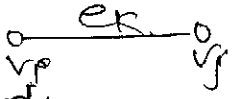
Graph Theory:-

$G = (V, E)$

V – vertex set $= \{v_1, v_2, \ldots v_n\}$

E – Edge set $= \{e_1, e_2 \ldots e_m\}$

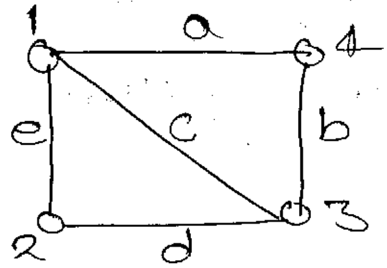
in which $e_k \in E$ is $e_k = \{v_i, v_j\}$ 

is called undirected graph.

$e_k = \{v_i, v_j\}$ → undirected edge.

→ $|V|$ = order of the Graph.

→ $|E|$ = Size of the graph.

ex:- 

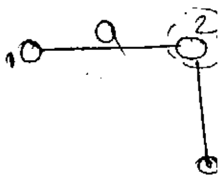
$G = (V, E)$

$V = \{1, 2, 3, 4\}$

$E = \{a, b, c, d, e\}$.

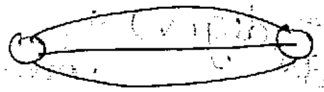
$a = \{1, 4\}$, $b = \{4, 3\}$, $c = \{1, 3\}$, $d = \{2, 3\}$, $e = \{1, 2\}$.

→ adjacent vertices:- vertices having common edg

→ Adjacent edges:- edge having common vertex 

So a & b are adjacent edges

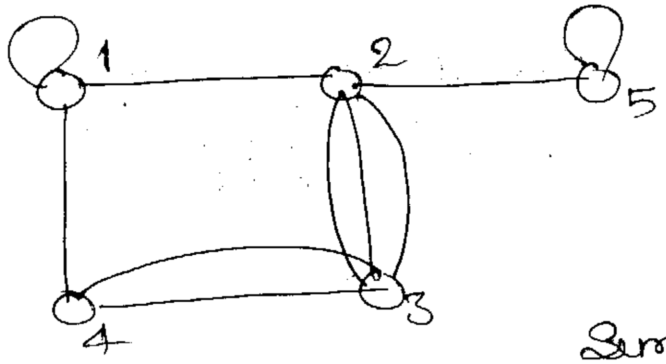
→ Self loop:-  edge joining a vertex to itself

→ multiedges:- (|| edges)  edge b/w same end points.

Graph	Self loop	Multiedges
General or Pseudograph	✓	✓
multigraph	X	✓
Simple Graph	X	X

→ Degree of a vertex: No. of edges incident on it (counting loops twice).

*Every simple Graph is multiGraph as well as Pseudo Graph. Similarly a multiGraph is pseudo Graph. But vice-versa is NOT true.



v	d(v)
1	4
2	5
3	5
4	3
5	3

Sum of degree = 20 = 2(10).
= 2(No. of edges).

→ First Theorem of Graph Theory (Handshaking Lemma)

→ In any Graph $G=(V,E)$

→ The sum of degrees of vertices is twice the no. of edges

$$**\ \boxed{\sum_{v\in V} d(v) = 2|E|}$$

Pf: Since each edge contribute 2 ~~edges~~ degrees.

Result

* The no. of odd vertices in any graph is always even.

$$\sum \deg(v) = 2|E|$$

$$\sum_{odd} \deg(v) + \underbrace{\sum_{even} \deg(v)}_{even} = even$$

$$\sum_{odd} \deg(v) = even - even = even$$

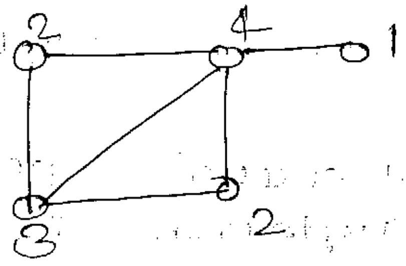
$$\underbrace{\sum_{odd} \deg(v)}_{even} = even$$

only if we have even # of odd degrees.

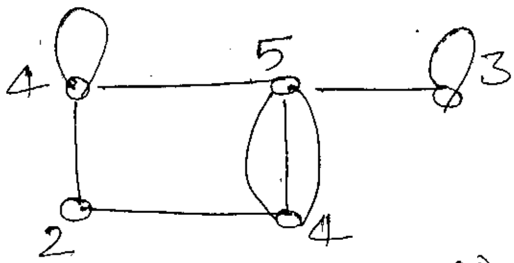
→ Degree sequence

The arrangement of degree in non-ascend (or non-descending) order.

Ex:-



(4, 3, 2, 2, 1) ≡ Degree sequence



(5, 4, 4, 3, 2)

But to derive original graph from a given degree sequence is NOT easy. It is very difficult as many possibilities exist.

→ But Given a degree sequence, we can verify whether there exists a simple graph corresponding to it or NOT using Havel-Hakimi algorithm.

→ Havel-Hakimi Procedures

$(\cancel{2}, \underbrace{2, 2}_{-1}, 2) \longrightarrow (1, 1, 2)$

$(\cancel{2}, \underbrace{1, 1}_{-1}) \longrightarrow (0, 0).$

$\boxed{\text{STOP}}$ ≡ Simple graph exists

① → Put degree sequence in non-~~descending~~ ascending order.

② → Remove highest degree. (let it is k).

③ → Subtract 1 from next k ~~degrees~~ entries (degrees

④ → Repeat step 1, 2, 3

& STOP when

→ If we get all zero entries → simple graph exists

→ If we get atleast one negative entry → NO → simple graph doesn't exist

→ NOT enough degrees (entries) → Simple graph doesn't exist

ex:- (3, 2, 1, 1, 0) → 3 odd degrees
So No Graph possible

ex:- ($\cancel{3}$, $\underbrace{2, 1, 0}_{-1}$, 0) → (1, 0, -1, 0)
simple Graph doesn't exist.

ex:- (6, $\underline{5, 4, 3, 3, 1}$)? → Not enough degrees
Simple Graph does Not exist.

ex:- which of the following degree sequence doesn't correspond to a simple Graph

i) → 7, 6, 5, 4, 4, 3, 2, 1
ii) → 6, 6, 6, 6, 3, 3, 2, 2
iii) → 7, 6, 6, 4, 4, 3, 2, 2
iv) → 8, 7, 7, 6, 4, 2, 1, 1.

a) → 1 & 2 only
b) → 3 & 4 only
c) → 4 only
d) → 2 & 4 only

i) → ($\cancel{7}$, $\underbrace{6, 5, 4, 4, 3, 2, 1}_{-1}$) → ($\cancel{5}$, $\underbrace{4, 3, 3, 2, 1}_{-1}$, 0)
→ ($\cancel{3}$, $\underbrace{2, 2, 1}_{-1}$, 0, 0) → ($\cancel{1}$, $\underbrace{1}_{-1}$, 0, 0, 0) → (0, 0, 0, 0).
So simple Graph exist

ii) → ($\cancel{6}$, $\underbrace{6, 6, 6, 3, 3, 2}_{-1}$, 2) → ($\cancel{5}$, $\underbrace{5, 5, 2, 2, 2}_{-1}$, 1)
→ ($\cancel{4}$, $\underbrace{4, 4, 1, 1}_{-1}$, 0) → (3, $\underbrace{1, 0, 0}_{-1}$, 0)
→ (0, -1, -1)
So No exist.

iii) → ($\cancel{7}$, $\underbrace{6, 6, 4, 4, 3, 2, 2}_{-1}$) → ($\cancel{5}$, $\underbrace{5, 3, 3, 2, 1}_{-1}$, 1)
→ (4, $\underbrace{2, 2, 1, 0}_{-1}$) → ($\cancel{1}$, $\underbrace{1}_{-1}$, 0, 0, 0)
→ (0, 0, 0, 0)
→ Yes. It is simple.
So ~~simple Graph doesn't exist~~

(IV) → $(8, \underbrace{7, 7, 6, 4, 2, 1, 1}_{-1})$? → Not enough entries
So No
ie simple graph doesn't exist.

(*) → min degree → "δ"
max degree → "Δ"

Result
*) G is a graph with v-vertices and e-edges.

***
$$\boxed{\delta \le \frac{2e}{v} \le \Delta}$$

Proof

$$\sum \deg(v) = 2e$$

→ Replacing each degree with min. degree

$$\underbrace{\delta+\delta+\delta+ \cdots +\delta}_{v \text{ times}} \le 2e$$

$$\boxed{v \cdot \delta \le 2e} \quad \text{— (I)}$$

→ Similarly, replacing each degree with max degree

$$\boxed{2e \le \Delta \cdot v} \quad \text{— (II)}$$

from (I) & (II)

$$\boxed{v \cdot \delta \le 2e \le v \cdot \Delta}$$

Dividing with v

$$\boxed{\delta \le \frac{2e}{v} \le \Delta}$$

exe — G is a graph with 11 edges & min degree = 3. what is max # of vertices?

a) 6 ; b) 7 ; c) 8 ; d) 9

$\delta \le \frac{2e}{v}$; $\delta v \le 2e$

$\delta v \le 2 \times 11$

$3 \cdot v \le 2 \times 11$

$v \le \frac{22}{3}$; $v \le \lfloor 7\text{.something} \rfloor$

max vertex = $\left\lfloor \frac{22}{3} \right\rfloor = 7$.

Ex:- G is a graph with 12 vertices & max degree = 4. The max # of edge. The max. # of edges in G

a) 22 ; b) 23 ; c) 24 ; d) 25

$2e \leq v \cdot \Delta$

$2e \leq 12 \times 4$

$2e \leq 48.$

$e \leq 24.$ So max = 24.

⇒ Special Graphs:-

→ **Null Graph (N_n):-** A graph having no edge and n-vertices

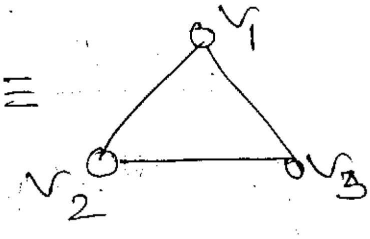
ex: $N_1 \equiv$ 

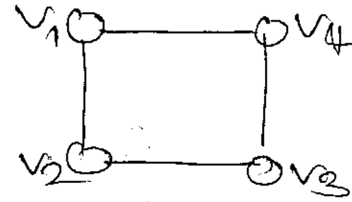
$N_2 \equiv$ 

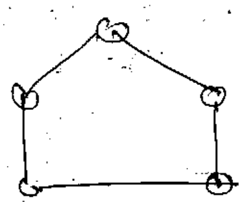
$N_3 \equiv$ 

	v	e	d(v)
N_n	n.	0.	0.

→ **Cycle Graph (C_n, $n \geq 3$):-** The cycle Graph is a simple graph with n-vertices $\{v_1, v_2, \dots v_n\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\} \dots \dots \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

$C_3 \equiv$ 

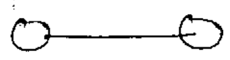
$C_4 \equiv$ 

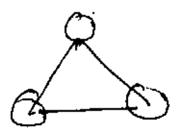
$C_5 \equiv$ 

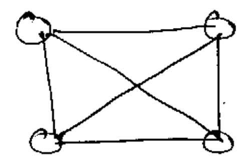
	v	e	d(v)
C_n	n.	n.	2.

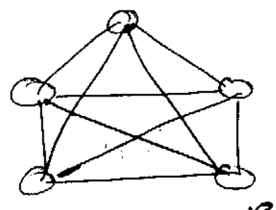
→ <u>Complete Graph (K_n)</u>: Complete Graph K_n, is a simple Graph in which every pair of vertices are adjacent.

