S6 Improve Sensitivity II

Reduce Variance: Stratification, Control Variables (CUPED)

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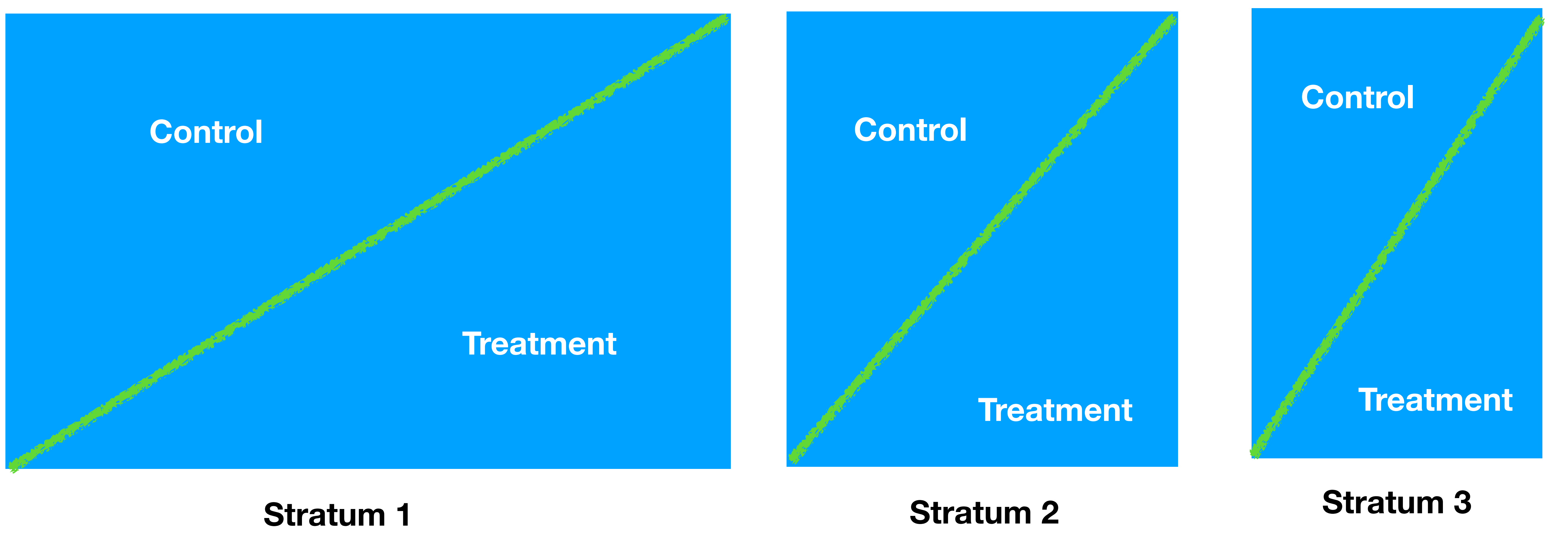
Reduce Variance σ^2

- Stratification *at assignment*
  - Post Stratification
- Control Variates
  - Regression with Control Variates
  - CUPED

Stratification

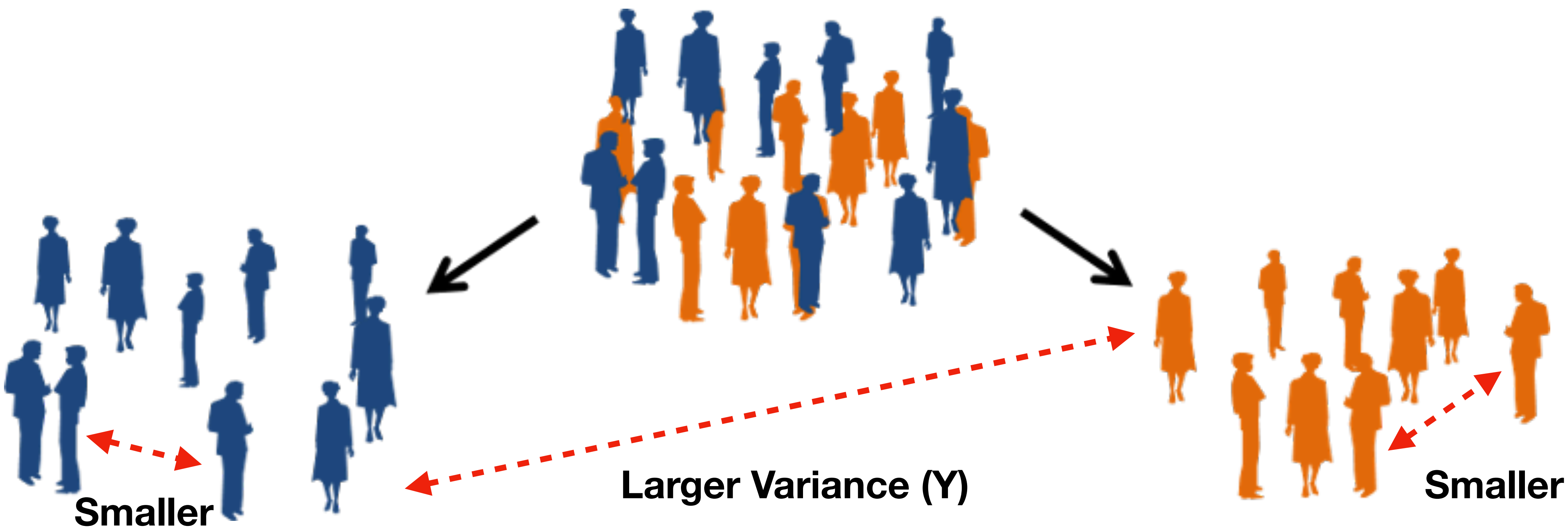
- The most well-known at-assignment variance reduction technique.
  - Block Design
- Basic procedure:
  - Divide the population into strata
  - Sample from each stratum independently
  - Randomize within each stratum
  - Combine treatment effects from strata to give an overall estimate

Block Design



Stratification

- Idea: Remove between-strata variance in metrics (OECs)
- Choose a categorical variable to divide population into different strata (categories) with different OEC (Y)
- The value of OECs are different across strata.



How to Divide Population into Strata

- Stratified sampling aims to remove the between-strata variance in OEC (Y).

$$Var(\bar{Y}^s) = Var(\bar{Y}) - \frac{1}{n}\sum_{1}^{K} p_k(\mu_k - \mu)^2$$

- Which variable will you choose to divide the population into strata?

