Functions of T

Shannon Pileggi

STAT 417

OUTLINE

The survival function: $S(t)$

$F(t)$ $\rightarrow$ gives the probability that the time to event is (less than/greater than) time t

$\rightarrow$ i.e., the probability that an individual (does/does not) survive *beyond* time t

$S(t)$ $\rightarrow$ gives the probability that the time to event is (less than/greater than) time t

$\rightarrow$ i.e., the probability that an individual (does/does not) survive *beyond* time t

$S(t) =$

Interpretations of $S(t)$

Two primary interpretations of $S(t)$:

1. The unconditional probability that a randomly selected individual in the population will survive beyond (not experience the event of interest after) time t, and;
2. The proportion of subjects in a population who have yet to experience the event of interest after time t.

Two **incorrect** interpretations of $S(t)$:

1. $S(t) \neq$ the probability that a subject experiences the event after time t
2. $S(t) \neq$ the probability that the subject experiences the event at time t

Group Exercise

Suppose that the lifetime (in years) of a particular brand of light bulb, T, follows an exponential distribution with parameter $\lambda = 5$. The probability that a randomly selected light bulb lasts at least 7 years is given by...

1. $F(7)$
2. $S(7)$
3. $F(5)$
4. $S(5)$

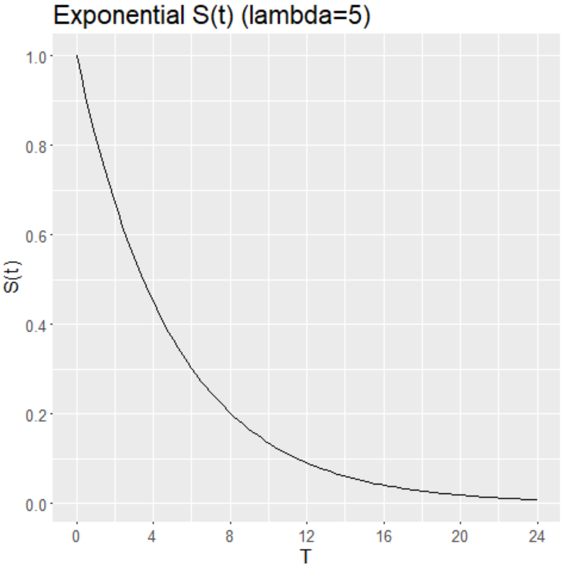
Example of survival function (exponential distribution)

Suppose that the lifetime (in years) of a particular brand of light bulb, T, follows an exponential distribution with parameter $\lambda = 5$.

1. Find the survival function $S(t)$.

2. Calculate the probability that a randomly selected light bulb lasts at least 7 years.

3. Calculate the proportion of light bulbs that last longer than 6 years.


Exponential S(t) (lambda=5)
S(t)
1.0
0.8
0.6
0.4
0.2
0.0
0
4
8
12
16
20
24
T

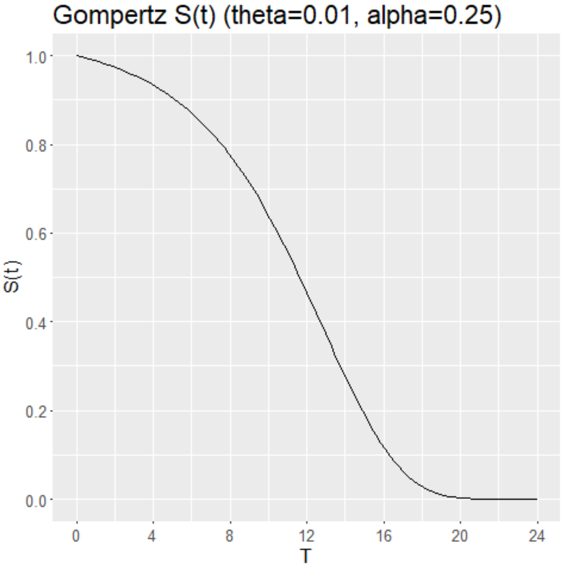
Example of survival function (Gompertz distribution)

Recall the radiation exposure example. The time until death in months for mice exposed to a high dose of radiation follows a Gompertz distribution with parameters $\theta = .01$ and $\alpha = .25$.

