Comment on "How to observe and quantify quantum-discord states via correlations"

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The protocol proposed in the commented-upon article can certify zero discord; however, it fails to *quantify* non-zero discord for some states. Further, the protocol is less efficient than calculating discord via performing full quantum state tomography.

Recently Ref. [1] proposed an interferometric protocol for observing and quantifying quantum discord. The basic idea of the protocol is that the shape of the zero visibility lines can distinguish discorded states from the non-discorded ones, and (possibly) quantify the amount of discord present in the state. Another advantage of the protocol is that it maps the discord, a non-linear function of the system density matrix, into direct results of measurements (which are linear in the density matrix).

Two claims are made by Ref. [1] regarding the proposed protocol. (i) That the protocol enables one "to detect and characterize quantum discord of any unknown mixed state of a generic nonentangled bipartite system", namely to construct a "discord quantifier [that] is qualitatively consistent and quantitatively very close to the original measure". (ii) That the protocol is more efficient than calculating the discord via full-state tomography of the system density matrix. This statement follows, e.g., from the following passage in the introduction: "Despite increasing evidence for the relevance of quantum discord, quantifying it in a given quantum state is a challenge. Even full quantum state tomography would not suffice since determining discord requires minimizing a conditional mutual entropy over a full set of projective measurements. Moreover, even computing discord is very difficult (it has been proven to be NP complete [18]). ... In this paper, we propose an alternative discord quantifier which would overcome these fundamental difficulties and render quantum discord to be experiment friendly for many-body electronic systems."

In this comment, we show that the protocol fails to be a universal discord quantifier. The comment is structured into three sections. We first show, that the proposed protocol does indeed allow for idenitfying non-discorded states. We then demonstrate that statement (i) is not valid. Namely, we consider the example of the Werner state in which the shape of the zero-visibility lines does not reflect the *size* of discord. Finally, we show that statement (ii) is not valid. Namely, we demonstrate that the protocol is less efficient than calculating the discord via full state tomography.

Section I. Identifying non-discorded states.—The protocol of Ref. [1] is guaranteed to enable identification of non-discorded states. This is stated in Ref. [1] and proven in the PhD thesis of M. Hunt [2, Sections 5.3, 5.4]. We find it useful, however, to briefly explain this fact here as it elucidates the essence of the protocol discussed.

Consider a system comprising two parts, A and B, in a state characterized by density matrix ρ^{AB} . Similarly to Ref. [1], we assume that the Hilbert spaces of each A and B are two-dimensional. It is known [3] that the quantum state is not A-discorded if and only if the density matrix can be presented in the following form:

$$\rho^{AB} = \sum_{i} p_i |\psi_i^A\rangle\langle\psi_i^A| \otimes \rho_i^B, \qquad (1)$$

where states $|\psi_i^A\rangle$ form an orthonormal basis in the Hilbert space of subsystem A, ρ_i^B are arbitrary density matrices of subsystem B, and $p_i \geq 0$ are probabilities such that $\sum_i p_i = 1$.

Omitting unnecessary technicalities, the protocol of Ref. [1] involves the following elements: evolving $\rho^{AB} \rightarrow$ $S\rho^{AB}S^{\dagger}$ with unitary $S=S^{A}\otimes S^{B}$, applying extra unitary $S_d \propto \exp(i\phi_d\sigma_z^A/2)$, and measuring some observable involving subsystems A and B and investigating its dependence on ϕ_d , S^A , and S^B . Assuming that ρ^{AB} is of the form (1), one sees that there should exist S_0^A such that $S_0^A | \psi_1^A \rangle = |\uparrow\rangle$ and $S_0^A | \psi_2^A \rangle = |\downarrow\rangle$. Since $S_d |\uparrow\rangle \langle\uparrow| S_d^{\dagger} = |\uparrow\rangle \langle\uparrow| \text{ and } S_d |\downarrow\rangle \langle\downarrow| S_d^{\dagger} = |\downarrow\rangle \langle\downarrow|, \text{ it is evident that } S_d S_0 \rho^{AB} S_0^{\dagger} S_d^{\dagger}, \text{ where } S_0 = S_0^A \otimes S^B, \text{ does not}$ depend on ϕ_d (i.e., any observable, e.g., K_{ϕ_d} from Eq.(5) of Ref. [1], will have zero visibility). Note that this does not depend on S^B used. Therefore, in line with Ref. [1], if the state is non-discorded, the zero visibility lines in the protocol do not depend on S^B . The correctness of the converse statement is shown in the PhD thesis of M. Hunt [2, Sections 5.3, 5.4].

