Notations

ϕ : empty set

$\mathbb{Z}$: integers, i.e. $\{0, \pm 1, \pm 2, \pm 3, \cdots\}$

$\mathbb{Q}$: rationals, i.e. $\{\underbrace{\tfrac{a}{b}}_{\text{"the set of all } \frac{a}{b}} \underbrace{\mid}_{\text{such that}} \underbrace{a, b \in \mathbb{Z},\ b \neq 0}_{\text{a and b are integers, with b being nonzero"}}\}$

$\mathbb{R}$: reals

$\mathbb{C}$: complex numbers, i.e. $\{a + bi \mid a, b \in \mathbb{R}\}$

$\mathbb{R}^2$: the Cartesian plane, i.e. $\{(x, y) \mid x, y \in \mathbb{R}\}$

S^n : the set of all ordered n-tuples with all entries in S, i.e. $\{(s_1, s_2, \cdots, s_n) \mid s_1, s_2, \cdots, s_n \in S\}$

$\forall$: for all $\qquad \exists$: there exists $\qquad \in$: belongs to

"s.t.": such that $\qquad$ "iff": if and only if

$P \Rightarrow Q$: if P is true, then Q is true

$P \Leftrightarrow Q$: $P \Rightarrow Q$ and $Q \Rightarrow P$ (P is true iff Q is true)

Linear Algebra: an example

A cow costs \$5, a sheep costs \$1, and a rabbit costs 5 cents. A farmer bought 100 animals for \$100, including 18 more cows than sheep. How many of each did the farmer buy?

let x = # of cows $\qquad$ then $5x + y + 0.05z = 100$

y = # of sheep $\qquad x + y + z = 100$

z = # of rabbits $\qquad x - y = 18$

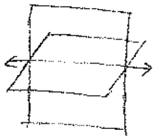
We want to solve for x, y, z. Questions : (1). Do solutions exist?

(2). If so, how many solutions are there?

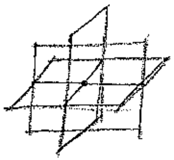
It helps to think geometrically. Each equation describes a plane in $\mathbb{R}^3$, so the set of solutions is exactly the intersection of the 3 planes.

Possibilities :

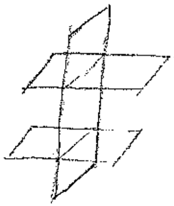
Plane



Line



Point



ϕ (2 or more planes are parallel)

In both of the first two cases, we have infinitely many solutions. What's the difference? How do we generalize to higher dimensions when we can't draw pictures?

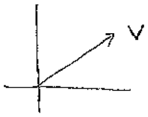
This is the beginning of Linear Algebra ("linear" because all variables appear in 1^{st} powers and different variables are not multiplied together).

7.6 Vector Spaces

An n-dimensional vector is

(1). Algebraically, an ordered n-tuple of numbers

(2). Geometrically, an arrow from the origin



$v = (v_1, v_2)$

you'll probably see me use pointy brackets $\langle v_1, v_2 \rangle$ for emphasis

A <u>vector space</u> V over $\mathbb{R}$ is a set of vectors s.t.

(1). $\forall u, v \in V,\ u+v \in V$ (closure under vector addition)

remark: vector addition is only defined between vectors of the same dimension

(2). $\forall u, v \in V,\ u+v = v+u$ (Commutativity)

(3). $\forall u, v, w \in V,\ (u+v)+w = u+(v+w)$ (Associativity)

(4). $\exists\ \vec{0}$ vector s.t. $\forall v \in V,\ v+\vec{0} = \vec{0}+v = v$ (Identity)

(5). $\forall v \in V,\ \exists\, u \in V$ s.t. $v+u = u+v = \vec{0}$ (Inverse)

namely, take $u = -v$

Axioms for vector addition

(6). $\forall k \in \mathbb{R},\ \forall v \in V,\ kv \in V$ (closure under scalar multiplication)

(7). $\forall k \in \mathbb{R},\ \forall u, v \in V,\ k(u+v) = ku+kv$

(8). $\forall k_1, k_2 \in \mathbb{R},\ \forall v \in V,\ (k_1+k_2)v = k_1v+k_2v$

(7)–(8): (distributivity)

(9). $\forall k_1, k_2 \in \mathbb{R},\ \forall v \in V,\ (k_1k_2)v = k_1(k_2v)$

(10). $\forall v \in V,\ 1v = v$

Axioms for scalar multiplication

Example: (1) $\mathbb{R}^2$ is a vector space

(2) $\{(v_1, v_2) \mid v_1, v_2 \in \mathbb{R}^+\}$ is not a vector space (Axioms 5, 6 not met)

Remark: (1) we can replace $\mathbb{R}$ with $\mathbb{C}$, $\mathbb{Q}$, or many other sets (but for technical reasons, not $\mathbb{Z}$)

(2) we will work with mostly $\mathbb{R}^n$ and $\mathbb{C}^n$

<u>Subspace</u>: if V is a vector space, $W \subseteq V$ is itself a vector space under the operations of vector addition and scalar multiplication defined on V, then W is a subspace of V.

Example: (1) the xy plane in $\mathbb{R}^3$

(2) every vector space has at least 2 subspaces: itself and the zero subspace $\{\vec{0}\}$

Luckily, it is not necessary to check all axioms of a vector space. Since $W \subseteq V$, W already inherits most of the properties of V. It suffices to only check closure.

<u>Theorem</u>: Let V be a vector space and $W \subseteq V$. Then W is a subspace of V iff

(1). $\forall u, v \in W,\ u+v \in W$, and

(2). $\forall v \in W$ and any scalar k, $kv \in W$

Example: the plane $x=2$, i.e. $\{(2, y, z) \mid y, z \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^3$ (not closed under scalar multiplication)

Linear Independence: vectors $v_1, v_2, \ldots, v_n$ are said to be linearly independent if

$$k_1 v_1 + k_2 v_2 + \ldots + k_n v_n = \vec{0} \iff k_1 = k_2 = \ldots = k_n = 0$$

Otherwise they are said to be linearly dependent.

example: $u = \langle 1, 1, 1 \rangle$, $v = \langle 1, 3, 5 \rangle$, $w = \langle 1, 2, 3 \rangle$ are linearly dependent $(u + v - 2w = 0)$

$i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$ are linearly independent

Note that not only are i, j, k linearly independent, every vector in $\mathbb{R}^3$ can be expressed as a linear combination of them. This motivates the following definition...

Basis: let V be a vector space and let $v_1, v_2, \ldots, v_n \in V$. If the v_i's are linearly independent and every vector of V can be expressed as a linear combination of the v_i's, i.e. $\forall v \in V, \exists k_1, \ldots, k_n$ s.t. $v = k_1 v_1 + \ldots + k_n v_n$, then we call the v_i's a basis for V.

Remark: (1) $\{i, j, k\}$ is called the standard basis of $\mathbb{R}^3$

(2) in general, the standard basis of $\mathbb{R}^n$ is $\{\langle 1, 0, \ldots, 0 \rangle, \langle 0, 1, 0, \ldots, 0 \rangle, \ldots, \langle 0, \ldots, 0, 1 \rangle\}$

(3) when we say that $v = \langle 1, 2, 3 \rangle$, we really mean that $1, 2, 3$ are the coordinates of v relative to the standard basis in $\mathbb{R}^3$. without knowing the basis, the coordinates are meaningness !

For instance, $\langle 2, 0, 0 \rangle$, $\langle 0, 2, 0 \rangle$, $\langle 0, 0, 2 \rangle$ also form a basis for $\mathbb{R}^3$, and with respect to this basis (which is itself defined in terms of the standard basis), $v = \langle \frac{1}{2}, 1, \frac{3}{2} \rangle$

Fact: (1). Every vector space has a basis

(2). Bases are usually not unique, but if one basis of V contains n vectors, then every basis of V contains n vectors. Thus the # of vectors in a basis is an invariant of the vector space

Dimension: the # of vectors in a basis for a vector space V is called the dimension of V.

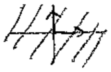
Example: (1). $\dim(\mathbb{R}^n) = n$

(2). we will define the dimension of $\{\vec{0}\}$ to be 0.

(3). consider P_n, the set of polynomials in 1 variable of degree $\leq n$. (check that P_n satisfies the axioms of a vector space). A basis for P_n is $\{1, x, x^2, \ldots, x^n\}$, so $\dim(P_n) = n+1$. We will have more to say later about the linear (in)dependence of functions.

Span: for vectors $v_1, v_2, \ldots, v_n$, their span is the set of all linear combinations $k_1 v_1 + \ldots + k_n v_n$

Example: $\{i, j\}$ spans $\mathbb{R}^2$



$\{i, i+j\}$ also spans $\mathbb{R}^2$ (since j can be obtained as a linear combination of i and $i+j$)

$\{i, 2i\}$ does not span $\mathbb{R}^2$, only $\mathbb{R}$

Remark: Vectors $v_1, v_2, \ldots, v_n$ form a basis for V iff the v_i's are linearly independent and span V.

8.1 Matrix Algebra

A matrix is a rectangular array of numbers (or functions)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

An $m \times n$ matrix has m rows and n columns

$$a_{\underset{\text{row}}{i}\underset{\text{column}}{j}}$$

vectors are in fact special matrices: an $n \times 1$ matrix e.g. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a column vector

an $1 \times n$ matrix e.g. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a row vector

Matrix addition is defined by addition of corresponding entries (matrices must have the same size)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+3 \\ 3+2 & 4+4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix}$$

Scalar multiplication is defined by multiplying every entry by the scalar

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2(1) & 2(2) \\ 2(3) & 2(4) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

Properties of matrix addition & scalar multiplication

Let A, B, C be $m \times n$ matrices and k_1, k_2 be scalars. Then

(1). $A + B = B + A$ (Commutativity)

(2). $(A+B) + C = A + (B+C)$ (Associativity)

(3). $(k_1 k_2)A = k_1(k_2 A)$

$$\left.\begin{array}{l} (4).\ k_1(A+B) = k_1 A + k_2 B \\ (5).\ (k_1+k_2)A = k_1 A + k_2 A \end{array}\right\} \text{(Distributivity)}$$

Matrix multiplication can be defined on A and B if # of columns of A = # of rows of B

Think of A as a vertical array of row vectors and B as a horizontal array of column vectors

$$\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}}_{A\ (m\times p)} \underbrace{\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}}_{B\ (p\times n)} = \underbrace{\begin{pmatrix} \overset{\text{dot product}}{u_1 \cdot v_1} & u_1 \cdot v_2 & \cdots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \cdots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \cdots & u_m \cdot v_n \end{pmatrix}}_{AB\ (m\times n)}$$

(all u_i's and v_j's must have the same dimension as vectors)

Ex: $$\overset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}} \overset{B}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}} = \begin{pmatrix} 1(1)+2(2) & 1(2)+2(3) & 1(3)+2(4) \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(4) \end{pmatrix} = \begin{pmatrix} 5 & 8 & 11 \\ 8 & 13 & 18 \end{pmatrix}$$

Remark: matrix multiplication is in general not commutative!

In fact, BA in the above example is not even defined.

Even when A and B are both square matrices, usually $AB \neq BA$.

But, matrix multiplication is associative ($(AB)C = A(BC)$) and satisfies the distributive laws ($A(B+C) = AB+AC$ and $(B+C)A = BA+CA$) provided everything's defined.

<u>Transpose</u> The transpose of an $m\times n$ matrix A is the $n\times m$ matrix A^T whose columns are the rows of A.

ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

Properties : (1). $(A^T)^T = A$ (3). $(AB)^T = B^T A^T$

(2). $(A+B)^T = A^T + B^T$ (4). $(kA)^T = kA^T$

<u>Special Matrices</u> (among square matrices)

(1). the zero matrix (every entry is 0) plays the role of, well, the 0.

(2). The identity matrix (a square matrix with 1 in every entry along the main diagonal and zero everywhere else) plays the role of 1.

ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(3). upper triangular matrix

all entries below the main diagonal are 0 $\quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Lower triangular matrix

all entries below the main diagonal are 0 $\quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

(4). Diagonal matrix : every entry outside the main diagonal is 0, ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(5). Symmetric matrix : $A = A^T$ (entries are symmetric wrt the main diagonal)

ex: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

Remark : Sometimes, we will think of a matrix as a function between vector spaces.

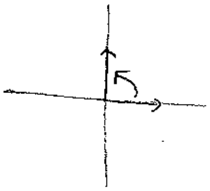
ex: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a function from $\mathbb{R}^3$ to $\mathbb{R}^2$

A square matrix can be viewed as a function from a vector space to itself.

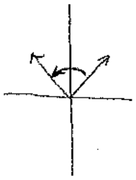
ex: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ takes one vector in $\mathbb{R}^2$ to another vector in $\mathbb{R}^2$

in fact, it rotates a vector by 90° counterclockwise

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



8.2 Systems of Linear Algebraic Equations

Recall the farmer's problem from yesterday. Now let's solve it by eliminating variables.

$$\left.\begin{array}{ll} ① & 5x + y + 0.05z = 100 \\ ② & x + y + z = 100 \\ ③ & x - y = 18 \end{array}\right\} \quad \begin{array}{l} ③ - ② : \ ④ \ -2y - z = -82 \\ 5② - ① : \ ⑤ \ 4y + 4.95z = 400 \\ ⑤ + 2④ : \ ⑥ \ 2.95z = 236 \\ \qquad\qquad\qquad z = 80 \end{array}$$

→ back sub. into ④ : $y = 1$

back sub. into ③ : $x = 19$

Remark : a system of LE's with at least 1 solution is said to be consistent

no solution inconsistent

<u>Goal</u> : to systematically solve systems of LE's by representing them as matrices.

