Type Safety

Concepts of Programming Languages
Lecture 19

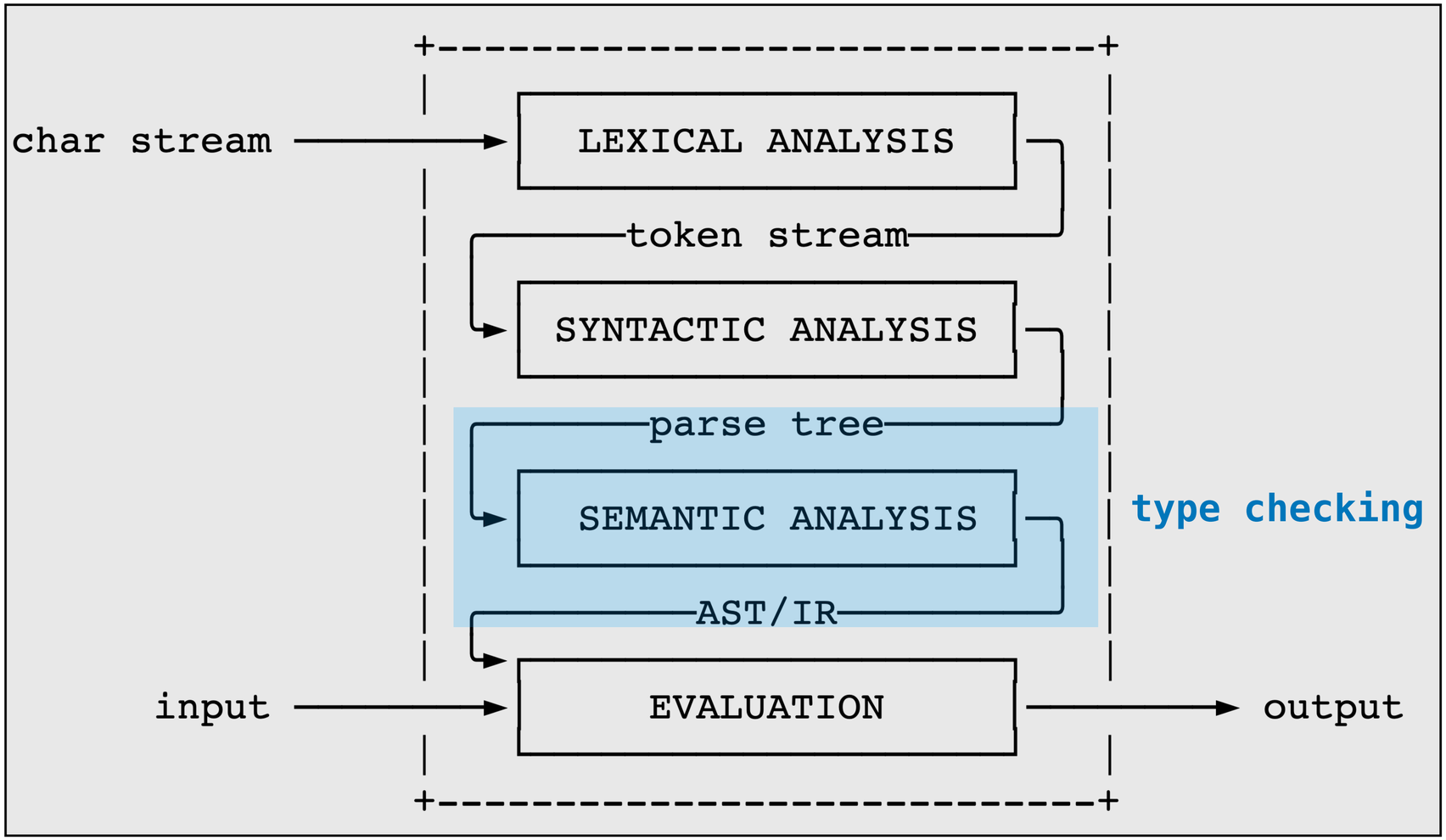
CAS CS 320

Outline

- Demo an **implementation** of the simply typed lambda calculus and desugaring
- Discuss **induction** over derivations
- Show that STLC satisfies **progress** and **preservation**

Recap

Recall: The Picture



Recall: Type Checking/Inference

```
type_check : expr -> ty -> bool

type_of : expr -> ty option
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Type checking: determining whether an expression can be typed

Type inference: *synthesizing* a type for an expression

Theoretically, these two problems can be very different. *For STLC, they are both easy*

Recall: Syntax (STLC)

fun $(x : \tau) \to e$

$$e ::= \bullet \mid x \mid \lambda x^{\tau}.e \mid ee$$

$$\tau ::= \top \mid \tau \to \tau$$

Types

$$x ::= variables$$

The syntax is the same as that of the lambda calculus except:

- we include a unit expression
