Shannon Dow

AMath 585 Homework #2 Due Thursday, January 23, 2019

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5223966`

Problem 1.

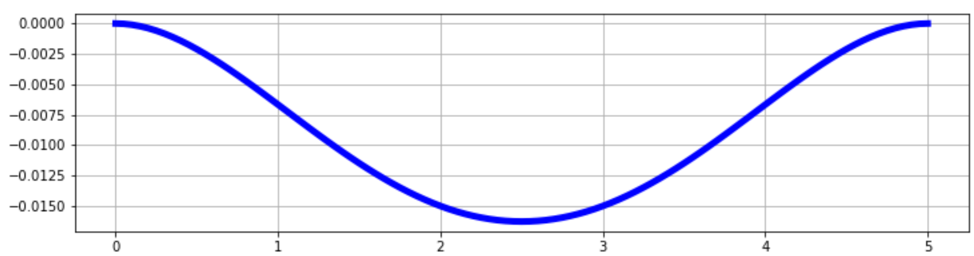
Consider a "rigid" beam of length L that is supported at both ends and sags by a very small amount in the center due to gravity acting on the mass. The small deflection can be modeled by the Euler-Bernoulli beam equation described for example at
`https://en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory`,
which in the simplest case where the beam has constant cross-sectional area and is made of a uniform material takes the simple form

$$u''''(x) = \gamma, \qquad \text{for } 0 \le x \le L,$$

where γ is a constant depending on the material properties. If both ends are embedded in walls that hold their position constant (at $u = 0$, say) and also hold them horizontal at the ends, then the boundary conditions are

$$u(0) = u'(0) = 0, \qquad u(L) = u'(L) = 0.$$

The deflection then looks something like this (with a greatly exagerated vertical scale):



(a) The exact solution to this problem is easy to compute as a quartic function that satisfies the four boundary conditions. Compute this for $L = 5$ and $\gamma = 0.01$, and confirm that it looks like the figure above.

(b) Write a computer program to solve this problem. Use the second-order accurate formula for the fourth derivative that you derived in Homework 1, together with formulas for boundary conditions that preserve the second order accuracy. There is more than one way to do this that would be correct.

Test your program for a series of grids and produce a log-log plot to verify the expected accuracy.

Solution:

(a)

$$u''''(x) = \gamma$$

$$u'''(x) = \gamma x + c_1$$

$$u''(x) = \frac{1}{2}\gamma x^2 + c_1 x + c_2$$

$$u'(x) = \frac{1}{6}\gamma x^3 + \frac{1}{2}c_1 x^2 + c_2 x + c_3$$

$$u(x) = \frac{1}{24}\gamma x^4 + \frac{1}{6}c_1 x^3 + \frac{1}{2}c_2 x^2 + c_3 x + c_4$$

Implementing the first boundary conditions:

$$u(0) = c_4 = 0$$

$$u'(0) = c_3 = 0$$

Implementing the second boundary conditions at L:

$$u(L) = \frac{1}{24}\gamma L^4 + \frac{1}{6}c_1 L^3 + \frac{1}{2}c_2 L^2 = 0$$

$$u'(L) = \frac{1}{6}\gamma L^3 + \frac{1}{2}c_1 L^2 + c_2 L = 0$$

Dividing the $u'(L)$ equation by L and solving it for c_2 we get:

$$c_2 = -\frac{1}{6}\gamma L^2 - \frac{1}{2}c_1 L$$

Substituting c_2 into the first equation and dividing by L^2we get:

$$u(L) = \frac{1}{24}\gamma L^2 + \frac{1}{6}c_1 L + \frac{1}{2}[-\frac{1}{6}\gamma L^2 - \frac{1}{2}c_1 L] = 0$$

$$u(L) = \frac{1}{24}\gamma L + \frac{1}{6}c_1 - \frac{1}{12}\gamma L - \frac{1}{4}c_1] = 0$$

$$u(L) = c_1[\frac{1}{6} - \frac{1}{4}] = \frac{1}{12}\gamma L - \frac{1}{24}\gamma L$$

$$u(L) = c_1[-\frac{1}{12}] = \frac{1}{24}\gamma L$$

$$c_1 = -\frac{1}{2}\gamma L$$

$$c_2 = \frac{1}{12}\gamma L$$

$$u(x) = \frac{1}{24}[\gamma x^4 - 2\gamma L x^3 + \gamma L^2 x^2]$$

Plugging in $\gamma = -0.01$ and$L = 5$ we get: $c_1 = \frac{1}{40}$ and $c_2 = \frac{1}{240}$and

$$u(x) = \frac{-1}{2400}x^4 + \frac{1}{240}Lx^3 - \frac{1}{96}x^2$$

In the Jupyter Notebook, I verify that when you plug in $\gamma = -0.01$ and $L = 5$, it does look like the figure above.