Given

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

Form the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$\underbrace{\phantom{a_{11}\ a_{12}\ \cdots\ a_{1n}}}_{A}$ $\underbrace{}_{B}$

ex : $\left(\begin{array}{ccc|c} 5 & 1 & 0.05 & 100 \\ 1 & 1 & 1 & 100 \\ 1 & -1 & 0 & 18 \end{array}\right)$ want to solve $AX = B$, where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

<u>Elementary row operations</u>

(1). multiply a row with a nonzero constant

(2). interchange any two rows

(3). add a multiple of one row to another row

<u>Gaussian Elimination</u> : perform elementary row operations until we get row-echelon form

<u>Row-echelon form</u> : (1). rows with all zeros are at the bottom

(2). in a nonzero row, the first nonzero entry is a 1

(3). in consecutive nonzero rows, the first 1 in the lower row appears to the right of the first 1 in the higher row

ex : $\left(\begin{array}{ccc|c} 0 & 1 & 2 & 4 \\ 1 & 0 & 3 & 5 \end{array}\right)$ not in row-echelon form

$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right)$ in row-echelon form

ex. of Gaussian Elimination :

$$\left(\begin{array}{ccc|c} 5 & 1 & 0.05 & 100 \\ 1 & 1 & 1 & 100 \\ 1 & -1 & 0 & 18 \end{array}\right) \xrightarrow{r3-r2} \left(\begin{array}{ccc|c} 5 & 1 & 0.05 & 100 \\ 1 & 1 & 1 & 100 \\ 0 & -2 & -1 & -82 \end{array}\right) \xrightarrow{5r2-r1} \left(\begin{array}{ccc|c} 5 & 1 & 0.05 & 100 \\ 0 & 4 & 4.95 & 400 \\ 0 & -2 & -1 & -82 \end{array}\right)$$

$$\xrightarrow{r2+2r3} \left(\begin{array}{ccc|c} 5 & 1 & 0.05 & 100 \\ 0 & 0 & 2.95 & 236 \\ 0 & -2 & -1 & -82 \end{array}\right) \xrightarrow[\boxed{\text{divide each row by leading constant}}]{\text{switch } r2 \ \& \ r3} \left(\begin{array}{ccc|c} 1 & 0.2 & 0.01 & 20 \\ 0 & 1 & 0.5 & 41 \\ 0 & 0 & 1 & 80 \end{array}\right) \begin{array}{l} \leftarrow x + 0.2 + 0.8 = 20, \ x = 19 \\ \leftarrow y + 0.5(80) = 41, \ y = 1 \\ \leftarrow z = 80 \end{array}$$

↑ you don't really have to do this

<u>Gauss-Jordan Elimination</u> : perform elementary row operations until we get <u>reduced row-echelon form</u>, which is a row-echelon matrix s.t. if a column contains a leading 1, then all other entries in the column are zero.

ex: $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right)$ in row-echelon form, but not in reduced row-echelon form

$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right)$ in reduced row-echelon form

Continuing our previous example:

$$\left(\begin{array}{ccc|c} 1 & 0.2 & 0.01 & 20 \\ 0 & 1 & 0.5 & 41 \\ 0 & 0 & 1 & 80 \end{array}\right) \underset{r1-0.01r3}{\xrightarrow{r2-0.5r3}} \left(\begin{array}{ccc|c} 1 & 0.2 & 0 & 19.2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 80 \end{array}\right) \xrightarrow{r1-0.2r2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 80 \end{array}\right) \begin{array}{l} \leftarrow x=19 \\ \leftarrow y=1 \\ \leftarrow z=80 \end{array}$$

remark: If you are asked to just solve a system of LE's, I don't care how you do it. But row-echelon form is useful to determine if a system is consistent and (later) how many solutions there are.

ex: solve $\left.\begin{array}{l} x+y=1 \\ 4x-y=-6 \\ 2x-3y=8 \end{array}\right\}$ $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right) \underset{r3-2r1}{\xrightarrow{r2-4r1}} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & -10 \\ 0 & -5 & 6 \end{array}\right) \xrightarrow{r3-r2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & -10 \\ 0 & 0 & 16 \end{array}\right)$

Since the final row says that $0=16$, this is an inconsistent system.

<u>Homogeneous systems</u>: when solving $AX=B$, if $B=\vec{0}$, we say the system is homogeneous. Observe that a homogeneous system always has a solution, namely $X=\vec{0}$.

Question: does it have a nontrivial solution?

<u>Thm</u>: A homogeneous system has a nontrivial solution if # equations < # unknowns

ex: $\left.\begin{array}{l} x+y+z=0 \\ x+y+2z=0 \end{array}\right\}$ $z=0$, $x+y=0$, so solutions are of the form $(t,-t,0)$

In fact, when # equations < # unknowns, the homogeneous system always has infinitely many solutions (more on this tomorrow)

<u>Nonhomogeneous systems</u>: if $B\neq\vec{0}$, we call $AX=\vec{0}$ the associated homogeneous system of $AX=B$. If X_h solves $AX=\vec{0}$ and X_p (called a particular solution) solves $AX=B$, then X_h+X_p is a solution of $AX=B$.

<u>Proof</u>: $A(X_h+X_p) = AX_h + AX_p = \vec{0}+B = B$.

ex: $\left.\begin{array}{l} x+y+z=0 \\ x+y+2z=1 \end{array}\right\}$ a particular solution is of the form $(s,-1-s,1)$, e.g. $(0,-1,1)$

Since $(3,-3,0)$ is a solution of the associated homogeneous system, $(3,-4,1)$ is another solution of $AX=B$.

In fact, all solutions of $AX+B$ arise this way, i.e. as the sum of a solution of $AX=\vec{0}$ and a solution of $AX=B$. Thus the solution set is $\{(t,-t,0)+(s,-1-s,1) \mid t,s\in\mathbb{R}\}$

8.3 Rank of a Matrix

<u>Rank</u> : The rank of a matrix is the maximum number of linearly independent rows.

Ex : $A = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}$ $\quad r_1 = (1 \; 1 \; -1 \; 3)$, $r_2 = (2 \; -2 \; 6 \; 8)$, $r_3 = (3 \; 5 \; -7 \; 8)$

$4r_1 - \frac{1}{2}r_2 - r_3 = (0 \; 0 \; 0 \; 0) \Rightarrow \text{rank}(A) < 3$

r_2 is not a scalar multiple of $r_1 \Rightarrow \text{rank}(A) \geq 2$

$\Rightarrow \text{rank}(A) = 2$

Definition : we say that A and B are equivalent if B can be attained from A by performing elementary row operations.

<u>Theorem</u> : If A is equivalent to a matrix B that is in row-echelon form, then rank (A) = rank(B) = # nonzero rows in B

Ex :

$$A = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \xrightarrow{r_3 + \frac{1}{2}r_2} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rank}(A) = 2$$

Ex : Determine if $u = \langle 2, 1, 1 \rangle$, $v = \langle 0, 3, 0 \rangle$, $w = \langle 2, 1, 2 \rangle$ are linearly independent.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 2 & 1 & 2 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

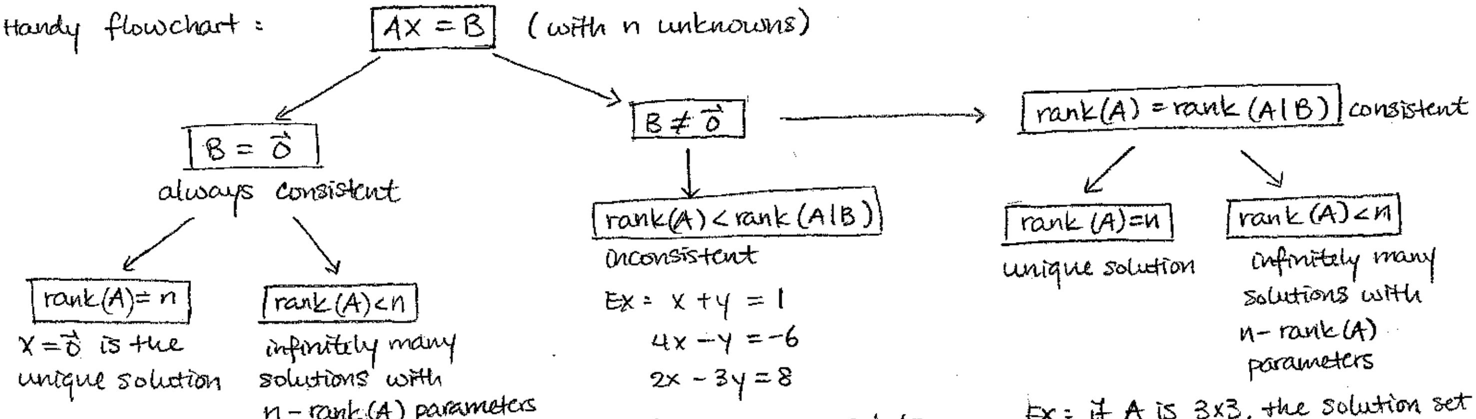
since rank (A) = 3, u, v, w are linearly independent

Remark : let $r_1, \cdots, r_n$ be the row vectors of A. The span of $r_1, \cdots, r_n$ is called the <u>row space</u> of A. If B is the row-echelon form of A, then the row space of A is equal to the row space of B. Furthermore, the nonzero rows of B form a basis for this row space. So rank (A) is the dimension of the row space of A.

Interesting fact : for any matrix, # linearly independent rows = # lin. indep. columns, so for instance $\langle 2, 0, 2 \rangle$, $\langle 1, 3, 1 \rangle$, $\langle 1, 0, 2 \rangle$ are also lin. indep.

What is the significance of rank? In solving AX = B, rank (A) tells us # solutions !

Handy flowchart :



Ex : $x + y = 1$

$4x - y = -6$

$2x - 3y = 8$

(A|B) in row-echelon form is $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & -10 \\ 0 & 0 & 16 \end{array}\right)$

rank(A) = 2

rank (A|B) = 3

Ex : $x + y + z + w = 0$

$x + y + z + 2w = 0$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ $n = 4$, rank(A) = 2

$w = 0$, $x + y + z = 0$, so solution set is $\{(s, t, -s-t, 0) \mid s, t \in \mathbb{R}\}$

s and t are the parameters

Ex : if A is 3x3, the solution set is the intersection of 3 planes.

If rank (A) < rank (A|B), no solution

else rank (A) = 1, solution set a plane

rank (A) = 2, solution set a line

rank (A) = 3, unique solution (a point)

8.4 Determinants

Recall that ranks of equivalent matrices are the same, so rank is in some sense an invariant of a matrix (it is unchanged by elementary row operations).

The determinant is another matrix invariant, unchanged by certain things we do to the matrix.
↑ only defined for square matrices

1x1 matrix (i.e. a number): determinant is just the number itself.

2x2 matrix : $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad - bc$ ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = 2$

3x3 matrix : $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Ex: $u = \langle 1, 2, 3 \rangle$
$v = \langle 2, 3, 4 \rangle$

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = (8-9)i - (4-6)j + (3-4)k = \langle 1, 2, -1 \rangle$$

Definition: the minor determinant M_{ij} (sometimes just called the minor) of a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and j^{th} column.

the cofactor C_{ij} of a_{ij} is defined to be $(-1)^{i+j} M_{ij}$

Ex: for $\begin{pmatrix} 1 & 2 & \cancel{3} & 4 \\ \cancel{5} & \cancel{6} & \cancel{\textcircled{7}} & \cancel{8} \\ 9 & 10 & \cancel{11} & 12 \\ 13 & 14 & \cancel{15} & 16 \end{pmatrix}$ $C_{23} = (-1)^{2+3} \underbrace{\begin{vmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{vmatrix}}_{M_{23}}$

Note that the determinant of a 3x3 matrix can be defined in terms of cofactors:

$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

We can generalize this to any $n \times n$ matrix A: pick any row i. then

$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$. Alternatively, we can pick any column j, and

$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$.

Ex: $A = \begin{pmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 & 0 \end{pmatrix}$

if we always expand along the first row,

$$\det A = \underbrace{1(-1)^2 \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 1 & 0 \end{vmatrix}}_{\shortparallel} + 2(-1)^3 \begin{vmatrix} 2 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{vmatrix} + 2(-1)^5 \begin{vmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{vmatrix}$$

$$1(-1)^2 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 0 & 3 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 0 \end{vmatrix}$$

and so on
Yuck!

we can instead expand along the 5^{th} column:

$$\det A = 1(-1)^9 \begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ \boxed{0 \quad 0 \quad 3 \quad 1} \\ 1 & 2 & 3 & 1 \end{vmatrix} = -\left(3(-1)^6 \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} + 1(-1)^7 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \right)$$

still annoying, but much better.

8.5 Properties of Determinants

Theorem : $\det(A) = \det(A^T)$

Ex : $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\det(A) = 4-6 = -2$ $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $\det A^T = 4-6 = -2$

Theorem : if B is obtained from A by switching any two rows (or columns) of A, then $\det(B) = -\det(A)$

Ex : $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\det(A) = -2$ $B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ $\det(B) = 2 = -\det(A)$ $C = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ $\det(C) = 2 = -\det(A)$

Theorem : if B is obtained from A by multiplying an entire row (or column) by a constant k, then $\det(B) = k\det(A)$

Ex : $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\det(A) = -3$ $B = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$ $\det(B) = -24 = (2)^3 \det(A)$

Theorem : Adding a constant multiple of one row to another row does not change the determinant

Ex : $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 6 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$\det(A) = -3$

$$\begin{aligned}\det(B) &= 4(12-15) - 6(6-12) + 9(5-8) \\ &= -12 + 36 - 27 \\ &= -3 = \det(A)\end{aligned}$$

Theorem/Observation : the determinant of a triangular matrix is simply the product of the entries along the main diagonal.

Ex : $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix} = (1)(2)(3)(4) = 24 \qquad \begin{vmatrix} 7 & 5 & 3 & 1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (7)(5)(3)(1) = 105$$

Because triangular matrices are nice, we want to make matrices triangular!

Ex : $$\begin{vmatrix} 1 & -2 & 2 & 1 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{vmatrix} \underset{\substack{r_2-2r_1 \\ r_3-3r_1 \\ r_4-3r_1}}{=} \begin{vmatrix} 1 & -2 & 2 & 1 \\ 0 & 5 & -6 & 1 \\ 0 & 10 & -14 & -2 \\ 0 & -5 & 6 & -1 \end{vmatrix} \underset{\substack{r_3-2r_2 \\ r_4+r_2}}{=} \begin{vmatrix} 1 & -2 & 2 & 1 \\ 0 & 5 & -6 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Theorem : Let A be an $n \times n$ matrix. Then $\det(A) = 0$ iff $\operatorname{rank}(A) < n$. In particular, matrices with repeating rows or zero rows have determinant zero.

Theorem : $\det(AB) = \det(A)\det(B)$. note that even though $AB \neq BA$, $\det(AB) = \det(BA)$!

Ex : $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ $AB = \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}$ $BA = \begin{pmatrix} 7 & 11 \\ 5 & 8 \end{pmatrix}$ $\det(A) = -1$ $\det(AB) = 1$ $\det(B) = -1$ $\det(BA) = 1$

8.6 Inverse of a Matrix

$\forall$ nonzero real number a, $\exists\, b \in \mathbb{R}$ s.t. $ab = 1$ (i.e. let $b = \frac{1}{a}$).

Is there something analogous for matrices?

Inverse: Let A be an $n \times n$ matrix, if $\exists\, B$ s.t. $AB = BA = I$, then B is called the inverse of A, and A is said to be invertible (or nonsingular).

