Course: Master of Science in Quantitative Finance

Subject: QF 620 Stochastic Modelling in Finance

Project Report

Group Number: Group 7

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PART I: Analytical Option Formulae

Assume S_0 is the stock price at t = 0, and K, σ, and T are the strike price, volatility, and maturity of the option respectively, with risk-free rate r. Divide the option into three classes: vanilla European option, digital cash-or-nothing option and digital asset-or-nothing option. Use these parameters to calculate the price of call option V_c and the price of put option V_P in 4 different models: Black-Scholes Model, Bachelier Model, Black 76 Model and Displaced Diffusion Model.

1.1 Black-Scholes Model

Let:

$$d_1 = \frac{log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}; \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Option Classification	Call (V_c)	Put (V_P)
Vanilla Option	$S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$	$Ke^{-rT}\Phi(-d_2) - S_0\Phi(- d_1)$
Cash-or-Nothing	$e^{-rT}\Phi(d_2)$	$e^{-rT}\Phi(-d_2)$
Asset-or-Nothing	$S_0\ \Phi(d_1)$	$S_0\ \Phi(-d_1)$

1.2 Bachelier Model

Let:

$$d = \frac{S_0 - K}{\sigma\sqrt{T}}.$$

Option Classification	Call (V_c)	Put (V_P)
Vanilla Option	$e^{-rT}[(S_0 - K)\Phi(d) + \sigma\sqrt{T}\Phi(d)]$	$e^{-rT}[(K - S_0)\Phi(-d) + \sigma\sqrt{T}\phi(d)]$
Cash-or-Nothing	$e^{-rT}\Phi(d)$	$e^{-rT}\Phi(-d)$
Asset-or-Nothing	$S_0(\Phi(d) + \sigma\sqrt{T}\Phi(d))$	$S_0(\Phi(-d) + \sigma\sqrt{T}\phi(-d))$

1.3 Black 76 Model

Let:

$$d_1 = \frac{log\frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}; \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{log\frac{F_0}{K} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}.$$

Option Classification	Call (V_c)	Put (V_P)
Vanilla Option	$e^{-rT}[F_0\Phi(d_1) - K\Phi(d_2)]$	$e^{-rT}[K\Phi(-d_2) - F_0\Phi(-d_1)]$
Cash-or-Nothing	$e^{-rT}\Phi(d_2)$	$e^{-rT}\Phi(-d_2)$
Asset-or-Nothing	$e^{-rT}F\Phi(d_1)$	$e^{-rT}F\Phi(-d_1)$

1.4 Displaced Diffusion Model

Let:

$$d_1 = \frac{\log \frac{\frac{F_0}{\beta}}{K + \frac{1-\beta}{\beta}F_0} + \frac{\sigma^2\beta^2}{2}T}{\sigma\beta\sqrt{T}}; \qquad d_2 = d_1 - \sigma\beta\sqrt{T} = \frac{\log \frac{\frac{F_0}{\beta}}{K + \frac{1-\beta}{\beta}F_0} - \frac{\sigma^2\beta^2}{2}T}{\sigma\beta\sqrt{T}}.$$

Option Classification	Call (V_c)	Put (V_P)
Vanilla Option	$e^{-rT}\left[\frac{F_0}{\beta}\Phi(d_1) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$	$e^{-rT}\left[\left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(-d_2) - \frac{F_0}{\beta}\Phi(-d_1)\right]$
Cash-or-Nothing	$e^{-rT}\Phi(d_2)$	$e^{-rT}\Phi(-d_2)$
Asset-or-Nothing	$e^{-rT}\frac{F_0}{\beta}\Phi(d_1)$	$e^{-rT}\frac{F_0}{\beta}\Phi(-d_1)$

PART II: Model Calibration

2.1 Data Explanation

In this case, the datasets provided are the prices of six option assets with three different maturity dates (2020-12-18, 2021-01-15, 2021-02-19) based on two underlying assets, which are S&P500 index (SPX) and SPDR S&P500 Exchange Traded Fund (SPY). Assume the initial prices of two underlying assets are \$3662.45 and \$366.02 respectively on 2020-12-01, then calculating the market implied volatility of the six options based on Black-Scholes pricing model.