Section II. Failing to quantify discord by zero visibility lines.—The above consideration shows that for non-discorded states the zero visibility lines will not depend on manipulations with subsystem B. In Section IV, Ref. [1] suggests to quantify discord (approximately) by looking at the deviation of the zero visibility lines from horizontal (the x axis is a parameter controlling S^B). This suggestion is supported by a study of a small number of example states. Here we provide an example of a family of states that have different discord values yet all exhibit the same zero visibility lines, which makes it completely impossible to quantify the discord even approximately using the quantifier proposed by Ref. [1].

Consider the Werner state

$$\rho_W(c) = c|\Psi^-\rangle\langle\Psi^-| + \frac{1-c}{4}\mathbb{I}, \qquad (2)$$

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where $|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is the singlet Bell state, I is the identity matrix, and $c \in [0,1]$. It is known [4, 5] that $\rho_W(c)$ is separable (or non-entangled) for $0 \le c \le 1/3$ and inseparable (or entangled) for $1/3 < c \le 1$. In other words, for $c \le 1/3$, the Werner state can be represented as a convex sum of product states [6, 7], cf. Eq. (1) of Ref. [1]. The states have non-zero discord for any c > 0; the discord is given by [8, 9]

$$Q = \frac{1-c}{4}\log_2(1-c) - \frac{1+c}{2}\log_2(1+c) + \frac{1+3c}{4}\log_2(1+3c).$$
(3)

What is important for us is that the dependence of the discord on c is non-trivial.

The visibility in the protocol of Ref. [1] applied to the Werner state is

$$\mathcal{V}(c) = c\sqrt{(\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)\cos(\phi_A - \phi_B))^2 + \sin^2(\beta)\sin^2(\phi_A - \phi_B)},\tag{4}$$

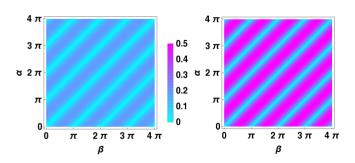


Figure 1. The dependence of the visibility $\mathcal{V}(c)$, cf. Eq. (4), for the Werner state on the parameters α and β controlling unitaries S^A and S^B (at $\phi_A = \phi_B$; c = 0.2 (left) and c = 0.5 (right)). The shape of the lines of zero visibility does not depend on c > 0.

where α , ϕ_A and β , ϕ_B are the parameters of S^A and S^B respectively, cf. Eq. (3) in Ref. [1] and the paragraph containing it. Observe that the visibility depends on c multiplicatively. Therefore, at c>0, the shape of the lines of zero visibility does not depend on c. In particular, for $\phi_A=\phi_B$, the zero visibility lines in the (α,β) plane always correspond to $\alpha=\beta+\pi n$ with integer n, cf. Fig. 1. This shows that the shape of the zero visibility lines does not in general allow for quantifying the discord even approximately.

Section III. The protocol efficiency.—One of the motivations mentioned in Ref. [1] for proposing the discussed protocol is finding an efficient way of estimating quantum discord. Here we argue that the protocol of Ref. [1] is computationally less efficient than calculating the discord via full state tomography. We start with the two-qubit case, considered in Ref. [1], and then provide estimates for the protocol complexity in the general many-body context.

