Simple Types

Concepts of Programming Languages
Lecture 18

CAS CS 320

Outline

Have a high-level discussion of **type theory** in general

Introduce and analyze the **simply-typed lambda calculus** (STLC)

Demo an **implementation** of the STLC

Recap

Recall: The Environment Model

$$\langle \mathcal{E}, e \rangle \Downarrow v$$

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Now the **configurations** in our semantics have nonempty state

Recall: Closures

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The environment *captures* bindings which a function needs

Functions need to *remember* what the environment looks like in order to behave correctly according to lexical scoping

Recall: Named Closures

$$(\text{name}, \mathscr{E}, \lambda x \,.\, e)$$

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The idea. Named closures will put themselves into their environment *when they're called*

Recall: Lambda Calculus^{++} (Syntax)

```
<expr> ::= λ<var>.<expr>
         | <var>
         | <expr><expr>
         | let <var> = <expr>
           in <expr>
         | let rec <var> <var> = <expr>
           in <expr>
         | <num>
```

Recall: Lambda Calculus^{++} (Semantics)

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values and variables

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application (unnamed closure)

$$\frac{\langle \mathscr{E}, e_1 \rangle \Downarrow (\mathscr{E}', \lambda x \,.\, e) \qquad \langle \mathscr{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathscr{E}'[x \mapsto v_2], e \rangle \Downarrow v}{\langle \mathscr{E}, e_1 e_2 \rangle \Downarrow v}$$

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let expressions

$$\frac{\langle \mathscr{E}, e_1 \rangle \Downarrow v_1 \qquad \langle \mathscr{E}[x \mapsto v_1], e_2 \rangle \Downarrow v_2}{\langle \mathscr{E}, \text{let } x = e_1 \text{ in } e_2 \rangle \Downarrow v_2} \qquad \frac{\langle \mathscr{E}[f \mapsto (f, \mathscr{E}, \lambda x . e_1)], e_2 \rangle \Downarrow v_2}{\langle \mathscr{E}, \text{let rec } f\ x = e_1 \text{ in } e_2 \rangle \Downarrow v_2}$$

Practice Problem

```
let x = 0 in
let g = fun y -> x + 1 in
let x = 1 in
let f = fun y -> g x in
let x = 2 in
f
```

What (closure) does the following expression evaluate to? You don't need to give the derivation

Answer

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let x = 0 in
let g = fun y -> x + 1 in
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demo

Type Theory

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Types help us delineate "well-behaved" programs

Trade-offs

$$(\lambda x\,.\,xx)(\lambda x\,.\,xx)$$

`lambda term called Ω`

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<u>The goal is to balance:</u>

» Simplicity/Usability
» Expressivity
» Safety/Theoretical Guarantees

OCaml

```
# let big_omega =
    let little_omega x = x x in
    little_omega little_omega;;
Error: This expression has type 'a -> 'b
       but an expression was expected of type 'a
       The type variable 'a occurs inside 'a -> 'b
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The more expressive, the more complex the the type system, designing programming languages is finding the balance that works for you

Recall: Typing Judgments

$$\Gamma \vdash e : \tau$$

This judgment reads:

e has type τ in the context Γ

We say that e is **well-typed** if $\cdot \vdash e : \tau$ for some type τ

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e *has type* τ *in the context* Γ

We say that e is **well-typed** if $\cdot \vdash e : \tau$ for some type τ

Most of what type theorists do is come up with rules for deriving typing judgments

Recall: Contexts

$$\begin{aligned} \Gamma &::= \cdot \mid \Gamma, x : \tau \\ x &::= \texttt{vars} \\ \tau &::= \texttt{types} \end{aligned}$$

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`In Theory:` A context is an inductively-defined syntactic object, just like a type or a expression

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In Practice: A context is a set (or ordered list, in some cases) of **variable declarations**

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In Theory: A context is an inductively-defined syntactic object, just like a type or a expression

In Practice: A context is a set (or ordered list, in some cases) of **variable declarations**

(a variable declaration is a variable together with a type)

Recall: Inference Rules

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \ldots \qquad \Gamma \vdash e_k : \tau_k}{\Gamma \vdash e : \tau}$$

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» How do we know what rules to include?

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The questions we need to answer:

» How do we know what rules to include?

» How do we know if we've chosen *good* rules?

Simply-Typed Lambda Calculus

STLC Syntax

