⇛ LINEAR ALGEBRA ⇚

Lecture-3
07/11/2012

1) - Types of Matrices.
2) - MATRIX operations.
3) - Solution of Linear Systems.
4) - Eigen values & Eigen vectors.

1) Types of Matrices: Square, Diagonal, Scalars, UNIT, Null, Singular, Non-singular, Symmetric, Skew-Symm, orthogonal, Hermitian, Skew-HERM. UNITARY, IDEMPOTENT, NILLPOTENT, INVOLUTORY,

2) Operations: $\pm$, $*$, A^{-1}, cof(A), Adj(A), A^t, |A|, rank(A), Eig(A), eigvec(A)

3) Solutions of Linear Systems

- → Homogenious — consistent
  - → UNIQE
  - → Infinite
- → Non-Homogeneous
  - Consistent
    - → Unique
    - → Infinite
  - Inconsistent

4) EIGEN values & EIGEN VECTORS: $A_{n \times n}$

→ Evaluate Evalue, Evector.
→ Properties of Evalue, Evector.
→ Cayley-Hamilton Theorem and its applications
  - → A^{-1}
  - → A^n

→ Diagonalization → : A^n

⇒ Types of Matrices

$$A_{m\times n} = [a_{ij}]$$

$A_{n\times n}$ = a matrix A of order n.

⇒ Square Matrix $A_{m\times n}$ where m = n.

⇒ Cof(A), adj(A), A^{-1}, |A|, Eignv(A), Eigenv(A) are defined only for square matrix.

⇒ There are some opns that can be applied on all matrix = rank(A), A^t, $A \pm B$, $A \times B$.

⇒ Diagonal Matrix

Square + $\boxed{a_{ij} = 0 \quad ; i \neq j}$ ie all off-diagonal elements are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{Diag}\,[1, 2, -3]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix} = \text{Diag}\,[1, -6]$$

⇒ $\text{Diag}(a, b, c) \pm \text{Diag}(d, e, f) = \text{Diag}(a \pm d, b \pm e, c \pm f)$

⇒ $\text{Diag}(a, b, c) \cdot \text{Diag}(d, e, f) = \text{Diag}(ad, be, cf)$.

⇒ $|\text{Diag}(a, b, c)| = abc$

⇒ $\text{Diag}(a, b, c)^{-1} = \text{Diag}(1/a, 1/b, 1/c)$
& $a, b, c \neq 0$

⇒ $\text{Eig}(\text{Diag}(a, b, c)) = a, b, c$

⇒ $[\text{Diag}(a, b, c)]^n = \text{Diag}(a^n, b^n, c^n)$

⇒ Upper Triangular $a_{ij} = 0$ if $i > j$ ≡ U

⇒ Lower Triangular $a_{ij} = 0$ if $i < j$ ≡ L

Diagonal matrix(D) ~~is~~ Both U & L.

Gauss Jordan ≡ $AX = B \xrightarrow{\text{Diagonal}} DX = B' \Rightarrow O(n^3)$

Gauss elimination ≡ $AX = B \xrightarrow{\text{Upper}} UX = B' \Rightarrow O(n^3)$.

⇒ Scalar matrix ≡ It is a ~~scalar~~ diagonal matrix with all diagonal elements same -

Scalar ≡ Diag (K, K, K).

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$AX = 2X$$

So, it serve as an scalar in multiplication

⇒ Unit matrix ≡ It is scalar with all diagonals as 1.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad I_1 = [1].$$

unit matrix of order 3

I_3 = Diag (1, 1, 1).

property ≡ $AI = IA = A$

This matrix serves as multiplicative identity

$$\boxed{I^{-1} = I}$$

Eigen (I_3) = 1, 1, 1.

i.e. Eigen value of identity matrix is 1.

⇒ <u>Null Matrix</u>

$$O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ It need not be square

$$O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Null matrix serves as additive identity.

⇒ properties ⇒ $AO = O$

$|O| = 0$

O^{-1} = Not exist.

A^{-1} exists iff $|A| \neq 0$

i.e. iff A is Non-singular

⇒ <u>Singular matrix</u>

⇒ A matrix is singular iff $|A| = 0$ otherwise it is Non-singular.

⇒ For every singular matrix, atleast one eigen value ~~....~~ must be zero.

as $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = |A| = 0$

So atleast one of λ_i must be zero.

⇒ For Real matrix

⇒ Symmetric ⇒

$$A^t = A$$

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -1 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -1 & -5 \\ 3 & -5 & 6 \end{bmatrix} = A$$

⇒ Skew symmetric

$$A^t = -A$$

$$A \begin{bmatrix} 0 & -2 & 3 \\ +2 & 0 & -6 \\ -3 & 6 & 0 \end{bmatrix}$$ Skew Symmetric

⇒ So, for skew symmetric Diagonal elements should all be zero.

⇒ Upper △ & lower △ mirror image should be same but ~~not~~ sign is different

⇒ orthogonal matrix

$$A^t = A^{-1}$$ But it is time consuming to check.

So, we do $\boxed{AA^t = I}$

⇒ For complex matrix.

