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AMath 585
Homework #1
Due Thursday, January 16, 2019

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5199959`

If you haven't done so already, clone the class repository and read about how to use it on the class webpage `http://staff.washington.edu/rjl/classes/am585w2020/class_repos.html`.

Also, this would be a good time to try to figure out how to install Python and use Jupyter notebooks from the `notebooks` directory of the repository. See `http://staff.washington.edu/rjl/classes/am585w2020/code.html`. Contact the instructor or TA if you need help.

Problem 1. Suppose we wish to approximate the fourth derivative $u^{(4)}(x_0)$ using a 5-point stencil

$$u^{(4)}(x_0) \approx c_{-2}u(x_{-2}) + c_{-1}u(x_{-1}) + c_0u(x_0) + c_1u(x_1) + c_2u(x_2)$$

in the case of equally spaced points, $x_j = x_0 + jh$ for some $h = \Delta x$.

By hand, work out the Vandermonde system of equations to be solved for the coefficients and confirm that the coefficients produced by the `fdcoeffV` function gives coefficients that satisfy this linear system. (Use either the Python or Matlab version, but I suggest you try using Python and the Jupyter notebook.)

Solution:

Using Taylor Expansion, it can be determined that:

$$c_{-2} * [u(x_{-2}) = u(x_0) - 2hu'(x_0) + 2h^2u''(x_0) - \frac{4}{3}h^3u'''(x_0) + \frac{2}{3}h^4u^{(4)}(x_0) + ...]$$

$$c_{-1} * [u(x_{-1}) = u(x_0) - hu'(x_0) + \frac{1}{2}h^2u''(x_0) - \frac{1}{6}h^3u'''(x_0) + \frac{1}{24}h^4u^{(4)}(x_0) + ...]$$

$$c_0 * [u(x_0) = u(x_0)]$$

$$c_1 * [u(x_1) = u(x_0) + hu'(x_0) + \frac{1}{2}h^2u''(x_0) + \frac{1}{6}h^3u'''(x_0) + \frac{1}{24}h^4u^{(4)}(x_0) + ...]$$

$$c_2 * [u(x_2) = u(x_0) + 2hu'(x_0) + 2h^2u''(x_0) + \frac{4}{3}h^3u'''(x_0) + \frac{2}{3}h^4u^{(4)}(x_0) + ...]$$

Next, use a matrix system to solve for the coefficients. Since we are looking for a 4th order approximation, we want all other terms to have the coefficient of zero. Hence, the system looks like this:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving for c results in:

$$c = \frac{1}{h^4}\begin{pmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{pmatrix}$$

This result was verified in the Jupyter Notebook by solving the linear system, and by using the fcoeffV.

Problem 2. (a) Determine the order of accuracy and leading term of the asymptotic error for the approximation derived above. In other words, if the function $u(x)$ is sufficiently smooth, show the approximation has an error that is of the form $Ch^p + \mathcal{O}(h^{p+1})$ and determine p and C. The value of C will depend on higher-order derivative(s) of u at x_0.

(b) Test your result by computing the error for $u(x) = \sin(2x)$ at the point $x_0 = 1$ and various choices of h and show this is consistent with what you derived. Produce a log-log plot of the absolute error vs. h and the expected error to show that it has the expected form (similar to the plots produced in the `fdstencil_errors.ipynb` notebook). How good an approximation can you get before rounding error takes over?

Solution:

(a) Expanding up to 6th derivatives:

$$\frac{1}{h^2}*[u(x_{-2}) = u(x_0) - 2hu'(x_0) + 2h^2u''(x_0) - \frac{4}{3}h^3u'''(x_0) + \frac{2}{3}h^4u^{(4)}(x_0) - \frac{4}{15}h^5u^{(5)}(x_0) + \frac{4}{45}h^6u^{(6)}(x_0)]$$

$$\frac{1}{h^2}*[-4*u(x_{-1}) = -4*u(x_0) + 4hu'(x_0) - 2h^2u''(x_0) + \frac{2}{3}h^3u'''(x_0) - \frac{1}{6}h^4u^{(4)}(x_0) + \frac{1}{30}h^5u^{(5)}(x_0) - \frac{1}{180}h^6u^{(6)}(x_0)]$$

$$\frac{1}{h^2}*[6*u(x_0) = 6u(x_0)]$$

$$\frac{1}{h^2}*[-4*u(x_1) = -4*u(x_0) - 4hu'(x_0) - 2h^2u''(x_0) - \frac{2}{3}h^3u'''(x_0) - \frac{1}{6}h^4u^{(4)}(x_0) - \frac{1}{30}h^5u^{(5)}(x_0) - \frac{1}{180}h^6u^{(6)}(x_0)]$$

$$\frac{1}{h^2}*[u(x_2) = u(x_0) + 2hu'(x_0) + 2h^2u''(x_0) + \frac{4}{3}h^3u'''(x_0) + \frac{2}{3}h^4u^{(4)}(x_0) + \frac{4}{15}h^5u^{(5)}(x_0) + \frac{4}{45}h^6u^{(6)}(x_0)]$$

Adding all the equations together and cancelling terms:

$$u^{(4)}(x_0) = u^{(4)}(x_0) + \frac{1}{6}h^2u^{(6)}(x_0) + \mathcal{O}(h^3)$$

(b) The 6th derivative of u(x) = sin(2x) is:

$$u^{(6)}(x) = -2^6 sin(2x)$$

The loglog plot is included in the jupyter notebook. The error approximation is fairly accurate until the roundoff error begins to take effect. This occurs between $h = 10^-2$ and 10^-3.