$K_1 \equiv$ 

$K_2 \equiv$ 

$K_3 \equiv$ 

$K_4 \equiv$ 

$K_5 \equiv$ 

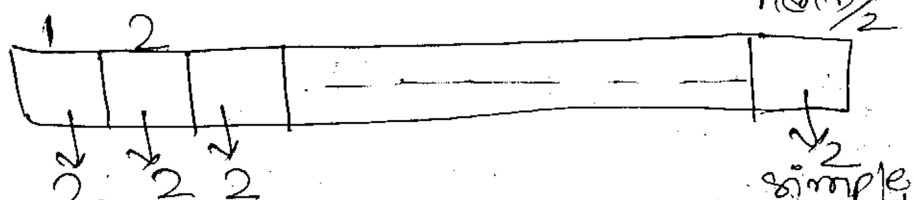
	v	e	deg(v)
K_n	n	${}^nC_2 = \frac{n(n-1)}{2}$	$n-1$.

	v	e	deg(v)
K_1	1	0	0
K_2	2	1	1
K_3	3	3	2
K_4	4	6	3
K_5	5	10.	4

(*) max. no. of edges in a simple Graph with n-vertices ${}^nC_2 = \frac{n(n-1)}{2}$

max # edges in a simple Graph will occur in case of Complete Graph [Every pair of vertices must be adjacent]

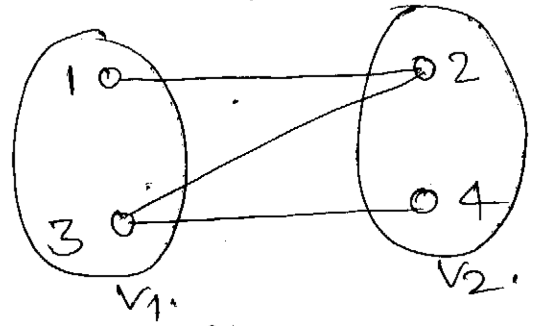
<u>Ques (3)</u>: $V = \{V_1, V_2, \text{———} V_n\}$ of n-vertices.



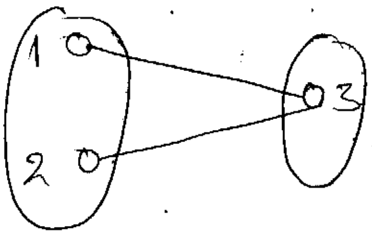
$\Rightarrow$ So total possible simple graphs $= \boxed{2^{n(n-1)/2}}$

→ **Bipartite Graphs** — A graph $G=(V,E)$ is bipartite if the vertex set can be partitioned into two sets V_1 and V_2 such that every edge is from, b/w a vertex of V_1 to a vertex of V_2

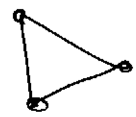
ex:-



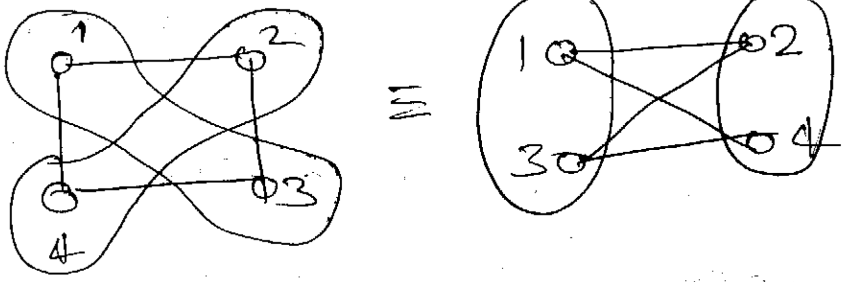
$V_1=\{1,3\}$
$V_2=\{2,4\}$



≡ Bipartite



NOT Bipartite

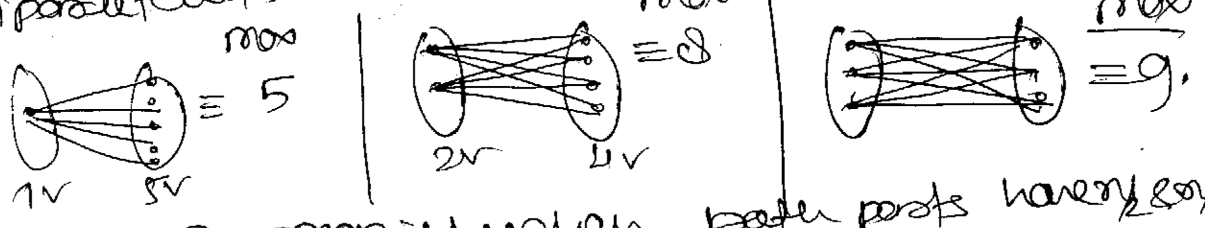


$\boxed{C_n \text{ is bipartite iff } n \text{ is even}}$

C_3, C_5 --- are NOT Bipartite

(*) which of the following is the max no of edges in bipartite graph with n-vertices
a) → n^2 b) → $n^2/2$ c) → $n^2/4$ d) → $n/2$.

ex: Bipartite with 6 V.

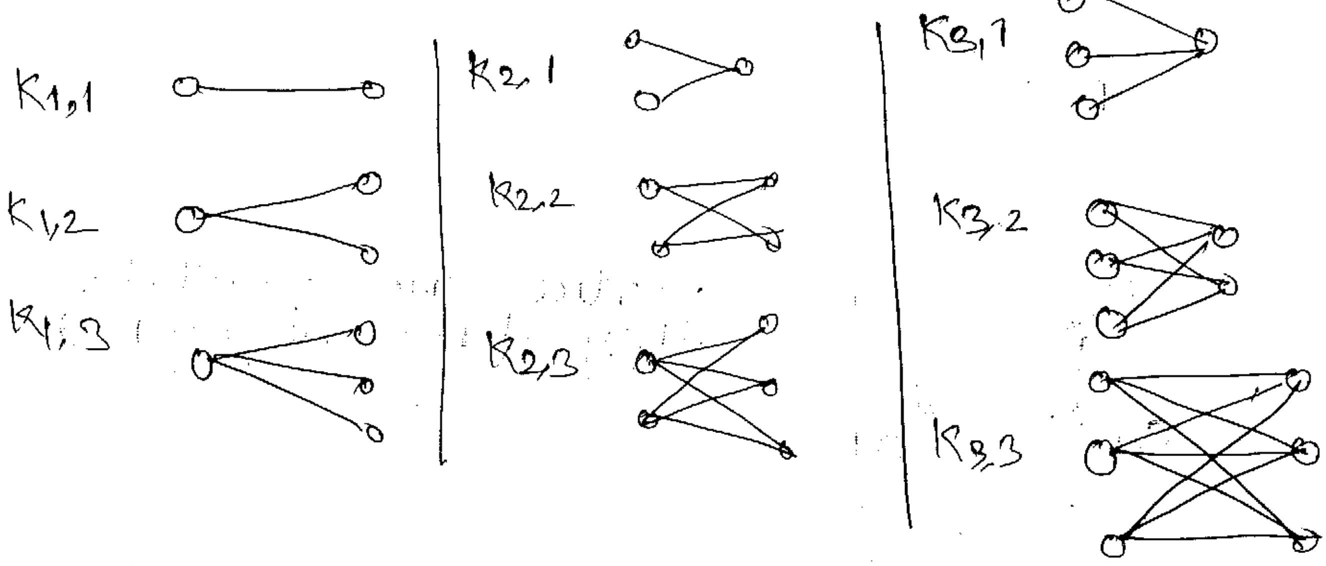


max ≡ 5 | max ≡ 8 | max ≡ 9.

So max is when both parts have $n/2$ & $n/2$ vertices so max # edge $= (n/2)(n/2) = n^2/4$.

**) The maximum no. of edges in a bipartite graph with n-vertices is atmost $n^2/4$.

→ Complete Bipartite Graph ($K_{m,n}$): A bipartite graph $G = (V_1 \cup V_2, E)$ in which every vertex in V_1 is adjacent to every vertex in V_2 is called complete Bipartite Graph



	V	e	$d(v)$
$K_{m,n}$	$m+n$	$m \cdot n$	$\begin{cases} m : v \in V_2 \\ n : v \in V_1 \end{cases}$

$|V_1| = m$
$|V_2| = n$

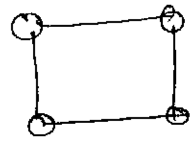
→ Regular Graph: A Graph in which every vertex has same degree

$\forall v \in V \; deg(v) = K \Rightarrow K$-regular Graph

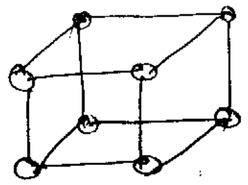
ex: N_n : 0-regular Graph.
C_n : 2-Regular Graph.
8 : 2- " "
K_n : $(n-1)$ Regular Graph.

N-cube :-

1-cube



2cube

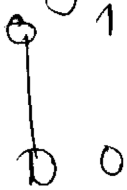


$\#$ vertices $= 2^N$

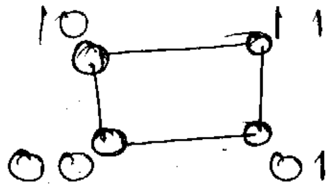
$\#$ edges $= N \times 2^{N-1}$

$d(v) = N$.

Binary string of length 1.

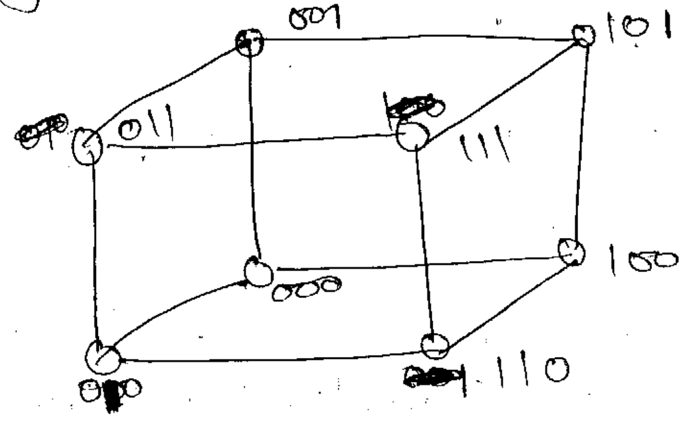


Binary string of length 2.



vertices are connected If they differ only by 1 bit ie Hamming distance $= 1$

Binary string of length 3.



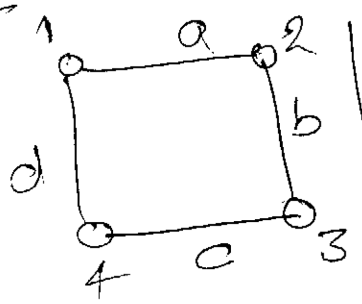
Subgraph

A graph $H = (V', E')$ is Subgraph of $G = (V, E)$ if

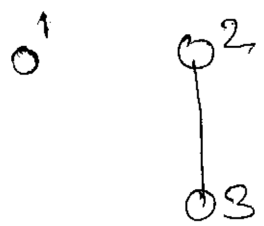
$V' \subseteq V$.

$E' \subseteq E$.

ex:-



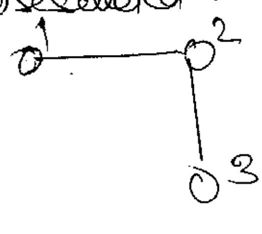
H_1



H_2

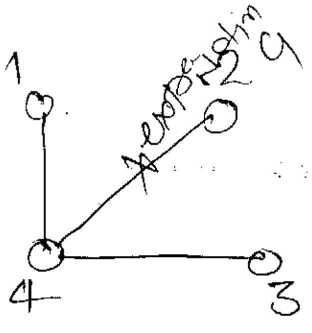
Subgraph formed by $\{1,2,3\}$

Along with vertices all possible edges are taken from G for those selected vertices

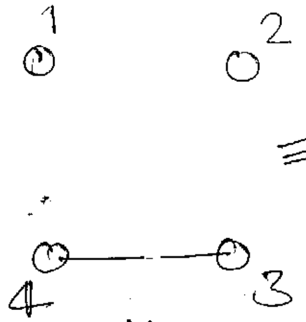


H_3

Subgraph induced by $\{1,2,3\}$.



= H4
Not Subgraph



= Spanning subgraph.

