Chapter 2

$\boxed{{}^{A}\mathbf{P} = {}^{A}R_B\,{}^{B}\mathbf{P} + {}^{A}\mathbf{P}_{BORG}}$

$${}^{A}T_B = \left[\begin{array}{c:c} {}^{A}R_B & {}^{A}\mathbf{P}_{BORG} \\ \hdashline \mathbf{0}^T & 1 \end{array}\right]$$: **Homogeneous transform**

- Cayley's formula: $R = (I_3 - S)^{-1}(I_3 + S)$ (where S is a skew-symmetric matrix; $S = -S^T$)
- $S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$ → $\therefore$ R: 3 independent parameters

X-Y-Z Fixed Angle

$$\boxed{{}^{A}R_{BXYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)}$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Z-Y-X Euler Angle

$$\boxed{{}^{A}R_{BZ'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)}$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Angle-Axis

- Equivalent rotation matrix for ${}^{A}\hat{K} = [k_x \; k_y \; k_z]^T$

$$R_K(\theta) = {}^{A}R_B(\hat{K},\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_y k_x v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_z k_x v\theta - k_y s\theta & k_z k_y v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

(versed sine: $\text{versine}(\theta) = \text{vers}(\theta) = v\theta = 1 - c\theta$)

Rodriques' formula: $Q' = R_K(\theta)Q = Q\cos\theta + \sin\theta\left(\hat{K}\times Q\right) + \left(1-\cos\theta\right)\left(\hat{K}\bullet Q\right)\hat{K}$

Chapter 3

DH Table

Joint i	θ_i	d_i	a_i	α_i	Joint variable $\mathbf{q}$
Revolute	$\theta_i = \tilde{\theta}_i + q_i$	d_i	a_i	α_i	q_i
Prismatic	θ_i	$d_i = \tilde{d}_i + q_i$	a_i	α_i	q_i

$$\boxed{{}^{i-1}T_i = \left[\begin{array}{ccc|c} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}$$

Chapter 4

- Let $\boxed{u = \tan\frac{\theta}{2}}$ and substitute $\cos\theta = \frac{1-u^2}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$ (Weierstrass Substitution)

- Two-argument arctangent function $\phi = \operatorname{atan2}(y,x)$
  Defined on all four quadrants ($-\pi \le \phi < \pi$)

Case	Quadrants	$\phi = \operatorname{atan2}(y,x)$
$x > 0$	1, 4	$\phi = \arctan(y/x)$
$x = 0$	1, 4	$\phi = \underbrace{\operatorname{sgn}(y)}_{=\pm 1}(\pi/2)$
$x < 0$	2, 3	$\phi = \arctan(y/x) + \operatorname{sgn}(y)\cdot\pi$

Law of Cosines: $a^2 + b^2 - 2ab\cos C = c^2$

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Chapter 5

$$ {}^{B}\mathbf{V}_{Q} = \frac{d}{dt}\,{}^{B}\mathbf{Q} = \lim_{\Delta t \to 0} \frac{{}^{B}\mathbf{Q}(t+\Delta t) - {}^{B}\mathbf{Q}(t)}{\Delta t} = {}^{B}({}^{B}\mathbf{V}_{Q}) $$

$$ \boxed{{}^{A}\mathbf{V}_{Q} = {}^{A}\mathbf{V}_{BORG} + {}^{A}R_{B}\,{}^{B}\mathbf{V}_{Q} + {}^{A}\mathbf{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}\mathbf{Q}} $$

$$ \boxed{S = \dot{R}R^{T} = \dot{R}R^{-1}} $$

$$ \boxed{\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{\mathbf{K}}} $$

$$ \mathbf{\Omega} = E_{Z'Y'Z'}(\mathbf{\Theta}_{Z'Y'Z'})\dot{\mathbf{\Theta}}_{Z'Y'Z'} $$

$$ E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix} $$

Revolute:

$$ \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_{i} = {}^{i}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ \boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}R_{i}\,({}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_{i})} $$

$$ \boxed{{}^{i+1}\upsilon_{i+1} = {}^{i+1}R_{i}\,({}^{i}\upsilon_{i} + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1})} $$

Prismatic:

$$ \boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}R_{i}\,{}^{i}\omega_{i}} $$

$$ \boxed{{}^{i+1}\upsilon_{i+1} = {}^{i+1}R_{i}\,({}^{i}\upsilon_{i} + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1} + \dot{d}_{i+1}\,{}^{i}\hat{Z}_{i})} $$

$$ \boxed{\omega_{n} = \sum_{i=1}^{n} \dot{\theta}_{i}\hat{Z}_{i-1}} $$

$$ \boxed{\upsilon_{n} = \sum_{i=1}^{n} [\dot{\theta}_{i}\hat{Z}_{i-1} \times (P_{n} - P_{i-1}) + \dot{d}_{i}\hat{Z}_{i-1}]} $$

$$ {}^{A}J(\mathbf{q}) = \left[\begin{array}{c:c} {}^{A}R_{B} & 0 \\ \hdashline 0 & {}^{A}R_{B} \end{array}\right] {}^{B}J(\mathbf{q}) $$

$$ \boxed{J(\mathbf{X}) = \frac{\partial \mathbf{F}_{(m\times 1)}}{\partial \mathbf{X}_{(n\times 1)}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{(m\times n)}} $$

$$ \boxed{\dot{\mathbf{Y}} = J(\mathbf{X})\dot{\mathbf{X}}} $$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

$$\boxed{J_i(\mathbf{q})_{(6\times1)} = \begin{bmatrix} J_{P,i}(\mathbf{q})_{(3\times1)} \\ \hdashline J_{O,i}(\mathbf{q})_{(3\times1)} \end{bmatrix} = \begin{cases} \begin{bmatrix} \hat{Z}_{i-1} \\ \hdashline \mathbf{0} \end{bmatrix} & \leftarrow \text{Prismatic joint } i \\ \begin{bmatrix} \hat{Z}_{i-1} \times (P_n - P_{i-1}) \\ \hdashline \hat{Z}_{i-1} \end{bmatrix} & \leftarrow \text{Revolute joint } i \end{cases}}$$

$$\boxed{{}^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1},\ {}^{i}n_i = {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} - {}^{i}P_{i-1} \times {}^{i}f_i} \qquad \boxed{\boldsymbol{\tau} = J^T \mathbf{F}}$$