Problem 3. Suppose we want to solve a 2-point boundary value problem of the form

$$u^{(4)}(x) = 3u''(x) + 4u(x) + f(x)$$

for $a \leq x \leq b$ with prescribed boundary conditions

$$u(a) = \alpha_0, \quad u'(a) = \alpha_1, \qquad u(b) = \beta_0, \quad u'(b) = \beta_1.$$

Set up the linear system of equations that would be solved to find a finite-difference solution to this problem. For this homework you do not need to program this or solve the system, just write out the system in a way that is clear what the banded matrix is, and what system needs to be solved. In particular, write out explicitly at least the first three and last three rows of the matrix and elements of the right-hand side to show how the function values $f(x_j)$ and boundary conditions come into these.

Use a uniform grid $x_j = a + jh$ with $h = (b-a)/(m+1)$, Use the approximation to $u^{(4)}$ from Problem 1 and the standard centered approximation for $u''(x)$. For the boundary conditions on u', use one-sided approximations that are second-order accurate.

Since $u(a)$ and $u(b)$ are both known from the boundary conditions, you can set this up as a system of m equations for the interior unknowns $U_1, \ \ldots, \ U_m$.

Solution:

First, isolate f(x) on the right hand side:

$$u^{(4)}(x) - 3u''(x) + -4u(x) = f(x)$$

Using the approximation previously calculated for $u^{(4)}(x)$:

$$u^{(4)}(\bar{x}) = \frac{1}{h^4}u(\bar{x}-2h) - \frac{4}{h^4}u(\bar{x}-h) + \frac{6}{h^4}u(\bar{x}) - \frac{4}{h^4}u(\bar{x}+h) + \frac{1}{h^4}u(\bar{x}+2h)$$

To approximate $u''(x)$ use the standard centered approximation:

$$u''(\bar{x}) = \frac{1}{h^2}u(\bar{x}-h) - \frac{2}{h^2}u(\bar{x}) + \frac{1}{h^2}u(\bar{x}+h)$$

Substituting both into $u^{(4)}(x) = 3u''(x) + 4u(x) + f(x)$ we get:

$$\frac{1}{h^4}u(\bar{x}-2h) - \frac{4+3h^2}{h^4}u(\bar{x}-h) + \frac{6+6h^2-4h^4}{h^4}u(\bar{x}) - \frac{4+3h^2}{h^4}u(\bar{x}+h) + \frac{1}{h^4}u(\bar{x}+2h) = f(x)$$

To set up the system of equations, we must consider the boundary conditions. For the Neumann conditions, we will use one sided approximations that are second order accurate so:

$$u'(\bar{x}) = \frac{-3}{2h}u(\bar{x}) + \frac{2}{h}u(\bar{x}+h) - \frac{1}{2h}u(\bar{x}+2h)$$

and

$$u'(\bar{x}) = \frac{3}{2h}u(\bar{x}) + \frac{-2}{h}u(\bar{x}-h) + \frac{1}{2h}u(\bar{x}-2h)$$

So, for this problem, since $u'(a) = \alpha_1$, we get:

$$u'(a) = \frac{-3}{2h}u(a) + \frac{2}{h}u(x_1) - \frac{1}{2h}u(x_2) = \alpha_1$$

Since $u'(b) = \beta_1$, we get:

$$u'(b) = \frac{3}{2h}u(b) - \frac{2}{h}u(x_m) + \frac{1}{2h}u(x_{m-1}) = \beta_1$$

Using this, we get the system:

$$\frac{1}{h^4}\begin{pmatrix} 2h^3 & \frac{-h^3}{2} & & & & \\ -(4+3h^2) & 6+6h^2-4h^4 & -(4+3h^2) & 1 & & \\ 1 & -(4+3h^2) & 6+6h^2-4h^4 & -(4+3h^2) & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 1 & -(4+3h^2) & 6+6h^2-4h^4 & -(4+3h^2) & 1 \\ & & 1 & -(4+3h^2) & 6+6h^2-4h^4 & -(4+3h^2) \\ & & & & \frac{h^3}{2} & -2h^3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-2} \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \alpha_1 + \frac{3\alpha_0}{2h} \\ f(x_2) - \frac{\alpha_0}{h^4} \\ f(x_3) \\ \vdots \\ f(x_{m-2}) \\ f(x_{m-1}) - \frac{\beta_0}{h^4} \\ \beta_1 - \frac{3\beta_0}{2h} \end{pmatrix}$$