OEC is d(purchase) of Taobao

1. Age Groups
2. Genders
3. #visits in the last week
4. $ spending in the last week
5. d(purchase) during the week before the experiment (high, low)

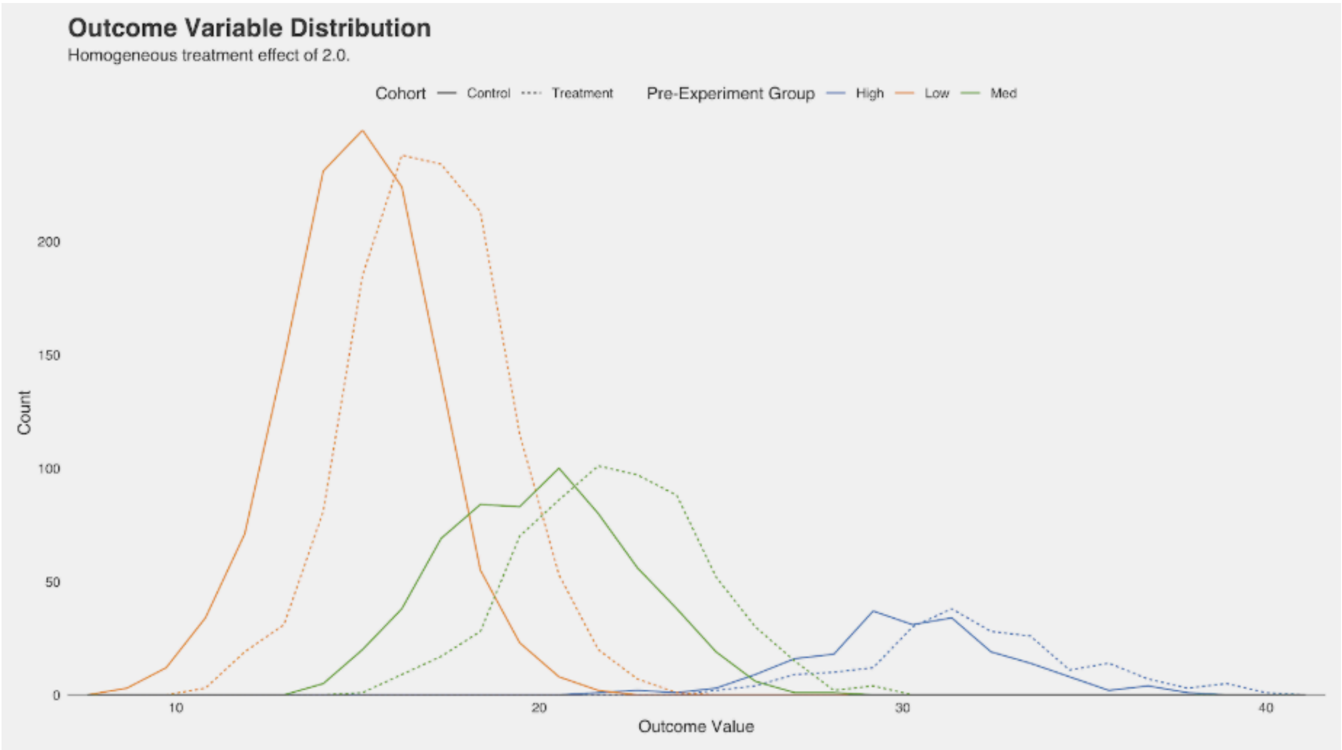
With the largest correlation with post-experiment OEC to remove the largest between-strata variance

How to Divide Population into Strata

- Categorize population into different Strata
  - $n_k = p_k n$

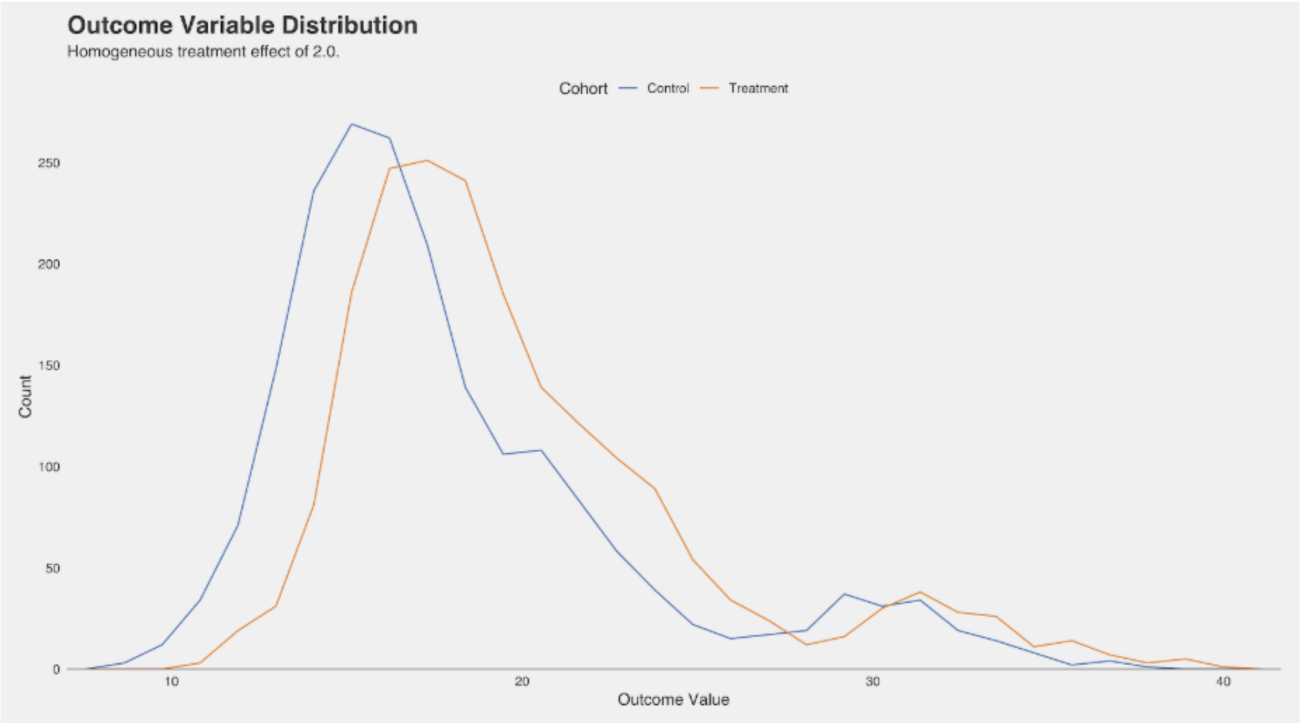
Proportion of the stratum in the population

