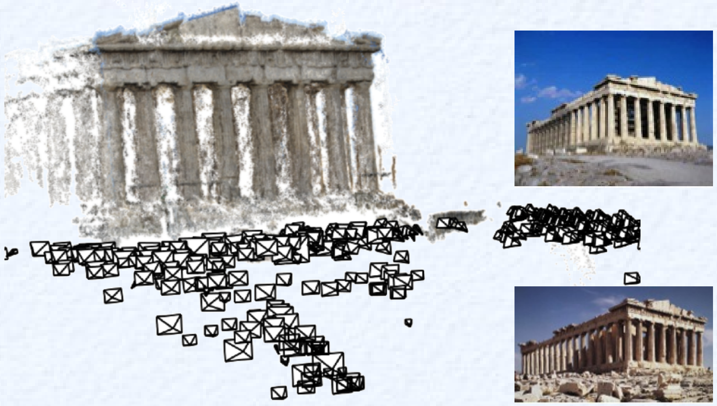
ICML 2014, Beijing

Near-Optimal Joint Object Matching via Convex Relaxation

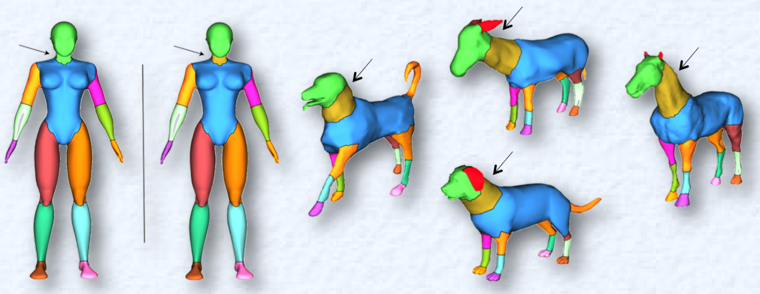
Yuxin Chen, Leonidas Guibas, and Qixing Huang

Stanford University

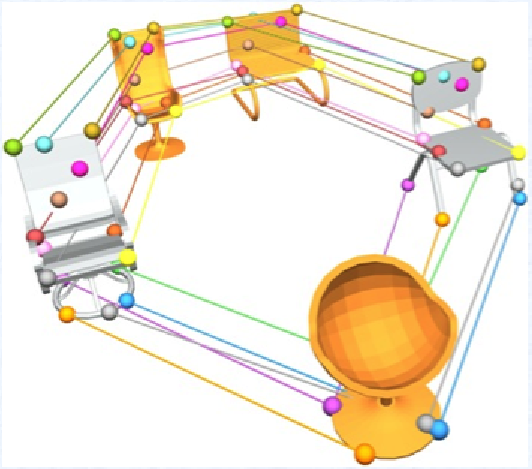
Joint Matching



Structure from Motion



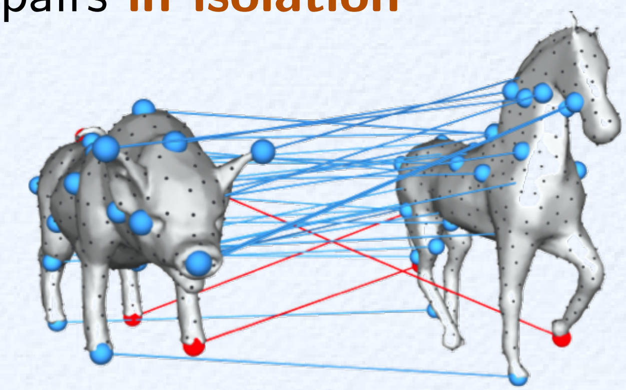
Joint Segmentation



- **Given**: n instances (graphs), each containing a few elements (vertices)
- **Goal**: *jointly* match all similar elements across objects

Popular Approach: 2-Stage Method

- **Stage 1: Pairwise Matching**
  - Compute pairwise matching across a few pairs **in isolation**
  - *off-the-shelf pairwise methods*



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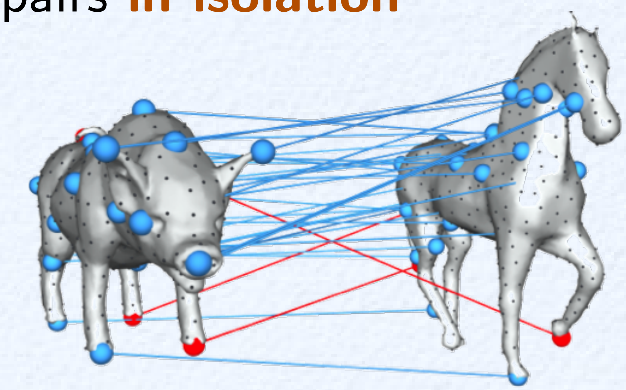
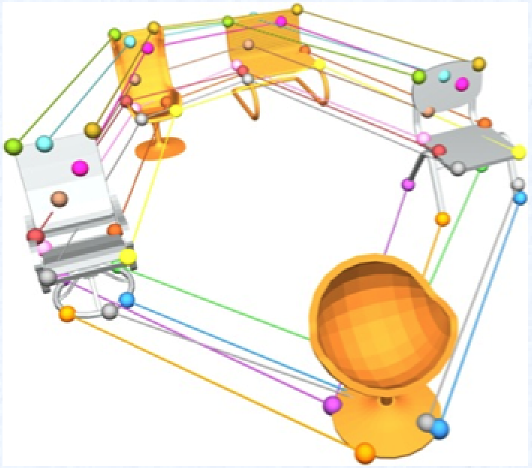




