Mathematics for Robotics (ROB-GY 6013 Section A)

- **Week 7:**
  - Norm(s)
  - Inner Product(s)

Review Complex Numbers

Inspiration: Length of a Vector

- Pythagorean Theorem

$$x \in (\mathbb{R}^2, \mathbb{R}) \longrightarrow \sqrt{x_1^2 + x_2^2}$$

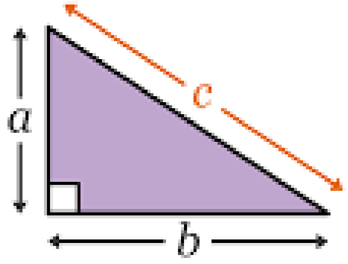
vector in 2-D space **length**

Inspiration: Length of a Vector

- Pythagorean Theorem

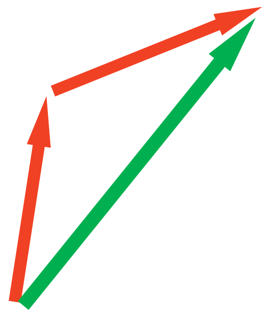
$$x \in (\mathbb{R}^2, \mathbb{R}) \longrightarrow \sqrt{x_1^2 + x_2^2}$$

vector in 2-D space **length**



$a^2 + b^2 = c^2$

- Properties of the length:
  a) Length is non-negative *and* only zero when x is zero.
  b) Length of the **sum of two vectors** ≤ sum of the lengths of the **two vectors**
  c) Scaling the vector by α also scales its length by α



Definition: Norm

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space where the field $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

  A **function** $\| \cdot \|: \mathcal{X} \to \mathbb{R}$ is a **norm** if it satisfies:

  a) **Non-negativity:** $\|x\| \geq 0, \forall x \in \mathcal{X}$ *and* $\|x\| = 0 \iff x = 0$

  b) **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathcal{X}$

  c) **Scaling:** $\|\alpha x\| = |\alpha| \cdot \|x\|, \forall x \in \mathcal{X}, \alpha \in \mathcal{F}$

     If $\alpha \in \mathbb{R}$, $|\alpha|$ means the absolute value
     If $\alpha \in \mathbb{C}$, $|\alpha|$ means the magnitude (modulus) $z \cdot \overline{z} = |z|^2$

Definition: Norm

- Let $(\mathcal{X},\mathcal{F})$ be a vector space where the field $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

  A **function** $\|\cdot\|: \mathcal{X} \to \mathbb{R}$ is a **norm** if it satisfies:

  a) **Non-negativity:** $\|x\| \geq 0, \forall x \in \mathcal{X}$ *and* $\|x\| = 0 \iff x = 0$

  b) **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathcal{X}$

  c) **Scaling:** $\|\alpha x\| = |\alpha| \cdot \|x\|, \forall x \in \mathcal{X}, \alpha \in \mathcal{F}$

     If $\alpha \in \mathbb{R}$, $|\alpha|$ means the absolute value
     If $\alpha \in \mathbb{C}$, $|\alpha|$ means the magnitude (modulus) $z \cdot \overline{z} = |z|^2$

- Is the Pythagorean Length a valid norm according to the above properties?
- Are there other norms?

Examples: Norms for n-tuples

- Given the vector space $(\mathcal{F}^n, \mathcal{F})$, where $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- Possible norms that satisfy our definition

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

Euclidean norm or 2-**norm** extends Pythagorean length to n-dimensional space

If $x_i \in \mathbb{R}$, $|x_i|$ means the absolute value
If $x_i \in \mathbb{C}$, $|x_i|$ means the magnitude (modulus)

Examples: Norms for n-tuples

- Given the vector space $(\mathcal{F}^n, \mathcal{F})$, where $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- Possible norms that satisfy our definition

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

Euclidean norm or 2-**norm** extends Pythagorean length to n-dimensional space

If $x_i \in \mathbb{R}$, $|x_i|$ means the absolute value
If $x_i \in \mathbb{C}$, $|x_i|$ means the magnitude (modulus)

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, p-**norm**

Examples: Norms for n-tuples

- Given the vector space $(\mathcal{F}^n, \mathcal{F})$, where $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- Possible norms that satisfy our definition

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

Euclidean norm or 2-**norm** extends Pythagorean length to n-dimensional space

If $x_i \in \mathbb{R}$, $|x_i|$ means the absolute value
If $x_i \in \mathbb{C}$, $|x_i|$ means the magnitude (modulus)

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, p-**norm**

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$$

max-norm, sup-norm or ∞-**norm**

Examples: Norms for functions

- Given the vector space $(\mathcal{X}, \mathcal{F})$, where $\mathcal{F}$ is $\mathbb{R}$, $\mathcal{D} \subset \mathbb{R}$, $\mathcal{D} := [a, b]$, $-\infty < a < b < \infty$, and $\mathcal{X} := \{f : \mathcal{D} \to \mathbb{R} \mid f \text{ is continuous}\}$
- Possible norms that satisfy our definition:
  - **Takes a function as input and gives a real value as output**

