AMath 585
Homework #3
Due Thursday, February 6, 2020

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5237966`

Problem 1. Consider the nonlinear boundary value problem

$$\sin(u''(x)) = u(x)\exp(u'(x)) + f(x)$$

for $0 \le x \le 1$, with Dirichlet boundary conditions $u(0) = \alpha,\ u(1) = \beta$.

(a) Discretize using the standard second-order centered approximations for $u'(x_i)$ and $u''(x_i)$, giving a nonlinear system of equation $G(U) = 0$ where $G : \mathbb{R}^m \to \mathbb{R}^m$ and U is the vector of interior unknowns. What is the i'th component $G_i(U)$?

(b) What is the (i, j) element of the Jacobian matix $G'(U)$ needed to implement Newton's method for this system?

(c) Is the Jacobian matrix symmetric in general?

Solution:

(a) Using the second order centered approximations:

$$u''(x_i) = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

$$u'(x_i) = \frac{U_{i+1} - U_{i-1}}{2h}$$

Substituting these into the BVP, we get:

$$G_i(U) = \sin\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) - U_i \exp\left(\frac{U_{i+1} - U_{i-1}}{2h}\right) - f(x) = 0$$

Additionally,

$$G_0(U) = U_0 - \alpha = 0$$

$$G_{m+1}(U) = U_{m+1} - \beta = 0$$

$$U = \begin{bmatrix} \alpha \\ U_1 \\ U_2 \\ \vdots \\ U_m \\ \beta \end{bmatrix}$$

(b) The Jacobian matrix :

$$J_{i,j} = \frac{\partial G_i(U)}{\partial U_j}$$

$$J_{i,j} = \begin{cases} \frac{1}{h^2}\cos\left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}\right) + \frac{U_i}{2h}e^{\left(\frac{U_{i+1}-U_{i-1}}{2h}\right)} & j = i-1 \\ \frac{-2}{h^2}\cos\left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}\right) - e^{\left(\frac{U_{i+1}-U_{i-1}}{2h}\right)} & j = i \\ \frac{1}{h^2}\cos\left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}\right) - \frac{U_i}{2h}e^{\left(\frac{U_{i+1}-U_{i-1}}{2h}\right)} & j = i+1 \end{cases}$$

(c) The Jacobian is not necessarily symmetric, and is not actually symmetric in this case. It depends heavily on the chosen approximations of the derivative.

Problem 2.

Consider the linearized pendulum problem of Section 2.16, which we will rewrite with x in place of t and now calling the angle $u(x)$ for consistency with our other BVPs:

$$u''(x) = -\gamma^2 u(x)$$

on the interval $0 \le x \le 1$, with $\gamma^2 = g/L$.

(a) Confirm that the general solution to this ODE is $u(x) = c_1 \sin(\gamma x) + c_2 \cos(\gamma x)$.

(b) If we impose Dirichlet boundary conditions $u(0) = \alpha$ and $u(1) = \beta$, show that this leads to a 2×2 linear system of equations to determine the coefficients c_1 and c_2. Solve this system for the case $\gamma = \pi/2$ to find c_1 and c_2 in terms of α and β.

(c) Show that if $\gamma = \pi$ then the system for c_1, c_2 is singular. For what choices of α, β does the system have a solution? In such cases it has infinitely many solutions, what are they?

(d) Recall that $-\pi^2$ is an eigenvalue of the operator ∂_x^2 on the interval $[0, 1]$ with homogeneous boundary conditions $u(0) = u(1) = 0$. The linear pendulum equation can be written as $\mathcal{L}u = 0$, where $L = \partial_x^2 + \gamma^2$. What are the eigenvalues and eigenfunctions of this operator and how do these relate to your answer to part (c)?

(e) Suppose we discretize this BVP with the usual second-order centered approximation, giving a tridiagonal matrix A for the system for interior unknowns $[U_1, \ \ldots, \ U_m]$. What are the eigenvalues and eigenvectors of this matrix? Hint: if T is the tridiagonal matrix of (2.10) then $A = T + \gamma^2 I$.