```
<e>  ::= () | <v> | <e> <e>
       | fun ( <v> : <ty> ) -> <e>
<ty> ::= unit | <ty> -> <ty>
<v>  ::= a | ... | z
```

STLC Syntax

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This is the first time that **types are a part of our syntax**

Syntax

$$e ::= \bullet \mid x \mid \lambda x^{\tau}.\, e \mid ee$$

$$\tau ::= \top \mid \tau \to \tau$$

$$x ::= \textit{variables}$$

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$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

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These rules enforce that a function can only be applied if we *know* that it's a function

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No, but it does change the way typing works

If we include annotations we're using **Church-style typing.** If we drop annotations, we're using **Curry-style typing**

Aside: Church vs. Curry Typing

```
fun x -> x

fun (x : unit) -> x
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What is the type of the first expression? How about the second?

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In **Curry-style typing**, the type of an expression is *extrinsic*, the expression is just an expression in the lambda calculus

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In **Curry-style typing**, the type of an expression is *extrinsic*, the expression is just an expression in the lambda calculus

In **Church-style typing**, it's *intrinsic*, built into the expression and the semantics

Using Curry-style typing is not the same as having polymorphism

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Lemma. If $\Gamma \vdash e : \tau_1$ and $\Gamma \vdash e : \tau_2$ then $\tau_1 = \tau_2$

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Proof. The rough idea is to do induction *on the derivations themselves* (whoa)

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In the simply typed lambda calculus with Church-style typing, every expression has a *unique type*

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In the simply typed lambda calculus with Church-style typing, every expression has a *unique type*

In particular, the function `type_of` is well-defined

STLC Semantics (Review)

$$\frac{}{\langle \mathcal{E}, \lambda x^{\tau} . e \rangle \Downarrow (\mathcal{E}, \lambda x . e)}\ \texttt{fun} \qquad \frac{}{\langle \mathcal{E}, \bullet \rangle \Downarrow \bullet}\ \texttt{unit} \qquad \frac{}{\langle \mathcal{E}, x \rangle \Downarrow \mathcal{E}(x)}\ \texttt{variable}$$

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This is part of the point. Type-checking only determines *whether* we go on to evaluate the program (whether it makes sense to)

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The semantics are <u>identical</u>

This is part of the point. Type-checking only determines *whether* we go on to evaluate the program (whether it makes sense to)

It doesn't determine **how** we evaluate the program

Example (Church)

$$\lambda x^{\tau}.\, xx$$

What happens if we try to give a type to the above expression? What should τ be?

Practice Problem

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

$$\cdot \vdash \lambda f^{\top \to \top} . \lambda x^{\top} . f x : (\top \to \top) \to \top \to \top$$

Give a derivation for the above judgment

Answer

$$\cdot \vdash \lambda f^{\top \to \top}.\, \lambda x^{\top}.\, fx : (\top \to \top) \to \top \to \top$$

How do we know if we've defined a "good" programming language?

Type Safety

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With small-step semantics, we can give a finer-grained analysis:

> **Theorem.** If $\cdot \vdash e : \tau$, then
>
> » *(progress)* either e is a value or there is an e' such that $e \longrightarrow e'$
>
> » *(preservation)* If $\cdot \vdash e : \tau$ and $e \longrightarrow e'$ then $\cdot \vdash e' : \tau$

Type Safety

> **Theorem.** If $\cdot \vdash e : \tau$ then there is a value v such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