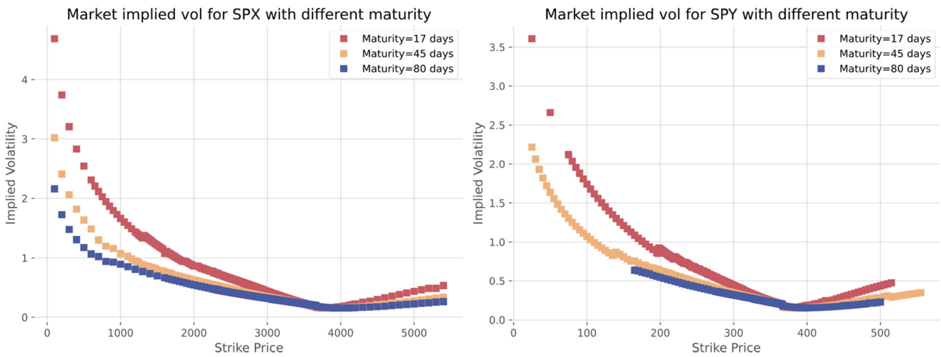

Figure 2-1 Implied market volatility of options among different maturities

Based on Figure 2-1, the bottom of the volatility smile is at-the-money options, which means for all kinds of options in the market, in-the-money options, and out-of-the-money options have higher implied volatility than at-the-money options.

Besides, near events and policy will bring a high level of uncertainty to short-term options, which leads investors to pay more premium for increased volatility in short term. On the other hand, short-term options are more sensitive to the market's immediate change because of the time limit. What's more, Gamma value will increase when an option is closer to the maturity date, which indicates the value of Delta will change rapidly, thus Delta hedging investors will adjust their portfolios frequently in a short term. As a result, shorter expiries will have higher implied volatility, and volatility smile is steeper for options with shorter maturities and flatter for options with longer maturities.

2.2 Model Calibration

2.2.1 Displaced-Diffusion model

One assumption of Black-Scholes model is the change of underlying asset price complies with lognormal distribution. However, Displaced-Diffusion model introduces a concept of β:

$$dF_t = \sigma[\beta F_t + (1 - \beta F_0)]dW_t.$$

Where σ is the volatility of options, F_0 is the forward price at time 0, F_t is the forward price in time t, and $\beta \in [0, 1]$ is the weight of the lognormal price F_t.

In reality, the price distribution of most financial assets has fat tail characteristic instead of complying lognormal distribution. Therefore, a change of β will make displaced-diffusion model fit asset prices with different distributions in a relatively accurate way.

Because Displaced-Diffusion model is a transformation of Black-Sholes model and Black 76 model in PART I, the pricing function can be rewritten from:

$$Black\ 76: F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

to:

$$Displaced\ Diffusion: F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2 \sigma^2 T}{2} + \beta \sigma W_T} - \frac{1 - \beta}{\beta} F_0.$$

2.2.2 SABR model

The SABR Model (Stochastic Alpha-Beta-Rho) is used to describe the evolution of the underlying asset price S_t and its associated volatility α_t. The basic equation is as follows:

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t = \nu \alpha_t dW_t^{\alpha} \end{cases},$$

where $dW_t^F dW_t^{\alpha} = \rho dt$, and ν is the speed at which volatility evolves over time, β represents how sensitive volatility is to variations in the underlying asset's price.

Compared with Displaced-Diffusion model, SABR model inherently captures the asymmetry in volatility movements and leverage effect, as known as the tendency for volatility to increase when the underlying asset price falls. These are important characteristics of financial markets that are not explicitly accounted for in the displaced-diffusion model. Meanwhile, SABR model has different parameters ρ, ν, α, β that have specific economic significance. This feature ensures SABR model's flexibility and fitting accuracy. Moreover, SABR model directly provides an explicit solution for the implied volatility of options, rather than an option pricing formula, so it is more convenient to apply SABR model when measuring implied volatility.

The superiority of SABR lies in its ability to interpret the volatility smile well, and it can correctly move the entire volatility smile when the underlying asset price moves. Consequently, SABR model has more considerable fitting results than Displaced-Diffusion model.

