Assignment 1 - Introduction to Algorithms

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P1 $T(n) = \begin{cases} 2 & \text{if } n=2, \\ 2T(n/2)+n & \text{if } n=2^k, \text{for } k>1 \end{cases}$

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is $T(n) = n \lg n$

Base case $n=2$

$T(n) = 2$

$T(2) = 2 \log_2 2$

Mathematical Induction - If true for $k-1$ then it's true for k as well.

$T(n) = 2T(n/2)+n$ if $n=2^k$, for $k>1$

$$T(2^k) = 2T\left(\frac{2^k}{2}\right) + 2^k$$

$$= 2T\left(2^{k-1}\right) + 2^k$$

$$= 2\left[2^{k-1}\log\left(2^{k-1}\right)\right] + 2^k$$

$$= 2^k\left[\log\left(2^{k-1}\right)\right] + 2^k$$

$$= 2^k\left[\log\left(2^{k-1}\right) + 1\right]$$

$$= 2^k\left[\log\left(2^{k-1}\right) + \log_2 2\right]$$

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$$= 2^k[(k-1)\log_2 2 + \log_2 2]$$

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$$= 2^k[\log_2 2\,(k-1+1)]$$

$$= 2^k[k \log_2 2]$$

$$= 2^k[\log_2 2^k]$$

$$= n \lg n \quad (\because n = 2^k)$$

$$(\because \lg = \log_2)$$

P2 1. Given functions -

$\lg(\lg^* n)$ $\quad n^{\lg \lg n}$ $\quad n^{\log_6 5}$ $\quad n^2$ $\quad n!$ $\quad 2^{2n}$

$\left(\frac{4}{3}\right)^n$ $\quad n^2+n$ $\quad \left(\frac{3}{4}\right)^n$ $\quad \lg(n!)$ $\quad 2^{2^n}$ $\quad n^{1/\lg n}$

$\ln \ln n$ $\quad 2^n$ $\quad n2^n$ $\quad n^n$ $\quad \ln n$ $\quad n \lg n$

$2^{\lg n}$ $\quad (\lg n)^{\lg n}$ $\quad n$ $\quad \sqrt{\lg n}$ $\quad 1$ $\quad \lg^*(\lg n)$

Rearranging functions by type

Constant - 1

Logarithmic - $\ln \ln n$, $\lg(\lg^* n)$, $\ln n$, $\lg^*(\lg n)$, $(\lg n)^{\lg n}$, $\sqrt{\lg n}$, $\lg(n!)$

