CHAPTER 6. MANIPULATOR DYNAMICS

- Two types of dynamics problems:
  $\boldsymbol{\tau} \rightarrow \boxed{\text{Forward dynamics}} \rightarrow \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ (E.g., Simulation)
  $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \rightarrow \boxed{\text{Inverse dynamics}} \rightarrow \boldsymbol{\tau}$ (E.g., Control)

- Two types of kinematics problems (recall):
  $\mathbf{q} \rightarrow \boxed{\text{Forward kinematics}} \rightarrow {}^0T_n$
  ${}^0T_n \rightarrow \boxed{\text{Inverse kinematics}} \rightarrow \mathbf{q}$

Acceleration of a Rigid Body

- $^B\dot{\mathbf{V}}_Q = \frac{d}{dt}\,{}^B\mathbf{V}_Q = \lim_{\Delta t \to 0} \frac{^B\mathbf{V}_Q(t+\Delta t) - {}^B\mathbf{V}_Q(t)}{\Delta t}$ $\qquad$ $^A\dot{\boldsymbol{\Omega}}_B = \frac{d}{dt}\,{}^A\boldsymbol{\Omega}_B = \lim_{\Delta t \to 0} \frac{^A\boldsymbol{\Omega}_B(t+\Delta t) - {}^A\boldsymbol{\Omega}_B(t)}{\Delta t}$
- When the reference frame of differentiation is universal (= global) reference frame $\{U\}$:
  $\dot{\mathbf{v}}_A = {}^U\dot{\mathbf{V}}_{AORG}$ and $\dot{\boldsymbol{\omega}}_A = {}^U\dot{\boldsymbol{\Omega}}_A$

Linear Acceleration

- Recall (Chapter 5) when origins coincide:
  $$^A\mathbf{V}_Q = {}^AR_B\,{}^B\mathbf{V}_Q + {}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q} = \frac{d}{dt}({}^AR_B\,{}^B\mathbf{Q}) \rightarrow \text{time-derivative of } {}^A\mathbf{Q}$$
  - Acceleration of $^B\mathbf{Q}$ expressed in $\{A\}$: $^A\dot{\mathbf{V}}_Q = \frac{d}{dt}({}^AR_B\,{}^B\mathbf{V}_Q) + {}^A\dot{\boldsymbol{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q} + {}^A\boldsymbol{\Omega}_B \times \frac{d}{dt}({}^AR_B\,{}^B\mathbf{Q})$
  => $^A\dot{\mathbf{V}}_Q = {}^AR_B\,{}^B\dot{\mathbf{V}}_Q + 2\,{}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{V}_Q + {}^A\dot{\boldsymbol{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q} + {}^A\boldsymbol{\Omega}_B \times ({}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q})$
  (from undergraduate dynamics)

- General case (origins are not coincident):
  $$\boxed{^A\dot{\mathbf{V}}_Q = {}^A\dot{\mathbf{V}}_{BORG} + {}^AR_B\,{}^B\dot{\mathbf{V}}_Q + 2\,{}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{V}_Q + {}^A\dot{\boldsymbol{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q} + {}^A\boldsymbol{\Omega}_B \times ({}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q})}$$

- Coriolis acceleration ($2\,{}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{V}_Q$): contains the product of two different joint velocities
  → Combined effect of the relative motion of a point and of the rotation of the frame, due to:
  (a) Changing direction in space of the velocity of the point relative to the moving frame
  (b) Changing magnitude or direction of the point's position vector relative to the moving frame

- When $^B\mathbf{Q}$ is constant ($^B\mathbf{V}_Q = {}^B\dot{\mathbf{V}}_Q = \mathbf{0}$): $^A\dot{\mathbf{V}}_Q = {}^A\dot{\mathbf{V}}_{BORG} + {}^A\boldsymbol{\Omega}_B \times ({}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q}) + {}^A\dot{\boldsymbol{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q}$