(f) Note that if $\gamma = \pi$ then this matrix is not singular (all the eigenvalues are nonzero) for $h > 0$, so the discrete system has a unique solution for any $h > 0$. However, we expect trouble as $h \to 0$, since we are approximating a BVP that does not have a unique solution for this choice of γ. Show that in the 2-norm the method is *not stable* in the sense of Definition 2.1 in this case. How rapidly does $\|A^{-1}\|_2$ grows as $h \to 0$. (E.g. like $1/h$? or $1/h^2$?)

(g) On the other hand, for any γ that is not an integer multiple of π, show that this method is stable in the 2-norm.

Solution:

(a) To confirm that the general solution to the ODE is $u(x) = c_1 \sin(\gamma x) + c_2 \cos(\gamma x)$
$u'(x) = c_1 \gamma \cos(\gamma x) - c_2 \gamma \sin(\gamma x)$
$u''(x) = -c_1 \gamma^2 \sin(\gamma x) - c_2 \gamma^2 \cos(\gamma x)$

$$-c_1 \gamma^2 \sin(\gamma x) - c_2 \gamma^2 \cos(\gamma x) = (-\gamma^2)(c_1 \sin(\gamma x) + c_2 \cos(\gamma x))$$

(b) Impose the boundary conditions: $u(0) = \alpha$ and $u(1) = \beta$

$$u(0) = c_2 = \alpha$$

$$u(1) = c_1 \sin(\gamma) + c_2 \cos(\gamma) = \beta$$

This leads to the linear system of 2 equations with the coefficients c_1 and c_2

$$\begin{bmatrix} 0 & 1 \\ sin(\gamma) & \cos(\gamma) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

In the case for $\gamma = \frac{\pi}{2}$,

$$c_1 = \beta$$
$$c_2 = \alpha$$

(c) In the case for $\gamma = \pi$, the system becomes:

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

In this case, we have that $c_2 = \alpha$ and $c_2 = -\beta$ so there are infinitely many solutions. c_1 can be anything, but there are only infinitely many solutions if $\alpha = -\beta$.

(d)

$$Lu = \lambda u$$
$$u'' + \gamma^2 u = \lambda u$$
$$u'' + (\gamma^2 - \lambda)u = 0$$

Applying the Boundary conditions: The general solution is: $y = Asin(\sqrt{\gamma^2 - \lambda}x) + Bcos(\sqrt{\gamma^2 - \lambda}x)$

$$y(0) = B = 0$$
$$y(1) = Asin(\sqrt{\gamma^2 - \lambda}) = 0$$
$$\sqrt{\gamma^2 - \lambda} = n\pi$$
$$\lambda = \gamma^2 - n^2\pi^2$$

Thus, our eigienfunction is $\Phi_n = A_n sin(\sqrt{\gamma^2 - \lambda}x)$ We can solve forA_n by normalizing the eigenfunction:

$$\|\Phi\|^2 = \int_0^1 A_n^2 sin^2(\sqrt{\gamma^2 - \lambda}x)dx = 1$$
$$\|\Phi\|^2 = \frac{A_n^2}{2}\int_0^1 1 - cos(2\sqrt{\gamma^2 - \lambda}x)dx = 1$$
$$\|\Phi\|^2 = \frac{A_n^2}{2}[x - \frac{1}{2\gamma}sin(2\sqrt{\gamma^2 - \lambda}x)]|_0^1 = 1$$
$$\|\Phi\|^2 = \frac{A_n^2}{2}[1 - \frac{1}{2\gamma}sin(2\sqrt{\gamma^2 - \lambda})] = 1$$

Since $sin(\sqrt{\gamma^2 - \lambda}) = 0$

$$A_n = \sqrt{2}$$
$$\Phi = \sqrt{2}sin(\sqrt{\gamma^2 - \lambda}x)$$

The boundary conditions are homogeneous, so, in (c) it was determined that, there were only solutions if $\alpha = -\beta$ and that is satisfied if both boundary conditions are zero.

