding edges joining pairs of vertices whose distance in G is 2, then G' is 2-connected.

Solution. Let us first note that G' is connected since it is made by adding edges to the connected graph G.

To show that G' is 2-connected we must show that any vertex cut must have at least two elements. Suppose not. Then G' has a cut vertex v and $G' - v$ is disconnected.

Now, since G' is connected, $G' - v$ is disconnected if and only if some of the neighbors of v in G' are in one component of $G' - v$ and some are in another component of $G' - v$.

By definition of G', the neighbors of v in G are adjacent in G' and therefore they are in the same component of $G' - v$. Hence the neighbors of v in G' are in the same component of $G' - v$ and hence $G' - v$ is connected, a contradiction.

(6) Show that if a connected graph G has blocks $B_1, B_2, \ldots, B_k$, then

$$|G| = \left(\sum_{i=1}^{k} |B_i| \right) - k + 1.$$

Solution. Let us give two proofs.

First proof. It follows from corollary 5.9 on page 113 of the book that two distinct blocks have at most one vertex in common and that vertex is a cut vertex.

Another way to look at what corollary 5.9 is as follows. If we form a new graph G^* with one vertex for each block of G and an edge between vertices v_1 and v_2 if and only if $B_1 \cap B_2 \neq \varnothing$, where v_i corresponds to B_i, then the graph G^* is a tree on k vertices. This is because corollary 5.9(c) means that each edge of G^* is a bridge, and therefore G^* is a tree by Theorem 4.1 on page 86.

We use the fact that a tree on k vertices has $k - 1$ edges to see that there are $k - 1$ cut vertices.

Therefore if we form the sum $\sum_{i=1}^{k} |B_i|$ we have counted each cut vertex twice. So we subtract the number of cut vertices to obtain $(\sum_{i=1}^{k} |B_i|) - k + 1$, as required.

Second proof. We proceed by induction on k. If $k = 1$ then $|G| = |B_1|$. Assume the result is true for all graphs with $k - 1$ blocks. We must prove it for a graph G with k blocks.

The fact that $k > 1$ means that $\kappa(G) = 1$ because deleting any cut vertex will result in a disconnected graph. Thus G is not 2-connected, so there is a block B which contains only one of the cut vertices (this block corresponds to an end vertex of the tree G^* constructed in the first proof, and such end vertices exist in trees by 4.3 on page 89 of the book). Write v for this cut vertex. We may choose notation so that $B = B_k$ is the highest numbered block.

Put $G' := G - (V(B_k) - \{v\})$. The graph G' is connected and has blocks $B_1, \ldots, B_{k-1}$. By induction, we have $|G'| = (\sum_{i=1}^{k-1} |B_i|) - (k - 1) + 1$. Since we deleted $|B_k| - 1$ vertices from G to obtain G', we have $|G| = (\sum_{i=1}^{k} |B_i|) - k + 1$, as required.

(Secretly, this proof reproduces the inductive proof that a tree on n vertices has $n - 1$ edges.)