Angular Acceleration

- If $\{B\}$ rotates relative to $\{A\}$ with $^A\boldsymbol{\Omega}_B$ and $\{C\}$ rotates relative to $\{B\}$ with $^B\boldsymbol{\Omega}_C$:
  $^A\boldsymbol{\Omega}_C = {}^A\boldsymbol{\Omega}_B + {}^AR_B\,{}^B\boldsymbol{\Omega}_C$
  (Note that $^AR_{\cancel{B}}\,{}^{\cancel{B}}\boldsymbol{\Omega}_C \neq {}^A\boldsymbol{\Omega}_C$. $\because$ $^B\boldsymbol{\Omega}_C = {}^B({}^B\boldsymbol{\Omega}_C)$ and $^AR_B\,{}^B\boldsymbol{\Omega}_C = {}^AR_{\cancel{B}}\,{}^{\cancel{B}}({}^B\boldsymbol{\Omega}_C) = {}^A({}^B\boldsymbol{\Omega}_C)$.)
  => $^A\dot{\boldsymbol{\Omega}}_C = {}^A\dot{\boldsymbol{\Omega}}_B + \frac{d}{dt}({}^AR_B\,{}^B\boldsymbol{\Omega}_C)$ $\qquad$ $\therefore \boxed{^A\dot{\boldsymbol{\Omega}}_C = {}^A\dot{\boldsymbol{\Omega}}_B + {}^AR_B\,{}^B\dot{\boldsymbol{\Omega}}_C + {}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\boldsymbol{\Omega}_C}$

Mass Distribution (Review undergraduate dynamics!)
(Note: The conventions used in Craig 4th book are not consistent. Follow the conventions given below.)

- Inertia tensor defined relative to local frame $\{A\}$ attached to a rigid body:

$${}^{A}I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$ (Note: Craig 4th book has negative signs in all off-diagonal elements.)

  where
  - Mass moments of inertia (> 0): $I_{xx} = \int_V (y^2 + z^2)\rho dv$, $I_{yy} = \int_V (z^2 + x^2)\rho dv$, $I_{zz} = \int_V (x^2 + y^2)\rho dv$
  - Mass products of inertia (>, =, or < 0): $I_{xy} = -\int_V xy\rho dv$, $I_{yz} = -\int_V yz\rho dv$, $I_{zx} = -\int_V zx\rho dv$

    (Note: Craig 4th book does not have negative signs in the products of inertia definitions.)

- Reference frame that cause zero products of inertia → principal axes (eigenvectors of the inertia tensor); principal moments of inertia (the associated eigenvalues)

- Parallel axis theorem: If $\{A\}$ and $\{C\}$ have same orientation, and origin of $\{C\}$ is at the center of mass (i.e., center of mass coordinate relative to $\{A\}$ is $\mathbf{P}_c = [x_c, y_c, z_c]^T$),

  ${}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$, …… ${}^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$, ……

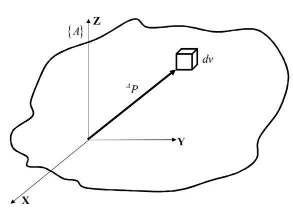
  In vector-matrix form: ${}^{A}I = {}^{C}I + m[\mathbf{P}_c^T\mathbf{P}_c I_3 - \mathbf{P}_c\mathbf{P}_c^T]$

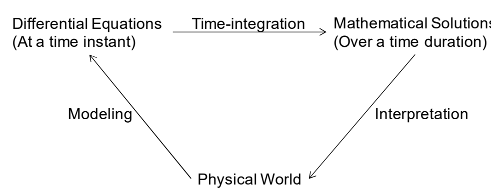
- Recall: Inertia tensors

  If two axes of reference frame form a plane of symmetry for the mass distribution, the products of inertia having as an index the coordinate which is normal to the plane of symmetry are zero.

  $I_{xx} + I_{yy} + I_{zz}$ is invariant under reference frame's orientation changes.

(Note: The frame of expression for inertia tensor is identical to that of the observer. This is because, unlike geometric vectors that can also be represented algebraically, inertia tensor has only algebraic representation.)





Newton-Euler Equation

Assume rigid body
$\dot{v}_C$: Center of mass acceleration (linear)
ω, $\dot{\omega}$: Angular velocity and acceleration of a rigid body
F, N: Resultant force and resultant moment about COM
${}^{C}I$: Inertia tensor written in COM frame $\{C\}$ whose origin is at COM

- Newton's equation: $\sum f = F = m\dot{v}_C$ Euler's equation: $\sum n = N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega$

Iterative Newton-Euler Dynamic Formulation

Given: joint variables $\mathbf{q}$, velocities $\dot{\mathbf{q}}$, and accelerations $\ddot{\mathbf{q}}$

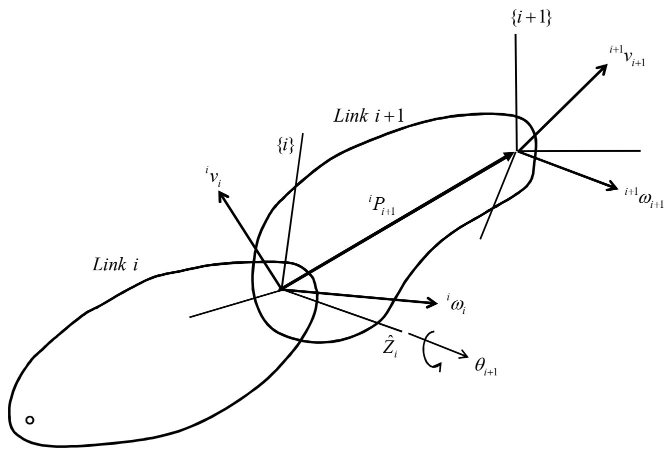
Compute: required joint actuator torques $\boldsymbol{\tau}$