H

Spanning subgraphs: A subgraph containing all vertices of the Graph

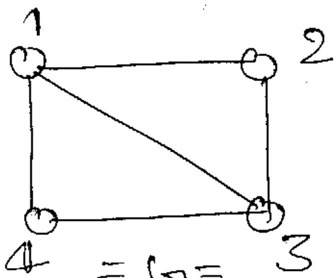
⇒ Adjacency ~~Graph~~ Matrix: $G=(V,E)$, $|V|=n$.
it can be represented as a $n \times n$-matrix.

$$A[G] = [a_{ij}]_{n \times n}$$

$$a_{ij} = \begin{cases} 0 & \text{if } \{i,j\} \notin E \\ 1 & \text{if } \{i,j\} \in E \end{cases}$$

→ No provision of ||el edges
→ Graph has no self loops iff the main diagonal entries are all zeros.
→ for Graph having no self loops $\deg(v_p)$ = sum of entries in pth row (or) pth column

Ex:



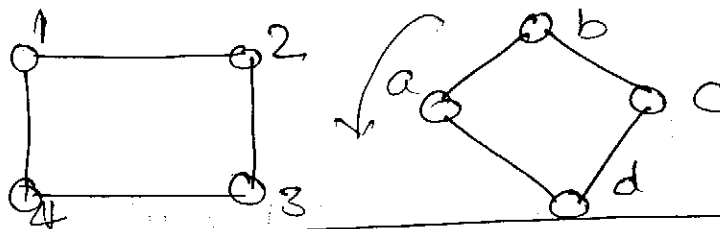
= G =

$$A[G] = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

→ It doesn't have self loop
→ $\deg(2)=1+1=2$
→ $\deg(3)=1+1+1=3$.

→ Isomorphism :- "Isomorphic Graphs are same graphs drawn differently."

ex :-



(*) complexity of checking whether two graphs are isomorphic = $O(n!)$

1 — a.
2 — b.
3 — c.
4 — d.

} vertices have 1-1 correspondence.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array} \equiv \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

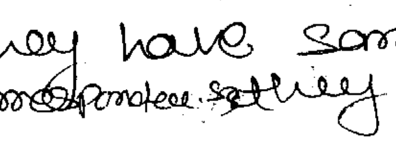
So, they have same adjacency matrix for this vertex correspondence so they are isomorphic.

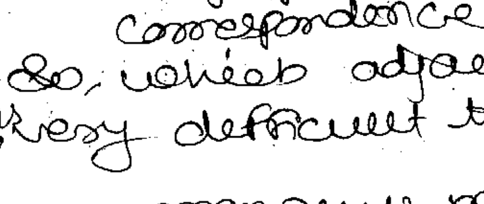
→ But ~~for a graph with~~ n vertices = $n!$ 1-1 correspondence exists b/w vertices.

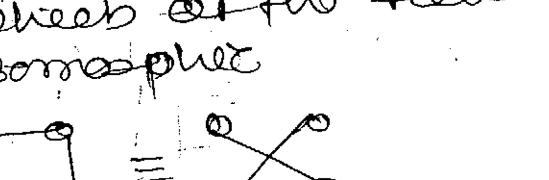
So, which adjacency matrix match is matching is very difficult to check $\equiv O(n!)$.

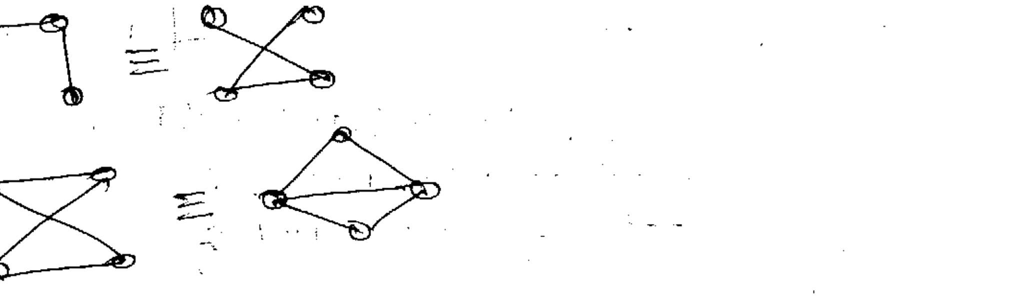
So, commonsense method is used to verify.

Ques which of the following graphs are isomorphic

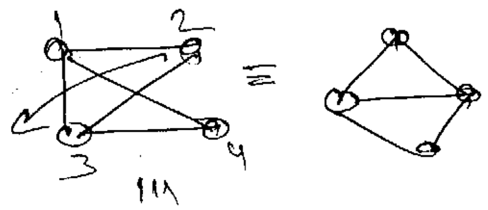
a) → 

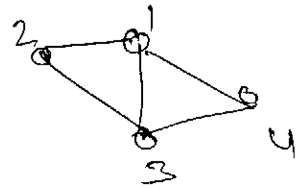
b) → 

c) → 

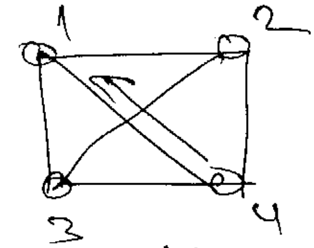
d) → 

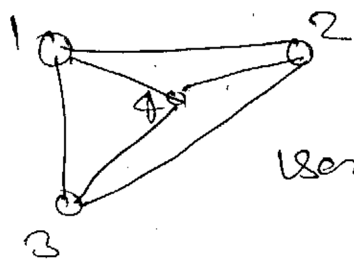
a) →  Isomorphic

b) → 



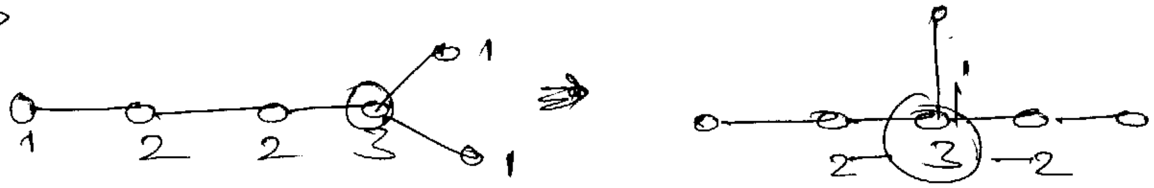
Isomorphic

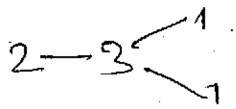
c) → 

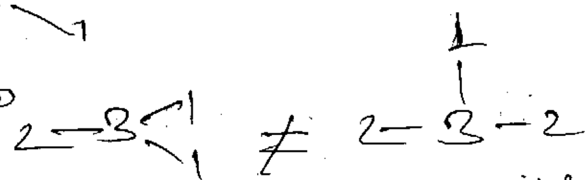


Isomorphic

c) →

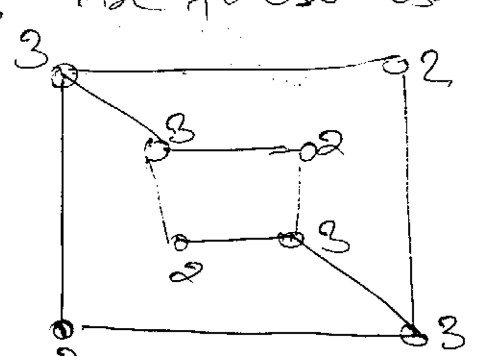


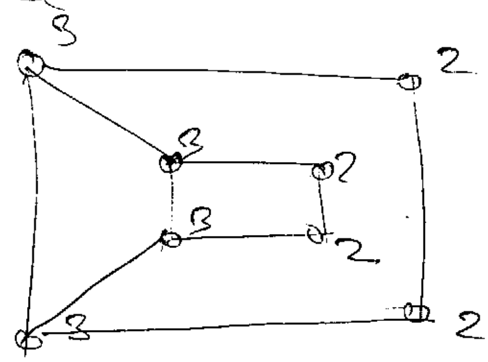


So 

as adjacency is NOT preserved
So, NOT isomorphic

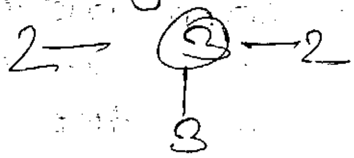
Ques: Are these isomorphic?



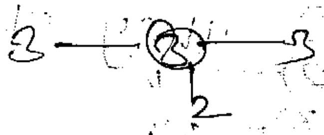


So, each vertex of degree 2 has adjacency

Here, each vertex of degree 2 has adjacency

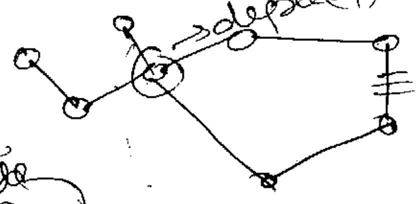


≠



So NOT isomorphic as adjacency is NOT preserved.

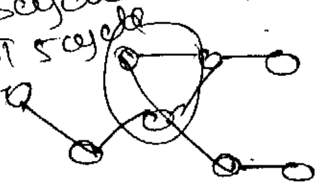
2012
Ques which of the following graphs is isomorphic to



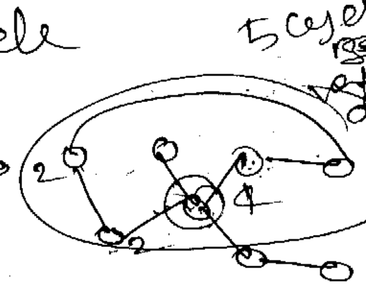
≡ 5 cycle

3 cycle
NOT 5 cycle

a) →



b) →

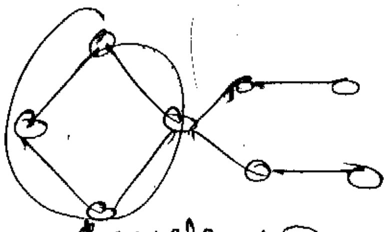


5 cycle & ~~no~~ vertex of degree 4 exists
So iso morphic

c) →

5 cycle
~~But in cycle No vertex has degree 4~~
But in cycle No vertex of degree 4 exists

d) →



4 cycle NOT 5 cycle

⇒ Two Graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic $G \cong G'$ if we can define a function

$$f : V \longrightarrow V'$$

Such that

1) → f is 1 − 1
2) → f is ONTO
3) → Adjacency ~~property~~ is preserved.

→ Two Graphs G and G' are isomorphic iff
$$A[G] = A[G'].$$
ie have same adjacency matrix.

→ Necessary Conditions for isomorphic isomorphism of two Graphs.

(1) → They should have same no. of vertices
(2) → " " " " " edges
(3) → " " " " " degree sequence

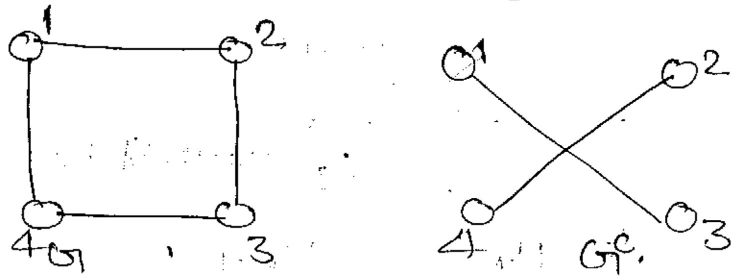
30/11/2012

⇒ Complement of a Graph

$G = (V, E)$ be a simple Graph

$G^c = (V, E^c)$ such that, two vertices are adjacent in G^c, iff they are NOT adjacent in G.

