

Mathematics for Robotics

ROB-GY 6103

Homework 3 Answers

October 11, 2023

Shantanu Ghodgaonkar

Univ ID: N11344563

Net ID: sng8399

Ph.No.: +1 (929) 922-0614

Question: 1.(a)

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{S} = (p_0 = 1, p_1 = x, p_2 = x^2)$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

\therefore the components of $r(x)$ in basis \mathcal{S} is

$$r_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad (1)$$

Question: 1.(b)

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{Q} = (q_0 = 1, q_1 = 1 - x, q_2 = x + x^2)$.

From the definition of the basis \mathcal{Q} , we can rewrite it in terms of basis \mathcal{S} as,

$$q_{0\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_{1\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } q_{2\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Q_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

$$\Rightarrow r_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad (1)$$

Now, we need to find numbers \tilde{r}_0, \tilde{r}_1 and \tilde{r}_2 such that,

$$r_{\mathcal{S}} = \tilde{r}_0 q_{0\mathcal{S}} + \tilde{r}_1 q_{1\mathcal{S}} + \tilde{r}_2 q_{2\mathcal{S}} \quad (2)$$

$$\Rightarrow \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \tilde{r}_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \tilde{r}_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \tilde{r}_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (3)$$

Eq^n can be rewritten as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad (4)$$

Applying the Gauss method,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (5)$$

$R_2 \rightarrow (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (6)$$

$R_1 \rightarrow R_1 + (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (7)$$

$R_2 \rightarrow R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (8)$$

$$R_1 \rightarrow R_1 + (-1)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (9)$$

Hence, the solution is,

$$r_q = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} \quad (10)$$

Question: 2.

Answer: Given the matrix,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (1)$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \Rightarrow (\lambda I - A)v = 0 \Leftrightarrow \det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 1 & -4 & -10 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \quad (2)$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - 1) \cdot (\lambda - 2) \cdot (\lambda - 3) = 0 \quad (3)$$

$$\therefore \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \quad (4)$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 1 - \lambda_i & 4 & 10 \\ 0 & 2 - \lambda_i & 0 \\ 0 & 0 & 3 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

From Eq^n 5, substituting $\lambda_1 = 1$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 4v_2^1 + 10v_3^1 = 0 \\ v_2^1 = 0 \\ 2v_3^1 = 0 \end{array} \Rightarrow \begin{array}{l} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{array} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -v_1^2 + 4v_2^2 + 10v_3^2 = 0 \\ 0 = 0 \\ v_3^2 = 0 \end{array} \Rightarrow \begin{array}{l} v_1^2 = 4v_2^2 \\ v_2^2 = \text{any value} \\ v_3^2 = 0 \end{array} \Rightarrow v^2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad (7)$$

From Eq^n 5, substituting $\lambda_3 = 3$

$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -2v_1^3 + 4v_2^3 + 10v_3^3 = 0 \\ -v_2^3 = 0 \\ 0 = 0 \end{array} \Rightarrow \begin{array}{l} v_1^3 = 5v_3^3 \\ v_2^3 = 0 \\ v_3^3 = \text{any value} \end{array} \Rightarrow v^3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \quad (8)$$

Now we shall verify that v^1 , v^2 and v^3 are *Linearly Independent*,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = 0 \quad (9)$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 0 \quad (10)$$

$$\alpha_1 + 4\alpha_2 + 5\alpha_3 = 0 \quad (11)$$

$$\alpha_2 = 0 \quad (12)$$

$$\alpha_3 = 0 \quad (13)$$

Solving above system of equations gives us $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$.
 $\therefore v^1, v^2$ and v^3 are *Linearly Independent*.

Question: 3. IS THE BASIS CORRECT? The basis is LD. So it is wrong

Answer: Given the matrix,

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (1)$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \Rightarrow (\lambda I - A)v = 0 \Leftrightarrow \det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 3 & -1 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \quad (2)$$

$$\Rightarrow \det(\lambda I - A) = (\lambda - 3) \cdot (\lambda - 3) \cdot (\lambda - 2) = 0 \quad (3)$$

$$\therefore \lambda_1 = 3, \lambda_2 = 2 \quad (4)$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 3 - \lambda_i & 1 & 0 \\ 0 & 3 - \lambda_i & 0 \\ 0 & 0 & 2 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

