Problem 1:

(a) By inspection,

$$\boldsymbol{r}(x) = 2\boldsymbol{p}_0 + 3\boldsymbol{p}_1 - \boldsymbol{p}_2$$

and thus

$$\big[\boldsymbol{r}(x)\big]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

(b) We seek coefficients $\alpha_0, \alpha_1, \alpha_2$ such that

$$\boldsymbol{r}(x) = \alpha_0 \boldsymbol{q}_0 + \alpha_1 \boldsymbol{q}_1 + \alpha_2 \boldsymbol{q}_2$$

hence

$$\begin{aligned} 2 + 3x - x^2 &= \alpha_0 + \alpha_1(1-x) + \alpha_2(x + x^2) \\ &= (\alpha_0 + \alpha_1) + (\alpha_2 - \alpha_1)x + \alpha_2 x^2 \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_0 + \alpha_1 &= 2 \\ \alpha_2 - \alpha_1 &= 3 \\ \alpha_2 &= -1 \end{aligned}$$

which yields

$$\begin{aligned} \alpha_0 &= 6 \\ \alpha_1 &= -4 \\ \alpha_2 &= -1 \end{aligned}$$

In other words,

$$\big[\boldsymbol{r}(x)\big]_{\mathcal{Q}} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

(c) We define $\bar{P} = P^{-1}$, and compute $\bar{P} = [\bar{P}_1, \bar{P}_2, \bar{P}_3]$, where

$$\bar{P}_1 = \big[\boldsymbol{q}_0\big]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{P}_2 = \big[\boldsymbol{q}_1\big]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{P}_3 = \big[\boldsymbol{q}_2\big]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2: By inspection, the e-values are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

$$(A_3 - \lambda_1 I) = \begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$0 = (A_3 - \lambda_1 I)v^1 \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A_3 - \lambda_2 I) = \begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$0 = (A_3 - \lambda_2 I)v^2 \Rightarrow v^2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$(A_3 - \lambda_3 I) = \begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$0 = (A_3 - \lambda_3 I)v^3 \Rightarrow v^3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

The e-vectors are NOT unique. Any non-zero multiples are also e-vectors.

Because the e-values are distinct, the e-vectors should be linearly independent.

$$0 = \alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = \left[\begin{array}{c|c|c} v^1 & v^2 & v^3 \end{array} \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$V = \left[\begin{array}{c|c|c} v^1 & v^2 & v^3 \end{array} \right] = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\det(V) = 1 \neq 0$. Hence, only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and indeed, the set $\{v^1, v^2, v^3\}$ is linearly independent.

```
>> A3=[1 4 10; 0 2 0; 0 0 3];
>> [V,D]=eig(A3)

V =

    1.0000    0.9701    0.9806
         0    0.2425         0
         0         0    0.1961


D =

     1     0     0
     0     2     0
     0     0     3
```

MATLAB normalizes the e-vectors so that the sum of the squares of the components equals 1.0. Here is how to check our answers

```
>> (A3-eye(3))*[1 0 0]'

ans =

     0
     0
     0

>> (A3-2*eye(3))*[4 1 0]'

ans =

     0
     0
     0

>> (A3-3*eye(3))*[5 0 1]'

ans =

     0
     0
     0
```

Problem 3: By inspection, $\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 2$ and we have repeated e-values.

$$(A_4 - \lambda_1 I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$0 = (A_4 - \lambda_1 I)v^1 \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Of course, the same is true for λ_2, and thus v^2 is not independent of v^1.

$\therefore$ No basis of e-vectors is possible.

Problem 4: We use two facts:

(i) $P^{-1} \cdot P = I$.

(ii) $\det(M_1 M_2) = \det(M_1)\det(M_2)$ for compatible matrices M_i.

$$\begin{aligned} \det(\lambda I - B) &= \det(\lambda I - P^{-1}AP) \\ &= \det(\lambda P^{-1}P - P^{-1}AP) \\ &= \det(P^{-1}[\lambda I - A]P) \\ &= \det(P^{-1})\det(\lambda I - A)\det(P) \end{aligned}$$

But $1 = \det(I) = \det(P^{-1}P) = \det(P^{-1})\det(P)$, and thus

$$\det(\lambda I - P^{-1}AP) = \det(\lambda I - A)$$

Problem 5: We translate $Av^i = \lambda_i v^i$ into the matrix formula

$$AP = P\Lambda$$

where $P = \left[\ v^1 \mid \cdots \mid v^n\ \right]$ and $\Lambda = \begin{bmatrix} \lambda_1 & & \text{\huge 0} \\ & \ddots & \\ \text{\huge 0} & & \lambda_n \end{bmatrix}$.

P is invertible if, and only if, its columns are linearly independent, which we know is the case when the e-values are distinct.

$$\therefore P^{-1}AP = \Lambda$$

is the general result, and the specific result for A_3 is

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

<u>Remark</u> You did not have to provide the general result!

Problem 6:

(a) Checking $L(M_1 + M_2) = L(M_1) + L(M_2)$ and $L(\alpha M) = \alpha L(M)$ is immediate and not given.

(b) $A = \left[\ A_1 \mid A_2 \mid A_3 \mid A_4\ \right]$ where

$$A_i = \left[L(u^i)\right]_{\{u\}}$$

$\{u\} = \{E^{11}, E^{12}, E^{21}, E^{22}\}$.

<u>We compute</u>

$$L(E^{11}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot E^{11} + 0 \cdot E^{12} + 0 \cdot E^{21} + 0 \cdot E^{22}$$

$$L(E^{12}) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = 0 \cdot E^{11} + \frac{1}{2} \cdot E^{12} + \frac{1}{2} \cdot E^{21} + 0 \cdot E^{22}$$

$$L(E^{21}) = L(E^{12})$$

$$L(E^{11}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \cdot E^{11} + 0 \cdot E^{12} + 0 \cdot E^{21} + 1 \cdot E^{22}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 7:

(a) We use the basis

$$\left\{ e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\hat{A}_i = \left[L(e^i)\right]_{\{e\}}$, $\hat{A} = \left[\begin{array}{c|c|c} \hat{A}_1 & \cdots & \hat{A}_n \end{array}\right]$, $L(e^i) = Ae^i =$ the i-th column of A.

Hence $\boxed{\hat{A} = A}$.

(b) Let $\{\lambda_1, \cdots, \lambda_n\}$ be the e-values, which are assumed to be distinct, and let $\{v^1, \cdots, v^n\}$ be the e-vectors, which are consequently linearly independent and form a basis for $(\mathbb{C}^n, \mathbb{C})$. We define

$$\hat{A}_i = \left[L(v^i)\right]_{\{v\}}$$

$L(v^i) = Av^i = \lambda_i v^i$,and thus

$$\hat{A}_i = \lambda_i e^i$$

and therefore

$$\hat{A} = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$