(b) From Homework 1, this is the 2nd order accurate 4th derivative approximation:

$$u^{(4)}(\bar{x}) = \frac{1}{h^4}u(\bar{x} - 2h) - \frac{4}{h^4}u(\bar{x} - h) + \frac{6}{h^4}u(\bar{x}) - \frac{4}{h^4}u(\bar{x} + h) + \frac{1}{h^4}u(\bar{x} + 2h)$$

Using the one-sided, second order accurate approximations for the first derivative:

$$u'(0) = \frac{-3}{2h}u(0) + \frac{2}{h}u(x_1) - \frac{1}{2h}u(x_2) = 0$$

$$u'(5) = \frac{3}{2h}u(5) - \frac{2}{h}u(x_m) + \frac{1}{2h}u(x_{m-1}) = 0$$

$$f(x) = -0.01$$

Additionally: $u(0) = 0$ and $u(5) = 0$

From this we obtain the system:

$$
\frac{1}{h^4}\begin{pmatrix}
2h^3 & \frac{-h^3}{2} & & & & & \\
-4 & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 1 & -4 & 6 & -4 & 1 & \\
 & & 1 & -4 & 6 & -4 & \\
 & & & & \frac{h^3}{2} & -2h^3 &
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-2} \\ u_{m-1} \\ u_m \end{pmatrix}
=
\begin{pmatrix} 0 \\ -0.01 \\ -0.01 \\ \vdots \\ -0.01 \\ -0.01 \\ 0 \end{pmatrix}
$$

Problem 2. The problem above doesn't fully test whether the boundary conditions are implemented properly since the values specified are all zero.

To test your code a bit more, adapt it to solve the problem $u''''(x) = f(x)$ on the interval $0 \leq x \leq 1$ with the function $f(x)$ and boundary conditions on $u(0)$, $u'(0)$, $u(1)$, and $u'(1)$ chosen so that the true solution is $u(x) = 2 + 3x + x^5$. (This is the method of manufactured solutions as discussed in the notebook `BVP1.ipynb`.)

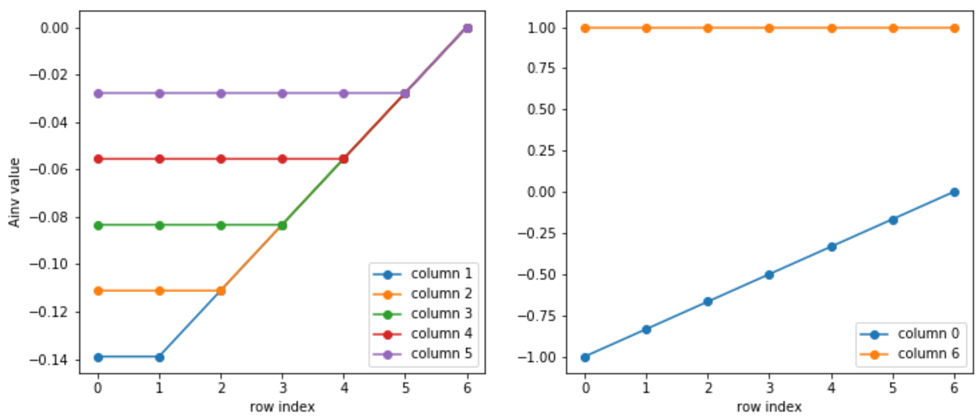
Solution:

Using the method of manufactured solutions and implementing the boundary conditions: $u(0) = 2$, $u(1) = 6$, $u'(0) = 3$,$u'(1) = 8$ To find f(x), take the 4th derivative to obtain: $f(x) = 120x$ Then, using the same code as was utilized in 1(b), we will solve the system:

$$
\frac{1}{h^4}\begin{pmatrix}
2h^3 & \frac{-h^3}{2} & & & & & \\
-4 & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 1 & -4 & 6 & -4 & 1 & \\
 & & 1 & -4 & 6 & -4 & \\
 & & & & \frac{h^3}{2} & -2h^3 &
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-2} \\ u_{m-1} \\ u_m \end{pmatrix}
=
\begin{pmatrix} 6 + \frac{3(2)}{2h} \\ f(x_2) - \frac{1}{h^4} \\ f(x_3) \\ \vdots \\ f(x_m) \\ f(x_{m-1}) - \frac{6}{h^4} \\ 8 - \frac{3(3)}{2h} \end{pmatrix}
$$

Problem 3. The notebook `BVP_stability.ipynb` (visible as a rendered webpage here) shows plots of the columns of A^{-1} that correspond to the discussion of Section 2.11 and Figure 2.1 in the book. Also shown are similar plots for the case of Neumann boundary conditions at the left boundary.

The plot below shows the columns in the simpler case where the matrix from (2.54) is used for the Neumann boundary condition, for the case $m = 5$.



(a) Explain why each of these has the form it does, and give a closed form expression for all elements of A^{-1} in the general case with m interior points (similar to (2.46) for the Dirichlet case).