- we have types, which annotate arguments

Recall: Typing (STLC)

$$\frac{}{\Gamma \vdash \bullet : \top}\ \textbf{unit}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau}.\, e : \tau \to \tau'}\ \textbf{abstraction}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textbf{variable}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

These rules enforce that a function can only be applied if we *know* that it's a function

Recall: Semantics (STLC)

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \texttt{leftEval}$$

$$\frac{}{(\lambda x\,.\,e)e' \longrightarrow [e'/x]e}\ \texttt{beta}$$

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We can use any semantics (this is small-step CBN)

This is part of the point. Type-checking only determines *whether* we go on to evaluate the program

It doesn't determine *how* we evaluate the program

Practice Problem

$$(\lambda x^{(\mathsf{T}\to\mathsf{T})\to\mathsf{T}} . x(\lambda z^{\mathsf{T}} . x(wz)))y$$

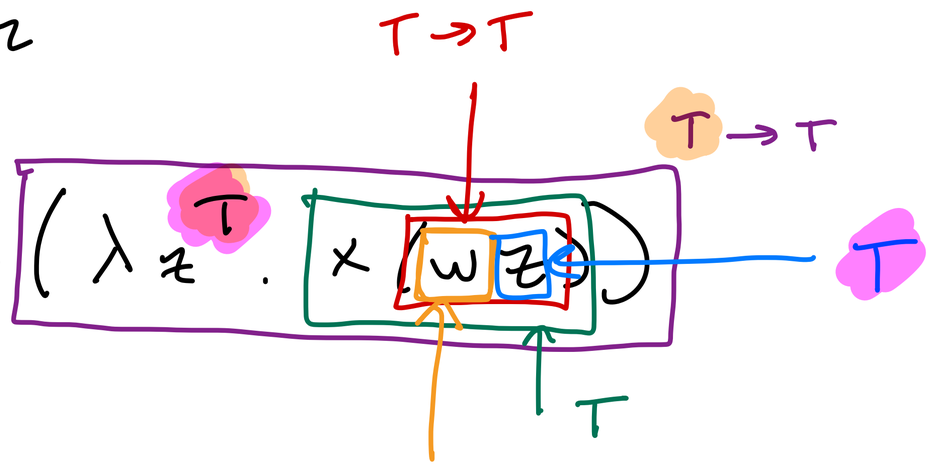
Determine the smallest context such that the above expression is typeable (also give its type)

Answer

$$(\lambda x^{(\mathsf{T}\to\mathsf{T})\to\mathsf{T}} . x(\lambda z^{\mathsf{T}} . x(wz)))y$$

$(\mathsf{T}\to\mathsf{T})\to\mathsf{T}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\, e_2 : \tau_2}$$



$\{w : ?,\ x : (\mathsf{T}\to\mathsf{T})\to\mathsf{T}\} \vdash x(\lambda z^{\mathsf{T}} . x(wz))$

do the derivation

$$w : \mathsf{T}\to(\mathsf{T}\to\mathsf{T}),\ y : (\mathsf{T}\to\mathsf{T})\to\mathsf{T} \vdash \lambda x^{(\mathsf{T}\to\mathsf{T})\to\mathsf{T}} . x(\lambda z^{\mathsf{T}} . x(wz)))y$$

demo

(STLC)

Induction over Derivations

The Key Idea

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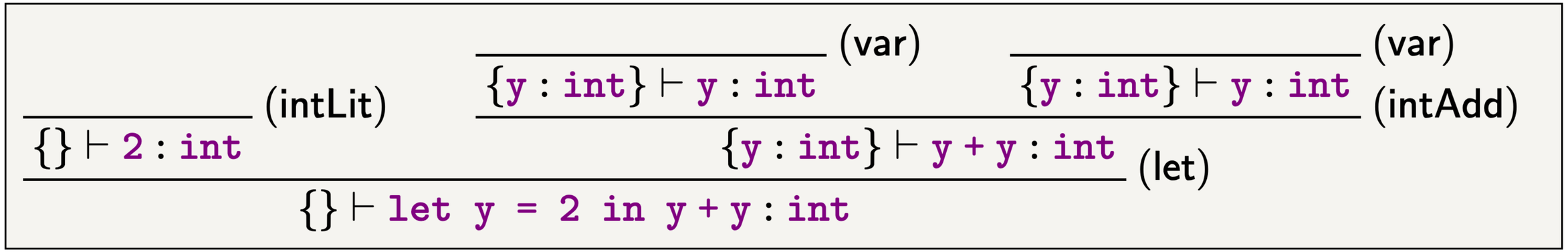

Derivations are *trees*

The Key Idea