- **Stage 2: Global Refinement**
  - Jointly refine all provided maps

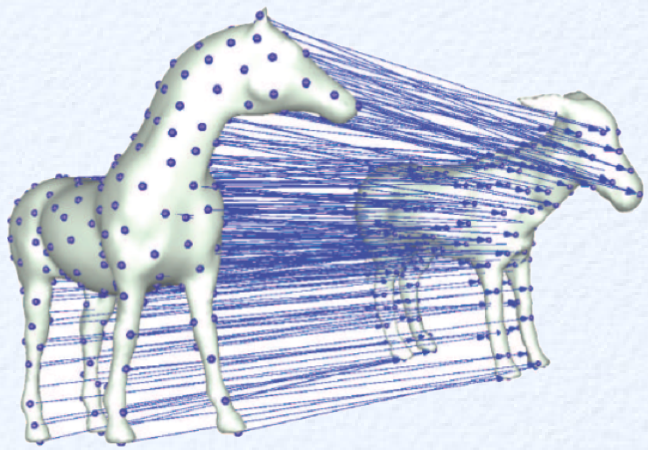
Representation

- **Objects**
  - a set of points
  - drawn from the same universe

Representation

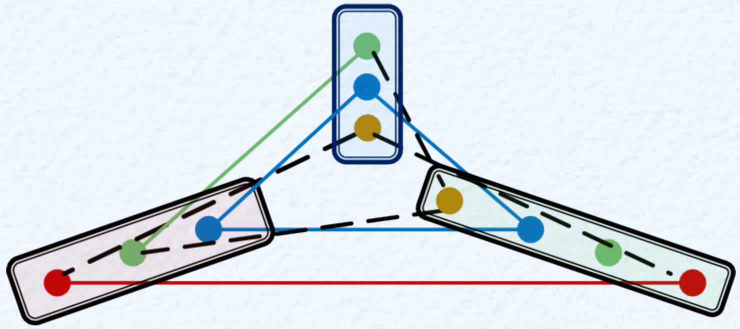
- **Objects**
  - a set of points
  - drawn from the same universe

- **Map**
  - point-to-point correspondence



Problem Formulation

- **Input:** a few pairwise matches computed in isolation



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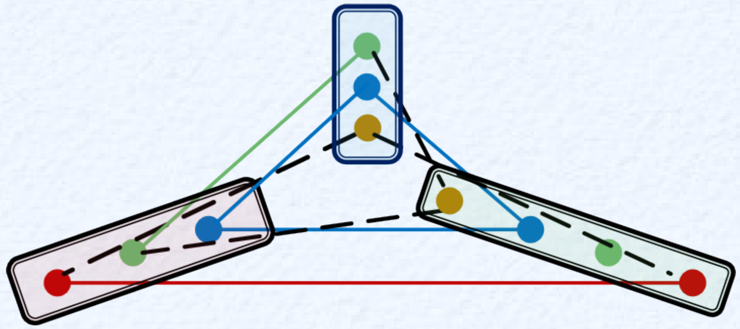
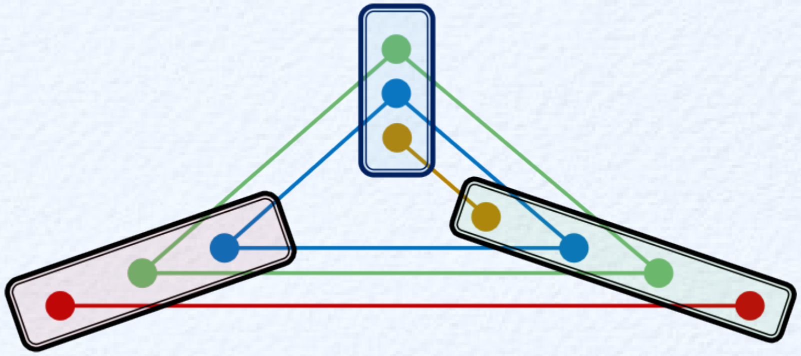


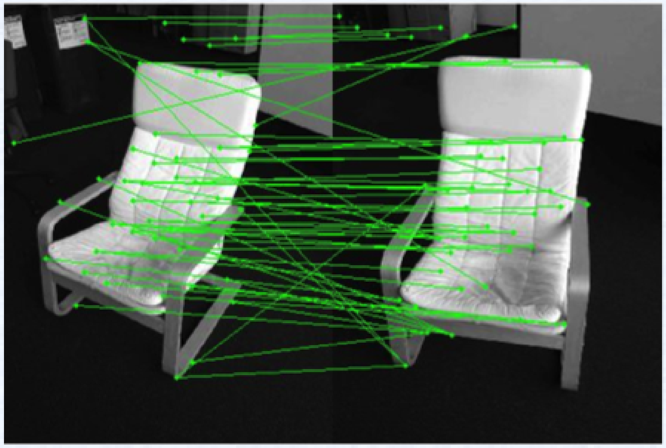
- **Output:** a collection of maps that are
  - close to the inputs
  - globally consistent



