Three Approaches to Keller's Conjecture

15-300, Fall 2021

Shannon Ding

October 27, 2021

1 Introduction

If you've ever covered a Scrabble board with tiles or attempted to form a shape out of a toy set of pentominoes, you've participated in a tiling problem. Tiling is a geometric technique in which a space is covered with shapes. There have been numerous interesting tiling problems proposed that restrict the types of shapes allowed and consider spaces in high dimensions. One such tiling variant asks for a n-dimensional space to be completely covered by identical n-dimensional cubes. In 1930, the German mathematician Eduard Ott-Heinrich Keller claimed any valid tiling under these constraints must have two cubes that completely share an n-1 dimension face. This claim is known as Keller's Conjecture and the proof has been an open problem until it was resolved in 2019. [1]

To get an initial understanding, consider the problem in dimension 2. Keller's Conjecture in n=2 asks if a flat plane can be completely covered by identical squares such that no two squares fully share an edge. It is fine for two squares to partially share an edge, but they cannot be perfectly matched up. Keller's conjecture in dimension 2 is true: to avoid the overhang of some square in the plane causing the space to be incompletely covered, there must exist two squares that fully share an edge.

Although Keller's Conjecture is ultimately an interesting math problem with no direct applications, the techniques used to solve the conjecture have wide applicability. For example, efficiently searching for cliques in a graph has applications in developing robust computer networks. The clever use of SAT solvers can be applied to optimize situations under a set of constraints. More directly, Keller's conjecture contributes to understanding certain types of error correcting codes that can make transmitting data faster and more accurate. [1]

The main hurdle that kept Keller's Conjecture an open problem was reasoning about the infinite nature of a tiling and needing to prove or disprove the statement for all values of n. Past dimension 3, it becomes difficult to think of the problem visually. Instead, researchers had to translate this geometric problem into different mathematical fields, namely graph theory, geometry, and propositional logic. We will examine the work of three different researchers: Corradi and Szabo, Kisielewicz and Lysakowska, and Brakensiek, Heule, Mackey, and Narvaez who reduced the tiling problems to each field respectively.

Corradi and Szabo noticed that the infinite, but repetitive structure of a tiling could be described by a finite set of vertices. Then, the relationship that two cubes in the tiling are adjacent can be described by a set of edges. They translated a tiling into a graph and thus could use graph theory to reason about the nature of the tiling. [2]

Kisielewicz and Lysakowska explored various geometric techniques to better understand the general nature of a tiling in the seventh dimension. [3]

Brakensiek, Heule, Mackey, and Narvaez began with a graph describing a tiling in dimension 7 and creatively molded the graph into a propositional formula. They proceeded use a SAT solver to determine a statement about the formula and thus about the tiling. [1]

2 Infinite tiling to finite Keller graph

2.1 Approach to solving the problem

Corradi and Szabo built off earlier work by Hajos [2], who reduced the problem to constructing a special group that satisfies a factorization property. A group is a set of elements along with an operation. Groups have certain circular properties, such as closure of the group operation, that make the mathematical construct of groups suitable to represent the repetitive nature of a tiling. Hajos constructed a group of finite size to represent a tiling covering an infinite space.

Hajos concluded that the conjecture holds if the group contains a certain subgroup, which is a subset of elements from the original group. Subgroups have the unique property of being ”self contained” so Hajos proved the existence of the subgroup meant there was a way to arrange individual cubes into a larger ”self contained” super cube structure. This super cube structure can easily tile the entire space without sharing any faces. Therefore, the existence of the subgroup means there is a tiling without face sharing and Keller's conjecture is false by counterexample.

Corradi and Szabo felt Hajos' result was valuable, but could be best utilized if applied to a field outside group theory. They also aimed to make the reduction easier to work with and also more dimension specific.

In their paper, they define a Keller's graph which consists of a mapping from the elements of the group to vertices in a graph and an edge if the elements differ in a specific way. More specifically, the vertices are all distinct vectors of length n such that the value at each index is within $\{0, 1, 2, 3\}$. For Keller's Conjecture in dimension n, this graph has 4^n vertices. We draw an edge between two vertices a and b if there exists some index j such that $a_j - b_j = 2 \mod 4$. They claim the process of finding Hajos' subgroup has become finding a 2^n sized clique in the Keller graph. A clique is a set of vertices in the graph who have edges to all other vertices in the group as well. The verticies describe the elements in the subgroup and since they are all adjacent to each other, the subgroup possesses the desired factorization property.

2.2 Summary of results

The authors conclude that proving a counterexample for Keller's conjecture for dimension n can be done by finding a 2^n sized clique in a Keller's graph, because this clique maps back to Hajos' special subgroup in the group. They provided a proof that analyzed the mapping so that the clique corresponds to the subgroup.