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ $AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Notation: $B = A^{-1}$

Note: if A and B are $n \times n$ matrices s.t. $AB = I$, then $BA = I$

Not every matrix have an inverse!

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ if $A \cdot B = 0$, where $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

then $AB = \begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ impossible!

So A has no inverse. We call A non-invertible (or singular)

In general, how do we know if inverses exist and how to compute them?

Adjoint Method: let C_{ij} be the cofactor of the entry a_{ij} in A.

The adjoint of A is

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

To summarize, to find the adjoint, we have to ① find the minor determinant of each entry, ② multiply by 1 or −1, ③ put these in a matrix, ④ take the transpose.

Let's try an example ...

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \quad C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad C_{12} = -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5 \quad C_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2 \quad C_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \quad C_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \quad C_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6$$

$$\operatorname{adj} A = \begin{pmatrix} 1 & 5 & -3 \\ -2 & 2 & 6 \\ 2 & -2 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$$

Theorem if $\det A \neq 0$, then A^{-1} exists and $A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$

Ex: let A be as above. $\det(A) = 2(1-0) - 2(-2-3)$

$= 12$

so $A^{-1} = \begin{pmatrix} \frac{1}{12} & \frac{-2}{12} & \frac{2}{12} \\ \frac{5}{12} & \frac{2}{12} & \frac{-2}{12} \\ \frac{-3}{12} & \frac{6}{12} & \frac{6}{12} \end{pmatrix}$

Note that when we have a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $C_{11} = d$ $\quad C_{12} = -c$
$C_{21} = -b$ $\quad C_{22} = a$

so $\text{adj}\, A = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

and $A^{-1} = \det A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Remark: we said that if $\det A \neq 0$, then A^{-1} exists. what about the converse?

If A^{-1} exists, then $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1 \Rightarrow \det(A) \neq 0$!

Thus A is invertible iff $\det(A) \neq 0$. Ex: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ has determinant 0, so is non-invertible.

Elimination method: perform elementary row operations on $(A \mid I)$, get $(I \mid A^{-1})$

Ex:

$$\left(\underbrace{\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \xrightarrow{r_3 - 2r_2} \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{matrix}\right)$$

(if it's impossible to get I on the left side, then A is non-invertible)

$$\xrightarrow[r_2 + r_3]{r_1 + r_3} \left(\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix} \middle| \begin{matrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 1 \end{matrix}\right) \xrightarrow[r_3 \times (-1)]{r_1 - r_2} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \overbrace{\begin{matrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{matrix}}^{A^{-1}}\right)$$

Properties of A^{-1}: if A and B are invertible matrices, then

(1). $(A^{-1})^{-1} = A$

(2). $(AB)^{-1} = B^{-1}A^{-1}$

(3). $(A^T)^{-1} = (A^{-1})^T$

Ex of (3): $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad (A^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (A^{-1})^T$

So why do we care about the inverse?

For one, it helps us solve systems of linear equations!

In the system $Ax = B$, if A is invertible, then $A^{-1}AX = A^{-1}B$ and $X = A^{-1}B$.

so solution exists and is unique! $\uparrow$ equivalently, if $\det(A) \neq 0$

Ex:

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 2 \\ 2y + z &= 3 \end{aligned} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

$$A^{-1}B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

what about when A is not invertible (equivalently, when $\det(A)=0$)?

If $B=\vec{0}$, there are infinitely many solutions (so nontrivial solutions exist iff $\det(A)=0$).

If $B\neq\vec{0}$, we have either no solution (if $\text{rank}(A) < \text{rank}(A|B)$) or infinitely many solutions (if $\text{rank}(A) = \text{rank}(A|B)$).

Application to Cryptography

Suppose I want to secretly send a message to my friend. I want to encrypt my message so that it will make no sense to others who may intercept it, but that my friend can decode it.

One way to do this is to assign a number to every letter. $a \leftrightarrow 1$, $b \leftrightarrow 2$, ---, $z \leftrightarrow 26$. Translate the message into a string of numbers, and put these numbers into an $m\times n$ matrix M. Ex: SECRET can be represented by the 2X3 matrix $\begin{pmatrix} 19 & 5 & 3 \\ 18 & 5 & 20 \end{pmatrix}$
(pad with 0 if there are extra entries at the end)

Pick an invertible $m\times m$ matrix A. Give this to your friend ahead of the time.

To encrypt your message, multiply A and M. AM will look very different from M!

All your friend has to do is to multiply by A^{-1}. $A^{-1}AM = M$, so the original message is recovered!

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ $\quad AM = \begin{pmatrix} 31 & 17 & 28 & 39 & 28 & 12 \\ 23 & 16 & 6 & 34 & 27 & 5 \\ 41 & 21 & 7 & 54 & 50 & 10 \end{pmatrix}$

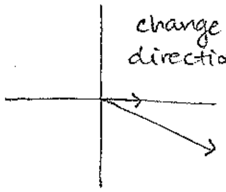
$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix}$ $\quad A^{-1}AM = M = \begin{pmatrix} 8 & 1 & 22 & 5 & 1 & 7 \\ 18 & 5 & 1 & 20 & 23 & 5 \\ 5 & 11 & 5 & 14 & 4 & 0 \end{pmatrix}$

HAVEAGREATWEEKEND

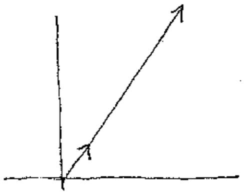
8.8 The Eigenvalue Problem

Recall that an $n \times n$ matrix can be viewed as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$; it takes one n-dim. vector to another. Sometimes it changes both the direction and magnitude of the vector, but sometimes it changes only the magnitude.

Ex: $\begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$



change in both direction and magnitude



change only in magnitude, direction is preserved

We can rewrite the second equation as $\begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We call $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ an <u>eigenvector</u> of the matrix and 5 an <u>eigenvalue</u>.

<u>Formal Definition</u>: Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if $\exists$ <u>nonzero</u> vector K s.t. $AK = \lambda K$. The vector K is called an eigenvector corresponding to λ.

<u>Finding eigenvalues and eigenvectors</u>: Since $K = IK$, we can rewrite $AK = \lambda K$ as $(AK - \lambda I K) = (A - \lambda I)K = \vec{0}$. Now there is a nontrivial solution iff $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is a degree-n polynomial in λ. It is called the <u>characteristic equation</u> of A. The eigenvalues of A are the roots of the characteristic equation.

Ex: $A = \begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}$ $\quad$ $A - \lambda I = \begin{pmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{pmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= (6-\lambda)(-4-\lambda) - (-16) \\ &= -24 - 2\lambda + \lambda^2 + 16 \\ &= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) \end{aligned}$$

So the eigenvalues are 4 and -2.

Now we want to find the corresponding eigenvectors.

<u>$\lambda = 4$</u>: solve $(A - 4I)K = \vec{0}$.

$$\begin{pmatrix} 2 & 16 \\ -1 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2x_1 + 16x_2 &= 0 \\ -x_1 - 8x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_2 &= 1 \\ x_1 &= -8 \end{aligned} \qquad K = \begin{pmatrix} -8 \\ 1 \end{pmatrix}$$

<u>$\lambda = -2$</u>: solve $(A + 2I)K = \vec{0}$

$$\begin{pmatrix} 8 & 16 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 8x_1 + 16x_2 &= 0 \\ -x_1 - 2x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_2 &= 1 \\ x_1 &= -2 \end{aligned} \qquad K = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

<u>Remark</u>: for every eigenvalue λ, there are infinitely many solutions to $(A - \lambda I)K = \vec{0}$. We are only interested in <u>linearly independent</u> solutions, i.e. a <u>basis</u> for the set of solutions.

Ex: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad$ $A - \lambda I = \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) \\ &= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda \\ &= -\lambda^3 + 3\lambda + 2 \\ &= -(\lambda + 1)^2(\lambda - 2) \end{aligned}$$

$\lambda = -1$ (multiplicity 2), 2

$\underline{\lambda = 2}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow{\substack{r_2+\frac{1}{2}r_1 \\ r_3+\frac{1}{2}r_1}} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right) \xrightarrow{r_3+r_2} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

the general solution is $(t,t,t) = t(1,1,1)$, so the eigenvector is $(1,1,1)$.

$\underline{\lambda = -1}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

the general solution is $(t, s, -t-s) = t(1,0,-1) + s(0,1,-1)$

so the eigenvectors are $\underbrace{(1,0,-1)}_{(\text{letting } t=1,\, s=0)}$ and $\underbrace{(0,1,-1)}_{(\text{letting } t=0,\, s=1)}$.

<u>Remark</u> : (1). The number of <u>linearly independent</u> eigenvectors corresponding to λ is at most the multiplicity of λ.

(2). The eigenvectors corresponding to different eigenvalues are always linearly independent, since a degree-n polynomial has n roots (counting multiplicity), an $n \times n$ matrix has at most n linearly independent eigenvectors.

Ex :

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = (1-\lambda)(1-\lambda)(2-\lambda)$

$\lambda = 1$ (multiplicity 2), 2

in general, the eigenvalues of a triangular matrix is just the entries on the diagonal.

$\underline{\lambda = 2}$:

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

general solution is $(0, t, t)$

so the eigenvector is $(0,1,1)$

$\underline{\lambda = 1}$:

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

general solution is $(t, 0, 0)$

so the eigenvector is $(1,0,0)$

Here though the multiplicity of $\lambda = 1$ is 2, there is only 1 (linearly independent) corresponding eigenvector. A has a total of 2 (linearly independent) eigenvectors.

<u>Question</u> : Does $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which rotates every vector in $\mathbb{R}^2$ by 90°, have an eigenvector?

$$A - \lambda I = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \qquad \det(A - \lambda I) = \lambda^2 + 1 = (\lambda + i)(\lambda - i) \qquad \lambda = \pm i$$

$\underline{\lambda = i}$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} -ix_1 - x_2 &= 0 \\ x_1 - ix_2 &= 0 \end{aligned}$$

multiply the second eq. by $(-i)$ and we get the first eq. so the eigenvector is $(1, -i)$

$\underline{\lambda = -i}$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} ix_1 - x_2 &= 0 \\ x_1 + ix_2 &= 0 \end{aligned}$$

again, these eq's are constant multiples, so the eigenvector is $(1, i)$

<u>Remark</u> : (1). Every matrix has n <u>complex</u> eigenvalues (counting multiplicity)

(2). Since complex roots come in pairs, if λ is an eigenvalue with eigenvector K, then $\bar{\lambda}$ (the complex conjugate of λ) is also an eigenvalue with eigenvector $\bar{K}$.

(3). What happens if 0 is an eigenvalue? Then $AX = \vec{0}$ has a nontrivial solution, so $\det(A) = 0$. In general, the product of all eigenvalues of A (counting multiplicity) is equal to $\det(A)$!

8.12 Diagonalization

For an $n \times n$ matrix A, does there exist an $n \times n$ invertible matrix P s.t. $P^{-1}AP$ is a diagonal matrix? If so, we say that A is <u>diagonalizable</u> and that P diagonalizes A.

Suppose $P^{-1}AP = D$, where D is a diagonal matrix. Then $AP = PD$,

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{p_{n1}}_{\vec{c}_1} & \underbrace{p_{n2}}_{\vec{c}_2} & \cdots & \underbrace{p_{nn}}_{\vec{c}_n} \end{pmatrix}}_{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \underbrace{\begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}}_{D}$$

which means $\left(A\vec{c}_1 \;\; A\vec{c}_2 \;\; \cdots \;\; A\vec{c}_n\right) = \left(d_1\vec{c}_1 \;\; d_2\vec{c}_2 \;\; \cdots \;\; d_n\vec{c}_n\right)$

So the d_i's are exactly the eigenvalues of A and P_i's are the corresponding eigenvectors. Note that the c_i's must be linearly independent, because otherwise P is non-invertible. Thus a necessary condition for diagonalizing A is that A must have n linearly independent eigenvectors. It is easy to see that this is a sufficient condition as well. To summarize:

<u>Theorem</u>: for an $n \times n$ matrix A, $\exists$ invertible matrix P s.t. $P^{-1}AP = D$ for some diagonal matrix D iff A has n linearly independent eigenvectors. In this case, the entries of D are the eigenvalues of A and the columns of P are the corresponding eigenvectors.

Ex: $A = \begin{pmatrix} -5 & 9 \\ -6 & 10 \end{pmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= (-5-\lambda)(10-\lambda) + 54 \\ &= -50 - 5\lambda + \lambda^2 + 54 \\ &= \lambda^2 - 5\lambda - 4 \\ &= (\lambda-4)(\lambda-1) \\ \lambda &= 4, 1 \end{aligned}$$

Since the eigenvalues are all distinct, there are $n=2$ linearly independent eigenvectors, so A is diagonalizable.

<u>$\lambda = 4$</u>

$$\begin{pmatrix} -9 & 9 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{c}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>$\lambda = 1$</u>

$$\begin{pmatrix} -6 & 9 \\ -6 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{c}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Thus $P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ $\left(\text{or } P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right)$

We have

$$\underbrace{\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -5 & 9 \\ -6 & 10 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}}_{D}$$

Ex: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

from before, we found that the eigenvalues of A are 1 and 2. the eigenvector corresponding to 2 is $(1, 1, 1)$ and the eigenvectors corresponding to -1 are $(1, 0, -1)$ and $(0, 1, -1)$. Here even though the eigenvalues are not distinct, we have $n = 3$ linearly independent eigenvalues and thus A is diagonalizable.

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Exercise: compute P^{-1}

can switch these 2 columns

Ex:
$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
From before, the eigenvalues are 1 and 2. $\lambda = 1$ has multiplicity 2, but only 1 corresponding eigenvector, so A is not diagonalizable.

Question: suppose A is diagonalizable. How can we quickly compute powers of A?

Since $P^{-1}AP = D$, $A = PDP^{-1}$, and $A^m = \underbrace{(PD\cancel{P^{-1}}PDP^{-1}P\cdots P^{-1}PD\cancel{P^{-1}}PDP^{-1})}_{m \text{ times}} = PD^mP^{-1}$

↑ just raise each entry to the mth power

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$ $\quad \lambda = \frac{1 \pm \sqrt{1-(-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\quad K = \begin{pmatrix} \frac{1 \pm \sqrt{5}}{2} \\ 1 \end{pmatrix}$

$$A^m = \underbrace{\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^m & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^m \end{pmatrix}}_{D^m} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-1+\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}}_{P^{-1}}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} (\lambda_1)^{m+1} - (\lambda_2)^{m+1} & (\lambda_1)^m - (\lambda_2)^m \\ (\lambda_1)^m - (\lambda_2)^m & (\lambda_1)^{m-1} - (\lambda_2)^{m-1} \end{pmatrix} \quad \text{where } \lambda_1 = \frac{1+\sqrt{5}}{2}, \ \lambda_2 = \frac{1-\sqrt{5}}{2}$$

That was a lot of busywork! Now, the payoff:

Consider the Fibonacci sequence $\overset{F_0}{0}, \overset{F_1}{1}, \overset{F_2}{1}, \overset{\cdots}{2}, 3, 5, 8, \ldots$ where $F_{m+2} = F_m + F_{m+1}$.