⇒ Hermitian $A^\theta = A$.

ex- $A = \begin{bmatrix} 3+i & -3+6i \\ +i & -3 \end{bmatrix}$

$$A^\theta = (\bar{A})^t$$

$$\bar{A} = \begin{bmatrix} 3-i & -3-6i \\ -i & -3 \end{bmatrix}$$

$$(\bar{A})^t = A^\theta = \begin{bmatrix} 3-i & -i \\ -3-6i & -3 \end{bmatrix}.$$

So this is Not hermitian.

⇒ Skew Hermitian

$$A^\theta = -A$$

⇒ Unitary $A^\theta = A^{-1}$

This is also diff. to check

So, for ease, we check as

$$\boxed{AA^\theta = I}$$

⇒ Properties of Symmetric & Skew Symmetric

under addition

Any Matrix can be broken down into 2 pieces, such that one of them is symmetric & other is skew symmetric

$$A = \text{Sym} + \text{Skew Sym}$$

$$A = \underbrace{\left[\frac{A+A^t}{2}\right]}_{\text{Symmetric}} + \underbrace{\left(\frac{A-A^t}{2}\right)}_{\text{Skew symmetric}}.$$

for any matrix A,

$\left[\frac{A+A^t}{2}\right]$ is symmetric,

$\left[\frac{A-A^t}{2}\right]$ is skew-symmetric.

ex:- $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix} =$

$\left[\frac{A+A^t}{2}\right] = \quad A^t = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \quad \left[\frac{A+A^t}{2}\right] = \begin{bmatrix} 1 & +3/2 \\ 3/2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 9/2 \\ 9/2 & 0 \end{bmatrix}$

$\left[\frac{A+A^t}{2}\right] \quad \left[\frac{A-A^t}{2}\right]$

⇒ Also, this property holds for hermitian & skew-hermitian,

$$A = \underbrace{\left[\frac{A+A^\theta}{2}\right]}_{\text{Hermitian}} + \underbrace{\left[\frac{A-A^\theta}{2}\right]}_{\text{Skew Hermitian}}$$

~~Symmetric~~ ~~Skew Symmetric~~

⇒ ~~other properties~~

$(A^t)^t = A$	$(A^\theta)^\theta = A$
$(A+B)^t = A^t + B^t$	$(A+B)^\theta = A^\theta + B^\theta$
$(AB)^t = B^t A^t$	$(AB)^\theta = B^\theta A^\theta$
$(KA)^t = K \cdot A^t$	$(KA)^\theta = KA^\theta$ if K is real; $\bar{K} A^\theta$ if K is complex
$(A^t)^{-1} = (A^{-1})^t$	$(A^\theta)^{-1} = (A^{-1})^\theta$

$(A+A^t)^t = A^t + (A^t)^t = A^t + A$ = So, $(A+A^t)$ is Symmetric

$\cancel{(A-A^t)}^t = A$

⇒ If A and B are symmetric
Then which of following is symmetric

i) $= A+B$ $\quad = $ So $(A+B)^t$ should be $= (A+B)$
$(A+B)^t = A^t + B^t = \underline{\underline{A+B}}$

ii) $= A-B$

iii) $= AB$ } Not necessary
iv) $= BA$ }

⇒ If A and B are skew-sym. which of following is skew-symmetric

i) $= A+B$ $\quad = (A+B)^t = \boxed{-(A+B)}$
ii) $= A-B$ $\quad A^t + B^t = -A - B = -(A+B)$ = Prove

iii) $= AB$
iv) $= BA$

⇒ If A & B are orthogonal. Then which of following is orthogonal

i) $= A+B$
ii) $= A-B$
iii) $= AB$ $\Rightarrow = (AB)^t = \boxed{(AB)^{-1}}$
iv) $= BA$ $\quad = B^t A^t = B^{-1} A^{-1} = \boxed{(AB)^{-1}}$

⇒ Every Eigen value of symmetric & hermitian matrix is always real. =

Eigen V (S) = real $= a$
Eigen (H) = real $= a$
Eigen (skew sym) = imaginary $= bi$ or $a=$
Eigen (skew-Hermit) = imaginary $= bi$ or a

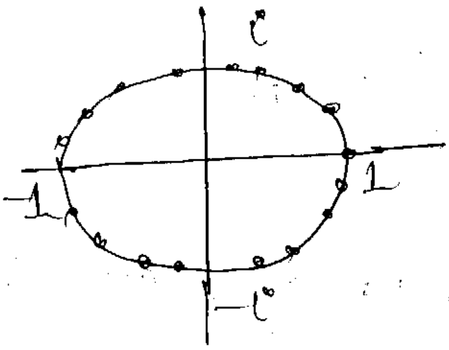
⇒ So Eigen values of Skew Sym. and Skew hermit matrices is either zero or purely Imaginary i.e.

|eigen v (orthogonal)| = 1 i.e. $1, 1, \frac{1}{2} + \frac{\sqrt{3}}{2} i$

|Eigen V (Unitary)| = 1.

i.e.

