Polymorphism

Concepts of Programming Languages
Lecture 20

CAS CS 320

Outline

- Discuss **polymorphism** in general
- Discuss **System F**, a type system with parametric polymorphism
- Demo an implementation of **System F**

Practice Problem

```
fun f -> fun x -> f (x + 1)

let rec f x = f (f (x + 1)) in f
```

What are the types of the above OCaml expressions?

Answer

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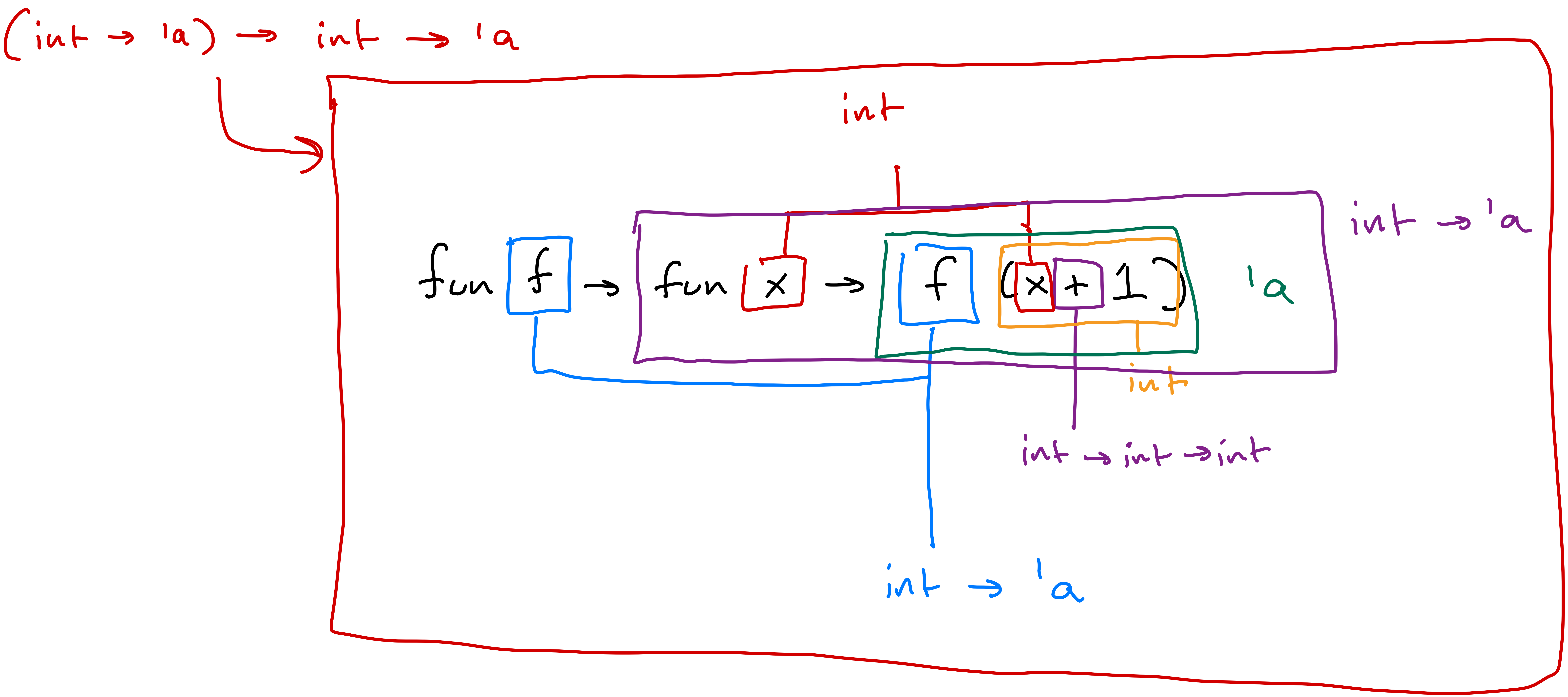

