Course: Master of Science in Quantitative Finance

Subject: QF 605 Fixed Income Securities

Project Report

Group Number: Group 6

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PART I: Bootstrapping Swap Curves

1.1 Bootstrap OIS Discount Factor

Given Overnight Index Swap (OIS) fixed rates, calculating OIS discount factor follows the following procedure:

1. Calculate the $6m$ and $1y$ **overnight rate** and **discount factor**.

   By deriving the formula in *Project Note*:

$$D_o(0,6m) = 1 \div \left(1 + \frac{f_0}{360}\right)^{360\times 0.5}$$

$$D_o(0,1y) = \left(1 + \frac{f_0}{360}\right)^{-180} \left(1 + \frac{f_1}{360}\right)^{-180}$$

2. Given the earlier discount factors, bootstrapping new discount factor for the following year 2 to 30, with the following pseudo-code.

Algorithm 1
1: **for** every year between year 2 ~ 30:
2: **if** rate is present **and** current discount factor is not yet present **and** previous discount factor is present:
3: calculate the discount factor using root search, such that $PV_{flt} = PV_{fix}$
4: prepare for future years by calculating sum of discount factor, PV_{flt} and PV_{fix}

3. Use linear interpolation to interpolate the discount factors where OIS rate is not given.

1.2 Bootstrap LIBOR Discount Factor

Obtaining LIBOR discount factor is similar to the OIS discount factor. Though the forward appreciation or accrual rate (uncollateralized LIBOR rate) is different from the backward discounting rate (collateralized OIS rate). Thus, the floating rate calculation is different.

1. Calculate the $6m$ and $1y$ **discount factor**.

   For $\widetilde{D}(0,6m)$:

$$\widetilde{D}(0{,}6m) = \frac{1}{1 + 0.5 \times L(0,6m)}$$

   For $\widetilde{D}(0,1y)$:

$$\widetilde{D}(0{,}1y) = \widetilde{D}(0{,}6m) \div \left(\frac{L(6m,1y)}{2} + 1\right)$$

2. Calculate new discount factor given earlier discount factors (Bootstrapping)

Algorithm 2
1: **for** every year between year 0.5 ~ 30:
2: **if** IRS data is present:
3: find the $\widetilde{D}(0,t)$ such that $PV_{flt} = PV_{fix}$

3. Use linear interpolation to interpolate the discount factors where IRS rate is not given.

Below is the graph for both discount factor curve.