1. Find the survival function $S(t)$.

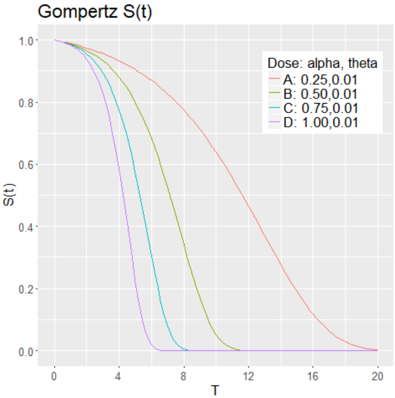
2. Calculate the probability that a randomly chose mouse will live at least 18 months.

3. Calculate the probability that a randomly chosen mouse will die before 6 months.


Gompertz S(t) (theta=0.01, alpha=0.25)
S(t)
1.0
0.8
0.6
0.4
0.2
0.0
0
4
8
12
16
20
24
T

Shape of survival function

Suppose the survival curves correspond to the lifetimes of mice exposed to different doses of radiation; hence, the varying α's (T = time until death from radiation exposure).

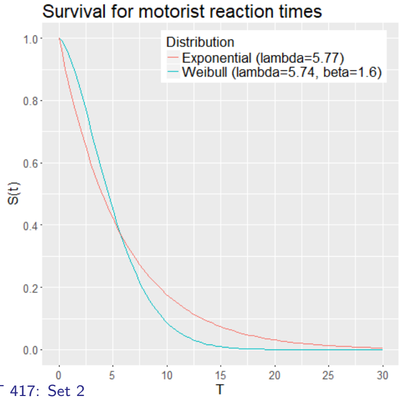


Which dose appear to be most lethal?

1. Dose A
2. Dose B
3. Dose C
4. Dose D

Shape of survival function

Recall the motorist reaction time data (seconds until a blocked driver honks the horn or flashes the high beams). Define $T =$ seconds until motorist reacts aggressively, and consider two possible probability models for T: Weibull($\lambda = 5.74$, $\beta = 1.6$) and exponential($\lambda = 5.77$).

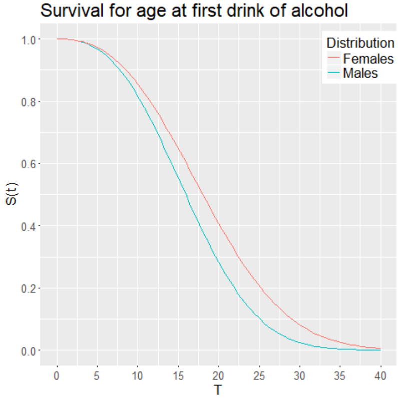


Estimate the probability of survival for

Weibull:	Exponential:
$S(10) =$	$S(10) =$
$S(20) =$	$S(20) =$

Comparing survival experiences

Consider the data on the age when individuals took their first drink of alcohol - do the "survival" experiences differ by gender? Assume the probability model for males is Weibull($\lambda = 18.28$, $\beta = 2.64$) and for females is Weibull($\lambda = 20.85$, $\beta = 2.52$).



1. Compare the survival experiences of males and females.

2. Estimate $S(20)$

   Males:

   Females:

Survival function

Hazard function

Cumulative hazard function

Summary

Motorist reaction time example

Let T (time until motorist reacts aggressively) follow a Weibull distribution with parameters $\lambda = 5.74$ and $\beta = 1.6$. The survival function is given by

$$S(t) = \exp\left[-\left(\frac{t}{5.74}\right)^{1.6}\right]$$

What is the chance that a motorist blocked behind a vehicle does not act aggressively for at least 2 seconds?