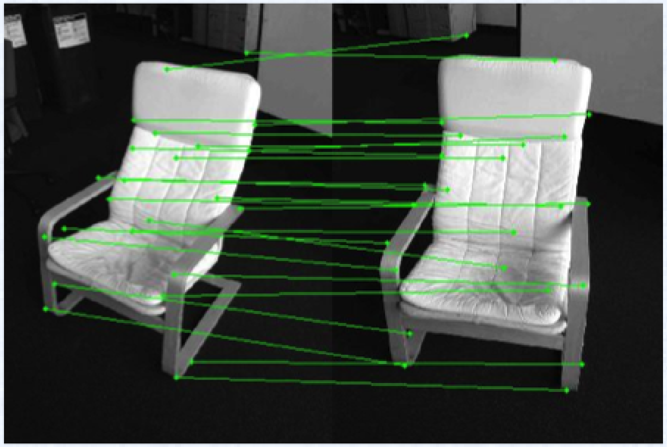
- **NP-Hard! [Huber 02]**

Challenge 1: Dense Input Errors

- **Input Errors**
  - A significant fraction of inputs are corrupted



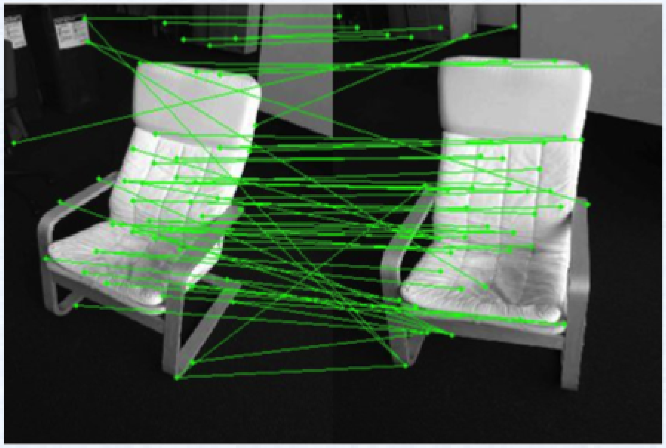
Input Maps



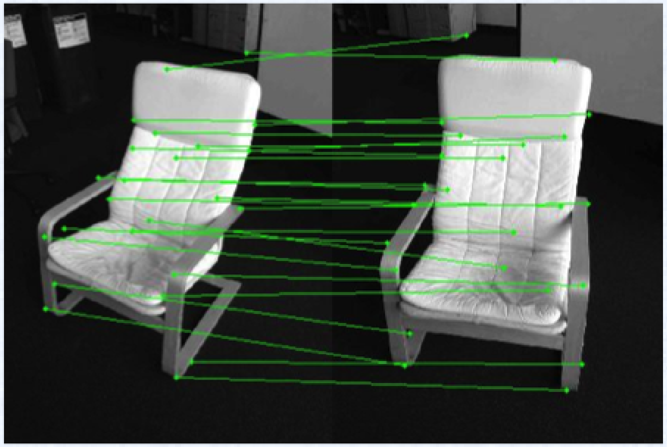
Ground Truth

Challenge 1: Dense Input Errors

- **Input Errors**
  - A significant fraction of inputs are corrupted
  - Prior art:
    — tolerate **50%** input errors



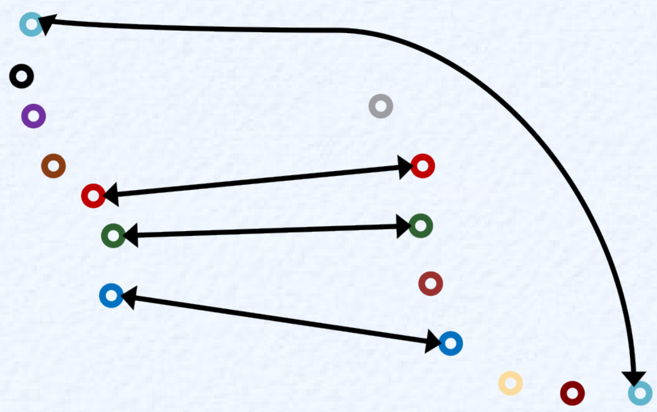
Input Maps



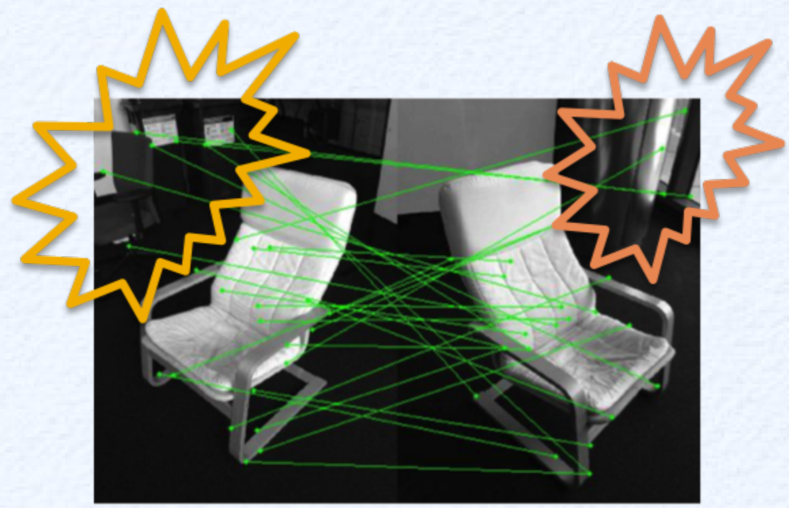
Ground Truth

Challenge 2: Partial Similarity

- **Partial Similarity**
  - Objects might only be partially similar to each other.



