AMATH 586 SPRING 2020
HOMEWORK 2 — DUE APRIL 24 ON GITHUB BY 11PM
SHANNON DOW

Be sure to do a `git pull` to update your local version of the `amath-586-2020` repository.

Problem 1: Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\mathrm{cn}(x,k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\mathrm{cn}(x,m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 7 — produce a plot and and a table.

Solution:
The general form of a Runge-Kutta method is:

$$Y_1 = U^n + k\sum_{j=1}^{3} a_{1j} f(Y_j, t_n + c_j k)$$

$$Y_2 = U^n + k\sum_{j=1}^{3} a_{2j} f(Y_j, t_n + c_j k)$$

$$Y_3 = U^n + k\sum_{j=1}^{3} a_{3j} f(Y_j, t_n + c_j k)$$

$$U^{n+1} = U^n + k\sum_{j=1}^{3} b_j f(Y_j, t_n + c_j k)$$

To be 3rd order accurate, we must satisfy the following conditions:

$$\sum_{j=1}^{3} b_j = 1$$

$$\sum_{j=1}^{3} b_j c_j = \frac{1}{2}$$

$$\sum_{j=1}^{3} b_j c_j^2 = \frac{1}{3}$$

$$\sum_{j=1}^{3} a_{ij} = c_i$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} b_i a_{ij} c_j = \frac{1}{6}$$

To start, let us assume that $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{3}$

$$b_2 c_2 + b_3 c_3 = \frac{1}{2} b_2 + \frac{1}{3} b_3 = \frac{1}{2}$$

$$b_2 = 1 - \frac{2}{3} b_3$$

$$b_2 c_2^2 + b_3 c_3^2 = (1 - \frac{2}{3} b_3)(\frac{1}{4}) + b_3 \frac{1}{9} = \frac{1}{3}$$

$$b_3 = -\frac{3}{2}$$

$$b_2 = 2$$

$$b_1 = \frac{1}{2}$$

Then since we are letting $c_1 = 0$, then $a_{11} = a_{12} = a_{13} = 0$
Also let $a_{22} = a_{23} = 0 \implies a_{21} = c_2 = \frac{1}{2}$
And finally let $a_{31} + a_{32} = c_3 = \frac{1}{3}$
Thus:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} b_i a_{ij} c_j = b_3 a_{32} c_2 = -\frac{3}{2} a_{32} \frac{1}{2} = \frac{1}{6} \implies a_{32} = -\frac{2}{9}$$

$$a_{31} = \frac{5}{9}$$

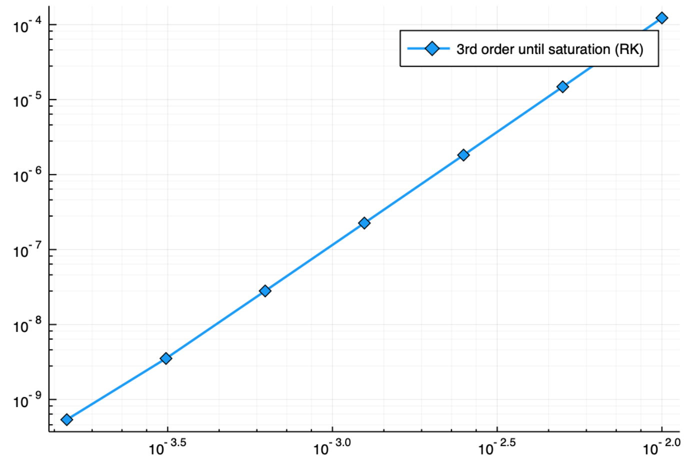
Thus the 3rd order Runge-Kutta method is:

$$Y_1 = U^n$$

$$Y_2 = U^n + \frac{k}{2} f(Y_1, t_n)$$

$$Y_3 = U^n + \frac{5k}{9} f(Y_1, t_n) - \frac{2k}{9} f(Y_2.tn + \frac{1}{2})$$

$$U^{n+1} = U^n + \frac{k}{2} f(Y_1.t_n) + 2k f(Y_2, t_n + \frac{k}{2}) - \frac{3k}{2} f(Y_3, t_n + \frac{k}{3})$$



k	RK 3rd Order
0.005000	8.2853
0.002500	8.1448
0.001250	8.0731
0.000625	8.0377
0.000313	7.9528
0.000156	6.5505

