National Taiwan University
Department of Electrical Engineering
Algorithms, Fall 2019

Handout #34
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Sample Solutions to Quiz #5

1. (15) The Edmonds-Karp algorithm.

   The corresponding residual networks are shown in Figure 1 step by step as follows.

   (1) Figure 1(a) shows the first augmenting path $s \to c \to t$ and the residual capacity is 3.

   (2) Figure 1(b) shows the second augmenting path $s \to a \to d \to t$ and the residual capacity is 3.

   (3) Figure 1(c) shows the third augmenting path $s \to b \to d \to t$ and the residual capacity is 1.

   (4) Figure 1(d) shows the final residual network and the maximum flow is $3 + 3 + 1 = 7$.

   Figure 1(e) shows a cut with capacity $3 + 3 + 1 = 7$. By the max-flow min-cut theorem, the capacity of the cut is 7 which is an upper bound of flow. Thus, the maximum flow is 7.

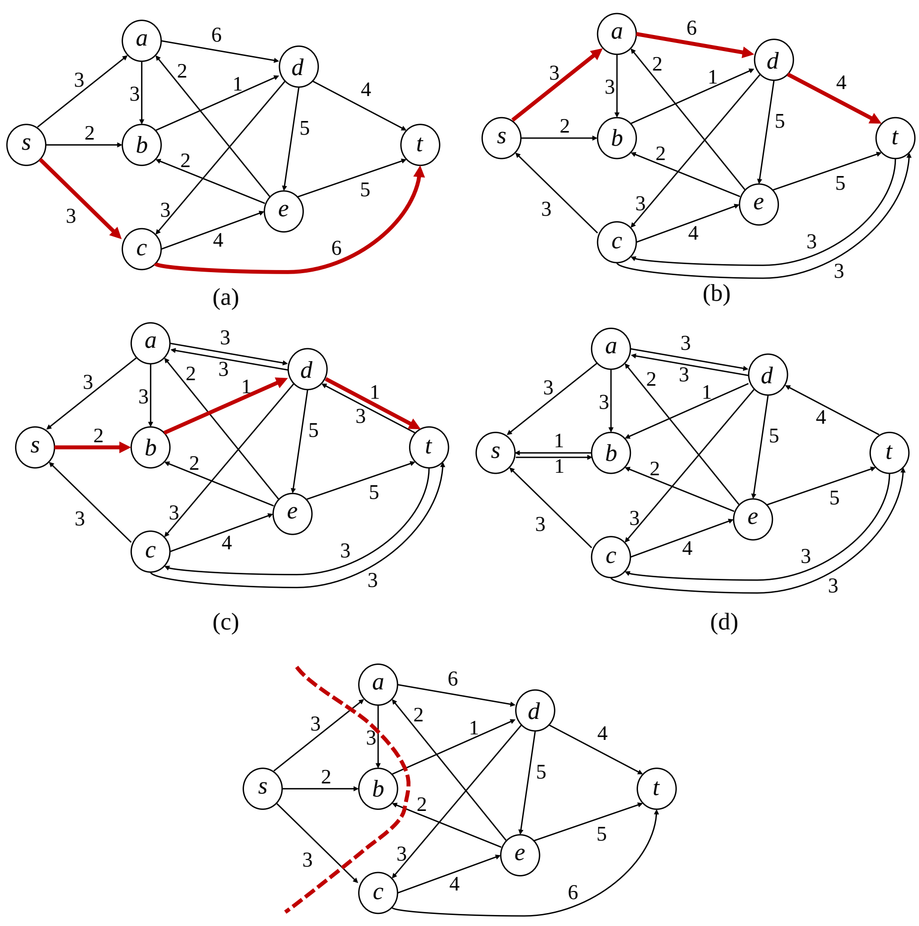

Figure 1: (a) The augmenting path with residual capacity 3. (b) The augmenting path with residual capacity 3. (c) The augmenting path with residual capacity 1. (d) Final residual network. (e) A cut with capacity 7.

2. (10)

   - S2: Only some particular instances of HP can be reduced to a shortest path problem.
   - S2: The transformation (reduction) does not take polynomial time.

3. (15)

   We can check a given tour (certificate) by checking (1) if the tour visits every city exactly once, (2) if the tour returns to the start, and (3) if the total distance $\leq B$. The verification algorithm takes $O(n)$ time. Therefore, TSP $\in$ NP.

   To prove that TSP is NP-hard, we first define a reduction function f for HC $\leq_p$ TSP. Given an HC instance $G = (V, E)$ with n vertices, create a set of n cities labeled with names in V. Assign distance between u and v as follows:

   $$d(u, v) = \begin{cases} 1 & \text{, if } (u, v) \in E \\ 2 & \text{, if } (u, v) \notin E \end{cases}$$

   Set bound $B = n$. The reduction function f takes $O(V^2)$ time, which is polynomial-time.

   Next, we claim that G has an HC *iff* the reduced instance has a TSP with distance $\leq B$.

   - $x \in$ HC $\Rightarrow f(x) \in$ TSP:

     Suppose the HC is $h =< v_1, v_2, \ldots, v_n, v_1 >$. Then, h is also a tour in the transformed TSP instance. The distance of the tour h is $n = B$ since there are n consecutive edges in E, and so has distance 1 in $f(x)$. Thus, $f(x) \in$ TSP ($f(x)$ has a TSP tour with distance $\leq B$).

   - $f(x) \in$ TSP $\Rightarrow x \in$ HC:

     Suppose there is a TSP tour with distance $\leq n = B$. Let it be $< v_1, v_2, \ldots, v_n, v_1 >$. Since the distance of the tour $\leq n$ and there are n edges in the TSP tour, the tour contains only edges in E since all edge weights are equal to 1. Thus, $< v_1, v_2, \ldots, v_n, v_1 >$ is an HC ($x \in$ HC).

   So, we prove that TSP is NP-complete.