Mathematics for Robotics
ROB-GY 6103
Homework 2 Answers

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Question: 1. Given two finite subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ in a vector space $\mathcal{V}$ show that

$$Span(\mathcal{S}_1 \cup \mathcal{S}_2) = Span(\mathcal{S}_1) + Span(\mathcal{S}_2)$$

Answer: Given, Two finite subsets S_1, S_2 in a vector space V having Span

$$Span\{S_1\} = \{x_1 \in \mathcal{X} | \exists n \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F}, v_1^1, \ldots, v_1^n \in S_1, s.t.\ x_1 = \alpha_1 \cdot v_1^1 + \alpha_2 \cdot v_1^2 + \ldots + \alpha_n \cdot v_1^n\} \tag{1}$$

$$Span\{S_2\} = \{x_2 \in \mathcal{X} | \exists m \geq 1, \beta_1, \ldots, \beta_m \in \mathcal{F}, v_2^1, \ldots, v_2^m \in S_2, s.t.\ x_2 = \beta_1 \cdot v_2^1 + \beta_2 \cdot v_2^2 + \ldots + \beta_m \cdot v_2^m\} \tag{2}$$

Combining subspaces S_1 and S_2 *i.e.* combining $Eq^n(1)$ and $Eq^n(2)$, we get,

$$\begin{aligned} Span\{S_1 \cup S_2\} = \big\{x_1 + x_2 \in \mathcal{X} \mid \exists\, n, m \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F}, \beta_1, \ldots, \beta_m \in \mathcal{F}, \\ v_1^1, \ldots, v_1^n \in S_1, v_2^1, \ldots, v_2^m \in S_2 \\ s.t.\ x_1 + x_2 = (\alpha_1 \cdot v_1^1 + \beta_1 \cdot v_2^1) + \\ (\alpha_2 \cdot v_1^2 + \beta_2 \cdot v_2^2) \\ . \\ . \\ . \\ (\alpha_n \cdot v_1^n + \beta_m \cdot v_2^m)\big\} \end{aligned} \tag{3}$$

So, from $Eq^n(3)$, we get,

$$x_1 + x_2 = (\alpha_1 \cdot v_1^1 + \beta_1 \cdot v_2^1) + (\alpha_2 \cdot v_1^2 + \beta_2 \cdot v_2^2) \cdot \cdot \cdot (\alpha_n \cdot v_1^n + \beta_m \cdot v_2^m) \tag{4}$$

$$= (\alpha_1 \cdot v_1^1 + \alpha_2 \cdot v_1^2 + \ldots + \alpha_n \cdot v_1^n) + (\beta_1 \cdot v_2^1 + \beta_2 \cdot v_2^2 + \ldots + \beta_m \cdot v_2^m) \tag{5}$$

Upon observation, we can deduce that $Eq^n(5) = Eq^n(1) + Eq^n(2)$, i.e,

$$Span(S_1 \cup S_2) = Span\{S_1\} + Span\{S_2\} \tag{6}$$

Q.E.D.

Question: 2.(a)

Answer: Given set,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \right\} \tag{1}$$

To check for Linear Dependance,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 0 \tag{2}$$

This gives us three equations,

$$\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \tag{3}$$

$$2\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \tag{4}$$

$$3\alpha_1 + 9\alpha_3 = 0 \tag{5}$$

Substituting $\alpha_1 = -3$, $\alpha_2 = 1$ and $\alpha_3 = 1$ in above $Eq^n(3), (4) \& (5)$

$$Eq^n(3) \Rightarrow -3 + 2 + 1 = 0 \tag{6}$$

$$Eq^n(4) \Rightarrow -6 + 1 + 5 = 0 \tag{7}$$

$$Eq^n(5) \Rightarrow -9 + 9 = 0 \tag{8}$$

$\therefore$ the given set is *Linearly Dependent*.