$$\dfrac{\dfrac{}{\{\} \vdash \texttt{2} : \texttt{int}}\ (\text{intLit}) \qquad \dfrac{\dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\ (\text{var}) \qquad \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\ (\text{var})}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} + \texttt{y} : \texttt{int}}\ (\text{intAdd})}{\{\} \vdash \texttt{let y = 2 in y + y} : \texttt{int}}\ (\text{let})$$

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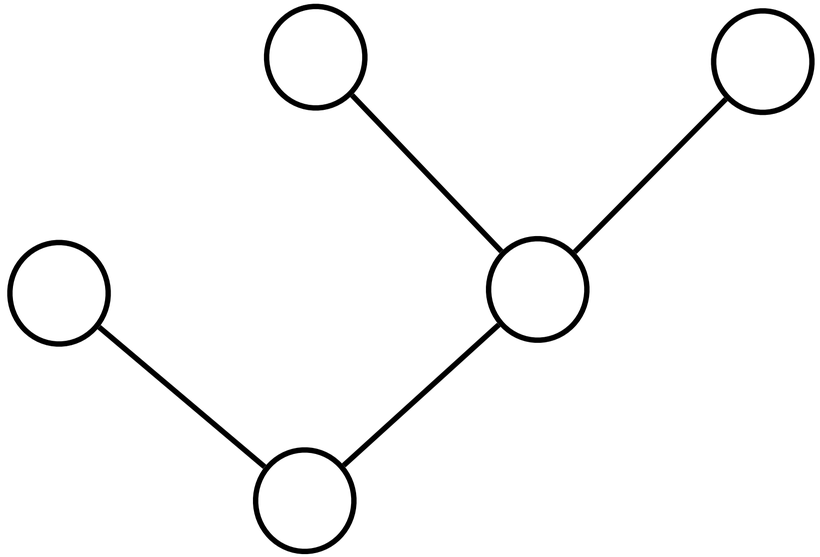
We can prove things about trees using induction



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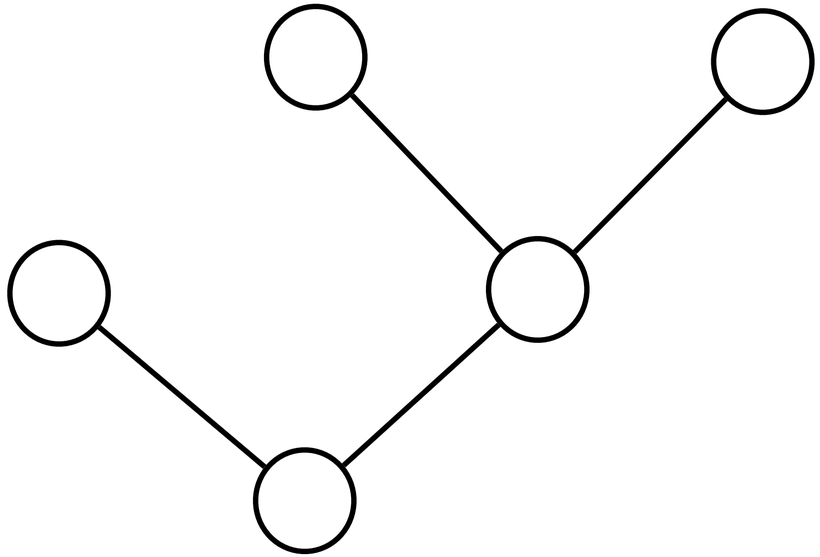
We can prove things about *derivable judgments* using induction



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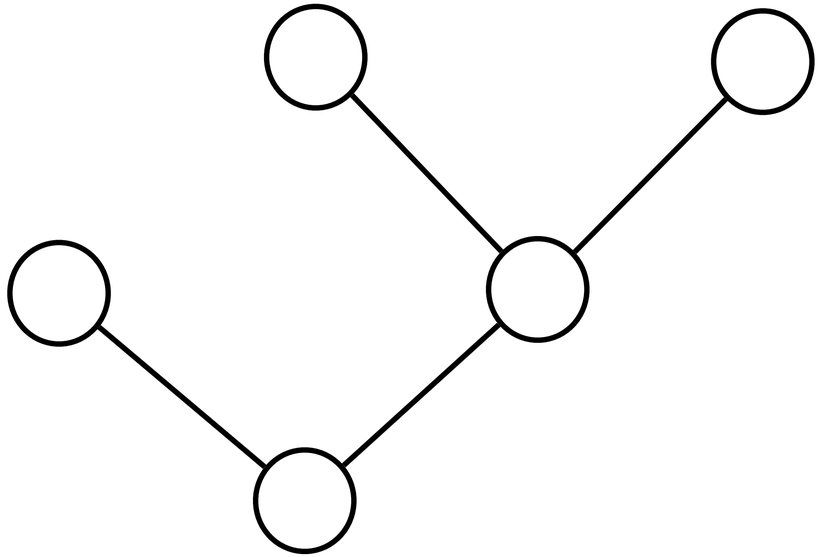
We can prove things about *derivable judgments* using induction

Important: Every derivable judgment corresponds to a derivation



Warm-up: Binary Trees