(e) If we discretize, we get

$$A = \begin{bmatrix} \frac{-2}{h^2} + \gamma^2 & \frac{1}{h^2} & 0 & 0 & \\ \frac{1}{h^2} + & \frac{-2}{h^2} + \gamma^2 & \frac{1}{h^2} & 0 & \\ 0 & \frac{1}{h^2} & \frac{-2}{h^2} + \gamma^2 & \frac{1}{h^2} & \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

To find the eigenvalues:

$$det(A - \lambda I) = 0$$

$$det(T + \gamma^2 I - \lambda I) = 0$$

$$det(T - (\lambda + \gamma^2)I) = 0$$

From the book, the eigenvalues of the T matrix are:

$$\lambda_p = \frac{2}{h^2}(cos(p\pi h) - 1)$$

Thus the eigenvalues for our matrix are the same but with an extra γ^2

$$\lambda_p = \frac{2}{h^2}(cos(p\pi h) - 1) + \gamma^2$$

To solve for the corresponding eigenvectors:

$$A\vec{v} = \lambda\vec{v}$$

Since: $A = T + \gamma^2 I$ and $\lambda = \lambda + \gamma^2$

$$(T + \gamma^2 I)\vec{v} = (\lambda + \gamma^2)\vec{v}$$

$$(T)\vec{v} = (\lambda)\vec{v}$$

Thus the eigenvectors of T are the same as $T + \gamma^2 I$ From the textbook, we know these to be:

$$u_j^p = sin(p\pi jh)$$

(f) If we want to get a bound for $\|A^{-1}\|_2$, we want to find the smallest eigenvalue of A, since the 2-norm of A is equal to the maximum eigenvalue and the 2-norm of A inverse is (1/2-norm of A) So:

$$\|A^{-1}\|_2 \leq \frac{1}{\lambda 1}$$

The smallest (in magnitude) eigenvalue of A is:

$$\lambda_1 = \frac{2}{h^2}(cos(\pi h) - 1) + \gamma^2$$

Using Taylor expansion of $cos(\pi h)$, we get:

$$\lambda_1 = \frac{2}{h^2}(\frac{-1}{2}\pi^2 h^2 + \frac{1}{24}\pi^4 h^4 + O(6)) + \gamma^2$$

$$\lambda_1 = -\pi^2 + \gamma^2 + O(h^2)$$

To get a bound on the error, we want to show that:

$$\|E\| \leq \|A^{-1}\|_2 \|\tau\|_2 \approx \frac{1}{\lambda 1}\|\tau\|_2$$

$$\|E\| \leq \|A^{-1}\|_2 \|\tau\|_2 \approx \frac{1}{-\pi^2 + \gamma^2 + O(h^2)}\|\tau\|_2$$

If $\gamma = \pi$, then there is no bound and it is not stable in the sense of Definition 2.1.

$$\|E\| \leq \|A^{-1}\|_2 \|\tau\|_2 \approx \frac{1}{O(h^2)}\|\tau\|_2$$

So, $\|A^{-1}\|_2$ grows at a factor of $\frac{1}{h^2}$

(g) In the case where $\gamma \neq \pi$ or an integer multiple of π, we do not have the cancellation of the bottom terms and thus we have an upper bound on $\|A^{-1}\|_2$ that does not "blow up". Thus this method is stable in the 2-norm in the sense of definition 2.2.

$$\|A^{-1}\|_2 \leq \frac{1}{-\pi^2 + \gamma^2 + O(h^2)}$$

Problem 3.

If $A \in \mathbb{R}^{m \times m}$ and $u \in \mathbb{R}^m$ is any nonzero vector, then the scalar value $Q(u) = u^T Au/u^T u$ is called the Rayleigh quotient.