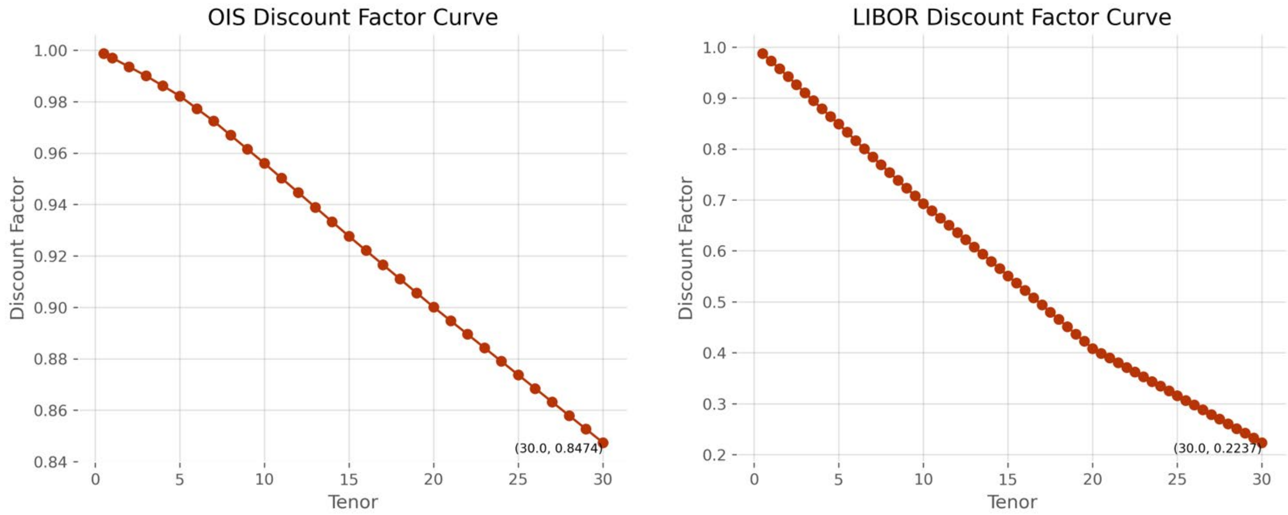

Figure 1-1 Graph for OIS Discount Factor Curve and LIBOR Discount Factor Curve

1.3 Calculate Forward Swap Rates

Par swap rate, K is determined such that $PV_{flt} = PV_{fix}$,

where

$$PV_{flt} = 0.5 \sum_{0.5}^{n} D_o(0, T_i)\, LIBOR(0, T_i)$$

$$PV_{fix} = 0.5K \sum_{0.5}^{n} D_o(0, T_i)$$

while making use of the previous discount factor table.

By calculating the PV in the specified interval, the K can be obtained accordingly. The result is shown below.

Table 1-1 Forward Swap Rates

Expiry / Tenor (Year)	1	2	3	5	10
1	0.032007	0.033259	0.034011	0.035255	0.038420
5	0.039274	0.040075	0.040050	0.041061	0.043576
10	0.042084	0.043006	0.043982	0.046124	0.053280

PART II: Swaption Calibration

2.1 Data Explanation

In this part, the dataset provided insights into the implied volatilities of swaptions, which means significantly in trading and investment strategies for market participants. It was focusing on the various components as expiry, tenor, and strike levels, featuring expiries of 1 year, 5 years, and 10 years, tenors of 1 year, 2 years, 3 years, 5 years, and 10 years and the range of strike levels from 200 basis points below the forward rate to the above 200 basis points.

The choices of strike levels covered types of At-the-money (ATM), Out-of-the-money (OTM) and In-the-money (ITM) relative to the forward rate of the underlying swap, expressed in basis points. It tested the sensitivity of implied volatilities at different strike levels elucidating market participants' views on potential interest rate movements.

2.2 Model Calibration

2.2.1 Calibrated Displaced-Diffusion Model

Displaced-Diffusion model is a transformation of Black 76 model, similar to the option pricing, the swaption can be valued as, for payer swaption,

$$V_{n,N}^{payer}(0) = P_{n+1,N}(0)\left[S_{n,N}(0)\Phi(d_1) - K\Phi(d_2)\right]$$

And if the implied volatility skew is between normal and log normal, the pricing formula of payer swaption under displaced-diffusion model is,

$$V_{n,N}^{payer}(0) = P_{n+1,N}(0)Black76Call\left(\frac{S_{n,N}(0)}{\beta},\ K + \frac{1-\beta}{\beta}S_{n,N}(0),\ \sigma\beta,\ T\right)$$

The β in the Displaced-Diffusion model reflected how changes in the forward rates affected the current interest rate. In this project, it was considered to implement the least square method to find the best fitting parameter with a reasonable initial guess and bound.

Table 2-1 Fitted Parameters in Calibrated Displaced Diffusion Model

Sigma, σ					
Expiry/ Tenor (Year)	**1**	**2**	**3**	**5**	**10**
1	0.2250	0.2872	0.2978	0.2607	0.2447
5	0.2726	0.2983	0.2998	0.2660	0.2451
10	0.2854	0.2928	0.2940	0.2674	0.2437
Beta, β					
Expiry/ Tenor (Year)	**1**	**2**	**3**	**5**	**10**
1	1.6351×10^{-10}	6.2099×10^{-8}	9.1014×10^{-13}	3.1328×10^{-6}	2.4892×10^{-5}
5	8.9731×10^{-7}	1.5710×10^{-7}	1.5791×10^{-5}	7.3193×10^{-6}	5.7790×10^{-2}
10	5.1749×10^{-7}	4.0861×10^{-8}	1.6204×10^{-6}	6.1306×10^{-5}	6.4590×10^{-4}

2.2.2 Calibrated SABR Model Parameters

The SABR Model (Stochastic Alpha-Beta-Rho) is used to describe the evolution of the underlying asset price S_t and its associated volatility α_t. The basic equation is as follows:

$$dF_t = \alpha_t F_t^{\beta} dW_t^F$$

$$d\alpha_t = \nu\alpha_t dW_t^{\alpha},$$

where $dW_t^F dW_t^{\alpha} = \rho dt$, where ρ is the correlation between two Brownian motions and ν in this stochastic process is called volatility of volatility.