Motorist reaction time example, continued

Let T (time until motorist reacts aggressively) follow a Weibull distribution with parameters $\lambda = 5.74$ and $\beta = 1.6$. The survival function is given by

$$S(t) = \exp\left[-\left(\frac{t}{5.74}\right)^{1.6}\right]$$

Suppose that a motorist has been waiting for two seconds behind a stopped vehicle. What is the chance that the motorist will react aggressively within the next 1 second?

Motorist reaction time example, continued

Motorist reaction time example, summary

- The survival function is useful for examining the probability that a randomly selected individual survives beyond time t.

- The survival function does not assess what will happen to an individual immediately beyond time t *conditional* on surviving to time t.

Hazard function

- Viewed in a slightly different way, we are attempting to assess the *risk* or *potential* that an individual who has not done so already, will experience the event of interest in the next instant (or "very small") amount of time.

- This (conditional) *risk* is the basis for the concept called **hazard** and for the function that describes changes in hazard over time called the **hazard function**.

- If we are trying to assess the risk in the next "instant" of time, given that the target event has yet to occur, then let's try to develop this idea using conditional probability...

Motorist reaction time: assessing risk

Now let's see what happens when the subsequent interval of time decreases from 1 second to $1/2$ second, and then to $1/4$ of a second:

1. If a motorist has been blocked for at least 2 seconds, then find the probability that the motorist will react aggressively within the next half second. Compare your answer to the conditional probability found in part (2) of the previous example above.
2. Repeat (1) for the next quarter second.
3. What appears to be happening as the amount of elapsed time after 2 seconds becomes very small?
4. Now generalize: If a motorist has not reacted aggressively for at least t seconds, then as the next interval of time shrinks to 0 seconds, what is the chance that a motorist reacts aggressively in that interval?

Motorist reaction time: assessing risk, cont.

Motorist reaction time: assessing risk, cont.

Hazard function

- So we cannot assess instantaneous risk of event occurrence at time t using conditional probability, since it will always approach 0 as Δt approaches 0.
- But if we scale the limiting conditional probability by dividing by the change in time, Δt, then we get the following *instantaneous rate*, known as the **hazard function**, denoted $h(t)$:

Hazard function

- The hazard function $h(t)$ describes the rate of "failure" or "event experience" *per unit of time* in an instant after time t (sometimes called the *instantaneous rate of failure*), *conditional* on surviving to time t.
- For *continuous* T, $h(t)$ is *not* a probability and should never be interpreted as such. The only restriction on values of $h(t)$ is that $h(t) \geq 0$.
- In other fields, $h(t)$ is known as:
  - Conditional failure rate (reliability theory).
  - Force of mortality (demography).
  - Intensity function (stochastic processes).
  - Age-specific failure rate (epidemiology).

General connection between the hazard function $h(t)$ and the survival function $S(t)$:

Formal definition of $h(t)$

- The formal definition for the hazard function $h(t)$ is:

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | T \geq t)}{\Delta t}$$

where Δt represents a *small* elapsed amount of time.

- From this definition, we can derive another useful mathematical expression for $h(t)$:

Motorist reaction time: Weibull hazard function

Suppose that the time until the motorist reacts aggressively follows a Weibull probability model with parameters $\beta = 1.6$ and $\lambda = 5.74$. Note that:

$$f(t) = \frac{1.6t^{.6}}{(5.74)^{1.6}} \exp\left[-\left(\frac{t}{5.74}\right)^{1.6}\right]$$