Linear - n, $n!$

Linearithmic - $n \lg n$

Polynomial - n^2, n^2+n

exponential - $2^{2n}, 2^{2^n}, 2^n, n2^n, n^n$ Date: / / /

$\rightarrow 2^{\lg n} = n$

Let $y = 2^{\lg n}$

$\lg y = \lg 2^{\lg n}$

$\lg y = \lg n \lg 2$

$\lg y = \lg n$

$y = n$

$\rightarrow n^{\lg \lg n} = (\lg n)^{\lg n}$

$\rightarrow n^{1/\lg n}$

$x = n^{\frac{1}{\lg n}}$

applying log on both sides

$\lg x = \lg n^{\frac{1}{\lg n}}$

$= \frac{1}{\lg n} \times \lg n$

$\lg x = 1$

$x = 2$

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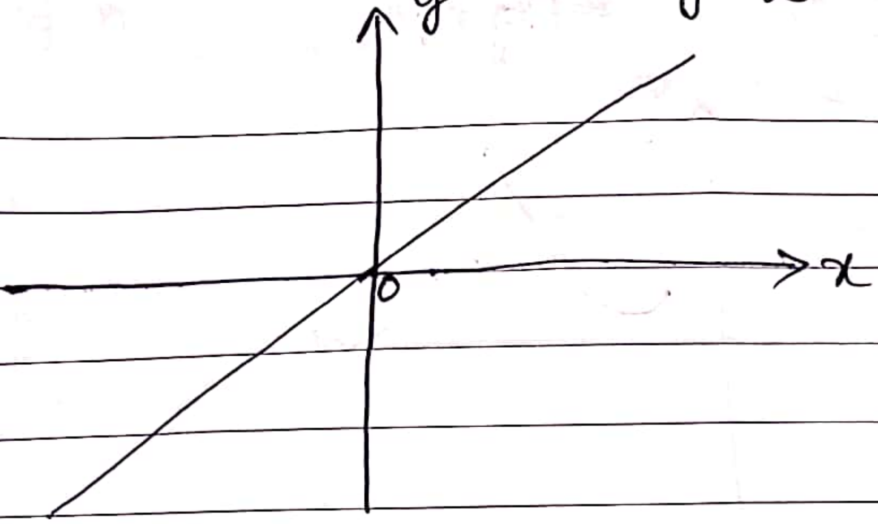
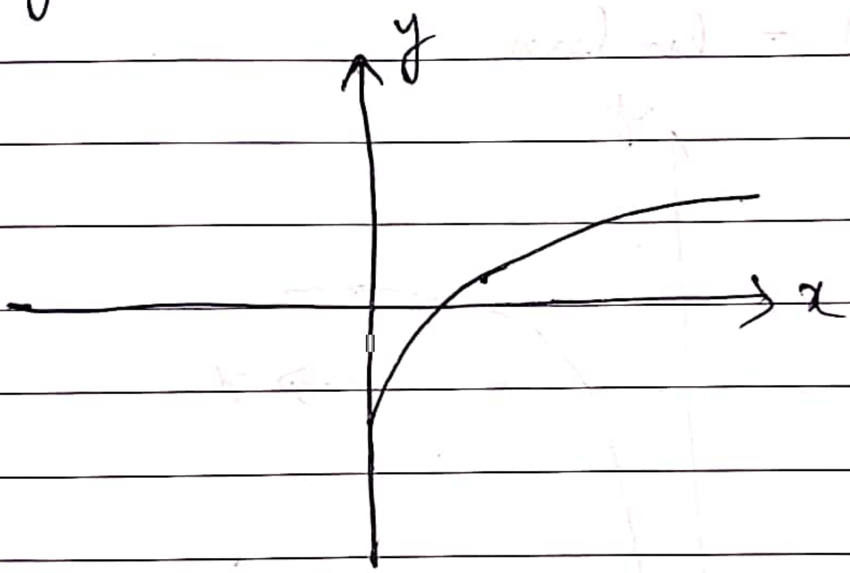
$y = 2^{\lg x} = 2^{\lg n}$



