AMath 585
Homework #5
Due Thursday, March 6th, 2020
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Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5253097`

Problem 1.

Suppose $A \in \mathbb{R}^{m\times m}$ is strictly row diagonally dominant, i.e.,

$$\sum_{j\neq i} |a_{ij}| < |a_{ii}|, \quad \text{for } i = 1,\ 2,\ \ldots,\ m. \tag{1}$$

where the sum goes from $j = 1$ to m omitting the diagonal term.

Let G be the Jacobi iteration matrix for this matrix A.

(a) Show that $\|G\|_\infty < 1$ and hence the Jacobi iteration converges, since $\|e_k\|_\infty \le \|G\|_\infty^k \|e_0\|_\infty$. (Here and below subscript k refers to the kth iteration).

(b) Use the Gershgorin Theorem of Appendix C.8 to show that we also have $\rho(G) < 1$ in the case when (1) holds.

(c) Suppose G is a normal matrix ($G^TG = GG^T$) and suppose something stronger than (1) holds, namely,

$$\sum_{j\neq i} |a_{ij}| \le \beta |a_{ii}|, \quad \text{for } i = 1,\ 2,\ \ldots,\ m \tag{2}$$

for $\beta = 1/2$. Show that in this case Jacobi would reduce the 2-norm of the error $\|e_k\|_2$ by a factor of at least 10^6 in only 20 iterations, i.e., $\|e_k\|_2 \le 10^{-6}\|e_0\|_2$.

(d) More generally, suppose the factor β in (2) is some value satisfying $\beta < 1$, and suppose the matrix is not necessarily normal but we have an upper bound $\tilde{\kappa}$ on the 2-norm condition number of R, the matrix of right eigenvectors of G. In terms of β and $\tilde{\kappa}$, what is the maximum number of iterations that would be required to reduce the 2-norm error by a factor of 10^6? (In exact arithmetic.)

Solution:
(a)

$$A = D - (L + U)$$

Subtract true solution from approximation, and $e = u - u^*$

$$Du^{[k+1]} = (L + U)u^{[k]} + f$$

$$Du^* = (L + U)u^* + f$$
$$De^{[k+1]} = (L + U)u^{[k]}$$
$$e^{[k+1]} = D^{-1}(L + U)u^{[k]}$$

Thus the Jacobi iteration matrix$G = D^{-1}(L + U)$

$$\|G\|_\infty \le \|D^{-1}\|_\infty \|(L + U)\|_\infty$$

$$\|G\|_\infty \le \frac{1}{\|D\|_\infty} \|(L + U)\|_\infty$$

Recall that the infinity norm is the maximum row sum.

$$\|G\|_\infty \leq \max_{\{1\leq i\leq m\}} \frac{\sum_{j\neq i}|a_{ij}|}{|x_{ii}|}$$

From above and since it is strictly diagonally row dominant, we know,

$$\|G\|_\infty \leq \max_{\{1\leq i\leq m\}} \frac{\sum_{j\neq i}|a_{ij}|}{|x_{ii}|} < 1$$

Each iteration then, $\|e_k\|_\infty \leq \|G\|_\infty^k \|e_0\|_\infty$
If $\|G\|_\infty\| < 1$, and $\|e_0\|_\infty$ is constant, then each iteration, it gets smaller and smaller, since $\|G\|_\infty$ gets smaller as k gets larger.

(b)

$$G = D^{-1}(L+U)$$

Thus G has the form:

$$G = \begin{cases} 0 & if\ i=j \\ \frac{a_{ij}}{a_{ii}} & if\ i\neq j \end{cases}$$

Using Gershgorin's theorem and letting each eigenvalue be λ_i

$$|\lambda_i - 0| \leq \sum_{j\neq i} \frac{|a_{ij}|}{|a_{ii}|} < 1$$

$$|\lambda_i| < 1$$

Thus, any $\lambda_i < 1$ so the spectral radious $\rho(G) < 1$

(c) When you diagonalize G, you get $G = R\Gamma R^{-1}$ Since the eigenvectors are mutually orthogonal you can choose R such that $R^{-1} = R^T$ Then, $\|R\|_2 = \|R^{-1}\|_2 = 1$
Thus

$$\|e^{[k]}\|_2 = \|G^{[k]}e^{[0]}\|_2$$

$$\|e^{[k]}\|_2 \leq \|R\|_2\|\Gamma\|_2\|R^{-1}\|_2\|e^{[0]}\|_2$$

$$\|e^{[k]}\|_2 \leq max|\gamma_p^k|\|e^{[0]}\|_2$$

$$\|e^{[k]}\|_2 \leq \rho(G)^k\|e^{[0]}\|_2$$

if $\sum_{j\neq i}|a_{ij}| \leq \beta|a_{ii}|$ holds, then when you use Gershgorin's theorem: Using Gershgorin's theorem and letting each eigenvalue be λ_i

$$|\lambda_i - 0| \leq \sum_{j\neq i} \frac{|a_{ij}|}{|a_{ii}|} \leq \beta$$

$$\rho(G) \leq \beta$$

If $\beta = \frac{1}{2}$, after 20 iterations:

$$\|e^{[k]}\|_2 \leq (\frac{1}{2})^{20}\|e^{[0]}\|_2$$

$$\|e^{[k]}\|_2 \leq 10^{-6}\|e^{[0]}\|_2$$

Problem 2.

