AMATH 586 SPRING 2020
HOMEWORK 3 — DUE MAY 8 ON GITHUB BY 11PM

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Problem 1: It is natural to ask if a neighborhood of $z = 0$ can be in the absolute stability region S for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$. Show:

- Consistency implies that $\pi(1; 0) = 0$.
- Stability implies that $\rho'(1) \neq 0$.
- Suppose $\xi = 1 + \eta(z)$ for z near zero so that $\pi(\xi; z) = \pi(1 + \eta(z); z) = 0$. Compute $\eta'(0)$. Why does this imply that there must be an interval $(0, \epsilon]$ for some small $\epsilon > 0$ that does not lie in the absolute stability region S.

Solution:
First, recall that: $\rho(\xi) = \sum_{j=0}^{r} \alpha_j \xi^j$ and $\sigma(\xi) = \sum_{j=0}^{r} \beta_j \xi^j$ Additionally recall that, for a LMM to be consistent, the following things must be satisfied: $\sum_{j=0}^{r} \alpha_j = 0$ and $\sum_{j=0}^{r} j\alpha_j = \sum_{j=0}^{r} \beta_j$

(a)

$$\pi(1; 0) = \rho(1) = \sum_{j=0}^{r} \alpha_j = 0 \text{ if the method is consistent} \therefore \text{ Consistency implies that } \pi(1; 0) = 0.$$

(b) Using Taylor Expansion:

$$\pi(\xi; z = 0) = \rho(\xi) = \rho(1) + \rho'(1)(\xi - 1) + \frac{1}{2}\rho''(1)(\xi - 1)^2 + \frac{1}{6}\rho'''(1)(\xi - 1)^3 + \frac{1}{14}\rho^{(4)}(\xi - 1)^4 + \dots$$

From consistency, we know $\rho(1) = 0$ (from part (a)) If we assume that $\rho'(1) = 0$ then:

$$\pi(\xi; z = 0) = \rho(\xi) = \frac{1}{2}\rho''(1)(\xi - 1)^2 + \frac{1}{6}\rho'''(1)(\xi - 1)^3 + \frac{1}{14}\rho^{(4)}(\xi - 1)^4 + \dots$$

For an r-step LMM to be stable, if must satisfy the root condition:
$|\xi_j| \leq 1$ for j = 0,1...r. If ξ_j is a repeated root, then $|\xi_j| < 1$
From above, we can see that $\xi = 1$ is a repeated root, so the root condition is violated. Thus, stability implies that $\rho'(1) \neq 0$ **(c)** First,

$$\pi(1 + \eta(z); z) = \rho(1 + \eta(z)) + z\sigma(1 + \eta(z))$$

$$\pi(1 + \eta(z); z) = \sum_{j=0}^{r} \alpha_j (1 + \eta(z))^j + z \sum_{j=0}^{r} \beta_j (1 + \eta(z))^j$$

$$\pi(1+\eta(z);z) = \sum_{j=0}^{r}(\alpha_j - z\beta_j)(1+\eta(z))^j = 0$$

$$\pi'(1+\eta(z);z) = \sum_{j=0}^{r}(\alpha_j - z\beta_j)j(1+\eta(z))^{j-1}\eta'(z) - \beta_j(1+\eta(z))^j = 0$$

$$\pi'(1+\eta(0);0) = \eta'(0) = \frac{\sum_{j=0}^{r}\beta_j(1+\eta(0))^j}{\sum_{j=0}^{r}\alpha_j j(1+\eta(0))^{j-1}}$$

To enforce consistency, we can wee that $\sum_{j=0}^{r}\beta_j = \sum_{j=0}^{r}\alpha_j j$
Thus, $\eta'(0) = 1+\eta(0)$ Additionally, we need $\pi(1;0) = 0$ for consistency. So let us consider $\eta(0) = 0$. Thus means that $\eta'(0) > 0$ Since the derivative is positive near z=0, $\xi = 1+\eta(\epsilon) > 1$, since $\eta(\epsilon) > 0$. This violates the root condition which means that there is an interval $(0,\epsilon]$ that does not lie in the absolute stability region S.