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type 'a tree =
  | Empty
  | Node of 'a tree * 'a * 'a tree
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Let $\text{size}(T)$ denote the number of `Node`s and let $\text{height}(T)$ denote the length of the *longest* path from the root to `Empty`

Theorem. $\text{size}(T) \leq 2^{\text{height}(T)}$ for any tree T

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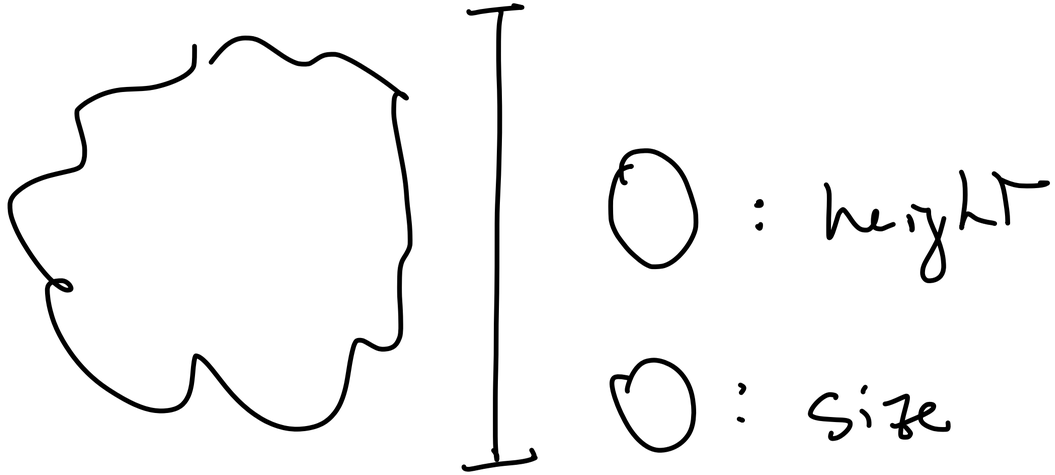
Theorem. $\text{size}(T) \leq 2^{\text{height}(T)}$ for any tree T

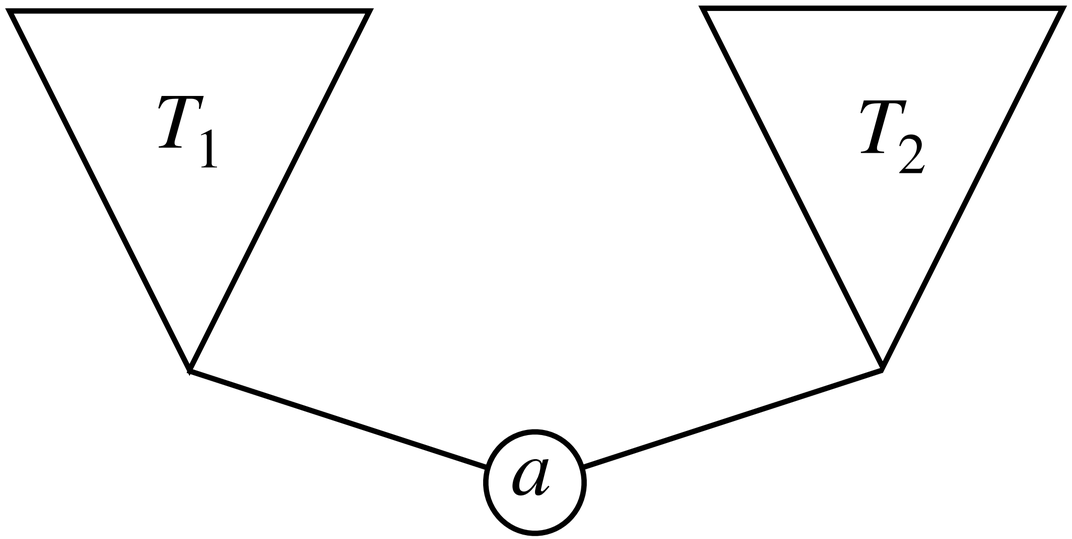
Proof. By induction on the structure of trees

Base Case: Empty



$$0 = size(Empty) \leq 2^{height(T)} = 2^0 = 1$$

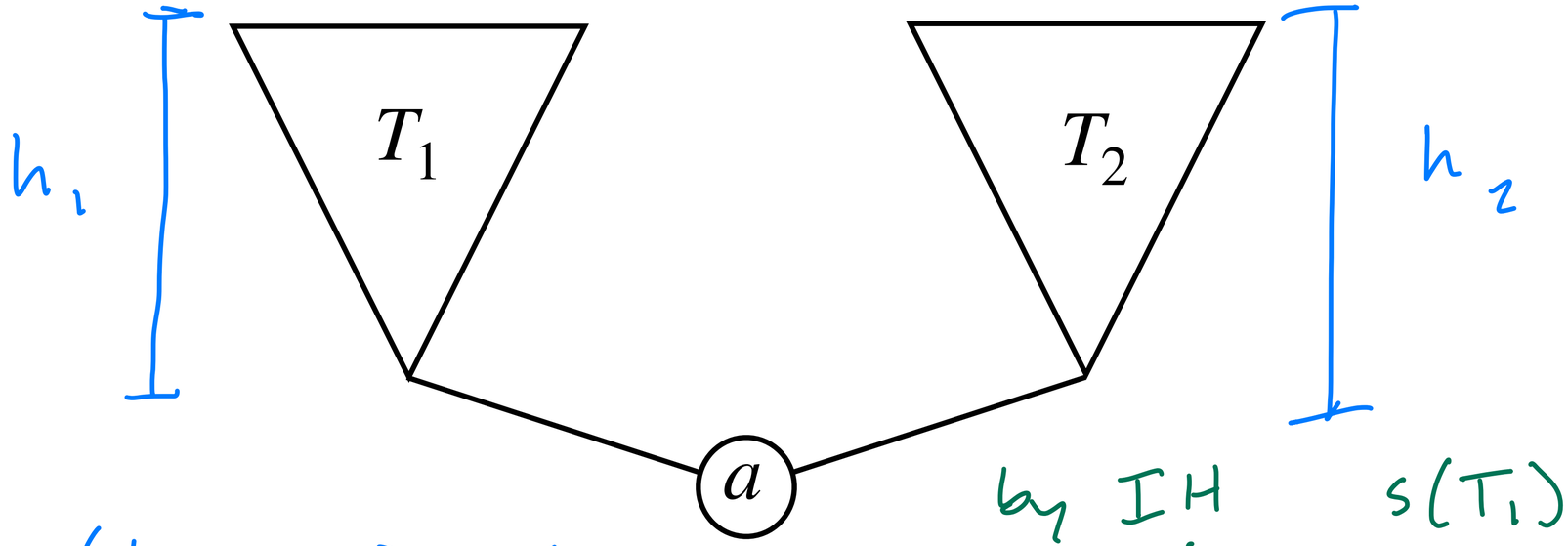
Inductive Hypothesis



If T is of the form Node (T_1, a, T_2) then $\text{size}(T_1) \leq 2^{\text{height}(T_1)}$ and $\text{size}(T_2) \leq 2^{\text{height}(T_2)}$

That is, we get to assume that what we want holds of our subtrees