→ Ex:



*) → Ques → If No. of edges in $G = 13$ & No of edges in G^c = the No. of vertices in G

a) → 6 ; b) → 7 ; c) → 8 ; d) → 9.

$$\frac{n(n-1)}{2} = 13 + 15 = 28$$

$$n(n-1) = 56$$

$$n = 8.$$

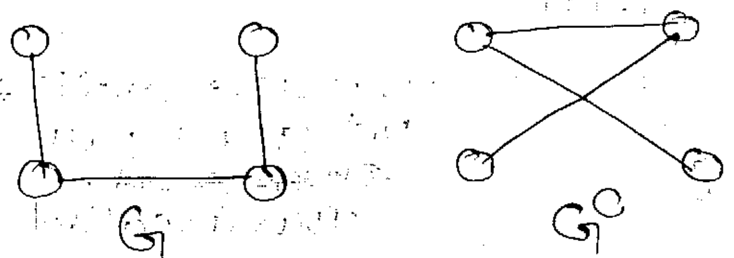
*) Result: If G is a simple Graph with n-vertices.

⇒ The No of edges in G + # edges in G^c = # edges in Complete Graph

⇒ $$\boxed{e + e^c = \frac{n(n-1)}{2}}$$

⇒ Self Complementary Graph: A graph which is isomorphic to its complement, is called self-complementary Graph.

eg:-



$G \equiv G^c$

So, G is self complementary Graph

(**) → which of the following can not be no. of vertices in a self complementary graph

a) → 4 ; b) → 5 ; c) → 9 ; d) → 10

edges $= \frac{4 \times 3}{2} = 6$; $\frac{5 \times 4}{2} = 10$; $\frac{9 \times 8}{2} = 36$; $\frac{10 \times 9}{2} = 45$

3 3 ; 5 5 ; 18 18 equal ; 22 2

*>- Result>: The no. of vertices in a self complementary graph is of the form 4K or 4K+1

Soln: $e + e^c = \frac{n(n-1)}{2}$

In self complementary graph $e = e^c$

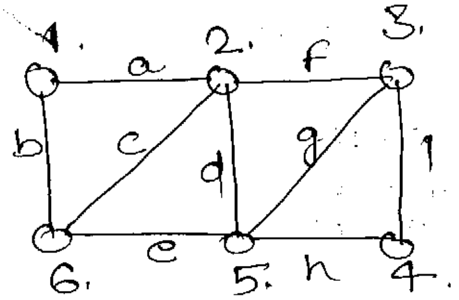
$\Rightarrow 2e = \frac{n(n-1)}{2}$

$\Rightarrow e = \boxed{\frac{n(n-1)}{2}}$ must be integer.

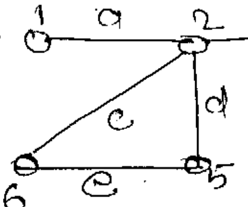
$\Rightarrow 4 | n$ (or) $4 | n-1$.

$\Rightarrow n = 4K$ (or) $n = 4K+1$.

Ex:

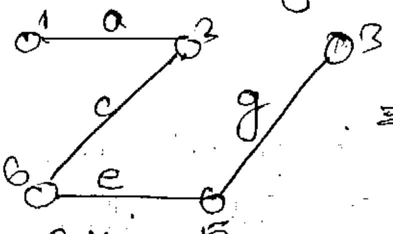


⇒ 1a2d5e6c2f3 = edge sequence



= In this sequence, No edge is repeated. This is called walk. More specifically, it is called open walk.

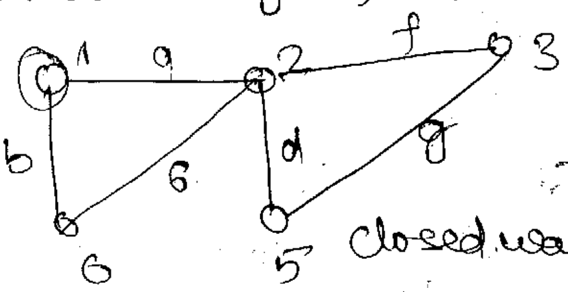
⇒ 1a2c6e5g3



Path 5

⇒ In this edge sequence, No vertex is repeated along with edges. Such an edge sequence is called path ~~[illegible]~~.

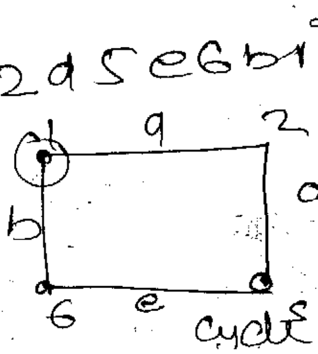
⇒ 1a2d5g3f2c6b1



⇒ This is a walk that ends and starts with same vertex ~~[illegible]~~. This is called Closed walk

Closed walk if NOT closed = open walk

1a2d5e6b1



cycle

⇒ This is a path starting & ending with same vertex and No other vertex is repeated in this edge sequence. This is called closed path. cycle

⇒ Edge Sequence: Sequence of edges, starting & ending with a vertex.

*) Walk: Edge sequence in which no edge is repeated.

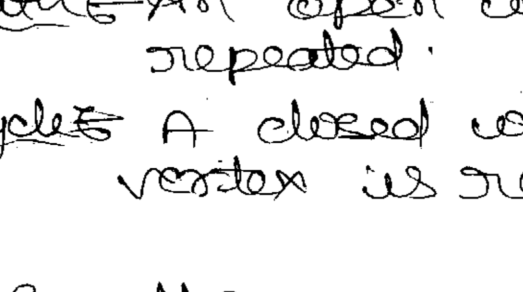
*) Closed walk: A walk which start and end at same vertex is called closed walk. Otherwise, it is an open walk.

*) Path: An open walk in which no vertex is repeated.

*) Cycle: A closed walk in which no other vertex is repeated.

*) Result: A graph is bipartite iff every cycle in the graph is even cycle.

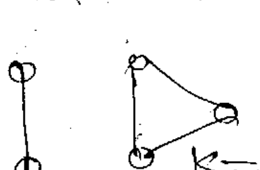
*) Connected: Two vertices are said to be connected in a graph iff there is atleast one path b/w them.



components k=1

(Connected Graph)

A graph is connected if every pair of vertices are connected.



k=2

(G_1)

Disconnected Graph

k=3

(G_2)

Disconnected Graph

→ The maximal connected subgraph is called component

(*) Result> A simple Graph with n-vertices and k-components has atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

→ ∴ if a G is a graph with
n - vertices
k - components
$$e \leq \frac{(n-k)(n-k+1)}{2}.$$

Ex: G is a graph with 10 V and 3 components. Then maximum no of edges in G

a) → 26 b) → 27 c) → 28 d) → 29.

$$e \leq \frac{(10-3)(10-3+1)}{2} = \frac{7 \times 8}{2} = 28.$$

Result> [Sufficient condition for connectedness.]
*)- G is a simple Graph with n-vertices and no of edges $> \frac{(n-1)(n-2)}{2}$
Then G is connected.

[if p then q
⇒ p is sufficient condition for q]
& q is necessary for p.

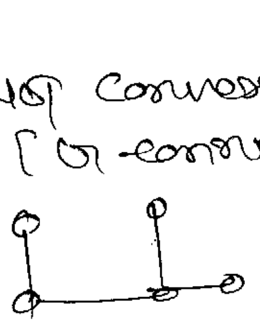
Pf: Let G disconnected.
Then n vertices atleast 2 - components

$v = n$; $k = 2$
$$e \leq \frac{(n-2)(n-2+1)}{2}$$
$$\boxed{e \leq \frac{(n-1)(n-2)}{2}}$$ But this is contradiction
our assumption is wrong.

So for $e > \frac{(n-1)(n-2)}{2}$ is connected.

Note → NOT Conversely
[G - connected ⇏ condition].

ex:



$v = 5$ $e > \frac{(5-1)(5-2)}{2} = \frac{4 \times 3}{2} = 12$ [sic]
$e = 4$ $e \not> 6$
So still it is connected but

Ques: Which of the following simple graph are always connected.

$\frac{(5-1)(5-2)}{2} = \frac{12}{2} = 6$

a) G with 5V & 5 edges → 5 < 6 so Not sure

b) ,, ,, 6V & 9 ,, → $\frac{5\times4}{2} = 10$ 9 < 10 so Not sure

c) ,, ,, 7V & 15 edges → $\frac{6\times5}{2} =$ 15 = 15 but 15 > 15 So Not sure

d) ,, ,, 8V & 22 edges → $\frac{7\times6}{2} = 21$ | 22 > 21 so surely connected

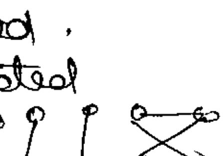
*) Result

At least one of the graph G or G^c is connected

Ques: Which of the following is always true.

S_1: G connected, then G^c disconnected.

S_2: G disconnected, then G^c is connected

a) only S_1 true (Not necessarily true) as 

G G^c

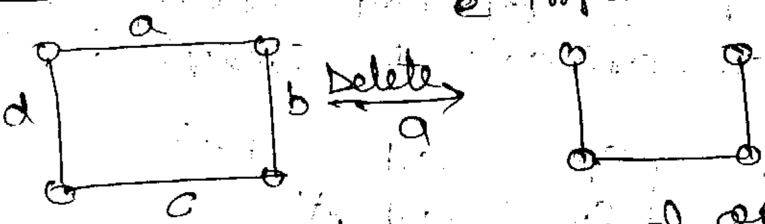
Both are connected

b) only S_2 true

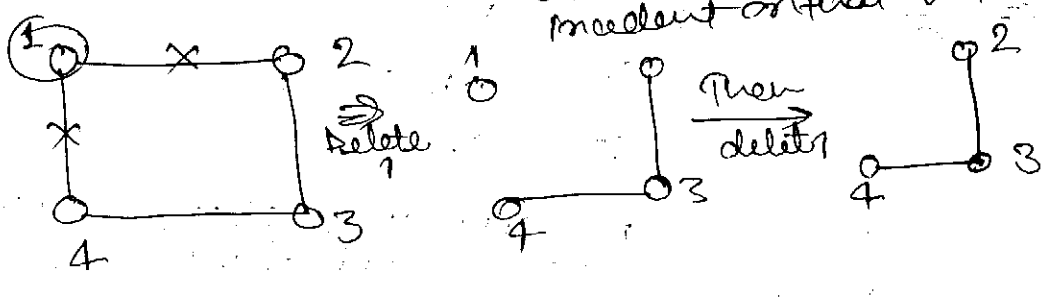
c) Both

d) None

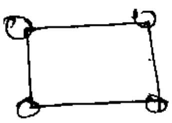
→ Removal of an edge: ie Removal of an edge e implies removal of that edge only.



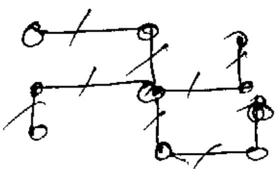
→ Removal of a vertex: Removal of vertex implies removal of edges incident on that vertex also



→ Cut edge (or) bridge: A single edge whose removal disconnects the connected graph is called cut edge.

ex:-  → No of cut edges: a) 0 b) 1 c) 2 d) 3

→

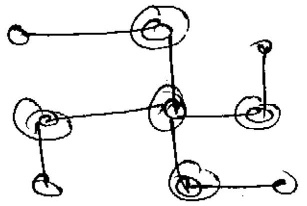
 → No of cut edges: a) 6, b) 2, c) 0 d) 3

* Cut vertex (or) Articulation point:

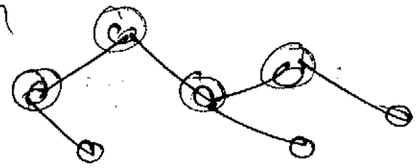
A single vertex whose removal disconnects the connected graph is called cut vertex (or) A

→ No. of articulation in the graph are

a) 4 b) 8, c) 6 d) 7



→ No. of AP in

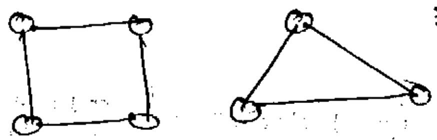


a) 3, b) 4

c) 5, d) 6,

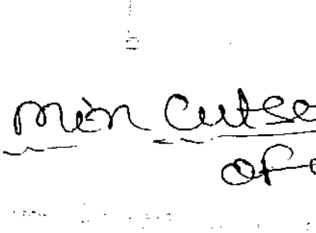
* A graph having no articulation points is called a biconnected graph

eg:



= all cycles graphs are biconnected.