Problem 2: Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+3} = U^{n+1} + 2kf(U^n)$,

(b) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$,

(c) $U^{n+1} = U^n$,

(d) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$,

(e) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

Solution:
The general form of a linear multistep method is:

$$\sum_{j=0}^{r} \alpha_j U^{n+r} = k \sum_{j=0}^{r} \beta_j f(U^{n+j})$$

To be consistent, it must satisfy:

$$\sum_{j=0}^{r} \alpha_j = 0, \sum_{j=0}^{r} (j\alpha_j - \beta_j) = 0$$

(a)
Verify Consistency:

$$U^{n+3} - U^{n+1} = 2kf(U^n)$$
$$\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 0, \alpha_3 = 1$$
$$\beta_0 = 2$$
$$\sum_{j=0}^{r} (j\alpha_j - \beta_j) = -2 - 1 + 3 = 0$$

The method is consistent Verify Zero Stability:

$$p(\xi) = \xi^3 - \xi = \xi(\xi^2 - \xi) = 0 \implies \xi = 0, -1, 1$$
$$\||\xi_j| <= 1$$

and there are no repeated roots, so it is zero stable, therefore convergent.
(b)

$$U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$$

Verify Consistency:

$$\alpha_0 = -\frac{1}{2}, \alpha_1 = -\frac{1}{2}, \alpha_2 = 1,$$
$$\beta_1 = 2$$
$$\sum_{j=0}^{r} (j\alpha_j - \beta_j) = -\frac{1}{2} - 2 + 2 \neq 0$$

The method is inconsistent Verify Zero Stability:

$$p(\xi) = \xi^2 - \frac{1}{2}xi - \frac{1}{2} = (\xi - 1)(\xi + \frac{1}{2}) = 0 \implies \xi = 1, -\frac{1}{2}$$
$$|\xi_j| <= 1$$

and there are no repeated roots, so it is zero stable. It is inconsistent, therefore not convergent.
(c)

$$U^{n+1} - U^n = 0$$

Verify Consistency:

$$\alpha_0 = -1, \alpha_1 = 1$$
$$\beta_j = 0$$
$$\sum_{j=0}^{r} (j\alpha_j - \beta_j) = 1 \neq 0$$

The method is inconsistent Verify Zero Stability:

$$p(\xi) = \xi - 1 = 0, \xi = 1$$
$$|\xi_j| <= 1$$

and there are no repeated roots, so it is zero stable. It is inconsistent, therefore not convergent.
(d)

$$U^{n+4} - U^n = \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$$

Verify Consistency:

$$\alpha_0 = -1, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 1$$

$$\beta_0 = 0, \beta_1 = \frac{4}{3}, \beta_2 = \frac{4}{3}, \beta_3 = \frac{4}{3}, \beta_4 = 0,$$

$$\sum_{j=0}^{r}(j\alpha_j - \beta_j) = -\frac{4}{3} - \frac{4}{3} - \frac{4}{3} + 4 = 0$$

The method is consistent.
Verify Zero Stability:

$$p(\xi) = \xi^4 - 1 = 0, \xi = 1, -1$$

$$|\xi_j| <= 1$$

and there are no repeated roots, so it is zero stable. It is consistent, and zero stable, therefore consistent.
(e)$U^{n+3} + U^{n+2} - U^{n+1} - U^n = 2k(f(U^{n+2}) + f(U^{n+1}))$
Verify Consistency:

$$\alpha_0 = -1, \alpha_1 = -1, \alpha_2 = 1, \alpha_3 = 1$$

$$\beta_0 = 0, \beta_1 = 2, \beta_2 = 2, \beta_3 = 0$$

$$\sum_{j=0}^{r}(j\alpha_j - \beta_j) = -3 + 2 - 2 + 3 = 0$$

The method is consistent.
Verify Zero Stability:

$$p(\xi) = \xi^3 + \xi^2 - \xi - 1 = (x-1)(x+1)^2 = 0 \implies \xi = 1, -1, -1$$

$$|\xi_j| = 1$$

and there are repeated roots, so it is not zero stable.
It is consistent, and zero stable, therefore consistent.