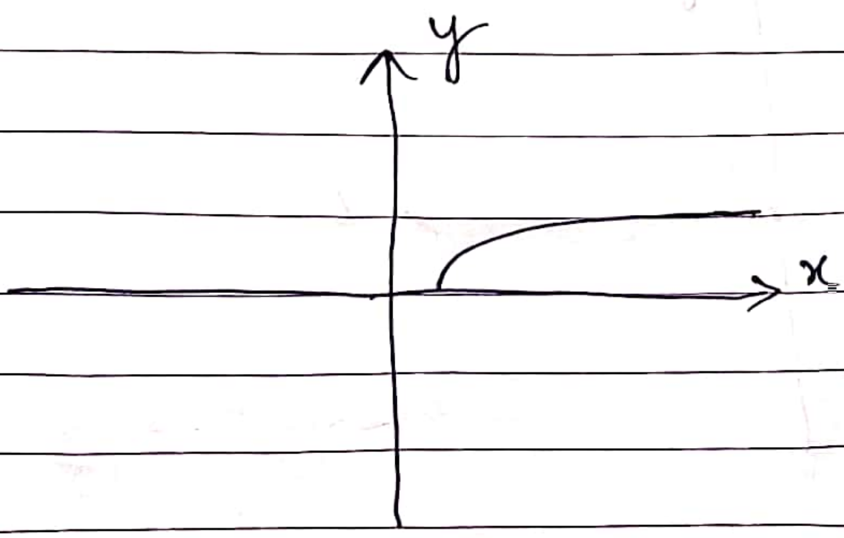
x	y
0	0
1	1

$y = \ln x = \ln n$



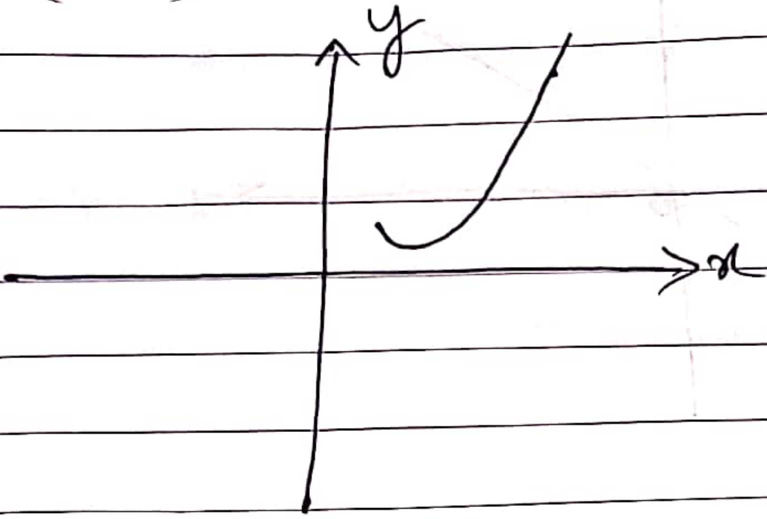
x	y
1	0
2	0.693
3	1.099

$y = \sqrt{\log_2 x} = \sqrt{\lg n}$

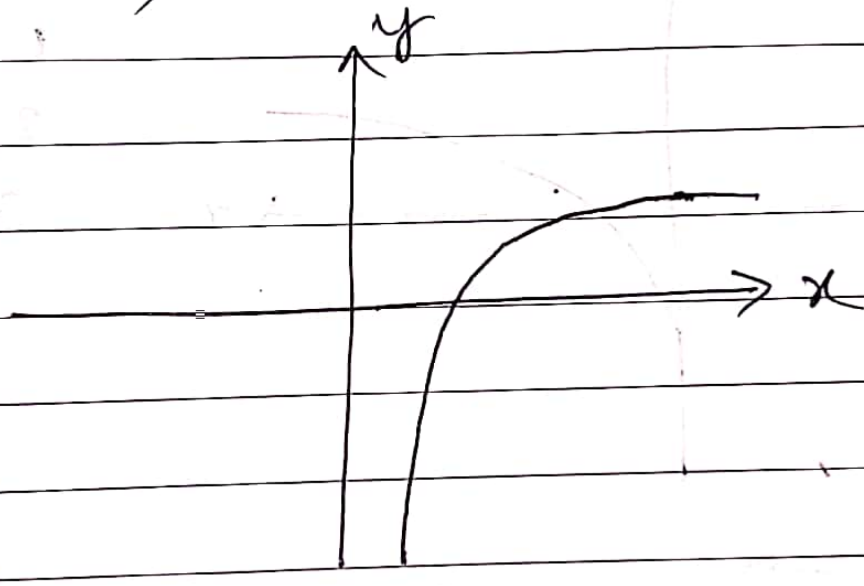


x	y
0	undefined
1	0
2	1

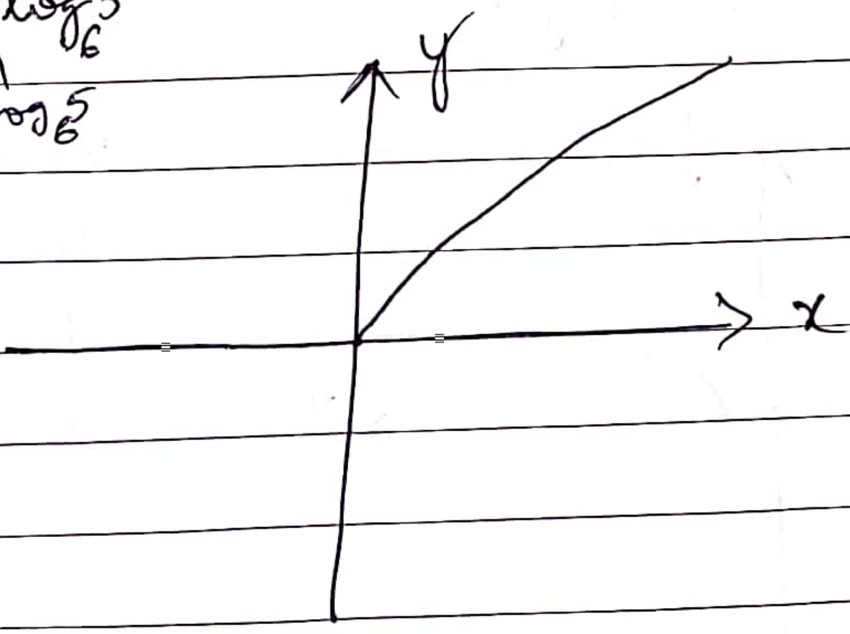
$y = \left(\log_2 x\right)^{\log_2 x} = (\lg n)^{\lg n}$ Date: / / /



