⇒ ~~Groups~~ $S \neq \phi$.
$*$ be an operation

1> — Closure: (S is closed w.r.to $*$)
(or) $*$ is closed on S.

i.e. $a*b \in S \quad \forall a,b \in S$.

2> — Associative: (S is associative w.r.to $*$)
or $*$ is associative on S.

i.e $a*(b*c) = (a*b)*c \quad \forall a,b \in S$.

3> — Existence of Identity:— (Identity exists in S w.r.to $*$)
$\exists e \in S$; $\forall a \in S$.
such that
$e*a = a*e = a \quad \forall a \in S$.

4> — Existence of inverse:— (Inverse exists in S w.r.to $*$)
Let e be identity in S,
$\forall a \in S$, $\exists b \in S$. inverse
$a*b = b*a = e$.

5> — Commutative:—
$a*b = b*a \quad \forall a,b \in S$.

*> — closure —— $(S,*)$ is algebraic structure
$*$ is called binary opn on S

*> — Closure + Associative —— $(S,*)$: Semigroup.

*> — Closure + Associative + Identity exist } → $(S,*)$: monoid.

*> — Closure + Associative + identity exists + inverse exists. } → $(S,*)$: Group.

*> — Group + commutative → $(S,*)$: Abelian Group
(or) Commutative Group.

	Closure	Associative	Identity	Inverse	Commutative	So we have
(i) $(Z,+)$	✓	✓	$a+0=a$ identity; 0 : Identity	$a+(-a)$ inverse ✓; $-a$: Inverse, $-a \in Z$	✓	Abelian Group.
(ii) $(Z,\times)$	✓	✓	$a \cdot 1 = a$ Identity $= 1$ ✓	$\frac{1}{a} \notin Z$ inverse Not exists ✗		Monoid
(iii) $(R,+)$	✓	✓	"0" identity ✓	$-a \in R$ inverse ✓	✓	Abelian
(iv) $(R,\times)$	✓	✓	"1" identity ✓	$\frac{1}{0} \notin R$ inverse not exists ✗		Monoid
(v) $(R-\{0\},\times)$	✓	✓	"1" is identity ✓	$\frac{1}{a} \in R-\{0\}$ ✓	✓	Abelian
(vi) $(A_{n\times n},\times)$ Set of non-singular real matrix with matrix multiplication	$A_{n\times n}\times B_{n\times n} = C_{n\times n}$ ✓	✓	$I_n \times A_{n\times n} = A_{n\times n}$ ✓	inverse as $\lvert A\rvert \neq 0$ $\Leftrightarrow$ A^{-1} exists ✓	✗	Group

$\Rightarrow G = \{1,-1,i,-i\}, \times$ Set of fourth roots of unity

→ composition Table

$\boxed{i^2 = -1}$

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

— Identity

→ Closure :- elements of Table $\in G$.
→ Associative :- Complex number multiplication is associative
→ Identity :- "1" identity
→ Inverse :-

element	inverse
1	1
-1	-1
i	$-i$
$-i$	i

So $\forall a \in G$, $\exists a^{-1}$

→ Commutative :- ✓
So, it is abelian Group.

Ques:- $a * b = \frac{ab}{5}$ $(R-\{0\}, *)$.

→ Closure :- $a*b \in R-\{0\}$ $\forall a,b \in R-\{0\}$
→ Associative :- $a*(b*c) = a*\left(\frac{bc}{5}\right) = \frac{abc}{25}$

$$= \frac{\left(\frac{ab}{5}\right)c}{5} = \left(\frac{ab}{5}\right)*c = (a*b)*c.$$

→ Identity :-
$a*e = a$
$\frac{ae}{5} = a$; $e = \frac{a5}{a} = 5$
So $5 \in R-\{0\}$ is identity

→ Inverse :- $a*b = e$
$\frac{ab}{5} = 5$, $\boxed{b = \frac{25}{a}} \in R-\{0\}$

→ Commutative :-
$\frac{a*b}{5} = \frac{b*a}{5}$
$(R-\{0\}, *)$ is abelian Group

Ques $a \circ b = a+b+1$, $(R, \circ)$.

→ Identity

$$a \circ e = a$$
$$a+e+1 = a$$
$$e = a - \cancel{a} - 1 = -1$$
$$\text{So } e = -1 \in R.$$

→ inverse:-

$$a \circ b = a+b+1 = -1$$
$$a+b+1 = -1$$
$$b = -1-1-a$$
$$b = -(2+a)$$

Ques $a * b = a+b+ab$. $(R - \{-1\}, *)$

→ Identity : $a * e = a$

$$\cancel{a} + e + ae = \cancel{a}$$
$$e(1+a) = 0$$
$$e = 0 :$$

→ inverse :

$$a * b = 0$$
$$a+b+ab = 0$$
$$b = -a$$
$$\boxed{b = \frac{-a}{(1+a)}}$$

⇒ modulo - m

$a +_m b$ = The remainder when $m | a+b$; $m \in Z$ $a, b \in Z$

$a \times_m b$ = The remainder when $m | a \times b$;

ex:- $2 +_4 7 = 1$.

$11 +_7 12 = 2$.

$A \times_3 3 = 4$.

mod m → remainder possible
0, 1, 2, --- m-1

ex: $G = \{0, 1, 2, 3\}, +_4$

$+_4$	0	1	2	3
0	$\boxed{0}$	1	2	3
1	1	2	3	$\boxed{0}$
2	2	3	$\boxed{0}$	1
3	3	$\boxed{0}$	1	2.

— Identity

Closed ✓
Associative: ✓
Identity :- ✓
Inverse :- 0→0, 1→3, 2→2, 3→1.
Commulative ✓ ∴ abelian.

(*) → $(G = \{0, 1, 2, 3, --- m-1\}, +_m)$
is an abelian Group.
Identity → "0"
Inverse → "m-a" mod m.

Ques: $G = \{0, 1, 2, 3\}, \times_4$

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

0 → No inverse
1 → Identity
2 → No inverse

Closed ✓
Associative ✓
Identity ✓ "1"
inverse — Not exists for 0 & 2

$G = \{1, 2, 3\}$

$\times_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

$\rightarrow 0 \notin G$

So, NOT closed
So NOT an algebraic structure

ex:- $G = \{1, 2, 3, 4\}, \times_5$

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

→ closed ✓
→ Associative ✓
→ Identity ✓
→ inverse :- exists
$1 \rightarrow 1 \mid 2 \rightarrow 3 \mid 3 \rightarrow 2 \mid 4 \rightarrow 4$

Result:
$G = \{1, 2, 3, \text{———}\ m-1\}$, $\times_m$ is abelian
if m is prime Number.
identity $\rightarrow 1$

$*) \rightarrow (G, *)$ is Group
$|G| = O(G)$ = order of group.

ex:- ($G = \{1, -1, i, -i\}, \times$) abelian group
$O(G) = 4$.
Identity $= 1$.
smallest possible
$(-1)^2 = 1$ = order of $-1 = 2$
$(1)^1 = 1$ = order of $1 = 1$
$(i)^4 = 1$ = order of $i = 4$.
$(-i)^4 = 1$. = order of $-i = 4$.