Examples: Norms for functions

- Given the vector space $(\mathcal{X},\mathcal{F})$, where $\mathcal{F}$ is $\mathbb{R}$, $\mathcal{D} \subset \mathbb{R}$, $\mathcal{D} := [a, b]$, $-\infty < a < b < \infty$, and $\mathcal{X} := \{f : \mathcal{D} \to \mathbb{R} \mid f \text{ is continuous}\}$
- Possible norms that satisfy our definition:
  - **Takes a function as input and gives a real value as output**

$$\|f\|_2 := \left(\int_a^b |f(t)|^2 \, dt \right)^{\frac{1}{2}}$$

Examples: Norms for functions

- Given the vector space $(\mathcal{X},\mathcal{F})$, where $\mathcal{F}$ is $\mathbb{R}$, $\mathcal{D} \subset \mathbb{R}$, $\mathcal{D} := [a, b]$, $-\infty < a < b < \infty$, and $\mathcal{X} := \{f : \mathcal{D} \to \mathbb{R} \mid f \text{ is continuous}\}$
- Possible norms that satisfy our definition:
  - **Takes a function as input and gives a real value as output**

$$\|f\|_2 := \left(\int_a^b |f(t)|^2 \, dt \right)^{\frac{1}{2}}$$

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\|f\|_\infty := \max_{a \le t \le b} |f(t)|$$ also written as $$\|f\|_\infty := \sup_{a \le t \le b} |f(t)|$$

Examples: Norms for functions

- Given the vector space $(\mathcal{X},\mathcal{F})$, where $\mathcal{F}$ is $\mathbb{R}$, $\boxed{\mathcal{D} \subset \mathbb{R},\ \mathcal{D} := [a, b],\ -\infty < a < b < \infty,}$ and $\mathcal{X} := \{f : \mathcal{D} \to \mathbb{R} \mid f \text{ is continuous}\}$

Closed interval avoids improper integrals, etc.

- Possible norms that satisfy our definition:
  - **Takes a function as input and gives a real value as output**

$$\|f\|_2 := \left(\int_a^b |f(t)|^2 \, dt \right)^{\frac{1}{2}}$$

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\|f\|_\infty := \max_{a \le t \le b} |f(t)|$$ also written as $$\|f\|_\infty := \sup_{a \le t \le b} |f(t)|$$

Definition: Normed Space

- $(\mathcal{X}, \mathcal{F}, \|\cdot\|)$ is called a **normed space**.

Definition: Normed Space

- $(\mathcal{X}, \mathcal{F}, \|\cdot\|)$ is called a **normed space**.
- Often we will say normed vector space is a vector space ***equipped*** with a norm

More Definitions: Normed Space

- $(\mathcal{X}, \mathcal{F}, \|\cdot\|)$ is called a **normed space**.
- Often we will say normed vector space is a vector space ***equipped*** with a norm

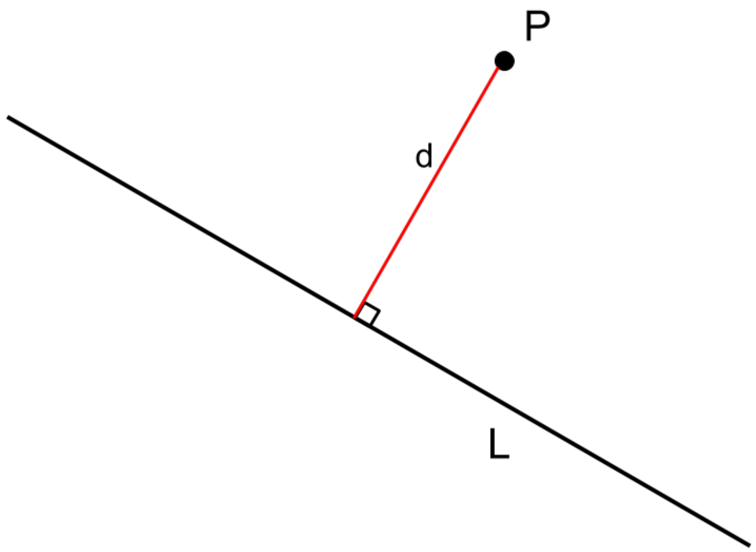
- For $x, y \in \mathcal{X}$, $d(x, y) := \|x - y\|$ is called the **distance from x to y**.

More Definitions: Normed Space

- $(\mathcal{X}, \mathcal{F}, \|\cdot\|)$ is called a **normed space**.
- Often we will say normed vector space is a vector space ***equipped*** with a norm

- For $x, y \in \mathcal{X}$, $d(x, y) := \|x - y\|$ is called the **distance from x to y**.

- How about distance from x **to some subset of $\mathcal{X}$** ?
  - Think about the distance from a point to a line.

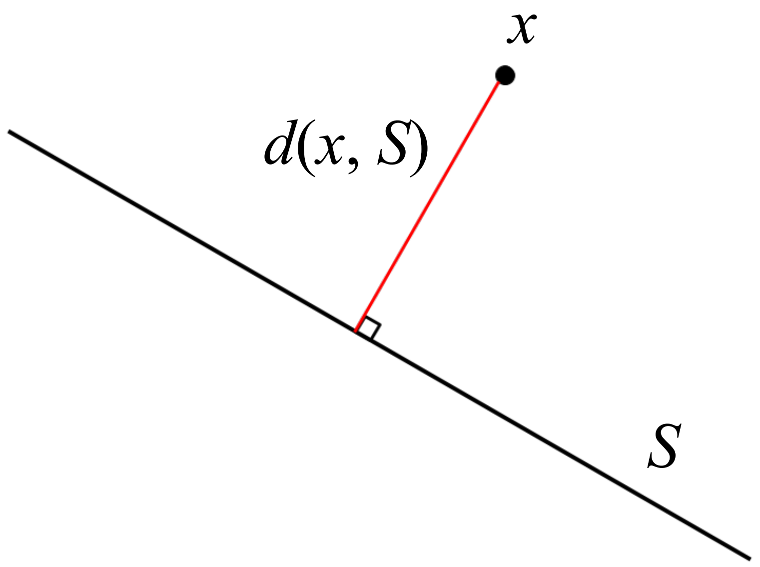


Distance to a set

- Let $S \subset \mathcal{X}$ be a subset.