→ Cutset: A set of edges whose removal disconnects the graph is cutset.



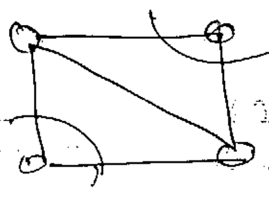
$C_1 = \{a, b, c, d, e\}$
$C_2 = \{a, b, d\}$

$\left.\begin{array}{l} C_3 = \{a, e\} \\ C_4 = \{a, b, c\} \\ C_5 = \{c, d\} \end{array}\right\}$ min cutset.

min cutset: A cutset, no proper subset of which is a cutset, is called min cutset

→ Edge connectivity (λ):

The min. no of edges whose removal disconnects the connected graph.

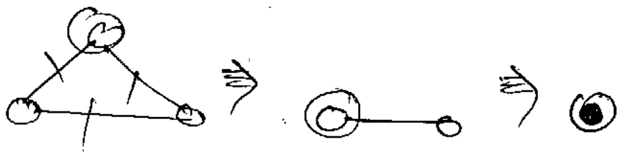


$\lambda = 2$.

→ **Vertex connectivity** (K – kappa): The minimum # of vertices whose removal disconnects the graph (or) leaves trivial graph, is called vertex connectivity.

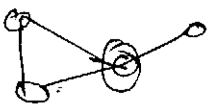
$\circ \rightarrow$ Trivial Graph $\equiv (K_1)$.

Ex:



= Not disconnected But Trivial. So $K=2$.

Ex:



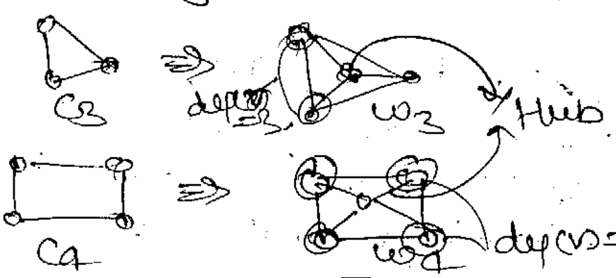
$K = 1$.

G	λ	K
C_n	2.	2
W_n	3	3
K_n	$n-1$	$n-1$
$K_{m,n}$	min (m,n)	min (m,n).

→ **Wheel Graph (W_n); $n \geq 3$.**

Wheel Graph is a graph formed from a cycle graph by adding a new vertex and connecting that vertex with every vertex of C_n.

Ex:-



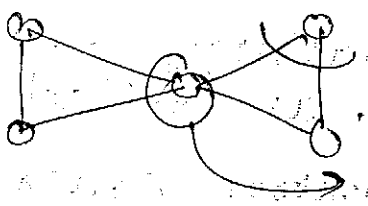
	V	e	$d(v)$
W_n	$n+1$	$2n$.	$\begin{cases} n - v \in \text{Hub} \\ 3 - v \notin \text{Hub} \end{cases}$

Result

※ – In any Graph → Whitney Theorem

$$\boxed{K \leq \lambda \leq \delta}$$

Ex:-



$\delta = 2$

$\lambda = 2$.

$K = 1$

Ques G is a graph with 11 edges & min degree is 4. what is max value of vertex connectivity

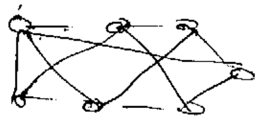
$$\kappa \le \lambda \le \delta$$

$\delta \to 4$

So $\kappa \le \delta = 4$

So at most, it is 4

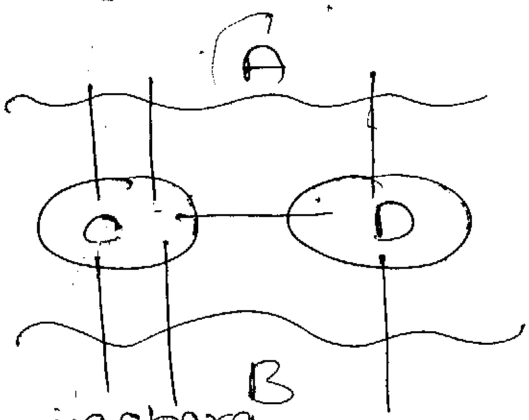
→ G is a graph with 11 edges & 7 vertices what is the max value of K?



$$K \le \delta \le \frac{2e}{V}$$

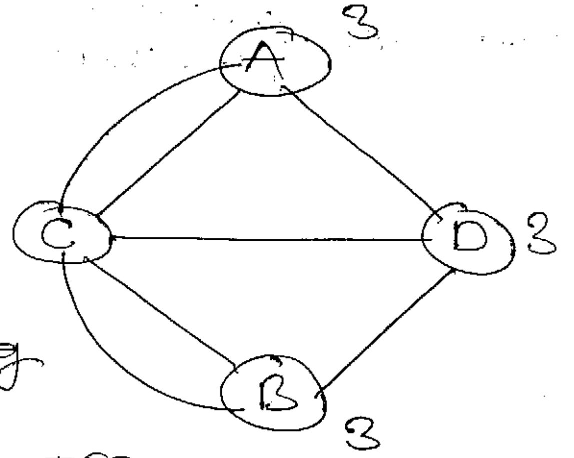
$K \le \frac{2e}{V}$ || $K \le \frac{2 \times 11}{7}$ || $K \le 3$.

⇛ Euler's Graph



Konigsberg Bridge Problem

Start at any land area cross every bridge exactly once and come back to starting position.



Euler's Paper on Konigsberg Problem

~~A Graph is Euler ckt~~

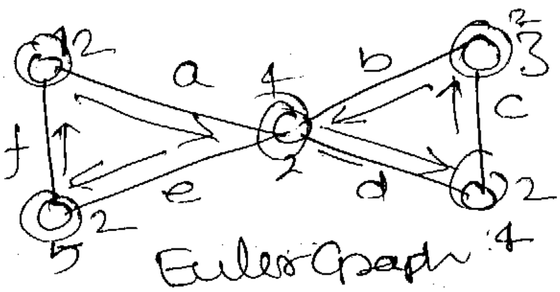
Def → A closed walk containing every edge of a multigraph is called Euler circuit.

Def A multi graph containing Euler circuit is called Euler graph

Def: An open walk containing every edge of a multigraph is called Universal line (or) Open Euler walk.

Def: A multigraph containing universal line is called universal graph

Ex:



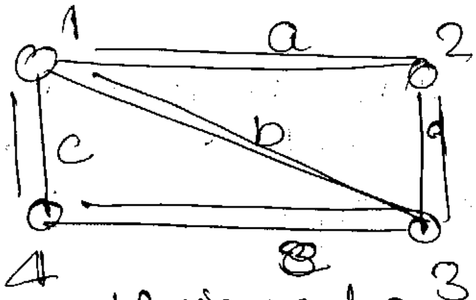
Euler Graph

$\boxed{1a2b3c4d2e5f1}$

$\boxed{1a2d4c3b2e5f1}$

Euler circuit.
i.e. closed walk containing all the edges.

Ex:

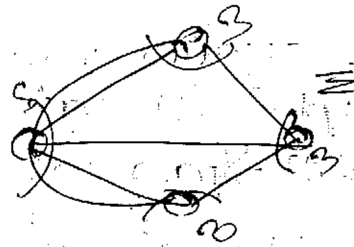


Universal Graph

$\boxed{1\,b3e4c1a2d3}$

open walk containing all edges → Universal line (Euler walks)

Ex:

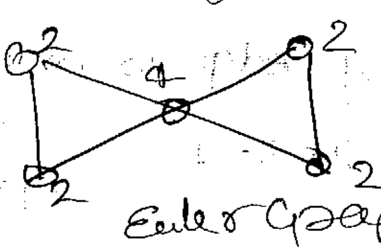


= Neither Euler Graph nor Universal.

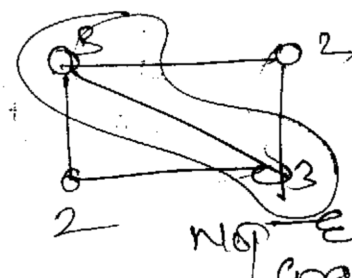
→ Euler Graph	Universal Graph
✗	✗
✓	✗
✗	✓

So, a graph can't be both.

Result **: A multigraph is Euler Graph iff degree iff degree of every vertex is even.

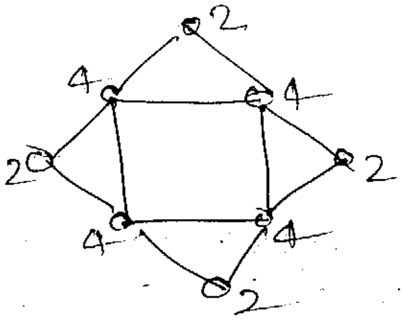


Euler Graph



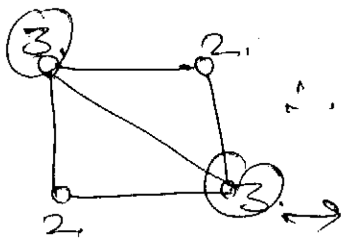
Not Euler Graph.

eg :-



≡ ∴ Euler Graph.

*) Result: A multigraph is Unicursal Graph iff there are exactly 2 vertices of odd degree.



∴ Unicursal Graph (Bcoz exactly 2 vertices of odd degree).

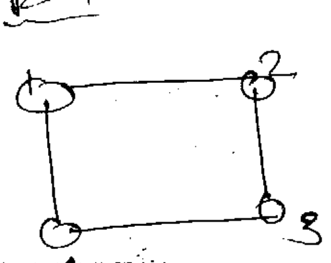
→ To trace, start at odd degree vertex (it will end at another odd degree vertex).

⇒ Hamiltonian Graph: A cycle containing all the vertices of a graph is called Hamiltonian cycle (or) Spanning cycle.

* A graph containing Hamiltonian cycle is called Hamiltonian Graph.

* A path containing all the vertices of the graph is called Hamiltonian path.