⇒ Let $(G, *)$ be a multiplicative Group.
The smallest positive integer m such that
$a^m = e$ (e is identity in G).
is called order of element a
i.e. we write $o(a) = m$.

eg:- $o(i) = 4 = o(G)$.

$(i)^1 = i$
$(i)^2 = -1$
$(i)^3 = -i$ — Generator of the Group.
$(i)^4 = 1$

⇒ An element a such that
$o(a) = o(G)$
is called generator of the group.
∴ we write $G = \langle a \rangle$.

eg: $(G = \{1, -1, i, -i\}, \times)$

$G = \langle i \rangle$ ∵ $o(i) = o(G)$
$G = \langle -i \rangle$ ∵ $o(-i) = o(G)$.
G is cyclic Group.

⇒ A group having atleast one generator is called cyclic group.

eg: $\langle G = \{0, 1, 2, 3\}, +_4 \rangle$ Abelian Group.
is an additive Group.
Identity
$4(1) = 0 \rightarrow o(1) = 4$ i.e. $1+1+1+1 = 0$ under $+_4$
$2(2) = 0 \rightarrow o(2) = 2$
$4(3) = 0 \rightarrow o(3) = 4$
$1 \cdot (0) = 0 \rightarrow o(0) = 1$

⇒ Let $(G, *)$ be an additive group. The smallest positive integer m such that
$m \cdot a = e$ (e is identity in G)
m is called order of element a.
∴ we write $O(a) = m$.

$O(1) = O(3) = 4 = O(G)$.

∴ 1 and 3 are generators of group.
($G = \{0, 1, 2, 3\}, +_4$).
and G is cycle group.

⇒ Subgroups

(A non-empty subset H of G)
⇒ $\phi \neq H \subseteq G$ is subgroup of $(G, *)$ if H is group w.r.to $*$.

eg:- $(G = \{1, -1, i, -i\}, \times)$
$(H = \{1\}, \times)$ Subgroup
$(H = \{1, -1\}, \times)$ Subgroup.

→ A subset $H \neq \phi$ of G is a subgroup of the group $\langle G, * \rangle$ iff, for every pair elements $a, b \in H$, $a * b^{-1} \in H$.

→ ~~Bijective function is an abelian group.~~
→ Two bijective fn under composition
→ closed
→ Associative $f \circ (g \circ h) = (f \circ g) \circ h$
→ Identity $= I(x) = x$
$f \circ I(x) = f(I(x)) = f(x)$.
→ Inverse :- $f \circ f^{-1} = I(x)$.

So, it is a group under composition

WB/Ques-13/Pop016 $\qquad$ 15 625

$$x * y = \overline{x+y}$$
$$z = x * y$$
$$= z * x$$
$$= \overline{(\overline{x+y}) + x}$$
$$= (\overline{\bar{x}\bar{y}}) + x \quad \text{(Absorption)}$$
$$= x$$

Ques-(16) → $m * n = \max(m, n) \rightarrow (Z, *)$

closed ✓
associative ✓
Identity $\quad n * e = n$

$$\max(n, e) = n$$

So No Identity exists.

Ques 34 → $\qquad$ Identity $= 1$

$(1)^1 = 1$.
$(2)^3 = 1$
$(3)^6 = 1$
$(4)^3 = 1$
$(5)^6 = 1$
$(6)^2 = 1$

$O(A) = 6$.

So, $O(3) = O(5) = O(A)$
So 3 & 5 are Generators & it is a cyclic group.

(44) → $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ac & ad \\ 0 & 0 \end{bmatrix}$ closed

associative.

Identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$.

∴ Identity fails

44 $\{1,3,5\}$ $\{1,3,5\}$

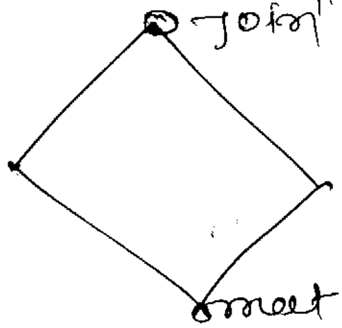
$R = \{(1,3)\ (3,5)\}$

$S = \{(1,3), (1,5), (1,1), (3,3), (3,5), (5,5)\}$

$R \circ S = \{(1,3), (1,5)(3,5)$

⇒ 39) $A = \{1, 2, 3, 4, 5\}$.

$\{\{1,2,3\}, \{4\}, \{5\}\}$ ⇒ Take union $\{\{1,2,3\}, \{4,5\}\}$



$a = \{\{1,2,3\}, \{4,5\}\}$ $b = \{\{1\}, \{2,3\}, \{4,5\}\}$

meet : $\{\{1\}, \{2,3\}\ \{4\}, \{5\}\}$.

Take intersection of each element of a with that of b.

REFINEMENT OP^n:

∴ P_0 refines P_1 & P_2

ie. every element of P_0 is a subset of at least one ele. of P_1 and also of P_2

Join $\{\{1,2,3\}, \{4,5\}\}$

Take UNION

$\{1,2,3\} \cup \{1\} = \{1,2,3\}$

$\{1,2,3\} \cup \{2,3\} = \{1,2,3\}$

$\{4\} \cup \{4,5\} = \{4,5\}$

$\{5\} \cup \{4,5\} = \{4,5\}$.

01/12

⇒ Sequence :-

$$\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_2, a_3, \ldots$$

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

$$f: N \rightarrow R.$$

$$\{2^n\}_{n=1}^{\infty} : 2, 2^2, 2^3, \ldots$$

$$\{2n+1\}_{n=0}^{\infty} : 1, 3, 5, \ldots$$

$$\{n\}_{n=0}^{\infty} : 0, 1, 2, 3 \ldots$$

$$\{2n\}_{n=0}^{\infty} : 0, 2, 4, 6, \ldots$$

→ Arithmetic Progression :- (A.P.)

$a, a+d, a+2d, a+3d, \ldots$

$a \rightarrow$ first element.