2.2.3 Calibration Outcome

To get the best-fitted model, use the least square method to calculate essential parameters in models. The estimated result of at-the-money volatility σ and β in Displaced-Diffusion model, and $\rho,\ \nu,\ \alpha$ in SABR model, with $\beta = 0.7$ are demonstrated in Table 2-1

Table 2-1 Fitted parameters in Displaced-Diffusion model and SABR model

Assets	Maturity	Displaced-Diffusion Model		SABR Model		
		$\boldsymbol{\beta}$	$\boldsymbol{\sigma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\rho}$	$\boldsymbol{\nu}$
SPX	**17 days**	4.050×10^{-8}	0.174	1.212	-0.301	5.460
	45 days	9.862×10^{-7}	0.186	1.817	-0.404	2.790
	80 days	1.165×10^{-6}	0.195	2.140	-0.575	1.842
SPY	**17 days**	1.080×10^{-7}	0.201	0.665	-0.412	5.250
	45 days	5.000×10^{-11}	0.198	0.908	-0.489	2.729
	80 days	5.600×10^{-10}	0.201	1.121	-0.633	1.742

After that, use the best-fitting SABR and Displaced-Diffusion model to fit the market implied volatility smile curve.

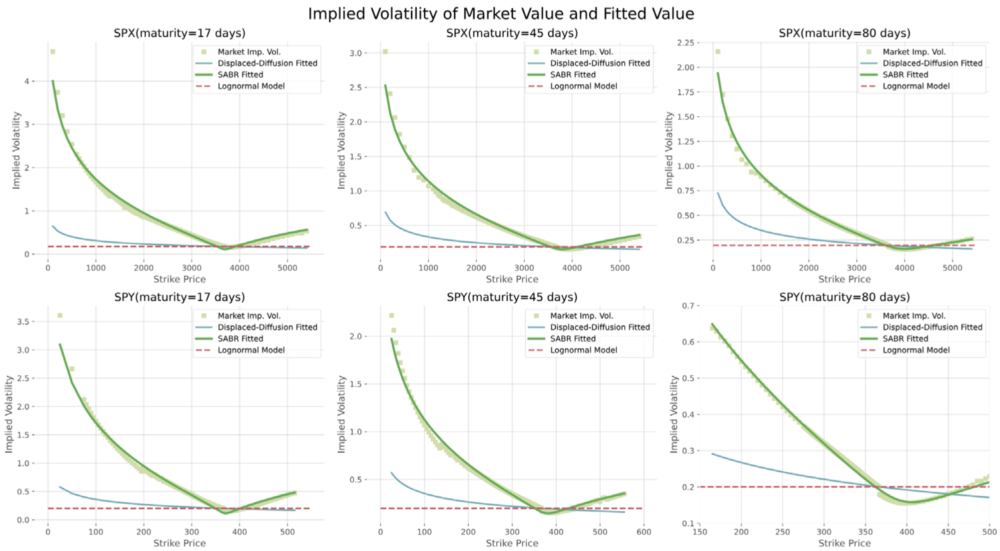

Figure 2-2 Implied volatility of market value and fitted value

Figure 2 illustrates how well the SABR model fits the market implied volatility, while Displaced-Diffusion model performs poorly.

To elucidate the outcomes, initially employ the Black Scholes Model as the reference model for calculating implied volatility. In this scenario, the volatility remains constant. Additionally, considering the quantity of model parameters, interpretations for the disparities in performance between the two models during calibration could be offered.

As indicated in Table 2-1 Fitted parameters in Displaced-Diffusion model and SABR model, the best-fitted beta, representing the displacement of asset prices in the Displaced-Diffusion Model, is very close to zero. A parameter proximity to zero implies that drift and volatility are highly influenced by the current price rather than the past price, potentially impacting accuracy during implied market volatility fitting. Consequently, the Displaced-Diffusion Model exhibits relatively poorer performance in this instance.

Contrastingly, the SABR Model encompasses more tunable parameters (α, ρ, ν, and β). Despite the increased complexity, all parameters yield reasonable results in this case, contributing to a more comprehensive explanation for market volatility.