Problem 2: Recall the test problem

$$v'''(t) + v'(t)v(t) - \frac{\beta_1+\beta_2+\beta_3}{3}v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3-\beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3-\beta_1}{12}}t, \sqrt{\frac{\beta_3-\beta_2}{\beta_3-\beta_1}}\right)$$

is a solution where $\mathrm{cn}(x,k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\mathrm{cn}(x,m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3-\beta_1)(\beta_3-\beta_2)}{6}.$$

Write the equation as a system and compute the Jacobian. For $\beta_1 = 0, \beta_2 = 1, \beta_3 = 10$, based on an analysis of the Jacobian, suggest methods to solve the problem.
Solution

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3-\beta_1)(\beta_3-\beta_2)}{6}.$$

Create the system:

$$u_1'(t) = v'(t) = u_2(t), u_2'(t) = v''(t) = u_3(t), u_3''(t) = \frac{\beta_1+\beta_2+\beta_3}{3}u_2(t) - u_2(t)u_1(t).$$

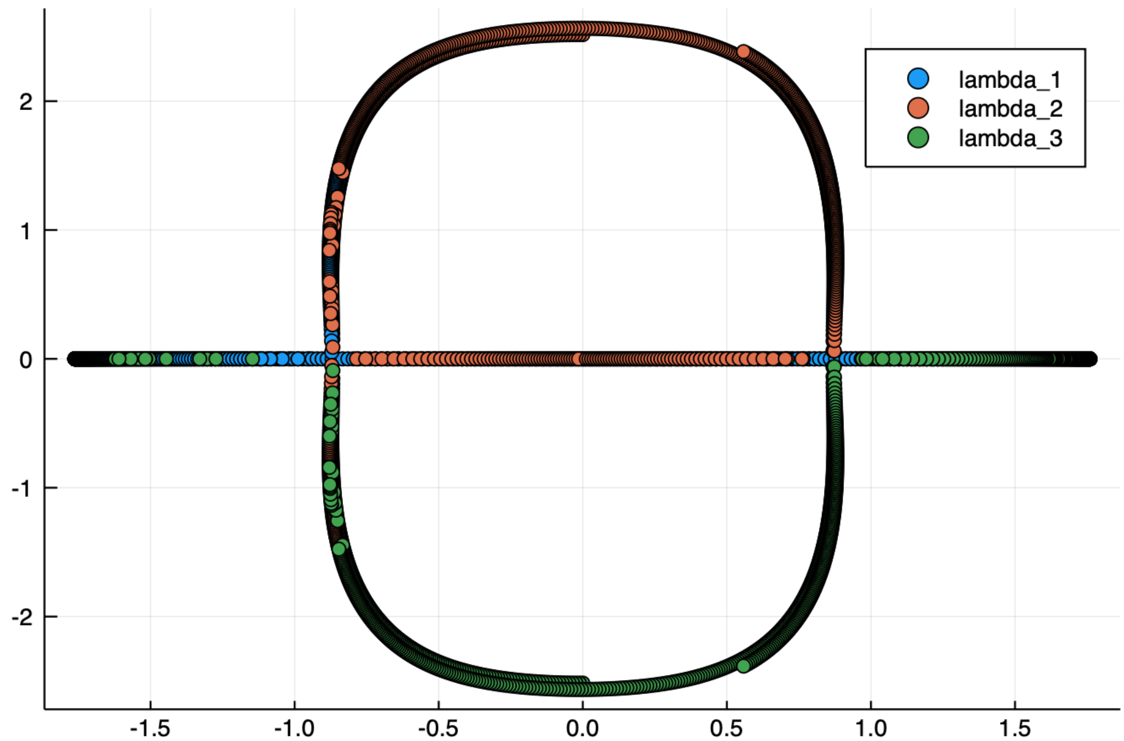
So, set $c = \frac{(\beta_1+\beta_2+\beta_3)}{3} = \frac{11}{3}$

$$f(u) = \begin{bmatrix} u_2 \\ u_3 \\ u_1(c-u_2) \end{bmatrix}.$$

The Jacobian is:

$$D_u f(u) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -u_2 & c-u_1 & 0 \end{bmatrix}.$$

Since the Jacobian is changing with the values of u, which are dependent on the values of t we can find the eigenvaues numerically changing the values of u.