Inductive Step: Nodes



$h(T) = \max(h_1, h_2) + 1$

$s(T) = s(T_1) + s(T_2) + 1$

$s(T_1) + s(T_2) + 1 \leq 2^{\max(h_1, h_2) + 1}$

by IH $s(T_1) + s(T_2) + 1 \leq$

$\hookrightarrow 2^{h_1} + 2^{h_2} + 1 \leq$

$2(2^{\max(h_1, h_2)}) + 1 =$

$2^{\max(h_1, h_2)+1} + 1$

Another Warm-up: Well-Scopedness

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An expression e is **well-scoped** with respect to a context Γ if $x \in FV(e)$ implies x appears in Γ

> Theorem. If e is well-typed in Γ, then e is well-scoped

Proof. By induction on derivations

Base Case: Axioms

$$\frac{}{\Gamma \vdash \bullet : \top}\ \textbf{unit}$$

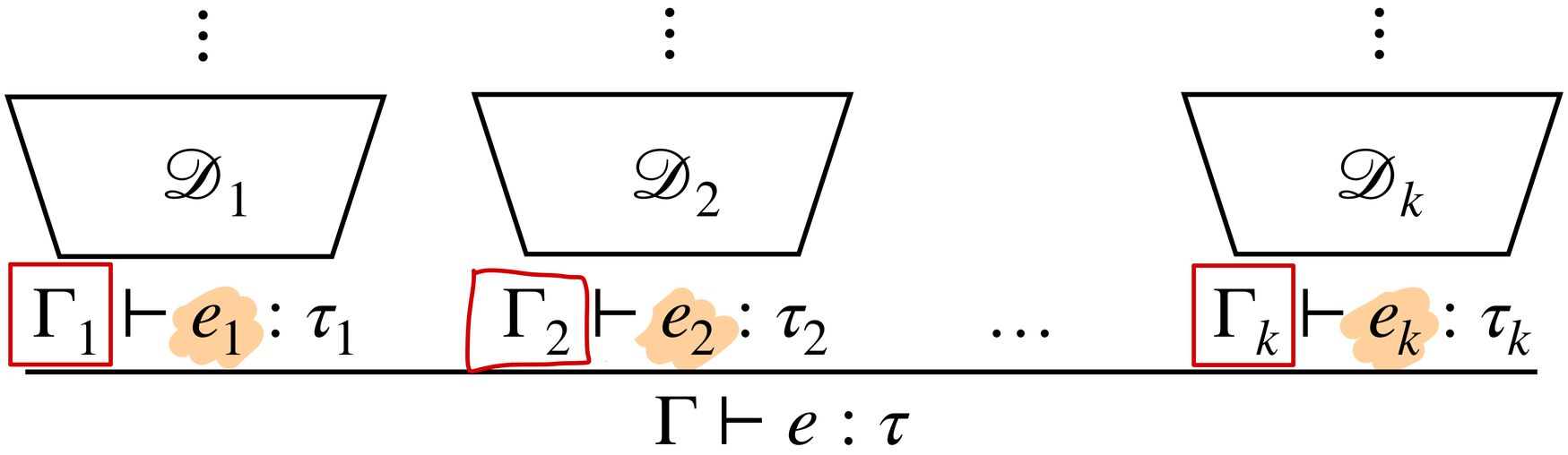
$FV(\bullet) = \{\}$ ✓

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textbf{variable}$$

$FV(x) = \{x\}$ ✓

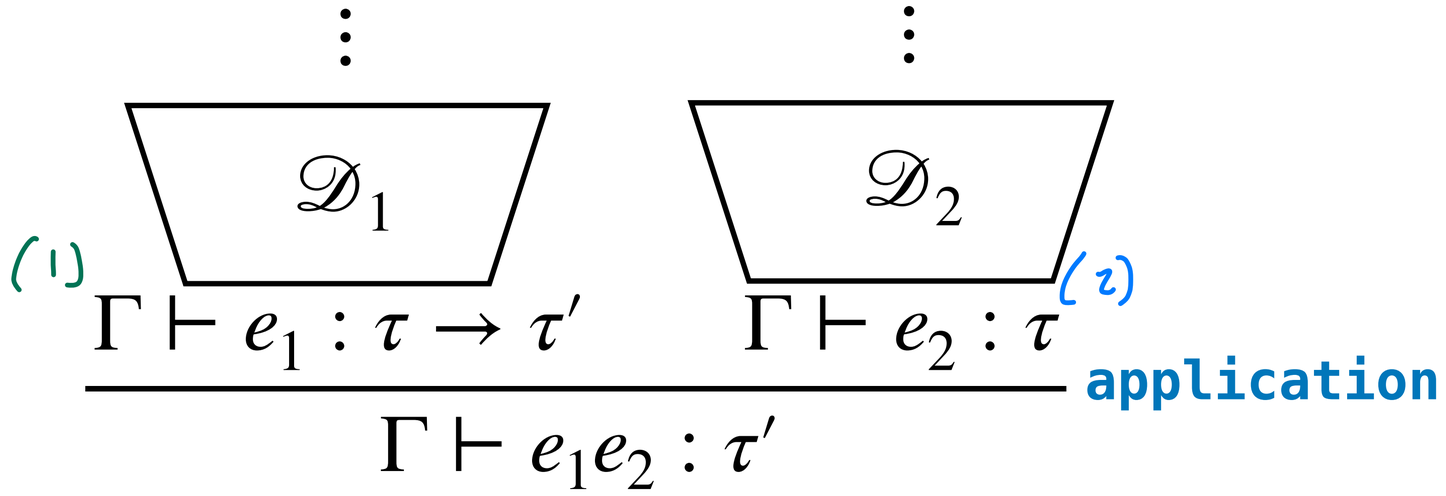
We need to show that expressions typed using just axioms satisfy well-scopedness

Inductive Hypothesis



