Distribution of Sample Proportions and Confidence Interval for a Population Proportion

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STAT 217

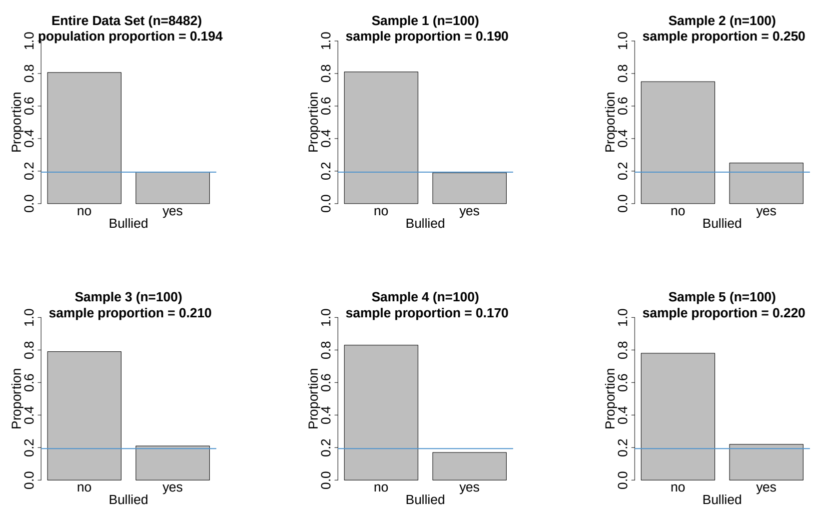
OUTLINE

The Data

From the CDC's 2013 Youth Risk Behavior Surveillance System

	gender	height_m	weight_kg	bmi	carried_weapon	bullied	days_drink
1	female	1.73	84.37	28.2	yes	no	30
2	female	1.6	55.79	21.8	no	yes	1
3	female	1.5	46.72	20.8	no	yes	0
4	female	1.57	67.13	27.2	no	yes	0
5	female	1.68	69.85	24.7	no	no	0
6	female	1.65	66.68	24.5	no	no	1
7	male	1.85	74.39	21.7	no	no	0
8	male	1.78	70.31	22.2	yes	no	0
9	male	1.73	73.48	24.6	no	yes	0
10	male	1.83	67.59	20.2	no	no	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8482	male	1.73	68.95	23	no	no	0

The idea



Parameters vs statistics

A **parameter** is a numerical summary of a population. It is usually *unknown*, although we can make *assumptions* about parameter values for population distributions. We generally use Greek letters (without bars or hats) to denote population parameters:

population mean	μ
population standard deviation	σ
population proportion	p

A **statistic** is a numerical summary of the sample. It is estimated from observed data. We generally use lower case letters (with bars or hats) to denote sample statistics:

sample mean	$\bar{x}$
sample standard deviation	s
sample proportion	$\hat{p}$

Describing distributions

Group Exercise

When describing distributions, what are the three features you should address?

1.

2.

3.

Three distributions to keep in mind

1. The **population distribution** refers to the actual distribution of a variable in a population.
2. The **data distribution** refers to the distribution of observed values from a *single* sample.
3. The **sampling distribution** refers to the distribution of a statistic from *many* samples.

sampling distribution of the sample proportion	=	the distribution of sample proportions ($\hat{p}$) from many samples
sampling distribution of the sample means	=	the distribution of sample means ($\bar{x}$) from many samples

Population and data distribution of `bullied`

Consider the 8,482 observations from the YRBSS data set to be the *entire* population of interest. Now let's describe the **population distribution** of `bullied`.

- True population proportion:

Example data distributions from `bullied`

Now let's take three random samples of size $n = 10$ from the population distribution of `bullied`. Each random sample represents a **data distribution**.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$\hat{p}$
Sample 1											
Sample 2											
Sample 3											

Many samples from `bullied`

Let's repeat the process and take 1000 random samples of size $n = 10$ from the population distribution of `bullied`.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$\hat{p}$
Sample1	no	yes	no	no	no	no	no	no	no	yes	0.2
Sample2	no	no	no	no	no	no	no	no	no	yes	0.1
Sample3	no	no	yes	no	no	yes	yes	no	no	no	0.3
Sample4	no	no	no	yes	yes	no	no	no	yes	no	0.3
Sample5	no	yes	no	no	no	no	no	yes	no	no	0.2
Sample6	no	n	no	yes	yes	yes	yes	no	no	no	0.4
Sample7	yes	no	no	no	no	no	no	no	no	yes	0.2
Sample8	no	yes	yes	yes	no	no	no	no	no	yes	0.4
Sample9	no	yes	no	no	no	no	no	yes	no	no	0.2
Sample10	no	yes	no	no	no	no	no	no	no	no	0.1
Sample11	yes	no	no	no	no	no	no	no	no	no	0.1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sample1000	no	no	no	no	no	no	no	no	no	no	0

Group Exercise

What do you think will be the shape of the distribution of the 1000 sample proportions?