$d \rightarrow$ common difference.

$$n\text{th term} = l = a+(n-1)d.$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[a + a+(n-1)d\right]$$

$$= \frac{n}{2}\left[a+l\right]$$

ex:- $1, 2, 3, 4, \ldots$

$$n\text{th term} = 1+(n-1)$$

$$= \cancel{1}+n-\cancel{1}$$

$$= n.$$

$$S_n = \frac{n}{2}\left[1+n\right]$$

$$S_n = \frac{n(n+1)}{2}.$$

WB/ Pg21/4:

m^{th} term $= a+(m-1)d = n$

n^{th} term $= a+(n-1)d = m$

$(m-n)d = (n-m)$

$d = -1.$

→ Putting d-value

$a+(m-1)d = n$

$a - m + 1 = n$

$a = (n+m-1)$

→ $(m+n)^{th}$ term $= a + (m+n-1)d$

$= (m+n-1) + (m+n-1)(-1)$

$= (m+n-1) - (m+n-1)$

$= 0$

(5) → $S_n = \frac{n}{2}(t_1 + t_n) = 0$ [Given]

$t_1 + t_n = 0$

$t_1 = -t_n$

(6) → $(3+n)/4$: = n^{th} term

$(n+1)^{th}$ term $= \frac{3+(n+1)}{4}$

$(n+1)^{th}$ term $-$ n^{th} term $= \frac{3+(n+1)}{4} - \frac{3+n}{4}$

$= \frac{\cancel{3}+\cancel{n}+1-\cancel{3}-\cancel{n}}{4} = \frac{1}{4} = d.$

as difference is constant

So, series is A.P. with $d = 1/4$

$S_{105} = \frac{n}{2}[a+l]$; $a_1 = \left(\frac{3+1}{4}\right) = 1$

$a_{105} = \frac{(3+105)}{4} = 27$

$S_{105} = \frac{105}{2}[1+27] = 105 \times 14 = 1470$

⇒ Geometric Progressions

$a,\ ar,\ ar^2,\ ar^3 \ldots$

$a \to$ first term

$r \to$ common ratio.

$\boxed{n^{th}\ \text{term} = l = a \cdot r^{n-1}}$.

$\rightarrow S_n = \dfrac{a(r^n - 1)}{(r-1)}$ (if $r>1$).

$= \dfrac{a(1-r^n)}{(1-r)}$ (if $r<1$)

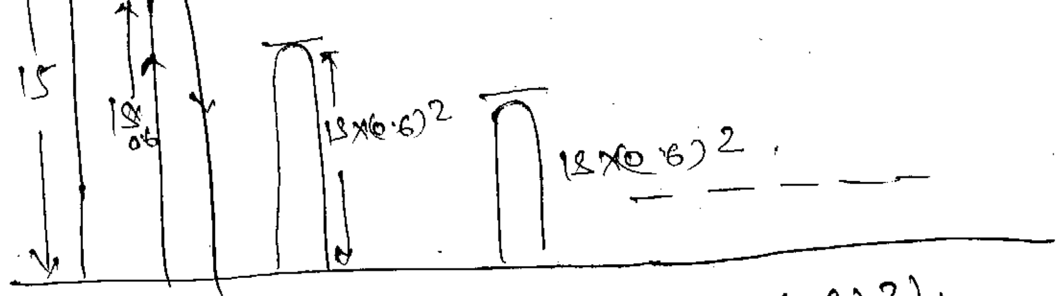
$\rightarrow S_\infty = \dfrac{a}{1-r}$ $|r|<1$. ie $-1<r<1$

eg:- $1+3+3^2+3^3+ \ldots + 3^{n-1} = \dfrac{3^n - 1}{3-1}$

$1+\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2+ \ldots + \left(\tfrac{1}{2}\right)^{n-1} = \dfrac{1-\left(\tfrac{1}{2}\right)^n}{1-\tfrac{1}{2}}$

$1+\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2+ \ldots \infty = \dfrac{1}{1-\tfrac{1}{2}} = \dfrac{1}{1/2} = 2$

WB/Pg-28/ques-29



$= 15 + (2(15)\times(0.6)) + (2\times 15\times(0.6)^2) + \ldots$

$= 15 + 2\times 15\left[0.6 + (0.6)^2 + (0.6)^3 + \ldots \infty\right]$

$$15 + 30\left[\frac{0.6}{1-0.6}\right] = 15+30\left[\frac{0.6}{0.4}\right]$$
$$= 15+\cancel{30}^{15}\left[\frac{\cancel{6}^{3}}{\cancel{4}}\right]$$
$$= 15+45 = 60$$

→ **Summation Notation**:

$$\Rightarrow \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\boxed{\sum n = \frac{n(n+1)}{2}}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\cancel{\sum n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\sum n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum n^3 = \frac{n^2(n+1)^2}{4}$$

Ques 30:-

$1\times3 + 2\times4 + 3\times5 + \cdots$

1^{st} term $= 1, 2, 3, \cdots$ n^{th} term $= n$.

2^{nd} term $= 3, 4, 5, \cdots$ n^{th} term $= (n+2)$.

$$\Rightarrow \sum n(n+2)$$
$$\Rightarrow \sum n^2+2n$$
$$\Rightarrow \sum n^2 + 2\sum n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{\cancel{2}n(n+1)}{\cancel{2}}$$
$$= \frac{n(n+1)(2n+1)+6(n(n+1))}{6}$$

$$= \frac{n(n+1)}{6}\left[2n+1+6\right]$$

$$= \frac{n(n+1)(2n+7)}{6}.$$

⇒ 17 :

abc

3 $\begin{cases} a \\ b \\ c \end{cases}$

2 $\begin{cases} ab \\ bc \end{cases}$

1 $\{ abc$

ac is NOT substring
it is only subsequence

No. of substring of length 1 = n

" " " " " 2 = $n-1$

" " " " " 3 = $n-2$

| | | | | | |

" " " " " $n = 1$

Total No of substrings = $n+(n-1)+(n-2)+ \cdots +1$

$= \sum n = \frac{n(n+1)}{2}$

(36) :

1 2 3 4 5 6 7 8 9 10 11 12 13

A X I O M A T I Z A B L E

Total # of substrings = $\frac{n(n+1)}{2}$

$= \frac{13(14)}{2}$

$= 13 \times 7$

$= 91$

Substrings of length 1.

3a's A, A, A

2I's I, I

$= 91 - 3 = 88.$

⇒ Recurrence Relation :- A recurrence relation is a formula which relates nth term of the sequence $\{a_n\}_{n=0}^{\infty}$ with one or more terms from $a_0, a_1, \ldots a_{n-1}$

$$a + a+d + a+2d$$

$$a_n = a+(n-1)d$$
$$= a_{n-1}+d.$$

→ Formulation :-

ex:- In a colony bacteria double in every hour. If the initial population is 5, express this as recurrence relation.

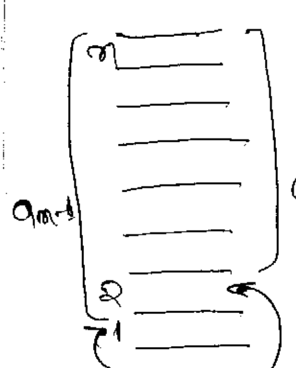
→ a_n = amount of bacteria present after n-hours.
a_{n-1} = " " " " $(n-1)$ hour

$$\boxed{a_n = 2 \times a_{n-1}}$$
$$a_0 = 5.$$

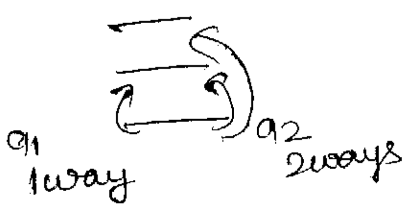
eg:- A person can climb n-steps such that
a) → he can skip at most one step.