Problem 3: For the one-step method (6.17), with Ψ given in (6.18), show that the Lipschitz constant is $L' = L + \frac{k}{2}L^2$ where L is the Lipschitz constant for f.

Solution:
If we know that f is Lipschitz continuous with Lipschitz constant L, then:

$$\|f(u) - f(v)\| \leq L\|u - v\|$$

Let $\Psi(u,t,k) = f(u + \frac{1}{2}kf(u))$

$$[\|\Psi(u) - \Psi(v)\| = \|f(u + \frac{1}{2}kf(u)) - f(v + \frac{1}{2}kf(v))\| \leq L\|u + \frac{1}{2}kf(u) - v - \frac{1}{2}kf(v)\|$$

Since f is Lipschitz, then:

$$\leq L\|u - v + \frac{1}{2}k(f(u) - f(v))\| \leq L\|u - v\| + L\|\frac{1}{2}k(f(u) - f(v))\|$$

by the triangle inequality, and then again by Lipschitz:

$$\|\Psi(u) - \Psi(v)\| \leq (L + \frac{1}{2}L^2)\|u - v\|$$

And thus Ψ is Lipschitz with constant $L' = L + \frac{1}{2}L^2$

Problem 4: The Fibonacci numbers

(a) Determine the general solution to the linear difference equation $U^{n+2} = U^{n+1} + U^n$.

(b) Determine the solution to this difference equation with the starting values $U^0 = 1$, $U^1 = 1$. Use this to determine U^{30}. (Note, these are the *Fibonacci numbers*, which of course should all be integers.)

(c) Show that for large n the ratio of successive Fibonacci numbers U^n/U^{n-1} approaches the "golden ratio" $\phi \approx 1.618034$.

Solution:

(a)

$$U^{n+2} - U^{n+1} - U^n = 0$$
$$\xi^2 - \xi - 1 = 0$$
$$\xi = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

The general form of the solution to the linear difference equation is:

$$U^n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$$

(b)

$$U^0 = c_1 + c_2 = 1$$
$$U^1 = c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1$$
$$(1 - c_2)(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1$$

Solving for c_2 and then c_1, we get:

$$c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$
$$U^n = \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$
$$U^{30} = \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{30} + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{30} = 1346269$$

(c)

$$\lim_{n\to\infty} U^n/U^{n-1} = \frac{\frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n}{\frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-1} + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n-1}}$$

Since $\frac{1-\sqrt{5}}{2} < 1$ those terms go to zero and we get:

$$\lim_{n\to\infty} U^n/U^{n-1} = \frac{\frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n}{\frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-1}} = \frac{1+\sqrt{5}}{2} \approx 1.618034$$

Problem 5: Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r. For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathrm{O}(z^{p+1}) \quad \text{as } z \to 0.$$

This indicates that the one-step error is $\mathrm{O}(z^{p+1})$ on this problem, as expected for a pth order accurate method.

The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to $u'(t) = \lambda u(t)$. Show that in fact when applied to this problem the method becomes $U^{n+1} = R(z)U^n$ where $R(z)$ is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of e^z through $O(z^4)$ terms.

We will see that this function $R(z)$ is also important in the study of absolute stability of a one-step method.