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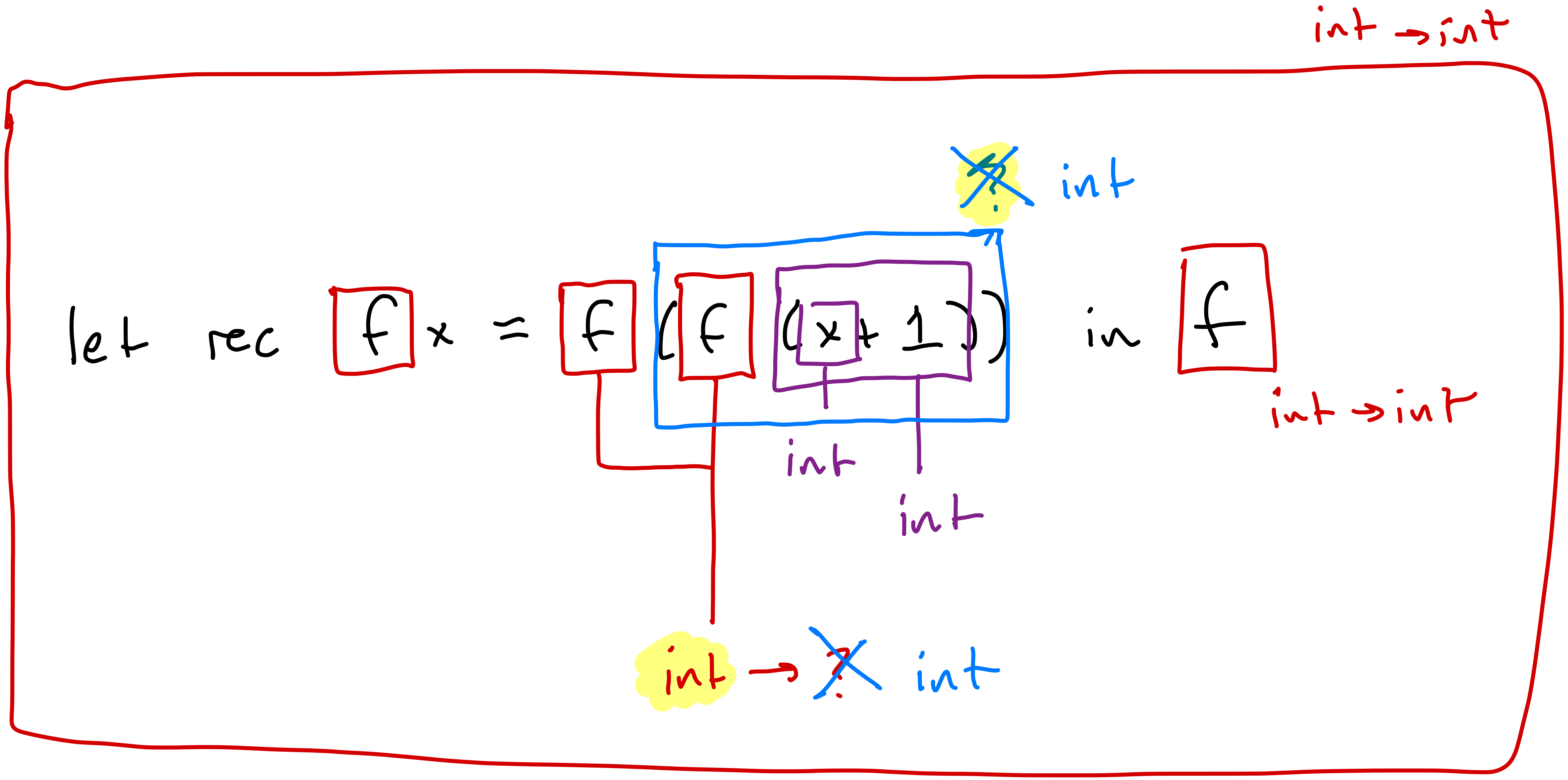