So, the form of eigen values will be either
$\sin\theta + i\cos\theta$
$\cos\theta + i\sin\theta$
or in any point on above circle



⇒ I: Every S is H. // bcoz every real # is complex.
⇒ II: A H may or may Not be S // as a complex Number is not necessarily ~~comp~~ real.
⇒ III: Every real Hermition matrix is Symmetric. as Symmetric is Nothing but real hermition.

⇒ Idempotent ≡ If $A^2 = A$

⇒ Involutory ≡ $A^2 = I$

} ⇒ Identity matrix is always idempotent as well as Involutory

⇒ Nillpotent ≡ $A^2 = 0$ & ~~...~~.

⇒ Nillpotent class x ≡ If $A^x = 0$ & $A^n \neq 0$, for $n < x$

Null matrix is always idempotent & Nullpotent But NOT involutionary.

$\Rightarrow A^0 = I$
$\Rightarrow O^0 = I$ also } i.e. power "0" of any matrix is always identity matrix.

$\Rightarrow$ Idempotent matrix
$\Rightarrow$ If $A^2 = A$
Then every higher power $A^x = A;\ x \geq 1$
$A^x = A^{-1}\quad x \leq -1$

$\Rightarrow$ Involutionary :-
If $A^2 = I$, Then $A^x = I$ x is even
$A^x = A$; x is odd.

$\Rightarrow$ Nullpotent of class-x
If $A^x = O$, then $\Rightarrow A^n = O$ for all $n \geq x$.

~~$A^n \neq O$ for $-(x+1) \leq n \leq (x+1)$~~
$\Rightarrow A^n \neq O \quad -(x+1) \leq n \leq (x-1)$
$\Rightarrow A^n$ Not exists $\quad n < -(x-1)$

$\Rightarrow$ (2) $\Rightarrow$ MATRIX operations

$\Rightarrow A_{m\times n} + B_{m\times n}$ Additive identity $\equiv O$; $A + O = A$
Additive inverse $= -A$; $A - A = O$;
commutative & Associative.

$\Rightarrow A_{m\times n} - B_{m\times n} \Rightarrow$ Neither com. & Nor Assoc.

$\Rightarrow A \times B \equiv A_{m\times n} * B_{n\times p} = C_{m\times p}$.
$\Rightarrow$ multiplicative identity $= I$; $AI = IA = A$
multiplicative inverse $= A^{-1} = A^{-1}A = AA^{-1} = I$
$(A^m)^n = (A^n)^m$.

$\Rightarrow$ Inverse $(A^{-1}) \Rightarrow$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{t} = (A^{t})^{-1}$$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$AA^{-1} = \frac{A\,Adj(A)}{|A|}$$

$$A\,Adj(A) = A^{-1}A|A| = |A| = Adj(A) \cdot A.$$

multiplication is NOT commutative
But associative.

$$\Rightarrow A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\rightarrow$ Inverse using this column because it has one element as zero so easy to calculate

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 3 & 1 & -1 \\ 6 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \end{array} = -1(6\times3-2) + 1(3\times2+6)$$

$$= -1(16) + 1(12)$$

$$= -4.$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$Cof(A) = \begin{array}{c} \\ - \\ \\ \end{array}\begin{bmatrix} 2 & -16 & -6 \\ -2 & +10 & 4 \\ 2 & -12 & -6 \end{bmatrix}\begin{array}{c} \\ - \\ \\ \end{array}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$Adj(A) = (Cof(A))^{t} = \begin{bmatrix} 2 & -2 & 2 \\ -16 & 10 & -12 \\ -6 & 4 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{Adj(A)}{|A|} = -\frac{1}{4}\begin{bmatrix} 2 & -2 & 2 \\ -16 & 10 & -12 \\ -6 & 4 & -6 \end{bmatrix}.$$

<u>Ques</u>: find X suchthat $AX=B$.

So

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B.$$

Same $YA = B$

$$YAA^{-1} = BA^{-1}$$
$$Y = BA^{-1}.$$

<u>Ques</u> $A + BC = D$. So, $C = ?$

$$BC = D - A.$$
$$C = B^{-1}(D-A)$$
$$C = B^{-1}D - B^{-1}A.$$

Inverse is distributive

$$\boxed{\begin{array}{l} \text{under } A(B+C) = AB + AC \\ \text{But } A + BC \neq (A+B)(A+C) \end{array}}$$

⇒ |A| ⇒ Properties of Determinant:-

Determinants unaffected by multiply

① ⇒ $|A| = \sum a_{1j} . \text{cof}(a_{1j})$
$= \sum a_{2j} \text{ cof } (a_{2j})$
$= \sum a_{i1} \text{ cof}(a_{i1})$.

ie. Same row & corresponding cofactors, or Same column & corresponding co-factors

Multiplying all elements of a row or column with its corresponding cofactor. And adding all of them results determinant.

$\sum a_{1j} * \text{cof}(a_{2j}) = 0$

ie. multiplication of one row with cofactors of some other row is always zero.