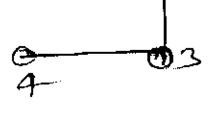
eg :-



1 - 2 - 3 - 4 - 1 : H-cycle

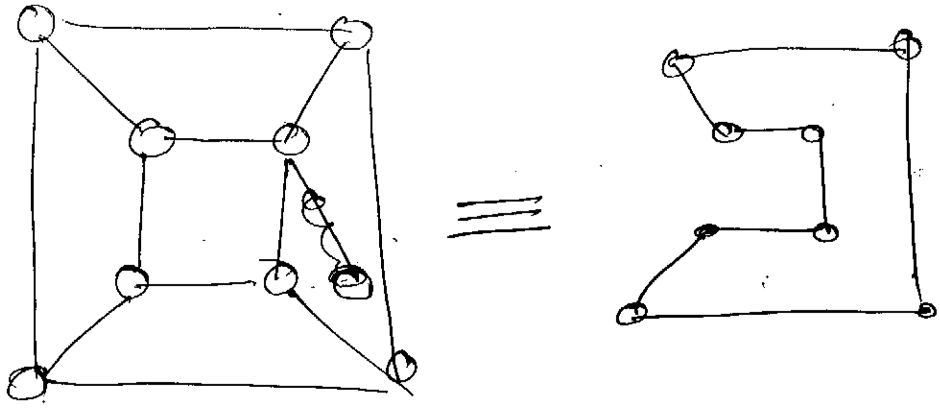
∴ H-graph

on Removing any edge from H-cycle



≡ 1 - 2 - 3 - 4 ≡ H-path

eg:-



H-cycle,

H-Graph

*) Results

*) If G is a ~~~~ hamiltonian Graph with n-vertices

i) → Hamiltonian cycle contain $\underline{n}$ vertices

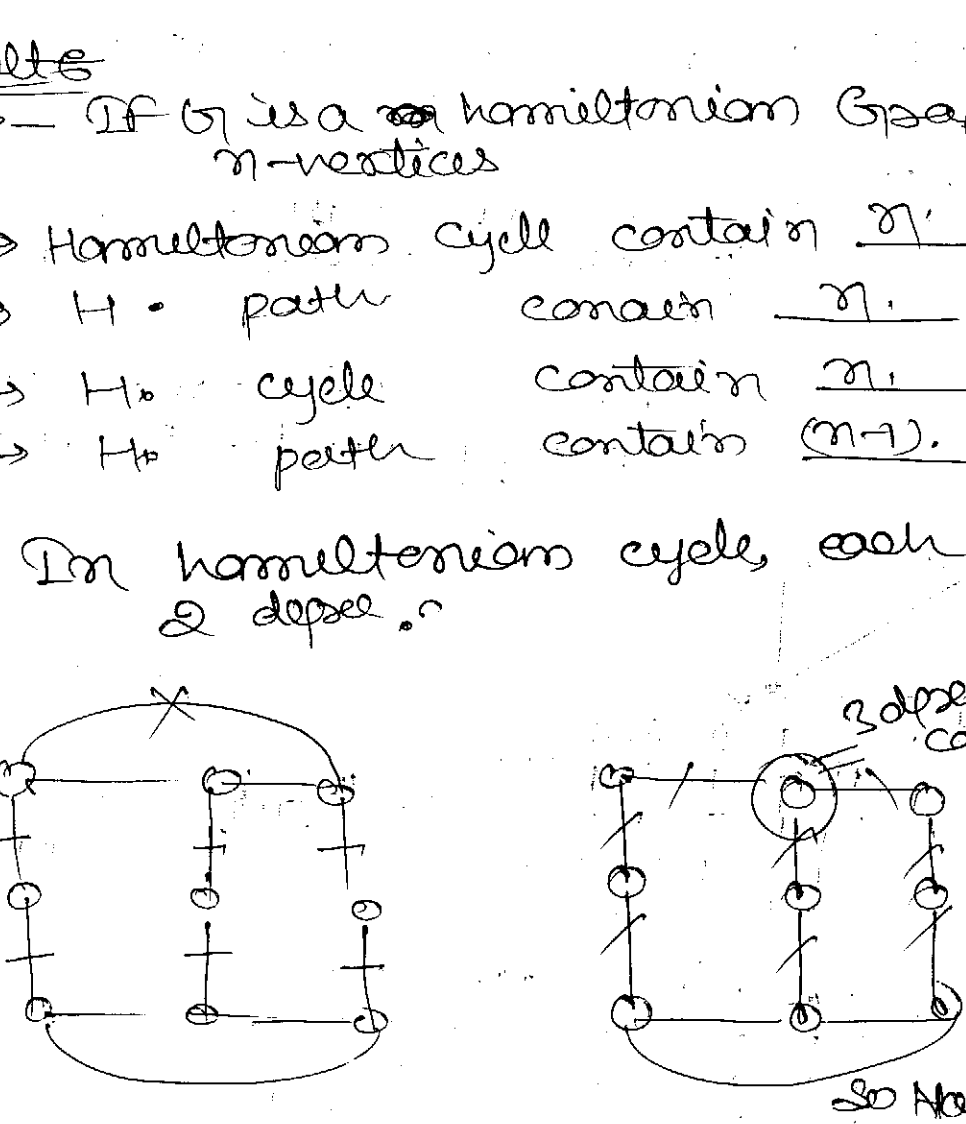
ii) → H. path conain $\underline{n}$ vertices

iii) → H. cycle contain $\underline{n}$ edges

iii) → H. path contain $\underline{(n-1)}$ edges

eg:- In hamiltonian cycle, each vertex has 2 degree.

3 degree & No edge can be remov



So, H-cycle is NOT poss

So Not H-Graph

⇒ Sufficient (Not Necessary)

(I): Dirac's Theorem (Simple Graph).

If min degree $(\delta) \geq n/2$

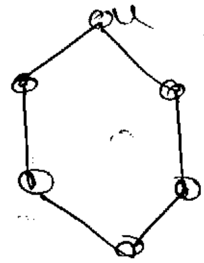
then G is Hamiltonian Graph

(II): Ore's Theorem: If $\deg(u) + \deg(v) \geq n$. (for $n \geq$

for every pair of non-adjacent vertices u

Then ~~~~ G is H-Graph.

These conditions are only sufficient, Not necessary
ie. Converse Need not be true.

eg. 

C_6

$= \delta = 2 \not\geq 6/2$

$\deg(u) + \deg(v) \not\geq 6$

$2+2 \not\geq 6$.

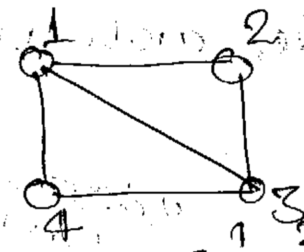
But still it is H-graph.

→ Powers of Adjacency Matrix

A(G) is an adjacency matrix of graph G.

$[A(G)]^n$ = No. of edge sequences of length n joining vertex i & vertex j

⇒ $(i,j)^{th}$ entry gives No. of edge sequence of length n joining vertex i and vertex j

eg:- 

$$\therefore A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \boxed{1} & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

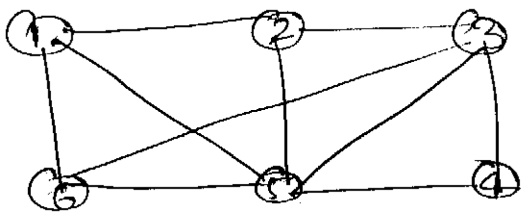
$(2,3)^{th}$ entry = No of edge sequence of length 1

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

*)- Eulers and H-graphs are defined only for connected graphs.

WB-Pg 18/Ques-33

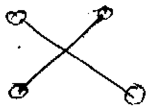


$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & & & \\ 2 & & & & & & \\ 3 & 0 & 1 & 0 & 1 & 1 & 1 \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array} \quad \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & & 1 & \\ 2 & & & & & 1 & \\ 3 & & & & & 1 & \\ 4 & & & & & 1 & \\ 5 & & & & & 0 & \\ 6 & & & & & 1 & \end{array} = 0+1+0+1+0+1 = 3.$$

$(3,5)^2$ So # of paths b/w 3rd & 5th vertex of length 2 =

* Planar Graph

→ Cross-over

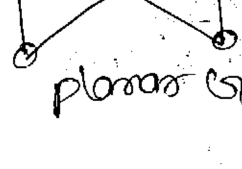


→ The planar representation of a graph is drawing the graph on plane without any cross-over.

→ A graph having planar representation is called planar Graph.

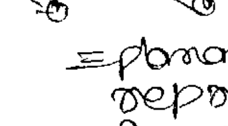
ex:

⇒



planar Graph

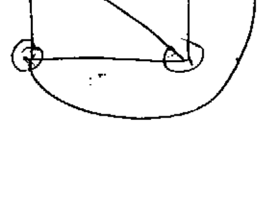
⇒



= planar representation of Graph. So, planar Graph.

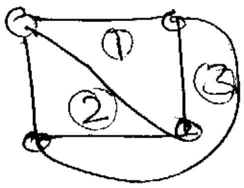
⇒

⇒

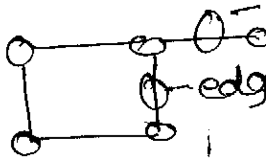


planar representation exists. So Graph is planar

→ The planar representation of planar graph divides entire plane into regions or faces

→ 

interior region - 1,2,3
exterior region - 4,

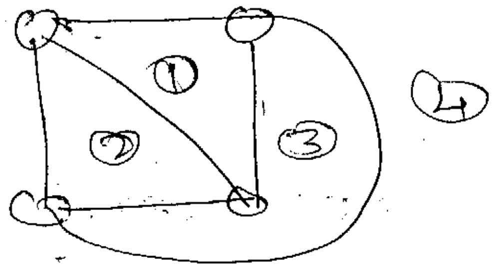
→ 

- edge in a single region
- edge b/w 2 regions (boundary edge).

→ **Degree of a region:** No. of boundary edges touching it.

ex:

r	deg(r)
1	3
2	3
3	3
4	3.

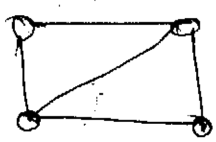


Result
*) - Sum of degrees of regions = Twice the no of boundary edges.

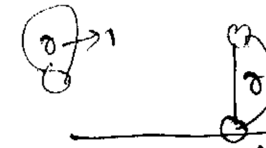
Proof: Since each boundary edge gives two degrees.

Euler's formula: G be a connected planar graph with
v - vertices
e - edges
r - regions

Then $\boxed{v-e+r=2}$

ex: 

$v=4$
$e=5$
$r=3$
$v-e+r = 4-5+3 = 2.$

→  for pseudographs degree 1 & 2 also possible

min degree of region in simple Graph = 3

⇒ Let G be a simple connected graph

$$\sum \deg(r) = 2e$$

replace each degree with min degree

$$\therefore \underbrace{3+3+3+\cdots+3}_{r\ \text{times}} \le 2e$$

→ $3r$ times $\le 2e$

i.e. $\boxed{3r \le 2e}$

⇒ $2 = v - e + r$

$$2 \le v - e + \frac{2e}{3}$$

$$\le v - \frac{e}{3}$$

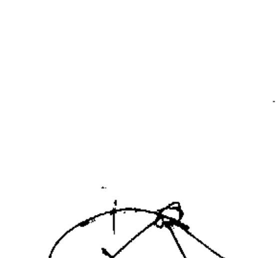
$$2 \le \frac{3v - e}{3}$$

$$6 \le 3v - e.$$

$$\boxed{e \le 3v - 6}$$

⇒ – In a simple connected planar graph with min degree of region = 3.

$$\boxed{\begin{array}{ll} \text{i)} & v - e + r = 2 \\ \text{ii)} & 3r \le 2e \\ \text{iii)} & e \le 3v - 6 \end{array}}$$



case K_5 is Non-planar

$v = 5$, $e = 10$; $r = ?$

So, if it is planar, $v - e + r = 2$

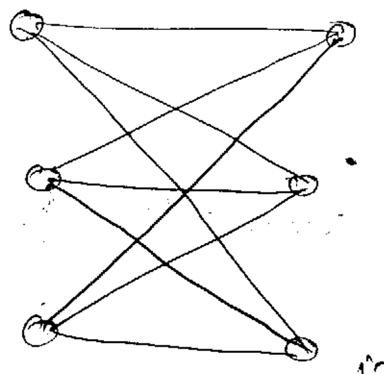
$5 - 10 + r = 2$; $\boxed{r = 2 + 5 = 7.}$

→ $3r \le 2e$

$3 * 7 \le 2 * 10$

$21 \not\le 20$ ✗ – So, K_5 is Non-planar

$\Rightarrow K_{3,3}$

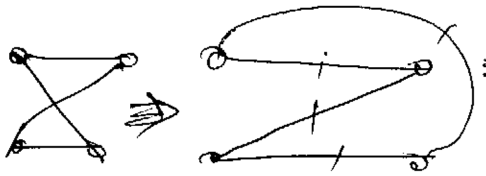


$V = 6.$
$e = 9.$

** ~~$v - e + r = 2$; $\therefore r = 5$ | $3r \le 2e$, $15 \le 18$ | $e \le 3v - 6$, $9 \le 18 - 6$, $9 \le 12$~~

~~But this analysis is~~ invalid. These are valid only if mindegree $(r) = 3$

But in any complete bipartite graph; min degree $(r) = 4$



$\equiv r$

More General formula.

*) — In a simple connected planar graph with minimum degree of region $= K$,

i) $v - e + r = 2 \equiv v - e + r = p + 1$ (# of components)

ii) $Kr \le 2e$.

iii) $e \le \dfrac{K(v-2)}{K-2}$

$\rightarrow K_{3,3} \rightarrow$ In any complete bipartite graph, The minimum degree of region $= 4$

So $K = 4$.

