Staggered PRF

Debby

I. Signal Model

Consider a pulse-Doppler radar transceiver that transmits T modulated pulse trains, each consisting of P equally spaced pulses. For $0 \leq t \leq P\tau$, the ith train is given by

$$x_T(t) = \sum_{p=0}^{P-1} h(t-p\tau)e^{-j\frac{2\pi c_i}{P\tau}t} \approx \sum_{p=0}^{P-1} h(t-p\tau)e^{-j\frac{2\pi c_i}{P}p}. \quad (1)$$

Here c_i reminds the ith coset of the multicoset system and represents the shift of the ith pulse train. The pulse-to-pulse delay τ is the PRI, and its reciprocal $1/\tau$ is the pulse repetition frequency (PRF). The entire span of the signal in (1) is called the CPI. The pulse $h(t)$ is a known time-limited baseband function with continuous-time Fourier transform (CTFT) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\mathrm{d}t$. We assume that $H(\omega)$ has negligible energy at frequencies beyond $B_h/2$ and we refer to B_h as the bandwidth of $h(t)$.

Now, consider L non-fluctuating point-targets, according to the Swerling-0 model, where L, or at least an upper bound for it, is assumed to be known. The pulses reflect off the L targets and propagate back to the transceiver. The lth target is defined by three parameters: a time delay $\tilde{\tau}_l$, proportional to the target distance to the radar; a Doppler radial frequency ν_l, proportional to the target closing velocity to the radar; and a complex amplitude α_l.

The time delay $\tilde{\tau}_l$ is not assumed to be less than the PRI τ, as in traditional pulse radar; we assume $0 \leq \tilde{\tau}_l < Q\tau$, where Q is a model parameter. For convenience, we decompose $\tilde{\tau}_l = \tau_l + q_l\tau$ with $0 \leq q_l < Q$ and $0 \leq q_l < \tau$. The received signal from the ith pulse train can be written as

$$x(t) = \sum_{p=0}^{P-1}\sum_{l=0}^{L-1} \alpha_l h(t-\tilde{\tau}_l-p\tau)e^{-j2\pi\nu_l(p+q_l)\tau}e^{-j\frac{2\pi c_i}{P}p}. \quad (2)$$

It will be convenient to express $x(t)$ as a sum of buckets b rather than a sum of pulses p. The bth bucket is the time interval $[b\tau, (b+1)\tau[$. Note that in the traditional pulse Doppler settings, namely under the assumption that $0 \leq \tilde{\tau}_l < \tau$, the pulses and buckets are the same. Here, the pth pulse reflected from the lth target is received in the $p + q_l$ bucket. We can then rewrite (2) as

$$x(t) = \sum_{b=0}^{P+Q-1}\sum_{l=0}^{L-1} \alpha_l h(t-\tau_l-b\tau)e^{-j2\pi\nu_l b\tau}e^{-j\frac{2\pi c_i}{P}(b-q_l)}, \quad (3)$$

where we used $\tilde{\tau}_l = \tau_l + q_l\tau$. Note that the first and last Q buckets may not contain reflections from all targets depending on their specific q_l. In the next section, when writing (3) in matrix form after Fourier transformation, we will remove these buckets.

For convenience, we write $x(t)$ as a sum of single frames

$$x(t) = \sum_{b=0}^{P+Q-1} x_b(t), \quad (4)$$

where

$$x_b(t) = \sum_{l=0}^{L-1} \alpha_l h(t-\tau_l-b\tau)e^{-j2\pi\nu_l b\tau}e^{-j\frac{2\pi c_i}{P}(b-q_l)}. \quad (5)$$

Consider the Fourier series representation of the aligned frames $x_b(t+b\tau)$:

$$X_b[k] = \frac{1}{\tau}H[k]\sum_{l=0}^{L-1} \alpha_l e^{-j2\pi k\tau_l/\tau}e^{-j2\pi b(\nu_l\tau+\frac{c_i}{P})}e^{j2\pi q_l\frac{c_i}{P}}, \quad (6)$$

for $0 \leq k \leq N-1$ with $N = \tau B_h$. From (6), we see that the unknown parameters $\{\alpha_l, q_l, \tau_l, \nu_l\}_{l=0}^{L-1}$ are embodied in the Fourier coefficients $X_b[k]$. The goal is then to recover these parameters from $X_b[k]$. Note that we consider Nyquist samples here, namely we have $N = B_h\tau$ samples from each of the P pulses from each of the $T = Q$ pulse trains. In the next part of this work, we will investigate the use of $K < N$ samples from $M < P$ pulses from $T \geq Q$ trains.

II. Example

Consider the following simple examples with $P = 5$ emitted pulses and $L = 2$ targets. We consider $Q = 2$, namely the target delays can be up to twice the PRI τ. The transmitted signal $x_T(t)$ and received signal $x(t)$ are shown in Fig. 1 along with the pulses and buckets indices p and b, respectively.