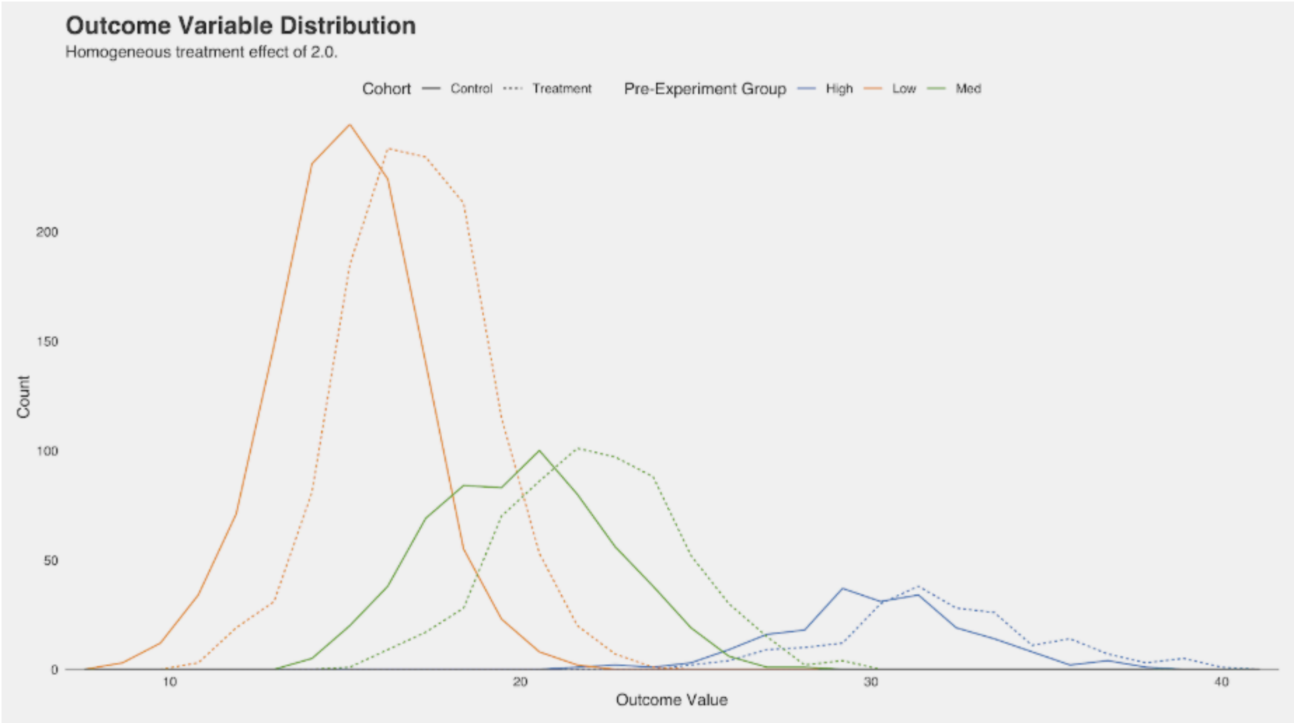
- Pre-experiment OEC is a good choice and correlates well with OEC
- If OEC is a continuous variable
  - Transform pre-experiment OEC into a categorical variable
  - e.g., High, Middle, Low
  - Based on A/A test or Historical Data



Stratification

- For example, divide into three strata based on pre-experiment Y (OEC):
  - High, Middle, Low
- Estimate three treatment effects and then combine them together (weighted average)





Reference: https://www.tripadvisor.com/engineering/reducing-a-b-test-measurement-variance-by-30/

How to Divide Population into Strata

- When pre-experiment data on OEC is not available
    - A new product
    - Experiment with new users
- Choose the one, most likely highly correlated with the OEC
    - Based on experience and data
    - e.g., two UIs for shopping cart and OEC is purchase rate
        - Age or Gender

Stratification

1. Pre-stratify the sample, $n_k = p_k n$
2. Randomize completely within each stratum
3. Combine the Treatment Effects with different weights

$$\bar{Y}^s = \sum_{k=1}^{K} p_k \bar{Y}_k$$

$$\Delta^s = \bar{Y}_1^s - \bar{Y}_0^s$$

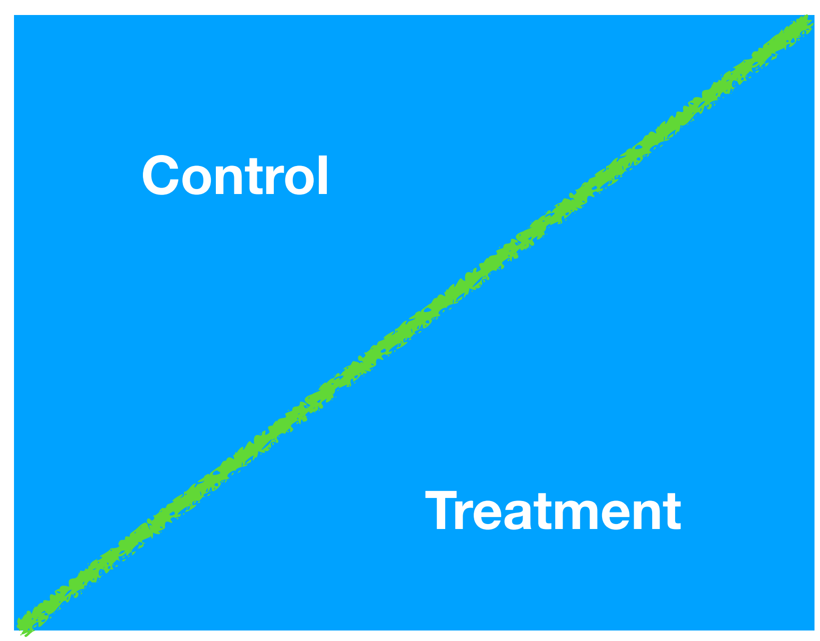
4. Calculate the variance

$$Var(\bar{Y}^s) = \frac{1}{n} \sum_{k=1}^{K} p_k \sigma_k^2$$

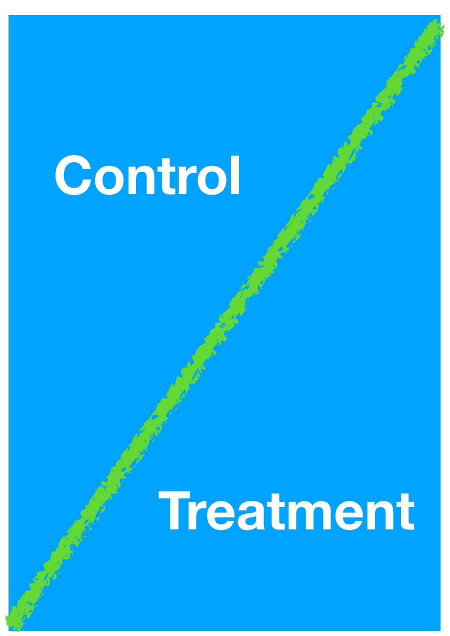
$$Var(\Delta^s) = Var(\bar{Y}_1^s) + Var(\bar{Y}_0^s)$$

