Unification

Concepts of Programming Languages
Lecture 22

CAS CS 320

Outline

- Finish up our discussion of **Hindley–Milner Light** (HM^-)
- Briefly discuss **let-polymorphism**
- Describe the **unification** algorithm used to determine the "actual" type of our expression, given a collection of constraints

Recap

Recall: Parametric Polymorphism

```
let rec rev = function
  | [] -> []
  | x :: xs -> rev xs @ [x]
```

Parametric polymorphism allows for functions which are agnostic to the types of its inputs

For example, we can write a single reverse function and use it in multiple contexts

Recall: Quantification

```
let id : 'a . 'a -> 'a = fun x -> x
```

`In reality, types variables in OCaml are` **`quantified`**

`We read this "`**`id`**` has type `**`t -> t`**` for any type `**`t`**`"`

Recall: Hindley-Milner Light

$$
\begin{aligned}
e ::= \;& \lambda x\,.\,e \mid e e \\
& \mid \text{let } x = e \text{ in } e \\
& \mid \text{if } e \text{ then } e \text{ else } e \\
& \mid e + e \mid e = e \\
& \mid n \mid x \\
\sigma ::= \;& \text{int} \mid \text{bool} \mid \alpha \mid \sigma \to \sigma \\
\tau ::= \;& \sigma \mid \forall \alpha\,.\,\tau
\end{aligned}
$$

Recall: Type Schemes

$\forall \alpha_1 \ldots \forall \alpha_1 . \boxed{(\text{int} \to \alpha_1) \to \text{bool} \ldots}$

monotype

$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \to \sigma$$

$$\tau ::= \sigma \mid \forall \alpha \, . \, \tau$$

σ represents **monotypes**, types with *no quantification*. A type is **monomorphic** if it is a monotype with no type variables

τ represents **type schemes**, which are types with some number of quantified type variables

We say a type is **polymorphic** if it is a *closed* type scheme

Recall: Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathscr{C}$$

Our typing rules well need to keep track of a set of **constraints**, which tell use what must hold for e to be well-typed

The idea: We're formalizing the idea of "collecting together" our constraints, as in our intuitive example

Recall: What is a constraint?

$$\tau_1 \doteq \tau_2$$

In general, a **type constraint** is a predicate on types. The only kind we will consider:

"τ_1 should be the same as τ_2"

Enforcing a constraint like this is called **unifying** τ_1 and τ_2

Recall: HM$^-$ (Typing)

$$\frac{n \texttt{ is an integer}}{\Gamma \vdash n : \mathsf{int} \dashv \varnothing}\ (\texttt{int})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2 \qquad \Gamma \vdash e_3 : \tau_3 \dashv \mathscr{C}_3}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \tau_3 \dashv \tau_1 \doteq \mathsf{bool}, \tau_2 \doteq \tau_3, \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3}\ (\texttt{if})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 = e_2 : \mathsf{bool} \dashv \tau_1 \doteq \tau_2, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{eq})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 + e_2 : \mathsf{int} \dashv \tau_1 \doteq \mathsf{int}, \tau_2 \doteq \mathsf{int}, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{add})$$

$$\frac{\boxed{\alpha \texttt{ is fresh}} \qquad \Gamma, x : \alpha \vdash e : \tau \dashv \mathscr{C}}{\Gamma \vdash \lambda x . e : \alpha \to \tau \dashv \mathscr{C}}\ (\texttt{fun})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2 \qquad \boxed{\alpha \texttt{ is fresh}}}{\Gamma \vdash e_1 e_2 : \alpha \dashv \tau_1 \doteq \tau_2 \to \alpha, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{app})$$

Recall: HM⁻ (Typing Variables)

polymorphic monotype

$$\frac{(x : \boxed{\forall \alpha_1 . \forall \alpha_2 \ldots \forall \alpha_k . \tau}) \in \Gamma \qquad \beta_1, \ldots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1/\alpha_1] \ldots [\beta_k/\alpha_k]\tau \dashv \varnothing} \text{ (var)}$$