⇒ GATE:-

$$\begin{vmatrix} 2 & -3 & 0 & 1 \\ 1 & -6 & 0 & -2 \\ \cancel{3} & \cancel{-1} & \cancel{2} & \cancel{+5} \\ 1 & -3 & 0 & 5 \end{vmatrix} = +2 * \begin{vmatrix} 2 & -3 & -1 \\ 1 & -6 & 2 \\ 1 & -3 & 5 \end{vmatrix}$$

But here zeros are given,
But for some problems, zeros are not those we will try to create zeros by seeing behaviour of matrix

$$\begin{vmatrix} 2 & 3 & 4 & 2 \\ 1 & -2 & -2 & 7 \\ 3 & 1 & 2 & 1 \\ 1 & 7 & 6 & 0 \end{vmatrix} \quad C_3 \to 2C_1 = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 1 & 2 & 0 & -1 \\ 3 & 1 & 0 & 1 \\ 1 & 7 & 6 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & 4 & 2 \\ 1 & -2 & -2 & -1 \\ 3 & 1 & 2 & 1 \\ 1 & -1 & 6 & 0 \end{vmatrix} \equiv \begin{matrix} R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{2}R_1 \\ R_4 - \frac{1}{2}R_1 \end{matrix} \equiv \begin{vmatrix} 2 & 3 & 4 & 2 \\ 0 & -2 & -2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 6 & 0 \end{vmatrix}$$

$$A \xrightarrow[C_i \pm KC_j]{R_i \pm KR_j} B.$$ This opn never change determinant ie $|A| = \underline{|B|}$.

(2) ⇒ If any row or column of matrix A is zero, then $|A| = 0$

(3) ⇒ If any 2 rows or columns are identical, then $|A| = 0$

(4) ⇒ If any row or column is multiple of any other row or column, then $|A| = 0$

⇒ (4.5) & $A \xrightarrow[C(i,j) \Rightarrow \text{exchange } C_i \& C_j]{R(i,j) = \text{exchange } R_i \& R_j} B$ For elementary opn on row/column then $|B| = -|A|$

ex = $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $B = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ Here $A \xrightarrow{R(2,3),\ R(2,1)} B$

so $|B| = -(-|A|) = |A|$

$\Rightarrow$ 5) -

Second elementary opⁿ on row / column

$$A \xrightarrow[KC_j]{KR_i} B \text{ Then } \Rightarrow B = K|A|$$

ex = $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$; $B = \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix}$

Then $\boxed{|B| = 2|A|}$.

$\Rightarrow$ 6). IF $B \Rightarrow KA$.

$$|KA| = K^n |A|.$$

ex: $|A_{4\times 4}| = 5$.

$$|3A| = 3^4 * 5.$$

$\Rightarrow$ 7.) $|A^n| = |A|^n$.

$\Rightarrow$ 8) $|AB| = |A||B|$

$$|ABC| = |A||B||C|.$$

$\Rightarrow$ 9) $|A^{-1}| = 1/|A|$ where A is Non-singular matrix

Bcoz A^{-1} doesn't exist for singular as $|A| = 0$.

$$|AA^{-1}| = |I|$$

$$|A||A^{-1}| = 1$$

$$|A^{-1}| = 1/|A|$$

$\Rightarrow$ 10) $|A^t| = |A|$.

$\Rightarrow$ 11) $A \xrightarrow[C_i \pm KC_j]{R_i \pm KR_j} B$ Third elementary opn on row/column

Then $|B| = |A|$.

$\Rightarrow$ Elementary opns on matrix doesn't affect the rank of a matrix. But determinant may change

$\Rightarrow$ 12) ~~$|A| =$~~

$$|Adj(A)| = |A|^{n-1}$$

So if $n = 2$. $\boxed{|Adj(A)| = |A|}$.

ex: $|A|_{4\times4} = 3$.

$$|Adj(A)| = 3^{4-1}$$
$$= 3^3$$
$$= 27.$$

$\Rightarrow$ $A \cdot Adj(A) = |A| \cdot I$

ex: $A = \begin{bmatrix} 1 & 3 \\ -1 & 6 \end{bmatrix}$

$$A \cdot Adj(A) = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

$$|A \cdot Adj(A)| = ||A| \cdot I|$$

$$|A| \cdot |Adj(A)| = |A|^n \cdot 1$$

$$|Adj(A)| = \frac{|A|^n}{|A|} = |A|^{n-1}.$$

Ques

$$|\operatorname{adj}(\operatorname{adj}(A))| = |\operatorname{Adj}(A)|^{n-1}$$
$$= \left| |A|^{n-1} \right|^{n-1}$$
$$= |A|^{(n-1)^2}.$$

So, if $|A| = 5$

So $|\operatorname{adj}(\operatorname{adj}(A))| = 5^{(3-1)^2}$
$= 5^4$
$= 625.$

Ques ~~A. Adj(A)~~

$\operatorname{Adj}(A) \cdot \operatorname{Adj}(\operatorname{adj}(A))$? as $A \cdot \operatorname{adj}(A) = |A| I$

$= |\operatorname{Adj}(A)| \cdot I$

$= |A|^{n-1} I.$

⇒ 137) A $\equiv$ B iff (similar)

$B = X^{-1} A X$ for some Nonsingular X.

exe $A \equiv A$

Bcoz $A = I^{-1} A I$

⇒ If $A \equiv B$ Then $B \equiv A$

as $B = X^{-1} A X \Rightarrow X B = X X^{-1} A X$

$X B X^{-1} = X X^{-1} A X X^{-1} = A$

So $A \equiv B$ iff $B \equiv A$.