$$d(x, S) := \inf_{y \in S} \|x - y\|$$

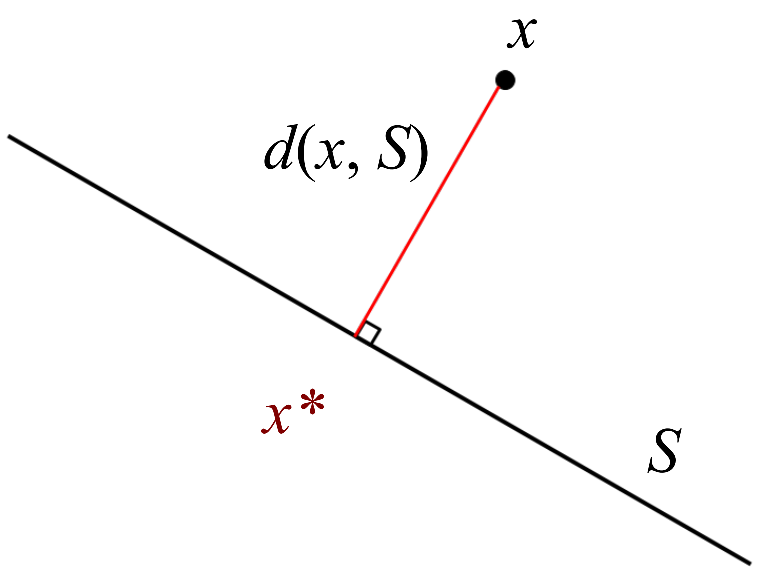
- Recall infimum is the greatest lower bound.



Distance to a set

- Let $S \subset X$ be a subset.

$$d(x,S) := \inf_{y \in S} \|x - y\|$$

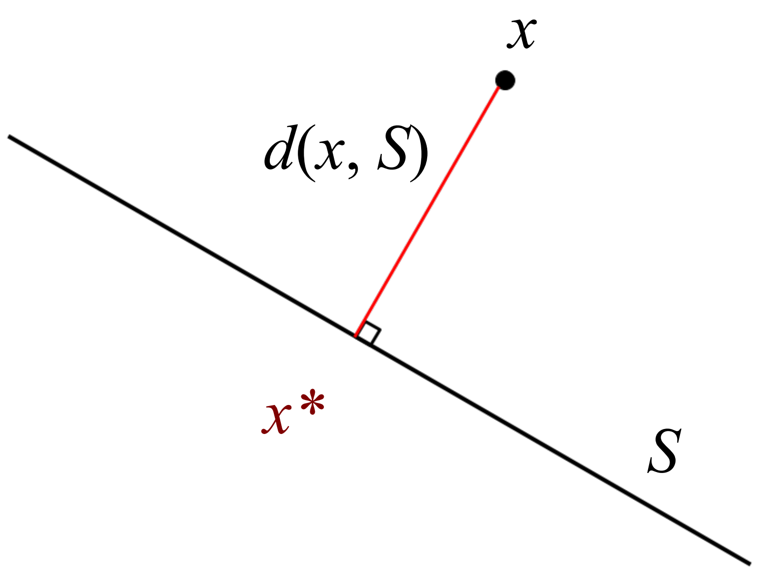


- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$?

Distance to a set

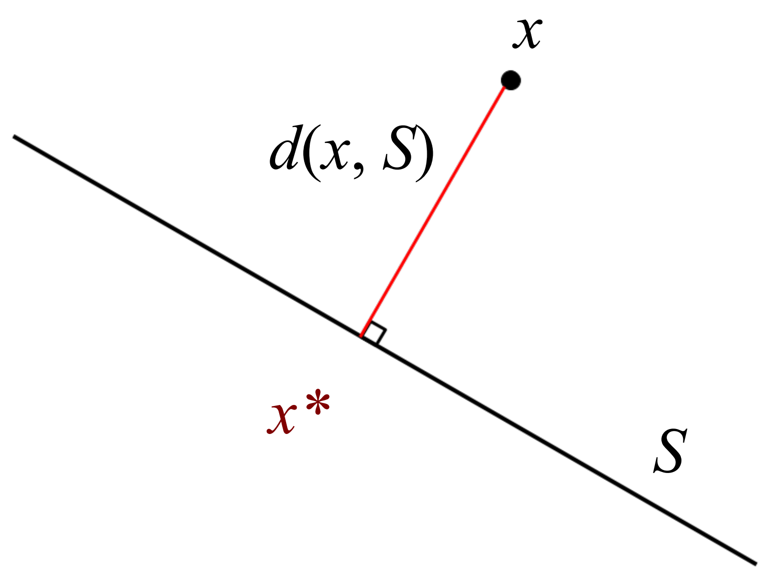
- Let $S \subset X$ be a subset.

$$d(x,S) := \inf_{y \in S} \|x - y\|$$



- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$? **(Existence)**

Distance to a set



- Let $S \subset X$ be a subset.