Chapter 6

$${}^{A}\dot{\mathbf{\Omega}}_B = \frac{d}{dt}\,{}^{A}\mathbf{\Omega}_B = \lim_{\Delta t \to 0} \frac{{}^{A}\mathbf{\Omega}_B(t+\Delta t) - {}^{A}\mathbf{\Omega}_B(t)}{\Delta t}$$

$$\boxed{{}^{A}\dot{\mathbf{V}}_Q = {}^{A}\dot{\mathbf{V}}_{BORG} + {}^{A}R_B\,{}^{B}\dot{\mathbf{V}}_Q + 2\,{}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{V}_Q + {}^{A}\dot{\mathbf{\Omega}}_B \times {}^{A}R_B\,{}^{B}\mathbf{Q} + {}^{A}\mathbf{\Omega}_B \times ({}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{Q})}$$

$$\boxed{{}^{A}\dot{\mathbf{\Omega}}_C = {}^{A}\dot{\mathbf{\Omega}}_B + {}^{A}R_B\,{}^{B}\dot{\mathbf{\Omega}}_C + {}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{\Omega}_C}$$

$${}^{A}I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where

- Mass moments of inertia (> 0): $I_{xx} = \int_V (y^2 + z^2)\rho dv$, $I_{yy} = \int_V (z^2 + x^2)\rho dv$, $I_{zz} = \int_V (x^2 + y^2)\rho dv$

- Mass products of inertia (>, =, or < 0): $I_{xy} = -\int_V xy\rho dv$, $I_{yz} = -\int_V yz\rho dv$, $I_{zx} = -\int_V zx\rho dv$

${}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$, …… ${}^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$, ……

In vector-matrix form: ${}^{A}I = {}^{C}I + m[\mathbf{P}_c^T \mathbf{P}_c I_3 - \mathbf{P}_c \mathbf{P}_c^T]$

Revolute: $\boxed{{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i({}^{i}\dot{\omega}_i + \dot{\theta}_{i+1}\,{}^{i}\omega_i \times {}^{i}\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^{i}\hat{Z}_i)}$

$$\boxed{{}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1})}$$

Prismatic: $\boxed{{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{\omega}_i}$

$$\boxed{{}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^{i}\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}\hat{Z}_i}$$

$$\boxed{{}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{C_i}) + {}^{i}\dot{v}_i}$$

$$\sum f = F_i = m\dot{v}_{C_i} \qquad \sum n = N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$$

$$\boxed{{}^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1} - m_i\,{}^{i}R_0\,{}^{0}\mathbf{g} - \sum_j {}^{i}R_0\,{}^{0}f_j^{ext} + {}^{i}F_i}$$

$$\boxed{\begin{aligned}{}^{i}n_i = {}&{}^{i}R_{i+1}\,{}^{i+1}n_{i+1} - ({}^{i}P_{i-1} - {}^{i}P_{c_i}) \times {}^{i}F_i - {}^{i}P_{i-1} \times {}^{i}R_{i+1}\,{}^{i+1}f_{i+1} + ({}^{i}P_{i-1} - {}^{i}P_{c_i}) \times m_i\,{}^{i}R_0\,{}^{0}\mathbf{g} \\ &- \sum_j [({}^{i}P_j - {}^{i}P_{i-1}) \times {}^{i}R_0\,{}^{0}f_j^{ext}] - \sum_k {}^{i}R_0\,{}^{0}n_k^{ext} + {}^{i}N_{i,C_i}\end{aligned}}$$

$$k_i = \underbrace{\frac{1}{2}m_i v_{C_i}^T v_{C_i}} + \underbrace{\frac{1}{2}\,{}^{i}\omega_i^T\,{}^{C_i}I_i\,{}^{i}\omega_i} \qquad k = \sum_{i=1}^{n} k_i = k(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}}$$

$$u_i = -m_i\,{}^{0}\mathbf{g}^T\,{}^{0}P_{C_i} + u_{ref_i} \qquad u = \sum_{i=1}^{n} u_i = u(\mathbf{q})$$

$$\boxed{L(\mathbf{q},\dot{\mathbf{q}}) = k(\mathbf{q},\dot{\mathbf{q}}) - u(\mathbf{q})} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau}}$$

$$\boldsymbol{\tau} = M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) - \sum_k J_k^T \begin{bmatrix} {}^{0}\mathbf{f}_k^{ext} \\ {}^{0}\mathbf{n}_k^{ext} \end{bmatrix} + \mathbf{F}(\mathbf{q},\dot{\mathbf{q}})$$

$$\ddot{\mathbf{q}} = M^{-1}(\mathbf{q})\left[\boldsymbol{\tau} - \mathbf{V}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{G}(\mathbf{q}) + \sum_k J_k^T \begin{bmatrix} {}^{0}\mathbf{f}_k^{ext} \\ {}^{0}\mathbf{n}_k^{ext} \end{bmatrix} - \mathbf{F}(\mathbf{q},\dot{\mathbf{q}})\right]$$