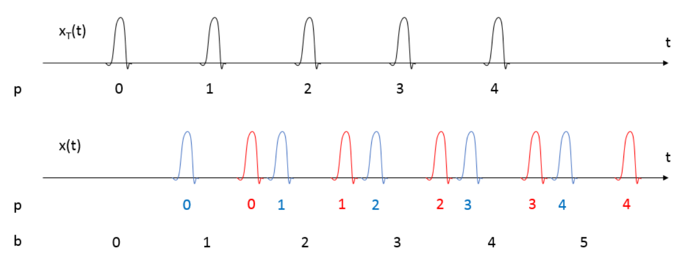

Fig. 1. Transmitted (top) and received (bottom) signals with two target, one ”close” and one ”far”.

The first target (blue) is a ”traditional” or close target, namely $\tilde{\tau}_0 = \tau_0 < \tau$ and $q_0 = 0$. The part of equation (2) for this target $l = 0$ becomes

$$x(t) = \sum_{p=0}^{P-1} \alpha_0 h(t-\tau_0-p\tau)e^{-j2\pi\nu_0 p\tau}e^{-j\frac{2\pi c_i}{P}p}. \quad (7)$$

The corresponding part of equation (3) for the target $l=0$ is

$$x(t)=\sum_{b=0}^{P+Q-1}\alpha_0 h(t-\tau_0-b\tau)e^{-j2\pi\nu_0 b\tau}e^{-j\frac{2\pi c_i}{P}b}, \quad (8)$$

Here the sum stops at $b=P-1$. Substituting $b=p+q_0=p$ (here the pulses and buckets obviously coincide), we see that the two equations above are identical.

The second target (red) is the more interesting target. Here, $\tilde{\tau}_1=\tau_1+\tau>\tau$ and $q_1=1$. The part of equation (2) for this target $l=1$ becomes

$$x(t)=\sum_{p=0}^{P-1}\alpha_1 h(t-\tau_1-\tau-p\tau)e^{-j2\pi\nu_1(p+1)\tau}e^{-j\frac{2\pi c_i}{P}p}. \quad (9)$$

The corresponding part of equation (3) for the target $l=1$ is

$$x(t)=\sum_{b=0}^{P+Q-1}\alpha_1 h(t-\tau_1-b\tau)e^{-j2\pi\nu_1 b\tau}e^{-j\frac{2\pi c_i}{P}(b-1)}, \quad (10)$$

Here the sum starts at $b=1$. Substituting $b=p+q_1=p+1$, we see that the two equations above are identical.

In order to avoid the issues with the first and last buckets which are not full, that is do not contain reflections from all targets, (here the first target does not appear in the 6th and last bucket and the second target does not appear in the first bucket), we can remove these edge buckets and only consider measurements from the "full" buckets. Here, we would only consider buckets $b=1,2,3,4$.

III. Possible Solutions

A. Simultaneous Matrix Approach

We now write (6) in matrix form. Suppose that we limit ourselves to the Nyquist grid so that $\tau_l/\tau=s_l/N$, where s_l is an integer satisfying $0\leq s_l\leq N-1$, and $\nu_l\tau=r_l/P$, where r_l is an integer in the range $0\leq r_l\leq P-1$. Let $\mathbf{X}_{c_i}$ be the $N\times(P-Q)$ matrix with bth column given by $X_b[k]$, $Q-1\leq b\leq P-1$. We can then write $\mathbf{X}_{c_i}$ as

$$\mathbf{X}_{c_i}=\mathbf{H}\mathbf{F}^1_{c_i}\mathbf{A}\mathbf{F}^2_{c_i}. \quad (11)$$

Here $\mathbf{H}=\frac{1}{\tau}\text{diag}\left(H[k]\right)$ The (m,b)th entry of the $P\times(P-Q)$ matrix $\mathbf{F}^2_{c_i}$ is given by $e^{-j2\pi b(m+\frac{c_i}{P})}$. The $N\times QN$ matrix $\mathbf{F}^1_{c_i}$ is composed of Q blocks of size $N\times N$. The (k,n)th entry of the qth block of $\mathbf{F}^1_{c_i}$ is given by $e^{-j2\pi k\frac{n}{N}}e^{-j2\pi q\frac{c_i}{P}}$. The matrix $\mathbf{A}$ is an $QN\times P$ sparse matrix that contains the values α_l at the corresponding L indices $\{s_l+Nq_l,r_l\}$.

The goal is to recover $\mathbf{A}$ from the T matrices $\mathbf{X}_{c_i}$. Of course, each equation (11) cannot be solved individually because of the ambiguity between the columns of $\mathbf{F}^1_{c_i}$. For example, we cannot distinguish between a target in the first block with value α_l and a target in the second block with value $\alpha_l e^{-j2\pi\frac{c_i}{P}}$.