1. bell-shaped
2. left-skewed
3. right-skewed
4. uniform

Simulated sampling distribution, example 1

The collection of the sample proportions from the 1000 samples of size $n = 10$ represents a simulated **sampling distribution** of the sample proportion.

- Shape of the sampling distribution:
- Mean of the sampling distribution:
- Standard deviation of the sampling distribution:

Re-cap, example 1

Population distribution

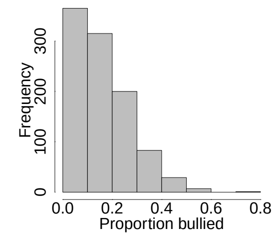
$p = 0.194$

Single data distribution ($n = 10$)

```
no yes no no no
no no no no yes
```

$\hat{p} = 0.2$

Distribution of 1000 sample proportions from samples of size $n = 10$



mean $= 0.202$

sd $= 0.127$

Group Exercise

What do you think will happen to the distribution of sample proportions if we increase the sample size for each individual sample from $n = 10$ to $n = 100$? (The number of samples will stay the same at 1000.)

The shape will be _____________, the mean will _____________, the standard deviation will _____________.

1. shape: right-skewed, left-skewed, approximately normal
2. mean: increase, decrease, remain the same
3. standard deviation: increase, decrease, remain the same

Simulated sampling distribution, example 2

The collection of the sample proportions from the 1000 samples of size $n = 100$ represents a simulated **sampling distribution** of the sample proportion.

- Shape of the sampling distribution:
- Mean of the sampling distribution:
- Standard deviation of the sampling distribution:

Re-cap, example 2

Population distribution

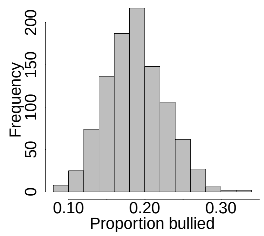
$p = 0.194$

Single data distribution ($n = 100$)

```
no no no no no no
no yes no no yes
no no no no no no
no no no no no no
no no no yes no
no no no no no no
no no no no no no
yes ...
```

$\hat{p} = 0.173$

Distribution of 1000 sample proportions from samples of size $n = 100$



mean $= 0.193$
sd $= 0.04$

Summary

Feature	Example 1 ($n = 10$)	Example 2 ($n = 100$)
Observed in simulation		
Shape		
Mean		
Std Dev		
According to theory		
Shape		
Mean		
Std Dev		

Distribution of Sample Proportions

OR: the sampling distribution of the sample proportion

For a random sample of size n from a population with population proportion p, the distribution of sample proportions has

$$\text{mean} = p, \text{ standard deviation} = \sqrt{\frac{p(1-p)}{n}}$$

Saying the same thing, but with more notation:

$$\text{mean}(\hat{p}) = p, \text{ sd}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

Distribution of Sample Proportions

OR: the sampling distribution of the sample proportion

When $np \geq 10$ and $n(1-p) \geq 10$, then the the distribution of sample proportions has an **approximately normal** shape. That is, when this condition is satisfied, the distribution of sample proportions has:

- shape = normal
- mean = p
- standard deviation = $\sqrt{\dfrac{p(1-p)}{n}}$

The condition $np \geq 10$ and $n(1-p) \geq 10$ means that you have at least 10 expected 'successes' and 10 expected 'failures.'

Group Exercise

Suppose that 80% of Americans prefer milk chocolate to dark chocolate. For which of the following sample sizes would the distribution of the sample proportions of Americans that prefers milk chocolate be approximately normally distributed? Mark all that apply.

1. $n = 20$
2. $n = 40$
3. $n = 60$
4. $n = 80$

Group Exercise

Which of the following affects the variability (or spread) in the distribution of the sample proportions? Select all that apply

1. the population mean
2. the population standard deviation
3. the sample size
4. the number of samples collected

Group Exercise

When discussing sampling distributions, what started off as KNOWN and UNKNOWN?

A. population proportion: (1) known OR (2) unknown

B. sample proportion: (1) known OR (2) unknown

Example

Suppose that 80% of Cal Poly students own a Mac laptop, and that we take samples of size $n = 100$ from the population of all Cal Poly students. Specify (with justification) the following features of the distribution of sample proportions and sketch the distribution of sample proportions.