Similarity is transitive also
as if $A \equiv B$ & $B \equiv C$.
Then $A \equiv C$.

⇒ If $A \equiv B$.
Then
⇒ $|A| = |B|$. similar determinant
⇒ Eigen(A) = Eigen(B). eigen value
⇒ Eigenvec(A) = Eigenvec(B). eigen vec
⇒ Rank(A) = Rank(B). Rank.

⇒ $|A| = |B|$
as $|A| = |X^{-1} B X|$
$= |X^{-1}| |B| |X|$
$= \frac{1}{\cancel{|X|}} |B| \cancel{|X|} = |B|$.

⇒ If $A \equiv B$
Then $A^{100} \equiv B^{100}$
as $A = X^{-1} B X$
$A^2 = X^{-1} B^2 X$

Proof = $A^2 = AA = X^{-1} B \underbrace{\cancel{X X^{-1}}}_{I} B X$
$= X^{-1} B^2 X$.

So $\boxed{A^n = X^{-1} B^n X}$

⇒ Rank (A) :-

Rank (A) → Size of the largest Non-zero minor.

Rank is used to check the # of independent eqns in linear system ≡ Rank

Rank

Rank(A) — if till $A_{3\times 3}$ - use direct method

\ if 3×4, 4×4 or greater — Then use Gauss elimination method.

Start with biggest square minor

⇒ If $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \neq 0$: Then Rank = 3.

~~Then~~ If it is zero, Then check for less size ie sub matrices of order 2,

we have $3\times 3 = 9$ subarrays of size 2×2

If any square submatrix of size 2×2 has determinant non-zero. Then Rank = 2.

If all submatrices have determinant = 0 Then check = size = 1.

Except Null matrix, each matrix has atleast (1) Rank.

⇒ Gauss-Elimination method :-

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -2 \\ 7 & 8 & 9 & 6 \end{bmatrix}.$$

$A_{3\times4}$. So applying Gaus elimenation method is easier

For $A_{m \times n}$: How many minors are of size r,

$$= {}^{m}C_{m-r} * {}^{n}C_{n-r}$$
$$= {}^{m}C_{r} * {}^{n}C_{r}.$$

But if $A_{4\times3}$ is Given Then $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 7 & 1 & 2 \end{bmatrix}$.

Then applying Gauss elimenation is defficult

So, first make its Transpose then apply Gauss elimenation for (A^t)

as Rank (A) = Rank (A^t).

Gauss Elimenation method change a matrix into upper Triangular matrix

Ques

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & +2 \\ 7 & 8 & 9 & -6 \end{bmatrix} \xrightarrow[R_3 - 7R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & +6 \\ 0 & -6 & -12 & +1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 2R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Rank(A) is also the # of Non-zero rows in its Echelon form.

So $R(A) = 3$.

In Echelon form, in a matrix, in each row, Below Row leader, everything must be zero.
i.e. matrix should be in step form.

leader

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ \boxed{0} & -3 & -6 & -6 \\ \boxed{0} & \boxed{0} & 0 & 11 \end{bmatrix}$$

So, in echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ \boxed{0} & 0 & -6 & -6 \\ \boxed{0} & 0 & \boxed{0} & 11 \end{bmatrix}$$

So, in echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ \boxed{0} & 0 & 6 & 8 \\ \boxed{0} & \boxed{0} & \boxed{6} & 11 \end{bmatrix}$$

Not zero. So Not in echelon form.

⇒ Properties of Rank(A):

1 → Rank is Not affected by elementary row/column opns.

2 → $r(A^t) = r(A)$

3 → If $A \equiv B$ then $r(A) = r(B)$.

4 → $r(A_{m \times n}) \leq \min(m, n)$.
i.e. max rank possible of $A_{m\times n}$ is minimum of (m, n)

5 → $r(AB) \leq r(A)$,
$\leq r(B)$,
i.e. $\leq \min(r(A), r(B))$.

Ex: $r(A_{5\times4}, B_{4\times10})$

So $r(AB) \le 4$

2.) $r(A+B) \le r(A) + r(B)$. If $\boxed{r(A)+r(B) \le}$

ex = if $A_{10\times10}$ & $r(A)=3$.

$B_{10\times10}$ & $r(B)=4$.

$C_{10\times10} = A_{10\times10} + B_{10\times10}$

$r(C) \le 7$ — strong & correct

$r(C) \le 10$ — week & correct

ex = if $r(A)=7$

$r(B)=6$.

Then $r(C) \le 10$ Bcoz $7+6=13$, so 13 NOT possible at all

$r(C) \le 10$ — strong & correct

$r(C) \le 13$ — week & correct

⇒ Lecture = 4~~6~~ 08/11/2012.