Based on Problem 1, you might think that the more diagonally dominant a matrix is, the better in terms of convergence rate. However, consider these two matrices:

$$A_1 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -3/4 \\ -1/12 & 1 \end{bmatrix}.$$

Determine the values β for each case (as in (2), and also the asymptotic convergence rates $\rho(G_1)$ and $\rho(G_2)$ in each case, where G_i is the Jacobi iteration matrix for A_i.

Solution:

For A_1, $\beta = \frac{1}{2}$

$$\sum_{j \neq i} |a_{ij}| \leq \frac{1}{2}|a_{ii}|$$

To find ρ, find the eigenvalues of A_1

$$\lambda_1^2 + \frac{1}{4}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

For A_2, $\beta = \frac{3}{4}$

$$\sum_{j \neq i} |a_{ij}| \leq \frac{3}{4}|a_{ii}|$$

To find ρ, find the eigenvalues of A_2

$$\lambda_1^2 + \frac{1}{16}$$

$$\lambda_1 = \lambda_2 = \frac{1}{4}$$

For this case, we can see that the convergence rate is faster for A_2 despite A_1 being more diagonally dominant. This is because as k gets larger, $(\rho(G) = \frac{1}{4})^k$ will decrease faster than $(\rho(G) = \frac{1}{2})^k$

Problem 3. Suppose A is a *singular* matrix and we do a splitting of the form $A = M - N$ in such a way that M is nonsingular (e.g. Jacobi iteration in a case where the diagonal elements of A are all nonzero).

Show that in spite of M being nonsingular, the iteration matrix $G = M^{-1}N$ can never satisfy $\rho(G) < 1$.

Thus we cannot expect an iterative method for such a system to converge in general, which makes sense if the matrix is singular. However, see the next problem...

Solution:

$$Av = \lambda v$$

Since we know that A is singular, 0 is an eigenvalue:

$$Av = 0$$

$$(M - N)v = 0$$

$$Mv - Nv = 0$$
$$Mv = Nv$$
$$v = M^{-1}Nv$$
$$v = Gv$$

Thus, 1 is an eigenvalue of G, so the spectral radius cannot satisfy $\rho(G) < 1$

Problem 4. Suppose we want to solve the boundary value problem $u''(x) = f(x)$ on $0 \leq x \leq 1$ with *periodic boundary conditions*: $u(0) = u(1)$. Recall that this has no solution unless a certain condition is satisfied by $f(x)$, in which case it has infinitely many solutions.

Recall also that if we discretize this with the standard centered second-order approximation, using a uniform grid with $h = 1/(m+1)$, we get a tridiagonal matrix with additional corner terms from the periodic boundary conditions. The matrix is singular and so the discrete problem has an analogous solvability condition.

(a) Suppose we use Jacobi iteration to solve this problem, in a case where the discrete solvability condition is satisfied. What is the iteration matrix G for this problem?

(b) Determine the eigenvalues and eigenvectors of G. Appendix C.7 of the text might be useful.

(c) You should find that one eigenvalue of G is equal to 1, and hence the Jacobi iteration does not appear to converge according to the theory of Section 4.2 (and consistent with Problem 3). But if the solvability condition is satisfied then in practice the method does converge to one of the infinitely many solutions of the linear system. Explain in what sense this is true and how the particular solution obtained is related to the initial guess $u^{[0]}$ used for the Jacobi iteration. **Hint:** Express the initial error relative to some particular solution as a linear combination of the eigenvectors and then observe the effect of iterating with the iteration matrix G.

(d) In the case it does converge, as described in (c), what is the expected convergence rate? (Note that since $\rho(G) = 1$, this is not it.)