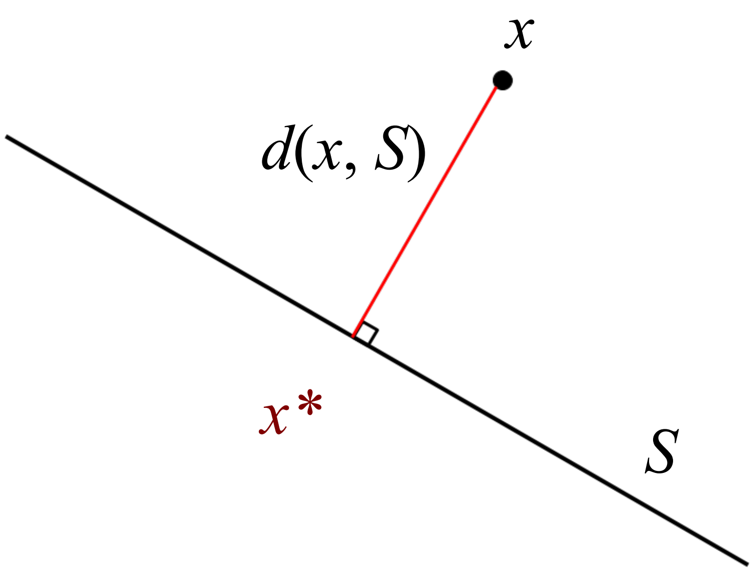
$$d(x,S) := \inf_{y \in S} \|x - y\|$$

- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$? **(Existence)**
  - Then x^* is the **best approximation of** x **by elements in** S.

Distance to a set

- Let $S \subset X$ be a subset.

$$d(x,S) := \inf_{y \in S} \|x - y\|$$

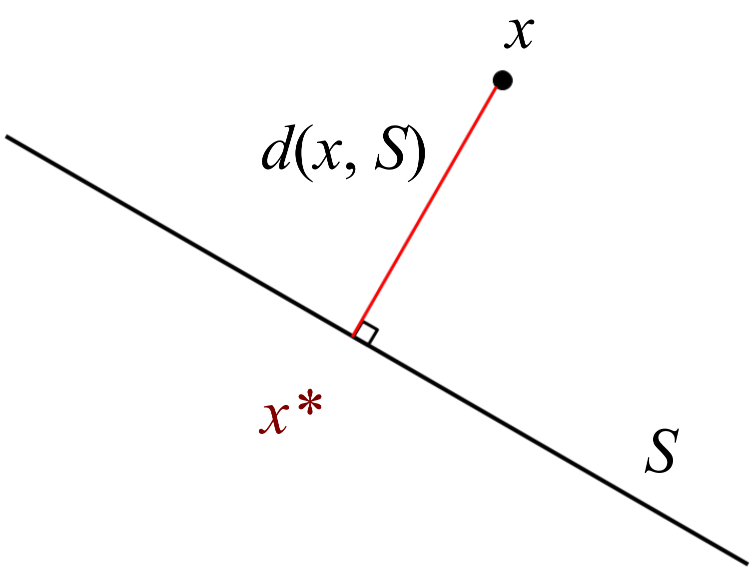


- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$? **(Existence)**
  - Then x^* is the **best approximation of** x **by elements in** S.
    - Is x^* unique? **(Uniqueness)**

Distance to a set

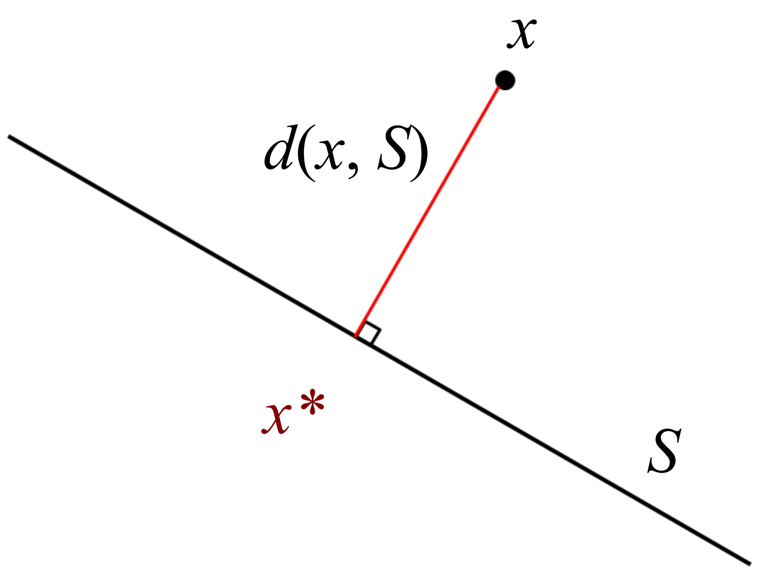
- Let $S \subset X$ be a subset.

$$d(x,S) := \inf_{y \in S} \|x - y\|$$



- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$? **(Existence)**
  - Then x^* is the **best approximation of x by elements in S.**
    - Is x^* unique? **(Uniqueness)**
      - Then you may write $x^* := \arg\min_{y \in S} \|x - y\|$

Distance to a set



- Let $S \subset X$ be a subset.

$$d(x,S) := \inf_{y \in S} \|x - y\|$$

- Recall infimum is the greatest lower bound.
- Can we find x^* such that $d(x,S) = \|x - x^*\|$? **(Existence)**
    - Then x^* is the **best approximation of x by elements in S.**
        - Is x^* unique? **(Uniqueness)**
            - Then you may write $x^* := \arg\min_{y \in S} \|x - y\|$

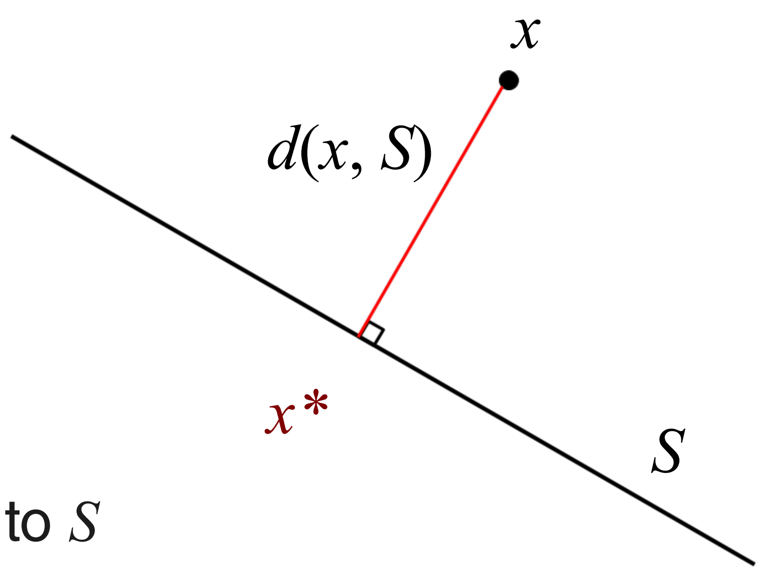
Ideas

- **Approximation:** Least-squares fitting, etc.