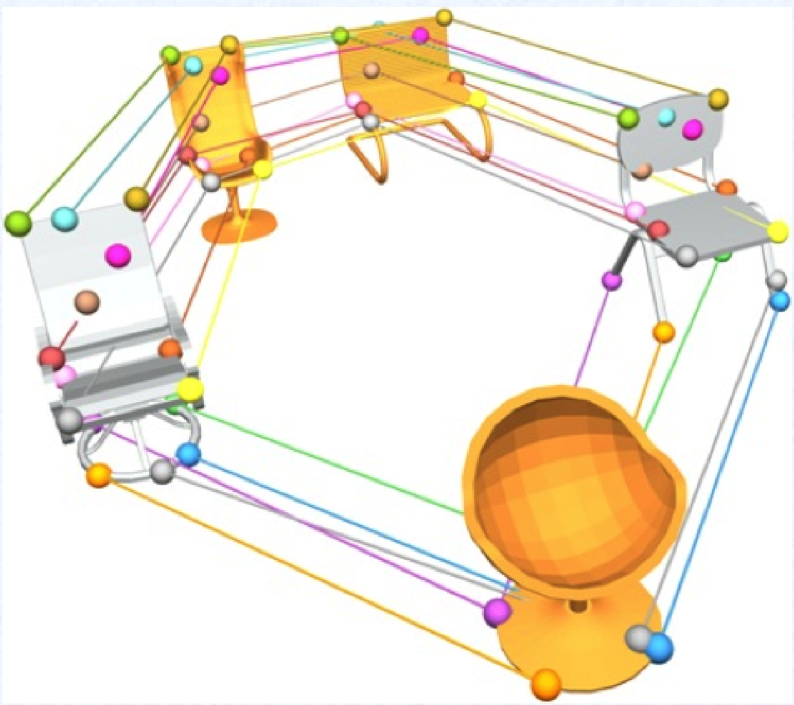
Subgraph Matching

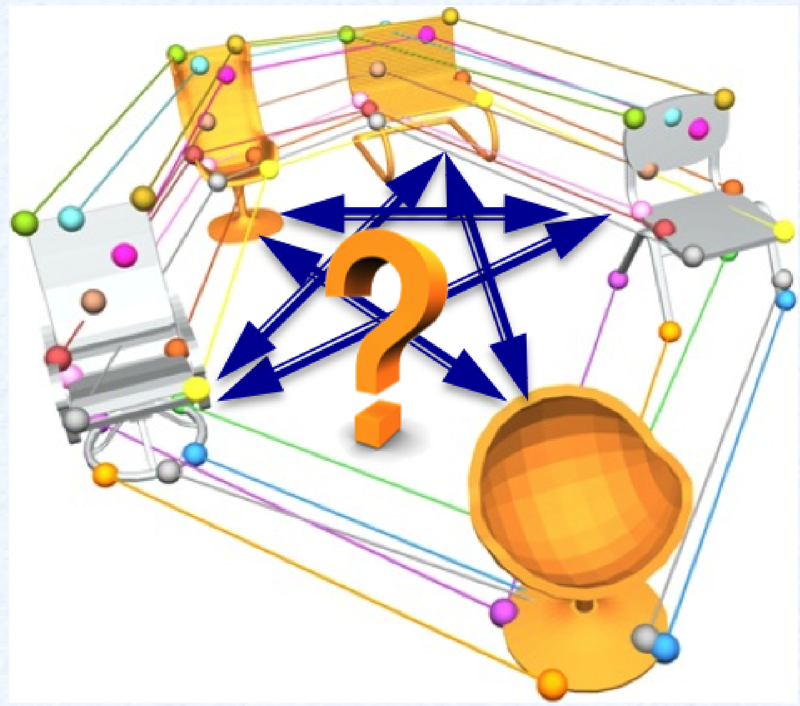


Input Maps

Challenge 3: Missing Data

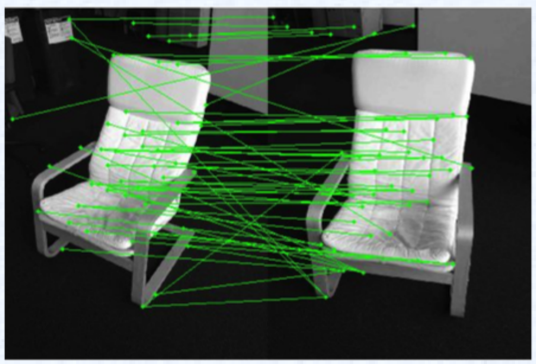
- **Incomplete Input Matches**



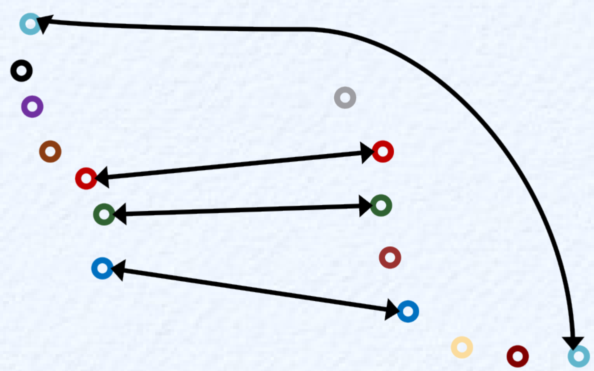


Our Goal

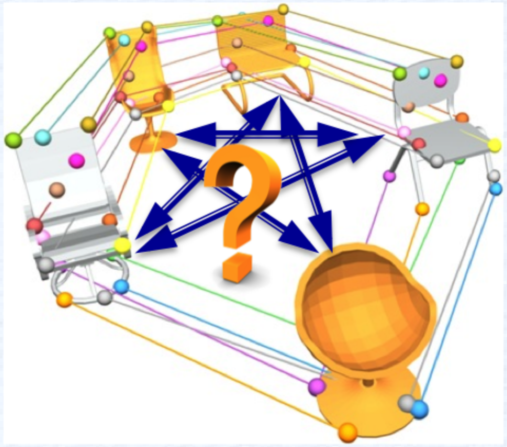
- **Develop an effective joint recovery method**
  - strong theoretical guarantee (*address the 3 challenges*)
  - parameter free
  - computationally feasible



tolerate dense errors



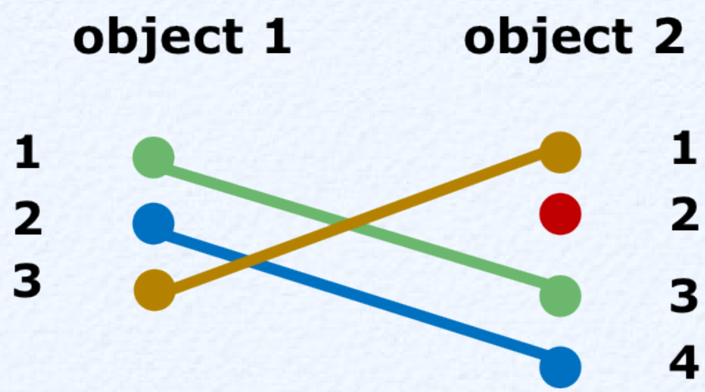
handle partial similarity



fill in missing matches

Matrix Representation

- All objects / sets are sub-sampled from the same universe (of size m).



$$X_{12} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix Representation

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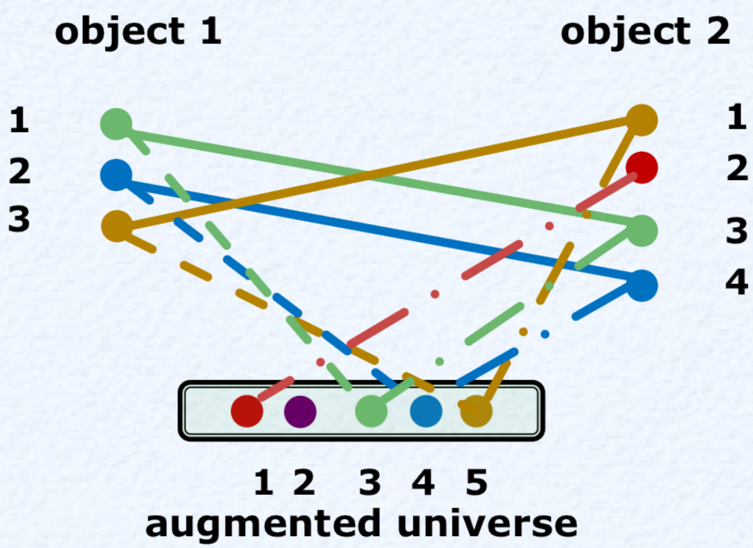
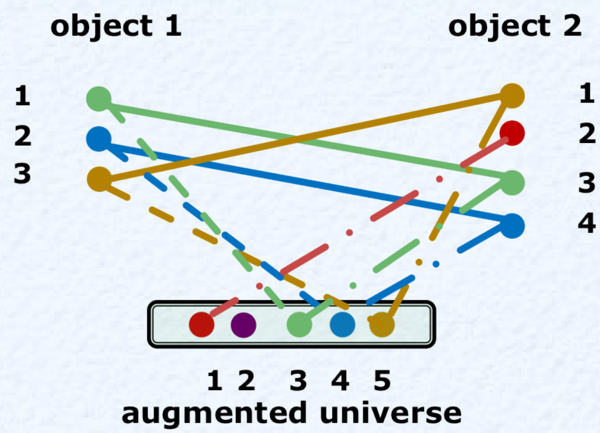