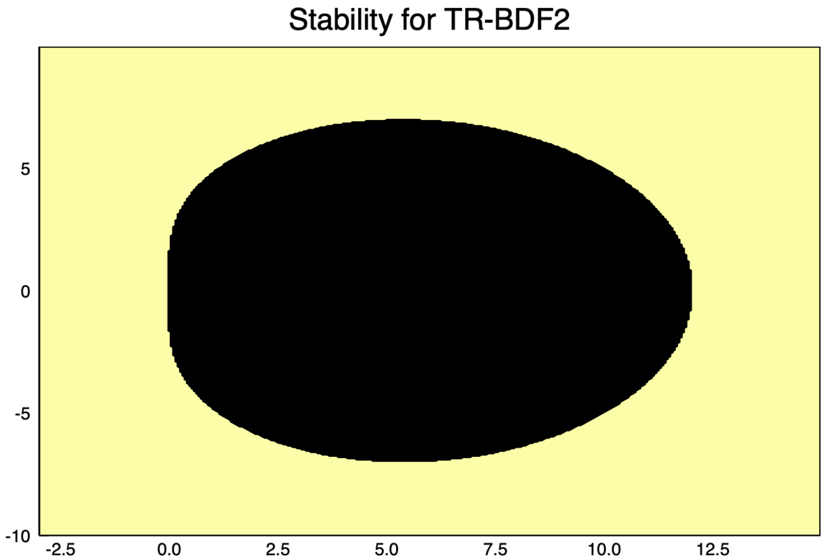
Unfortunately, the eigenvalues are on both sides of the real and imaginary axis and they are not contained in the stability regions of the methods we have seen plotted. Instead, we can choose a method that focuses on accuracy and tries to capture most of the eigenvalues. Since we can get the a lot of the eigenvalues with the third order AM method, I would choose to use this method.

Problem 3:

- Plot the absolute stability region for the TR-BDF2 method (8.6).
- By analyzing $R(z)$, show that the method is both A-stable and L-stable. Hint: To show A-stability, show that $|R(z)| \leq 1$ on the imaginary axis and explain why this is enough.

Solution

$$R(z) = \frac{3 + \frac{5}{4}z}{3(1 - \frac{z}{4})(1 - \frac{z}{3})}$$



To show that $|R(z)| \leq 1$ on the imaginary axis, let z = x+ iy and then let x = 0, so we are only considering the imaginary axis.

$$|R(z)| = \frac{1}{3}|\frac{-36+15iy}{(3i+y)(4i+y)}| = \frac{\sqrt{144+25y^2}}{\sqrt{144+25y^2+y^4}} \leq 1$$

We can use the maximum modulus principle to show that this is enough to ensure A-stability. Let Ω be the entire left plane. If $z = x+iy$, then $x \leq 0$ The poles of rational funciton R(z) are at 3 and 4. Both of these occur in the right half plane, so R(z) is analytic on ω which is bounded at infinity. Thus, by the maximum modulus principle, $|R(z)|$ is maximized on the boundary, which is the imaginary axis. We have already shown that this is between -1 and 1.

To show L-stability,

$$\lim_{z\to\infty} |R(z)| = \frac{3+\frac{5}{4}z}{3(1-\frac{z}{4})(1-\frac{z}{3})} = 0$$

Problem 4: Let $g(x) = 0$ represent a system of s nonlinear equations in s unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{1}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess x^0, compute

$$x^{j+1} = g(x^j) \tag{2}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).

(b) Suppose $g(x)$ is differentiable and let $D_x g(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(D_x(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$
\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n) \\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1 \\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j) \\
&\qquad \text{end} \\
&U^{n+1} = \hat{U}^N.
\end{aligned}
$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20,\ 50$ and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues λ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (c) to converge to as $N \to \infty$?