Soln :-
a_n = No. of ways to climb n-steps.

He takes first step without skipping any step or He skips the first step & start

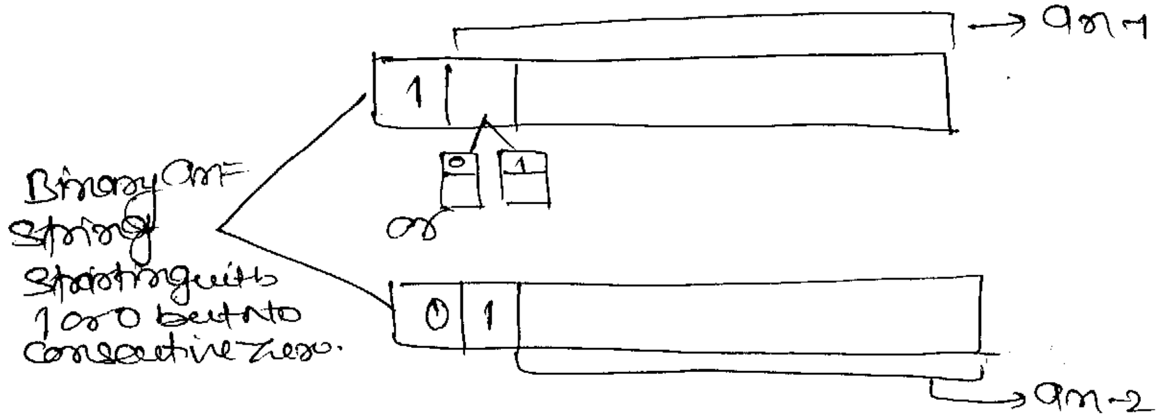


$$\boxed{a_n = a_{n-1} + a_{n-2}}$$
$$a_1 = 1;\ a_2 = 2.$$



⊛→ How many binary strings of length n that donot contain consecutive zeros.

→ a_n = No. of binary strings of length n that do not contain consecutive zero's.



$$\Rightarrow a_n = \underset{\text{starting with 1}}{a_{n-1}} + \underset{\text{starting with zero.}}{a_{n-2}}$$

$a_1 = 2$ — 1, 0

$a_2 = 3$ — ~~00~~, 10, 01, 11.

⇒ Solution of R.E

→ Substitution methods

→ $a_n = n \cdot a_{n-1}$.

$a_1 = 1$

$a_2 = 2 \cdot a_1$

$a_2 = 2 \cdot 1$

$a_3 = 3 \cdot a_2$

$= 3 \cdot 2 \cdot 1$

$a_4 = 4 \cdot a_3$

$= 4 \cdot 3 \cdot 2 \cdot 1$

⇓

$a_n = n(n-1) \cdots \{3 \cdot 2 \cdot 1\}$

$= n!$

Forward substitution

$a_n = n \cdot a_{n-1}$

$= n \cdot (n-1) \cdot a_{n-2}$

$= n(n-1)(n-2) a_{n-3}$

$= n(n-1)(n-2) \cdots 2 \cdot a_1$

$= n(n-1)(n-2) \cdots 2 \cdot 1$

$= n!$

WB/PG-24/38

a_n = ants present after n years

a_{n-1} = ,, ,, ,, (n-1) years.

$a_n = 2(2(a_{n-1}))$

$\boxed{a_n = 4a_{n-1}}$

$a_0 = 37$

→ Substitution method

$a_n = 4a_{n-1}$

$= 4\cdot(4\cdot a_{n-2})$

$= 4^2 a_{n-2}$

$= 4^2[4a_{n-3}]$

$= 4^3 a_{n-3}$

$= \boxed{4^n a_{n-n}}$

$a_n = 4^n a_0$

→ $a_5 = (4)^5 a_0$

$= (4)^5 * 37 = 2^{10} * 37$

$= 1024 \times 37.$

$= 37888$

Ques $a_n = 2a_{n-1} + 1$; $a_1 = 1$

$a_n = 2[a_{n-1}] + 1.$

$= 2[2a_{n-2} + 1] + 1$

$= 2^2 a_{n-2} + 2 + 1$

$= 2^2[2a_{n-3} + 1] + 2 + 1$

$= 2^3 a_{n-3} + 2^2 + 2 + 1$

$a_n = 2^{n-1} a_{(n-(n-1))} + 2^{n-2} + 2^{n-3} \cdots + 2 + 1$

$= 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1$

$\Rightarrow 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1$

$= \dfrac{2^n - 1}{2-1} = 2^n - 1$

$\boxed{a_n = 2^n - 1}$

Solving Linear Recurrence Relations

$\Rightarrow$ General form of Linear Recurrence Relations

$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$; $(n \geq k)$ forcing fn.

is called linear recurrence Reln of degree k.

$f(n) = 0 \Rightarrow$ Homogeneous Recurrence Relation

$f(n) \neq 0 \Rightarrow$ Non-homogeneous " "

$\rightarrow$ Characteristic Eqns

$C(t) = t^k + C_1 t^{k-1} + C_2 t^{k-2} + \cdots + C_k = 0$

$\Rightarrow$ Homogeneous Linear R.R.

* Characteristic Root method.

Ex:- $a_n - 3a_{n-1} = 0$

$a_0 = 1$

Degree $= 1$

$C(t) = t - 3 = 0$

Roots of $C(t)$

$t = 3$

$a_n = C_1 (3)^n$

$a_0 = C_1 (3)^0 = 1$

$\boxed{C_1 = 1}$

$\boxed{a_n = (3)^n}$

Ex!- $a_n + 5a_{n-1} = 0$; $a_0 = 3$.

degree = 1

$c(t) = t + 5 = 0$

Roots -

$t = -5$

Soln = $\boxed{(-5)^n C_1 = a_n}$

To calculate C_1,

$a_n = C_1(-5)^n$

$a_0 = C_1(-5)^0$

$\boxed{3 = C_1}$.

So, $a_n = 3(-5)^n$.

Ex-

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

$a_0 = 2$; $a_1 = 5$!

Degree = 2

$c(t) = t^2 - 3t + 2 = 0$

$\Rightarrow t^2 - 2t - t + 2$

$\Rightarrow t(t-2) - 1(t-2) = 0$

$(t-2)(t-1) = 0$

Roots $t = 1, 2, \ldots$

$\boxed{a_n = C_1(1)^n + C_2(2)^n}$.

$\boxed{a_n = C_1 + C_2(2)^n}$.