(b) Using this formula, obtain an upper bound on $\|A^{-1}\|_\infty$ that is independent of h in order to prove stability of this method.

(c) Determine the Green's function for the problem with the Neumann condition at the left boundary, i.e. the function $G(x; \bar{x})$ that solves

$$u''(x) = \delta(x - \bar{x}), \qquad \text{for } 0 \le x \le 1,$$

with boundary conditions

$$u'(0) = 0, u(1) = 0.$$

Solution:

(a) Similar to the example done in class, we need a formula for A^{-1} to get an upper bound on the norm so we can bound the error and determine stability. To do this, set $V = A^{-1}$ meaning $AV = I$

The matrix A (mxm) has Neumann conditions on the left boundary and Dirichlet on the right: (2.54) h = 1/(m+1)

$$\frac{1}{h^2}\begin{pmatrix} -h & -h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 0 & h^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-2} \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \sigma \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ f(x_{mm}) \\ \beta \end{pmatrix}$$

To find the columns of V, we want to solve A*(a column of V) = (a column of I) To begin, find the v values:

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{m-2} \\ v_{m-1} \\ v_{m+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, this must be the first column of A^{-1}

$$\begin{pmatrix} -1 \\ -h(m) \\ -h(m-1) \\ -h(m-2) \\ \vdots \\ -h \\ 0 \end{pmatrix}$$

Next, find the last column of A^{-1}

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{m-2} \\ v_{m-1} \\ v_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0\vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, this must be the last column of A^{-1}

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Continuing in this manner, we can obtain the internal vectors of A^{-1} . To get the j_{th} column of A^{-1} , we want to set $AV = e_j$

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{m-2} \\ v_{m-1} \\ v_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0\vdots \\ 0 \end{pmatrix}$$

where the 1 is in jth position. This results in the internal arrays of V:

$$\begin{pmatrix} -h^2 m \\ -h^2 m \\ -h^2(m-1) \\ -h^2(m-2) \\ \vdots \\ -h^2 \\ 0 \end{pmatrix}, \begin{pmatrix} -h^2(m-1) \\ -h^2(m-1) \\ -h^2(m-1) \\ -h^2(m-2) \\ \vdots \\ -h^2 \\ 0 \end{pmatrix}, \dots \begin{pmatrix} -h^2 \\ -h^2 \\ -h^2 \\ -h^2 \\ \vdots \\ -h^2 \\ 0 \end{pmatrix}$$

These pictures make sense, since you are starting with the Neumann conditions on the left, you satisfy the derivative being equal to zero. Then you eventually jump to the Dirichlet conditions on the right hand side, where the u has to be equal to zero.

The closed form expression of all the elements of A^{-1}

$$V_{i,j} = \begin{cases} -h(m+1-i) & j = 0 \\ -h^2(m-j+1) & 0 < i \leq j \\ -h^2(m+1-i) & j < i < m+1 \\ 1 & j = m+1 \end{cases}$$

(b)

$$\|A^{-1}\|_\infty = \|V\|_\infty = \|max \sum_j^m V_{i,j}\|_\infty$$

The infinity norm of a matrix is the sum of the absolute values of the rows.

$$\sum_j |V_{i,j}| = |V_{i,0}| + \sum_{j=1}^m |V_{i,j}| + |V_{i,m+1}|$$

$$\sum_j |V_{i,j}| = h + h^2(m^2 - m) + 1 \leq 2 + h^2 m(m+1) \leq 2 + \frac{m(m+1)}{(m+1)^2} \leq 3$$

since $h = \frac{1}{m+1} \leq 1$

$$\|A^{-1}\| \leq 3$$

(c)The Greens functions:

$$G_{xx} = \delta(x - \bar{x})$$

case 1: $x < \bar{x}$
$G_{xx} = 0$ and $G'(0) = 0$ So we can assume a linear solution
$G = Ax + B$
$G'(0) = 0$ implies $B = 0$ so $G = Ax$
case 2: $x > \bar{x}$
$G_{xx} = 0$ and $G(1) = 0$ So we can assume a linear solution
$G = Cx + D$
$G'(1) = 0$ implies $C + D = 0$ so $G = -Dx + D$

Since the Greens function must be continuous at $\bar{x}$, we know that:

$$B = -D\bar{x} + D$$

Thus:

$$\bar{x} = \frac{D - B}{D}$$

Since the derivative of the Greens function has a jump equal to 1 at $\bar{x}$, from the derivative of the heaviside function, we know that $-D - 0 = 1$ Thus $D = -1$
Solving for B in terms of $\bar{x}$, we get
$B = \bar{x} - 1$ so:

$$G(x, \bar{x}) = \begin{cases} \bar{x} - 1 & 0 \leq x \leq \bar{x} \\ x - 1 & \bar{x} \leq x \leq 1 \end{cases}$$