So, we can express each vector as a linear combination of the remaining vectors of the set. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \tag{9}$$

Question: 2.(b)

Answer: Given set,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{1}$$

To check for Linear Dependance,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + \alpha_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{2}$$

This gives us three equations,

$$\alpha_1 + \alpha_4 = 0 \tag{3}$$
$$2\alpha_1 + 4\alpha_2 + \alpha_4 = 0 \tag{4}$$
$$3\alpha_1 + 5\alpha_2 + 6\alpha_3 + \alpha_4 = 0 \tag{5}$$

Substituting $\alpha_1 = -1$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{8}$ and $\alpha_4 = 1$ in above $Eq^n(3), (4)\&(5)$

$$Eq^n(3) \Rightarrow 1 - 1 = 0 \tag{6}$$
$$Eq^n(4) \Rightarrow -2 + 1 + 1 = 0 \tag{7}$$
$$Eq^n(5) \Rightarrow -3 + \frac{5}{4} + \frac{6}{8} + 1 = 0 \tag{8}$$

$\therefore$ the given set is *Linearly Dependent.*

So, we can express each vector as a linear combination of the remaining vectors of the set. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

Question: 2.(c)

Answer: Given set,

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{1}$$

To check for Linear Dependance,

$$\alpha_1 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0 \tag{2}$$

This gives us three equations,

$$3\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \tag{3}$$
$$2\alpha_1 + \alpha_3 = 0 \tag{4}$$
$$\alpha_1 = 0 \tag{5}$$

The rearrangement of above $Eq^n(3), (4)\&(5)$ gives us $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ The given set is *Linearly Independant.*

Question: 3.

Answer: Given set,

$$\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \right\} \tag{1}$$

To check for Linear Dependance,

$$\alpha_1 \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = 0 \tag{2}$$

This gives us the following equations,

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = 0 \tag{3}$$
$$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \tag{4}$$
$$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \tag{5}$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{6}$$

Substituting $\alpha_1 = 1$, $\alpha_2 = -\frac{3}{2}$ and $\alpha_3 = \frac{1}{2}$ in above $Eq^n(3), (4)\&(6)$

$$Eq^n(3) \Rightarrow 1 - 3 + 2 = 0 \tag{7}$$
$$Eq^n(4) \Rightarrow 2 - \frac{3}{2} - \frac{1}{2} = 0 \tag{8}$$
$$Eq^n(6) \Rightarrow 1 - \frac{3}{2} + \frac{1}{2} = 0 \tag{9}$$

$\therefore$ The given set is *Linearly Dependent*.

Question: 4.

Answer: Given,

- $(\mathcal{X}, \mathcal{F})$ is a vector space
- $\mathcal{Y}$ is a subspace of $\mathcal{X} \Rightarrow$
  - $\mathcal{Y}$ is non-empty
  - $\mathcal{Y}$ is closed under vector addition
  - $\mathcal{Y}$ is closed under scalar multiplication
- $\mathcal{S} \subset \mathcal{X}$
- $\mathcal{S} \subset \mathcal{Y}$

Now consider the $span\{\mathcal{S}\}$. By definition,

$$span\{S\} = \left\{ \ x \in \mathcal{Y} \mid \exists n \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F} \ ; \ v^1, \ldots, v^n \in \mathcal{S} \ ; \ s.t. x = \alpha_1 v^1 + \ldots + \alpha_n v^n \ \right\} \tag{1}$$

So, the $span\{\mathcal{S}\}$ is a *linear combination* of all the elements of $\mathcal{S}$.

But seeing that $\mathcal{S} \subset \mathcal{Y}$ where $\mathcal{Y}$ is a subspace of $\mathcal{X} \Rightarrow \mathcal{Y}$ is closed under vector addition and scalar multiplication $\Rightarrow span\{\mathcal{S}\}$ is a part of $\mathcal{Y}$.

$\therefore$ $span\{\mathcal{S}\} \subset \mathcal{Y}$. **Q.E.D.**

Question: 5.