What is a formula for the mth term of this sequence, without having to compute the preceding terms?

We can represent the sequence as $\begin{pmatrix} F_{m+2} \\ F_{m+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix}$.

Note that
$$\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = A \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$
$$\begin{pmatrix} F_3 \\ F_2 \end{pmatrix} = A \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = A^2 \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$
$$\begin{pmatrix} F_4 \\ F_3 \end{pmatrix} = A \begin{pmatrix} F_3 \\ F_2 \end{pmatrix} = A^3 \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

In general, $\begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix} = A^m \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = A^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Thus
$$\begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} (\lambda_1)^{m+1} - (\lambda_2)^{m+1} \\ (\lambda_1)^m - (\lambda_2)^m \end{pmatrix}$$

In particular,
$$F_m = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m \right)$$

Sidenote: As $m \to \infty$, $\left(\frac{1+\sqrt{5}}{2}\right)^m$ dominates $\left(\frac{1-\sqrt{5}}{2}\right)^m$, so $\lim_{m \to \infty} \frac{F_{m+1}}{F_m} = \frac{1+\sqrt{5}}{2}$.

This is called the Golden Ratio.

3.1 Preliminary Theory : Linear (Differential) Equations

Differential equations are, unsurprisingly, equations involving derivatives.

Ex: $\underset{\substack{\uparrow \\ \text{Banker's Equation}}}{y' = 0.01y}$ $\qquad \underset{\substack{\uparrow \\ \text{Spring Equation}}}{3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 5y = \sin(2t)}$ $\qquad \underset{\substack{\uparrow \\ \text{Heat Equation (closely related to} \\ \text{the Black-Scholes equation in finance)}}}{\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0}$

(Notation: $y^{(n)}$ and $\frac{d^n y}{dt^n}$ will be used interchangeably.)

To <u>Solve</u> a DE means to write the dependent variable as a function of the independent variables. In this course we will study only <u>ordinary</u> (as opposed to partial) DE's, which have only one independent variable, x or t (t is usually used as time).

<u>In general, Solving DE's is very hard!</u> The theory of existence / uniqueness is the subject of very advanced mathematics, so our goal is to learn to recognize and solve ODE's with <u>known systematic methods of solutions.</u>

First, a <u>review of separation of variables</u>: solving the Banker's Equation

$$\frac{dy}{dt} = 0.01y \qquad \int \frac{dy}{0.01y} = \int dt \qquad 100\ln(y) = t + C \qquad y(t) = e^{\frac{t+C}{100}} = ke^{\frac{t}{100}}$$

the constant, k, depends on the <u>initial condition</u>

Ex: $y(0) = 100 \qquad y(0) = ke^{\frac{0}{100}} = k = 100 \qquad y(t) = 100e^{\frac{t}{100}} \leftarrow$ how much money you'll have at time t if you start with \$100

In general, an ODE has a family of solutions. The initial condition uniquely determines a specific solution in this family (kind of like parallel lines, each with a different y-intercept). Sometimes we will be asked to find the general family of solutions, sometimes we will be asked to solve the initial value problem.

<u>Linear Equations</u>: We will mostly study DE's of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

if $g(x) = 0$, we call the equation a <u>homogeneous</u> linear D.E.

The above equation is said to be linear because the derivatives of y appear in first powers and not mixed in products. Also, it's what we call a <u>linear operator</u>:

if $a_n(x)y_1^{(n)} + a_{n-1}(x)y_1^{(n-1)} + \dots a_0(x)y_1 = g_1(x)$ and $a_n(x)y_2^{(n)} + a_{n-1}(x)y_2^{(n-1)} + \dots + a_0(x)y_2 = g_2(x)$, written compactly as $L(y_1) = g_1(x)$ and $L(y_2) = g_2(x)$, then $L(\alpha y_1 + \beta y_2) = \alpha g_1(x) + \beta g_2(x)$.

(this is similar to $M(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha M\vec{v}_1 + \beta M\vec{v}_2$, where M is a matrix, α and β are scalars.)

<u>Superposition Principle for Homogeneous Equations</u>: if $y_1, y_2, \dots, y_n$ are solutions of a homogeneous DE $L(y)$, then $\forall$ all constants $c_1, c_2, \dots, c_n$, $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a solution of $L(y)$.

<u>Proof</u>: $L(c_1y_1 + c_2y_2 + \dots + c_ny_n) = c_1L(y_1) + c_2L(y_2) + \dots + c_nL(y_n) = c_1\cdot 0 + c_2\cdot 0 + \dots + c_n\cdot 0 = 0.$

Ex: x^2 and $x^2\ln x$ are both solutions of the DE $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$, so $3x^2 - 11x^2\ln x$ is also a solution on the interval $(0, \infty)$. Verify this!

As in linear algebra, we are interested in <u>linearly independent</u> solutions.

<u>Definition</u> : $f_1(x), f_2(x), \dots, f_n(x)$ are said to be linearly independent on an interval I if $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for all $x \in I \Rightarrow c_1 = c_2 = \dots = c_n = 0$.

Ex : $1, x, x^2$ are linearly independent on $(-\infty, \infty)$; $\sqrt{x} - 5,\ 1,\ 2\sqrt{x}$ are linearly dependent on $(0, \infty)$ because $\sqrt{x} - 5 + 5(1) - \frac{1}{2}(2\sqrt{x}) = 0$.

BTW, we will mostly ignore fine prints about intervals when we actually solve ODE'S

<u>The Wronksian</u> : Suppose $f_1(x), f_2(x), \dots, f_n(x)$ are all $(n-1)$ time differentiable.

The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the <u>Wronksian</u> of $f_1, f_2, \cdots f_n$.

Ex : $W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

<u>Criteria for LI</u> (the book is wrong on this!) : If $W(f_1, \dots, f_n) \neq 0$ for some $x \in I$, then $f_1, \dots, f_n$ are LI on the interval I.

Ex : x and x^2 are LI on $\mathbb{R}$ because $\exists\, x \in \mathbb{R}$ s.t. $W(x, x^2) \neq 0$.

<u>Theorem</u> : any n^{th} order linear homogeneous DE has n linearly independent solutions on some interval I. These solutions are said to be a <u>fundamental set of solutions</u> on I. The <u>general solution</u> on I is $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$, where $y_1, y_2, \dots, y_n$ are a fundamental set of solutions and $c_1, c_2, \dots, c_n$ are arbitrary constants.

Ex : $e^{3x} + e^{-3x}$ are both solutions of $y'' - 9y = 0$ on $(-\infty, \infty)$. Since this is a second order DE, and since $W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3e^{3x-3x} - 3e^{3x-3x} = -6 \neq 0$, e^{3x} and e^{-3x} form a fundamental set of solutions for $y'' - 9x = 0$, and the general solution of $y'' - 9y = 0$ on $(-\infty, \infty)$ is $c_1 e^{3x} + c_2 e^{-3x}$.

<u>Nonhomogeneous Equations</u> : If $a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = g(x) \neq 0$, we call the general solution of the associated homogeneous equation the <u>complementary function</u> of the DE and any solution of the DE a <u>particular solution</u>. The general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1 + c_2 y_2 + \dots + c_n y_n}_{\text{general solution of } a_n(x)y^{(n)} + \dots + a_0(x)y = 0} + \underbrace{y_p}_{\text{any solution of } a_n(x)y^{(n)} + \dots + a_0(x)y = g(x)}.$$

Ex : $y'' - 9y = 9x$.

From before, the general solution to $y'' - 9y = 0$ on $(-\infty, \infty)$ is $c_1 e^{3x} + c_2 e^{-3x}$. You can verify that $-x$ is a solution of $y'' - 9y = 9x$ on $(-\infty, \infty)$, so the general solution of $y'' - 9y = 9x$ on $(-\infty, \infty)$ is $y = c_1 e^{3x} + c_2 e^{-3x} - x$.

<u>Remark</u>

this works because of the linearity of the DE:

$$L(c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p) = L(c_1 y_1 + c_2 y_2 + \dots + c_n y_n) + L(y_p) = 0 + g(x) = g(x).$$

Also, this should remind you of nonhomogeneous systems from linear algebra.

Superposition Principle for nonhomogeneous equations

If y_i is a particular solution of $a_n(x)y^{(n)} + \dots + a_0(x)y = g_i(x)$, $1 \le i \le k$ then $y_1 + y_2 + \dots + y_k$ is a particular solution of

$$a_n(x)y^{(n)} + \dots + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x).$$

Ex: $-x$ is a particular solution of $y'' - 9y = 9x$ on $(-\infty, \infty)$

$-2x^2 - \frac{4}{9}$ is a particular solution of $y'' - 9y = 18x^2$ on $(-\infty, \infty)$

so $-x - 2x^2 - \frac{4}{9}$ is a particular solution of $y'' - 9y = 9x + 18x^2$ on $(-\infty, \infty)$.

Initial Value Problems

Generally, for an n-th order ODE, we are given n initial conditions, i.e.
$y(x_0) = k_1,\ y'(x_0) = k_2, \dots, y^{(n-1)}(x_0) = k_n$.

Ex: $3y''' + 5y'' - y' + 7y = 0$ $\quad y(1) = 0,\ y'(1) = 5,\ y''(1) = 12$ (note that x_0 does not have to be 0, but the initial conditions must be for the same x_0).

Theorem: Let $a_n(x), a_{n-1}(x), \dots, a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for each $x \in I$. If $x_0 \in I$, then a solution of the IVP exists and is unique on I.

Ex: $xy'' - y' = 0$ $\quad y(1) = 0,\ y'(1) = 1$ has a solution on $(0, \infty)$

$y(0) = 0,\ y'(0) = 1$ is not guaranteed to have a unique solution even on $(-\infty, \infty)$ because the coefficient x is 0 at $x_0 = 0$. In fact, this IVP has no solution.

$$\left.\begin{array}{l} y(0) = 0,\ y(1) = 1 \\ y(0) = 0,\ y'(1) = 1 \end{array}\right\}$$

These are called boundary value problems, and the above theorem does not apply. In general, BVP are more erratic than IVP, and existence/uniqueness of solutions isn't easily summarized.

3.3 Homogeneous Linear (Differential) Equations w/ Constant Coefficients

Linear differential equations are of the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x)$.
The easiest case is when $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ are all constants and $g(x) = 0$.

<u>n=1</u> : $ay' + by = 0$ this can be solved with separation of variables (e.g. Banker's Eq.)

<u>n=2</u> : $ay'' + by' + cy = 0$ guess $y = e^{mx}$ (m a constant) as a solution; the guess is reasonable because all derivatives of e^{mx} are its multiples.

$$\left.\begin{array}{l} y' = me^{mx} \\ y'' = m^2e^{mx} \end{array}\right\} \quad \begin{array}{l} am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \\ (am^2 + b + c)e^{mx} = 0 \qquad m = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \end{array}$$

3 cases: $e^{mx} \neq 0$, so $am^2 + b + c = 0$ (this is called the <u>auxiliary equation</u>)

<u>case 1</u> : real and distinct roots m_1 and m_2 ($b^2 - 4ac > 0$)
general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$

Ex: $2y'' - 5y' + 3y = 0$ auxiliary eq. is $2m^2 - 5m + 3 = 0$
$2m^2 - 5m + 3 = (2m-3)(m-1)$ $m = \frac{3}{2}, 1$
general solution is $y = c_1e^{\frac{3}{2}x} + c_2e^{x}$

<u>Case 2</u> : real and repeated root m ($b^2 - 4ac = 0$)
general solution is $y = c_1e^{mx} + c_2xe^{mx}$ (verify that xe^{mx} is a solution, and that e^{mx} and xe^{mx} are LI on $(-\infty, \infty)$!)

Ex: $y'' - 10y' + 25y = 0$ $y(0) = 0$ $y'(0) = 1$
auxiliary equation is $m^2 - 10m + 25 = 0$ $m^2 - 10m + 25 = (m-5)^2$ $m = 5$
general solution is $y = c_1e^{5x} + c_2xe^{5x}$
$y(0) = 0 \Rightarrow y(0) = c_1 = 0$ $y'(0) = 1$ $y' = 5c_1e^{5x} + c_2e^{5x} + 5c_2xe^{5x} \Rightarrow y'(0) = 5c_1 + c_2 = 1$
$c_2 = 1$
the solution to the IVP is $y = xe^{5x}$.

<u>Case 3</u> : complex roots $\alpha \pm \beta i$ ($b^2 - 4ac < 0$)
Like in case 1, the general solution is $y = c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha-\beta i)x}$. But we prefer to work with real functions. Using Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can rewrite the general solution as $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.

Ex: $y'' + 4y' + 7y = 0$ auxiliary eq. is $m^2 + 4m + 7 = 0$ $m = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm i\sqrt{3}$
general solution is $y = e^{-2x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)$.

What about <u>n>2</u>? Solve the auxiliary equation $a_nm^n + a_{n-1}m^{n-1} + \ldots + a_1m + a_0 = 0$ then form the general solution using e^{mx} and x^ke^{mx} ($k < n$), as necessary.

Ex: $y^{(4)} - 5y''' + 9y'' - 7y' + 2y = 0$ A.E. is $m^4 - 5m^3 + 9m^2 - 7m + 2 = (m-2)(m-1)^3 = 0$
general solution is $y = c_1e^{2x} + c_2e^{x} + c_3xe^{x} + c_4x^2e^{x}$.

Ex: $y^{(4)} + 2y'' + y = 0$ A.E. is $m^4 + 2m^2 + 1 = (m^2+1)^2 = 0$ $m = \pm i$ (each w/ multiplicity 2)
general solution is $y = e^{0\alpha}(c_1\cos x + c_2\sin x) + xe^{0\alpha}(c_3\cos x + c_4\sin x)$
$= c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x$.

3.4 Undetermined Coefficients

Recall that the general solution of a nonhomogeneous DE is of the form

$$y = \underbrace{c_1y_1 + c_2y_2 + \dots + c_ny_n}_{\substack{\text{general solution of the} \\ \text{associated homogeneous equation} \\ \text{(the complementary function, } y_c)}} + \underbrace{y_p}_{\text{any particular solution of the nonhomogeneous DE}}$$

Last time we learned to solve homogeneous DE with constant coefficients.