They also proved if a counterexample is found for n, it indicates a counterexample in all dimensions greater than n. As a result, Keller's conjecture is either true for all dimensions, or true up to a threshold dimension.

The authors creatively molded Hajos' reduction of Keller's Conjecture to group theory to a graph theory problem. The mapping is very simple and elegant and is a creative application of one theory subfield to another. The conclusions the paper drew contributes greatly to the research surrounding Keller's Conjecture, because it sets up a much easier framework under which to search for counterexamples.

The paper is widely cited by all future work around Keller's Conjecture, because it's the most straightforward way to find a counterexample. It has been used to prove Keller's Conjecture doesn't hold for $n \geq 8$ dimensions, which was a huge step in narrowing down the dimensions to prove Keller's Conjecture. In general, it has become a foundation for understanding this conjecture.

3 Deep dive into the geometry of a tiling in dimension 7

3.1 Approach to solving the problem

Kisielewicz and Lysakowska [3] honed in on Keller's Conjecture in dimension 7 in particular, because at that time it was the only dimension still unproven. The authors define two metrics, $r^-(T)$ and $r^+(T)$, which are related to the minimum and maximum number of adjacent cubes in tiling T over the n-dimensional space in a certain configuration. More specifically, the metric $L(T, x, i)$ = number of cubes in the tiling T that intersect the cube centered at coordinate x while being below the cube on the ith axis. They define:

$$r^-(T) = \min_{x \in R^d} \max_{1 \leq i \leq d} |L(T, x, i)|$$

$$r^+(T) = \max_{x \in R^d} \max_{1 \leq i \leq d} |L(T, x, i)|$$

We begin with the case we want to learn more about, $r^+(T) = 5$ and we examine the geometry to determine what specific kinds of tilings T can be. First, any T where $r^+(T) = 5$ can be transformed into a polycube, defined as a space formed by translating a unit cube inside a flat torus, such that the polycube has contains at least 2 valid tilings following several conditions. They reduce the possible polycubes to a set of smaller, simpler polycubes. Then, they show how we can find two cubes that completely share a face in the tiling the polycube originated from.

The authors use numerous techniques in the proof to reduce and compare polycubes: translating polycubes into polybox codes (a string from a special alphabet), geometrically slicing and casting shadows from the boxes, and comparing similarities of polybox codes through graphs. This chain of logic lets us connect the $r^-(T)$ and $r^+(T)$ metrics to important properties in Keller graphs.

3.2 Summary of results

They prove that Keller's Conjecture is also true for dimension 7 for tilings T such that $r^+(T) = 5$. Previously, it was only known that Keller's Conjecture is true for dimension 7 for T where $r^-(T) \leq 2$ or $r^+(T) \geq 6$. This discovery indicates we'll only need to check tilings where $r^+(T) < 5$ and $r^-(T) > 2$ for a counterexample, which reduces the search space.

The paper made large strides in understanding tiling in dimension 7. Previously, there were only several ways to prove Keller's Conjecture true in dimension 7, but the results from this paper gave several more methods, including providing Keller graphs for dimension 7 that are simpler and many orders of magnitude smaller. Future researchers directly used these simpler graphs to successfully prove the conjecture doesn't hold in dimension 7.

The paper also discussed the general behavior of tilings as the dimension increased and thus provided interesting insight into the underlying structure of any dimension tilings. For example, they discovered tiling algorithms can b used to partition a cube and find matchings on graphs constructed from d dimensional cubes.

4 Searching through tilings with computers

4.1 Approach to solving the problem

Overall, this paper by Brakensiek, Heule, Mackey, and Narvaez [1] proves the conjecture for dimension $n = 7$ by transforming the problem into a boolean satisfiability formula. They begin with Keller graph for dimension 7 and they encode the statement "there exists a clique of size 2^7 in the Keller graph" as a boolean formula.

In particular, the authors used the Keller graphs $G_{7,3}, G_{7,4}, G_{7,6}$ and hoped to find a clique in any one of them. The Keller graph $G_{n,s}$ is defined as a graph with $(2s)^n$ vertices. Each vertex is a vector of length n and each index can take on $2s$ values.

To encode the statement about the clique, the authors described the nature of cliques and Keller graphs propositionally. They partitioned the vertices into 2^n independent sets and stated that each independent set must contain exactly one vertex that is in the clique. Then, they added clauses stipulating the format of Keller graphs. For example, every pair of vertices must differ at some index and two vertices are considered connected if their vertices differ in a certain way.

This results in a formula that is very unwieldy and impossible to efficiently compute formula, because there are so many possible configurations of the graph to search through. Therefore, the bulk of their paper is applying techniques to drastically reduce the formula.