If x is declared in Γ, then x can be given the type τ *with all free variables replaced by* ***fresh variables***

This is where the polymorphism magic happens

Fresh variables can be unified with anything

$$\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash \text{if } f\ \text{true then } f\ 0 \text{ else } 1 : \text{int} \dashv \underbrace{\left\{\begin{array}{l} \gamma \doteq \text{bool} \\ \varepsilon \doteq \text{int} \\ \beta \to \beta \doteq \text{bool} \to \gamma \\ \eta \to \eta \doteq \text{int} \to \varepsilon \end{array}\right\}}_{C}$$

- $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash f\ \text{true} : \gamma \dashv \beta \to \beta \doteq \text{bool} \to \gamma$
  - $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash f : \beta \to \beta \dashv \emptyset$
  - $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash \text{true} : \text{bool} \dashv \emptyset$
- $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash f\ 0 : \varepsilon \dashv \eta \to \eta \doteq \text{int} \to \varepsilon$
  - $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash f : \eta \to \eta \dashv \emptyset$
  - $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash 0 : \text{int} \dashv \emptyset$
- $\{f : \forall\alpha.\,\alpha \to \alpha\} \vdash 1 : \text{int} \dashv \emptyset$

Practice Problem

not polymorphic

$$\{f : \boxed{\alpha \to \alpha}\} \vdash f\,(f\ 2 = 2) : \ \tau \dashv \mathscr{C}$$

Determine the type τ and constraints $\mathscr{C}$ such that the above judgment is derivable

$$\frac{n \texttt{ is an integer}}{\Gamma \vdash n : \mathsf{int} \dashv \varnothing}\ (\texttt{int})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 = e_2 : \mathsf{bool} \dashv \tau_1 \doteq \tau_2, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{eq})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2 \qquad \alpha \texttt{ is fresh}}{\Gamma \vdash e_1 e_2 : \alpha \dashv \tau_1 \doteq \tau_2 \to \alpha, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{app})$$

Answer

$$\{f : \alpha \to \alpha\} \vdash f\,(f\;2 = 2) : \tau \dashv \mathscr{C}$$

$$\{f : \alpha \to \alpha\} \vdash f\,(f\;2 = 2) : \gamma \dashv \mathcal{C}$$

$$\{f : \alpha \to \alpha\} \vdash f : \alpha \to \alpha \dashv \emptyset$$

$$\{f : \alpha \to \alpha\} \vdash f\;2 = 2 : \text{bool} \dashv \beta \doteq \text{int},\ \alpha \to \alpha \doteq \text{int} \to \beta$$

$$\{f : \alpha \to \alpha\} \vdash f\;2 : \beta \dashv \alpha \to \alpha \doteq \text{int} \to \beta$$

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$$\mathcal{C} = \{\alpha \to \alpha \doteq \text{bool} \to \gamma,\ \beta \doteq \text{int},\ \alpha \to \alpha \doteq \text{int} \to \beta\}$$

Let-Expressions

HM$^-$ (Typing Let-Expressions)

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash \textsf{let } x = e_1 \textsf{ in } e_2 : \tau_2 \dashv \mathscr{C}_1, \mathscr{C}_2} \; (\texttt{let})$$

The type of a let-expression is the same as the type of its body, relative to the constraints of typing the let-binding and the body (wordy...)