Solution:

(a) If we discretize with the standard centered second-order approximation with a uniform grid, we get the matrix:

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & \dots & & 1 \\ 1 & -2 & 1 & \dots & \\ 0 & 1 & -2 & 1 & \dots \\ & & \ddots & \ddots & \ddots \\ 1 & & \dots & 1 & -2 \end{bmatrix}$$

$$G = D^{-1}(L + U)$$

$$G = \frac{h^2}{-2}\begin{bmatrix} 1 & 0 & \dots & \\ 0 & 1 & 0 & \dots \\ & & \ddots & \ddots \end{bmatrix} * \frac{-1}{h^2}\begin{bmatrix} 0 & 1 & \dots & & 1 \\ 1 & 0 & 1 & \dots & \\ 0 & 1 & 0 & 1 & \dots \\ & & \ddots & \ddots & \ddots \\ 1 & & \dots & 1 & 0 \end{bmatrix}$$

$$G = \frac{1}{2}\begin{bmatrix} 0 & 1 & \dots & & 1 \\ 1 & 0 & 1 & \dots & \\ 0 & 1 & 0 & 1 & \dots \\ & & \ddots & \ddots & \ddots \\ 1 & & \dots & 1 & 0 \end{bmatrix}$$

(b) Using the appendix, the eigenvalues of a tridiagonal circulant matrix are:

$$\lambda_p = e^{-2\pi iph} + e^{2\pi iph}$$

$$\lambda_p = cos(2\pi ph)$$

The corresponding eigenvectors are:

$$r_{jp} = e^{2\pi ipjh}$$

(c) The largest eigenvalue (λ_1)

$$\lambda_1 = cos(0) = 1$$

This means that the $\rho(G)$ is not less than one and thus the theory of Section 4.2 would say it does not converge.

However, now we will express the initial error as a function of eigenvectors:

$$G = R\Gamma R^{-1}$$

$$G\vec{v}_i = \gamma_i\vec{v}_i$$

$$e^{[k]} = G^k e^{[0]}$$

$$e^{[k]} = c_1\gamma_1^k\vec{v}_1 + c_2\gamma_2^k\vec{v}_2 + \ldots + c_n\gamma_n^k\vec{v}_n$$

$$\|e^{[k]}\| \leq \|c_1\gamma_1^k\vec{v}_1\| + \|c_2\gamma_2^k\vec{v}_2\| + \ldots \|c_n\gamma_n^k\vec{v}_n\|$$

The stability condition is if $\int f dx = 0$. The norm of vectors are one, and the values of c_1 are determined by the initial error $e^{[0]}$. Additionally, we can see from above that γ_0 is 1 (this is the largest eigenvalue). The initial error $e^{[0]} = u^{[0]} - u_{true}$ So since the values of c depend on $e^{[0]}$and therefore $u^{[0]}$ The problem is is that since there are infinitely many solutions, we do not really know which u_{true}we are converging to.

(d) The convergence rate would be about the γ_2 or the second largest eigenvalue.

Problem 5.

Consider the problem

$$\frac{d}{dx}\left(\kappa(x)u'(x)\right) = f(x)$$

on $0 \leq x \leq 1$ with $\kappa(x) > 0$ everywhere.

(a) Determine the solution of this problem for the case of $f(x) = 0$ and a discontinuous $\kappa(x)$ given by

$$\kappa(x) = \begin{cases} 0.1 & \text{if } x < 0.5, \\ 1 & \text{if } x > 0.5. \end{cases}$$

with boundary conditions $u(0) = 0$ and $u(1) = 1$. The solution should be continuous and piecewise linear with a discontinuity in slope at $x = 0.5$ (Note this could model steady steady heat flow through a material that conducts heat better on the right half than on the left.)

(b) Now suppose we discretize using the symmetric matrix A of (2.73) in the text. Choose $m = 19$ (odd, so there is a grid point exactly at $x = 0.5$, in which case the solution to the discrete system should agree with the exact solution of the differential equation).

Write a computer code to solve this system using the SOR method for an arbitrary value of ω. Note that setting $\omega = 1$ should just reduce to the Gauss-Seidel method for comparison.

Test this code and estimate the rate of convergence with $\omega = 1$ and with $\omega = 0.7$, by doing a least squares fit of the convergence history as demonstrated in the notebook `IterativeMethods.ipynb`.

Solution:

(a)

$$\frac{d}{dx}\left(\kappa(x)u'(x)\right) = 0$$
$$\int \frac{d}{dx}\left(\kappa(x)u'(x)\right) = \int 0$$
$$\left(\kappa(x)u'(x)\right) = C$$

First assume $\kappa(x) = 0.1$, $x < 0.5$

$$\int (0.1u'(x)) = \int C$$
$$0.1u(x) = Cx + D$$
$$u(x) = 10Cx + 10D$$
$$u(0) = 10D = 0$$
$$u(x) = 10Cx$$

Now assume $\kappa(x) = 1$, $x > 0.5$

$$\int (u'(x)) = \int C$$
$$u(x) = Cx + B$$
$$u(1) = C + B = 1$$
$$u(x) = Cx + 1 - C$$

Since we know that the solution is continuous, the solution must be the same at x = 0.5

$$10C(0.5) = C(0.5) + 1 - C$$

$$5C = -0.5C + 1$$
$$\frac{11}{2}C = 1$$
$$C = \frac{2}{11}$$

$$u(x) = \begin{cases} \frac{20}{11}x & x \leq 0.5 \\ \frac{2}{11}x + \frac{9}{11} & x \geq 0.5 \end{cases}$$