We first estimate the number of measurements required to obtain the zero visibility lines via the protocol of Ref. [1]. Consider the case of the subsystems A and B having two-dimensional Hilbert spaces. For each value of parameters α , β , ϕ_A , ϕ_B , ϕ_d , a single measurement of

the observable yields a binary readout and the protocol utilizes the statistical average of these readouts. In order to estimate the average to accuracy $\epsilon \sim 1/\sqrt{m}$, one needs to repeat the measurement m times. Then, the same procedure has to be repeated for different values of parameters. Suppose each parameter is sampled at n points. This yields mn^5 measurements that need to be performed when executing the protocol.

We next estimate the cost of calculating the discord by means of full quantum state tomography. Once the state density matrix ρ^{AB} in a particular basis is known, quantum discord can be calculated on a computer. One has to minimize the expression for discord (Eq. (13) of Ref. [3]) over the sets of projectors $\Pi_i^A = |\psi_i^A\rangle\langle\psi_i^A|$ onto basis vectors of various bases in subsystem A Hilbert space. However, this operation can be performed using unitary rotations S^A applied to ρ^{AB} on a computer. Then it is enough to perform 15 different measurements on the system to recover ρ^{AB} . For example, these can be measurements of three expectation values of Pauli operators on subsystem A $(\langle \sigma_{i=x,y,z}^A \rangle),$ three expectation values for subsystem B $(\langle \sigma_{j=x,y,z}^B \rangle)$, and nine correlation functions $\langle \sigma_i^A \sigma_i^B \rangle$. Assuming that one needs to estimate each of these averages to accuracy $\epsilon \sim 1/\sqrt{m}$, one needs to performs 15m measurements to recover ρ^{AB} and perform the calculation of its discord on a computer.

Further, even if for some reason one does not want to perform the basis optimization on a computer, one can do it via repeating the tomography after applying various unitaries S^A , which is equivalent to sampling parameters α and ϕ_A . In this case, the number of required measurement operations becomes $15mn^2$, which is still less than mn^5 in the protocol of Ref. [1] for sufficiently large n. We thus conclude that estimating the location of zero visibility lines in the protocol of Ref. [1] is less efficient than quantifying the discord by means of full quantum state tomography.

Consider now the general case where the Hilbert spaces of subsystems A and B have dimensions d_A and d_B respectively. If they can be viewed as composed of N_A

and N_B qubits, then $d_A = 2^{N_A}$ and likewise for B. Estimates similar to the ones above show that the protocol of Ref. [1] would require $mn^{d_A^2+d_B^2-d_A-d_B+1}$ measurements, while performing the full tomography including the minimization via explicit sampling of S^A would require $\lceil (d_A + d_B)^2 - 1 \rceil m n^{d_A^2 - d_A}$ measurements. The last estimate does not count the operations with the inferred density matrix on a computer; counting them increases the last estimate by a factor which is only polynomial in $d_A + d_B$. Therefore, for sufficiently large n, the full tomography is more efficient than the protocol of Ref. [1]. Similarly, the protocol of Ref. [1] exhibits worse scaling with d_B . The only scenario in which the protocol of Ref. [1] may beat the full tomography is when keeping d_B constant and increasing d_A . However, even then the scaling is exponential in d_A (which is itself exponential in the subsystem size N_A) and thus the protocol is not "experiment friendly for many-body electronic systems".

Conclusion. We have analyzed the protocol of Ref. [1] for quantifying quantum discord. We have shown, that despite it is guaranteed to identify the states with zero discord correctly, it cannot serve as a universal discord quantifier. In particular, on the Werner state, any quantifier based solely on the shape of zero visibility lines in the protocol of Ref. [1] fails to capture the behavior of discord even qualitatively (apart from saying whether it vanishes). Further, we showed that even if one disregards this drawback, determining the shape of the zero-visibility lines in the protocol of Ref. [1] is less efficient than performing full state tomography to calculate the discord.

At the same time we do recognize the physical value of the protocol. While the discord is an abstract quantity, which is non-linear in terms of the system density matrix, the protocol of Ref. [1] works only with directly

observable quantities. Therefore, the protocol of Ref. [1] provides an intuitive illustration of what it means for a state to have zero or non-zero discord.

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