Let-Polymorphism

```
let f = fun x -> x in
let y = f 2 in
f true
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There are some interesting <u>debates</u> in the world of PL with regards to let-polymorphism...

<u>The Takeaway:</u> We will have to treat typing of top-level let-expressions as *different* from local let-expressions

Unification

High Level

$$a \doteq d \to e$$

$$c \doteq \mathsf{int} \to d$$

$$\mathsf{int} \to \mathsf{int} \to \mathsf{int} \doteq b \to c$$

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Unification is the process of solving a system of equations over *symbolic* expressions

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It's kind of like solving a system of linear equations, but instead of working over real numbers and addition, we work over *uninterpreted* operations

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Unification is the process of solving a system of equations over *symbolic* expressions

It's kind of like solving a system of linear equations, but instead of working over real numbers and addition, we work over *uninterpreted* operations

<u>The best way to think of it (in my opinion):</u> unification is solving a system of equations over *variables* and *ADT constructors*

ADT Unification Problem

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$$\begin{gathered} s_1 \doteq t_1 \\ s_2 \doteq t_2 \\ \vdots \\ s_k \doteq t_k \end{gathered}$$

ADT Unification Problem

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A **unification problem** is a collection of equations of the form

$$\begin{gathered} s_1 \doteq t_1 \\ s_2 \doteq t_2 \\ \vdots \\ s_k \doteq t_k \end{gathered}$$

where $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ are terms

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We write $\mathcal{S}t$ for $\underbrace{[t_n/x_n]\ldots[t_1/x_1]t}_{\text{reverse order}}$

A solution must have the property that it **satisfies** every equation

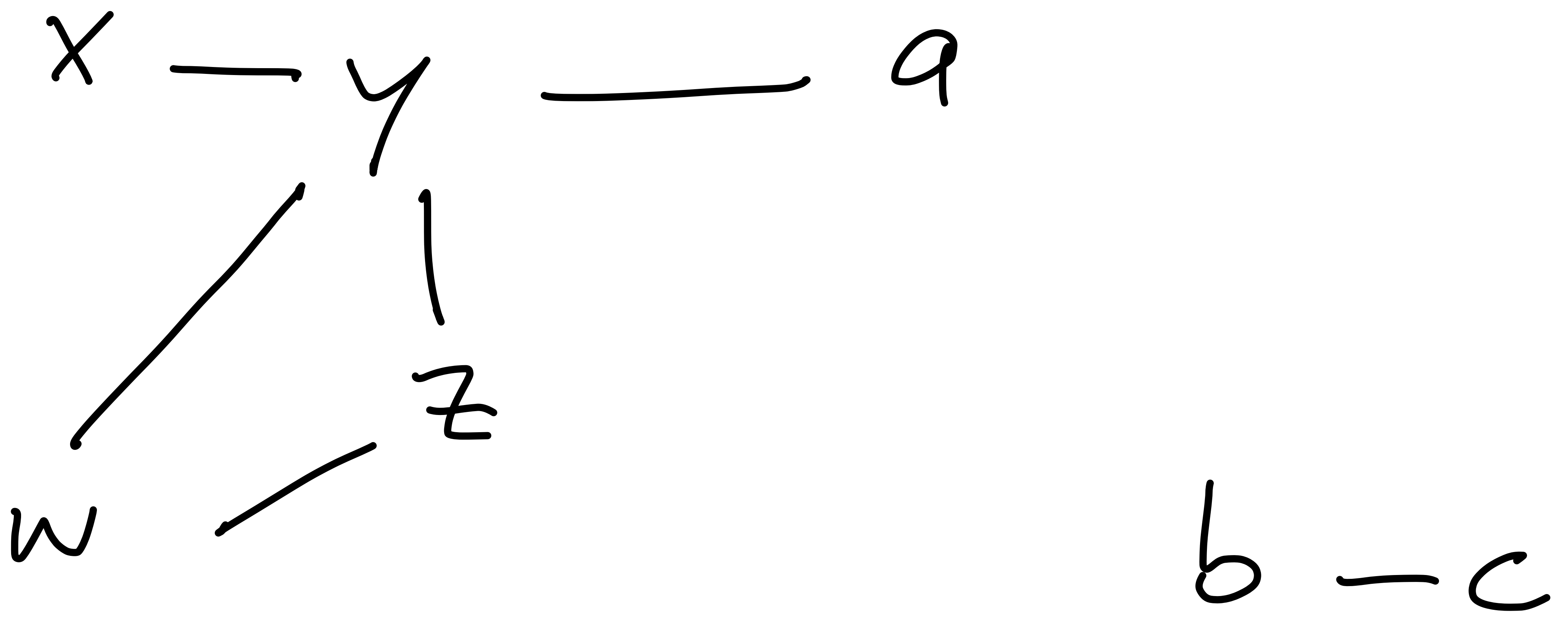
not $\doteq$

$$\begin{aligned} \mathcal{S}t_1 &= \mathcal{S}s_1 \\ \mathcal{S}s_2 &= \mathcal{S}t_2 \\ &\vdots \\ \mathcal{S}s_k &= \mathcal{S}t_k \end{aligned}$$

The Simple Case: Variables

Given a system of equations over *just* variables, the unification problem is equivalent to the **connected components** problem over undirected graphs

$$x \doteq \cancel{y} \quad x$$
$$x \quad \cancel{y} \doteq \cancel{z} \quad x$$
$$x \quad \cancel{z} \doteq \cancel{w} \quad x$$
$$x \quad \cancel{w} \doteq \cancel{y} \quad x$$
$$x \quad \cancel{a} \doteq \cancel{y} \quad x$$
$$c \quad \cancel{b} \doteq c$$



$$S = \{ y \mapsto x,\ z \mapsto x,\ w \mapsto x,\ a \mapsto x,\ b \mapsto c \}$$

Type Unification

$$\alpha \doteq int \to \beta$$

$$\beta \doteq \gamma \to int$$

$$\gamma \doteq int$$