Strategy to find particular solutions: make an educated guess!

Ex: $y'' + 4y' - 2y = 2x^2 - 3x + 6$

we know how to solve $y''+4y'-2y = 0$: $m^2 + 4m - 2 = 0$, $m = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \sqrt{6}$

so $y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$

Since the RHS is a degree 2 polynomial, we will guess that y_p is a degree 2 polynomial

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y'' &= 2A \end{aligned}\right\} \quad \begin{aligned} &\text{substitute into } y''+4y'-2y = 2x^2-3x+6 \\ &2A + 4(2Ax+B) - 2(Ax^2+Bx+C) \\ &= 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C \\ &= -2Ax^2 + (8A-2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6 \end{aligned}$$

So $-2A = 2 \Rightarrow A = -1$

$8A - 2B = -3$, $8(-1) - 2B = -3 \Rightarrow B = -\frac{5}{2}$

$2A + 4B - 2C = 6$, $2(-1) + 4(-\frac{5}{2}) - 2C = 6 \Rightarrow C = -9$

so $y_p = 6x^2 - \frac{5}{2}x - 9$ general solution is $y = c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x} + 6x^2 - \frac{5}{2}x - 9$

Ex: $y'' + 4y' - 2y = 265 \sin 3x$

we might guess $A\sin 3x$ for the particular solution, but in fact we must take into account not just $g(x)$, but derivatives of $g(x)$

since the derivative of $A\sin 3x$ is $B\cos 3x$, our guess for y_p is $A\sin 3x + B\cos 3x$

$$\left.\begin{aligned} y_p &= A\sin 3x + B\cos 3x \\ y_p' &= 3A\cos 3x - 3B\sin 3x \\ y_p'' &= -9A\sin 3x - 9B\cos 3x \end{aligned}\right\} \quad \begin{aligned} &-9A\sin 3x - 9B\cos 3x + 4(3A\cos 3x - 3B\sin 3x) - 2(A\sin 3x + B\cos 3x) \\ &= (-9A - 12B - 2A)\sin 3x + (-9B + 12A - 2B)\cos 3x \\ &= (-11A - 12B)\sin 3x + (-11B + 12A)\cos 3x = 265\sin 3x \end{aligned}$$

$$\left.\begin{aligned} -11A - 12B &= 265 \\ -11B + 12A &= 0 \end{aligned}\right\} \quad \begin{aligned} -132A - 144B &= 265(12) \\ 132A - 121B &= 0 \end{aligned} \quad \begin{aligned} &-265B = 265(12) \\ &B = -12,\ A = -11 \end{aligned}$$

$y_p = -11\sin 3x - 12\cos 3x$ general solution is $y = c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x} - 11\sin 3x - 12\cos 3x$

Ex: $y'' + 4y' - 2y = 2x^2 - 3x + 6 + 265\sin 3x$

By the superposition principle, a particular solution is $6x^2 - \frac{5}{2}x - 9 - 11\sin 3x - 12\cos 3x$

the general solution is $y = c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x} + 6x^2 - \frac{5}{2}x - 9 - 11\sin 3x - 12\cos 3x$

Alternatively, we could have guessed the particular solution to be of the form

$$\underbrace{Ax^2 + Bx + C}_{\substack{\text{corresponding to} \\ 2x^2 - 3x + 6}} + \underbrace{D\sin 3x + E\cos 3x}_{\substack{\text{corresponding to} \\ 265\sin 3x}} \quad \text{and go from there.}$$

Some more examples on the form of the particular solution:

(1). $y''-8y'+25y=5x^3e^{-x}-7e^{-x}$

we can write $5x^3e^{-x}-7e^{-x}$ as $(5x^3-7)e^{-x}$. In general, our guess for a product is the product of what we would guess for each term. Since we guess Ax^3+Bx^2+Cx+D for $5x^3-7$ and Ee^{-x} for e^{-x}, our guess for y_p is $(Ax^3+Bx^2+Cx+D)Ee^{-x}$. But note that the constant E can be combined with the other constants, so our final guess is $y_p=(Ax^3+Bx^2+Cx+D)e^{-x}$.

(2). $y''-9y'+14y=3x^2-5\sin 2x+7xe^{6x}$

guess for $3x^2$: Ax^2+Bx+C

guess for $-5\sin 2x$: $D\sin 2x+E\cos 2x$

guess for $7xe^{6x}$: $(Fx+G)e^{6x}$

$\Bigg\}$ guess $y_p=Ax^2+Bx+C+D\sin 2x+E\cos 2x+(Fx+G)e^{6x}$

see p. 130 in the book for a chart of guesses.

But sometimes a term in our guess for y_p already appears in y_c!

Ex: $y''-5y'+4y=8e^x$ we would guess $y_p=Ae^x$. but wait!

general solution of $y''-5y'+40=0$ is $c_1e^x+c_2e^{4x}$. a term of the form Ae^x already appears in the complementary function! when that happens we multiply our guess by the lowest power of x that creates a term not already in the complementary function.

In this case, we guess $y_p=Axe^x$. Then $y_p'=Ae^x+Axe^x$ and $y_p''=2Ae^x+Axe^x$, and substituting into $y''-5y'+4y=8e^x$ gives $-3Ae^x=8e^x$, so $A=-\frac{8}{3}$ and the general solution is $y=c_1e^x+c_2e^{4x}-\frac{8}{3}xe^x$. (BTW, why wouldn't Ae^x work?)

A few more examples:

(1). $y''+y=4x+10\sin x$ solution of $y''+y=0$ is $y_c=c_1\cos x+c_2\sin x$

guess for $4x$: $Ax+B$

guess for $10\sin x$: $C\sin x+D\cos x$, which already appear in y_c, so we have to multiply by x. Note that we don't multiply $Ax+B$ by x, only the terms which already appear in y_c.

guess $y_p=Ax+B+Cx\sin x+Dx\cos x$.

(2). $y''-6y'+9y=6x^2+2-12e^{3x}$ solution of $y''-6y'+9y=0$ is $y_c=c_1e^{3x}+c_2xe^{3x}$

guess for $6x^2+2$: Ax^2+Bx+C

guess for $-12e^{3x}$: De^{3x}, which we have to multiply by x^2 because both e^{3x} and xe^{3x} already appear in y_c.

guess $y_p=Ax^2+Bx+C+Dx^2e^{3x}$

(3). $y''-6y'+9y=(\sin x)(e^{3x})$ guess $y_p=(A\sin x+B\cos x)e^{3x}$, which is fine because even though e^{3x} appears in y_c, when multiplied out, $A(\sin x)e^{3x}+B(\cos x)e^{3x}$ are both OK.

(4). $y^{(4)}+y'''=x-x^2e^{-x}$ solution of $y^{(4)}+y'''=0$ is $y_c=c_1+c_2x+c_3x^2+c_4e^{-x}$

guess for x : $Ax+B$, which we have to multiply by x^3

guess for $-x^2e^{-x}$: $(Cx^2+Dx+E)e^{-x}$, which we have to multiply by x

guess $y_p=Ax^4+Bx^3+Cx^3e^{-x}+Dx^2e^{-x}+Exe^{-x}$

Remarks (1). this method works if $g(x)$ is some mix of polynomials, e^{mx}, and sin/cos.

(2). always solve the associated homogeneous equation first!!

(3). after solving the nonhomogeneous DE, use initial conditions (if given) to find the constants.

3.6 Cauchy- Euler Equation

A DE of the form $a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$ is called a Cauchy- Euler equation. The key is that the n^{th} power of x must match up with the n^{th} derivative of y. We will only study the case where $g(x)=0$.

n=1 : $axy' + by = 0$ can solve by separating variables.

n=2 : $ax^2y'' + bxy' + cy = 0$

Guess $y = x^m$. Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

Plug in : $ax^2 m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = x^m(am(m-1) + bm + c) = 0$

since $x^m \neq 0$, $am(m-1) + bm + c = am^2 + (b-a)m + c = 0$ } auxiliary eq. for Cauchy-Euler eq.

Case 1 : auxiliary eq has real distinct roots → general solution is $c_1x^{m_1} + c_2x^{m_2}$

Ex: $x^2y'' - 2xy' - 4y = 0$ $\quad y(1) = 0 \quad y'(1) = 1$

Note that since $a_n x^n = 0$ at zero, we will only consider IVP on $(0,\infty)$...

$am^2 + (b-a)m + c = m^2 + (-2-1)m - 4 = (m-4)(m+1) = 0 \qquad m = 4, -1$

$$\begin{array}{ll} y = c_1x^4 + c_2x^{-1} & y(1) = c_1 + c_2 = 0 \\ y' = 4c_1x^3 - c_2x^{-2} & y'(1) = 4c_1 - c_2 = 1 \end{array}\Bigg\} \begin{array}{l} 5c_1 = 1 \\ c_1 = 1/5 \quad c_2 = -1/5 \end{array} \qquad y = \tfrac{1}{5}x^4 - \tfrac{1}{5}x^{-1}$$

Case 2 : auxiliary eq has real repeated root → general solution is $c_1x^m + c_2x^m\ln x$

Ex : $4x^2y'' + 8xy' + y = 0$

$am^2 + (b-a)m + c = 4m^2 + (8-4)m + 1 = (2m+1)^2 = 0 \qquad m = -\frac{1}{2}$

$y = c_1x^{-\frac{1}{2}} + c_2x^{-\frac{1}{2}}\ln x$

Case 3 : auxiliary eq has complex roots $\alpha \pm \beta i$

we can take general solution to be $y = c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta}$, but like last time, we don't want to work with complex numbers, so we use Euler's formula $(e^{ix} = \cos x + i\sin x)$ to rewrite $y = c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta}$ as

$y = x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]$

Ex: $4x^2y'' + 17y = 0$

$am^2 + (b-a)m + c = 4m^2 - 4m + 17 = 0 \qquad m = \dfrac{4 \pm \sqrt{16-272}}{8} = \frac{1}{2} \pm 2i$

$y = x^{\frac{1}{2}}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$

n>2 : use the same technique, i.e. guess $y = x^m$.

Ex : $x^4y^{(4)} + 6x^3y''' = 0 \qquad y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$\downarrow \qquad y''' = m(m-1)(m-2)x^{m-3} \quad y^{(4)} = m(m-1)(m-2)(m-3)x^{m-4}$

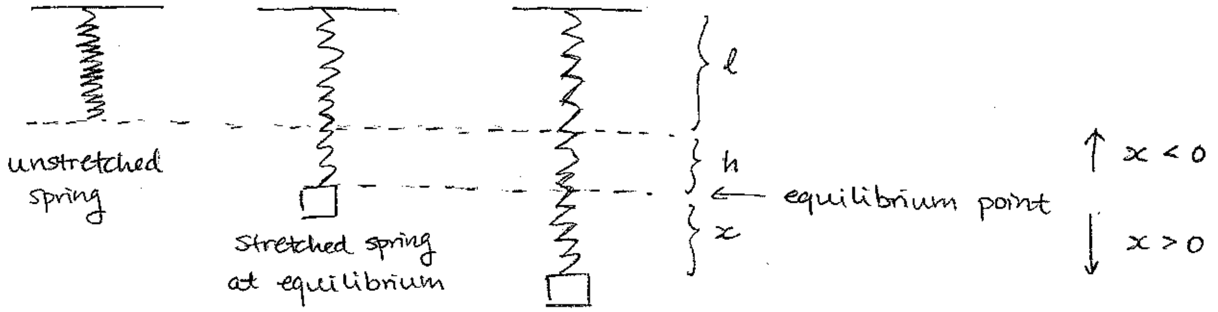
$$x^m\Big(\underbrace{m(m-1)(m-2)(m-3) + 6m(m-1)(m-2)}\Big) = 0$$

$(m-1)(m-2)(m(m-3) + 6m) = (m-1)(m-2)(m^2 + 3m) = (m-1)(m-2)(m)(m+3)$

$m = 1, 2, 0, -3 \qquad y = c_1x + c_2x^2 + c_3 + c_4x^{-3}$

Note: if the auxiliary equation is, say, $(m-1)(m-2)^3$, then the general solution would be $y = c_1x + c_2x^2 + c_3x^2\ln x + c_4x^2(\ln x)^2$.

3.8 : Linear Models : Initial-Value Problems (Spring / Mass Systems)



convention: downward direction is positive

Undamped Motion : Suppose we have a mass hanging from a spring, and suppose that the only forces present are gravity and the elasticity of the spring, then physicists have kindly used Hooke's Law and Newton's Second Law to model the spring-mass system for us:

$$\underset{\substack{\downarrow \\ \text{mass of the} \\ \text{object attached} \\ \text{to the spring}}}{m}\frac{d^2x}{dt^2} = \underset{\substack{\downarrow \\ \text{spring constant, } k = \frac{\text{weight of the object attached to the spring}}{\text{distance by which the object stretches the spring}}}}{-kx}$$

(here x indicates the position below equilibrium and is a funtion of time)

(the spring exerts a force opposite the downward stretch)

we rewrite this equation as $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. We know how to solve this!

auxiliary equation: $r^2 + \frac{k}{m} = 0$ (using r because we already use m to stand for mass)

$r = \pm i\sqrt{k/m}$, so letting $\omega = \sqrt{k/m}$, the general solution is $\underline{x(t) = c_1 \cos \omega t + c_2 \sin \omega t}$

Ex: An object weighing 2 pounds stretches a spring $\frac{1}{2}$ feet. The object is released at $\frac{2}{3}$ feet below equilibrium and given an initial upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of free motion ("free" means there are no external driving forces).

we just need to find m and k: $m = \frac{\text{weight}}{\text{acceleration due to gravity}} = \frac{2 \text{ pounds}}{32 \text{ ft}/s^2} = \frac{1}{16}$ "slug"

(recall that "kilogram" is a measure of mass while "pound" is a unit of force:

1 Newton = 1 kg · meter/s^2; 1 pound = 1 slug · ft/s^2)

$k = 2 \text{ pounds} / \frac{1}{2} \text{ ft} = 4 \text{ lb/ft}$ $\qquad k/m = \frac{4 \text{ lb/ft}}{\frac{1}{16} \frac{\text{lb}}{\text{ft}/s^2}} = 64/s^2, \quad \omega = 8/s$

so the DE which models the system is $\frac{d^2x}{dt^2} + 64x = 0$, with initial conditions $x(0) = \frac{2}{3}$ and $x'(0) = -\frac{4}{3}$.