$$x^* := \arg\min_{y \in S} \|x - y\|_2^2$$

- **Orthogonality:** $(x^* - x)$ is a vector perpendicular to S

$$(x^* - x) \perp S$$



*Often S is not just any subset, but a **subspace***

Inspiration: Dot Product

- Familiar to you $x, y \in \mathbb{R}^n, \quad x \cdot y = \sum_{i=1}^{n} x_i y_i$

- Possibly less familiar $x^T y = \sum_{i=1}^{n} x_i y_i$

- Properties:

  a) **Commutativity:** $x \cdot y = y \cdot x$

  b) **Linearity:** $(\alpha_1 x + \alpha_2 x_2) \cdot y = \alpha_1 (x_1 \cdot y) + \alpha_2 (x_2 \cdot y)$

  c) **Non-negativity:** $x \cdot x \geq 0$ for all x, *and* $x \cdot x = 0$ when x is zero

  Extra: $x \cdot y = 0$ means **orthogonality** $x \perp y$

Definition: Inner Product (Real)

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space where $\mathcal{F} = \mathbb{R}$.

  A **function** $\langle \cdot \, , \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is an **inner product** if:

  a) **Commutativity:** $\langle x, y \rangle = \langle y, x \rangle, \quad \forall\, x, y \in \mathcal{X}$

  b) **Linearity:** $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle, \quad \forall\, x_1, x_2 \in \mathcal{X}, \forall\, \alpha_1, \alpha_2 \in \mathbb{R}$

  c) **Non-negativity:** $\langle x, x \rangle \geq 0 \quad \forall\, x \in \mathcal{X}$

  and $\langle x, x \rangle = 0 \iff x = 0.$

Definition: Inner Product (Complex)

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space where $\mathcal{F} = \mathbb{C}$.

  A **function** $\langle \cdot , \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is an **inner product** if:

  a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\quad \forall\, x, y \in \mathcal{X}$

  b) Linearity: $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$, $\quad \forall\, x_1, x_2 \in \mathcal{X}, \forall\, \alpha_1, \alpha_2 \in \mathbb{C}$

  c) Non-negativity: $\langle x, x \rangle \geq 0$ $\quad \forall\, x \in \mathcal{X}$

  and $\langle x, x \rangle = 0 \iff x = 0$.

  $\langle x, x \rangle$ **is always real!**

Definition: Inner Product (Complex)

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space where $\mathcal{F} = \mathbb{C}$.

  A **function** $\langle \cdot , \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is an **inner product** if:

  a) $\langle x, y \rangle = \boxed{\overline{\langle y, x \rangle},}$ $\quad \forall\, x, y \in \mathcal{X}$

  b) Linearity: $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle,$ $\quad \forall\, x_1, x_2 \in \mathcal{X}, \forall\, \alpha_1, \alpha_2 \in \mathbb{C}$

  c) Non-negativity: $\boxed{\langle x, x \rangle \geq 0}$ $\quad \forall\, x \in \mathcal{X}$

  and $\langle x, x \rangle = 0 \iff x = 0.$

  $\langle x, x \rangle$ **is always real!**

Definition: Inner Product (Complex)

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space where $\mathcal{F} = \mathbb{C}$.

  A **function** $\langle \cdot \, , \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is an **inner product** if:

  a) $\langle x, y \rangle = \boxed{\overline{\langle y, x \rangle},}$ $\quad \forall\, x, y \in \mathcal{X}$

  b) Linearity: $\boxed{\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle} = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle, \quad \forall\, x_1, x_2 \in \mathcal{X}, \forall\, \alpha_1, \alpha_2 \in \mathbb{C}$

$$
\begin{aligned}
\langle x, \beta_1 y_1 + \beta_2 y_2 \rangle &= \overline{\langle \beta_1 y_1 + \beta_2 y_2, x \rangle} \\
&= \overline{\beta_1 \langle y_1, x \rangle + \beta_2 \langle y_2, x \rangle} \\
&= \overline{\beta_1} \; \overline{\langle y_1, x \rangle} + \overline{\beta_2} \; \overline{\langle y_2, x \rangle} \\
&= \overline{\beta_1} \; \langle x, y_1 \rangle + \overline{\beta_2} \; \langle x, y_2 \rangle
\end{aligned}
$$

Examples: Inner Products

- $(\mathbb{C}^n, \mathbb{C})$ $\quad \langle x, y \rangle = x^T \overline{y}$
- $(\mathbb{R}^n, \mathbb{R})$ $\quad \langle x, y \rangle = x^T y$
- $(\mathbb{R}^{n\times m}, \mathbb{R})$ $\quad \langle A, B \rangle = \mathrm{tr}(A^T B)$
- $(\mathcal{X}, \mathbb{R})$ $\quad \langle f, g \rangle = \int_a^b f(t)g(t)dt$

Given the vector space $(\mathcal{X}, \mathbb{R})$, where $\mathcal{D} \subset \mathbb{R}$, $\mathcal{D} := [a, b]$, $-\infty < a < b < \infty$, and $\mathcal{X} := \{f : \mathcal{D} \to \mathbb{R} \mid f \text{ is continuous}\}$

Definition: Inner Product Space

- $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ is called an **inner product space**.