2.3 Parameter Sensitivity Analysis

2.3.1 Displaced-Diffusion Model

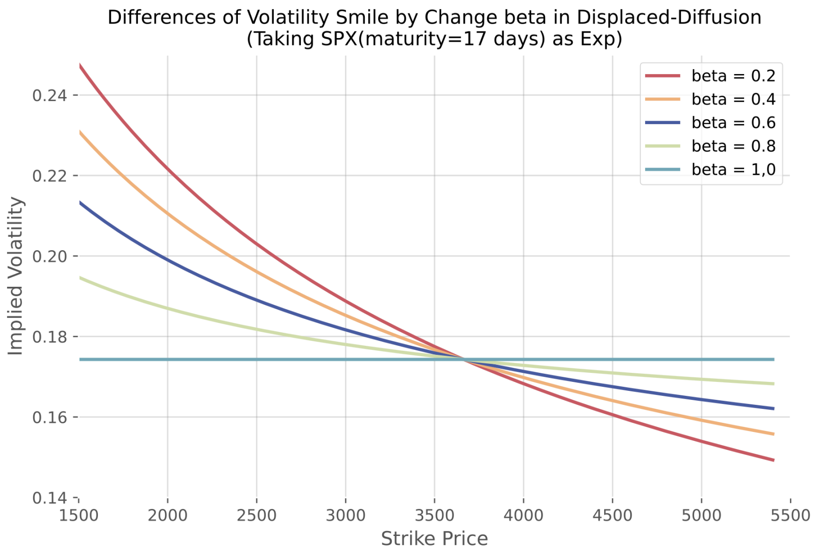

Figure 2-3 Differences of volatility smile with different β

In the Displaced-Diffusion model, the parameter β assumes a pivotal role in tailoring the implied volatility skew, facilitating a nuanced calibration of the observed market dynamics. A β close to zero implies a negligible displacement effect, rendering the model akin to the Bachelier Model, yielding a straight line on implied volatility. Conversely, a β nearing one aligns the model more closely with the Black-Scholes Model, and produces a volatility smile, capturing varying implied volatility with strike prices and potential fat tails.

The significance of β extends beyond its impact on the model's behavior; it serves as a potential indicator of external factors influencing asset prices. These external factors, encapsulated by the displacement term, might encompass variables such as interest rates, dividends, or other economic factors directly shaping pricing dynamics. The value of β also provides a window into investor perceptions and their incorporation of external information into pricing decisions. A higher β suggests that investors assign more weight to current economic conditions, reflecting an anticipation that external factors will exert an immediate impact on asset prices. In essence, β becomes a key metric for gauging market sentiment and expectations, shedding light on the temporal dynamics of asset pricing influenced by external economic variables.

2.3.2 SABR Model

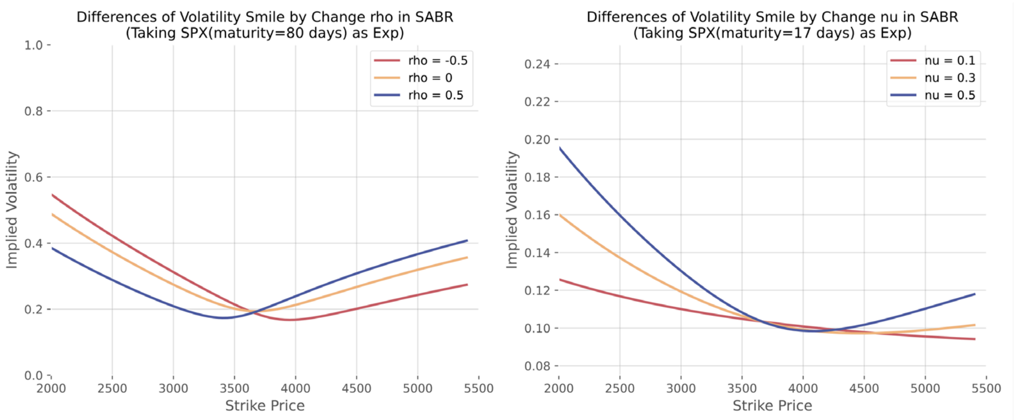

Figure 2-4 Differences of volatility smile with different ρ and ν

The ρ parameter signifies the correlation between an asset's volatility and the forward rate. When ρ is negative, it results in heightened volatility during declines in stock prices, expanding the left tail of the probability density. This leads to a fat left tail and a thin right tail in the return distribution. Conversely, a positive ρ produces the opposite effect. Because of this characteristic, determining ρ allows financial institutions to enhance their understanding and management of risks associated with interest rate derivative positions, given its sensitivity to volatility changes.

ρ contributes to more precise pricing of interest rate derivatives as it is specifically crafted to capture skewness in volatility, presenting a more realistic portrayal of market conditions. The inclusion of ρ in the SABR model facilitates the formulation of effective hedging strategies for interest rate derivatives. Financial institutions can leverage the model to identify suitable hedge ratios and dynamically adjust their hedges in response to changing market conditions.