1. shape
2. mean
3. std dev

Estimation

Given that we generally collect data from a sample (and not a population), how do we *estimate* population parameter values reliably? Examples of estimation:

- A study found that the average breath alcohol concentration (BrAC) was .091 when subjects drank alcohol mixed with a diet drink. By comparison, BrAC was .077 when the same subjects consumed the same amount of alcohol but with a sugary soda.
- Researchers estimate that domestic cats are responsible for the deaths of between 1.4 and 3.7 billion birds and 6.9-20.7 billion mammals annually.

Point estimate vs interval estimate

A **point estimate** is a **single number** that is our 'best guess' for the population parameter. Point estimates are given by sample statistics.

- the average number of hours of sleep on a typical night is 7.024 ($\bar{x} = 7.024$)
- 100/208 students indicated the Emory was their 1^{st} choice ($\hat{p} = 0.48$)

An **interval estimate** is an **interval of numbers** within which the population parameter value is believed to fall.

- What is a plausible range for the **population mean** (μ) number of hours of sleep of Emory students in a typical night?
- What is a plausible range for the **population proportion** (p) of students for whom Emory was their 1^{st} choice?

Confidence Interval

A **confidence interval** is an interval containing the most believable values for a population parameter.

- The probability that this method produces an interval that captures the true parameter value is called the **confidence level**.
- The confidence level is a number close to 1, and is most commonly 0.95.
- A confidence interval with a confidence level of 0.95 is called a 95% confidence interval.

General Form of Confidence Interval:

$$\text{point estimate} \pm \text{margin of error}$$

- the **point estimate** is your best guess of a population parameter, like $\bar{x}$ or $\hat{p}$
- the **margin of error** measures how accurate the point estimate is likely to be in estimated a parameter

Relationship between sampling distribution and confidence intervals

	Sampling Distribution	**Confidence Interval**
Population proportion	known	unknown
Sample proportion	unknown	known

The research question

How can we estimate the population proportion of Cal Poly students that own a Mac laptop?

In random sample of Cal Poly students, 50 out of 67 students reported that they own a Mac laptop.

CI for a population proportion

$$\hat{p} \pm z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

point estimate $\pm$ *critical value* $\times$ *standard error*

point estimate $\pm$ *margin of error*

- the **point estimate** is your best guess of a population parameter $\rightarrow \hat{p}$
- the **critical value** establishes your degree of confidence for that interval $\rightarrow$ use z
- the **standard error** allows for uncertainty in that point estimate $\rightarrow \sqrt{\hat{p}(1-\hat{p})/n}$
- the **margin of error** is the (critical value $\times$ standard error), and is everything after the $\pm \rightarrow z \times \sqrt{\hat{p}(1-\hat{p})/n}$

Choice of *approximate z*

point estimate $\pm z^* \times se \rightarrow$ general form of CI

point estimate $\pm 1 \times se \rightarrow$ approximate ____ CI

point estimate $\pm 2 \times se \rightarrow$ approximate <u>95%</u> CI

point estimate $\pm 3 \times se \rightarrow$ approximate ____ CI

Elements of an interpretation of a confidence interval

1. State the confidence level
2. Refer to the population
3. State the parameter being estimated
4. Utilize context
5. Include a range of values

At the __1__ % confidence level, we estimate that the __2__ __3__ of __4__ is in the interval __5__.

Evaluating claims

Different faculty members have different guesses on the percent of all Cal Poly students that own laptop. Based on the interval calculated, which of these claims are plausible? Mark all that apply.

1. Professor A claims that 60% of students own a Mac laptop.
2. Professor B claims that 70% of students own a Mac laptop.
3. Professor C claims that 80% of students own a Mac laptop.
4. Professor D claims that 90% of students own a Mac laptop.

Conditions required for a CI for p

1. The observations are independent. Discussion
2. $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$

   (at least 10 observed "successes" and 10 observed "failures")

Steps to constructing a confidence interval for a population proportion

1. Check your conditions.
2. Identify z^* for your specified level of confidence.

Confidence level	80%	90%	95%	99%
z^*	1.28	1.65	1.96	2.58

3. Calculate the interval: $\hat{p} \pm z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Example

In random sample of Cal Poly students, 50 out of 67 students reported that they own a Mac laptop. Compute and interpret a 95% CI for the population proportion of Cal Poly students that own a Mac laptop.

Group Exercise

A 95% confidence interval for the true proportion of US citizens who are opposed to issuing traffic tickets from traffic cameras is (0.57, 0.63) based on a sample of 1000 individuals.