Definition: Inner Product Space

- $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ is called an **inner product space**.

- Two vectors x and y are **orthogonal** if $\langle x, y \rangle = 0$. **Notation:** $x \perp y$

Definition: Inner Product Space

- $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ is called an **inner product space**.

- Two vectors x and y are **orthogonal** if $\langle x, y \rangle = 0$. **Notation:** $x \perp y$
- We would like to make a connection between the inner product and the norm.
- In fact, we can create a **norm** of a vector with the **inner product** of the vector with itself:

$$\| x \| := \langle x, x \rangle^{1/2} = \sqrt{\langle x, x \rangle}$$

- We will verify that the above is a norm.
  - Already satisfies **non-negativity** and easy to show **scaling**.
  - Harder to show the triangle inequality.

Cauchy-Schwarz Inequality

- Let $(\mathcal{X}, \mathbb{R}, \langle \cdot , \cdot \rangle)$ be an **inner product space**, with $\mathcal{F}$ either $\mathbb{R}$ or $\mathbb{C}$. Then, for all $x, y \in \mathcal{X}$

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$$

- Thm 3.14 in main text. See proof.

Cauchy-Schwarz Inequality

- Let $(\mathcal{X}, \mathbb{R}, \langle \cdot , \cdot \rangle)$ be an **inner product space**, with $\mathcal{F}$ either $\mathbb{R}$ or $\mathbb{C}$. Then, for all $x, y \in \mathcal{X}$

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$$

- Thm 3.14 in main text. See proof.
- Stepping stone to **triangle equality**.
- Show that every inner product ***induces*** a norm.

$$\| x \| := \langle x, x\rangle^{1/2} = \sqrt{\langle x, x\rangle}$$

- But not every norm gives rise to an inner product!

Triangle Inequality

- $\forall x, y \in \mathcal{X}$, does $\| x \| := \langle x, x \rangle^{1/2} = \sqrt{\langle x, x \rangle}$ satisfy $\|x + y\| \leq \|x\| + \|y\|$?

Proof: Triangle Inequality (Real)

Corollary 3.15 *Let* $(\mathcal{X}, \mathcal{F}, \langle \cdot\, , \cdot \rangle)$ *be an inner product space, with* $\mathcal{F}$ *either* $\mathbb{R}$ *or* $\mathbb{C}$*. Then,*

$$\|x\| := \langle x, x\rangle^{1/2} = \sqrt{\langle x, x\rangle}$$

*is a **norm**.*

Proof: As before, for clarity of exposition, we first assume $\mathcal{F} = \mathbb{R}$. We will only check the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$, which is equivalent to showing $\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$. The other parts are left as an exercise.

$$\begin{aligned}
\|x + y\|^2 &:= \langle x + y, x + y\rangle \\
&= \langle x, x + y\rangle + \langle y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \\
&\leq \|x\|^2 + \|y\|^2 + 2\,|\langle x, y\rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|
\end{aligned}$$

where the last step uses the Cauchy-Schwarz inequality.

Proof: Triangle Inequality (Complex)

We'll now quickly do the changes required to handle $\mathcal{F} = \mathbb{C}$. The triangle inequality is $\|x+y\| \leqslant \|x\| + \|y\|$, which is equivalent to showing $\|x+y\|^2 \leqslant \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$. Brute force computation yields,

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle \\
&= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} \\
&= \overline{\langle x, x\rangle + \langle y, x\rangle} + \overline{\langle x, y\rangle + \langle y, y\rangle} \\
&= \langle x, x\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\{\langle x, y\rangle\}
\end{aligned}$$

where $\mathrm{Re}\{\langle x, y\rangle\}$ denotes the real part of the complex number $\langle x, y\rangle$. However, for any complex number α, $\mathrm{Re}\{\alpha\} \leq |\alpha|$, and thus we have

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\{\langle x, y\rangle\} \\
&\leqslant \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\
&\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|,
\end{aligned}$$

where the last inequality is from the Cauchy-Schwarz Inequality.

$\blacksquare$

Definition: Orthogonal and Orthonormal vectors

- Two vectors x and y are **orthogonal** if $\langle x, y\rangle = 0$. **Notation:** $x \perp y$

- A set of vectors S is **orthogonal** if

$$\forall x,y \in S, x \neq y \Rightarrow \langle x, y\rangle = 0 \text{ (i.e. } x \perp y)$$

If in addition, $\|x\| = 1$ for all $x \in S$, then S is an **orthonormal set**.

Orthogonal Bases with Inner Products

- Use the inner product to construct an orthonormal basis out a set of linearly independent vectors:
    - **Gram-Schmidt Process**
- **Orthogonal Polynomial Bases**

Gram Schmidt Process

- Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be an **inner product space**, $\{y^1, \dots, y^k\}$ a **linearly independent set** and $\{v^1, \dots, v^{k-1}\}$ an **orthogonal set** satisfying

$$\operatorname{span}\{v^1, \dots, v^{k-1}\} = \operatorname{span}\{y^1, \dots, y^{k-1}\}$$

- Define

$$v^k = y^k - \sum_{i=1}^{k-1} \frac{\langle y^k, v^j \rangle}{\| v^j \|^2} \cdot v^j$$

- where $\|v^j\|^2 = \langle v^j, v^j \rangle$. Then $\{v^1, \dots, v^k\}$ is **orthogonal** and

$$\operatorname{span}\{v^1, \dots, v^k\} = \operatorname{span}\{y^1, \dots, y^k\}$$