- Propagation
  - Outward iterations: kinematics
  - Inward iterations: dynamics

Outward Iterations to Compute Velocities and Accelerations

Inertia forces at any given instant

Propagation: start with link 1 (${}^0\omega_0 = {}^0\dot{\omega}_0 = \mathbf{0}$), link by link, to link n



- Recall link velocities in standard DH convention (Chapter 5)
  - For revolute joint i+1: ${}^{i+1}\omega_{i+1} = {}^{i+1}R_i({}^i\omega_i + \dot{\theta}_{i+1}\,{}^i\hat{Z}_i)$ and ${}^{i+1}v_{i+1} = {}^{i+1}R_i({}^i v_i + {}^i\omega_{i+1} \times {}^i P_{i+1})$
  - For prismatic joint i+1: ${}^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^i\omega_i$ and ${}^{i+1}v_{i+1} = {}^{i+1}R_i({}^i v_i + {}^i\omega_{i+1} \times {}^i P_{i+1} + \dot{d}_{i+1}\,{}^i\hat{Z}_i)$

  → Apply successively from link 0 to link n → ${}^n\omega_n$, ${}^n v_n$

- Angular acceleration of link i+1 with respect to frame {i+1}
  - For revolute joint i+1: $\boxed{{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i({}^i\dot{\omega}_i + \dot{\theta}_{i+1}\,{}^i\omega_i \times {}^i\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^i\hat{Z}_i)}$
  - For prismatic joint i+1: $\boxed{{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^i\dot{\omega}_i}$

- Proofs

  1) For revolute joint i+1: In ${}^A\dot{\mathbf{\Omega}}_C = {}^A\dot{\mathbf{\Omega}}_B + {}^A R_B\,{}^B\dot{\mathbf{\Omega}}_C + {}^A\mathbf{\Omega}_B \times {}^A R_B\,{}^B\mathbf{\Omega}_C$ (textbook equation (6.15)), let {A} = {K}, {B} = {i}, and {C} = {i+1}.

  $${}^K({}^K\dot{\omega}_{i+1}) = {}^K({}^K\dot{\omega}_i) + {}^K R_i\,\underbrace{{}^i({}^i\dot{\omega}_{i+1})}_{=\ddot{\theta}_{i+1}\,{}^i\hat{Z}_i} + {}^K({}^K\omega_i) \times {}^K R_i\,\underbrace{{}^i({}^i\omega_{i+1})}_{=\dot{\theta}_{i+1}\,{}^i\hat{Z}_i}$$

  => ${}^K({}^K\dot{\omega}_{i+1}) = {}^K({}^K\dot{\omega}_i) + {}^K({}^K\omega_i) \times \dot{\theta}_{i+1}\,{}^K\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^K\hat{Z}_i$

  Pre-multiply by ${}^{i+1}R_k = {}^{i+1}R_i\,{}^iR_k$ → ${}^{i+1}({}^K\dot{\omega}_{i+1}) = {}^{i+1}R_i[\,{}^i({}^K\dot{\omega}_i) + {}^i({}^K\omega_i) \times \dot{\theta}_{i+1}\,{}^i\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^i\hat{Z}_i]$

  $\therefore\ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i({}^i\dot{\omega}_i + \dot{\theta}_{i+1}\,{}^i\omega_i \times {}^i\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^i\hat{Z}_i)$

  2) For prismatic joint i+1: ${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i({}^i\dot{\omega}_i + \cancel{\dot{\theta}_{i+1}}\,{}^i\omega_i \times {}^i\hat{Z}_i + \cancel{\ddot{\theta}_{i+1}}\,{}^i\hat{Z}_i)$ $\therefore\ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^i\dot{\omega}_i$

- Linear acceleration of origin of frame $\{i+1\}$ with respect to frame $\{i+1\}$

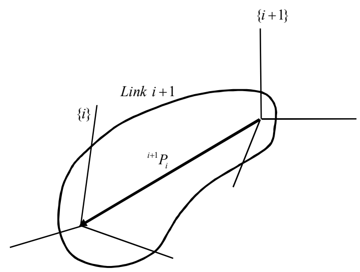
  For prismatic joint $i+1$:

  $$\boxed{{}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^i\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^i\hat{Z}_i}$$