If $e_1, \ldots, e_k$ are well-scoped (because they are typeable in the each of their contexts)

Inductive Step 1: Application



$FV(e_1\ e_2) = FV(e_1) \cup FV(e_2)$

(1) $FV(e_1) \subseteq \Gamma$ by IH

(2) $FV(e_2) \subseteq \Gamma$ by IH

$FV(e_1\ e_2) = FV(e_1) \cup FV(e_2) \subseteq \Gamma$

$e_1\ e_2$ is well-scoped

What if the last rule I applied was application?

$x : \text{int} \vdash y : \text{int}$

Inductive Step 2: Abstraction

$$\dfrac{\vdots\ \mathscr{D}}{\Gamma, x : \tau \vdash e : \tau'}\ (1)$$

$$\dfrac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau}.\, e : \tau \to \tau'}\ \textbf{abstraction}$$

$FV(\lambda x^{\tau}.e) = FV(e) \setminus \{x\}$. If $y \in FV(\lambda x^{\tau}.e)$ then $y \neq x$

(1) $FV(e) \subseteq \Gamma, x:\tau$ by IH so $y \in \Gamma, x:\tau$ and $y \neq x$ implies $y \in \Gamma$. So $FV(\lambda x^{\tau}.e) \subseteq \Gamma$

What if the last rule I applied was abstraction?