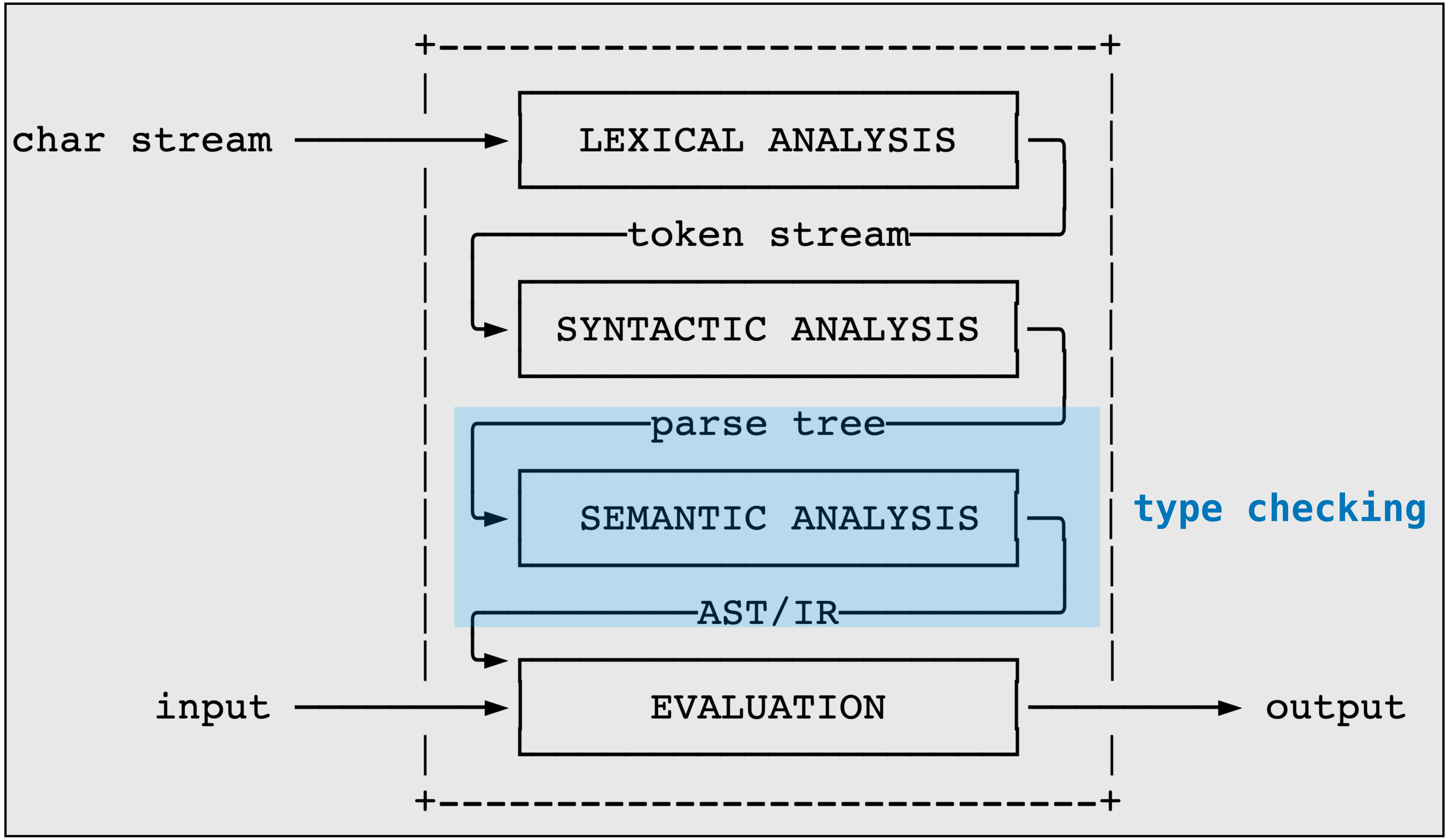
With small-step semantics, we can give a finer-grained analysis:

> **Theorem.** If $\cdot \vdash e : \tau$, then
>
> » *(progress)* either e is a value or there is an e' such that $e \longrightarrow e'$
>
> » *(preservation)* If $\cdot \vdash e : \tau$ and $e \longrightarrow e'$ then $\cdot \vdash e' : \tau$

These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

Type Checking

The Picture



char stream
LEXICAL ANALYSIS
token stream
SYNTACTIC ANALYSIS
parse tree
SEMANTIC ANALYSIS
type checking
AST/IR
input
EVALUATION
output

Type Checking vs. Type Inference

```
type_check : expr -> ty -> bool

type_of : expr -> ty option
```

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Type checking the problem of determining whether a given expression is a given type

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Type inference is the problem of *synthesizing* a type for a given expression, if possible

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Theoretically, these two problems can be very different

For STLC, they are both easy

The One Issue

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

The One Issue

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

How do we turn this into a type-checking procedure?

The One Issue

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It seems like we need to do *some* amount of inference because it's not immediately clear what type we should check e_1 to be

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Our solution: We'll just use type inference demo

Summary

Type systems delineate well-behaved expressions

Type inference can sometimes be easier to implement