From Eq^n 5, substituting $\lambda_1 = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_2^1 = 0 \\ 0 = 0 \\ -v_3^1 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{matrix} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ v_3^2 = \text{any value} \end{matrix} \Rightarrow v^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

Consider the set of eigenvectors $\mathcal{V} = \{v^1, v^2\}$. Firstly,

We know that any set of eigenvectors of a given matrix are Linearly Independent. So, v^1, v^2 are *L.I.* (8)

Secondly, consider a linear combination x such that,

$$\{x \in \mathbb{R} \mid \exists \alpha_1, \alpha_2 \in \mathbb{R}, v^1, v^2 \in \mathcal{V} ; \text{ s.t. } x = \alpha_1 v^1 + \alpha_2 v^2\} \quad (9)$$

By observing above Eq^n 9 we can see that it holds true for all cases.

\therefore based on the Eq^n 8 and 9 we can say that \mathcal{V} forms a basis for \mathbb{R}^3 .

Question: 4.

Answer: We are given two similar square matrices A and B such that,

$$B = P^{-1}AP \quad (1)$$

Consider the characteristic relation of matrix B ,

$$\det(\lambda I - B) \quad (2)$$

Apply Eqⁿ 1 in Eqⁿ ??,

$$\Rightarrow \det(\lambda I - B) = \det(\lambda I - P^{-1}AP) \quad (3)$$

The identity matrix I can also be written as $P^{-1}IP$. Substituting in Eqⁿ 3,

$$\Rightarrow \det(\lambda I - B) = \det(\lambda P^{-1}IP - P^{-1}AP) \quad (4)$$

Now we shall take P^{-1} and P out common,

$$\Rightarrow \det(\lambda I - B) = \det(P^{-1}(\lambda I - A)P) \quad (5)$$

We know that for compatible square matrices A and B , $\det(AB) = \det(A)\det(B)$. So Eqⁿ 5 becomes,

$$\Rightarrow \det(\lambda I - B) = \det(P^{-1})\det(\lambda I - A)\det(P) \quad (6)$$

Cancelling out $\det(P^{-1})$ by $\det(P)$

$$\Rightarrow \det(\lambda I - B) = \det(\lambda I - A) \quad (7)$$

When the relation in Eqⁿ 8 is equated to zero, we prove that the two matrices have the same eigenvalues as well as the characteristic equations. I.E.,

$$\Rightarrow \det(\lambda I - B) = \det(\lambda I - A) = 0 \quad (8)$$

Q.E.D.

Question: 5.

Answer: Given a matrix A_3 such that,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (1)$$

From Question: 2. We found the eigenvalues to be $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Using these eigenvalues, we can create the diagonal matrix, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (2)$$

To show similarity between A and Λ , we need to find a matrix P such that $\Lambda = PAP^{-1}$

Consider a matrix $P = [v^1 \ v^2 \ v^3]$ where v^1, v^2 and v^3 are the eigenvectors of A .

$$P = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Testing for invertibility, $\det(P) = 1$. As the determinant is non-zero, we can conclude that P is invertible.

In Question: 2. we already proved that v^1, v^2 and v^3 are *Linearly Independent*.

As these two conditions are satisfied, we can therefore test for similarity,

$$A_3 = P\Lambda P^{-1} \quad (4)$$

$$A_3 = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \quad (5)$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.333 \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} 1 & 8 & 15 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.333 \end{bmatrix} \quad (7)$$

$$= \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (8)$$

$\therefore \text{LHS} = \text{RHS}$

Thus, we can say that matrix A is similar to a diagonal matrix Λ . **Q.E.D.**

Question: 6.(a)

Answer: Given, a vector space $(\mathcal{X}, \mathbb{R})$ where \mathcal{X} is a set of 2 x 2 matrices with real coefficients.

An operation is defined $L : \mathcal{X} \rightarrow \mathcal{X}$ by

$$L(M) = \frac{1}{2}(M + M^T) \quad (1)$$

Where $M \in \mathcal{X}$ is a 2 x 2 real matrix

The operator L will be considered a *Linear Operator* if,

$$\forall x, y \in \mathcal{X}, \alpha, \beta \in \mathbb{R} \mid L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \quad (2)$$

Applying above Statement 2 to Eqⁿ 1,

$$L(\alpha x + \beta y) = \frac{1}{2}((\alpha x + \beta y) + (\alpha x + \beta y)^T) \quad (3)$$

Applying the property of sum of transpose of two matrices
(i.e. for two matrices A and $B \rightarrow (A + B)^T = A^T + B^T$)

$$L(\alpha x + \beta y) = \frac{1}{2}(\alpha x + \beta y + \alpha x^T + \beta y^T) \quad (4)$$

$$= \frac{1}{2}(\alpha x + \alpha x^T + \beta y + \beta y^T) \quad (5)$$

$$= \alpha\left(\frac{1}{2}(x + x^T)\right) + \beta\left(\frac{1}{2}(y + y^T)\right) \quad (6)$$

$$= \alpha L(x) + \beta L(y) \quad (7)$$

Thus, Eqⁿ 7 proves Statement 2. \therefore the given operator L is actually a *Linear Operator*.