Solution:

The fourth order RK:

$$Y_1 = U^n$$
$$Y_2 = \frac{1}{2}kf(Y_1, t_n)$$
$$Y_3 = \frac{1}{2}kf(Y_2, t_n + \frac{k}{2})$$

$$Y_4 = kf(Y_3, t_n + \frac{k}{2})$$

$$U^{n+1} = U^n + \frac{k}{6}[f(Y_1, t_n) + 2f(Y_2, t_n + \frac{k}{2}) + 2f(Y_3, t_n + \frac{k}{2}) + kf(Y_4, t_n + k)]$$

$$u'(t) = \lambda u(t)$$

$$Y_1 = U^n$$

$$Y_2 = U^n + \frac{1}{2}k\lambda U^n = (1 + \frac{1}{2}k\lambda)U^n$$

$$Y_3 = U^n + \frac{1}{2}k\lambda(1 + \frac{1}{2}k\lambda)U^n = (1 + \frac{1}{2}k\lambda + \frac{1}{4}(k\lambda)^2)U^n$$

$$Y_4 = U^n + k\lambda((1 + \frac{1}{2}k\lambda + \frac{1}{4}(k\lambda)^2U^n = (1 + k\lambda + \frac{1}{2}(k\lambda)^2 + \frac{1}{4}(k\lambda)^3)U^n$$

$$U^{n+1} = U^n + \frac{k}{6}[\lambda U^n + 2\lambda(1+\frac{1}{2}k\lambda)U^n + 2\lambda(1+\frac{1}{2}k\lambda+\frac{1}{4}(k\lambda)^2)U^n + \lambda(1+k\lambda+\frac{1}{2}(k\lambda)^2+\frac{1}{4}(k\lambda)^3)U^n]$$

Let $z = \lambda k$

$$U^{n+1} = U^n + \frac{z}{6}[1 + 2 + z + 2 + z + \frac{1}{2}z^2 + 1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3]U^n$$

$$U^{n+1} = (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4)U^n$$

Thus:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

which agrees with the Taylor expansion of e^z and is a 4th order polynomial.

Problem 6: Determine the function $R(z)$ described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for $u'(t) = \lambda u(t)$. (You might want to convince yourself these are the same method).

Confirm that $R(z)$ agrees with e^z to the expected order.

Note that for this implicit method $R(z)$ will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1 - \epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots.$$

Solution:

$$U^* = U^n + \frac{k}{4}(f(U^n) + f(U^*))$$

$$U^{n+1} = \frac{1}{3}(4U^* - U^n + kf(U^{n+1}))$$

$$U^* = U^n + \frac{k}{4}(\lambda U^n + \lambda U^*) = (1 + \frac{z}{4})U^n + \frac{z}{4}U^*$$

$$U^* = \frac{(1 + \frac{z}{4})}{(1 - \frac{z}{4})}U^n$$

$$U^{n+1} = \frac{1}{3}(4\frac{(1 + \frac{z}{4})}{(1 - \frac{z}{4})}U^n - U^n + zU^{n+1})$$

$$U^{n+1} = \frac{1}{3}(\frac{4 + z - 1 + \frac{z}{4}}{1 - \frac{z}{4}})U^n + \frac{z}{3}U^{n+1}$$

$$U^{n+1} = \frac{3 + \frac{5}{4}z}{3(1 - \frac{z}{4})(1 - \frac{z}{3})} U^n$$

Using the Neumann series:

$$U^{n+1} = (1 + \frac{5}{12}z)(1 + \frac{z}{4} + \frac{z^2}{16} + O(z^3))(1 + \frac{z}{3} + \frac{z^2}{9} + O(z^3))U^n$$

$$U^{n+1} = (1 + \frac{z}{4} + \frac{z^2}{16} + \frac{5z}{12} + \frac{5z^2}{48} + O(z^3))(1 + \frac{z}{3} + \frac{z^2}{9} + O(z^3))U^n$$

$$U^{n+1} = (1 + \frac{2z}{3} + \frac{z^2}{6} + O(z^3))(1 + \frac{z}{3} + \frac{z^2}{9} + O(z^3))U^n$$

$$U^{n+1} = 1 + z + \frac{1}{2}z^2 + O(z^3)U^n$$

Since this method is second order accurate, we expect it to match the Taylor series up toz^2 which is true.