$$\boldsymbol{X}_{12} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Map matrix $\boldsymbol{Y}_i$ between object i and the universe

$$\boldsymbol{Y}_1 := \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{m \text{ columns}}, \quad \boldsymbol{Y}_2 := \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{m \text{ columns}} \quad \Rightarrow \quad \boldsymbol{X}_{12} = \boldsymbol{Y}_1 \boldsymbol{Y}_2^\top$$

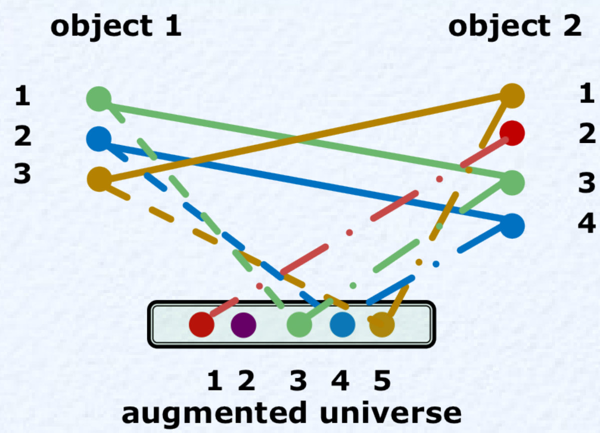
P.S.D. and Low-Rank Structure



- **Consider all n instances:**

$$\boldsymbol{X} := \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X}_{12} & \cdots & \boldsymbol{X}_{1n} \\ \boldsymbol{X}_{21} & \boldsymbol{I} & \cdots & \boldsymbol{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{n1} & \boldsymbol{X}_{n2} & \cdots & \boldsymbol{I} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \vdots \\ \boldsymbol{Y}_n \end{bmatrix}}_{m \text{ columns}} \begin{bmatrix} \boldsymbol{Y}_1^\top & \boldsymbol{Y}_2^\top & \cdots & \boldsymbol{Y}_n^\top \end{bmatrix} \succeq \boldsymbol{0};$$

P.S.D. and Low-Rank Structure



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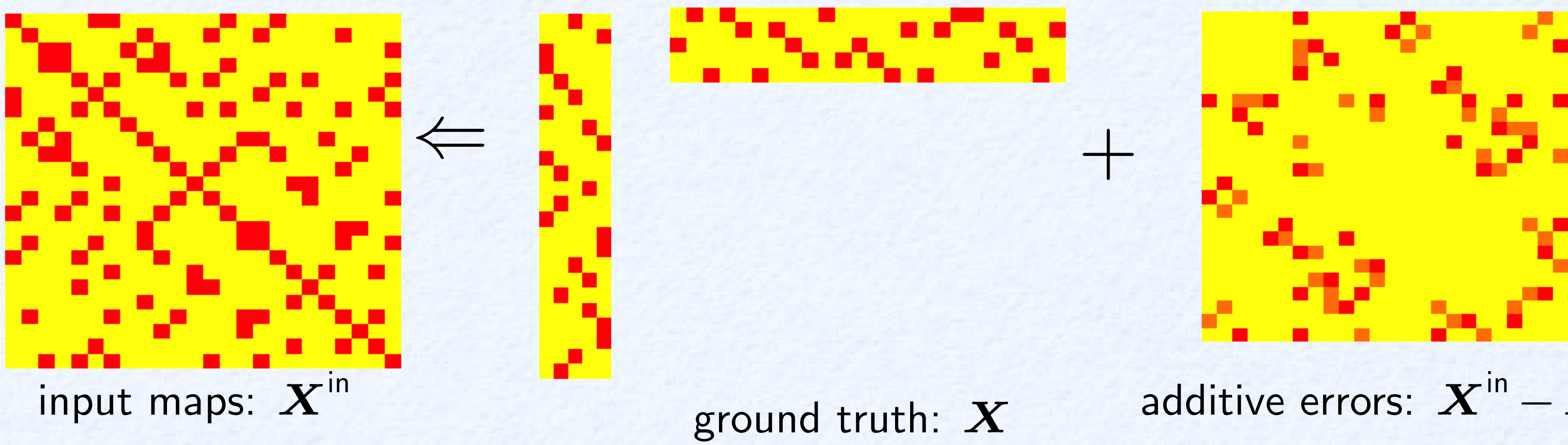
- **low rank:** $\text{rank}(\boldsymbol{X}) \leq m$.
    - m: universe size

Low Rank + Sparse Matrix Separation?