$$S(t) = \exp\left[-\left(\frac{t}{5.74}\right)^{1.6}\right]$$

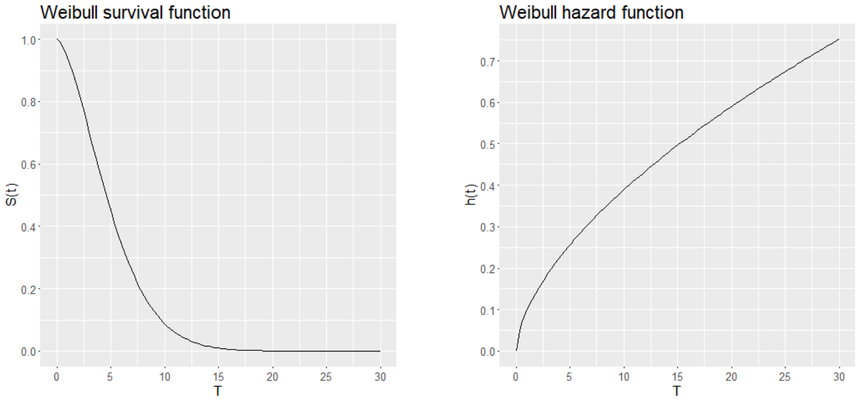
Motorist reaction time: Weibull hazard function

1. Derive the hazard function using the expression $h(t) = \frac{f(t)}{S(t)}$

Motorist reaction time: Weibull hazard function

2. Derive the hazard function using the expression $h(t) = -\frac{d}{dt}\ln[S(t)]$

Motorist reaction time: Weibull hazard function



Motorist reaction time: Weibull hazard function

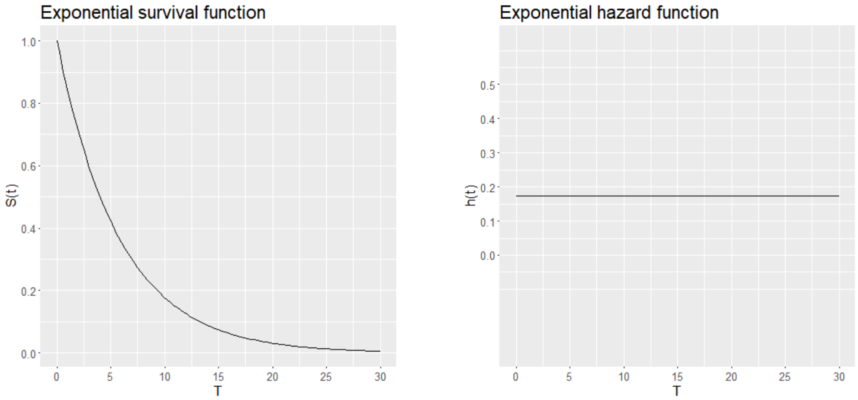
3. Examine the survival and hazard plots and comment on aggressive behavior over time.

Motorist reaction time: exponential hazard function

Assume that the time until a motorist acts aggressively follows an exponential distribution with $\lambda = 5.77$.

1. Find the hazard function $h(t)$.

Motorist reaction time: exponential hazard function



Motorist reaction time: exponential hazard function, cont.

2. What does this result imply about the risk of a randomly selected motorist acting aggressively?

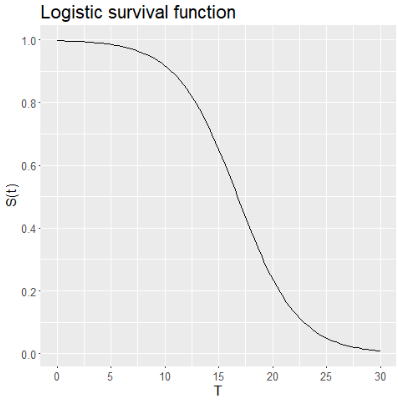
Age at first drink: logistic hazard function

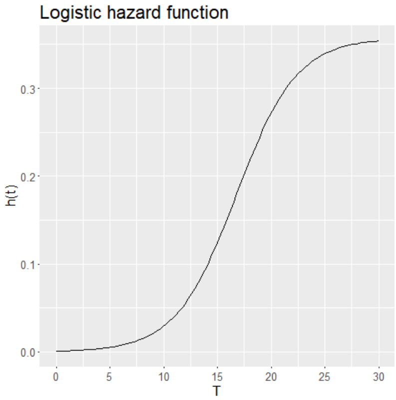
Another probability model used for describing time-to-event random variables is the **logistic distribution** with parameters α and β, which has probability density function given by:

$$f(t) = \frac{\exp^{-(t-\alpha)/\beta}}{\beta\left[1+\exp^{-(t-\alpha)/\beta}\right]^2}, \ \beta > 0$$