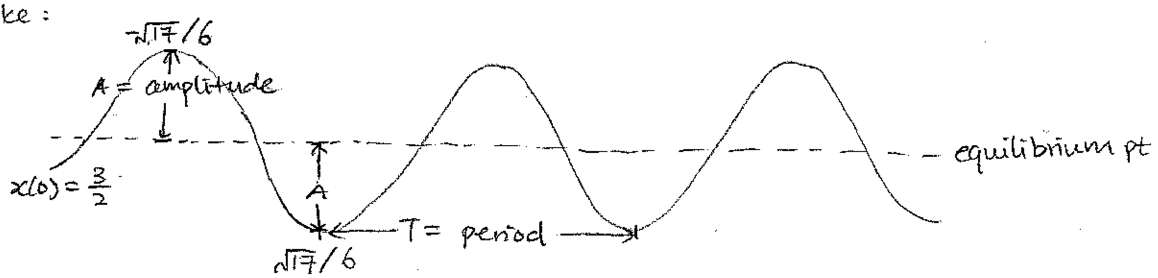
the general solution is $x(t) = c_1 \cos 8t + c_2 \sin 8t$. the initial conditions give

$x(0) = c_1 = \frac{2}{3}$ and $x'(0) = -8c_1 \sin(8 \cdot 0) + 8c_2 \cos(8 \cdot 0) = 8c_2 = -\frac{4}{3}$, so $c_2 = -\frac{1}{6}$

thus the equation of free motion is $x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$ (BTW, the units work out!)

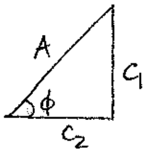
what this motion looks like:

this is harmonic motion (amplitude and period don't change)



T is given by $2\pi/\omega$ (in the example, $T = 2\pi/\frac{8}{3} = \frac{\pi}{4}$ s); Frequency $= \frac{1}{T}$

A is given by $\sqrt{c_1^2 + c_2^2}$ (in the example, $A = \sqrt{(\frac{2}{3})^2 + (-\frac{1}{6})^2} = \frac{\sqrt{17}}{6}$)



also there is something called the phase angle, ϕ, which is angle in a right triangle with c_1 as the opposite side and c_2 as the adjacent side.

So $\phi = \tan^{-1}(\frac{c_1}{c_2})$, but careful with the quadrants!

in the previous example, $\phi = \tan^{-1}(\frac{2/3}{-1/6}) = -1.326$ rad. but this is in the 4th quadrant while $(-\frac{1}{6}, \frac{2}{3})$ is in the 2nd quadrant! so $\phi = -1.326 + \pi = 1.816$ rad.

in general, since the range of $\tan^{-1}$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, we must add by π if ϕ is in the 2nd or 3rd quadrant.

with A and ϕ, we can rewrite the equation of motion as $\underline{x(t) = A\sin(\omega t + \phi)}$.
This equation is useful because it tells us when the spring is at equilibrium pt.
in the previous example, $x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816)$, so the spring returns to equilibrium pt whenever $\sin(8t + 1.816) = 0$, i.e. whenever $8t + 1.816 = n\pi$ $(t \geq 0)$, and the first time the spring reaches equilibrium pt is $t = \frac{\pi - 1.816}{8} \approx 0.166$ s.

Damped Motion

undamped motion can only occur in a vacuum; from practice we know that air resistance will slow down the spring until eventually the spring actually stops at equilibrium point.
This is called the damping force, which is given as a constant (β) multiple of instantaneous velocity.
The new model is now $m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$ (damping force acts in direction opposite to motion)
rewriting as $\underline{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$, this is again a DE with constant coefficients.
auxilliary equation is $r^2 + \frac{\beta}{m}r + \frac{k}{m} = 0$, with roots $-\frac{\beta}{2m} \pm \sqrt{(\frac{\beta}{2m})^2 - \frac{k}{m}}$
let $\lambda = \frac{\beta}{2m}$ and $\omega = \sqrt{\frac{k}{m}}$, the roots are $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$

Ex: an 8-pound object stretches a spring by 2 feet. A damping force numerically equal to 2 times the instantaneous velocity acts on the system. The object is released from the equilibrium position with an upward velocity of 3 ft/s. Determine the equation of free damped motion.

$\beta = 2$ $\quad m = \frac{8}{32} = \frac{1}{4}$ $\quad k = \frac{8}{2} = 4$ $\quad$ so $\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ becomes

$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$ $\quad x(0) = 0$ $\quad x'(0) = -3$ $\quad$ BTW, these can't be positive!

which we solve to get the general solution $x(t) = c_1e^{\boxed{-4}t} + c_2te^{\boxed{-4}t}$
initial conditions give $c_1 = 0$ and $c_2 = -3$, so $x(t) = -3te^{-4t}$

(Question: what is the maximum height the spring reaches above the equilibrium point?)

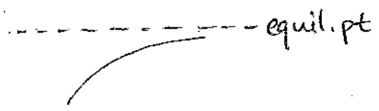
As always, there are 3 cases when it comes to 2nd order ODE's, depending on the roots $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$:

case 1: $\lambda^2 - \omega^2 > 0$

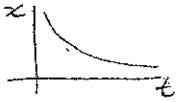
$x(t) = c_1e^{-\lambda - \sqrt{\lambda^2-\omega^2}} + c_2e^{-\lambda + \sqrt{\lambda^2-\omega^2}}$

"overdamped", i.e. no oscillatory motion

Spring trajectory looks like



note: the book graphs x as a fn of t, w/down as positive:

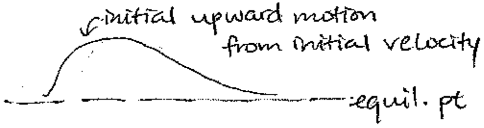


case 2 $\lambda^2 - \omega^2 = 0$

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

"critically damped"
still no oscillatory motion,
but any less resistance force,
there would oscillation

spring trajectory from ex:



case 3 $\lambda^2 - \omega^2 < 0$

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\, t + c_2 \sin\sqrt{\omega^2-\lambda^2}\, t\right)$$

"underdamped" – there is oscillation, but it weakens with time

spring trajectory:

in this case, we can write $x(t)$ in a form that emphasizes the oscillatory nature of the spring:

$$x(t) = \underbrace{Ae^{-\lambda t}}_{\text{"damped amplitude"}} \sin\left(\sqrt{\omega^2-\lambda^2}\, t + \phi\right)$$ where as before, $A = \sqrt{c_1^2 + c_2^2}$

$2\pi / \sqrt{\omega^2-\lambda^2}$ is sometimes called "quasi period"

$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$ w/ attention to the appropriate quadrant

(in the homework, you'll be asked when a spring described by an equation like this first crosses the equilibrium in an upward motion. that's when $\sin(\sqrt{\omega^2-\lambda^2}\, t + \phi) = 0$, though not necessarily the first time, depending the initial position / velocity.)

Driven Motion

Now imagine that in addition to the damping force, there is an external force $f(t)$ being exerted on the system (say I'm shaking the spring). Now the model becomes

$$m \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt} + f(t)$$

which we rewrite as

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{f(t)}{m}$$

again, some people prefer the slightly cleaner expression

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t) \quad \text{where } 2\lambda = \frac{\beta}{m},\ \omega^2 = \frac{k}{m},\ F(t) = \frac{f(t)}{m}$$

this is now a nonhomogeneous DE with constant coefficients, which we solve using the method of undetermined coefficients.

Ex: suppose that in the previous example, the spring/mass system is driven by an external force $f(t) = 2e^{-t}$. then the DE is $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 2e^{-t}$.

since the general solution of the associated homogeneous system is $c_1 e^{-4t} + c_2 t e^{-4t}$, we can guess a particular solution of the form $x_p = Ae^{-t}$

$$\left.\begin{aligned} x_p' &= -Ae^{-t} \\ x_p'' &= Ae^{-t} \end{aligned}\right\} \quad Ae^{-t} - 8Ae^{-t} + 16Ae^{-t} = 2e^{-t} \qquad 9A = 2 \quad A = \frac{2}{9}$$

this gives us $x(t) = c_1 e^{-4t} + c_2 t e^{-4t} + \frac{2}{9}e^{-t}$

$x(0) = 0 \Rightarrow c_1 + \frac{2}{9} = 0,\ c_1 = -\frac{2}{9}$

$x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t} - \frac{2}{9}e^{-t}$

$x'(0) = -3 \Rightarrow \frac{8}{9} + c_2 - \frac{2}{9} = -3,\ c_2 = -\frac{33}{9} = -\frac{11}{3}$

equation of driven motion is $x(t) = -\frac{2}{9}e^{-4t} + \frac{11}{3}te^{-4t} + \frac{2}{9}e^{-t}$.

10.1 Preliminary Theory: Systems of Differential Equations

suppose $x(t) = e^{-2t}$ and $y(t) = -e^{-2t}$

Note that $\left.\begin{aligned} x(t) + 3y(t) &= -2e^{-2t} = x'(t) \\ 5x(t) + 3y(t) &= 2e^{-2t} = y'(t) \end{aligned}\right\}$ write as $\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

In general, we will study systems of differential equations of the form

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2 \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n \end{aligned}\right\} \quad \text{write as} \quad \underbrace{\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}}_{F}$$

where each x_i and f_i is a function of t
and a_{ij}'s are constants

Ex: $$\left.\begin{aligned} x' &= 7x + 5y - 9z + 8e^{-2t} \\ y' &= 4x + y + z \\ z' &= -2y + 3z - 3e^{5t} + t \end{aligned}\right\} \quad \underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{X'} = \begin{pmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} + \begin{pmatrix} 8e^{-2t} \\ 0 \\ -3e^{5t} + t \end{pmatrix}$$

The system is said to be homogeneous if $F = \vec{0}$. We will only learn to solve homogeneous systems

Initial value problems: we would be given $x_i(t_0) = \gamma_i$ for each $1 \le i \le n$ (e.g. $X(\vec{0}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, which means $x(0) = 1$, $y(0) = 2$, and $z(0) = 3$.)

For linear systems with constant coefficients, an IVP always has a unique solution.

Superposition Principle: if $X_1, X_2, \dots, X_n$ are solutions of $AX = X'$, then any linear combination $c_1X_1 + c_2X_2 + \dots + c_nX_n$, with $c_1, c_2, \dots, c_n$ being constants, is also a solution.

Ex: we saw above that $\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ is a solution to $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$. $\begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ is another solution,
so $c_1\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ is a solution for any constants c_1 and c_2.

Linear independence: as always, we only care about linearly independent solutions. Here $X_1, X_2, \dots, X_n$ are said to be linearly independent on some interval I if

$$c_1X_1 + c_2X_2 + \dots + c_nX_n = \vec{0} \text{ for every } t \in I \implies c_1 = c_2 = \dots = c_n = 0.$$

Criteria for linear independence: let $X_1, X_2, \dots, X_n$ be solutions of $AX = X'$, where A is an $n \times n$ matrix. The Wronksian is now defined as the determinant

$$W(X_1, X_2, \dots, X_n) = \left| X_1 \ X_2 \ \cdots \ X_n \right| = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

with the columns marked $\underbrace{}_{X_1}$, $\underbrace{}_{X_2}$, …, $\underbrace{}_{X_n}$.

$X_1, X_2, \dots X_n$ are L.I. on I iff $W(X_1, X_2, \dots, X_n) \neq 0$ for some (equivalently every) $t \in I$.

Ex: $X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ $\quad W(X_1, X_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix}$

$X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ $\quad = 5e^{4t} + 3e^{4t} = 8e^{4t} \neq 0$, so X_1 and X_2 are L.I. solutions of $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$.

Any n L.I. solutions $X_1, X_2, \dots X_n$ of an n-dimensional system is called a fundamental set of solutions. The general solution is then $X = c_1X_1 + c_2X_2 + \dots + c_nX_n$.

10.2 Homogeneous Linear Systems

Goal : to solve homogeneous linear systems of the form $AX=X'$.

recall that when solving a single DE with constant coefficients, solutions are of the form $e^{\lambda t}$.

It's reasonable to guess that solutions of $AX=X'$ are of the form $Ke^{\lambda t}$, where K is a vector of constants (example from last time: $\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t}$ and $\begin{pmatrix}3\\5\end{pmatrix}e^{6t}$ are solutions of $X'=\begin{pmatrix}1&3\\5&3\end{pmatrix}X$).

Suppose $X=Ke^{\lambda t}$ then $AX=X'$ becomes $AKe^{\lambda t}=\lambda Ke^{\lambda t} \Rightarrow AK=\lambda K$. Looks familiar?

Theorem : If A has n linearly independent eigenvectors, then the general solution of $AX=X'$ is $X=c_1K_1e^{\lambda_1 t}+c_2K_2e^{\lambda_2 t}+\ldots+c_nK_ne^{\lambda_n t}$,
where $\lambda_1,\ldots,\lambda_n$ are eigenvalues of A (not necessarily distinct), and
$K_1,\ldots,K_n$ are corresponding eigenvectors.

Ex: $\left.\begin{matrix}x'=2x+3y\\y'=2x+y\end{matrix}\right\}$ $\begin{pmatrix}2&3\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$

eigenvalues of A

$(2-\lambda)(1-\lambda)-6=2-3\lambda+\lambda^2-6$

$=\lambda^2-3\lambda-4=(\lambda-4)(\lambda+1)$ $\lambda=4,-1$

$\lambda=4$

$\begin{pmatrix}-2&3\\2&-3\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $K=\begin{pmatrix}3\\2\end{pmatrix}$

$\lambda=-1$

$\begin{pmatrix}3&3\\2&2\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $K=\begin{pmatrix}1\\-1\end{pmatrix}$

general solution : $X=c_1\begin{pmatrix}3\\2\end{pmatrix}e^{4t}+c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$

Ex : $\left.\begin{matrix}x'=x-2y+2z\\y'=-2x+y-2z\\z'=2x-2y+z\end{matrix}\right\}$ $\begin{pmatrix}1&-2&2\\-2&1&-2\\2&-2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$

eigenvalues of A $(1-\lambda)[(1-\lambda)(1-\lambda)-4]+2[-2(1-\lambda)+4]+2[4-(1-\lambda)2]$

$=-(\lambda^3-3\lambda^2-9\lambda-5)=-(\lambda-5)(\lambda+1)^2$ $\lambda=5,-1$

eigenvector corresponding to 5: $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

eigenvectors corresponding to -1: $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}$

general solution : $X=c_1\begin{pmatrix}1\\-1\\1\end{pmatrix}e^{5t}+c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{-t}+c_3\begin{pmatrix}0\\1\\1\end{pmatrix}e^{-t}$

note that it's the number of linearly independent eigenvectors, <u>NOT</u> number of distinct eigenvalues, that matters.