In $K_{3,3}$: $v = 6$; $e = 9$.

i) $\rightarrow$ $v - e + r = 2$.
$6 - 9 + r = 2$.
$r = 5$.

ii) $4r \le 2e$
$20 \nleq 18$ ✗. } So NOT planar

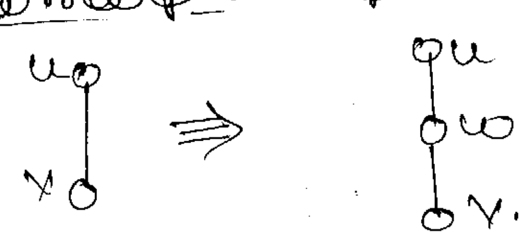
iii) $\rightarrow$ $9 \le \dfrac{4(6-2)}{4-2}$ $9 \le \dfrac{16}{2}$ $9 \nleq 8$

⇒ K_5 and $K_{3,3}$ are called Kuratowski's graphs.

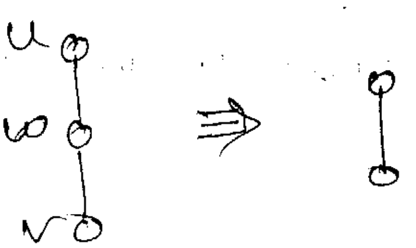
1.) Both are simple graph
2.) ,, ,, non-planar Graph
3.) K_5 is non-planar graph with min # of vert
4.) $K_{3,3}$ is non-planar graph with min # of edges.

So K_n is planar iff $n \leq 4$.
$K_{m,n}$ is planar iff $m \leq 2$ or $n \leq 2$.

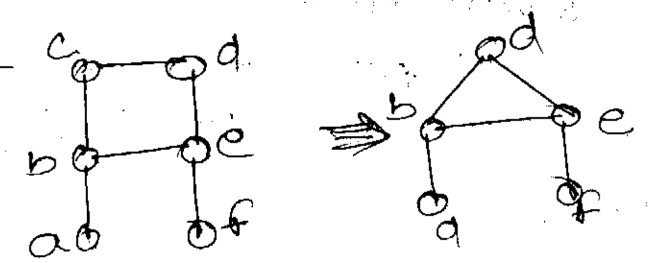
⇒ Homeomorphic Operations

I →: 

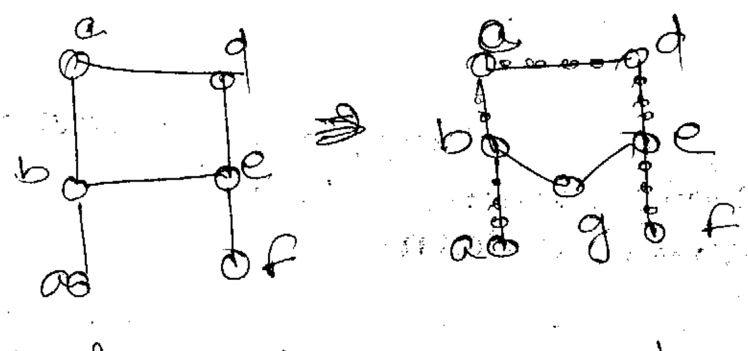
Inserting vertex of degree in series

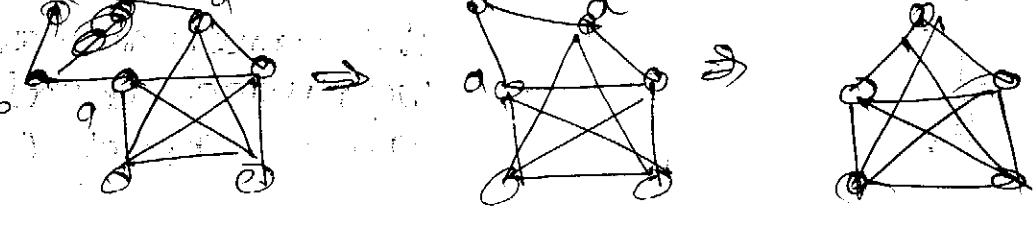
II →: 

merging a 2degree vertex in series.

ex:- 

merging



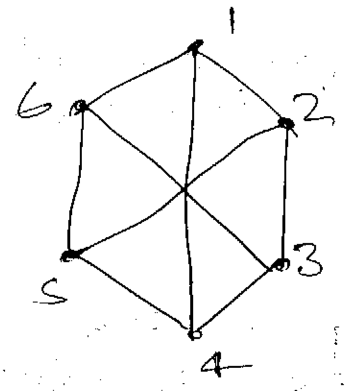
⇒ ex:- 

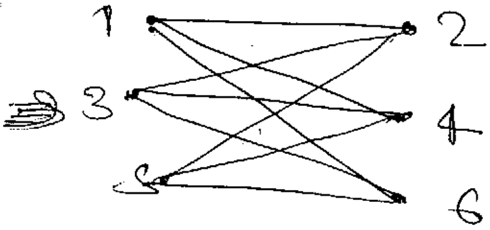
⇒ Kuratowski's Theorem

A connected ~~planar~~ graph is planar iff it is not homeomorphic to K_5 or $K_{3,3}$.

⇒ Ques WB/28/1 pg

planar





isomorphic to $K_{3,3}$ so not planar

15:- $V=20$

degree$(v)=3$ for all $v \in V$.

So sum of degree $= 3\times 20 = 60 = 2e$

So $e=30$

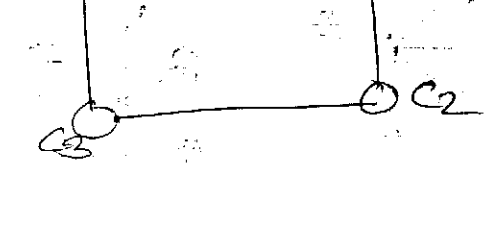
i.e. 30 edges exist

By $v-e+r=2$

$20-30+r=2$

$r=12$

⇒ Graph Colouring: Coloring the vertices of a graph such that no two adjacent vertices have the same colour



⇒ Chromatic Number ($\chi(G)$)

The min no. of colours required to colour the given graph G.

G	$\chi(G)$
(Null) N_n	1
(Cycle) C_n	$\begin{cases} 2 & : \text{if } n \text{ is even.} \\ 3 & : \text{if } n \text{ is odd.} \end{cases}$
(Wheel) W_n	$\begin{cases} 3 & : \text{if } n \text{ is even,} \\ 4 & : \text{if } n \text{ is odd.} \end{cases}$
K_n	n
$K_{m,n}$	2
Any Bipartite	2.

Properties

*). G is a graph with n-vertices
$\chi(G) \leq n$.

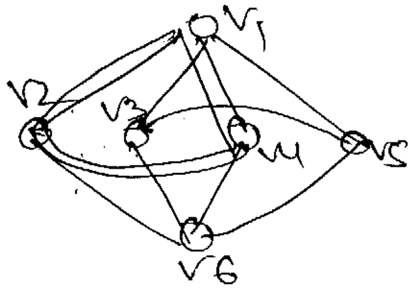
*). K_n is subgraph of graph G.
$\chi(G) \geq n$.

*). $\chi(G) \leq 1+\Delta$.

*). $\chi(G) \geq \frac{|V|}{|V|-\delta}$

*) The following statements are equivalent
(i) → G is bipartite
(ii) → G is 2-colourable.
(iii) → Every cycle in G is even cycle

⇒ Welsh-powell algorithm ⇐



Arrange in non-~~ascending~~ descending order degree

V1	V6	V2	V3	V4	V5
C1	C1	C2	C2	C3	C3.

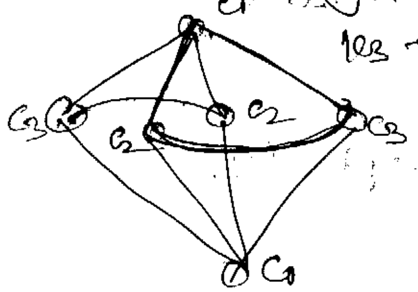
→ So $\boxed{\chi(G) \leq 3}$

→ K_3 is subgraph of G

$\chi(G) \geq 3.$

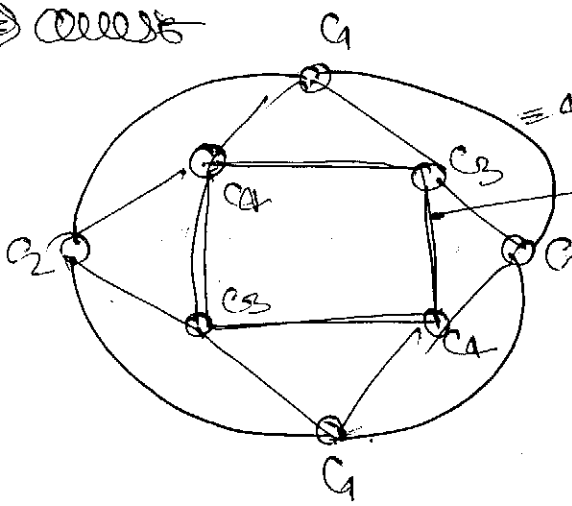
⇒ So $\chi(G) = 3.$

General coloring



K_3 so atleast 3 colors must be used

⇒ eullest

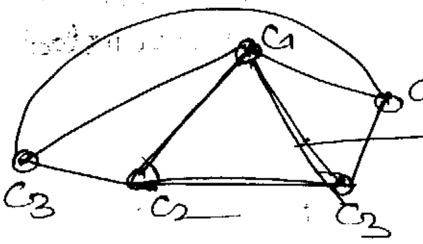


⇒ So 4 colors are needed

$\boxed{\chi(G) = 4}$

⇒ $\boxed{\text{No. of chromatic partitions} = \text{chromatic Number}}$

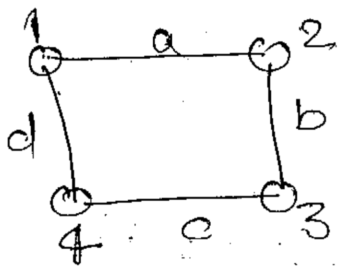
⇒ WB/Pg26/Q-12



$\chi(G)=3$.

⇒ **Matching**: Set of non-adjacent edges.

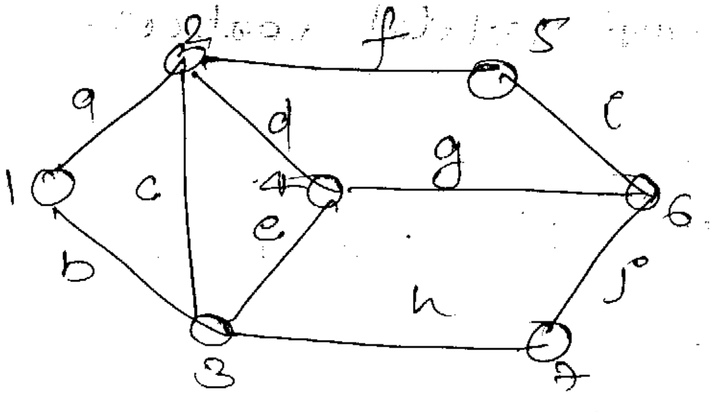
ex:-



$M_1=\{a\}$.
$M_2=\{a,c\}$.
$M_3=\{b,d\}$.

→ Matching No:- $[\alpha'(G)]$
Max No. of Non-adjacent edges.
So for above Graph $\alpha'(G)=2$.

ex:

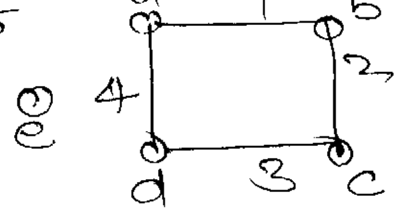


$M_1=$ ~~{...}~~ $\{a, i, h\}$
$M_2=\{f, g, h\}$
$M_3=\{i, e, a\}$.

$\boxed{\alpha'(G)=3}$

Edge covering: Set of edges which can cover all the vertices of positive degree. (i.e. Not isolated vertex).