Gram Schmidt Process

- Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be an **inner product space**, $\{y^1, \ldots, y^k\}$ a **linearly independent set** and $\{v^1, \ldots, v^{k-1}\}$ an **orthogonal set** satisfying

$$\operatorname{span}\{v^1, \ldots, v^{k-1}\} = \operatorname{span}\{y^1, \ldots, y^{k-1}\}$$

- Define

$$v^k = y^k - \sum_{i=1}^{k-1} \frac{\langle y^k, v^j \rangle}{\| v^j \|^2} \cdot v^j$$

- where $\|v^j\|^2 = \langle v^j, v^j \rangle$. Then $\{v^1, \ldots, v^k\}$ is **orthogonal** and

$$\operatorname{span}\{v^1, \ldots, v^k\} = \operatorname{span}\{y^1, \ldots, y^k\}$$

- This is a recipe for "growing" an orthogonal set out of a linearly independent set of vectors

Gram Schmidt Process: Example

- Orthogonalize then normalize

Example 3.21 *Given the following data in* $(\mathbb{R}^3, \mathbb{R})$,

$$\{y^1, y^2, y^3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

and inner product $< p, q >:= p^T q = \sum_{i=1}^{3} p_i q_i$, *apply Gram-Schmidt to produce an orthogonal basis. Normalize to produce an orthonormal basis.*

Gram Schmidt Process: Example

$$v^1 = y^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|v^1\|^2 = (v^1)^T v^1 = 2;$$

$$v^2 = y^2 - \frac{< v^1, y^2 >}{\|v^1\|^2} v^1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$\|v^2\|^2 = 9\frac{1}{2} = \frac{19}{2};$$

Gram Schmidt Process: Example

$$v^3 = y^3 - \frac{< v^1, y^3 >}{\|v^1\|^2} v^1 - \frac{< v^2, y^3 >}{\|v^2\|^2} v^2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{1} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{3\frac{1}{2}} \frac{1}{\frac{19}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{7}{38} \\ \frac{7}{38} \\ \frac{21}{19} \end{bmatrix} = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}.$$

Gram Schmidt Process: Example

- Normalize at the end

$$\tilde{v}_1 = \frac{v^1}{\|v^1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

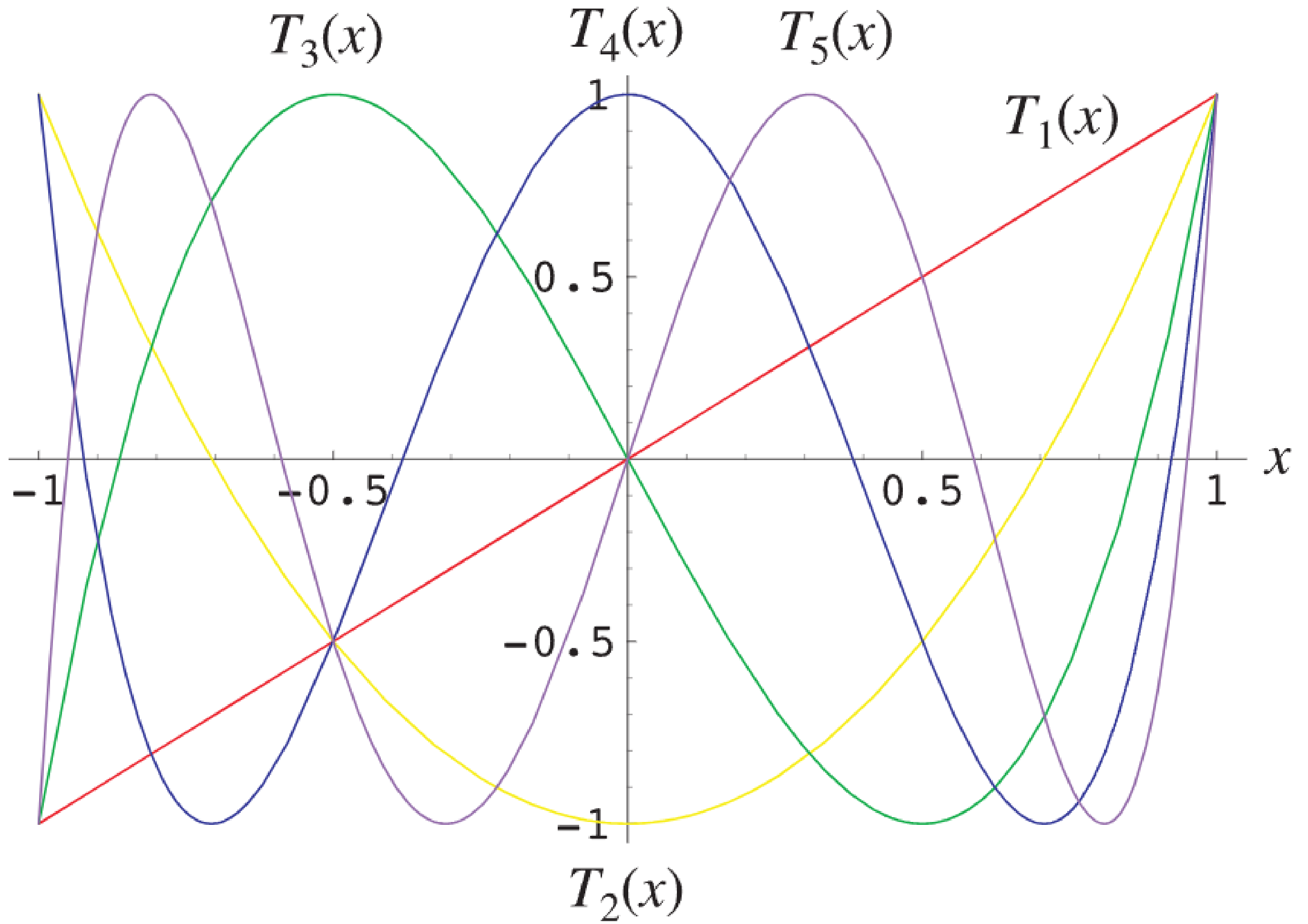
$$\tilde{v}_2 = \frac{v^2}{\|v^2\|} = \begin{bmatrix} \frac{-1}{\sqrt{38}} \\ \frac{1}{\sqrt{38}} \\ 3\sqrt{\frac{2}{19}} \end{bmatrix}$$