To get the best-fitted model, also used the least square method to calculate essential parameters ρ, ν, α in SABR model, with demonstrated $\beta = 0.9$. Notably, selecting an appropriate initial guess is crucial for calibration. Different initial guess values would yield diverse calibration results, and the bounded nature of the three parameters adds uncertainty to the outcomes. Thus, it was necessary to involve averaging results from multiple iterations to identify a guess value that offered relatively accurate calibration.

Table 2-2 Fitted Parameters in Calibrated SABR model

			Alpha, α		
Expiry/ Tenor (Year)	**1**	**2**	**3**	**5**	**10**
1	0.1391	0.1847	0.1969	0.1781	0.1700
5	0.1666	0.1995	0.2103	0.1902	0.1748
10	0.1783	0.1948	0.2083	0.2016	0.1802
			Nu, ν		
Expiry / Tenor (Year)	**1**	**2**	**3**	**5**	**10**
1	2.0494	1.6774	1.4381	1.0648	0.7922
5	1.3404	1.0620	0.9368	0.6745	0.5113
10	1.0103	0.9286	0.8717	0.7225	0.5804
			Rho, ρ		
Expiry/Tenor(Year)	**1**	**2**	**3**	**5**	**10**
1	-0.6332	-0.5251	-0.4828	-0.4145	-0.2557
5	-0.5857	-0.5470	-0.5497	-0.5068	-0.4157
10	-0.5484	-0.5475	-0.5536	-0.5637	-0.5084