  For revolute joint $i+1$: $\boxed{{}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^i\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1})}$

- Proofs

  1) For prismatic joint $i+1$:



In ${}^A\dot{\mathbf{V}}_Q = {}^A\dot{\mathbf{V}}_{BORG} + {}^AR_B\,{}^B\dot{\mathbf{V}}_Q + 2\,{}^A\mathbf{\Omega}_B \times {}^AR_B\,{}^B\mathbf{V}_Q + {}^A\dot{\mathbf{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q} + {}^A\mathbf{\Omega}_B \times ({}^A\mathbf{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q})$ (textbook Equation (6.10)), let $\{A\} = \{K\}$, $\{B\} = \{i+1\}$, $\mathbf{Q}$ = origin of $\{i\}$, ${}^AP_{BORG} = {}^KP_{i+1}$, and ${}^B\mathbf{Q} = {}^{i+1}P_i$ for any other previous frame $\{K\}$, e.g., $\{K\} = \{0\}$.

$${}^K({}^K\dot{v}_i) = {}^K({}^K\dot{v}_{i+1}) + {}^KR_{i+1}\,{}^{i+1}({}^{i+1}\dot{v}_i) + 2\,{}^K({}^K\omega_{i+1}) \times {}^KR_{i+1}\,{}^{i+1}({}^{i+1}v_i)$$
$$+ {}^K({}^K\dot{\omega}_{i+1}) \times {}^KR_{i+1}\,{}^{i+1}P_i + {}^K({}^K\omega_{i+1}) \times ({}^K({}^K\omega_{i+1}) \times {}^KR_{i+1}\,{}^{i+1}P_i)$$
$$\Rightarrow {}^K({}^K\dot{v}_{i+1}) = {}^K({}^K\dot{v}_i) + {}^K({}^K\dot{\omega}_{i+1}) \times (-{}^KR_{i+1}\,{}^{i+1}P_i) + {}^K({}^K\omega_{i+1}) \times ({}^K({}^K\omega_{i+1}) \times (-{}^KR_{i+1}\,{}^{i+1}P_i))$$
$$- {}^KR_{i+1}\,{}^{i+1}({}^{i+1}\dot{v}_i) - 2\,{}^K({}^K\omega_{i+1}) \times {}^KR_{i+1}\,{}^{i+1}({}^{i+1}v_i)$$

Since ${}^i({}^iv_{i+1}) = \dot{d}_{i+1}\,{}^i\hat{Z}_i$ and ${}^i({}^i\dot{v}_{i+1}) = \ddot{d}_{i+1}\,{}^i\hat{Z}_i$

$$\rightarrow {}^{i+1}({}^{i+1}v_i) = -{}^{i+1}R_i\,{}^i({}^iv_{i+1}) = -\dot{d}_{i+1}\,{}^{i+1}R_i\,{}^i\hat{Z}_i \text{ and } {}^{i+1}({}^{i+1}\dot{v}_i) = -{}^{i+1}R_i\,{}^i({}^i\dot{v}_{i+1}) = -\ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^i\hat{Z}_i$$
$$\Rightarrow {}^K({}^K\dot{v}_{i+1}) = {}^K({}^K\dot{v}_i) + {}^K({}^K\dot{\omega}_{i+1}) \times (-{}^KR_{i+1}\,{}^{i+1}P_i) + {}^K({}^K\omega_{i+1}) \times ({}^K({}^K\omega_{i+1}) \times (-{}^KR_{i+1}\,{}^{i+1}P_i))$$
$$+ \ddot{d}_{i+1}\,{}^KR_i\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^K({}^K\omega_{i+1}) \times {}^KR_i\,{}^i\hat{Z}_i$$

Pre-multiply by ${}^iR_K$

$$\Rightarrow {}^i({}^K\dot{v}_{i+1}) = {}^i({}^K\dot{v}_i) + {}^i({}^K\dot{\omega}_{i+1}) \times (-{}^iR_{i+1}\,{}^{i+1}P_i) + {}^i({}^K\omega_{i+1}) \times [{}^i({}^K\omega_{i+1}) \times \underbrace{(-{}^iR_{i+1}\,{}^{i+1}P_i)}_{={}^iP_{i+1}}]$$
$$+ \ddot{d}_{i+1}\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^i({}^K\omega_{i+1}) \times {}^i\hat{Z}_i$$

$$=> {}^i({}^K\dot{v}_{i+1}) = {}^i({}^K\dot{v}_i) + {}^i({}^K\dot{\omega}_{i+1}) \times {}^iP_{i+1} + {}^i({}^K\omega_{i+1}) \times ({}^i({}^K\omega_{i+1}) \times {}^iP_{i+1}) + \ddot{d}_{i+1}\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^i({}^K\omega_{i+1}) \times {}^i\hat{Z}_i$$