We first propose the following minimization problem

$$\min_{\mathbf{A}}\sum_{i=0}^{T-1}||\mathbf{X}_{c_i}-\mathbf{H}\mathbf{F}^1_{c_i}\mathbf{A}\mathbf{F}^2_{c_i}||_F^2+\lambda||\mathbf{A}||_1. \quad (12)$$

B. Kronecker-based Vector Approach

In this section, we propose to vectorize (11) in order to concatenate the T resulting equations. Using Kronecker product properties, we have

$$\text{vec}(\mathbf{H}^{-1}\mathbf{X}_{c_i})=\left(\left(\mathbf{F}^2_{c_i}\right)^T\otimes\mathbf{F}^1_{c_i}\right)\text{vec}(\mathbf{A}). \quad (13)$$

After concatenation, we obtain

$$\begin{bmatrix}\text{vec}(\mathbf{H}^{-1}\mathbf{X}_{c_0})\\ \text{vec}(\mathbf{H}^{-1}\mathbf{X}_{c_2})\\ \vdots\\ \text{vec}(\mathbf{H}^{-1}\mathbf{X}_{c_{T-1}})\end{bmatrix}=\begin{bmatrix}\left(\left(\mathbf{F}^2_{c_0}\right)^T\otimes\mathbf{F}^1_{c_0}\right)\\ \left(\left(\mathbf{F}^2_{c_2}\right)^T\otimes\mathbf{F}^1_{c_2}\right)\\ \vdots\\ \left(\left(\mathbf{F}^2_{c_{T-1}}\right)^T\otimes\mathbf{F}^1_{c_{T-1}}\right)\end{bmatrix}\text{vec}(\mathbf{A}). \quad (14)$$

Consider the dimensions of (14). The left side is composed of T vertically concatenated $N(P-Q)$ vectors and thus is a $TN(P-Q)\times 1$ vector. The unknown vector is a vectorized version of the matrix $\mathbf{A}$ and therefore is of size $QNP\times 1$. Last, the measurement matrix is composed of T vertically concatenated Kronecker products of size $N(P-Q)\times QNP$ and is thus of size $TN(P-Q)\times QNP$ We solve for vec($\mathbf{A}$) by using the pseudo-inverse of the matrix composed of the Kronecker products.

C. Doppler Focusing Approach

We now consider a Doppler Focusing approach similar to the one presented in [1]. Consider the aligned frames $x_b(t+b\tau)$

$$x_b(t+b\tau)=\sum_{l=0}^{L-1}\alpha_l h(t-\tau_l)e^{-j2\pi\nu_l b\tau}e^{-j\frac{2\pi c_i}{P}(b-q_l)}. \quad (15)$$

The Doppler focusing, including a c_i-shift for each pulse train, is given by

$$\begin{aligned}\phi(t,\nu) &= \sum_{b=0}^{P+Q-1}x_b(t+b\tau)e^{j2\pi b\left(\nu\tau+\frac{c_i}{P}\right)}\\ &= \sum_{l=0}^{L-1}\alpha_l h(t-\tau_l)e^{j\frac{2\pi c_i}{P}q_l}\sum_{b=0}^{P+Q-1}e^{-j\nu_l b\tau(\nu-\nu_l)}\\ &\approx P\sum_{l=0}^{L-1}\alpha_l h(t-\tau_l)e^{j\frac{2\pi c_i}{P}q_l}. \end{aligned} \quad (16)$$

In the Fourier domain, we obtain

$$\Phi_\nu[k]=\frac{P}{\tau}H[k]\sum_{l=0}^{L-1}\alpha_l e^{-j2\pi k\frac{\tau_l}{\tau}}e^{j2\pi q_l\frac{c_i}{P}}. \quad (17)$$

Concatenating the Fourier coefficients of all pulse trains in matrix form, we obtain, for each focused Doppler ν

$$\mathbf{\Phi}=\mathbf{H}\mathbf{F}_\tau\mathbf{B}\mathbf{F}_q. \quad (18)$$

Here, $\mathbf{B}$ is a $N\times Q$ sparse unknown matrix that contains the values α_l at the corresponding L indices $\{s_l,q_l\}$. The (k,i)th entry of the $N\times T$ matrix $\mathbf{\Phi}$ is given by the Fourier coefficient $\Phi_\nu[k]$ from the ith pulse train. The (k,n)th entry of the $N\times N$ matrix $\mathbf{F}_\tau$ is given by $e^{-j2\pi k\frac{n}{N}}$ and the (q,i)th entry of the $Q\times T$ matrix $\mathbf{F}_q$ is given by $e^{-j2\pi q\frac{c_i}{P}}$. To recover the sparse matrix $\mathbf{B}$, we adopt the Doppler focusing approach from [1] with the extended OMP algorithm to matrix recovery[2].

References

[1] O. Bar-Ilan and Y. C. Eldar, "Sub-Nyquist radar via Doppler focusing," *IEEE Trans. Sig. Proc.*, vol. 62, pp. 1796–1811, Apr. 2014.

[2] T. Wimalajeewa, Y. C. Eldar, and P. K. Varshney, "Recovery of sparse matrices via matrix sketching," *CoRR*, vol. abs/1311.2448, 2013. [Online]. Available: http://arxiv.org/abs/1311.2448