What is the point estimate for the proportion of sampled individuals who are opposed to issuing traffic tickets from traffic cameras?

1. 0.57
2. 0.63
3. 0.60
4. 0.95
5. not enough information to determine

Group Exercise

$$\hat{p} \pm z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \ 90\%\ z^* = \_\_\_\_\_\_, \ 95\%\ z^* = \_\_\_\_\_\_$$

The value of z for a 95% CI is __________ than the value of z^* for a 90% CI. This means that higher confidence levels correspond to __________ confidence intervals.

1. greater; wider
2. greater; narrower
3. less; narrower
4. less; wider

What factors affect the width of the CI? Constructing a confidence interval is a *compromise* between an acceptable width of your confidence interval and the desired level of confidence in correct inference.

Group Exercise

Will a 95% confidence interval always contain the estimate of the population proportion ($\hat{p}$)?

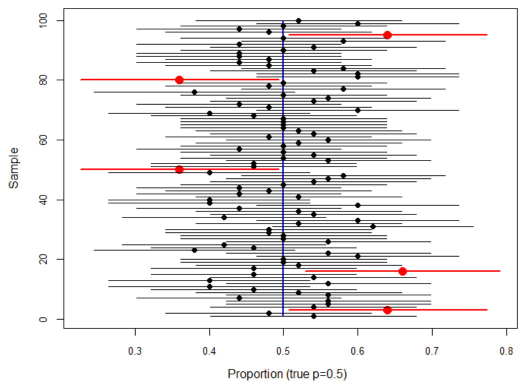
1. Yes
2. No

Will a 95% confidence interval always contain the population proportion (p)?

1. Yes
2. No

The meaning of a 95% CI

If we were to repeatedly sample over and over again, *in the long run* 95% of our confidence intervals would make correct inference. That is, if we took 100 random samples and calculated 100 confidence intervals, we would expect 95 of those confidence intervals to capture the true parameter value.



This figures shows 100 samples of size $n = 50$ where the true $p = 0.50$. Note that 5 of the 100 CI's don't actually capture the true p.

The meaning of a 95% CI, continued

- 95% of samples of this size will produce confidence intervals that capture the true proportion.
- This can be longwinded, so we say "We are 95% confident that the true proportion lies in this interval."
- In reality, we cannot know whether or our sample is one of the 95% that captured p, or one of the unlucky 5% that did not catch p.
- 95% confident means that we arrived at this interval by a method that gives us correct results 95% of the time.

Group Exercise

Based on data from the Winter 2016 STAT 217 class, a 95% confidence interval for the proportion of Cal Poly students who own a Mac laptop is 0.64 to 0.85.

Which of the following is a correct interpretation?

1. We are 95% confident that the proportion of Cal Poly students from this sample who own a Mac laptop is between 0.64 and 0.85.
2. We are 95% confident that the population proportion of Cal Poly students who own a Mac laptop is between 0.64 and 0.85.
3. 95% of the time the proportion of Cal Poly students who own a Mac laptop is between 0.64 and 0.85.
4. More than one statement is correct.

Interpreting the CI

- Interpreting a CI can be challenging
- Students often try to use their own words, and get the interpretation incorrect (just use my words).
- Confidence intervals are about values of <u>population parameters</u>, so *both* pieces of this information **must** be included in the interpretation.

Group Exercise

Sample 1	Sample 2
$\hat{p} = 0.35$	$\hat{p} = 0.35$
$n = 50$	$n = 100$

Suppose we construct a 95% confidence interval for p for both samples. How will the confidence intervals compare?

1. The width of the intervals from the two samples will be the same.
2. The CI for sample 1 will be wider than the CI for sample 2.
3. The CI for sample 2 will be wider than the CI for sample 1.
4. There is not enough information to determine.

Group Exercise

For each of the following scenarios, indicate if the research question should be answered with a confidence interval for a mean or proportion. We want to know something about...

1. how many times a day a baby laughs
2. whether or not babies are born with blue eyes
3. the number of months until the baby first sleeps through the night
4. if babies can sit up independently by age 6 months
5. the heart rate of baby immediately after birth

When are observations not independent?

Observations are not independent when they are correlated with each other. This occurs when observations are related and are more similar to each other than other observations in the data set.

- Measurements made on the same subject are typically not independent.
  - circumference of right and left thigh
  - blood pressure before and after a treatment
- Observations from different subjects may not always be independent.
  - siblings, twin pairs, husband-wife
- Observations from groups of subjects may not always be independent.
  - children that attend the same elementary school, patients that attend the same in-town clinic

Back