Pre-multiply by ${}^{i+1}R_i$

$$\Rightarrow {}^{i+1}({}^K\dot{v}_{i+1}) = {}^{i+1}R_i\,{}^i({}^K\dot{v}_i) + {}^{i+1}({}^K\dot{\omega}_{i+1}) \times {}^{i+1}R_i\,{}^iP_{i+1} + {}^{i+1}({}^K\omega_{i+1}) \times ({}^{i+1}({}^K\omega_{i+1}) \times {}^{i+1}R_i\,{}^iP_{i+1})$$
$$+ \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}({}^K\omega_{i+1}) \times {}^{i+1}R_i\,{}^i\hat{Z}_i$$
$$\therefore {}^{i+1}\dot{v}_{i+1} = {}^{i+1}\mathbf{R}_i\,{}^i\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1})$$
$$+ \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^i\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^i\hat{Z}_i$$

2) For revolute joint $i+1$: Since ${}^{i+1}({}^{i+1}v_i) = 0$ and ${}^{i+1}({}^{i+1}\dot{v}_i) = 0$,

$$\rightarrow {}^i({}^K\dot{v}_{i+1}) = {}^i({}^K\dot{v}_i) + {}^i({}^K\dot{\omega}_{i+1}) \times {}^iP_{i+1} + {}^i({}^K\omega_{i+1}) \times ({}^i({}^K\omega_{i+1}) \times {}^iP_{i+1})$$
$$\therefore {}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^i\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^iP_{i+1})$$

- Linear acceleration of COM frame $\{C_i\}$ origin of link i (for both revolute and prismatic joint i+1):
  $\boxed{{}^i\dot{v}_{C_i} = {}^i\dot{\omega}_i \times {}^iP_{C_i} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{C_i}) + {}^i\dot{v}_i}$

  Proof: In ${}^A\dot{\mathbf{V}}_Q = {}^A\dot{\mathbf{V}}_{BORG} + {}^AR_B\,{}^B\dot{\mathbf{V}}_Q + 2\,{}^A\mathbf{\Omega}_B \times {}^AR_B\,{}^B\mathbf{V}_Q + {}^A\dot{\mathbf{\Omega}}_B \times {}^AR_B\,{}^B\mathbf{Q} + {}^A\mathbf{\Omega}_B \times ({}^A\mathbf{\Omega}_B \times {}^AR_B\,{}^B\mathbf{Q})$ (textbook equation (6.10)), let $\{A\} = \{K\}$, $\{B\} = \{i\}$, $\mathbf{Q}$ = center of mass C_i, ${}^AP_{BORG} = {}^KP_i$, and ${}^B\mathbf{Q} = {}^iP_{C_i}$ for any other previous frame $\{K\}$, e.g., $\{K\} = \{0\}$.

  $\Rightarrow {}^K({}^K\dot{v}_{C_i}) = {}^K({}^K\dot{v}_i) + {}^KR_i\,\cancel{{}^i({}^i\dot{v}_{C_i})} + 2\,{}^K({}^K\omega_i) \times {}^KR_i\,\cancel{{}^i({}^iv_{C_i})}$

  $\quad + {}^K({}^K\dot{\omega}_i) \times {}^KR_i\,{}^iP_{C_i} + {}^K({}^K\omega_i) \times ({}^K({}^K\omega_i) \times {}^KR_i\,{}^iP_{C_i})$

  => ${}^K({}^K\dot{v}_{C_i}) = {}^K({}^K\dot{v}_i) + {}^K({}^K\dot{\omega}_i) \times {}^KR_i\,{}^iP_{C_i} + {}^K({}^K\omega_i) \times ({}^K({}^K\omega_i) \times {}^KR_i\,{}^iP_{C_i})$

  Pre-multiply by ${}^iR_K$ $\rightarrow$ ${}^i({}^K\dot{v}_{C_i}) = {}^i({}^K\dot{v}_i) + {}^i({}^K\dot{\omega}_i) \times {}^iP_{C_i} + {}^i({}^K\omega_i) \times ({}^i({}^K\omega_i) \times {}^iP_{C_i})$

  $\therefore\ {}^i\dot{v}_{C_i} = {}^i\dot{\omega}_i \times {}^iP_{C_i} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{C_i}) + {}^i\dot{v}_i$