Polymorphism

High Level

```
let rec rev_int (l : int list) : int list =
  match l with
  | [] -> []
  | x :: l -> rev l @ [x]

let rec rev_string (l : string list) : string list =
  match l with
  | [] -> []
  | x :: l -> rev l @ [x]

let _ = assert (rev_int [1;2;3] = [3;2;1])
let _ = assert (rev_string ["1";"2";"3"] = ["3";"2";"1"])
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Copy/pasting code is *time consuming* and *error prone*

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                               'a              'a
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Copy/pasting code is *time consuming* and *error prone*

Polymorphism allows for better code reuse. The *same* function can be applied in multiple contexts

Basic Example

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let a = id 0
let b = id (0 = 0)
let c = id id
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We want to be able to define functions that can be used in multiple contexts *and* that we can type check

<u>Important:</u> We can evaluate this if we *don't* type check

*But if we type-check, what should be the type of **id**?*

Polymorphism

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1. **Ad Hoc Polymorphism:** The ability to overload function names so that different types can share interfaces

our focus

2. **Parametric polymorphism:** The ability to define functions that are *agnostic* to (parts of) the types, giving it more reusability

Aside: Ad Hoc Polymorphism

```
let add (x : float) (y : float) = x +. y
let add (x : string) (y : string) = x ^ y
(* This doesn't work in OCaml... *)
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Ad hoc polymorphism is essentially **function overloading**

Functions can be defined and used for different types of inputs

Then we can define code against *interfaces* (this is common in object oriented programming)

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For example, we can write a single identity function and use it in multiple contexts

There are many subtleties to this...

Subtlety 1: Type Annotations

```
let rec rev ('a list) : 'a list =
  match l with
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let id : 'a -> 'a = fun x -> x
```

Subtlety 1: Type Annotations

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There are type systems *without* polymorphism *or* type annotations

There are type systems *with* polymorphism that *require* type annotations

Subtlety 2: Type Inference

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We will take up this topic next week

Subtlety 3: Dispatch

```
let to_string (x : 'a) : string = ...
(* This is not possible in OCaml *)
```

Parametric polymorphism cannot be used for *dispatch*

We can't write a polymorphic function that "checks the type" to see what to do

The point: Implementing polymorphism means fundamentally changing the type system

System F

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- **OCaml (Hindley-Milner):** Infer the "most general" polymorphic type
- **System F (2nd-Order λ-Calculus):** take types as arguments!

Implementing Polymorphism

There are a couple approaches to implementing parametric polymorphism:

- **OCaml (Hindley-Milner):** Infer the "most general" polymorphic type
- **System F (2nd-Order λ-Calculus):** take types as arguments!

Either way, we have to introduce the notion of a *type variable*

Type Variables

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The "parametric" part is the fact that types have *variables*

Type variables are instantiated at particular types according to the context

They are very similar to expression variables, e.g., we need to define *type-level capture avoiding substitution*

Quantification

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let id : 'a . 'a -> 'a = fun x -> x
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We read this "**id** has type **t -> t** for any type **t**"

System F

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let id_int : int -> int = fun (x : int) -> x
let id : 'a . 'a -> 'a = fun 'a -> fun (x : 'a) -> x

let test1 = id_int 2
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System F

$\forall \alpha (\alpha \to \alpha)$

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type quant.

type abs.

$\Lambda \alpha. \lambda f^{int \to \alpha}. \lambda x^{int}. f(x+1) :$

$\forall \alpha. (int \to \alpha) \to int \to \alpha$

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The basic idea: Introduce types into the language itself so we can *pass them as arguments to functions*

System F (Caution)

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There are very few languages that implement this kind of polymorphism

System F (Syntax)

bound term var.

type abstraction

fun α → e

STLC

$$
\begin{aligned}
e &::= \boxed{\bullet \mid x \mid \lambda x^{\tau}.\, e \mid ee} \mid \boxed{\Lambda \alpha .\, e} \mid \boxed{e\tau} \\
\tau &::= \top \mid \tau \to \tau \mid \boxed{\alpha} \mid \boxed{\forall \alpha .\, \tau} \\
x &::= \textit{variables} \\
\alpha &::= \textit{type variables}
\end{aligned}
$$

typ. app.

quantification

type variables

bound type variable

The syntax for SOLC is the same as the that of STLC but with:

» constructs for abstracting over and applying to *types*

» constructs for quantifying (or generalizing) over type variables

System F (Typing)

System F (Typing)

STLC

$$\frac{}{\Gamma \vdash \bullet : \top} \qquad \frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'} \qquad \frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

We add <u>two new rules</u> to STLC to deal with our new constructs for polymorphism:

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We add <u>two new rules</u> to STLC to deal with our new constructs for polymorphism:

1. We can generalize over a type variable if our context doesn't depend on it

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instantiate types

We add <u>two new rules</u> to STLC to deal with our new constructs for polymorphism:

1. We can generalize over a type variable if our context doesn't depend on it

2. We can apply an expression e to a type τ, but we have to *substitute* the type into the type of e

Type Substitution

$$[\tau/\alpha]\,\top \;=\; \top$$

$$[\tau/\alpha]\alpha' = \begin{cases} \tau & \alpha' = \alpha \\ \alpha' & \texttt{else} \end{cases}$$

$$[\tau/\alpha](\tau_1 \to \tau_2) = [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_2$$

$$[\tau/\alpha](\forall \alpha' .\, \tau') = \begin{cases} \forall \alpha' .\, \tau' & \alpha' = \alpha \\ \forall \beta .\, [\tau/\alpha][\beta/\alpha']\tau' & \texttt{else (}\beta\texttt{ is fresh)} \end{cases}$$

α-renaming

If we have variables in types, we also need to define *substitution* in types

And we have to deal with capture avoidance!

Example (Substitution)

$$[(\top \to \alpha)/\beta](\forall \alpha. \beta \to \alpha) =$$

$$[\top \to \alpha/\beta](\forall \eta. \beta \to \eta) =$$

$$\forall \eta. (\top \to \alpha) \to \eta$$

Example (Derivation)

$$(T \to T) \to (T \to T) = [T \to T/\alpha]\,\alpha \to \alpha$$

$$\dfrac{\overline{\{x:\alpha\} \vdash x:\alpha}}{\{x:\alpha\} \vdash \Lambda\alpha.\,x : \forall\alpha.\,\alpha}$$

not possible

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{x:\alpha\} \vdash x:\alpha}}{\cdot \vdash \lambda x^{\alpha}.\,x : \alpha \to \alpha}}{\cdot \vdash \Lambda\alpha.\,\lambda x^{\alpha}.\,x : \forall\alpha.\,\alpha \to \alpha}}{\cdot \vdash (\Lambda\alpha.\,\lambda x^{\alpha}.\,x)(T \to T) : (T \to T) \to (T \to T)} \qquad \dfrac{\dfrac{}{\{x:T\} \vdash x:T}}{\cdot \vdash \lambda x^{T}.\,x : T \to T}$$

$$\cdot \vdash \underbrace{\boxed{(\Lambda\alpha.\,\underbrace{\boxed{\lambda x^{\alpha} x}}_{\text{identity}})}}_{\text{polymorphic id.}}(T \to T)\underbrace{\boxed{\lambda x^{T}.\,x}}_{\text{identity}} : T \to T$$

Drawbacks

```
let k = fun 'a 'b (x : 'a) (y : 'b) -> x
let out = k int (bool -> int) 4 (fun b -> if b then 0 else 1)
```

Explicitly passing types as arguments is *clunky*

And maybe we should be able to "tell from context" what the instantiated types are...

<u>OCaml's approach:</u> *we'll figure out the "most general" type you need to pass in from context*

demo

(System F)

Comparison with Curry-Typing

Does dropping type annotations automatically give use polymorphism? (No)

Comparison with Curry-Typing

STLC (Curry)

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no type annotations

$\cdot \vdash \lambda x.x : \text{int} \to \text{int}$ ✓

$\cdot \vdash \lambda x.x : \text{bool} \to \text{bool}$

design this

let id = λx.x in
let a = id 0 in
let b = id • in
•

✗

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Summary

- Implementing **parametric polymorphism** means fundamentally changing our type system
- Polymorphism requires the introduction of **type variables** and **type quantification** in order to *generalize* over possible types