$y = \ln(\ln x) = \ln \ln n$



$y = n^{\log_6 5}$
$= x^{\log_6 5}$



$\left(\frac{3}{4}\right)^n < 1 < n^{1/\lg n} < \lg(\lg^* n) < \lg^*(\lg n) < \ln \ln n$

$< \sqrt{\lg n} < \ln n < n^{\log_6 5} < n = 2^{\lg n} < \left(\frac{4}{3}\right)^n < n\lg n =$

$\lg(n!) < n^2 < n^2+n < n^{\lg \lg n} = (\lg n)^{\lg n} < 2^n < n2^n <$

$2^{2n} < n! < n^n < 2^{2^n}$

2. $f(n) = \Theta(g(n))$

$n, 2^{\lg n}$

$n \lg n, \lg(n!)$

$n^{\lg \lg n}, (\lg n)^{\lg n}$

$\underrightarrow{P_3 \, a}$ $T(n) = 4T(n/3) + n \lg n$

$a = 4, b = 3, f(n) = n \lg n$

$$a f\left(\frac{n}{b}\right) = 4\left[\frac{n}{3} \lg\left(\frac{n}{3}\right)\right]$$

$$= \frac{4n}{3}\left(\lg n - \lg 3\right)$$

$$\frac{4}{3} n \lg n > \frac{4n}{3} \lg n - \frac{4n}{3} \lg 3 > \frac{4n}{3} \lg n - 0.3 n \lg n$$

$$\frac{\frac{4n}{3} \lg n}{n \lg n} > \frac{\frac{4n}{3} \lg n - \frac{4n}{3} \lg 3}{n \lg n} > \frac{n \lg n}{n \lg n}$$

$$\frac{4}{3} > \frac{a f\left(\frac{n}{b}\right)}{f(n)} > 1$$

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$$a f\left(\frac{n}{b}\right) = c f(n)$$

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$$\frac{4}{3} > c > 1 \Rightarrow c > 1$$

According to master theorem Case 3

$$T(n) = \Theta\left(n^{\log_b a}\right) \cancel{= \Theta\left(n^{\log_3 4}\right)}$$