- Robust PCA ? (Candes et al, Chandrasekahran et al)

Low Rank + Sparse Matrix Separation?

input maps: $\boldsymbol{X}^{\text{in}}$ ground truth: $\boldsymbol{X}$ additive errors: $\boldsymbol{X}^{\text{in}} - \boldsymbol{X}$

- Robust PCA ? (Candes et al, Chandrasekahran et al)

- Input errors are highly biased
  - *Cannot apply Chen et al (need random sign pattern)*

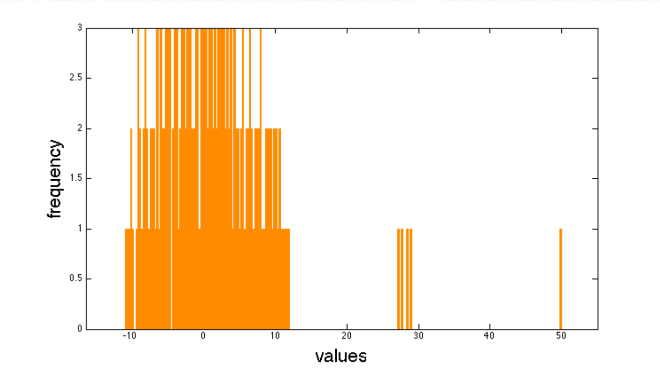
Debias the Error Components

Augmented Form

$$\begin{bmatrix} m & \mathbf{1}^\top \\ \mathbf{1} & \boldsymbol{X} \end{bmatrix} := \begin{bmatrix} \mathbf{1}^\top \\ \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \vdots \\ \boldsymbol{Y}_n \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{Y}_1^\top & \cdots & \boldsymbol{Y}_n^\top \end{bmatrix} \succeq 0$$

- Equivalently,

$$\boldsymbol{X} - \underbrace{\frac{1}{m}\mathbf{1}\mathbf{1}^\top}_{\text{debiasing}} \succeq 0.$$



- m **(universe size) can be estimated via spectral method!**

MatchLift: tractable convex programming

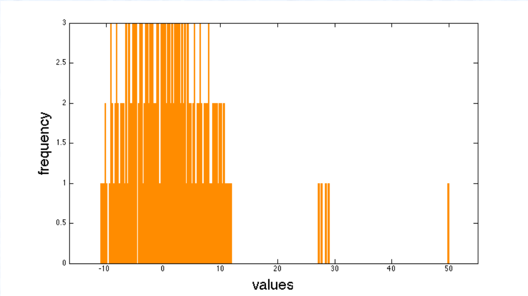
MatchLift

$$
\begin{aligned}
\text{minimize}_{\boldsymbol{X}} \quad & -\underbrace{\left\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \right\rangle}_{\text{agreement to inputs}} + \lambda \underbrace{\left\langle \boldsymbol{X}, \boldsymbol{1}\boldsymbol{1}^{\top} \right\rangle}_{\text{promote sparsity}} \\
\text{subject to} \quad & \boldsymbol{X} \geq \boldsymbol{0}, \\
& \begin{bmatrix} m & \boldsymbol{1}^{\top} \\ \boldsymbol{1} & \boldsymbol{X} \end{bmatrix} \succeq \boldsymbol{0}, \\
& \boldsymbol{X}_{ii} = \boldsymbol{I}.
\end{aligned}
$$

- Efficient Semidefinite Program

Two-Step Procedure: MatchLift

1. **Pre-Estimate m:**



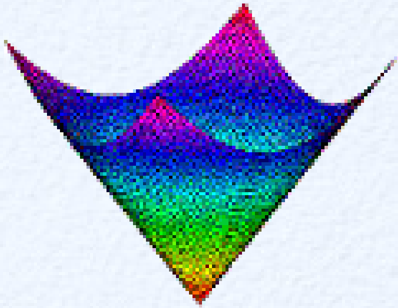
Spectral Method

- $m \longleftarrow$ # dominant eigenvalues of $\boldsymbol{X}^{\text{in}}$

2. **Joint Matching via Convex Relaxation:**

Convex Programming

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{X}} & -\left\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \right\rangle + \lambda \left\langle \boldsymbol{X}, \boldsymbol{1}\boldsymbol{1}^\top \right\rangle \\ \text{subject to} & \boldsymbol{X} \geq \boldsymbol{0}, \\ & \begin{bmatrix} \boldsymbol{m} & \boldsymbol{1}^\top \\ \boldsymbol{1} & \boldsymbol{X} \end{bmatrix} \succeq \boldsymbol{0}, \\ & \boldsymbol{X}_{ii} = \boldsymbol{I}. \end{array}$$



Exact Recovery under Randomized Model

$$\text{minimize}_{\boldsymbol{X}} \quad -\langle \boldsymbol{X}, \boldsymbol{X}^{\text{in}} \rangle + \lambda \langle \boldsymbol{X}, \mathbf{1}\mathbf{1}^{\top} \rangle, \qquad \text{s.t. feasible}$$

Theorem

MatchLift with $\boldsymbol{\lambda} = \sqrt{p_{\text{obs}}}$ is exact with high probability if

$$p_{\text{true}} \gtrsim \frac{\log^2(mn)}{p_{\text{set}}^2 \sqrt{p_{\text{obs}}} n}$$

- **Parameter-free**
  - *MatchLift is insensitive to* λ

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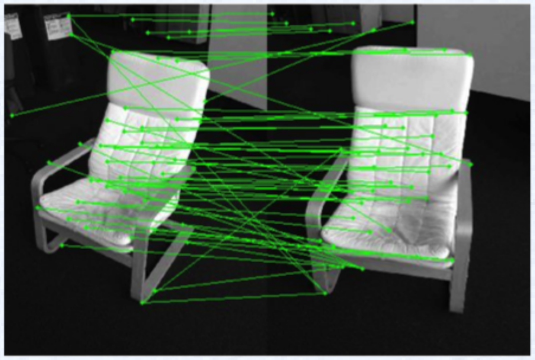
MatchLift with $\lambda = \sqrt{p_{\text{obs}}}$ is exact with high probability if

$$\underbrace{p_{\text{true}}}_{\text{non-corruption rate}} \gtrsim \frac{\log^2(mn)}{p_{\text{set}}^2 \sqrt{p_{\text{obs}}} \boldsymbol{n}}$$

- **Parameter-free**
  - *MatchLift is insensitive to* λ



- **Dense Error Correction**

$$\text{error correction ability} \approx 1 - 1/\sqrt{n}$$

when p_{set} and p_{obs} are constants.