5. Calculate t (z) stat

$$t = \frac{\Delta^s}{\sqrt{Var(\Delta^s)}}$$



High Pre-OEC



Low Pre-OEC

Stratification

- Assume there are k strata:

$$\bar{Y}^s = \sum_{k=1}^{K} p_k \bar{Y}_k$$

p_k: proportion of the population in the k-th stratum

n_k : number of users from the k-th stratum

$$n_k = p_k n$$

$$E(\bar{Y}^s) = \sum_{k=1}^{K} p_k E(\bar{Y}_k) = \sum_{k=1}^{K} p_k \mu_k = \mu$$

- Δ^s under stratified sampling is an unbiased estimate of δ

$$\Delta^s = \bar{Y}_1^s - \bar{Y}_0^s$$

$$E(\Delta^s) = E(\bar{Y}_1^s - \bar{Y}_0^s) = \mu_1 - \mu_0 = \delta$$

- However, $\mathrm{Var}(\Delta^s) < \mathrm{Var}(\Delta)$

$$Var(Y) = \sum_{k=1}^{K} p_k \sigma_k^2 + \sum_{1}^{K} p_k(\mu_k - \mu)^2 = Var(Y^s) + \sum_{1}^{K} p_k(\mu_k - \mu)^2$$

Within-strata variance+
Between-strata variance

$$Var(\bar{Y}^s) = Var(\bar{Y}) - \frac{1}{n}\sum_{1}^{K} p_k(\mu_k - \mu)^2$$

Remove Between-Strata Variance

$$Var(\Delta^s) = Var(\bar{Y}_1^s) + Var(\bar{Y}_0^s)$$

Class Exercise

OEC is the $Purchase/week

p(male)=p(female)=0.5

n=1000

Sample Characteristics:

- n(male)=n(female)=0.5*1000=500
- $\bar{Y}$(male)=20, $\bar{Y}$(female)=50
- Var(male)=20, Var(female)=10

A. What are the mean and variance for Y?

B. What are the stratification mean and its variance?

Class Exercise

A. What are the mean and variance for the whole sample?

- $\bar{Y}^s = \bar{Y} = \sum_{k=1}^{K} p_k \bar{Y}_k$ = 0.5*20+0.5*50=35
- $Var(Y) = \sum_{k=1}^{K} p_k \sigma_k^2 + \sum_{1}^{K} p_k(\mu_k - \mu)^2 = Var(Y^s) + \sum_{1}^{K} p_k(\mu_k - \mu)^2$

= 15 + 0.5*(20-35)^2 + 0.5*(50-35)^2

B. What are the stratification mean and its variance?

- $\bar{Y}^s = \sum_{k=1}^{K} p_k \bar{Y}_k$ = 0.5*20+0.5*50=35
- $Var(\bar{Y}^s) = \frac{1}{n} \sum_{k=1}^{K} p_k \sigma_k^2$ = 1/1000*(0.5*20+0.5*10)=0.015

Case: Netflix's Implementation of Stratification

A. Randomize users into different variants for each stratum in real-time.

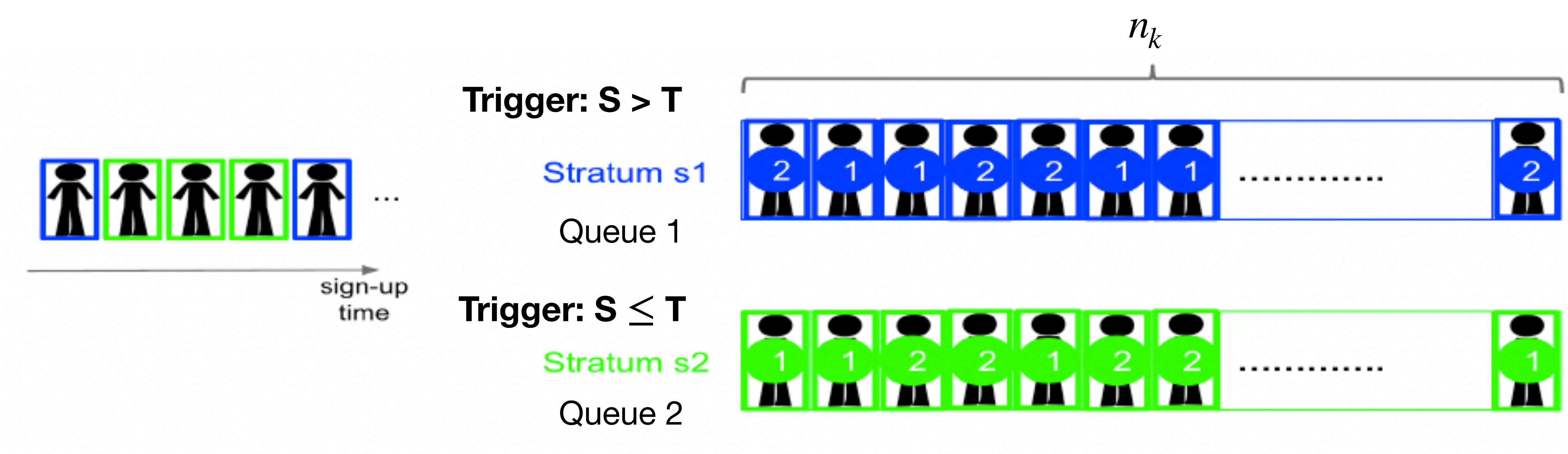
1. Define strata factor: pre-experiment streaming hours
   - Stratum (s) = high if factor > threshold (T)
   - Different T leads to different ρ = correlation (s, OEC)
     - $\rho > 0.5$
   - Only on existing users, not on new users WHY?
2. Pre-define n_k for each k (stratum) based on p_k

Reference: Xie, H., & Aurisset, J. (2016, August). Improving the sensitivity of online controlled experiments: Case studies at netflix. In *Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (pp. 645-654).