Question: what happens if there are less than n linearly independent eigenvectors?

Before we answer this question, let's look at an "application":

Romeo's love for Juliet grows proportionally to her love for him. Juliet's love for Romeo cools proportionally to his love for her, though it's also growing proportionally to her existing love for him. She's more into him than him on her at the beginning. What are their long term prospects?

x: Romeo

y: Juliet

$x'=2y$

$y'=-x+2y$

$x(0)=-1$

$y(0)=0$

$\begin{pmatrix}0&2\\-1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$

$(-\lambda)(2-\lambda)+2=-2\lambda+\lambda^2+2=\lambda^2-2\lambda+2$

$\lambda=\frac{2\pm\sqrt{4-8}}{2}=1\pm i$

for $\lambda = 1+i$, $\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad (-1-i)k_1 + 2k_2 = 0$, $\quad -k_1 + (1-i)k_2 = 0$ $\quad$ note that eq 1 = (1+i) eq 2

$k_2 = 1$, $(-1-i)k_1 = -2$

$k_1 = \frac{-2}{-1-i} = 1-i$

since $K = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$, the eigenvector corresponding to $1-i$ is $\bar{K} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

$$X = c_1\begin{pmatrix} 1-i \\ 1 \end{pmatrix}e^{(1+i)t} + c_2\begin{pmatrix} 1+i \\ 1 \end{pmatrix}e^{(1-i)t}$$

again, it would be nice to get rid of the complex numbers

In general, if K is the eigenvector for $\alpha + \beta i$, then the general solution can be rewritten as

$$X = c_1\left[\mathrm{Re}(K)\cos\beta t - \mathrm{Im}(K)\sin\beta t\right]e^{\alpha t} + c_2\left[\mathrm{Im}(K)\cos\beta t + \mathrm{Re}(K)\sin\beta t\right]e^{\alpha t}$$

$$X = c_1\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right]e^t + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t\right]e^t$$

$$\left.\begin{aligned} x(0) &= -1 = c_1[\cos(0) - (-1)\sin(0)]e^0 + c_2[(-1)\cos(0) + \sin(0)]e^0 = c_1 - c_2 \\ y(0) &= 0 = c_1\cos(0)e^0 + c_2\sin(0)e^0 = c_1 \end{aligned}\right\} \begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

$$X = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t\right]e^t$$

$x(t) = (-\cos t + \sin t)e^t$

$y(t) = (\sin t)e^t$

e^t blows up while the trig terms oscillate, so they alternate between love and hate, and both emotions grow more intense with time

Repeated Eigenvalues

Suppose an eigenvalue λ w/ multiplicity 2 only has 1 eigenvector K.

So we only get one solution to $AX = X'$: $Ke^{\lambda t}$. A second linearly independent solution is found by solving for P in $(A - \lambda I)P = K$ and forming $Kte^{\lambda t} + Pe^{\lambda t}$.

Ex: $X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}X \qquad (3-\lambda)(-9-\lambda) + 36 = \lambda^2 + 6\lambda + 9$

$= (\lambda+3)^2 \qquad \lambda = -3$

Solving $(A - \lambda I)K = 0$:

$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Solving $(A - \lambda I)P = K$:

$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad 6p_1 - 18p_2 = 3, \quad 2p_1 - 6p_2 = 1 \qquad P = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

general solution: $X = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix}te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}e^{-3t}\right]$

(P just needs to be <u>any</u> solution)

what if λ has multiplicity 3, but only 1 eigenvector K?

Solve $(A - \lambda I)P = K$, get solution $Kte^{\lambda t} + Pe^{\lambda t}$, then solve $(A - \lambda I)Q = P$, get solution $K\frac{t^2}{2}e^{\lambda t} + Pte^{\lambda t} + Qe^{\lambda t}$

Ex: $X' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}X$

$\lambda = 2$ (multiplicity 3)

<u>$(A - \lambda I)K = 0$</u>

$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ only one eigenvector

<u>$(A - \lambda I)P = K$</u>

$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$(A - \lambda I)Q = P$

$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}$

$$X = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t}\right] + c_3\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}e^{2t}\right]$$

10.3 Solution by Diagonalization

Another method to solve $X' = AX$: suppose A is diagonalizable, i.e. there exist P and D s.t. $P^{-1}AP = D$, where D is a diagonal matrix. Make the substitution $X = PY$. then

$(PY)' = APY$, so $PY' = PDY$ (since $AP = PD$)

P is invertible, so $P^{-1}PY' = P'PDY \Rightarrow Y' = DY$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

(recall λ_i's are eigenvalues of A)

this gives: $y_1' = \lambda_1 y_1$ so $y_1 = c_1 e^{\lambda_1 t}$

$y_2' = \lambda_2 y_2 \qquad y_2 = c_2 e^{\lambda_2 t}$

$\vdots \qquad \vdots$

$y_n' = \lambda_n y_n \qquad y_n = c_n e^{\lambda_n t}$

$$X = PY = P\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

where the λ_i's are the eigenvalues of A and P is the matrix of eigenvectors

Ex: $$X' = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} X$$

Solving for the eigenvalues: $(-2-\lambda)[(-3-\lambda)(9-\lambda)+32]$

$= (-2-\lambda)(-27-6\lambda+\lambda^2+32)$

$= (-2-\lambda)(\lambda^2-6\lambda+5) = (-2-\lambda)(\lambda-5)(\lambda-1)$; $\lambda = -2, 5, 1$

$\underline{\lambda = -2}$

$$\begin{pmatrix} 0 & -1 & 8 \\ 0 & -1 & 8 \\ 0 & -4 & 11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 5}$

$$\begin{pmatrix} -7 & -1 & 8 \\ 0 & -8 & 8 \\ 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad K = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} -3 & -1 & 8 \\ 0 & -4 & 8 \\ 0 & -4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad K = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{5t} \\ c_3 e^{t} \end{pmatrix}$$

$$PY = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{5t} + 2c_3 e^{t} \\ c_2 e^{5t} + 2c_3 e^{t} \\ c_2 e^{5t} + c_3 e^{t} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{t}$$

Note this this method isn't really "new", in the sense that it's essentially the same work as forming solutions from $Ke^{\lambda t}$. But it gives a different perspective on why it's the number of linearly independent eigenvectors that matters.

Review of Power Series

$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots$ is called a power series centered at x_0

$\sum_{n=0}^{\infty} c_n(x-x_0)^n$ is convergent at an x if $\lim_{N\to\infty} \sum_{n=0}^{N} c_n(x-x_0)^n$ exists and is finite.

Every series has a radius of convergence R s.t. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$. What happens at $x-x_0=R$ must be examined on a case-by-case basis!

Ratio Test: let $L = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} \right|$ then at x the series $\begin{cases} \text{converges (abs.) if } L<1 \\ \text{diverges if } L>1 \\ \text{inconclusive if } L=1 \end{cases}$

Ex: determine the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}/2^{n+1}(n+1)}{(x-3)^n/2^n n} \right| = \lim_{n\to\infty} \left| \frac{x-3}{2}\left(\frac{n}{n+1}\right) \right| = \left| \frac{x-3}{2} \right|$$

$\left| \frac{x-3}{2} \right| < 1$ when $x<5$ and $x>1$, so we know the series converges on $(1,5)$.

What about at 1 and 5?

when $x=1$, the series is $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is known to converge

$x=1$, the series is $\sum_{n=1}^{\infty} \frac{2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is known to diverge

So the radius of convergence is 2 and the interval of convergence is $[1,5)$.

Important : $\sum_{n=0}^{\infty} c_n(x-x_0) = 0$ for all x on its interval of convergence $(R>0)$

iff $c_n = 0$ for all n.

Shifting the summation index

we are going to be combining power series a lot ...

Ex: write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as one series

first the 2 series must be made to start at the same power:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2(2-1)c_2 x^0 + \underset{①}{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}} + \underset{②}{\sum_{n=0}^{\infty} c_n x^{n+1}}$$

for ①, let $k=n-2$, $n=k+2$, rewrite as $\sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k$

for ②, let $k=n+1$, $n=k-1$, rewrite as $\sum_{k=1}^{\infty} c_{k-1}x^k$

finally, add everything:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + c_{k-1} \right] x^k$$

Background for 5.1 & 5.2

We know how to solve differential equations of the form $ay''+by'+cy=0$ and $ax^2y''+bxy'+cy=0$.

What about $a(x)y''+b(x)y'+c(x)y=0$, where $a(x)$, $b(x)$, $c(x)$ are arbitrary polynomials?

Divide the equation by $a(x)$ to put into standard form $y''+P(x)y'+Q(x)y=0$, where $P(x)=\frac{b(x)}{a(x)}$ and $Q(x)=\frac{c(x)}{a(x)}$.

Ordinary vs. Singular points : x_0 is an ordinary point of the D.E. if $P(x)$ and $Q(x)$ are both analytic at x_0 (i.e. can be represented by a power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$).

x_0 is a singular point if it is not an ordinary point.

Since $a(x)$, $b(x)$, $c(x)$ are polynomials, x_0 is an ordinary point iff the denominators of $P(x)$ and $Q(x)$ are nonzero.

Ex: $(x^2-4)^2y''+3(x-2)y'+5y=0 \quad \rightarrow \quad y''+\frac{3(x-2)}{(x^2-4)^2}y'+\frac{5}{(x^2-4)^2}y=0$

singular points : ± 2; every other point is an ordinary point.

Regular vs. Irregular points : a singular point x_0 is called a regular singular point if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are both analytic at x_0. Otherwise it is an irregular singular point.

Since $a(x)$, $b(x)$, $c(x)$ are polynomials, a singular point x_0 is regular iff $x-x_0$ appears at most to the 1st power in the denominator of $P(x)$ and at most to the 2nd power in the denominator of $Q(x)$.

Ex: rewrite above example as $y''+\frac{3}{(x+2)^2(x-2)}y'+\frac{5}{(x+2)^2(x-2)^2}y=0$ (every fraction appears in simplified terms)

$x_0=2$ is a regular singular point; $x_0=-2$ is an irregular singular point.

Existence of power series solutions to $a(x)y''+b(x)y'+c(x)y=0$:

(1). If x_0 is an ordinary point, then $\exists$ 2 linearly independent solutions in the form of power series $y=\sum_{n=0}^{\infty} c_n(x-x_0)^n$. Furthermore, the radius of convergence of y is at least the distance between x_0 and the nearest singular point (in $\mathbb{C}$).
(i.e. lower bound)

Ex: RoC of power series solutions about $x_0=0$ in the above example is at least 2.

(2). If x_0 is a regular singular point, then $\exists$ at least one solution of the form $y=\sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}$, where r is a number to be determined. (r may not be an integer, in which case y is not technically a power series.)

We will only find series solutions centered at $x_0=0$.

5.1 : Solutions about Ordinary Points

Goal: to solve differential equations $a(x)y'' + b(x)y' + c(x)y = 0$, where $a(x)$, $b(x)$, $c(x)$ are polynomials. y will be in the form of a power series centered at 0, i.e. $\sum_{n=0}^{\infty} c_n x^n$.

For now we will work with DE's for which 0 is an ordinary point, which roughly speaking means $\frac{b(x)}{a(x)}$ and $\frac{c(x)}{a(x)}$ are defined at 0.

Example: solve $y'' + xy = 0$

let $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4 + \dots$

then $y' = c_1 + c_2 2n + c_3 3n^2 + c_4 4n^3 + \dots$

$y'' = c_2 2n^0 + c_3 (3)(2) n + c_4 (4)(3) n^2 + \dots = \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2}$

plug into $y'' + xy = 0$ and combine into one sum:

$$y'' + xy = \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = c_2 + \sum_{n=3}^{\infty} c_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$k = n-2$, $n = k+2$ $\qquad$ $k = n+1$, $n = k-1$

$$= c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1) x^k + \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= c_2 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) + c_{k-1} \right] x^k = 0$$

in order for an infinite series to be everywhere zero, all coefficients must be zero, i.e. $c_2 = 0$ and $c_{k+2}(k+2)(k+1) + c_{k-1} = 0$ for all $k \geq 1$.

this gives the <u>recurrence relation</u> $c_{k+2} = \frac{-c_{k-1}}{(k+2)(k+1)}$ $\quad k \geq 1$

$k=1$: $c_3 = \frac{-c_0}{(3)(2)}$ $\qquad$ $k=2$: $c_4 = \frac{-c_1}{(4)(3)}$ $\qquad$ $k=3$: $c_5 = \frac{-c_2}{(5)(4)} = 0$

$k=4$: $c_6 = \frac{-c_3}{(6)(5)} = \frac{c_0}{(6)(5)(3)(2)}$ $\qquad$ $k=5$: $c_7 = \frac{-c_4}{(7)(6)} = \frac{c_1}{(7)(6)(4)(3)}$ $\qquad$ $k=6$: $c_8 = \frac{-c_5}{(8)(7)} = 0$

$k=7$: $c_9 = \frac{-c_6}{(9)(8)} = \frac{-c_0}{(9)(8)(6)(5)(3)(2)}$ $\qquad$ $k=8$: $c_{10} = \frac{-c_7}{(10)(9)} = \frac{-c_1}{(10)(9)(7)(6)(4)(3)}$ $\qquad$ $k=9$: $c_{11} = \frac{-c_8}{(11)(10)} = 0$

So $y(x) = c_0 + c_1 x - \frac{c_0}{(3)(2)} x^3 - \frac{c_1}{(4)(3)} x^4 + \frac{c_0}{(6)(5)(3)(2)} x^6 + \frac{c_1}{(7)(6)(4)(3)} x^7 \dots$

$$= c_0 \underbrace{\left[1 - \frac{x^3}{(3)(2)} + \frac{x^6}{(6)(5)(3)(2)} - \frac{x^9}{(9)(8)(6)(5)(3)(2)} \dots \right]}_{y_0(x)}$$

$$+ c_1 \underbrace{\left[x - \frac{x^4}{(4)(3)} + \frac{x^7}{(7)(6)(4)(3)} - \frac{x^{10}}{(10)(9)(7)(6)(4)(3)} \right]}_{y_1(x)}$$

y_0 and y_1 are linearly independent solutions, c_0 and c_1 are arbitrary constants

$y = c_0 y_0 + c_1 y_1$ is the general solution

Remark: since $y'' + xy = 0$ has no singular points, both y_0 and y_1 converges on $(-\infty, \infty)$.