Solution of linear systems

- H — C
  - UNIQUE
  - Infinite
- IH
  - C
    - UNIQUE
    - Infinite
  - IC — No soln

eg:—

$x + y = 10$

$x - y = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$A \quad \hat{x} \quad B.$

⇒ ie (Cofficient matrix → A, B)

eg: $x + 2y + 3z = 5$

$4x + 5y + 6z = -2$

$7x + 8y + 9z = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

$X_{m \times n} \; X_{n \times 1} = B_{m \times 1}$

ie $A\hat{x} = B$.

*)— If B is Null matrix, the system of eqn is homogeneous.
Homogeneous have atleast one soln ie at $x = 0$, $y = 0$ (trivial soln)

ex:

$x + y = 10$
$x - y = 2$
C - unique.

$x + y = 10$
$2x + 2y = 20$
C - Infinite

$x + y = 10$
$2x + 2y = 30$
IC ie soln doesn't exist.

⇒ Augmented matrix [A|B].
Using Gauss elimination method: check rank of (A) and rank [A|B]

- Inhomogeneous
  - $r(A) = r[A|B] = r$ → C
    - $r = n$ → UNIQUE (ie if # of independent eqns is equal to # of variables.)
    - $r < n$ → INFINITE.
  - $r(A) \neq r[A|B]$ → IC — No soln

Homogenous ~~Homogeneous~~ $r(A)=r(A|B)=r$

$r=n \rightarrow$ Unique = Trivial

$r<n \rightarrow$ Infinite.

where r is called rank of system & n is # of variables.
(and assume know r cont ex and n)

<u>Ques</u>

$$A=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad B=\begin{bmatrix}10\\ 2\end{bmatrix}$$

$$(A|B)=\left[\begin{array}{cc|c}1 & 1 & 10\\ 1 & -1 & 20\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c}1 & 1 & 10\\ 0 & -2 & -8\end{array}\right]$$

$r(A)=2$

$r(A|B)=2=n.$

Solution is Unique & consistent.

<u>Ques</u>

$$(A|B)=\left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 4 & 5 & 6 & -2\\ 7 & 8 & 9 & 3\end{array}\right] \xrightarrow[R_3-7R_1]{R_2-4R_1} \left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 0 & -3 & -6 & -22\\ 0 & -6 & -12 & -32\end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 0 & -3 & -6 & -22\\ 0 & 0 & 0 & 12\end{array}\right]$$

$r(A)=2$

$r(A|B)=3.$

$r(A) < r(A|B).$ Inconsistent = No solution

So ~~It inconsistent~~.

for what value of K = System has unique →

Infinite → ?

No soln →

$$(A|B)\left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 4 & 5 & 6 & -2\\ 7 & 8 & K & 3\end{array}\right]\xrightarrow[\text{previous ones}]{\text{using}}\left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 0 & -3 & -6 & -22\\ 0 & 0 & K-9 & 12\end{array}\right]$$

So for ~~Infinite~~ unique $K-9\neq 0$ So $K\neq 9$

Infinite = Not possible

for No soln $\Rightarrow K=9$

Ques =
$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 4 & 5 & 6 & -2\\ 7 & 8 & K_1 & K_2\end{array}\right]\xrightarrow{\text{using previous examples}}\left[\begin{array}{ccc|c}1 & 2 & 3 & 5\\ 0 & -3 & -6 & -22\\ 0 & 0 & K_1-9 & K_2+9\end{array}\right]$$

For → unique soln = $K_1\neq 9$ ⇒

→ Infinite soln ⇒ $K_1-9=0$ & $K_2+9=0$

⇒ $K_1=9$ & $K_2=-9$

→ No soln ⇒ $K_1-9=0$ & $K_2+9\neq 0$

⇒ $K_1=9$ & $K_2\neq -9$

for $[A|B]_{m\times n}$

For $B\neq O$:

* $m>n$ < CU, CIF, IC

* $m<n$ < CIF, IC

* $m=n$ < CU, CIF, IC

for $B=O$:

* $m=n$ < CU, CIF

* $m>n$ < CU, CIF

* $m<n$ → IF

Solution of linear system involves two steps =

$FE + BS$ or : $BE + FS$

FE → Forward elimination (Gauss elimination). BS → Back substitution | BE → Backward Elimination, FS → Forward Substitution

⇒ Time complexity of Gauss elimination $= O(n^3) = n^3/3$ multiplications & divisions

of Gauss Jordan $= O(n^3) = n^3/2$ "

Although, T.C of both are same but Gauss Jordan requires more # of multiplications & divisions.

Rank (System) = $r(A) = r(A|B) = r$.

Nullity = No. of unknown parameters in infinite soln

⇒ $n - r$

\# of variables − rank.

⇒ for unique solution ⇒ Nullity (system) = 0
as $n - n = 0$

⇒ for infinite solutions ⇒ ~~nu~~ Nullity > 0.

⇛ Eigen values and Eigen vectors — E.V.P.

⇛ EVP (Eigen value problem)

⇛ Given $A_{n \times n}$ $\boxed{A\hat{x} = \lambda \hat{x}}$

find λ & $\hat{x}$ such that $A\hat{x} = \lambda\hat{x}$
(for some vector $\hat{x}$ and scalar λ).