$a_0 = C_1 + C_2 = 2$

$a_1 = \cancel{C_1} + 2C_2 = 5$

$-C_2 = -3$

$C_2 = +3$

$C_1 = -1$

$a_n = -1 + 3(2)^n$

$\boxed{a_n = 3(2)^n - 1}$.

ex: $a_n - 4a_{n-1} + 4a_{n-2} = 0$ Degree $=2$; $a_0 = 1$; $a_1 = 2$

$C(t) = t^2 - 4t + 4 = 0$

$= t^2 - 2t - 2t + 4 = 0$

$t(t-2) - 2(t-2) = 0$

$t = 2, 2$

Soln $= \boxed{a_n = C_1(2^n) + C_2 \cdot n(2^n)}$

$\Rightarrow [C_1 + C_2 n] 2^n$

$a_0 = (C_1 + 0) 2^0 = C_1 = 1.$

$a_1 = (C_1 + C_2) 2 = 2$

$(1 + C_2)\cancel{2} = \cancel{2} 1$

$C_2 = 1 - 1 = 0$

$\boxed{a_n = (2)^n}$

ROOTS	SOLUTION
1.)– b_1 and b_2 (real & distinct)...	$a_n = C_1 (b_1)^n + C_2 (b_2)^n$.
2.)– b, b (real & equal)...	$a_n = [C_1 + C_2 n] (b)^n$.
3.)– $\alpha \pm \beta i$ (complex Root)	$a_n = \alpha^n [C_1 \cos n\beta + C_2 \sin n\beta]$
(4.)– b_1, b_2, b_3	$a_n = C_1(b_1)^n + C_2(b_2)^n + C_3(b_3)^n$.
(5.)– b_1, b_1, b_2	$a_n = [C_1 + C_2 n] b_1^n + C_3 b_2^n$
(6.)– b, b, b Roots of multiplicity 3	$a_n = [C_1 + C_2 n + C_3 n^2] (b)^n$.

⇒ Non-homogenious R.R.E ie $\boxed{f(n) \neq 0}$

$$a_n = \underbrace{a_n^H}_{\text{Solⁿ to homogenious linear R.R}} + \underbrace{a_n^P}_{\text{Particular solⁿ of Non-homogeneous linear R-R}}$$

→ a_n^P depends on the form $f(n)$.

eg:- $\boxed{a_n = 2a_{n-1} + 1}$ $a_1 = 1$.

⇒ $\boxed{a_n - 2a_{n-1} = 1}$.

$a_n = a_n^H + a_n^P$.

⇒ a_n^H is solⁿ of $\boxed{a_n - 2a_{n-1} = 0}$

→ characteristic eqⁿ $t - 2 = 0$

roots → 2

Solⁿ = $\boxed{a_n^H = C_1(2)^n}$

→ a_n^P

$\boxed{f(n) = C.(b)^n \text{ ; } b \text{ is not root of } c(t) \text{ then } a_n^P = D(b)^n}$

for question $f(n) = 1$.

$f(n) = (1)^n$: 1 is NOT root $c(t)$

$a_n^P = D(1)^n$

$\boxed{a_n^P = D}$, Solⁿ to Non H.L.R.R.

$a_n = D$.

Substitute in main RR $a_{n-1} = D$.

$a_n = 2a_{n-1} + 1$

$D = 2D + 1 \quad = \quad -D = 1$.

$\boxed{D = -1}$.

Solⁿ = $a_n = a_n^H + a_n^P = \boxed{C_1(2)^n - 1}$

$a_1 = 2c_1 - 1 = 1$

$c_1 = 1$

$$\boxed{a_n = 2^n - 1}$$

Ques:- (2):- $a_n - 3a_{n-1} = 2(3)^n$

$$\boxed{a_n = a_n^H + a_n^P}$$

$\rightarrow a_n^H =$ $t - 3 = 0$

$t = 3.$

$a_n^H = C_1(3)^n$

$\rightarrow a_n^P:$ $f(n) = 2(3)^n$: & 3 is root of c(t) of multiplicity 1.

$\boxed{a_n^P = Dn(3)^n} \Rightarrow$ Soln of N.H.L.R.R

Substituting it in our main R.R.

$$Dn(3)^n - 3D(n-1)3^{n-1} = 2(3^n)$$

$$\cancel{3^{n-1}}\left[3nD - 3nD + 3D\right] = 2(3^n)$$

$$3D = 6$$

$$D = 2.$$

So, $a_n^P = 2n(3)^n$

Soln: $a_n = a_n^H + a_n^P$

$= \boxed{C_1(3)^n + 2n(3)^n}.$

$a_1 = 2c_1 - 1 = 1$

$c_1 = 1$

$$\boxed{a_n = 2^n - 1}$$

Ques :- (2) :- $a_n - 3a_{n-1} = 2(3)^n$

$$\boxed{a_n = a_n^H + a_n^P}$$

$\rightarrow a_n^H =$ $t - 3 = 0$

$t = 3$

$a_n^H = C_1(3)^n$

$\rightarrow a_n^P$: $f(n) = 2(3)^n$: & 3 is root of C.E of multiplicity 1.

$$\boxed{a_n^P = Dn(3)^n} \Rightarrow \text{Soln of N.H.L.R.R}$$

Substituting it in our main R.R.

$$Dn(3)^n - 3D(n-1)3^{n-1} = 2(3^n)$$

$$\cancel{3^{n-1}}\left[3nD - 3nD + 3D\right] = 2(3^{n})$$

$$3D = 6$$

$$D = 2.$$

So, $a_n^P = 2n(3)^n$

Soln $a_n = a_n^H + a_n^P$

$$= \boxed{C_1(3)^n + 2n(3)^n}.$$

$\Rightarrow$

I $\to$ $f(n) = C(b)^n$; b is NOT root of c(t)

$a_n^p = D(b)^n$

II $\to$ $f(n) = C(b)^n$: b is root of c(t) of multiplicity m.

$a_n^p = D n^m (b)^n$

III $\to$ $f(n) = (C_1 n^2 + C_2 n + C_3)(b)^n$; b is NOT root of c(t).

$a_n^p = [D_1 n^2 + D_2 n + D_3](b)^n$.

IV $\to$ $f(n) = (C_1 n^2 + C_2 n + C_3)(b^n)$; b is root of c(t) of multiplicity m.

$$\boxed{a_n^p = [D_1 n^2 + D_2 n + D_3] n^m (b)^n}$$

eg: $a_n - 4a_{n-1} + 4a_{n-2} = n^2 (3)^n$

$f(n) = n^2 (3)^n$; 3 is NOT root of c(t).

$a_n^p = (D_1 n^2 + D_2 n + D_3)(3)^n$.

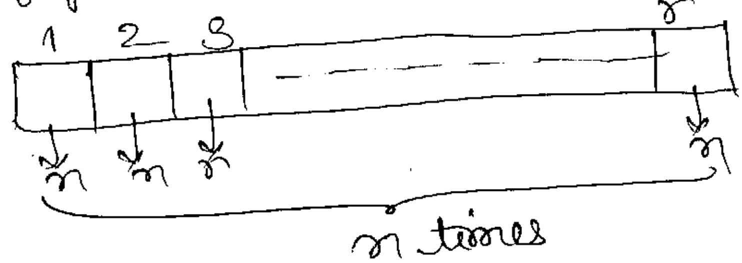
eg:- $a_n - 4a_{n-1} + 4a_{n-2} = n^2 (2)^n$

$a_n^p =$ $f(n) = n^2 (2)^n$; 2 is root of c(t) of multiplicity 2.