Exact Recovery under Randomized Model

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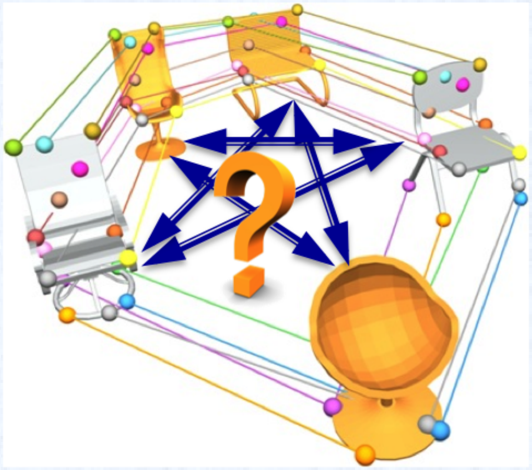
$$p_{\text{true}} \gtrsim \frac{\log^2(mn)}{p_{\text{set}}^2 \sqrt{\underbrace{p_{\text{obs}}}_{\text{observation ratio}} n}}$$

- **Incomplete Input Matches**
  - *Error correction ability decays at rate* $1/\sqrt{p_{obs}}$



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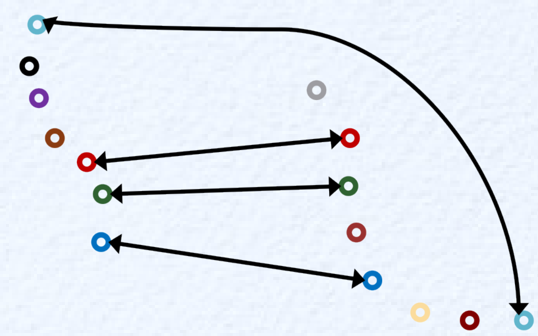
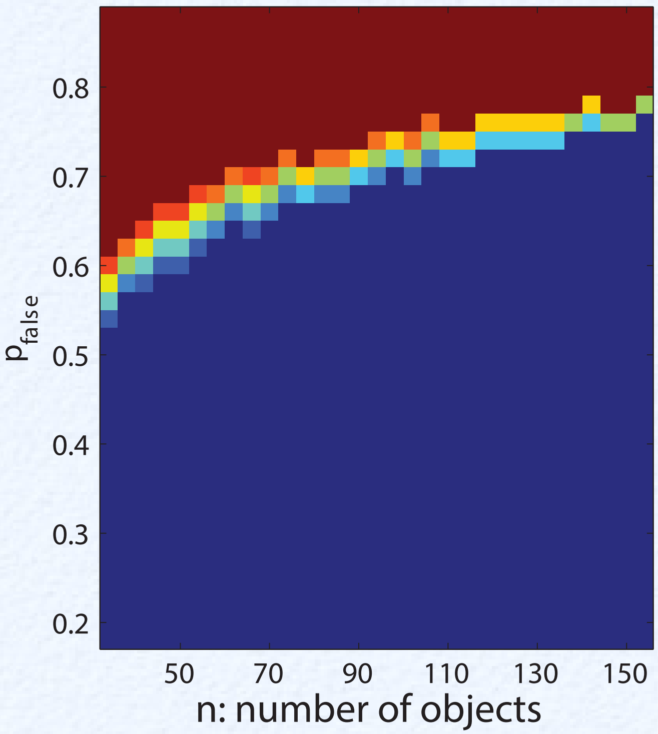
$$p_{\text{true}} \gtrsim \frac{\log^2(mn)}{\underbrace{\boldsymbol{p}_{\text{set}}^2}_{\text{fraction of each object being disclosed}} \sqrt{p_{\text{obs}} n}}$$

- **Incomplete Input Matches**
  - *Error correction ability decays at rate* $1/\sqrt{p_{obs}}$
- **Partial Similarity**
  - *Error correction ability decays at rate* $1/p_{\text{set}}^2$