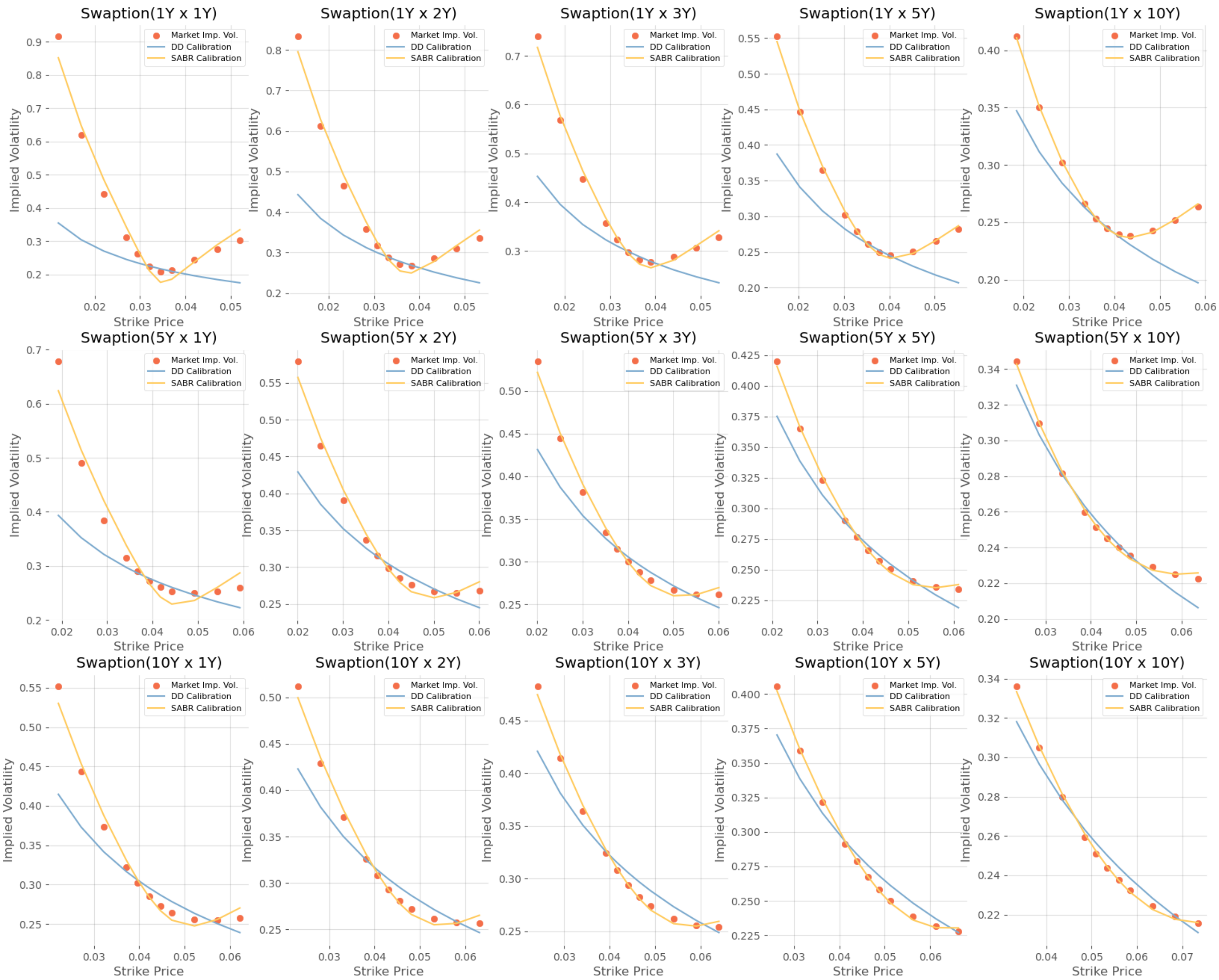

Figure 2-1 Swaption Volatility Smile for Calibrated Models

In the Figure 2-1, there is an insight of swaption volatility smiles for each calibrated model of both Displaced-Diffusion model and SABR model, compared with the market implied volatility. When determining the optimality of the pricing model, standard market practice is to check how close can it fit observable market prices [1]. As it showed in the Figure 2-1, SABR model performed almost perfectly market price fitting, while Displaced-Diffusion model performed closer to the market as the growth of the tenor years.

2.3 Model Pricing

2.3.1 Payer Swap

Based on the calibrated Displaced-Diffusion model and the SABR model, pricing the swaption with varying strike rates (1% to 8%) could be achieved. For the payer swaption, given $2y \times 10y$, the swaption price was shown in the Table 2-3.

Table 2-3 Payer Swaption under the calibrated Displaced-Diffusion and SABR model

Strikes	**1%**	**2%**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**
Displaced Diffusion	0.2881	0.1949	0.1123	0.0513	0.0174	0.0041	0.0007	6.8×10^{-5}
SABR	0.2897	0.1984	0.1151	0.0520	0.0215	0.0110	0.0070	0.0050

Comparing the results from both models, it was observed similar pricing trends, with slight variations in the calculated prices for each strike rate. And the swaption would approach to zero as the increase of the strike level. The reason behind this is the rise of strike level would yield to the diminish of likelihood of the market floating rate falling below the fixed rate. Consequently, exercising the swaption became less advantageous compared to remaining in the market. Based on the result, the Displaced-Diffusion model generally provided slightly lower prices compared to the SABR model.

2.3.2 Receiver Swap

For receiver swaptions, similarly, the results are shown in Table 2-4.

Table 2-4 Receiver Swaption under the calibrated Displaced-Diffusion and SABR model

Strikes	**1%**	**2%**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**
Displaced Diffusion	0.0190	0.0339	0.0566	0.0890	0.1321	0.1861	0.2506	0.3240
SABR	0.0197	0.0390	0.0616	0.0905	0.1304	0.1865	0.2581	0.3396

In contrast, as the strike level increases, it means that the fixed rate at which the receiver will receive in the swap is higher, while the market floating rate must be lower than the fixed rate at the time of exercise. Thus, the swaption moved away from zero, as the probability of exercising the swaption and receiving payments greater than the market rate decreases.