Case: Netflix's Implementation of Stratification

B. Rely on a Queue System for (Random) Assignment

1. Assign users into different queues based on their strata factor (e.g., pre-OEC) - trigger conditions



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Case: Netflix’s Implementation of Stratification

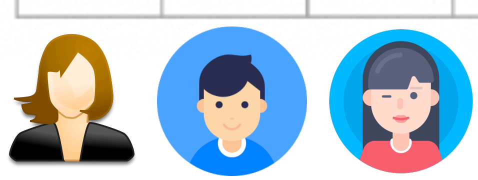
B. Rely on a Queue System for (Random) Assignment

1. Assign users into a different queue based on their strata factor (e.g., pre-OEC) - a trigger
2. A queue consists of many 100-slot segments
3. Randomly assign users into different variants within each queue
    A. Assign a number to each cell [1, 100]
    B. Shuffle the numbers for the cells
    C. Map 1 (control) to the cell with the number [1, 50], and 2 (treatment) with [51, 100]

(a)	1	2	3	4	5	6	.	.	.	.	.	.	.	100

(b)	25	57	9	12	95	64	.	.	.	.	.	.	.	43

(c)	1	2	1	1	2	2	.	.	.	.	.	.	.	1



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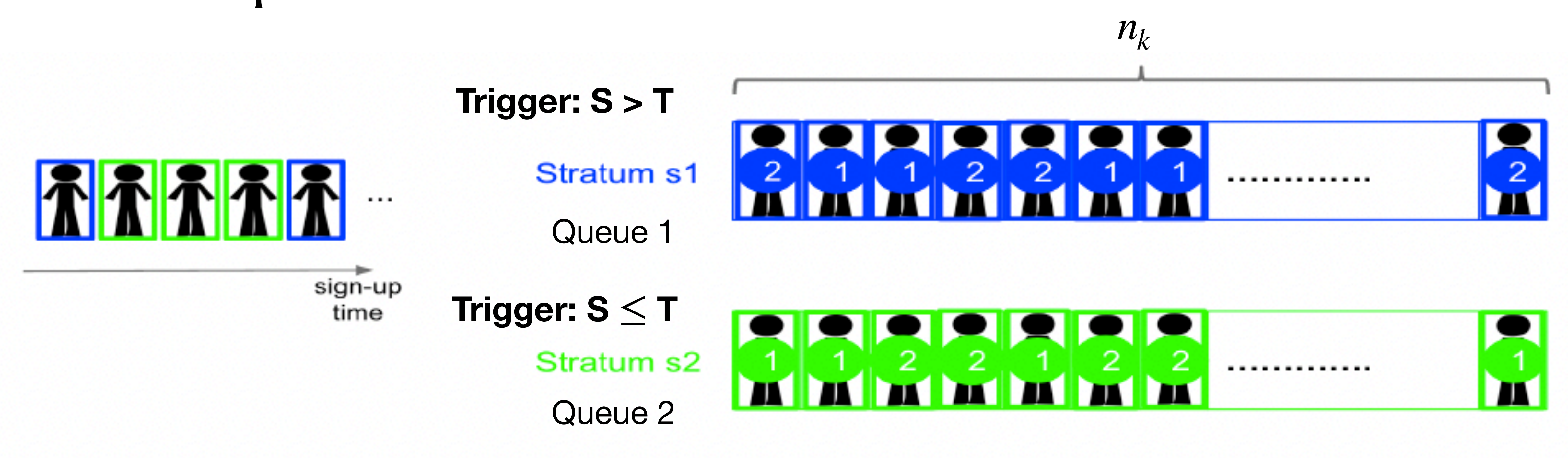

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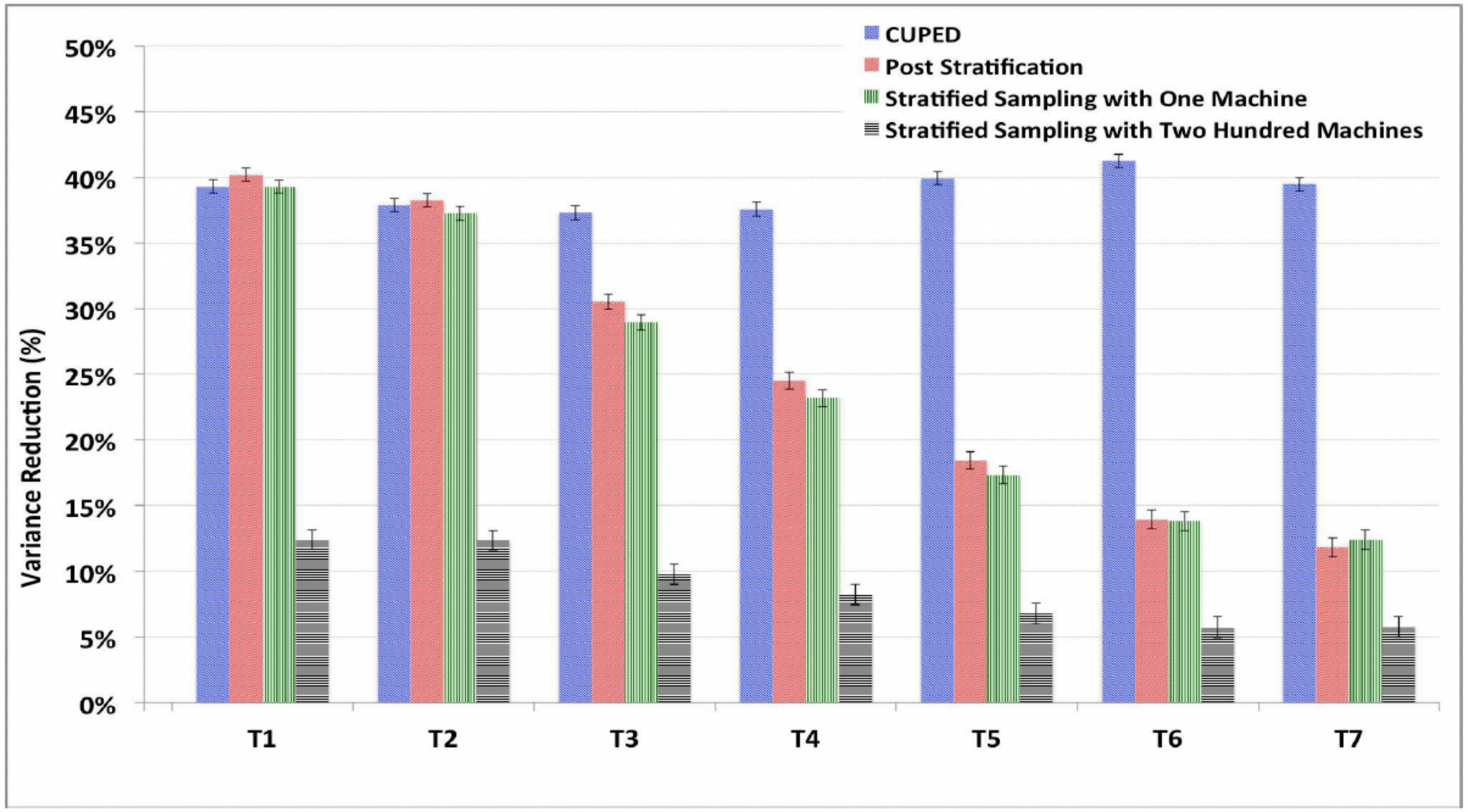
Compare Stratification, Post Stratification, and CUPED at Netflix

- All the techniques achieve much larger variance reduction on existing users.
- Theoretically, CUPED, Stratification, and Post-Stratification should achieve similar performance.
- CUPED and Post Stratification do better than Stratification at Assignment.
- The implementation of stratification is difficult and introduces noises.