Moreover, the SABR model, with its ρ parameter, empowers institutions to simulate and analyze the potential impacts of various scenarios involving changes in interest rates and volatility on their derivative positions. This analytical capability is crucial for stress testing and robust risk management practices.

ν is a key factor in capturing the market's perception of volatility dynamics. Financial markets experience varying levels of volatility over time, and the inclusion of ν allows the SABR model to accurately represent these fluctuations. This parameter is particularly significant in assessing the tails of return distributions. Higher values of ν contribute to fatter tails, indicating an increased probability of extreme market events, while lower values suggest a reduced likelihood of such occurrences.

Financial institutions rely on ν to effectively manage risks associated with their derivative portfolios. The volatility of volatility directly influences the potential for significant and unforeseen price movements, and incorporating ν into risk models enables institutions to better estimate the potential impact of extreme market events on their positions. Moreover, changes in the ν can serve as a valuable indicator of shifts in market sentiment and uncertainty. A sudden increase in the volatility of volatility may signal heightened market nervousness, prompting financial institutions to adapt their risk management strategies accordingly.

PART III: Static Replication

3.1 Static Replication

Static Replication Formula:

$$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\,dK$$

To adeptly execute the Carr-Madan static replication framework using Python, it is imperative to possess adeptness in executing numerical integrations, specifically pertaining to the integrals associated with put and call options. The initial phase entails the computation of the implied volatility parameter within the Black-Scholes model and Bachelier model, quantified as 0.1889 and 0.1978 for SPX and SPY respectively in this context. Subsequently, we proceed to the evaluation of the integrated variance, colloquially known as a realized volatility, for both the proposed models. This step is crucial in delineating the nuanced financial mechanisms at play and is integral to the precision and efficacy of the static replication strategy.

The obtained results are shown as follow:

Table 3-1 Payoff Function Result of the Three Models

	Black-Scholes Model	**Bachelier Model**	**SABR Model**
SPX	37.715	37.715	37.715
SPY	26.002	26.002	26.002

The consistency in results across the Black-Scholes, Bachelier, and SABR models for a given payoff function can be attributed to the nature of the payoff and conditions of convergence. If the payoff function is less sensitive to the distributional assumptions or market conditions and parameter settings lead to similar pricing outcomes, all three models may yield comparable results.

When static replication returns the same price across different models, it implies that the replication portfolio, composed of simpler instruments, has been constructed in such a way that its value matches the value of the more complex instrument consistently across different pricing models.

3.2 "Model Free" Integrated Variance

The "Model Free" integrated variance:

$$\sigma_{mF}^2 T = E\left[\int_0^T \sigma_t^2\,dt\right]$$

The obtained results are shown as follow:

Table 3-2 Integrated Variance Result of the Three Models

	Black-Scholes Model	**Bachelier Model**	**SABR Model**
SPX	4.397×10^{-3}	4.397×10^{-3}	6.335×10^{-3}
SPY	4.822×10^{-3}	4.822×10^{-3}	6.016×10^{-3}

The SABR model exhibits higher volatility and variance than the Black-Scholes and Bachelier models, mainly due to its more complex approach to volatility. Unlike Black-Scholes, which assumes constant volatility, and Bachelier, based on normal distribution, SABR introduces stochastic volatility, where volatility itself is a dynamic, random process. This allows SABR to reflect market realities more accurately, particularly in accommodating the volatility smile—a variation of implied volatility with strike price and maturity. The model's beta parameter provides flexibility in mimicking various asset price distributions, ranging from geometric Brownian motion to normal processes. Additionally, the inclusion of a correlation factor (rho) between asset price and volatility can lead to higher variances, especially in volatile market conditions. Overall, SABR's advanced treatment of volatility and its responsiveness to market conditions, including extreme events and heavy distribution tails, result in higher volatility and variance measurements in model-free integrated variance assessments.