To summarize :

① let $y=\sum_{n=0}^{\infty} c_n x^n$, differentiate term by term to find y' and y''

② plug the series y, y', and y'' into the DE, combine into into one series

③ set all coefficients of the combined series to equal zero, get equivalence relation

④ use recurrence relation to solve for each c_n in terms of c_0 and/or c_1

⑤ one solution y_0 obtained by letting $c_0=1$, $c_1=0$ and another (linearly independent) solution y_1 is obtained by letting $c_0=1$, $c_1=0$.
if we let c_0 and c_1 be arbitrary constants, then $c_0 y_0 + c_1 y_1$ is the general solution

another example: $(x^2+1)y''+xy'-y=0$ (here the RoC is at least 1)

$$y=\sum_{n=0}^{\infty} c_n x^n \qquad y'=\sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + 6c_3 x + c_1 x - c_0 - c_1 x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2,\ n=k+2} + \underbrace{\sum_{n=2}^{\infty} n c_n x^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_n x^n}_{k=n}$$

$$= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}\left[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k\right]x^k$$

$$= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}\left[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}\right]x^k = 0 \longrightarrow \boxed{\begin{array}{ll} c_2 = \frac{c_0}{2} & c_3 = 0 \\ c_{k+2} = \frac{-(k+1)(k-1)c_k}{(k+2)(k+1)} = \frac{(1-k)c_k}{k+2} & \end{array}}$$

$k=2:\ c_4 = \frac{-c_2}{4} = \frac{-c_0}{(4)(2)}$ $\qquad$ $k=3:\ c_5 = \frac{-2c_3}{5} = 0$

$k=4:\ c_6 = \frac{-3c_4}{6} = \frac{3c_0}{(6)(4)(2)}$ $\qquad$ $k=5:\ c_7 = \frac{-4c_5}{7} = 0$

$k=6:\ c_8 = \frac{-5c_6}{8} = \frac{-(5)(3)c_0}{(8)(6)(4)(2)}$ $\qquad$ $k=7:\ c_9 = \frac{-6c_7}{9} = 0$

$k=8:\ c_{10} = \frac{-7c_8}{10} = \frac{(7)(5)(3)c_0}{(10)(8)(6)(4)(2)}$ $\qquad$ $k=9:\ c_{11} = \frac{-8c_9}{11} = 0$

$$y(x) = c_0 + c_1 x + \frac{c_0}{2}x^2 - \frac{c_0}{(4)(2)}x^4 + \frac{3c_0}{(6)(4)(2)}x^6 - \frac{(5)(3)c_0}{(8)(6)(4)(2)}x^8 + \frac{(7)(5)(3)c_0}{(10)(8)(6)(4)(2)}x^{10} \cdots$$

$$= \underset{\underset{y_1}{\downarrow}}{c_1 x} + c_0 \underbrace{\left[1 + \frac{x^2}{2} - \frac{x^4}{(4)(2)} + \frac{3x^6}{(6)(4)(2)} - \frac{(5)(3)x^8}{(8)(6)(4)(2)} + \frac{(7)(5)(3)x^{10}}{(10)(8)(6)(4)(2)}\right]}_{y_0}$$

y_0 can be concisely represented as $1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)(2n-1)\cdots(5)(3)}{2^n\, n!}x^{2n}$

I won't ask you to formulate such representations, but you should be able to write down the first several terms of the sum if you see something like this.

Remark : the technique here can be applied to higher order ODE's, ODE's with nonpolynomial (but still analytic) coefficients (just express the coefficient functions in power series as well), and even nonhomogeneous ODE's. but that might take the rest of our natural lifetimes!

As an exercise, use the power series method to solve $y''+y=0$. you already know the answer is $c_1\cos x + c_2 \sin x$, so just make sure to recognize the power series expansions of sine and cosine!

5.2 Solutions about Singular Points

Goal: solve DE's of the form $a(x)y''+b(x)y'+c(x)y=0$ when 0 is a regular singular point (roughly speaking, when $x\frac{b(x)}{a(x)}$ and $x^2\frac{c(x)}{a(x)}$ are both defined at 0).

Example: Solve $3xy''+y'-y=0$

method of Frobenius: try to find solutions of the form $y=\sum_{n=0}^{\infty}c_nx^{n+r}$, where r is a constant TBD

$$y=\sum_{n=0}^{\infty}c_nx^{n+r} \qquad y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} \qquad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$$

plug into $3xy''+y'-y=0$:

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=\sum_{n=0}^{\infty}(n+r)(3n+3r-3)c_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=\sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r}=x^r\left[\sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_nx^{n-1}-\sum_{n=0}^{\infty}c_nx^n\right]$$

$$=x^r\left[r(3r-2)c_0x^{-1}+\underbrace{\sum_{n=1}^{\infty}(n+r)(3n+3r-2)c_nx^{n-1}}_{k=n-1,\ n=k+1}-\underbrace{\sum_{n=0}^{\infty}c_nx^n}_{k=n}\right]$$

$$=x^r\left[r(3r-2)c_0x^{-1}+\sum_{k=0}^{\infty}\left((k+1+r)(3k+3r+1)c_{k+1}-c_k\right)x^k\right]=0$$

so: $r(3r-2)c_0=0$ and $c_{k+1}=\frac{c_k}{(k+1+r)(3k+3r+1)}$

observe that if $c_0=0$, then every term is 0, so assume $c_0\neq 0$

then $r(3r-2)=0$. this is called the indicial equation of the problem, obtained by setting the coefficient of the lowest power of x to 0 (assume the constant c_0 is nonzero)

the roots $r=0,\frac{2}{3}$ are called the indicial roots

when the difference of the 2 roots is not an integer, we can obtain 2 linearly independent solutions to the ODE, one corresponding to each root

$r=0$: $c_{k+1}=\frac{c_k}{(k+1)(3k+1)}$

$k=0$ $\quad c_1=c_0$

$k=1$ $\quad c_2=\frac{c_1}{(2)(4)}=\frac{c_0}{(2)(4)}$

$k=2$ $\quad c_3=\frac{c_2}{(3)(7)}=\frac{c_0}{(2)(3)(4)(7)}$

$k=3$ $\quad c_4=\frac{c_3}{(4)(10)}=\frac{c_0}{4!(4)(7)(10)}$

$k=4$ $\quad c_5=\frac{c_4}{(5)(13)}=\frac{c_0}{5!(4)(7)(10)(13)}$

$$y=x^0\left[c_0+c_0x+\frac{c_0}{2!(4)}x^2+\frac{c_0}{3!(4)(7)}x^3+\frac{c_0}{4!(4)(7)(10)}x^4+\frac{c_0}{5!(4)(7)(10)(13)}x^5\cdots\right]$$

$$=c_0\underbrace{\left[1+x+\frac{x^2}{2!(4)}+\frac{x^3}{3!(4)(7)}+\frac{x^4}{4!(4)(7)(10)}\cdots\right]}_{y_1}$$

$r=2/3$: $c_{k+1}=\frac{c_k}{(k+5/3)(3k+3)}=\frac{c_k}{(3k+5)(k+1)}$

$c_1=\frac{c_0}{5}$

$c_2=\frac{c_1}{(8)(2)}=\frac{c_0}{(2)(5)(8)}$

$c_3=\frac{c_2}{(11)(3)}=\frac{c_0}{3!(5)(8)(11)}$

$c_4=\frac{c_3}{(14)(4)}=\frac{c_0}{4!(5)(8)(11)(14)}$

$c_5=\frac{c_4}{(17)(5)}=\frac{c_0}{5!(5)(8)(11)(14)(17)}$

$$y=x^{2/3}\left[c_0+\frac{c_0}{5}x+\frac{c_0}{2!(5)(8)}x^2+\frac{c_0}{3!(5)(8)(11)}x^3+\frac{c_0}{4!(5)(8)(11)(14)}x^4\dots\right]$$

$$=c_0\underbrace{x^{2/3}\left[1+\frac{x}{5}+\frac{x^2}{2!(5)(8)}+\frac{x^3}{3!(5)(8)(11)}+\frac{x^4}{4!(5)(8)(11)(14)}\cdots\right]}_{y_2}$$

the general solution is $y = \alpha y_1 + \beta y_2$, where α, β are arbitrary constants, as always.

Summary of finding series solutions if 0 is a regular singular point:

① let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$

② plug y, y', and y'' into the DE, combine into one series.

③ get the indicial equation from the coefficient of the lowest power of x (this is often x^{-1}), get the recurrence relation from the other coefficients.

④ solve for the indicial roots. If they differ by a non-integer, plug each into the recurrence relation to get a series solution; every c_n will be in terms of c_0, and factoring out c_0 leaves us with a fundamental solution.

⑤ Pat yourself on the back.

What happens if the two indicial roots differ by an integer (or are the same)? Sometimes we can still get 2 solutions using the method of Frobenius. but sometimes the two roots (even if different) give us the same series (you should verify this for $xy'' + y = 0$). Even so, a second (linearly independent) solution can be found, but with somewhat sinister methods (and natural logs).

Quick note on 2 special equations

Bessel equation: $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$ (ν is some constant)
0 is a regular singular point

Legendre equation: $(1-x^2) y'' - 2xy' + n(n+1) y = 0$ (n is some constant)
0 is an ordinary point

these functions have lovely properties that you will study in Math 241

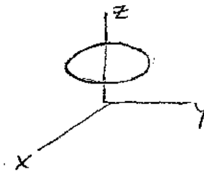
remember for Math 241: these DE's can be solved using the series methods, and the solutions are called Bessel and Legendre functions.

9.1 Vector Functions

Functions can be defined either in terms of independent variables, e.g. $z = f(x, y)$, or parametrically, e.g. $r(t) = \langle f(t), g(t), h(t) \rangle$. this is a vector function.
or $r(t) = f(t)i + g(t)j + h(t)k$ (book's notation)

"parametric" means all components are functions of the same variable, sometimes thought of as time. $r(t)$ traces out a path as t varies, e.g. at $t=0$, we have the point $(f(0), g(0), h(0))$.

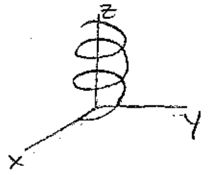
Ex: $r(t) = \langle 2\cos t, 2\sin t, 3 \rangle$



$(2\cos t)^2 + (2\sin t)^2 = 4$

so this describes a circle of radius 2 centered on the z-axis at height $z=3$

$r(t) = \langle 2\cos t, 2\sin t, t \rangle$

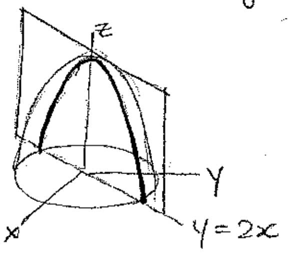


the x-y components still describe a circle, but z component is no longer a constant. this gives a helix.

Ex: Express the intersection of the plane $y = 2x$ and the paraboloid $z = 9 - x^2 - y^2$ as a vector function.

let $x = t$. then $y = 2t$ and $z = 9 - t^2 - 4t^2 = 9 - 5t^2$

so $r(t) = \langle t, 2t, 9 - 5t^2 \rangle$



Limit $\lim\limits_{t\to a} r(t) = \langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \rangle$, provided these limits exist

properties:

(1). $\lim\limits_{t\to a} c\,r(t) = c\left(\lim\limits_{t\to a} r(t)\right)$, where c is a constant

(2). $\lim\limits_{t\to a}(r_1(t) + r_2(t)) = \lim\limits_{t\to a} r_1(t) + \lim\limits_{t\to a} r_2(t)$

} these follow directly from properties of scalar function limits

(3). $\lim\limits_{t\to a} \left(r_1(t) \overset{\text{dot product}}{\cdot} r_2(t)\right) = \left(\lim\limits_{t\to a} r_1(t)\right) \cdot \left(\lim\limits_{t\to a} r_2(t)\right)$ } analogue of $\lim\limits_{x\to a} f(x)g(x) = (\lim\limits_{x\to a} f(x))(\lim\limits_{x\to a} g(x))$

Continuity recall that $f(x)$ is continuous at a if
(1) $f(x)$ is defined at a, (2). $\lim\limits_{x\to a} f(x)$ exists, and (3). $\lim\limits_{x\to a} f(x) = f(a)$

vector function $r(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at a if the components $f(t), g(t), h(t)$ are each continuous at a.

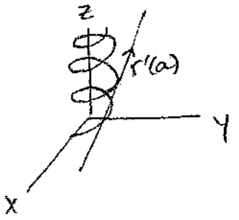
derivative recall that $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, wherever the limit exists

derivative of a vector function is obtained by differentiating the components:

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Ex: $r(t) = \langle 2\cos t, 2\sin t, t \rangle$

$r'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$



geometric interpretation: $r'(a)$ is the tangent vector to $r(t)$ at a. the tangent line to $r(t)$ at a is defined as the line parallel to $r'(a)$ that contains the point $r(a)$.

ex: tangent line to $r(t) = \langle 2\cos t, 2\sin t, t \rangle$ at $t = \pi$
is $L(s) = r(\pi) + r'(\pi)s = \langle -2, 0, \pi \rangle + \langle 0, -2, 1 \rangle s$
parametric equations: $x = -2 \quad y = -2s \quad z = \pi + s$

Chain rule for vector functions: if $r(s)$ is a vector function and $s = u(t)$ is itself a scalar function, then $\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = r'(s)\,u'(t)$

Ex: $r(s) = \langle s, s^2, s^3 \rangle$ $\quad s = t^2$

$$\frac{dr}{dt} = \langle 1, 2s, 3s^2 \rangle 2t = \langle 2t, 2(t^2)2t, 3(t^2)^2 2t \rangle = \langle 2t, 4t^3, 6t^5 \rangle$$

review: dot product $\langle a_1, b_1, c_1 \rangle \cdot \langle a_2, b_2, c_2 \rangle = a_1a_2 + b_1b_2 + c_1c_2$

cross product $\langle a_1, b_1, c_1 \rangle \times \langle a_2, b_2, c_2 \rangle = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

dot product of ⊥ vectors is 0
cross product of ∥ vectors is $\vec{0}$

properties of differentiation

(1). $(r_1(t) + r_2(t))' = r_1'(t) + r_2'(t)$

(2). $(u(t)\,r(t))' = u(t)\,r'(t) + u'(t)\,r(t)$, where $u(t)$ is a scalar function

(3). $(r_1(t) \cdot r_2(t))' = r_1(t)\,r_2'(t) + r_1'(t)\,r_2(t)$

(4). $(r_1(t) \times r_2(t))' = r_1(t) \times r_2'(t) + r_1'(t) \times r_2(t)$ (order matters here!)

Integration

integral of a vector function is obtained by integrating the components:

$$\int r(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

Ex: $r(t) = \langle t^2, e^t, \cos t \rangle$

$$\int r(t)\,dt = \left\langle \frac{t^3}{3} + c_1, e^t + c_2, \sin t + c_3 \right\rangle$$