```
type ty =
  | TInt
  | TBool
  | TFun of ty * ty
  | TVar of string
```

$$S = \left\{ \begin{array}{l} \alpha \mapsto int \to (int \to int) \\ \beta \mapsto int \to int \\ \gamma \mapsto int \end{array} \right\}$$

Type unification is the unification problem of an ADT of types (with type variables acting as variables in the unification problem)

Example

$$a \doteq d \to e$$
$$c \doteq \text{int} \to d$$
$$\text{int} \to (\text{int} \to \text{int}) \doteq b \to c$$

$\text{int} \to e = \text{int} \to e$ ✓

$\text{int} \to \text{int} = \text{int} \to \text{int}$ ✓

$\text{int} \to (\text{int} \to \text{int}) = \text{int} \to (\text{int} \to \text{int})$ ✓

$$S = \left\{ \begin{array}{l} b \mapsto \text{int} \\ c \mapsto \text{int} \to \text{int} \\ d \mapsto \text{int} \\ a \mapsto \text{int} \to e \end{array} \right\}$$

$c \mapsto \text{bool}$

Unification may Fail

Not all unification problems have solutions:

$$int \doteq bool$$

$$\alpha \to int \doteq bool \to \alpha$$

$$\alpha \doteq int \to \alpha$$

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The **most general unifier** of a unification problem is a solution $\mathscr{S}$ such that, for any solution $\mathscr{S}'$, there is another solution $\mathscr{S}''$ such that $\mathscr{S}' = \mathscr{S}\mathscr{S}''$

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<u>Ex.</u>

$$
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a &\doteq d \to e \\
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\mathsf{int} \to \mathsf{int} \to \mathsf{int} &\doteq b \to c
\end{aligned}
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$$\boxed{S = \left\{ \begin{array}{l} b \mapsto \mathsf{int} \\ c \mapsto \mathsf{int} \to \mathsf{int} \\ d \mapsto \mathsf{int} \\ a \mapsto \mathsf{int} \to e \end{array} \right\}}$$

$$S' = S \cup \{ e \mapsto \mathsf{int} \}$$

An Algorithm (High Level)

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When we see an assignment, it *becomes part of our solution*

And we're guaranteed to get the a most general unifier

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input: type unification problem 𝒰
output: most general unifier to 𝒰

𝒮 ← empty solution

WHILE eq ∈ 𝒰:  // 𝒰 is not empty
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    $t_1 = t_2 \implies \mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\}$ // if t_1 and t_2 are *syntactically* equal then remove eq from $\mathscr{U}$)

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    $s_1 \to t_1 \doteq s_2 \to t_2 \implies \mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\} \cup \{s_1 \doteq s_2, t_1 \doteq t_2\}$ // remove eq and add $s_1 \doteq s_2$ and $t_1 \doteq t_2$

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    t₁ ≐ t₂  ⟹  𝒰 ← 𝒰\{eq}   // if t₁ and t₂ are syntactically equal then remove eq from 𝒰)

    s₁ → t₁ ≐ s₂ → t₂  ⟹  𝒰 ← 𝒰\{eq} ∪ {s₁ ≐ s₂, t₁ ≐ t₂}  // remove eq and add s₁ ≐ s₂ and t₁ ≐ t₂

    α ≐ t  or  t ≐ α  where  α ∉ FV(t)  ⟹  // type variable α does not appear free in t
```