$$= \Theta\left(n^{\log_3 4}\right).$$

b) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{\lg n}$

$a=3$

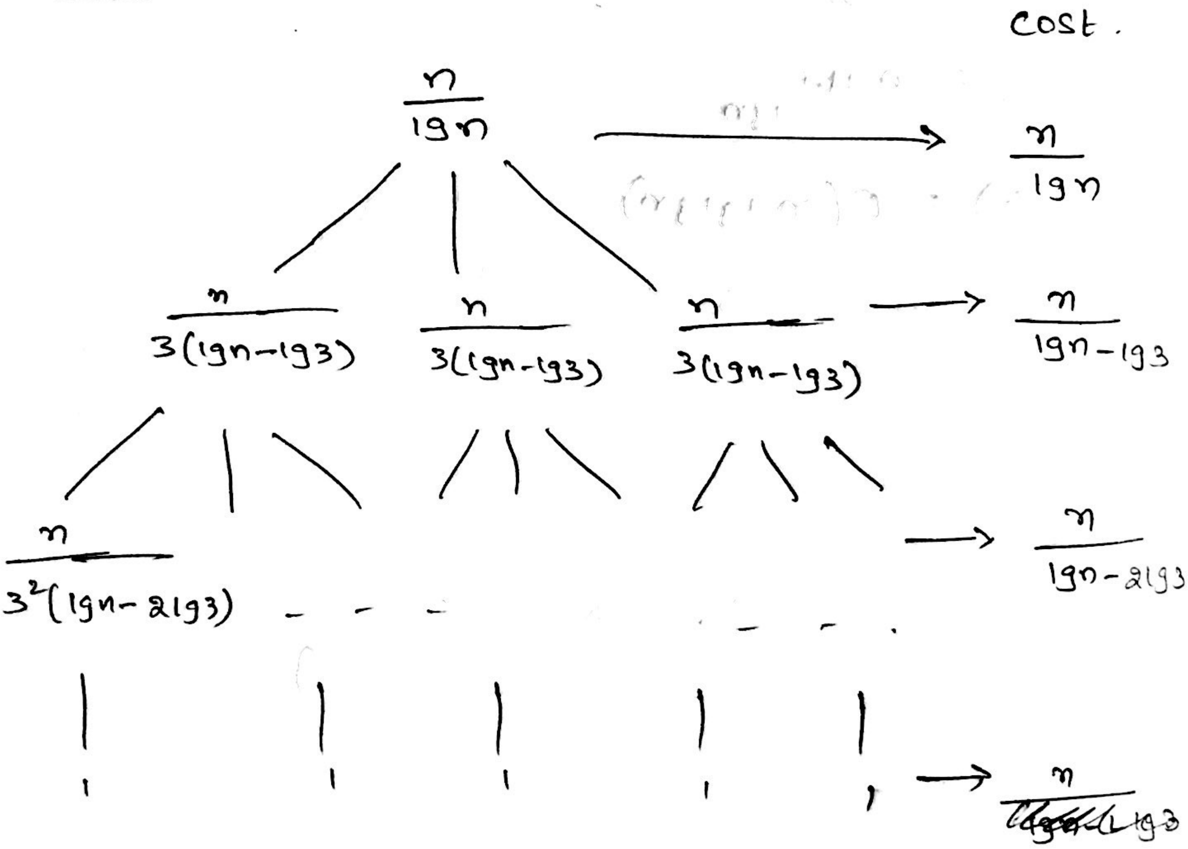
$b=3$

$f(n) = \frac{n}{\lg n}$

$af\left(\frac{n}{b}\right) = 3f\left(\frac{n}{3}\right) = \frac{3\left(\frac{n}{3}\right)}{\lg\left(\frac{n}{3}\right)} = \frac{n}{\lg n - \lg 3}$

when $n \to \alpha$, $af\left(\frac{n}{b}\right)$ almost same as $f(n)$

Recurrence tree:-



levels $L \Rightarrow$

$\frac{n}{3^L} = 1$

$n = 3^L$

$L = \log_3 n$

$\lg n - L\lg 3 = 1$

$L\lg 3 + 1 = \lg n$

$L\lg 3 = \lg n - 1$

$L = \frac{\lg n}{\lg 3} - \frac{1}{\lg 3} = \log_3 n - 2$

$L < \log_3 n - 2$

$$T(n) = \frac{n}{\lg n} + \frac{n}{\lg n - \lg 3} + \frac{n}{\lg n - 2\lg 3} + \cdots + n$$

$$= n\left\{ \frac{1}{\lg n} + \frac{1}{\lg n - \lg 3} + \cdots \frac{1}{1} \right\}$$

$$= n \sum_{i=0}^{\log_3 n - 1} \frac{1}{\lg n - i \lg 3}$$

$$= \frac{n}{\lg 3}\left[\frac{1}{\log_3 n} + \frac{1}{\log_3 n - 1} + \cdots + \frac{1}{1} \right]$$

$$\cancel{= \frac{n}{\lg 3} \sum_{i=1}^{\log_3 n} \frac{1}{i}}$$

$$\cancel{= \frac{n}{\lg 3} \Sigma}$$

$$= \frac{n}{\lg 3} \sum_{i=1}^{\log_3 n} \frac{1}{i}$$

$$= \frac{n}{\lg 3} \, H.P_{\log_3 n}$$

$$T(n) = O(n \lg \log_3 n)$$

$P_3 C \rightarrow$ $T(n) = 4T\left(\frac{n}{2}\right) + n^2\sqrt{n}$ Date: / / /