If $|\lambda| \geq 1$ ⇒ causes Divergence
or $|\lambda| < 1$ ⇒ causes convergence.

⇛ $n \times n$ system has max. 'n' Eigen values possible

⇛ Spectral radius of $A = \rho(A)$
$= \max(|\lambda_1|, |\lambda_2|, |\lambda_3|, \ldots |\lambda_n|)$.

$\rho(A) \geq 1$ Divergence
$\rho(A) < 1$ convergence

Ques-① Given $A_{n \times n}$, find Eigen value, Eigen vector.

$A\hat{x} = \lambda\hat{x}$

$$\left.\begin{aligned} A\hat{x} - \lambda\hat{x} &= 0 \\ [A - \lambda I]\hat{x} &= 0 \end{aligned}\right\} \text{as it is homogeneous}$$

So, $|A| \neq 0$ ⇒ Unique solution
⇒ Trivial solution ($x=0, y=0, z=0$)
Becoz $|A|\hat{x} = 0$; The If $A \neq 0$ Then $\hat{x}$ is 0 definitely.

$|A| = 0$ ⇛ Non-Trivial soln

So

$[A - \lambda I]\hat{x}$ → is defined as the Non-trivial non-zero solution for Eigen values for Nontrivial solution we need $|A-\lambda I|$ to be zero.

Characteristic eqn. $\boxed{|A-\lambda I|=0}$

$\Downarrow$

Characteristic polynomials

$\Rightarrow$ Eigen value is the characteristic Root / latent Root.

$\Downarrow$ Root of characteristic eqn.

$\Rightarrow$ If all n Eigen values are distinct. Then $\Rightarrow$ 'n' linearly independent E.vectors

$\Rightarrow$ If EV are repeated $\Rightarrow$ may or may Not have 'n' linearly independent E.vectors.

Ques: $A=\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$

$$|A-\lambda I|=0 \; ; \quad \left|\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}4-\lambda & 1\\ 2 & 3-\lambda\end{vmatrix}=0 \qquad (4-\lambda)(3-\lambda)-2=0$$

char. eqn. $\boxed{\lambda^2-7\lambda+10=0}$

Eigen value $\Rightarrow \lambda = 2, 5$

Now calculate Eigen vector $= [A-\lambda I]X=0$

$$\begin{bmatrix}4-\lambda & 1\\ 2 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

for $\lambda=\lambda_1$

$$\Rightarrow \begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \begin{matrix}2x_1+x_2=0\\ 2x_1+x_2=0\end{matrix}$$

So let $x_2=k$

$2x_1+k=0$

$x_1=-k/2$

$$\lambda_1 \Rightarrow 2 \; ; \quad \hat{X}_1=\begin{bmatrix}-k/2\\ k\end{bmatrix}$$

For $\lambda = 5 \Rightarrow$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$2x_1 + 2x_2 = 0$

let $x_2 = k$

$-x_1 + k = 0$

$x_1 = k$

for $\lambda_2 = 5 \Rightarrow$ $\hat{X}_2 = \begin{bmatrix} k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Eigenvector.

for $\lambda_1 = 2$ $\hat{X}_1 = \begin{bmatrix} -k/2 \\ k \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ we can convert it into ratio.

$\Downarrow$ while answering ratio matters,

Bcoz in exams rather than giving general answer, they may provide some specific answer. So, we will check by ratio

for above problem, what is/are eigen vectors.

a) $\Rightarrow$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $[1, 1]$. b) $\Rightarrow$ $[-2, 1]$, c) $\Rightarrow$ $[-1, 2]$

d) $\Rightarrow$ $[1, -2]$. e) $\Rightarrow$ $[0, 0]$

If, in above problem, both equations vanishes. Then Eigenvector will be $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

So, all combinations possible except $[0, 0]$.

$\Rightarrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are basis vectors for 2×2 matrix

So, only 2 distinct vectors possible in 2×2.

⇒ for $A_{3\times3}$ Similarly

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \& \begin{bmatrix}0\\0\\1\end{bmatrix}$$ are basis vectors.
So 3 independent Eigen vectors ~~possible~~ exist for $A_{3\times3}$

⇒ <u>Properties of Eigen Values & Eigen vectors</u>

1): If $\hat{X}$ is an Eigen vector corresponding to E.V. λ If $\hat{X}$ is an Eigen vector for λ.
⇒ Then $K\hat{X}$ is also another Eigen vector for λ for all real Numbers $\mathbb{R}$ except 0.

2): Two distinct Eigen values always create two linear independent Eigen vectors.

i.e. $\left.\begin{matrix}\lambda_1 \to \hat{X}_1 \\ \lambda_2 \to \hat{X}_2\end{matrix}\right\}$ linear independent

3): Every Eigen vector correspond to a unique E.V.
But converse is NOT true as one E.V. may ~~not~~ ~~be~~ not necessarily unique E. vector

$$\hat{X}_1 \longrightarrow \lambda_1$$
E.V E.vector

$$\lambda \to \hat{X}_1, \quad \lambda \to \hat{X}_2$$

4): Eigen values of symmetric hermitian matrices are always real.