Recall data on the age of first drink of alcohol, and suppose that $T =$ "age at first drink of alcohol" follows the logistic distribution with parameter values $\alpha = 16.74$ and $\beta = 2.8$.

Age at first drink: logistic hazard function





Age at first drink: logistic hazard function

Comment on how the risk of first alcoholic drink changes with age.

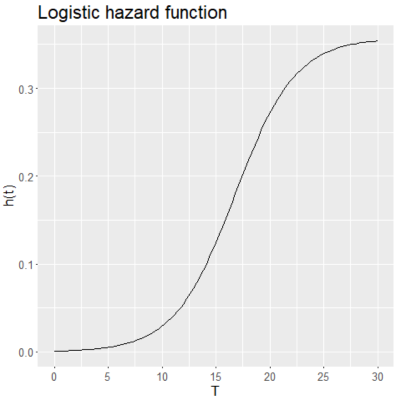
Cumulative hazard function

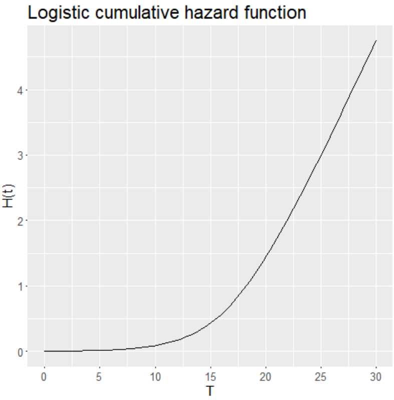
- Sometimes the changes in $h(t)$ are subtle, and it can be difficult to describe periods of increasing and decreasing risk.
- An alternative method to assess and describe how the hazard rates change over time is to investigate the accumulation of the hazard rates over time, and look for patterns in the **cumulative hazard**.
- The function that allows us to examine accumulated hazard over time is called the **cumulative hazard function**.
- The cumulative hazard function, denoted $H(t)$, is the accumulated risk (hazard) of experiencing an event up to time t, and is given by:

Cumulative hazard function, cont.

- An important point is that the cumulative hazard function $H(t)$ is neither a probability, nor a rate - it is simply an accumulation of (hazard) rates over time.
- Since $H(t)$ is *accumulating* rates over time, it never decreases (and rarely ever remains constant).
- Therefore, we can examine the nature of the increase in $H(t)$ (perhaps over specific time intervals of interest), i.e. whether the *rate* of increase in the curve is increasing or decreasing.
- How do we measure the rate of increase in $H(t)$ at a particular point in time?

Age at first drink: cumulative hazard function





Relationship between $H(t)$ and $h(t)$, cont.

- Period of time when the *rate* of increase in $H(t)$ is *increasing* indicates that:

- Period of time when the *rate* of increase in $H(t)$ is *decreasing* indicates that:

- Period of time when the *rate* of increase in $H(t)$ is *constant* indicates that:

- Period of time when the *rate* of increase in $H(t)$ is 0 indicates that:

Expressions for $H(t)$

Recall that the $H(t)$ is given by:

$$H(t) = \int_0^t h(y)dy$$

It can also be shown that:

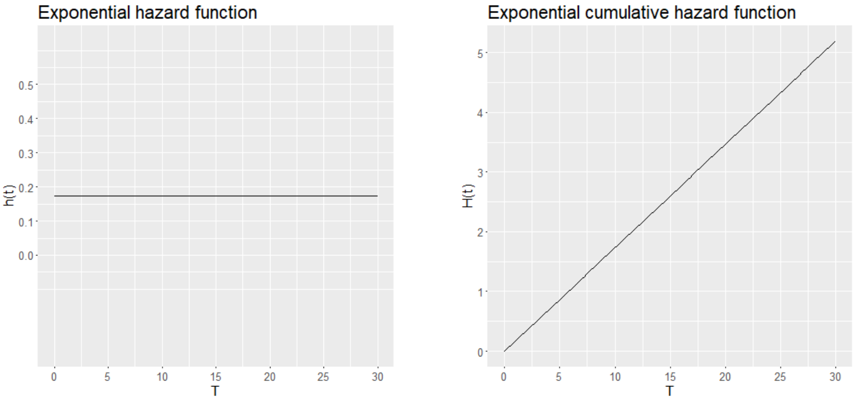
$$H(t) = -\ln[S(t)]$$

Motorist reaction time: exponential cumulative hazard

Consider an exponential($\lambda = 5.77$) probability model for the motorist reaction time variable. Derive $H(t)$.

$$h(t) = \frac{1}{5.77}$$

Motorist reaction time: exponential $h(t)$ and $H(t)$

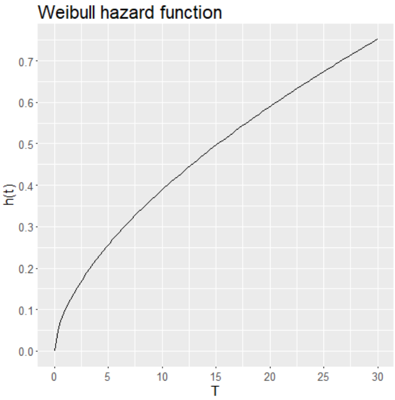


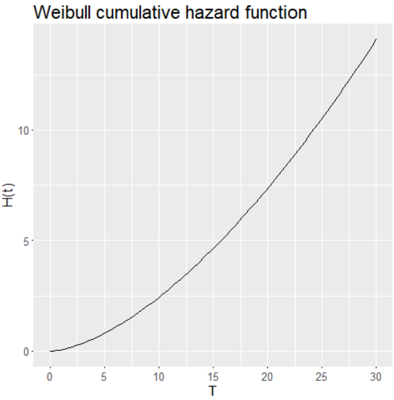
Motorist reaction time: Weibull cumulative hazard

Consider a Weibull($\beta = 1.6$ and $\lambda = 5.74$) probability model for the motorist reaction time variable. Derive $H(t)$.

$$S(t) = \exp\left[-\left(\frac{t}{5.74}\right)^{1.6}\right]$$

Motorist reaction time: Weibull $h(t)$ and $H(t)$





Probability models for T

Exponential $f(t) = \frac{1}{\lambda}\exp(-t/\lambda)$

Weibull $f(t) = \frac{\beta t^{\beta-1}}{\lambda^{\beta}}\exp(-(t/\lambda)^{\beta})$

Gompertz $f(t) = \theta e^{\alpha t}\exp\left[\frac{\theta}{\alpha}(1 - e^{\alpha t})\right]$

Lognormal $f(t) = \frac{\exp\left[-\frac{1}{2}\left(\frac{\ln(t)-\mu}{\sigma}\right)^2\right]}{t(2\pi)^{1/2}\sigma}$

Logistic $f(t) = \frac{\exp^{-(t-\alpha)/\beta}}{\beta\left[1+\exp^{-(t-\alpha)/\beta}\right]^2}$

Relationships between functions of T

Function	Relationships
PDF	$f(t) = \frac{d}{dt}F(t)$
CDF	$F(t) = \int_0^t f(y)dy$
Survival	$S(t) = 1 - F(t) = \exp[-H(t)] = \exp[-\int_0^t h(y)dy]$
Hazard	$h(t) = f(t)/S(t) = -\frac{d}{dt}\ln[S(t)]$
Cum. Haz.	$H(t) = \int_0^t h(y)dy = -\ln[S(t)]$