Ti are dummies with the different threshold of streaming hours

Existing Users

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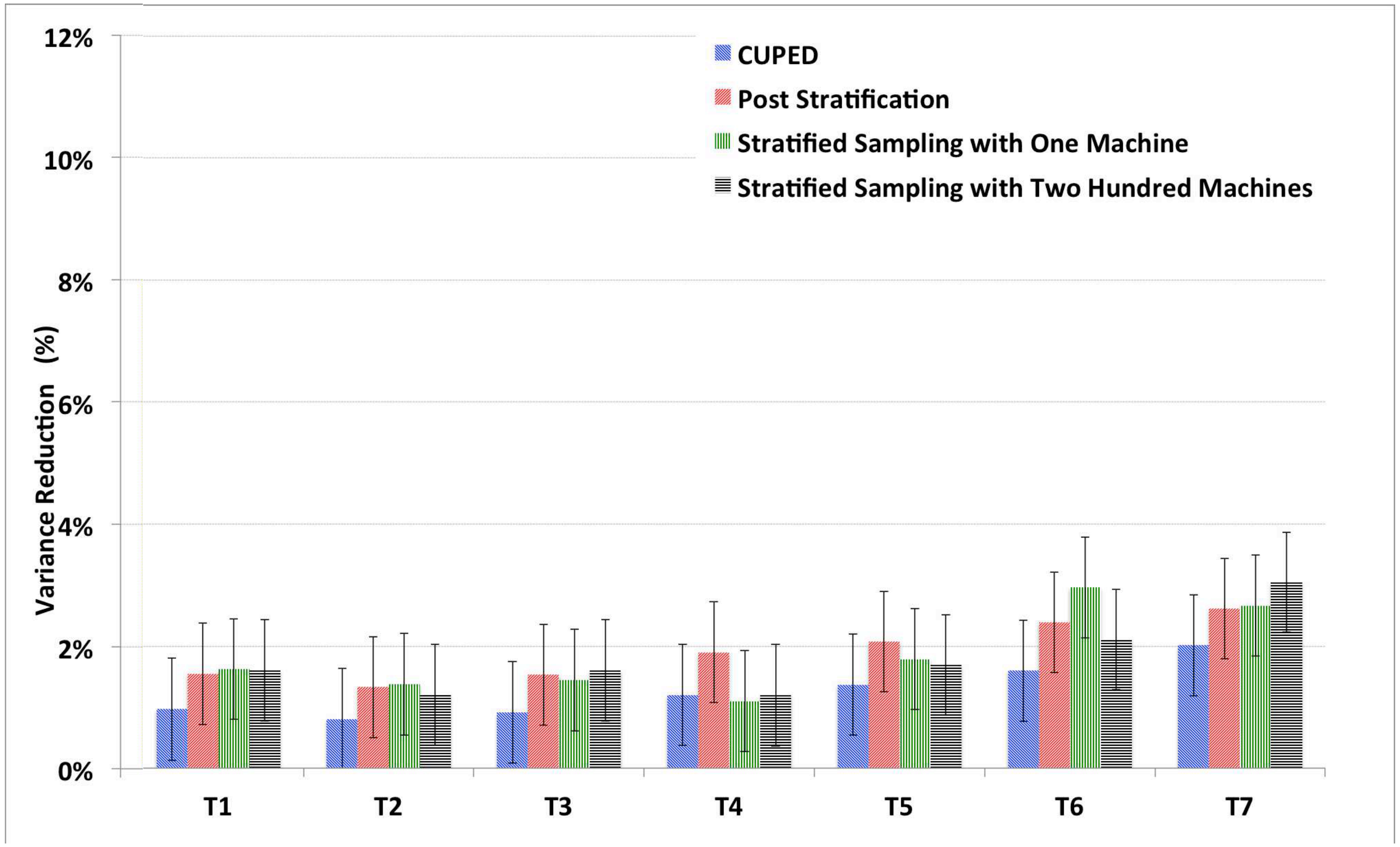
Compare Stratification, Post Stratification, and CUPED at Netflix

- Variance Reduction is not large on new Users with any techniques.
  - Insufficient pre-experiment data to construct Ti
  - Instead, Ti was constructed by predictions (Machine Learning Models)



Ti are dummies with the different threshold of streaming hours

New Users

Reference: Xie, H., & Aurisset, J. (2016, August). Improving the sensitivity of online controlled experiments: Case studies at netflix. In *Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (pp. 645-654).

Post Stratification

- A popular post-assignment variance reduction technique
- Assume simple random assignment but uses the estimate of stratification:

$$\bar{Y}^s_{post} = \sum_{k=1}^{K} p_k \bar{Y}_k$$

p_k: proportion of the population in the k-th stratum

n_k : number of users from the k-th stratum

$$n_k = p_k n$$

$$E(\bar{Y}^s_{post}) = \sum_{k=1}^{K} p_k E(\bar{Y}_k) = \sum_{k=1}^{K} p_k \mu_k = \mu$$

Post Stratification

1. Randomly assign users into control and treatment groups.
2. Collect data on OEC and S (strata factor)
3. Calculate $\bar{Y}_k$ for each stratum.
4. Calculate $\bar{Y^s_{post}} = \sum_{k=1}^{K} p_k \bar{Y}_k$
    - p_k is predefined according to "population" data
    - $\bar{Y}_k$ is sample mean for each stratum k
5. Calculate $\Delta^s_{post} = \bar{Y}^s_{post1} - \bar{Y}^s_{post0}$
6. Calculate Var (Δ^s_{post})

$$Var(\bar{Y}^s_{post}) = \frac{1}{n}\sum_{k=1}^{K} p_k \sigma_k^2 + \frac{1}{n^2}\sum_{k=1}^{K}(1-p_k)\sigma_k^2 + o(\frac{1}{n^2})$$

Larger than $Var(\bar{Y}^s)$

$$Var(\Delta^s_{post}) = Var(\bar{Y}^s_{post1}) + Var(\bar{Y}^s_{post0})$$

Post Stratification

- However, variance reduction is smaller than stratification at assignment
    - $n_k^{post} \neq p_k n$
- $var(\bar{Y}^s) \leq var(\bar{Y}^s_{post}) \leq var(\bar{Y})$
- When n is very large:
    - $var(\bar{Y}^s) \approx var(\bar{Y}^s_{post}) \leq var(\bar{Y})$
- Post stratification is less costly to implement.