$a_n^p = (D_1 n^2 + D_2 n + D_3) n^2 (2)^n$.

⇛ Permutations with repetitions:- r-permutation of n-objects with repetition

$U(n,r) :\equiv$ r-permutation of n-objects



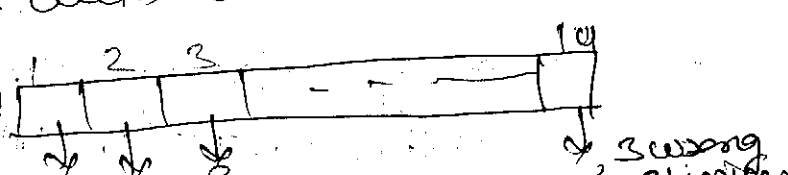
$\equiv n^r$.

Ques:- How many ways we can answer 25 T/F questions?

a) → 2^{25} b) → 25^2 c) → $^{25}P_2$ d) → $^{25}C_2$

Ques:- How many ways we can answer 20 multiple choice questions with 6 correct answers.

$= \underbrace{^{20}C_6}_{\text{6 correct out 20 question}} *$



$^{20}C_6 * (3)^{14}$.

→ Combinations with ~~repetitions~~

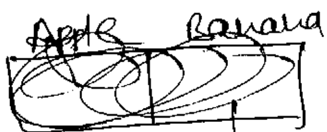
$V(n,r)$ = r combination of n-objects having unlimited repetitions.

= Non-negative integral sol^n of $x_1+x_2+x_3+ \cdots +x_n = r$
$(x_i \geq 0)$

= No. of ways of placing r balls in n-boxes

= No. of binary strings with $n-1$ 1's and r 0's

$$= {}^{n-1+r}C_r = {}^{n-1+r}C_{n-1}.$$



→ Binary strings of length 6 or those with exactly ~~4~~ 1's

$= {}^6C_4$

→ Binary strings of length $n-1+r$ with exactly $(n-1)$ 1's or exactly r 0's

$$\Rightarrow {}^{n-1+r}C_{n-1} = {}^{n-1+r}C_r.$$

In how many ways we can select 10 fruits from an unlimited supply of apples (A) & Banana (B).

$\rightarrow$ A B $= 10$

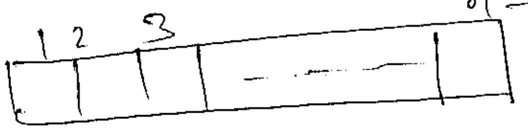
$\rightarrow 7 \quad 3$

$\rightarrow 6 \quad 4$

$\rightarrow 10 \quad 0$

$\boxed{x_1 + x_2 = 10}$

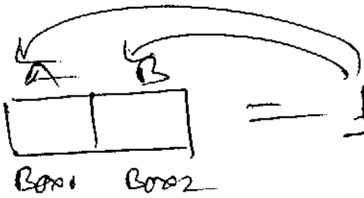
$x_1 \geq 0 \,;\, x_2 \geq 0$



n – n objects of unlimited supply

$x_1 + x_2 + x_3 + \dots + x_n = r$

$x_i \geq 0$



$\rightarrow$ A B $= 10$ balls

in how many ways we can place 10 balls in 2 boxes

$\rightarrow \quad x_1 + x_2 + x_3 = 10$

$\rightarrow \quad 4 \mid 2 \mid 4$

0 000 1 0 0 1 000 0

$\rightarrow \quad 5 + 2 + 3$

00000 1 00 1 000

$x_1 \quad x_2 \quad x_3$

x_1 0's 1 x_2 0's 1 x_3 0's

$x_1 + x_2 + x_3 + \dots + x_n = r$

1st 1 2nd 1 ... (n-1)th 1

So (n-1) 1's $\qquad$ r 0's.

Ques: How many ways 20 similar balls can be placed in 5 boxes.

$$x_1+x_2+x_3+x_4+x_5=20$$
$$x_i \geq 0$$

$$n=5;\ r=20$$

$$^{n-1+r}C_r = {}^{24}C_{20} = {}^{24}C_4.$$

Ques: How many ways 20 similar balls can be placed in 5 boxes so that each box is non-empty.

$x_1+x_2+x_3+x_4+x_5 = 20$; $x_i \geq 1$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ ← 20 balls.

20 − 5 = 15 balls.

$x_1+x_2+x_3+x_4+x_5 = 15$; $x_i \geq 0$ $\boxed{\ }$ $\boxed{\ }$ $\boxed{\ }$ $\boxed{\ }$ $\boxed{\ }$ ←

So, first place 1 ball each in each of boxes, so that non-empty condition is satisfied.

Now remaining problem is reduced to ways of placing 15 balls in 5 boxes

So $x_1+x_2+x_3+x_4+x_5=15$
$x_i \geq 0$ $= {}^{n-1+r}C_r = {}^{15-1+5}C_{15} =$

$\Rightarrow {}^{19}C_{15} = {}^{19}C_4$

Ques: WB|Pg [redacted] | quest [redacted] 39 sol

No of positive integral sol

$$x+y+z+t=20$$
$$x \geq 1;\ y \geq 1;\ z \geq 1,\ t \geq 1$$

$\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$
x y z t

$x+y+z+t = 16$ So $n=4$; $r=16$

${}^{19}C_3$

$x \geq 0;\ y \geq 0;\ z \geq 0;\ t \geq 0$

Scalaero Roses

28) → Pg 23

$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{} = 6$

$x_1 + x_2 + x_3 + x_4 + x_5 = 6$

$x_i \geq 0$

$n = 5;\ r = 6;\ {}^{10}C_4 = \underline{\underline{210}}$

Ques How many outcomes are possible when 10 similar coins are tossed.

If dissimilar coins are tossed
$2^{10} = 1024$

H T

10

$x_1\ x_2$

$x_1 + x_2 = 10$

$x_i \geq 0$

$n = 2;\ r = 10$

${}^{11}C_1 = 11.$

Ques How many outcomes are possible when 10 similar dices are rolled

if dissimilar dices
$= 6^{10}$

6 outcomes of dices

$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

$x_i \geq 0$

$n = 6;\ r = 10$

${}^{15}C_5 =$

⇒ Permutations with constrained repetitions:

$P(n : q_1, q_2, q_3, \ldots q_t)$ = n-permutations of objects in which q_1 are alike, q_2 are alike, ... q_t are alike.

$$= \frac{n!}{q_1!\, q_2! \cdots q_t!}$$

AAABB

No of permutations = x. (let assume).

A1 A2 A3 B1 B2.