PART III: Convexity Correction

3.1 Value the CMS Products

Constant maturity swap (CMS) pays a swap rate instead of a LIBOR rate on its floating leg. The value of a CMS payoff could be written as a function of the distribution of the swap rate. Therefore, it is reliable to consider the static replication method to figure out the value of the CMS products.

For static replication of any constant maturity swap (CMS) payoff is:

$$g(F) = F,$$

and a CMS contract paying can be expressed as:

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK,$$

where:

$$h(K) = \frac{g(K)}{IRR(K)}; IRR(K) = \sum_{i=1}^{N\times m} \frac{1}{(1+\frac{K}{m})^i}.$$

From the equations above, m is the payment frequency, which means $m = 2$ represents the semi-annual payment frequency. $N = T_N - T_n$ denotes the tenor of the swap. Further, we need to implement the payer and receiver IRR-settled swaption formulae into the valuation process:

$$V_{n,N}^{pay}(0) = D(0,T_n) \times IRR\left(S_{n,N}(0)\right) \times Black76Call\left(S_{n,N}(0), K, \sigma_{n,N}, T\right);$$

$$V_{n,N}^{rec}(0) = D(0,T_n) \times IRR\left(S_{n,N}(0)\right) \times Black76Put\left(S_{n,N}(0), K, \sigma_{n,N}, T\right),$$

$S_{n,N}(0) = F$ and K are today's forward swap rate and the strike price of the IRR-settled swaption respectively. Meanwhile, $\sigma_{n,N}$ is the implied volatility of the swaption calculated by the SABR model in Part II.

Consequently, we derive the present value of the two CMS products in Table 3-1 based on the valuation flow:

Table 3-1 PV of different types of CMS products

Leg type of CMS products	Present value
Receiving CMS10y semi-annually over the next 5 years	0.2187
Receiving CMS2y quarterly over the next 10 years	0.4318

3.2 Impact on Convexity Correction

It is widely known that CMS is an instrument having cashflows that are paid at the wrong time, and convexity correction is a valid method to gather the right price for financial products. In fact, convexity correction will raise differences in the swap rates with different tenors and maturities.

Forward swap rate VS. CMS rate

From Figure 3-1, we can find that the CMS rate is extremely close to but always higher than the forward swap rate. One mathematical explanation for this result is that we consider two additional integration terms in the CMS formula. However, from the financial perspective, we can interpret it by using the theory of convexity value. Assets with positive convexity are more sensitive to decreases in interest rates than increases. With interest rate changes, an asset with positive convexity will provide an additional return. The CMS rates involve longer-term rates, which typically have higher convexity value.

Interestingly, as the maturity of the swap becomes longer, the difference between the CMS rate and the forward swap rate will be larger. Besides, the gap between them keeps decreasing when the length of the tenor is shorter than 5 years. After that, the moving direction of it becomes inverse and increases as the expiry date is later. For very short tenors, the convexity effect is less pronounced since the cash flows are near-term and there's less uncertainty about them. As the tenor extends, the convexity value increases because the present value of future cash flows becomes more sensitive to changes in interest rates.