Example

OEC is the $Purchase/week

p(male)=p(female)=0.5

n=1000

Sample Characteristics:

- n(male)= 505
- n(female)= 495

Different from p_k

- $\bar{Y}$(male)=20, $\bar{Y}$(female)=50
- Var(male)=20, Var(female)=10

Stratification Mean and Variance (Mean)

- $\bar{Y}^s = \sum_{k=1}^{K} p_k \bar{Y}_k$ = 0.5*20+0.5*50=35
- $Var(\bar{Y}^s_{post}) = \frac{1}{n}\sum_{k=1}^{K} p_k \sigma_k^2 + \frac{1}{n^2}\sum_{k=1}^{K}(1-p_k)\sigma_k^2 + o(\frac{1}{n^2})$ =(0.5*20+0.5*10)/1000+(0.5*20+0.5*10)/1000^2+$o(\frac{1}{n^2})$

CUPED

CUPED: Controlled Experiments by Utilizing Pre-Experiment Data (Deng, Xu, Kohavi, & Walker, 2013)

- Remove variance in a metric that can be accounted for by pre-experiment information.
- Control Variates: Pre-experiment information X.
- $Y^{cuped} = Y - \theta X$ Remove the variance caused by X
- $\Delta^{cuped} = \Delta = m_1 - m_0 = \bar{Y}_1 - \bar{Y}_0 = \bar{Y}_1^{cuped} - \bar{Y}_0^{cuped}$

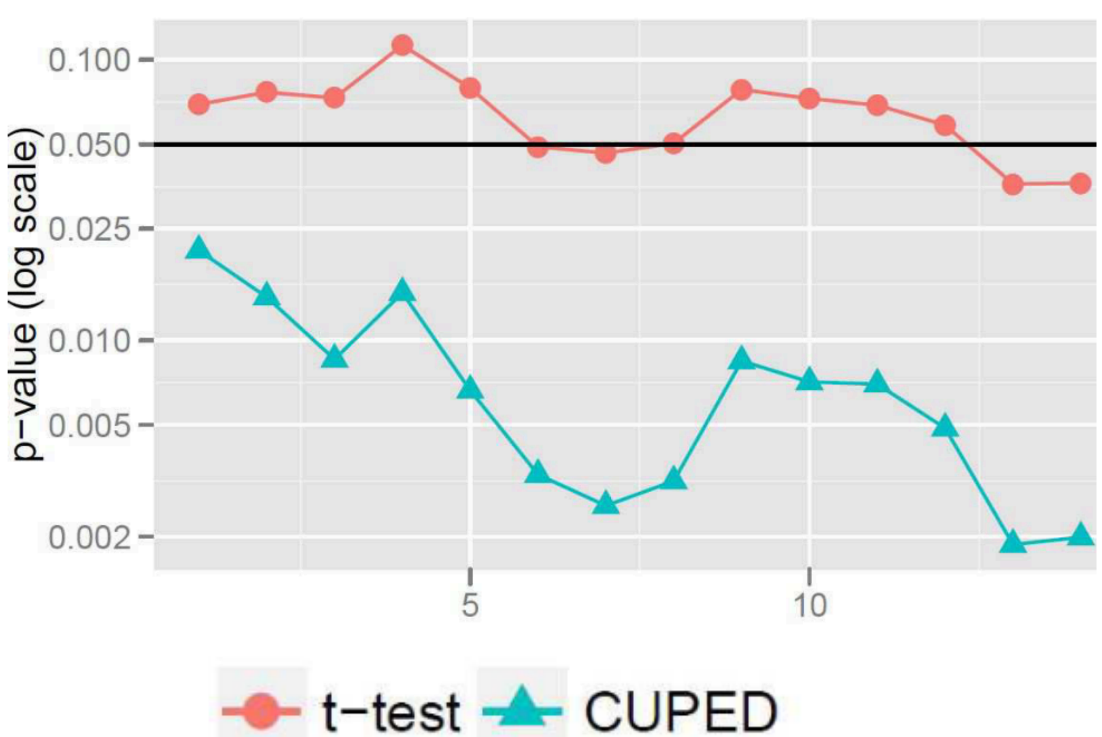
Deng, Alex, et al. "Improving the sensitivity of online controlled experiments by utilizing pre-experiment data." *Proceedings of the sixth ACM international conference on Web search and data mining*. 2013.

How to Choose Control Variables

- $Var(Y^{cuped}) = Var(Y) + \theta^2 Var(X) - 2\theta Cov(X, Y)$
- To Minimize $(Var(Y^{cuped}))$, $\theta = Cov(X, Y)/Var(X)$
- $Var(Y^{cuped})_{min} = Var(Y)(1 - \rho^2)$
  - $\rho =$ correlation(X,Y)
- Choose the X with the largest correlation with Y (e.g., post-experiment OEC).
- Empirically, pre-experiment OEC correlates well with Y.
- e.g., OEC is purchase amount/user, X is pre-experiment purchase amount

How to Choose Control Variables

"Across a large class of metrics, our results consistently showed that using the same variable from the pre- experiment period as the covariate tends to give the best variance reduction. " (Deng, Xu, Kohavi, & Walker, 2013)