An Algorithm (Pseudocode)

```
input: type unification problem 𝒰
output: most general unifier to 𝒰
```

$\mathcal{S} \leftarrow$ `empty solution`

WHILE $eq \in \mathcal{U}$: `// ` $\mathcal{U}$ `is not empty`

  MATCH eq:

    $t_1 \doteq t_2 \implies \mathcal{U} \leftarrow \mathcal{U} \setminus \{eq\}$ `// if` t_1 `and` t_2 `are` *`syntactically`* `equal then remove` eq `from` $\mathcal{U}$`)`

    $s_1 \to t_1 \doteq s_2 \to t_2 \implies \mathcal{U} \leftarrow \mathcal{U} \setminus \{eq\} \cup \{s_1 \doteq s_2, t_1 \doteq t_2\}$ `// remove` eq `and add` $s_1 \doteq s_2$ `and` $t_1 \doteq t_2$

    $\alpha \doteq t$ `or` $t \doteq \alpha$ `where` $\alpha \notin \mathsf{FV}(t) \implies$ `// type variable` α `does not appear free in` t

      $\mathcal{S} \leftarrow \mathcal{S} \cup \{\alpha \mapsto t\}$ `// add` $\alpha \mapsto t$ `to` $\mathcal{S}$

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input: type unification problem $\mathscr{U}$
output: most general unifier to $\mathscr{U}$

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    s₁ → t₁ ≐ s₂ → t₂  ⟹  𝒰 ← 𝒰\{eq} ∪ {s₁ ≐ s₂, t₁ ≐ t₂} // remove eq and add s₁ ≐ s₂ and t₁ ≐ t₂

    α ≐ t or t ≐ α where α ∉ FV(t)  ⟹  // type variable α does not appear free in t

      𝒮 ← 𝒮 ∪ {α ↦ t}   // add α ↦ t to 𝒮

      𝒰 ← 𝒰\{eq}
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WHILE $eq \in \mathscr{U}$: // $\mathscr{U}$ is not empty

MATCH eq:

$t_1 \doteq t_2 \implies \mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\}$ // if t_1 and t_2 are *syntactically* equal then remove eq from $\mathscr{U}$)

$s_1 \to t_1 \doteq s_2 \to t_2 \implies \mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\} \cup \{s_1 \doteq s_2, t_1 \doteq t_2\}$ // remove eq and add $s_1 \doteq s_2$ and $t_1 \doteq t_2$

$\alpha \doteq t$ or $t \doteq \alpha$ where $\alpha \notin \mathsf{FV}(t) \implies$ // type variable α does not appear free in t

$\mathscr{S} \leftarrow \mathscr{S} \cup \{\alpha \mapsto t\}$ // add $\alpha \mapsto t$ to $\mathscr{S}$