PART IV: Dynamic Hedging

In this part, simulate stock price processes under the Black-Scholes dynamic Delta hedging, ensuring that the portfolio is Delta neutral, and consequently hedging the exposure of call position using the underlying stock and the risk-free bond over time until maturity. Suppose S0 = \$100, σ = 0.2, r = 5%, T = 1/12 and K = \$100. The hedged portfolio is $V_t = \phi_t S_t + \psi_t B_t$, where:

$$\phi_t = \Delta_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

and

$$\psi_t B_t = -Ke^{-rT}\Phi\left(\frac{\log\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

Carry out simulations of the discrete hedging strategy over 50,000 different, randomly generated scenarios of future stock price evolution. In each scenario, there are 21 and 84 re-hedging trades spaced evenly in time over the life of the option. In the Figure 4-1, the left option is hedged once per trading day, while the right option is hedged 84 times, or four times a day as frequently.

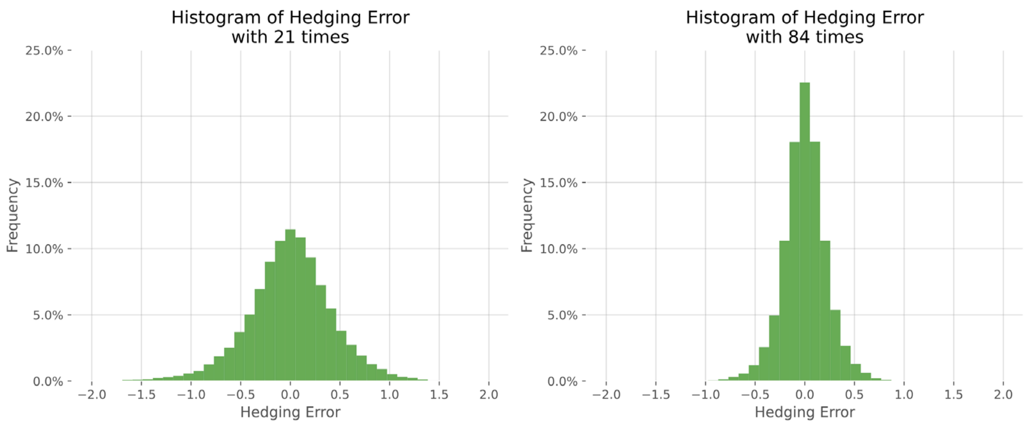

Figure 4-1 Hedging Error with N times in 21 trading days

Table 4-1 Statistical summary of the simulated profit/loss

Number of Trades	Mean Error	Standard Dev. of Error	Std of Error of option premium
21	0.0030	0.4264	16.97%
84	0.0001	0.2184	8.69%

From the above result, hedging more frequently reduces the standard deviation, as the standard deviation of the replication error is 0.4264 for N = 21, and 0.2184 for N = 84, with a fraction of the option premium is 16.97% and 8.69% separately, which is roughly halves the standard deviation of error. Furthermore, the Figure 4-1 exhibits that the final distribution of replication error resembles a perfect normal distribution, with a mean near 0. As the N increases, the higher frequency errors lay around the mean, showing the smaller standard deviations. Even though the standard deviation is around 17% of the premium for daily re-hedging, the probability of larger hedging errors goes up, both positively and negatively.

The simulation processes are a compilation of numerical results, derive the formula that agrees well with these results. As in the Black-Scholes formula, the number of hedging times is given by N.

- The standard deviation of error: $\sqrt{\frac{\pi}{4}}(\kappa)\frac{\sigma}{\sqrt{N}}$, where $\kappa = S_0\sqrt{T}\,\frac{\exp(-d_1^2)}{\sqrt{2\pi}}$ [2]
- Standard deviation of error of option premium: $\frac{\sigma_{\text{error}}}{premium} \approx \sqrt{\left(\frac{\pi}{4}\right)}\frac{1}{\sqrt{N}}$

Rebalancing four times as frequently halves the typical size of the hedging error. For N = 21, estimate $\sigma_{\text{error}} = 0.4409$, while $\sigma_{\text{error}} = 0.2205$ for N = 84, with option premium 2.512. These estimates are quite close to the simulation results of Table 4-1.

Simulation prevails the updating frequency increases per day, and the standard error of the hedging error reduces adhering closely to the $\sqrt{\Delta t}$ rule [1].

Bibliography

[1] Carr, Peter P. and Wu, Liuren, Static Hedging of Standard Options (May 21, 2004). NYU Tandon Research Paper No. 585451, http://dx.doi.org/10.2139/ssrn.585451

[2] http://pricing.free.fr/docs/when_you_cannot_hedge.pdf