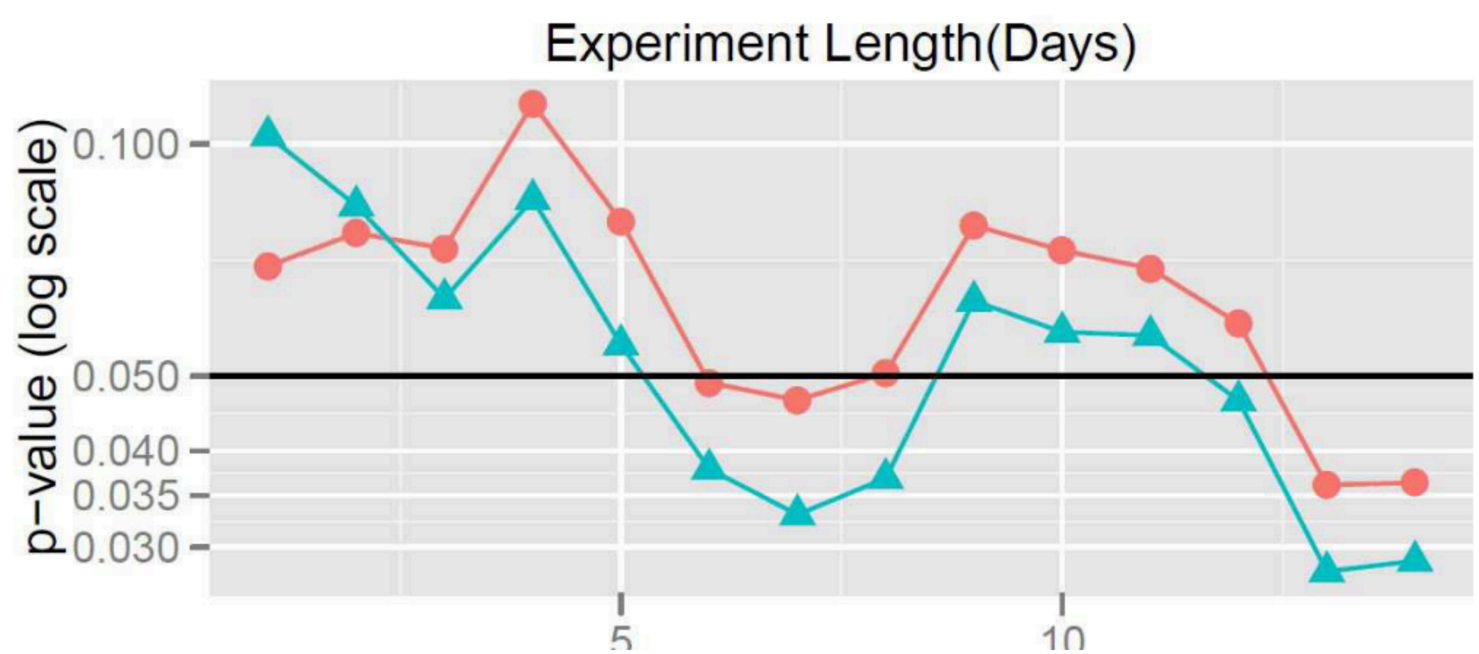


Figure 2: Slowdown experiment. Top: p-value. Bottom: p-value when using only half the users for CUPED.

Class Exercise

- Use exp_data_3.csv
- Find the Treatment Effects on Clicking Ads using CUPED
    - X: pre_click
    - $Y^{cuped} = Y - \theta X$
    - $\theta = Cov(X, Y)/Var(X)$
- Find the variance reduction by CUPED
    - Compare $Var(\Delta^{cuped})$ vs. $Var(\Delta)$
    - p value

Class Exercise

```
var_x=np.var(df.pre_click, ddof=1)
#np.cov returns a var-cov metrix
cov_xy = np.cov(df.pre_click,df.click, ddof=1)[0][1]
theta = cov_xy/var_x
df['theta']=theta
df['click_cuped']=df.click - df.pre_click*theta
```

With CUPED

```
d_0 = df[df['treat'] == 0]['click_cuped']
d_1 = df[df['treat'] == 1]['click_cuped']
diff = np.mean(d_1) - np.mean(d_0)
print(diff)
cm = sms.CompareMeans(sms.DescrStatsW(d_1), sms.DescrStatsW(d_0))
ttest = cm.ttest_ind(alternative = 'two-sided', usevar = 'unequal')
se = cm.std_meandiff_separatevar
print(se,ttest)
```

Class Exercise

Without CUPED

```
d_0 = df[df['treat'] == 0]['click']
d_1 = df[df['treat'] == 1]['click']
diff = np.mean(d_1) - np.mean(d_0)
print(diff)
cm = sms.CompareMeans(sms.DescrStatsW(d_1), sms.DescrStatsW(d_0))
ttest = cm.ttest_ind(alternative = 'two-sided', usevar = 'unequal')
se = cm.std_meandiff_separatevar
print(se,ttest)
```

Regression with Control Variables

- Remove the variances explained by X (covariates, e.g., age, gender, pre-experiment behaviors)
- X can be continuous variables
- Stratification: Strata Factor has to be categorical variable

Regression with Control Variates

$$Y_i = \beta_0 + \beta_1 . T_i + X_i\theta + \epsilon_i$$

$T_i = \{0,1\}$, Control or Treatment

Y_i is the OEC (or other metrics)

X are control variables

Regression with Control Variables

$$Y_i = \beta_0 + \beta_1 . T_i + X_i\theta + \epsilon_i$$

- Linear regression gives a consistent estimator for the average treatment effect
    - β_1 represents the Average Treatment Effects
- Reduces variance by controlling for X
- BUT, it makes assumption:
    - the conditional expectation of the outcome metric is linear in the treatment assignment and covariates.

Class Exercise

- Use exp_data_3.csv

1. Find the Treatment Effects of Clicking Ads using OLS:
   - $Y_i = \beta_0 + \beta_1 . T_i + \epsilon_i$
2. Find the correlations between X and Y.
   - X: gender and pre-click (i.e., whether clicking ads during the week before the experiment)
3. Adding Control Variables separately and then sequentially
   - $Y_i = \beta_0 + \beta_1 . T_i + X_i\theta + \epsilon_i$
4. What changes and What does not change after adding control variables?
5. Find the maximum variance reduction you can get by adding the control variables.

Class Exercise

```
import statsmodels.formula.api as smf

mod = smf.ols(formula='click ~ treat +
pre_click', data=df)

res = mod.fit()

print(res.summary())
```

Review

1. A/B testing Terminology and Overview
2. Statistics behind A/B testing
    1. Statistical tests (t, z, chi-square)
    2. Confidence intervals
    3. Type I error & Multiple Testing
    4. Type II error & Power Analysis
    5. Regression
3. Internal & External Validity
    1. Sanity Checks (SRM, Randomization checks, A/A tests)
    2. SUTVA (network interferences)
    3. Survivorship bias
    4. Heterogeneous Treatment Effects
    5. Novelty and Primacy Effects
4. Improve Sensitivity
    1. Estimate σ^2: ratio metrics (lift), Clustered SE (correlated observations)
    2. Increase N (pooled control group, split sample)
    3. Increase effect size (Triggering Experiments)
    4. Reduce variance (transform matrix and interleaving design)
    5. Stratification (post and at assignment)
    6. Regression with controls, CUPED
    7. Paired Design, Block Design

- **Compare the means (lift, median, etc) between treatment and control**
- **Interpret the results considering type I and II errors**

- **Two principles to be considered during the whole process of experiments**
- **Need to guarantee internal validity**
- **Consider external validity when generalizing the results**

- **Improve sensitivity means using the smaller sample to achieve larger power**
- **Always a desire to improve the power given a sample size**