$\mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\}$

perform the substitution $\alpha \mapsto t$ to *every* equation in $\mathscr{U}$

An Algorithm (Pseudocode)

<u>input:</u> type unification problem $\mathscr{U}$
<u>output:</u> most general unifier to $\mathscr{U}$

```
𝒮 ← empty solution

WHILE eq ∈ 𝒰:  // 𝒰 is not empty

  MATCH eq:

    t₁ ≐ t₂ ⟹ 𝒰 ← 𝒰\{eq}   // if t₁ and t₂ are syntactically equal then remove eq from 𝒰)

    s₁ → t₁ ≐ s₂ → t₂ ⟹ 𝒰 ← 𝒰\{eq} ∪ {s₁ ≐ s₂, t₁ ≐ t₂} // remove eq and add s₁ ≐ s₂ and t₁ ≐ t₂

    α ≐ t or t ≐ α where α ∉ FV(t) ⟹ // type variable α does not appear free in t

      𝒮 ← 𝒮 ∪ {α ↦ t}   // add α ↦ t to 𝒮

      𝒰 ← 𝒰\{eq}

      perform the substitution α ↦ t to every equation in 𝒰

    OTHERWISE ⟹ FAIL
```

An Algorithm (Pseudocode)

input: type unification problem $\mathscr{U}$
output: most general unifier to $\mathscr{U}$

$\mathscr{S} \leftarrow$ empty solution

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    $\alpha \doteq t$ or $t \doteq \alpha$ where $\alpha \notin \mathrm{FV}(t) \implies$ // type variable α does not appear free in t

      $\mathscr{S} \leftarrow \mathscr{S} \cup \{\alpha \mapsto t\}$ // add $\alpha \mapsto t$ to $\mathscr{S}$ $\alpha \doteq int \to \alpha$ X

      $\mathscr{U} \leftarrow \mathscr{U} \backslash \{eq\}$

      perform the substitution $\alpha \mapsto t$ to *every* equation in $\mathscr{U}$ α doesn't appear after subst.

  OTHERWISE $\implies$ **FAIL**

RETURN $\mathscr{S}$

Example

$$
\begin{aligned}
a &\doteq d \to e \\
c &\doteq \mathsf{int} \to d \\
\mathsf{int} \to \mathsf{int} \to \mathsf{int} &\doteq b \to c
\end{aligned}
$$

Another Example

$$\beta \doteq \eta$$

$$\alpha \to \beta \doteq \alpha \to \gamma$$

$$\alpha \to \beta \doteq \gamma \to \eta$$

$$\alpha \to \beta \doteq \text{int} \to \eta$$

Principle Type

$$\Gamma \vdash e : \tau \dashv \mathscr{C}$$

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$$\forall \alpha_1, \ldots, \alpha_k \,.\, \mathscr{S}\tau \text{ where } \mathsf{FV}(\mathscr{S}\tau) = \{\alpha_1, \ldots, \alpha_k\}$$

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This is called the **principle type** of e. Every type we *could* give e is a *specialization* $\forall \alpha_1, \ldots, \alpha_k . \mathscr{S}\tau$

Example

Determine the principle type of `fun f -> fun x -> f (x + 1)`

Example

Show that $f\ (f\ 2 = 2)$ *has no principle type in the context* $\{f : \alpha \to \alpha\}$

Putting everything together

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3. *Generalization:* Quantify over the free variables in $\mathscr{S}\tau$ to get the principle type $\forall \alpha_1 \ldots \forall \alpha_k . \mathscr{S}\tau$ of e

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4. Add $(x : \forall \alpha_1 \ldots \forall \alpha_k . \mathscr{S}\tau)$ to Γ

Summary

Unification is used to solve a collection of constraints generated by constraint-based inference

Not all unification problems have solutions. In the type unification problem, this indicates a type error

The **principle type** of an expression is the most general type we could give to an expression in our system