They started with general techniques that make solving general SAT formulas faster: removing any clauses that are always true/false and cleaning up redundant parts of clauses. In addition, if they identified characteristics that immediately let them rule out a vertex being in the clique, they stopped the computation and moved on to the next case. Another technique is taking advantage of the symmetry of the problem to avoid recomputing some values. Numerous cases share the same subproblems and therefore can directly look up the previously computed answer. Symmetry breaking was the most efficient at reducing the cases to search through.

Finally, the formula was sufficiently reduced. They used a computer cluster to run a SAT solver on the formula and received an answer after just half an hour.

4.2 Summary of results

The authors found a satisfying assignment to their SAT formula, which means there exists a counterexample to Keller's conjecture. Thus, Keller's Conjecture is false in dimension 7.

The authors have resolved Keller's Conjecture by contributing the answer to the last unknown dimension. The authors utilized new automated techniques to reduce the SAT formula rather than identifying ways to simplify/reduce clauses by hand. In addition, the paper uniquely relied on a computer to iterate through billions of cases to both reach the answer and generate a proof. Usually, theory research prefers to reason out an answer purely through math and hand checking. The paper received general, "popular science" fame for using a computer to generate a proof on such a large problem. It creates a thought provoking question about the validity of proofs where even the authors don't know the underlying reason why it works. Computers have great potential to automate mathematical problem solving and the effective application of a SAT solver in this paper may encourage future researchers in various subfields to apply it to their work.

5 Comparison of the Three Papers

All papers use the notion of a Keller graph in order to better understand the problem. This cements the Keller graph as the most convenient way to prove Keller's conjecture. They differ in their next steps after considering the Keller graph, because each chose to apply the problem to a different mathematical subfield. Usually, they chose their subfield because the authors had backgrounds in it. Although Corradi and Szabo motivated their choice of graph theory well by providing a very close mapping of the problem, it was not immediately apparent why the other authors decided on their subfields. After reading the paper, the choice of subfield makes more sense, but their initial proofs have weaker direction as the reader attempts to understand the purpose behind each transformation of Keller graphs to the subfield.

The authors also had different aims. Corradi and Szabo seemed to want to understand the overarching structure of the problem, rather than solve the conjecture for a specific dimension. As a result, their conclusion was more concise and plainly informative. The other two papers specifically wanted to prove the conjecture for a dimension. Their conclusions emphasized the importance of their findings more and suggested ideas for future work.

Corradi and Szabo's paper was comparatively much shorter. They only had one primary proof section, while the other two papers went through a multistage process to reach their conclusion. Each stage used a distinct technique (for instance, translating the graph into a formula, reducing the formula using three techniques, etc.), proved a set of lemmas, and ended with the overarching theorem at the end. In particular, Kisielewicz and Lysakowska's paper stepped through so many stages and different techniques that they ended up splitting their findings into 27 theorems. Brakensiek, Heule, Mackey, and Narvaez seemed to have acheived a happy medium in terms of paper length and organization. The paper started with a reasonable length roadmap of the proof and split each step into its own section with meaningful titles.

Each paper had a background section where they briefly reviewed concepts in the mathematical subfield. Kisielewicz and Lysakowska went a step forward and defined their own terminology and concepts. Their paper was a bit more distanced from existing work and thus they had to provide further explanation. The other two papers could directly cite results from other papers, while this paper had to write a full mathematical proof.

6 Conclusions

These three papers demonstrate the general structure to prove Keller's conjecture. First, they begin with a dimension, its corresponding Keller graph, and a goal of finding a clique in the graph. Unfortunately, the difficulty of finding a clique grows exponentially with dimension. As researchers began working on higher dimensions, they had to find creative and efficient ways to identify this clique. Often, they translate the Keller graph into some other structure like a polycube or propositional formula. Within this new mathematical structure, they have access to more techniques to analyze the Keller graph. Much of a paper will be devoted to this translation and subsequent analysis. If the conjecture is false, they will find the desired clique. Otherwise, they find a valid tiling without face sharing.

Tiling may be simply a fun, thought provoking math problem, but research in this area has led to contributions in various math subfields. In particular, some major lessons learned from tiling was the importance of reducing an infinite size problem (tiling an infinite space) into a finite problem (finding a clique in a graph). Then, researchers could further reduce this finite problem by finding smaller representative graphs. Finally, they can break the problem into a reasonably sized number of cases and individually check each one to reach a conclusion.

This trajectory has led to Keller's conjecture being solved as of Brakensiek, Heule, Mackey, and Narvaez's 2019 paper. Perhaps other open mathematical problems may follow in its steps.

References

[1] Mackey Brakensiek Heule and Narvaez. The resolution of keller's conjecture. *IJCAR 2020*.

[2] Corradi and Szabo. A combinatorial approach for keller's conjecture. *Periodica Mathematica Hungarica*, 21(2):95–100, 1990.

[3] Kisielewicz and Lysakowska. On keller's conjecture in dimension seven. *Electronic Journal of Combinatorics*, 22(1):1–16, January 2015.