<u>Newton-Euler Equations at Link *i*</u>

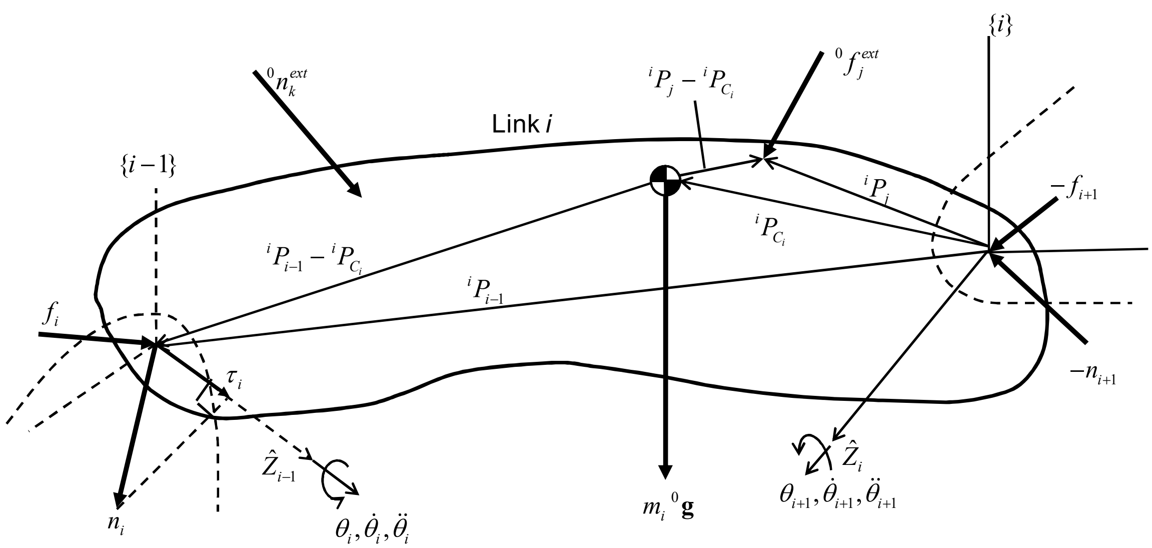
| Frame $\{C_i\}$ attached to link i with its origin at the link COM and <u>has same orientation as $\{i\}$</u>

- $\sum f = F_i = m\dot{v}_{C_i}$ $\qquad\qquad$ $\sum n = N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$

(Note: $\sum n = {}^iN_i = {}^{C_i}I\,{}^i\dot{\omega}_i + {}^i\omega_i \times {}^{C_i}I\,{}^i\omega_i$ is more complete expression, since ${}^{C_i}I$ is written in Frame $\{C_i\}$, and thus all vectors in the same equation should be written in Frame $\{i\}$ that moves with and has the same orientation as Frame $\{C_i\}$; see Craig's textbook equation (6.50).)

<u>Inward Iterations for Forces and Moments</u>

| f_i: force exerted ...... link i ...... link i-1
| n_i: moment exerted ...... link i ...... link i-1



- Force summation on link i: ${}^iF_i = {}^if_i - {}^iR_{i+1}\,{}^{i+1}f_{i+1} + m_i\,{}^iR_0\,{}^0\mathbf{g} + \sum_j {}^iR_0\,{}^0f_j^{ext}$
- Moment summation *about* COM on link i:
$${}^iN_{i,C_i} = {}^in_i - {}^in_{i+1} + ({}^iP_{i-1} - {}^iP_{c_i}) \times {}^if_i + (-{}^iP_{c_i}) \times (-{}^if_{i+1}) + \sum_k {}^iR_0\,{}^0n_k^{ext} + \sum_j [({}^iP_j - {}^iP_{c_i}) \times {}^iR_0\,{}^0f_j^{ext}]$$
$$\Rightarrow {}^iN_{i,C_i} = {}^in_i - {}^iR_{i+1}\,{}^{i+1}n_{i+1} + ({}^iP_{i-1} - {}^iP_{c_i}) \times {}^iF_i + {}^iP_{i-1} \times {}^iR_{i+1}\,{}^{i+1}f_{i+1} - ({}^iP_{i-1} - {}^iP_{c_i}) \times m_i\,{}^iR_0\,{}^0\mathbf{g}$$
$$+ \sum_j [({}^iP_j - {}^iP_{i-1}) \times {}^iR_0\,{}^0f_j^{ext}] + \sum_k {}^iR_0\,{}^0n_k^{ext}$$