$a = 4,\ b = 2,\ f(n) = n^2\sqrt{n}$

$$af\left(\frac{n}{b}\right) = 4f\left(\frac{n}{2}\right) = 4\left(\frac{n}{2}\right)^2\sqrt{\frac{n}{2}}$$

$$= \frac{n^2\sqrt{n}}{\sqrt{2}}$$

$$af\left(\frac{n}{b}\right) = cf(n)$$

$$c = \frac{af\left(\frac{n}{b}\right)}{f(n)} = \frac{\frac{n^2\sqrt{n}}{\sqrt{2}}}{n^2\sqrt{n}} = \frac{1}{\sqrt{2}}$$

$$c < 1$$

$$T(n) = \Theta(f(n)) = \Theta(n^2\sqrt{n})$$

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P_3 d $\rightarrow$ $$T(n) = 3T\left(\frac{n}{3} - 2\right) + \frac{n}{2}$$

-2 is ignored since for large values of n subtraction of -2 will not make a significant difference

$$a = 3, \; b = 3, \; f(n) = \frac{n}{2}$$

$$a f\left(\frac{n}{b}\right) = 3 f\left(\frac{n}{3}\right) = 3\left(\frac{\frac{n}{2}}{3}\right) = \frac{n}{2}$$

$$af\left(\frac{n}{b}\right) = cf(n)$$

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$$c = 1$$

According to master theorem Case 3

$$T(n) = \Theta\left(f(n) \log_b n\right)$$

$$= \Theta\left(\frac{n}{2} \log_3 n\right)$$

P3 e

$\rightarrow$ $T(n) = 2T(n/2) + n/\lg n$

$a=2,\ b=2,\ f(n) = n/\lg n$

$$a f\left(\frac{n}{b}\right) = 2 \frac{\left(\frac{n}{2}\right)}{\lg\left(\frac{n}{2}\right)} = \frac{n}{\lg n - 1}$$

It seems $f(n) > a f\left(\frac{n}{b}\right)$ since -1 in denominator makes $a f\left(\frac{n}{b}\right)$ lesser than $f(n)$. There is also a discussion that 1 being constant it doesn't affect and so $f(n) = a f\left(\frac{n}{b}\right)$ and it can go wrong while using master theorem. Hence using recurssive tree.

Recursion tree -

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$$\frac{n}{\lg n}$$

$$\frac{n}{2(\lg n - 1)} \qquad \frac{n}{2(\lg n - 1)}$$

$$\frac{n}{2^2(\lg n - 2)} \quad \cdots\cdots$$

$$\vdots$$

$$2^L = \frac{n}{\lg n}$$

$$\lg\left(\frac{n}{\lg n}\right) = L$$

$$L = \lg n - \lg\lg n$$

$$= \lg n - 1$$

$$\Sigma$$

$i=0 \quad \frac{n}{\lg n}$

$i=1 \quad \frac{n}{\lg n - 1}$

$i=2 \quad \frac{n}{\lg n - 2}$

$\vdots$

$i=L \quad \frac{n}{\lg n - L} = C$ constant

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$$T(n) = \sum_{i=0}^{\lg n - 1} \frac{n}{\lg n - i}$$

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$$\frac{n}{\lg n} + \frac{n}{\lg n - 1} + \cdots\cdots + \frac{n}{1}$$

$$\sum_{j=1}^{\lg n} \frac{n}{j} = n \sum_{j=1}^{\lg n} \frac{1}{j}$$

$n H_{\lg n} \leq n \lg \lg n$

$$= n H_{\lg n}$$

$$= \Theta(n \lg \lg n)$$

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$$

$H_{\lg n}$

$H_n \simeq \lg n + \gamma$

$H_{\lg n} \simeq \lg \lg n$