Solution:

(a)

If there exists a norm such that g(x) is Lipschitz continuous with constant $L < 1$in a neighborhood of $\bar{x}$, then :

$$\|g(x) - g(x')\| \le L\|x - x'\|, \{\bar{x} - \delta \le \bar{x} \le \bar{x} + \delta\}$$

Subtracting equation (1) from equation (2), and taking the norm of both sides we get:

$$\|g(x^j) - g(\bar{x})\| = \|x^{j+1} - \bar{x}\|$$

Based on the result above,

$$\|x^{j+1} - \bar{x}\| = \|g(x^j) - g(\bar{x})\| \le L\|x^j - \bar{x}\|$$

Continuing in this manner:

$$\|x^{j+1} - \bar{x}\| \le L\|x^j - \bar{x}\| \le L^2\|x^{j-1} - \bar{x}\|$$

And we get:

$$\|x^{j+1} - \bar{x}\| \le L^j\|x^0 - \bar{x}\|$$

Since $L < 1$, the limit as j approaches infinity, the error will go to zero. Thus fixed point iteration will converge for any starting value in this neighborhood.

(b)

Using multivariable Taylor expansion around the fixed point $\bar{x}$:

$$G(x^j) = G(\bar{x}) + D_x G(\bar{x})(x^j - \bar{x}) + \dots$$

This implies that:

$$\|G(x^j) - G(\bar{x})\| \le \|D_x G(\bar{x})(x^j - \bar{x})\|$$

$$\|G(x^j) - G(\bar{x})\| \le \|D_x G(\bar{x})\|\|(x^j - \bar{x})\|$$

From part (a), we know that: $\|(x^j - \bar{x})\| = \|g(x^j) - g(\bar{x})\|$ We get:

$$\|G(x^j) - G(\bar{x})\| \le \|D_x G(\bar{x})\|^j\|(x^0 - \bar{x})\|$$

Additionally, using the results from part (a), we know that:

$$\|D_x G(\bar{x})\| < 1$$

Since you can bound the spectral radius by the norm, we can conclude that:

$$\rho(D_x G(\bar{x})) < 1$$

(c)

Find the R(z) function for this method by examining the problem: $u' = \lambda u$

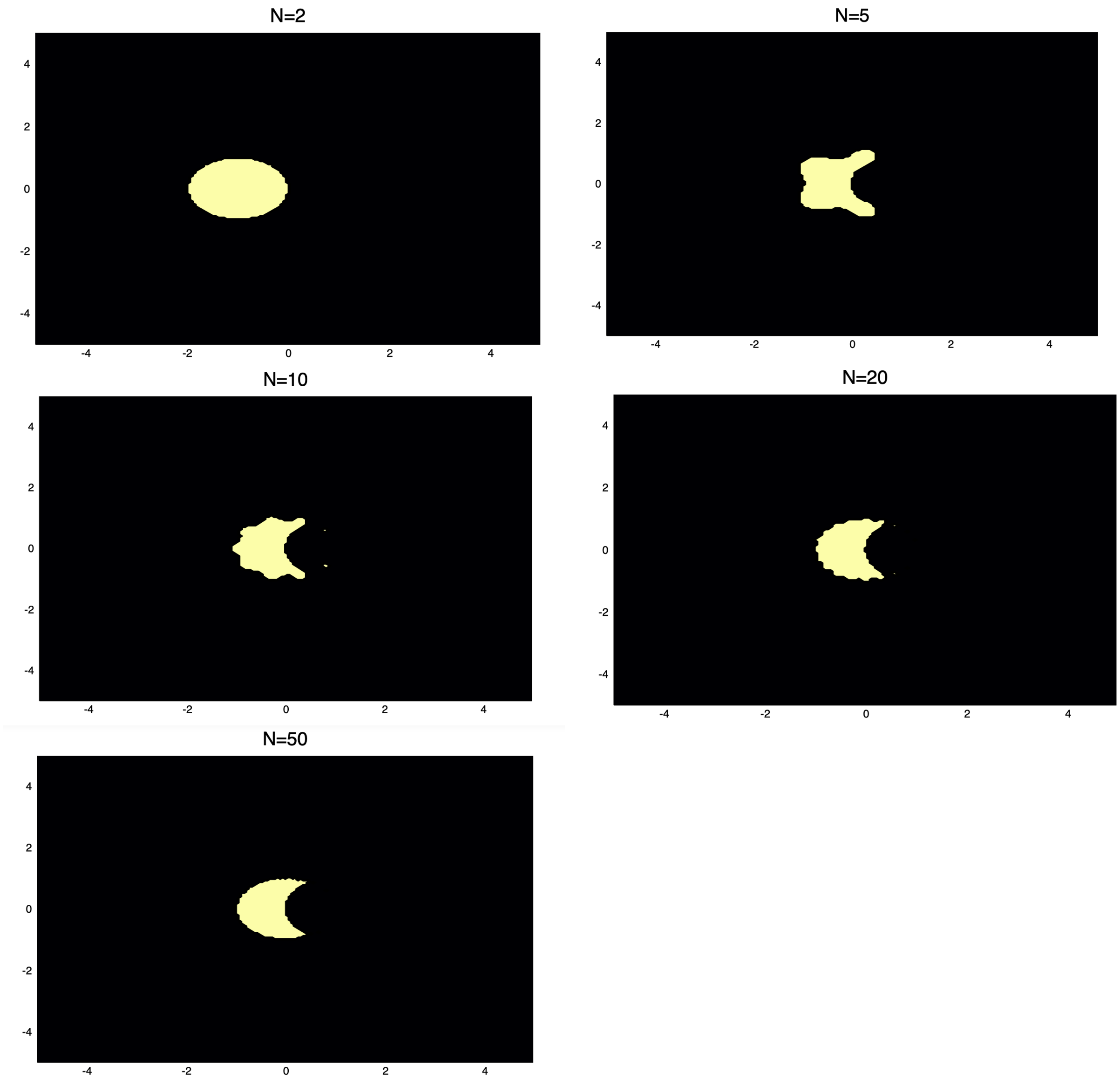
$$\hat{U}^0 = U^n + k\lambda U^n = (1 + k\lambda)U^n$$