- From higher to lower numbered:
$$\boxed{{}^if_i = {}^iR_{i+1}\,{}^{i+1}f_{i+1} - m_i\,{}^iR_0\,{}^0\mathbf{g} - \sum_j {}^iR_0\,{}^0f_j^{ext} + {}^iF_i}$$
$$\boxed{\begin{aligned}{}^in_i = {}^iR_{i+1}\,{}^{i+1}n_{i+1} - ({}^iP_{i-1} - {}^iP_{c_i}) \times {}^iF_i - {}^iP_{i-1} \times {}^iR_{i+1}\,{}^{i+1}f_{i+1} + ({}^iP_{i-1} - {}^iP_{c_i}) \times m_i\,{}^iR_0\,{}^0\mathbf{g} \\ - \sum_j [({}^iP_j - {}^iP_{i-1}) \times {}^iR_0\,{}^0f_j^{ext}] - \sum_k {}^iR_0\,{}^0n_k^{ext} + {}^iN_{i,C_i}\end{aligned}}$$
  Start from link n to link 0 (inward)
  ${}^{N+1}f_{N+1}$ and ${}^{N+1}n_{N+1}$ are zero or nonzero depending on contact
  (link n+1: environment or external object)

- Required actuation: $\tau_i = {}^in_i^T\,{}^i\hat{Z}_{i-1}$ (for revolute joint i) or $\tau_i = {}^if_i^T\,{}^i\hat{Z}_{i-1}$ (for prismatic joint i)

Iterative Newton-Euler Algorithm

Two parts:
1) Link velocities and accelerations from link 1 to n (outward) → F and N
2) Forces and moments from link n to 1 (inward) → $\boldsymbol{\tau}$

Closed Form Dynamics

- $\boldsymbol{\tau} = M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) - \sum_k J_k^T \begin{bmatrix} {}^0\mathbf{f}_k^{ext} \\ {}^0\mathbf{n}_k^{ext} \end{bmatrix} + \mathbf{F}(\mathbf{q},\dot{\mathbf{q}})$ (joint space equation)

  $\boldsymbol{\tau}$: nx1 actuator torque vector
  $\mathbf{q}$: nx1 joint variables vector
  $M(\mathbf{q})$: nxn mass matrix (symmetric, positive definite, invertible)
  $\mathbf{V}(\mathbf{q},\dot{\mathbf{q}})$: nx1 vector of centrifugal and Coriolis terms
  $\mathbf{G}(\mathbf{q})$: nx1 gravity vector (depends only on $\mathbf{q}$ and not on its derivatives)
  $\mathbf{F}(\mathbf{q},\dot{\mathbf{q}})$: nx1 friction (viscous and/or Coulomb) vector
  J_k: nxn Jacobian matrix of position of kth external load application
- Note: Workless constraint forces and moments (reactions) do not appear in the closed form equations. Only the actuator torques and forces are included.

Lagrangian Dynamics

Energy based, analytical dynamics (vs. force-balance based, vectorial Newton-Euler dynamics)
$\mathbf{q}$: nx1 joint variables vector
$\boldsymbol{\tau}$: nx1 actuator torque vector

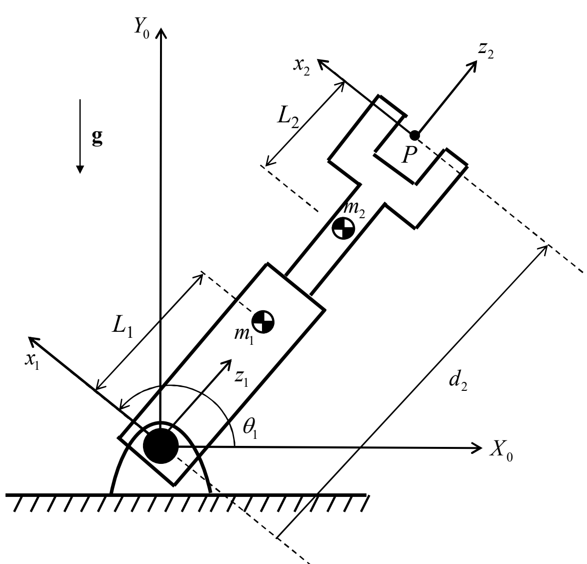
- Kinetic energy of link i: $k_i = \underbrace{\frac{1}{2} m_i v_{C_i}^T v_{C_i}}_{\substack{\text{due to link COM} \\ \text{linear velocity}}} + \underbrace{\frac{1}{2} {}^i\omega_i^T \, {}^{C_i}I_i \, {}^i\omega_i}_{\substack{\text{due to} \\ \text{link angular velocity}}}$
- Total kinetic energy of manipulator: $k = \sum_{i=1}^{n} k_i = k(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}}$ (quadratic form)
- Potential energy of link i: $u_i = -m_i \, {}^0\mathbf{g}^T \, {}^0P_{C_i} + u_{ref_i}$
  - ${}^0\mathbf{g}$: 3×1 gravity vector
  - ${}^0P_{C_i}$: position vector of link i COM
  - u_{ref_i} : reference constant
- Total potential energy of manipulator: $u = \sum_{i=1}^{n} u_i = u(\mathbf{q})$
- Lagrangian: $\boxed{L(\mathbf{q}, \dot{\mathbf{q}}) = k(\mathbf{q}, \dot{\mathbf{q}}) - u(\mathbf{q})}$ (scalar function)
- Lagrange's equations of motion: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau}}$ or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$ $(i = 1, 2, \ldots, n)$