(a) Show that if v is an eigenvector of A then $Q(v) = \lambda$, the corresponding eigenvalue.

(b) If $A = A^T$ is a symmetric matrix then the eigenvalues must be real. The matrix is called *symmetric positive definite* if the eigenvalues of A are all positive, or *symmetric negative definite* if they are all negative.

Show that if A is symmetric positive or negative definite then

$$\min_p \lambda_p \leq Q(u) \leq \max_p \lambda_p. \tag{1}$$

Hint: Recall that if A is symmetric then it is diagonalizable and the eigenvectors are mutually orthogonal, so we can write $A = V\Lambda V^T$ where V is the matrix of normalized eigenvectors (each column v_j has 2-norm equal to 1). So V is an "orthogonal matrix" with $V^{-1} = V^T$. Hence any vector $u \in \mathbb{R}^m$ can be written as $u = Vy$ for $y = V^T u$. (See also Appendix C.)

Conversely, show that if A is symmetric and if there are constants C_1, C_2 such that $C_1 \leq Q(u) \leq C_2$ for all nonzero vectors u, then the eigenvalues of A all satisfty $C_1 \leq \lambda \leq C_2$.

Investigating $Q(u)$ can sometimes help us to show that the eigenvalues of A are bounded away from 0, which can be useful in proving stability of a method.

The next problem illustrates one such case.

Solution:

(a) If $\vec{v}$ is and eigenvector of A, then $A\vec{v} = \lambda\vec{v}$

$$\vec{v}^T A\vec{v} = \vec{v}^T \lambda\vec{v}$$

$$\lambda = \frac{\vec{v}^T A\vec{v}}{\vec{v}^T \vec{v}} = Q(\vec{v})$$

(b)

$$Q(u) = \frac{u^T Au}{u^T u}$$

Then, we can write $A = V\Lambda V^T$ where V is the matrix of normalized eigenvectors (each column v_j has 2-norm equal to 1)

$$Q(u) = \frac{u^T V\Lambda V^T u}{u^T u}$$

Then, any $u = Vy$ so,

$$Q(u) = \frac{(Vy)^T V\Lambda V^T (Vy)}{(Vy)^T (Vy)}$$

Since $V^T = V^{-1}$ then, $V^{-1}V = I$

$$Q(u) = \frac{(Vy)^T V \Lambda V^T (Vy)}{(Vy)^T (Vy)}$$

$$Q(u) = \frac{y^T V^T V \Lambda V^T (Vy)}{y^T V^T (Vy)}$$

$$Q(u) = \frac{y^T \Lambda y}{y^T y}$$

$$Q(u) = \frac{\sum_i^p \lambda_i (y_i)^2}{\sum_i^p (y_i)^2}$$

Thus:

$$\frac{\lambda_{min} \sum_i^p (y_i)^2}{\sum_i^p (y_i)^2} \leq \frac{\sum_i^p \lambda_i (y_i)^2}{\sum_i^p (y_i)^2} \leq \frac{\lambda_{max} \sum_i^p (y_i)^2}{\sum_i^p (y_i)^2}$$

$$\min_p \lambda_p \leq Q(u) \leq \max_p \lambda_p.$$

If we know that A is symmetric and $\exists$ constants C_1 and C_2 such that $C_1 \leq Q(u) \leq C_2$ for all nonzero vectors u, and using what was derived above :

$$C_1 \leq \frac{y^T \Lambda y}{y^T y} \leq C_2$$

$$C_1 y^T y \leq y^T \Lambda y \leq C_2 y^T y$$

$$C_1 \sum_i^p (y_i)^2 \leq \sum_i^p \lambda_i (y_i)^2 \leq C_2 \sum_i^p (y_i)^2$$

$$C_1 \sum_i^p (y_i)^2 \leq \lambda_{min} \sum_i^p (y_i)^2 \leq \sum_i^p \lambda_i (y_i)^2 \leq \lambda_{max} \sum_i^p (y_i)^2 \leq C_2 \sum_i^p (y_i)^2$$

Thus for any given λ_i we can say that:

$$C_1 \leq \lambda \leq C_2$$

When we know that the eigenvalues are positive or negative definite, we can bound the Q(u) away from zero, the same in the opposite direction.