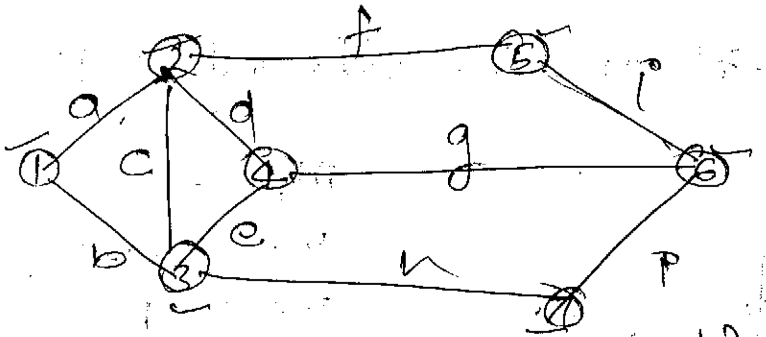
ex:



$E_1 = \{1, 2, 3, 4\}$

$\Rightarrow E_2 = \{1, 3\}$.

eg:-



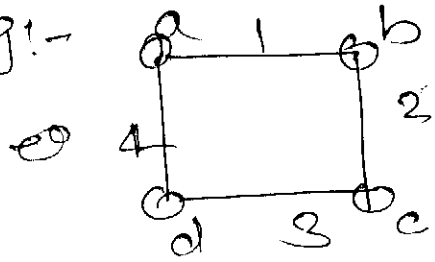
$\boxed{\beta'(G) = 4+0 = 4}$

$E_1 = \{a, b, c, d, e, f, g, h, i\}$

$E_2 = \{a, i, h, e\}$.

Edge covering No:- ($\beta'(G)$) min no. of edges which can cover all the vertices of positive degree + No. of isolated vertices.

eg:-

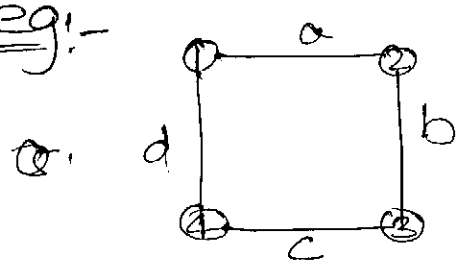


$\beta'(G) = 2+1 = 3$

For any Graph $\alpha'(G) + \beta'(G) = n$.

⇒ Independent Set: Set of non-adjacent vertices is called independent set.

eg:-

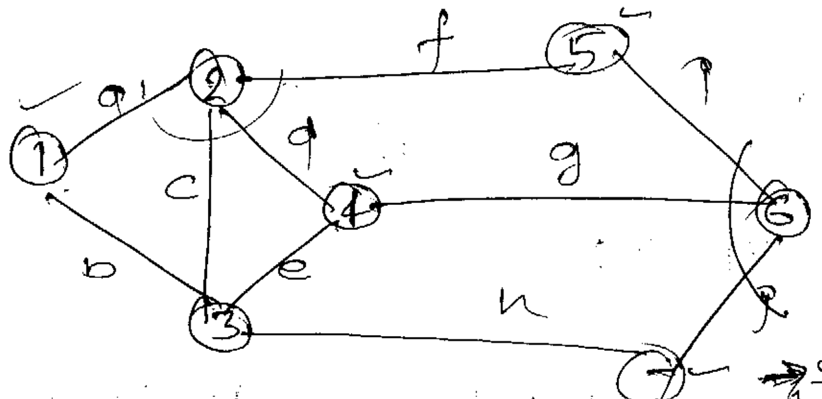


$I_1 = \{1, 5, 3\}$
$I_2 = \{5\}$
$I_3 = \{5, 2, 4\}$.

→ Independence No: $\alpha(G)$.
max. No. of nonadjacent vertices

for above graph $\alpha(G) = 3$.

eg:



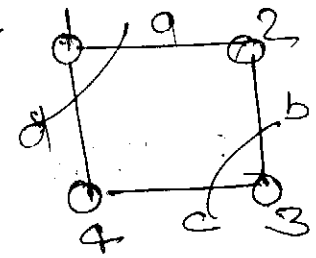
$I_1 = \{1, 4, 5, 7\}$
$I_2 = \{2, 6\}$

→ See for vertex with least degree & then greater & so on. This will give $\alpha(G)$

$\alpha(G) = 4$.

⇒ Vertex Cover: The set of vertices which can cover all the edges.

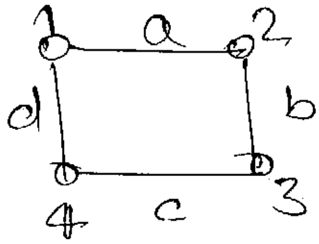
ex:-



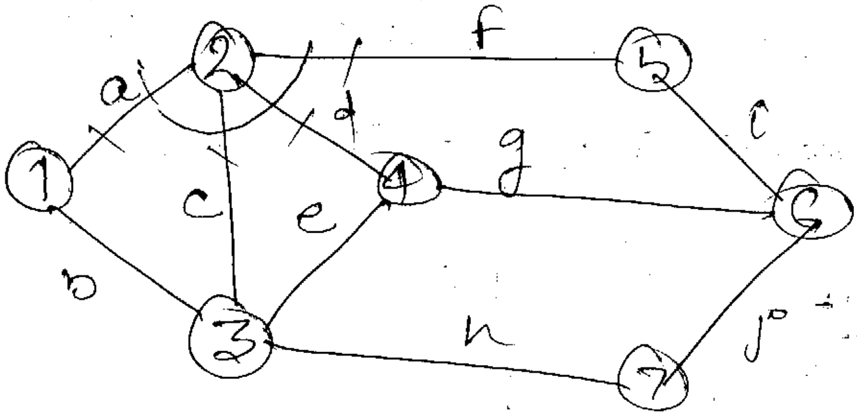
$V_1 = \{1, 2, 3, 4\}$
$V_2 = \{1, 3\}$
$V_3 = \{2, 4\}$.

Vertex Covering Number $[\beta(G)]$

The min^m number of vertices which can cover all the edges

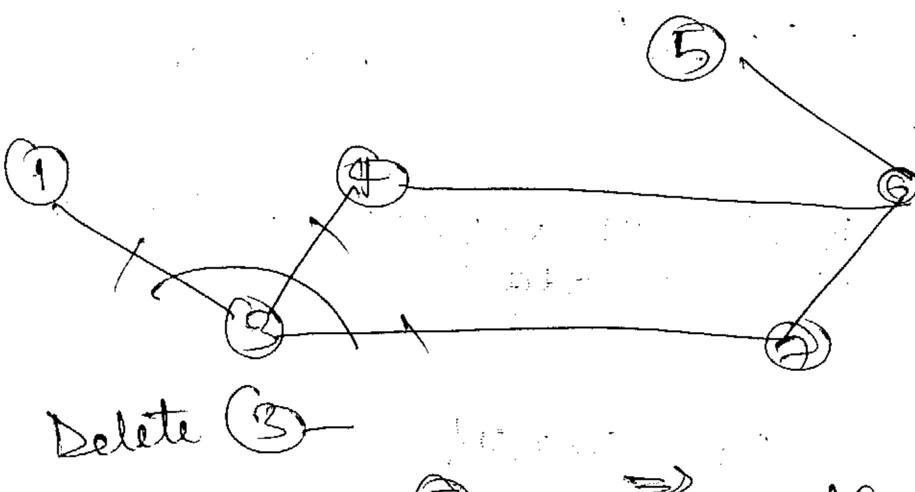


$\beta(G) = 2$



$V_1 = \{2\}$

Select vertex with highest degree – add it to V_1
Then delete it



⇒ Next highest degree vertex = 3
add it to V_1
$V_1 = \{2, 3\}$.

Delete (3) –

⇒ $V_1 = \{2, 3, 6\}$

$\boxed{\text{So } \beta(G) = 3}$.

** In any graph G with n-vertices

$$\boxed{\alpha(G)+\beta(G)=n}$$

⇒ Directed Graphs

$G=(V,E)$

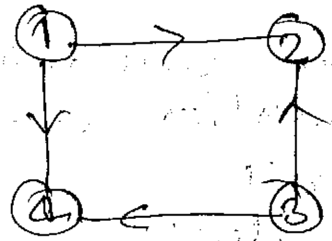
V – vertex set $=\{v_1, v_2, \cdots v_n\}$

E – edge set $=\{e_1, e_2 \cdots e_m\}$

where each $e_k \in E$ is

$e_k=(v_i, v_j)$

is called a directed graph (or) digraph.

⇒ 

→ in degree – No. of edges incident in.

→ out degree – No. of edges incident out.

V	in	out
1	0	2
2	2	0
3	0	2
4	2	0
Sum =	4	4

= No. of edges

Result

** In any digraph $G=(V,E)$

The sum of indegrees

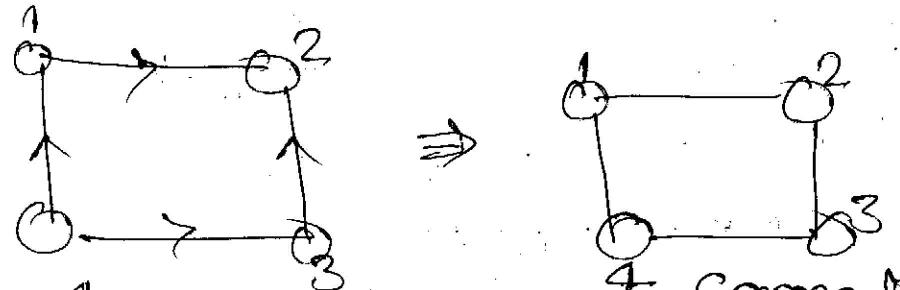
= Sum of out degrees

$=|E|=$ No. of edges in graph.

⇒ Two vertices are connected in a digraph iff there exists at least one directed path b/w them.

→ A digraph is weakly connected if the underlying undirected graph is connected.

ex:-

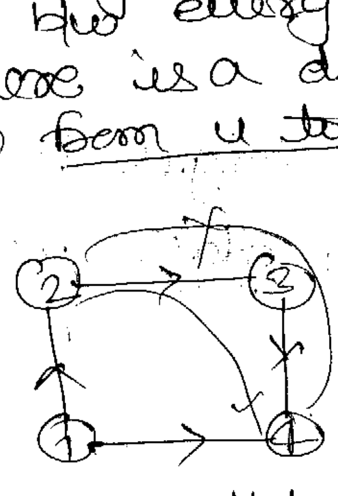


4 Connected

So it is weakly connected.

→ The digraph is ~~unilat~~ unilaterally connected if b/w every two vertices u & v. There is a directed path either from u to v (or) v to u

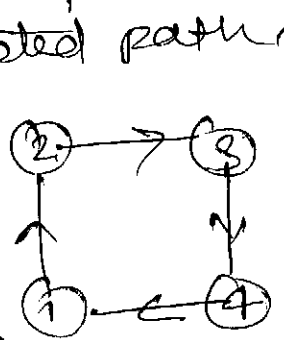
ex:-



Unilaterally connected

→ The digraph is strong connected if b/w every pair of vertices u & v, there is a directed path from u to v and v to u

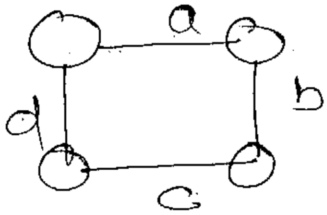
ex:-



⇒ Strongly connected ⇒ Unilaterally connected ⇒ weakly connected.

⇒ **Perfect Matching:** A matching which can cover all the vertices of a graph

eg:



$M_1 = \{a, c\}$ → can covers all vertices.
∴ Perfect matching.

⇒ A graph G has perfect matching only if $\underbrace{\text{G has even number of vertices.}}_{\text{Necessary cond}^n}$

⇒ No. of perfect matchings in K_{2n}

$$= \frac{(2n)!}{2^n (n!)}$$

Ques: Find No of perfect matchings in K_6

$n = 3$

$$= \frac{6!}{2^3 * 3!} = \frac{6*5*4*3!}{2*2*2 \; 3!} = 15.$$

Ques: No. of perfect matching in $K_{n,n} = n!$