Q.E.D.

Question: 6.(b) RECHECK THIS ANSWER

Answer: Given a linear transformation $L : \mathcal{X} \rightarrow \mathcal{X}$ and a basis E given by

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ used on both copies of } \mathcal{X}.$$

By the theorem, we know that $A_i = [L(u^i)]_v$. But in our case, $u = v = E$. So, we can rewrite it as $A_i = [L(E^{ij})]_E$ where $\forall i, j \in \mathbb{N} \mid 1 \leq i, j \leq 2$

But as it is given that $L : \mathcal{X} \rightarrow \mathcal{X} \Rightarrow L$ is the Identity Operation Id . Thus, $[L(E^{ij})]_E = [E^{ij}]_E$.

So, now we can form the columns of $A = [A_1 \ A_2 \ A_3 \ A_4]$ to be as,

$$A = [[E^{11}]_E \ [E^{12}]_E \ [E^{21}]_E \ [E^{22}]_E] \quad (1)$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

Question: 7.(a) Let A be an $n \times n$ matrix with possible complex coefficients. Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by $L(x) = Ax$. Note that the field is $\mathcal{F} = \mathbb{C}$. Compute the matrix representation of L when the natural basis is used in \mathbb{C}^n . Call your representation \hat{A} and find its relation to the original matrix A .

Answer: Given a linear operator $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $L(x) = Ax$ with the natural basis ($u = \{u^1 \dots u^n\}$) applied on both, the domain and codomain.

Consider $x \in \mathbb{C}^n$, such that $x = \alpha_1 u^1 + \dots + \alpha_n u^n$. It can be represented as,

$$[x]_u = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \in \mathbb{C}^n \quad (1)$$

Now let us compute the matrix representation,

$$[L(x)]_u = [L(\alpha_1 u^1 + \dots + \alpha_n u^n)]_u \quad (2)$$

$$[L(x)]_u = [L(\alpha_1 u^1) + \dots + L(\alpha_n u^n)]_u \quad (3)$$

$$[L(x)]_u = [\alpha_1 L(u^1) + \dots + \alpha_n L(u^n)]_u \quad (4)$$

But from the theorem on matrix representation, we know that $[L(x)]_v = A[x]_u$. But in our case $u = v$. So, above equation can be rewritten as,

$$\hat{A}[x]_u = [\alpha_1 L(u^1) + \dots + \alpha_n L(u^n)]_u \quad (5)$$

Where, \hat{A} is the Matrix Representation of L .

$$\hat{A}[x]_u = [\alpha_1 Au^1 + \dots + \alpha_n Au^n]_u \quad (6)$$

$$\hat{A}[x]_u = [Au^1 \ \dots \ Au^n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad (7)$$

We know that for the natural basis u , $A \times u = A$.

$$\hat{A}[x]_u = A[x]_u \quad (8)$$

$\therefore \hat{A} = A$. **Q.E.D.**

Question: 7.(b) Let A be an $n \times n$ matrix with possible complex coefficients. Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by $L(x) = Ax$. Note that the field is $\mathcal{F} = \mathbb{C}$. Suppose that the eigenvalues for A are distinct. Compute the matrix representation L with respect to a basis constructed from the eigenvectors of A . Call your representation \hat{A} .

Answer: As an extension to the answer of Question: 7.(a),

$$\hat{A}[x]_u = [L(x)]_v \quad (1)$$

Where, $v = \{v^1 \dots v^n\}$ is the basis consisting of all the distinct eigenvectors of A .

$$\hat{A}[x]_u = [\alpha_1 Au^1 + \dots + \alpha_n Au^n]_v \quad (2)$$

$$\hat{A}[x]_u = \alpha_1 [Au^1]_v + \dots + \alpha_n [Au^n]_v \quad (3)$$