If A1 A2 A3 are distinct we have 3! extra permutations

So # of permutations = $x \times 3!$

If B1 & B2 are also distinct = $x \times 2! \times 3!$.

If all are distinct total permutations = 5!

$$3! * 2! * x = 5!$$

$$\boxed{x = \frac{5!}{3! * 2!}}$$

⇒ MISSISSIPPI

No. of distinct permutations = $\frac{11!}{4!\,4!\,2!\,1!}$

M - 1
I - 4
S - 4
P - 2

eg:- How many ways 14 people can be partitioned into 4 teams where 1st team has 3 members
2nd team has 2 mem,
3rd " " 5 "
4th " " 4 ".

$$P(14 : 3,2,5,4) = \frac{14!}{3!\,2!\,5!\,4!}$$

eg:- In how many ways 12 of 14 people can be partitioned into 3 teams of ~~4~~
1st team has 6 mem
2nd " " 2 mem
3rd " " 4 mem

$$^{14}C_{12} \times P(12 : 6,2,4)$$

$$^{14}C_{12} \times \frac{12!}{6!\,2!\,4!}$$

ans - 116 $= {}^{8}C_2 \times {}^{6}C_2 \times {}^{4}C_2 \times {}^{2}C_2 = \frac{8!}{2!\,\cancel{6!}} \times \frac{\cancel{6!}}{2!\,\cancel{4!}} \times \frac{\cancel{4!}}{2!\,2!} \times \frac{2!}{2!\,0!}$

$= P(8 : 2,2,2,2) = \frac{8!}{(2!)^4} \qquad = \frac{8!}{(2!)^4}$

$\Rightarrow P(n : p, q, r, \text{---} n) = {}^{n}C_p \cdot {}^{n-p}C_q \cdot {}^{n-(p+q)}C_r \cdots$

$= C(n, q_1) \cdot C(n - q_1, q_2) \cdots$

$$= \frac{n!}{q_1!\,q_2!\,q_3! \cdots}$$

→ finde $x^2y^3z^2$ coefficient in $(x+y+z)^7$

$= P(7; 2,3,2)$

$= \frac{7!}{2!\,3!\,2!}$

→ find coefficient of $x^2y^3z^3$ in $(x-y+2z)^8$.

$= P(8; 2,3,3) * \underline{(1)^2(-1)^3(2)^3}$

$=$

TREESE A connected and acyclic graph is tree.

→ Resulte The following statements are equivalent for a graph with n-vertices

(a)→ T is a tree.

(b)→ T is connected and acyclic graph.

(c)→ T is connected and has n-1 edges

(d)→ T is acyclic and has n-1 edges

(e)→ There exists exactly one path b/w every pair of vertices.

(f)→ T is minimally connected. [every edge is cut-edge].

→ Every Tree is 2-colourable

→ Every Tree is bipartite graph

(*)→ No. of cut edges in a tree with n vertices $= n-1$.

(*)→ In a tree with n vertices
$$\sum \deg(v) = 2(n-1).$$

⇒ ex: T is a tree with 2 vertices of degree 3 & 5 vertices of degree 2 & remaining vertices of degree 1.

(1)→ How many vertices in T?

(2)→ " " " of degree 1 are there in T?

No. of V	degree
2	3
5	2
x	1

No of vertices $= (7+1)+x$

No of edges $= (7+x)-1 = 6+x$

$$\sum \text{degree}(v) = 2\times3 + 5\times2 + x\times1 = 6+10+x = 16+x$$

$$\boxed{\sum \deg(v) = 2e}$$

$$16+x = 2(6+x)$$
$$16+x = 12+2x$$
$$16-12 = x$$
$$x = 4.$$

No of vertices $7+x = 11$.

(17) → n_2 vertices of degree 2
n_3 ,, ,, ,, 3
n_k ,, ,, ,, k.
$x = ?$,, ,, ,, 1.

=

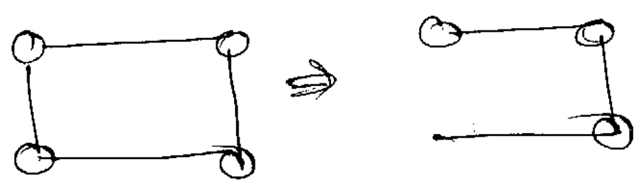
# of vertices	degree
n_2	2
n_3	3
|	|
n_k	k
x	1
No of V = $x + n_2 + n_3 + \cdots + n_k$	Sum of degree $= x + 2n_2 + 3n_3 + \cdots + kn_k$.

$$= x + 2n_2 + 3n_3 + 4n_4 \cdots + kn_k = 2\left[x + n_2 + n_3 + n_4 \cdots + n_k - 1\right]$$

$$\boxed{x = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2}$$

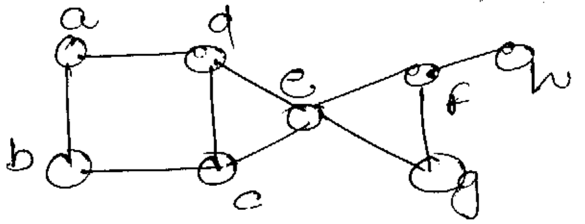
⇒ Spanning Tree
⇒ G be a connected graph.
→: A spanning subgraph T of G which is also tree is called spanning tree.