Example 6.5 (with standard DH convention)

Inertia tensors:

$${}^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}, \quad {}^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

The center of mass of link 1 and 2 is located at a distance L_1 and L_2 from the origin of link frame {1} and {2}, respectively. Use Lagrangian dynamics to determine the equations of motion for this robot.



Solution)

Recall that frame $\{C_i\}$ has its origin at the link mass center, and has the same orientation as the link frame $\{i\}$. Thus I_{yyi} should be used for the rotational kinetic energy as seen from the attached link frames. (Note: the textbook uses I_{zzi}, which is not consistent with standard DH convention.)

$$k_1 = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{yy1} \dot{\theta}_1^2$$

$$k_2 = \frac{1}{2} m_2 [(d_2 - L_2)^2 \dot{\theta}_1^2 + \{\frac{d}{dt}(d_2 - L_2)\}^2] + \frac{1}{2} I_{yy2} \dot{\theta}_1^2 = \frac{1}{2} m_2 [(d_2 - L_2)^2 \dot{\theta}_1^2 + \dot{d}_2^{\,2}] + \frac{1}{2} I_{yy2} \dot{\theta}_1^2$$

$$\Rightarrow \quad k(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} [m_1 L_1^2 + I_{yy1} + I_{yy2} + m_2 (d_2 - L_2)^2] \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^{\,2}$$

$u_1 = m_1 g L_1 \sin(\theta_1 - \pi/2) = -m_1 g L_1 \cos\theta_1$ and $u_2 = m_2 g (d_2 - L_2) \sin(\theta_1 - \pi/2) = -m_2 g (d_2 - L_2) \cos\theta_1$

$$\Rightarrow \quad u(\mathbf{q}) = -g[m_1 L_1 + m_2 (d_2 - L_2)] \cos\theta_1$$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial k}{\partial \dot{\mathbf{q}}} = \begin{bmatrix} [m_1 L_1^2 + I_{yy1} + I_{yy2} + m_2 (d_2 - L_2)^2] \dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \begin{bmatrix} [m_1 L_1^2 + I_{yy1} + I_{yy2} + m_2 (d_2 - L_2)^2] \ddot{\theta}_1 + 2 m_2 (d_2 - L_2) \dot{d}_2 \dot{\theta}_1 \\ m_2 \ddot{d}_2 \end{bmatrix}$$

$$\frac{\partial L}{\partial \mathbf{q}} = \begin{bmatrix} -g[m_1 L_1 + m_2 (d_2 - L_2)] \sin\theta_1 \\ m_2 (d_2 - L_2) \dot{\theta}_1^2 + g m_2 \cos\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}}$$

$$\Downarrow$$

$$\tau_1 = [m_1 L_1^2 + I_{yy1} + I_{yy2} + m_2 (d_2 - L_2)^2] \ddot{\theta}_1 + 2 m_2 (d_2 - L_2) \dot{d}_2 \dot{\theta}_1 + [m_1 L_1 + m_2 (d_2 - L_2)] g \sin\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 (d_2 - L_2) \dot{\theta}_1^2 - m_2 g \cos\theta_1$$

Dynamic Simulation

| Forward dynamics

- Solve the dynamic equations (ODE) for acceleration:

$$\ddot{\mathbf{q}} = M^{-1}(\mathbf{q}) \left[\boldsymbol{\tau} - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{G}(\mathbf{q}) + \sum_k J_k^T \begin{bmatrix} {}^0\mathbf{f}_k^{ext} \\ {}^0\mathbf{n}_k^{ext} \end{bmatrix} - \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}) \right]$$

- Numerical integration with initial conditions: $\mathbf{q}(0) = \mathbf{q}_0$, $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$ → positions and velocities
- Euler's method (with time step Δt):

$$\dot{\mathbf{q}}(t + \Delta t) = \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}(t) \Delta t \; ; \; \mathbf{q}(t + \Delta t) = \mathbf{q}(t) + \dot{\mathbf{q}}(t) \Delta t + \frac{1}{2} \ddot{\mathbf{q}}(t) \Delta t^2$$