$$\hat{U}^1 = U^n + k\lambda\hat{U}^0$$

$$\hat{U}^1 = U^n + k\lambda(1 + k\lambda)U^n = (1 + k\lambda + (k\lambda)^2)U^n$$

$$\hat{U}^N = (1 + z + z^2 + z^3 + ... + z^{N-1}U^n$$

$$R(z) = \sum_{j=0}^{N-1} z^j$$



(d)

Applying this to the problem, $u' = \lambda u$, we get: d

$$\hat{U}^{n+1} = U^n + k\lambda \hat{U}^{N-1}$$

and

$$U^{n+1} = U^n + k\lambda U^{n+1}$$

where $U^{n+1}$is the fixed point

$$\|\hat{U}^N - U^{n+1}\| = \|(U^n + k\lambda \hat{U}^{N-1}) - U^n + k\lambda U^{n+1}\| \leq |\lambda k| \|\hat{U}^{N-1} - U^{n+1}\|$$

Continuing in this matter, we get,

$$\|\hat{U}^N - U^{n+1}\| \leq |\lambda k|^{j+1} \|\hat{U}^0 - U^{n+1}\|$$

Now, we know that this will only converge if:

$$|\lambda k| < 1$$

(e) We would expect from part (d) that the stability region would be a circle of radius 1 around the origin since the method will only converge if $|k\lambda| < 1$ However, you can also see that on the right, the cut out section is in the shape of the non-stable region of backward Euler. So, as N $\to \infty$, The stability region S_N converges to the circle minus the unstable region of backward Euler.

Problem 5: Consider the matrix $M_r = I - rT$ where T is the $m \times m$ matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and $r \geq 0$. Find the largest value of c such that M_r is invertible for all $r \in [0, c)$.

Solution

$$M_r = \begin{bmatrix} 1+2r & -r & 0 & \dots & \\ -r & 1+2r & -r & 0 & \dots \\ 0 & -r & 1+2r & -r & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

Since this is a tridiagonal matrix, the eigenvalues are:

$$\lambda_p = 1 + 2r - 2r\cos\left(\frac{p\pi}{m+1}\right)$$

Based on this, the eigenvalues are bounded between 1 and 4r. This means that 0 is not an eigenvalue and therefore the matrix is invertible and c = ∞