$$\tilde{v}_3 = \frac{v^3}{\|v^3\|} = \frac{19}{\sqrt{76}} \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}$$

Chebyshev Polynomials (of the first kind)

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t) \frac{dt}{\sqrt{1-t^2}}$$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$



Laguerre Polynomials

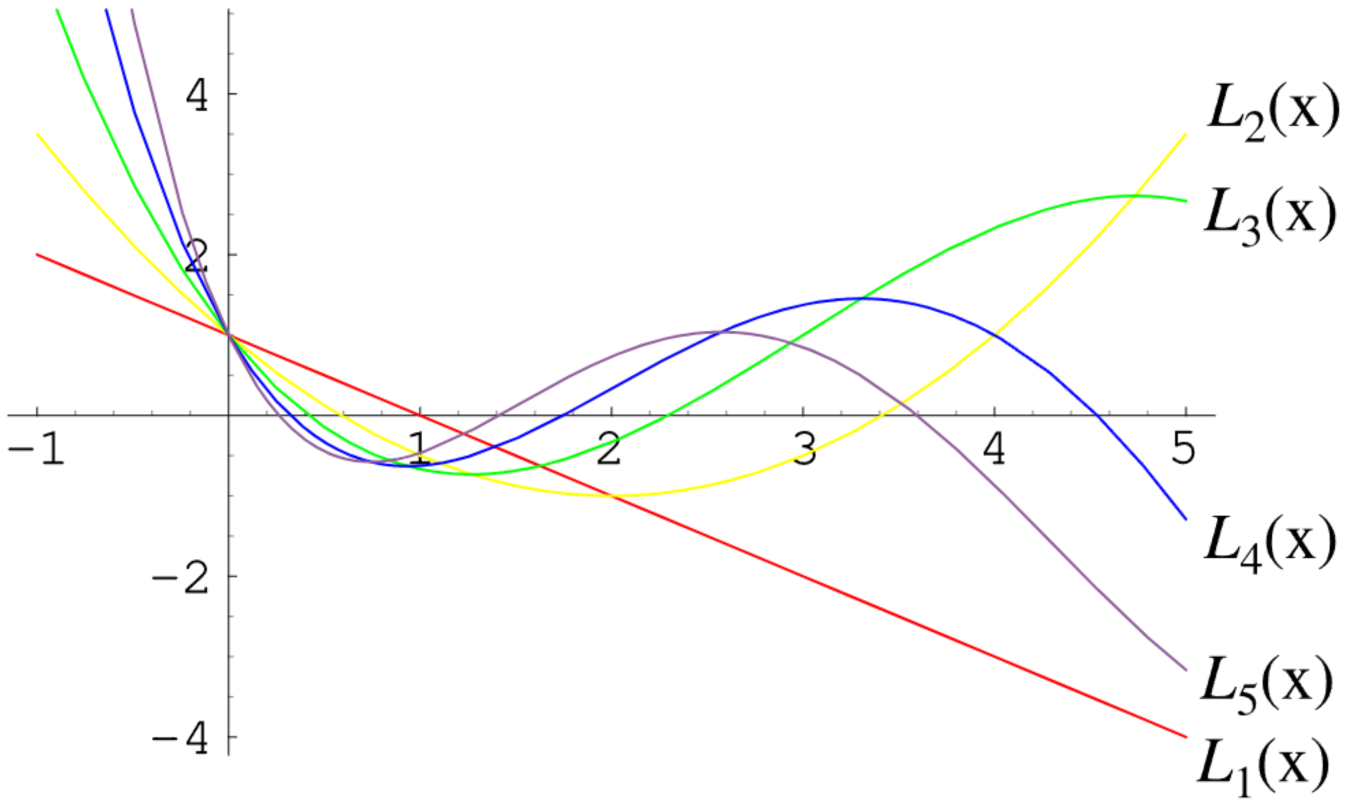
$$\langle f, g \rangle = \int_0^\infty f(t)g(t)e^{-t}dt$$

$$L_0(x) = 1$$

$$L_1(x) = -x + 1$$

$$L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$



You've seen them before

$$u := \{1, -t+1, t^2-4t+2, -t^3+9t^2-18t+6\}$$

$$v := \{1, t, 2t^2-1, 4t^3-3t\}$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$L_0(x) = 1$$

$$L_1(x) = -x + 1$$

$$L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

$$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\frac{dt}{\sqrt{1-t^2}}$$

$$\langle f, g\rangle = \int_{0}^{\infty} f(t)g(t)e^{-t}\,dt$$

Pythagorean Theorem

- If $x \perp y$, then

Pythagorean Theorem

- If $x \perp y$, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Pythagorean Theorem

- If $x \perp y$, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

- **Proof:** From the proof of the triangle inequality

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle.$$

Once we note that $\langle x, y\rangle = 0$ because $x \perp y$, we are done. $\blacksquare$

Next Week

- Numerical Issues with the Gram-Schmidt Process
- Projection Theorem