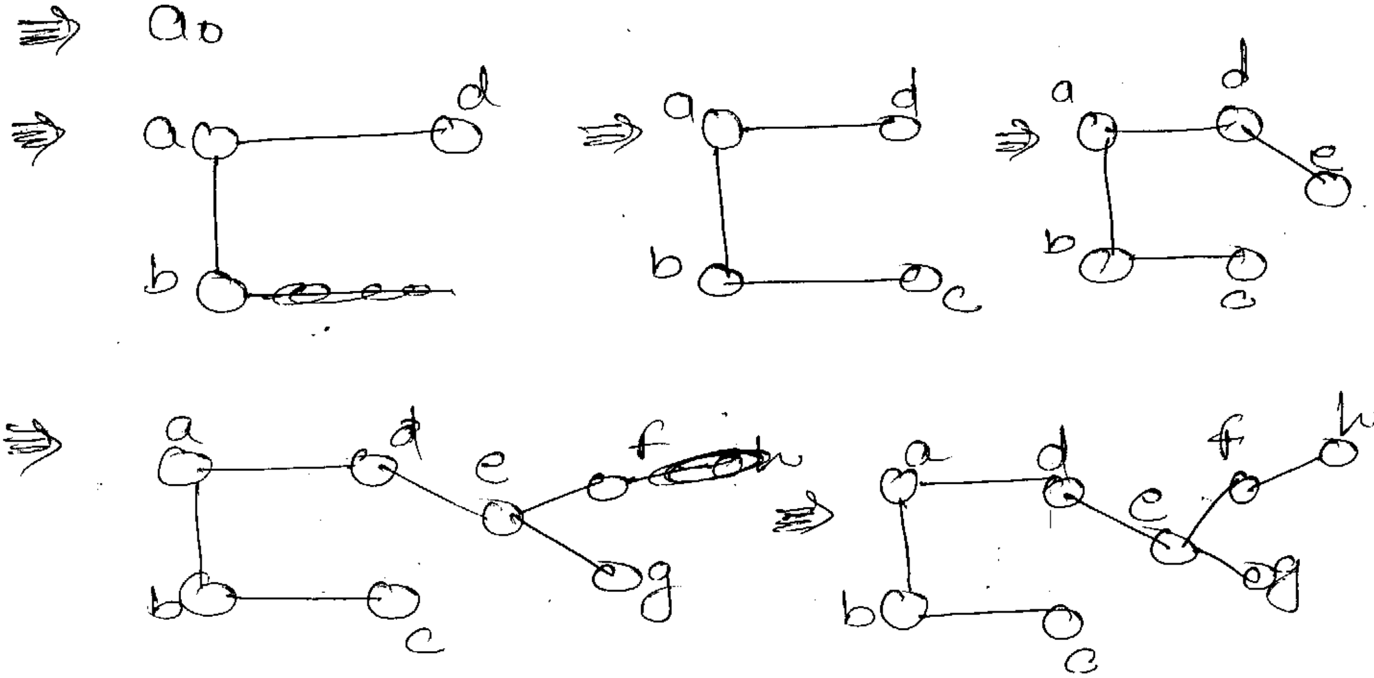
→: ex:-



Every H-path is spanning Tree. But every spanning tree need not be H-path.

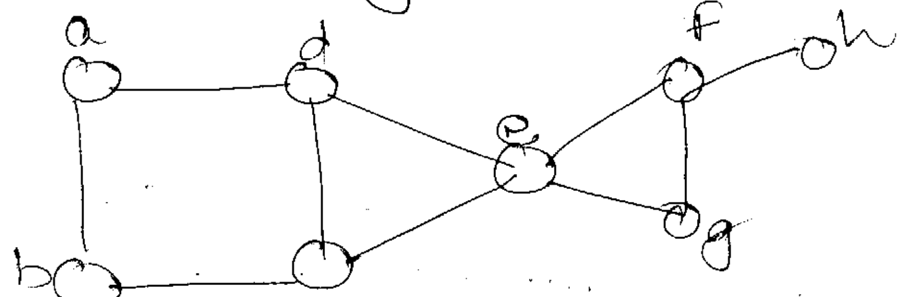
→ **BFS:** Visit all vertices in the next level at a given opportunity.

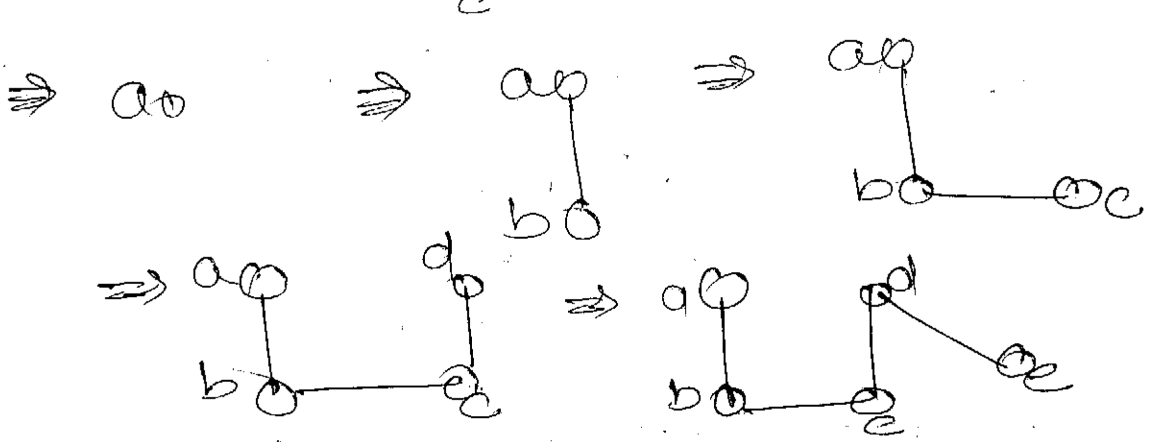


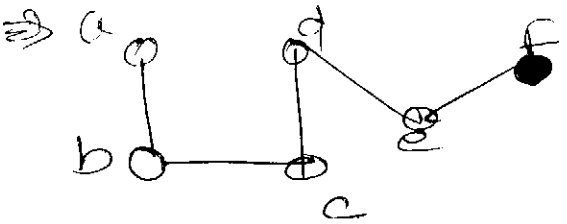
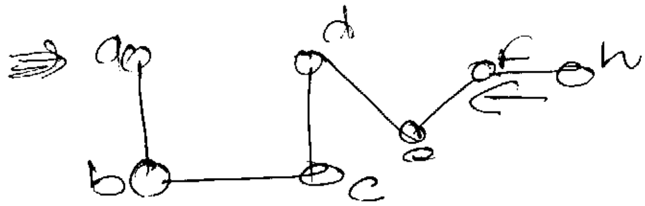


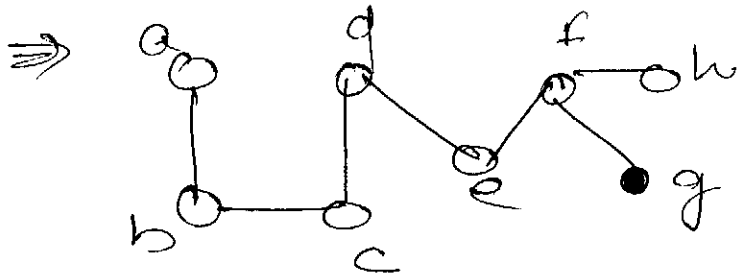
→ **DFS:** Visit a vertex in the next level and proceed further and backtrack if necessary.

eg:



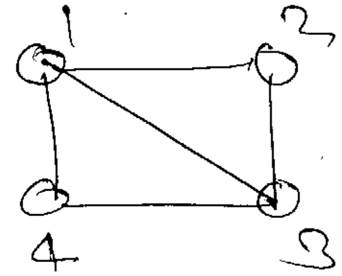


$\Rightarrow$  $\Rightarrow$ 

$\Rightarrow$ 

(*) → No. of spanning Trees in $K_n = n^{n-2}$ $(n \geq 2)$.

→ <u>Kirchoff's matrix method</u>



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

<u>Kirchoff's method</u>
Replace diagonal entries with degrees &
Replace 1's with -1's

Cofactor of any element in Kirchoff's matrix
= No. of spanning Trees.

Cofactor ~~of~~ $a_{ij} = (-1)^{i+j} M_{ij}$ = i.e. signed minor

Cofactor ~~of (3)~~ $= (-1)^{1+1} M_{11}$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$= 2(6-1) + 1(-2) + 0$

$= 10 - 2 = 8$