Progress and Preservation

Recall: Type Safety

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> Theorem. If $\cdot \vdash e : \tau$ then there is a value v such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

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With small-step semantics, we can give a finer-grained analysis:

> Theorem. If $\cdot \vdash e : \tau$, then
>
> » *(progress)* either e is a value or there is an e' such that $e \longrightarrow e'$
>
> » *(preservation)* If $\cdot \vdash e : \tau$ and $e \longrightarrow e'$ then $\cdot \vdash e' : \tau$

never stuck

Recall: Type Safety

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These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

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goal for today

> Theorem. If $\cdot \vdash e : \tau$, then
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These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

Disclaimer: We're gonna hand-wave liberally

Recall: STLC

$$e ::= \bullet \mid x \mid \lambda x^{\tau}.\, e \mid ee$$
$$\tau ::= \top \mid \tau \to \tau$$
$$x ::= \textit{variables}$$

Typing

$$\frac{}{\Gamma \vdash \bullet : \top}\ \textbf{unit}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau}.\, e : \tau \to \tau'}\ \textbf{abstraction}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textbf{variable}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

Semantics

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \textbf{leftEval}$$

$$\frac{}{(\lambda x.\, e) e' \longrightarrow [e'/x]e}\ \textbf{beta}$$

Progress (STLC)

> **Theorem.** If e is well-typed ($\cdot \vdash e : \tau$ for some type τ), then e is a value, or there is an expression e' such that $e \longrightarrow e'$

Proof. By induction over derivations

Base Case: Axioms

$$\frac{}{\cdot \vdash \bullet : \top}\ \textbf{unit}$$

value

$$\frac{(x : \tau) \in \emptyset}{\cdot \vdash x : \tau}\ \textbf{variable}$$

this is not possible by well-scopedness

We need to show that expressions typed using just axioms yield non-stuck terms

Inductive Step 1: Application

$$\frac{\cdot \vdash e_1 : \tau \to \tau' \qquad \cdot \vdash e_2 : \tau}{\cdot \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

(1) by IH either (a) e_1 is a value or (b) $e_1 \to e_1'$

(a) $e_1 = (\lambda x.e)$

(b) $\dfrac{e_1 \to e_1'}{e_1 e_2 \to e_1' e_2}$

$(\lambda x.e)\, e_2 \to [e_2/x]\, e$

*What do we know given that e_1 is either a **value** or **reducible**?*

Inductive Step 2: Abstraction

$$\frac{\{x : \tau\} \vdash e : \tau'}{\cdot \vdash \lambda x^{\tau} . e : \tau \to \tau'}\ \texttt{abstraction}$$

Our expression already a value if the last rule we applied was abstraction!

Preservation (STLC)

> Theorem. If e has type τ in Γ (i.e., $\Gamma \vdash e : \tau$ is derivable) and $e \longrightarrow e'$ then so is e' (i.e., $\Gamma \vdash e' : \tau$ is derivable)

Proof. By induction over derivations

This one is much tricker...

Base Case: Axioms

$$\frac{}{\Gamma \vdash \bullet : \top}\ \textbf{unit}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textbf{variable}$$

Expressions typed using just axioms cannot be reduced (nothing to do here)

Inductive Step 1: Abstraction

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}\ \textbf{abstraction}$$

Expressions derived using abstraction as the last rule is already a value (nothing to do here)

Inductive Step 2: Application

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

This is where the work comes in...

The trick: We do induction (inside our current induction) on the structure of *semantic* derivations!

What possible ways can $e_1 e_2$ *be reduced?*

Inductive Step 2.1: leftEval

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \texttt{leftEval} \qquad \frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \texttt{application}$$

What if our last rule was an application ***and*** $e_1 e_2$ *is reducible by leftEval?*

Inductive Step 2.1: leftEval

$$\frac{}{(\lambda x\,.\,e)e_2 \longrightarrow [e_2/x]e}\ \textbf{beta} \qquad \frac{\Gamma \vdash (\lambda x\,.\,e) : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash (\lambda x\,.\,e)e_2 : \tau'}\ \textbf{application}$$

*What if our last rule was an application **and** $e_1 e_2$ is reducible by beta?*

Substitution Lemma

Lemma. If $\Gamma \vdash e_2 : \tau_2$ and $\Gamma, x : \tau_2 \vdash e : \tau$ then

$$\Gamma \vdash [e_2/x]e : \tau$$

That is, if e is well-typed in a context with (x, τ) then we can substitute x with anything of type τ and it's still the same type

(we can prove this by, you guessed it, induction on derivations)

The Point

```
let rec eval env e =
  match e with
  | Var x -> Env.find x env
  ...
```

Progress and preservation tell us that **terms never get stuck during evaluation**

*This is **HUGE.** I can't emphasize this enough*

Our type system ensures we only evaluate programs that make sense!

Summary

- **Progress** and **preservation** are fundamental features of good programming languages
- We can prove things about well-typed expressions by performing **induction** over derivations