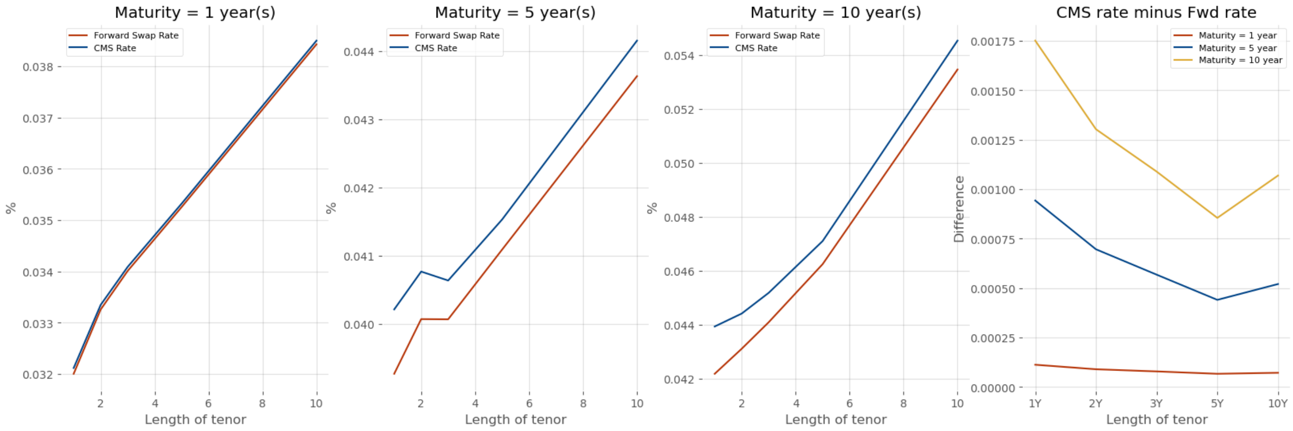

Figure 3-1 Differences between forward swap rates and CMS rates

Effect of the length of the tenor and maturity

Moreover, the CMS rate increases as the tenor of the swap turns longer, which is the same as the forward swap rate because term premium should be included to compensate for the increased uncertainty and risk.

The longer the maturity of CMS itself, the more time it will give for significant changes in interest rates. Therefore, convexity correction can bring more profits in this situation, which leads to a higher CMS rate.

Table 3-2 Differences between forward swap rates and CMS rates

Start	Tenor	Forward Rate	CMS Rate	CMS-Fwd
1Y	1Y	0.0320	0.0321	0.0001
1Y	2Y	0.0333	0.0333	8.99×10^{-5}
1Y	3Y	0.0340	0.0341	7.89×10^{-5}
1Y	5Y	0.0353	0.0353	6.69×10^{-5}
1Y	10Y	0.0384	0.0385	7.20×10^{-5}
5Y	1Y	0.0393	0.0402	0.0009
5Y	2Y	0.0401	0.0408	0.0007
5Y	3Y	0.0401	0.0406	0.0006
5Y	5Y	0.0411	0.0415	0.0004
5Y	10Y	0.0436	0.0442	0.0005
10Y	1Y	0.0422	0.0439	0.0018
10Y	2Y	0.0431	0.0444	0.0013
10Y	3Y	0.0441	0.0452	0.0011
10Y	5Y	0.0462	0.0471	0.0009
10Y	10Y	0.0535	0.0545	0.0011

PART IV: Decompounded Options

The decompounded option payoff function at $T = 5y$ is denoted as

$$CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$$

4.1 CMS Rate

From *lecture note Session 5 page 19*, the formula for CMS Rate is known as

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Let $g(F) = F$, we have the derivatives:

$$g(K) = K^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$$

$$g'(K) = \frac{1}{4} \cdot K^{-\frac{3}{4}}$$

$$g''(K) = -\frac{3}{16} \cdot K^{-\frac{7}{4}}$$

Using quotient rule, the first and second order derivatives of $h(K)$ are given by:

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2 \cdot IRR'(K)g'(K)}{IRR(K)^2} + \frac{2 \cdot IRR'(K)^2 g(K)}{IRR(K)^3}$$

Based on the valuation formula above and SABR parameters in Part 3, the CMS rate payment at $T = 5y$ is valued as **0.2497**.

4.2 CMS Caplet

From *lecture note Session 5 page 20*, the formula for CMS caplet is known as:

$$V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

where

$$F^{\frac{1}{4}} > 0.2$$

$$F > 0.2^4$$

$$F > 0.0016 = L$$

Based on the valuation formula above and SABR parameters in Part 3, the CMS caplet payment at $T = 5y$ is valued as **0.0349**.

Bibliography

[1] Neo, P. L., & Tee, C. W. (2019). Swaption Portfolio Risk Management: Optimal Model Selection in Different Interest Rate Regimes. *SSRN Electronic Journal.*