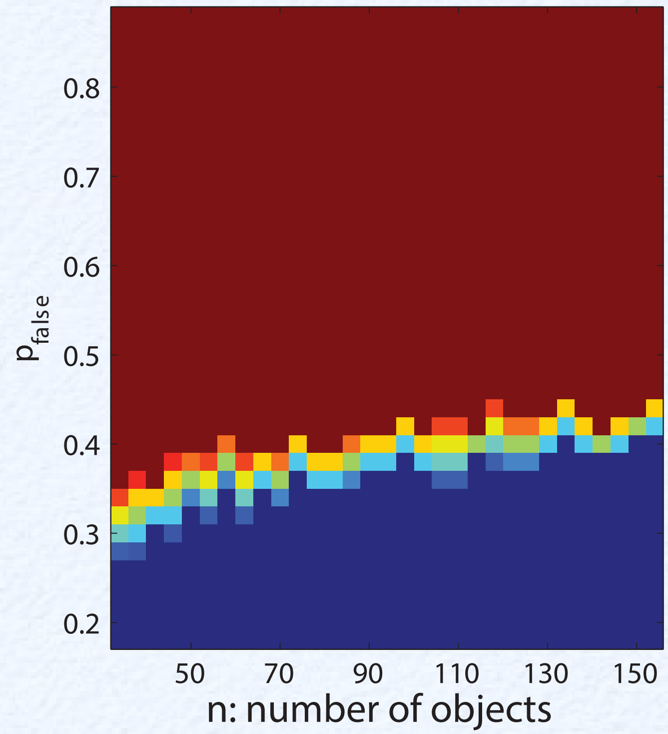


Phase Transition

- **Synthetic Data** **(input error rate v.s. # objects)**



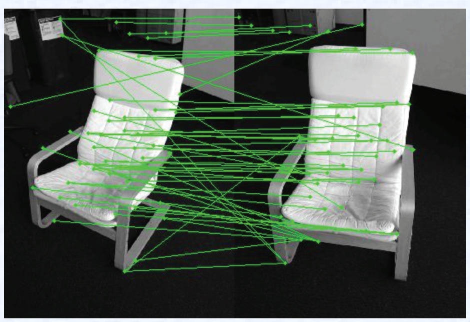
MatchLift



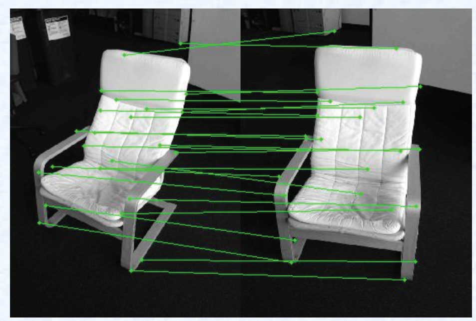
Robust PCA

Benchmark: Chairs

benchmark



initial maps

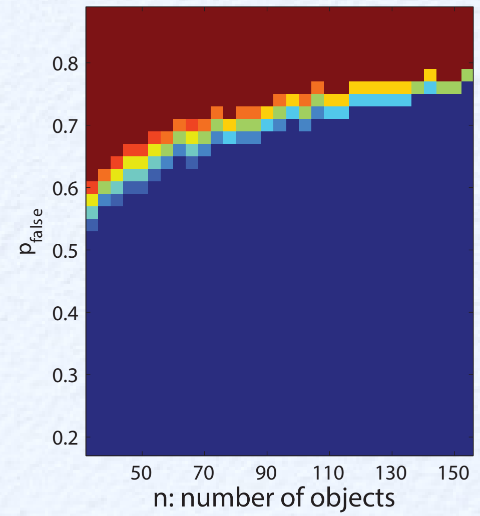


optimized maps

Input	MatchLift	RPCA	Leordeanu et al. 12
64.1%	100%	90.1%	94.8%

Concluding Remarks

- **MatchLift**
  - Parameter Free
  - Dense error correction (*near-optimal when* $m = \Theta(1)$)
  - Allow partial similarity
  - Missing inputs



- **Paper: Near-Optimal Joint Object Matching via Convex Relaxation**
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Thank You! Questions?

Optimality of MatchLift

Theorem (ChenGuibasHuang'14)

MatchLift with $\lambda = \sqrt{p_{\mathsf{obs}}}$ is exact with high probability if

$$p_{\mathsf{true}} \gtrsim \frac{\log^2(mn)}{p_{\mathsf{set}}^2 \sqrt{p_{\mathsf{obs}}} n}$$

- Is MatchLift Optimal?

Optimality of MatchLift

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- Is MatchLift Optimal?
- Fundamental Limits under Random Measurement Graphs
  - *Erdos-Renyi Graph* $\mathcal{G}(n, p_{\text{obs}})$

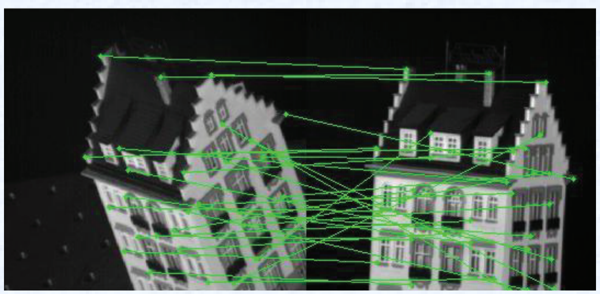
Theorem (ChenGoldsmith'14)

If the universe size m is a constant, then

$$\text{No method works if} \quad p_{\text{true}} \lesssim \frac{1}{\sqrt{p_{\text{obs}} n}} \left(\approx \frac{1}{\sqrt{\text{avg-degree}}}\right)$$

Benchmark: CMU Hotel

benchmark



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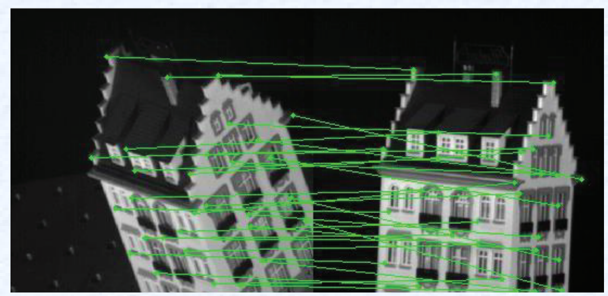

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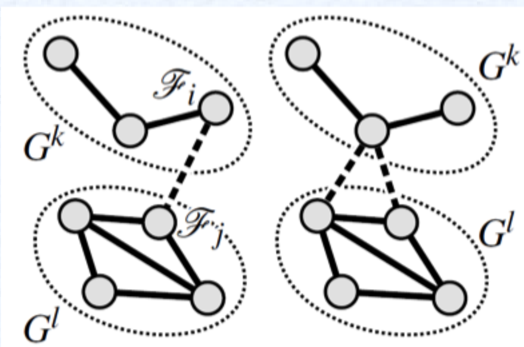
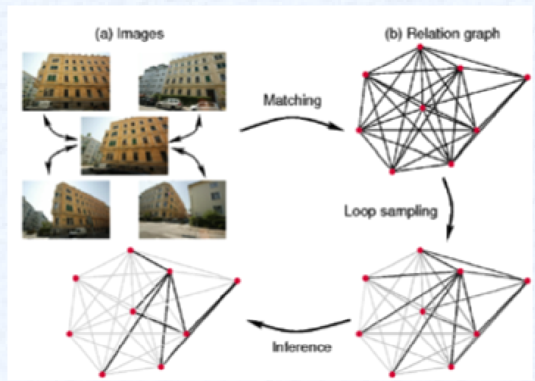
Prior Art

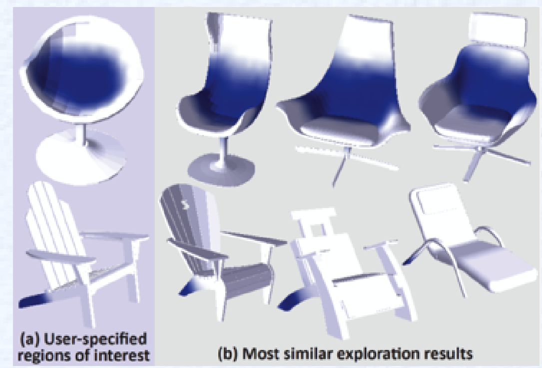
- Empirical Advances
  - little fundamental understanding
  - rely on hyper-parameter tuning



spanning tree opt [Huber'02]



detect inconsistent cycle [Zach'10, Ngu'11]



spectral [Kim'12, Huang'12]

- Fundamental Understanding
  - SDP (HuangGuibas'13), spectral methods (Pachauri et al'13)
  - generic graph clustering methods (Jalali et al, Chen et al, ...)