Problem 4.

Suppose we discretize the *nonlinear* pendulum problem $u''(x) = -\gamma^2 \sin(u(x))$ (again with Dirichlet boundary conditions) using the standard second order centered approximation to $u''(x_i)$, as discussed in Section 2.16.

The Jacobian matrix $G'(U)$ now has the form $G'(U) = T + D$, where T is the standard tridiagonal matrix of (2.10) and D is now a diagonal matrix that is no longer a scalar multiple of the identity matrix. It is similar to (2.82) but with an additional factor γ^2.

We can no longer compute the eigenvalues or eigenvectors of $A = T + D$ exactly.

Consider a problem for which $\gamma < \pi$ and suppose we also know there is a unique exact solution $u(x)$ to the boundary value problem being considered (there is in this case, but you don't need to prove it).

Then, using the results of Problem 3, show that as $h \to 0$ we can find a uniform bound on $\|(G'(U))^{-1}\|_2$ and hence this method for the nonlinear pendulum is stable in the sense of Definition 2.2.

Solution:

Using the second order centered approximation:

$$u''(x_i) = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

Substituting these into the BVP, we get:

$$G_i(U) = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + \gamma^2 sin(U_i)$$

Since we can assume a unique solution, this means that there are all of the eigenvalues are nonzero. As in problem 2, to get a bound on $\|G'(U)^{-1}\|_2$, we need to find a bound for the largest eigenvalue of G(U).

To do this using the results of problem 3, we need to show that G'(U) is positive or negative definite.

As in problem 2, we can rewrite this matrix into the sum of two matrices,

$$J = \begin{bmatrix} \frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 & \\ \frac{1}{h^2}+ & \frac{-2}{h^2} & \frac{1}{h^2} & 0 & \\ 0 & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} \gamma^2 cos(U_i) & 0 & 0 & 0 & \\ 0 & \gamma^2 cos(U_i) & 0 & 0 & \\ 0 & 0 & \gamma^2 cos(U_i) & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

J = T + D We can write G'(U) as the addition of The Rayleigh Quotient(RQ) of G'(U) = J is:

$$Q(J) = \frac{u^T(T+D)u}{u^t u} = \frac{u^T T u}{u^T u} + \frac{u^T D u}{u^T u}$$

From question 3, we can bound the RQ by the largest and smallest eigenvalues: Thus, we can bound Q(T)

$$-m^2\pi^2 \leq Q(T) \leq -\pi^2$$

For D, since it is diagonal, the eigenvalues are on the diagonal and additionally, cosine is bounded between -1 and 1.

$$-\gamma^2 \leq Q(D) \leq \gamma^2$$

This means we can bound Q(J):

$$-m^2\pi^2 - \gamma^2 \leq Q(J) \leq -\pi^2 + \gamma^2$$

If $\gamma < \pi$, then the RQ is bounded by 2 negative numbers, meaning the eigenvalues are all negative, so it is positive definite, and since we bounded the RQ away from zero, this confirms that zero is not an eigenvalue and we have a unique solution.

Using the results from problem 3, we know that if the eigenvalues are bounded, then:

$$\|G'(U)^{-1}\|_2 = \frac{1}{\|G'(U)\|_2} \leq \frac{1}{\gamma^2 - \pi^2 + O(h^2)} \leq C_2$$

$$\|G'(U)^{-1}\| \leq C_2 \ \forall \ h \leq h_0$$

This means we must be able to solve the resulting system and the iterative method (usually Newton's method) must converge, from definition 2.2, since it is stable.