Eigen value (Sym. Herm) = real.

* Eigen value will be 0 only for Singular matrix

while Eigen values of Skew-Symm & Skew-Hermitian is always pure imaginary or zero
i.e. EV(S-S, S-H) = imaginary

| Eigen value (orthogonal, unitary) | = 1.

Eigen(A) = 2,3,5

⇒ (II) ε 1.) Eig (KA) = K Eigen (A)   Eigen(2A) = 4, 6, 10

2.) Eigen $(A^n) = (\text{Eigen}(A))^n$ Eigen(A^2) = 4, 9, 25.

*) Eigen(D, U, L) = Diagonal element

*) Σ Eigen(A) = Tr(A) = Sum of diagonal elements

*) Π Eigen(A) = |A|.

*). max # of Eigenvalue = order of A.

*). For a singular matrix, atleast one eigenvalue must be zero.

*). For a Non-singular matrix, No Eigenvalue should be zero.

3) Eigen $(\underbrace{3A^2+5A+6I}_{\text{matrix Polynomial}}) = 3(\text{Eigen}(A))^2 + 5(\text{Eigen}(A)) + 6(\text{Eigen}(I))$.

ex- Eigen(A) = 2, 3, 5.

Eigen $(3A^2 + 5A + 6I) = 3*2^2 + 5*2 + 6,$
$3*3^2 + 5*3 + 6,$
$3*5^2 + 5*5 + 6,$

4) → $\text{Eigen}(A^t) = \text{Eigen}(A)$.

5) ⇒ $A \equiv B \Rightarrow \text{Eigen}(A) = \text{Eigen}(B)$.

6) ⇒ $\text{Eigen}(A^{-1}) = 1/\text{Eigen}(A)$.

7) ⇒ $\text{Eigen}(\text{Adj}(A)) = \dfrac{|A|}{\text{Eigen}(A)}$

as $A \cdot \text{Adj}(A) = |A| \cdot I$

$\text{Eg}(A \cdot \text{Adj}(A)) = \text{Eigen}(|A| \cdot I)$

$\text{Eigen}(A) \quad \text{Eigen}(\text{Adj}(A)) = |A| \cdot \text{Eigen}(I)$

$= |A|$

$\text{Eigen}(\text{Adj}(A)) = \dfrac{|A|}{\text{Eigen}(A)}$.

⇒ $\text{Eig}(\text{Adj}(\text{Adj}(A))) = ?$ $\quad \text{Eigen}(A) = \alpha, \beta, \gamma$

$$= \frac{|\text{Adj}(A)|}{\text{Eigen}(\text{Adj}(A))} = \frac{|A|^2}{\left(\dfrac{|A|}{\text{Eigen}(A)}\right)}$$

$\Rightarrow |A| \cdot \text{Eigen}(A)$

$\Rightarrow \alpha \cdot |A|, \ \beta |A|, \ \gamma |A|$

$= \alpha^2\beta\gamma, \ \alpha\beta^2\gamma, \ \alpha\beta\gamma^2$

⇒ CAYLEY Hamilton Theorem

Every square matrix satisfies its own char. eqn.

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\lambda^2 - 7\lambda + 10 = 0$$

Then $A^2 - 7A + 10I = 0$ — Null matrix

⇒ Uses $A^x_{n \times n}$ ⇒ can be solved using matrix of powers $x-1$.

we can also find A^{-1}

as $A^2 - 7A + 10I = 0$

$$I = \frac{7A}{10} - \frac{A^2}{10}$$

$$A^{-1} = \frac{7I}{10} - \frac{A}{10}$$

Unique $\boxed{A^{-1} = \frac{1}{10}\left[7I - A\right]}$

in terms of linear matrix.

⇒ for A^8

$A^2 = 7A - 10I$

$A^3 = 7A^2 - 10A$

$= 7(7A - 10I) - 10A = 39A - 70I$

$A^4 = 39A^2 - 70A$

⋮

$A^8 =$

or we can also do as by

$$A^2 = 7A - 10I$$

$$A^4 = A^2 \cdot A^2 = (7A - 10I)(7A - 10I)$$
$$= 49A^2 - 140A + 100I.$$
$$= 49(7A - 10I) - 140A + 100I.$$

$$A^8 = A^4 \cdot A^4.$$

⇒ Diagonalization :- It is another ~~part~~ method to find higher power of A

* we make $A = D$ ⟶ any Diagonal matrix.

$$A = X^{-1} D X \quad = \text{modular matrix}$$

⇒ $A_{n \times n}$ is diagonalizable iff A has n linearly independent eigen vectors.

$$\Rightarrow D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

i.e. D is constructed by placing its eigen values at diagonal places.

eg:- $\lambda = 2, 3$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$X = \begin{bmatrix} (\;) & (\;) \end{bmatrix}$$

Eigen vector   Eigen vector corresponding to this

$$DX = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} (\;) & (\;) \end{bmatrix}$$

vector corresponding to this.