Answer: Nagy Pg 115 Proof of Thm 4.1.14

→ Given that $X = V + W$. Suppose that $x \in V + W$.
$\Rightarrow \exists v \in V\ s.t.\ x = v + 0\ AND\ \exists w \in W\ s.t.\ x = 0 + w$
$\therefore\ v = w = 0 \Rightarrow V \cap W = \{0\}$

→ Given that $X = V + W \Rightarrow \forall x \in X$ there exist $v \in V$ and $w \in W$ s.t. $x = v + w$. Suppose there exists other vectors $v' \in V$ and $w' \in W$ s.t. $x = v' + w'$. Then,

$$0 = (v - v') + (w - w') \Leftrightarrow (v - v') = -(w - w')$$

$$\Rightarrow (v - v') \in W \Rightarrow (v - v') \in V \cap W$$

But, $V \cap W = \{0\}$. $\therefore v = v'\ AND\ w = w'$.

Q.E.D.

Question: 6.

Answer: Given set,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\} \tag{1}$$

Starting from the left and moving to the right, we shall discard a vector if it is linearly dependenton those preceding it.

So, considering the first two vectors, we shall check for linear dependence,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 0 \tag{2}$$

$Eq^n(2)$ resolves to $\alpha_1 = \alpha_2 = 0 \Rightarrow$ The considered set of vectors is *Linearly Independant.*

Now considering the first three vectors, we shall check for linear independence,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} = 0 \tag{3}$$

$Eq^n(3)$ resolves to $\alpha_1 = -4; \alpha_2 = 2; \alpha_3 = 1 \Rightarrow$ The considered set of vectors is *Linearly Dependant.*

So, let us discard the vector $\begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}$.

Now considering the first, second and fourth vectors, we shall check for linear independence,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{4}$$

$Eq^n(4)$ resolves to $\alpha_1 = \alpha_2 = \alpha_4 = 0 \Rightarrow$ The considered set of vectors is *Linearly Independant.*
Now considering the first, second, fourth and fifth vectors, we shall check for linear independence,

$$\alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_5 \cdot \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} = 0 \tag{5}$$

$Eq^n(5)$ resolves to $\alpha_1 = \alpha_2 = \alpha_4 = -\alpha_5 \Rightarrow$ The considered set of vectors is *Linearly Dependant.* So, let us discard the vector $\begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}$.

Finally, the basis of the given set can be found to be,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{6}$$

$\Rightarrow$ Number of elements in the basis = Dimension of the space = 3.

Question: 7.

Answer: Given,

$$v_s = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} \tag{1}$$

And the ordered basis,

$$\left(u_{1s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_{2s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, u_{3s} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \tag{2}$$

To find the components of v_s in the ordered basis described in $Eq^n 2$, we must put it in the form of a linear combination,

$$\alpha_1 u_{1s} + \alpha_2 u_{2s} + \alpha_3 u_{3s} = v_s \tag{3}$$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} \tag{4}$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 8 \tag{5}$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + 2\alpha_3 = 7 \tag{6}$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 4 \tag{7}$$

Solving above equations we get, $\alpha_1 = 9$, $\alpha_2 = 2$, and $\alpha_3 = -3$. Therefore,

$$\begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} = 9u_{1s} + 2u_{2s} - 3u_{3s} \iff [v_s]_{u_s} = \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix} \in \mathbb{R}^3$$

Question: 8.

Answer: Given, standard basis,

$$e = \left(e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \tag{1}$$

And the new basis,

$$u_s = \left(u_{1s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_{2s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, u_{3s} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \tag{2}$$

Now, look for a matrix P to switch from e to u_s:

$$[x]_{u_s} = P[x]_e$$

But as it is easier to find $\bar{P}$ first, we shall do it by working column by column,

$$\bar{\bar{P}} = \left[\begin{array}{c:c:c} \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \end{array} \right]$$

$$\bar{P}_1 = [u_{1s}]_e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ \bar{P}_2 = [u_{2s}]_e = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \ \bar{P}_3 = [u_{3s}]_e = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \bar{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{3}$$

As, $\bar{P} = P^{-1} \Rightarrow P = \bar{P}^{-1}$,

$$\Rightarrow P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{4}$$

Question: 9.

Answer: Consider below Figure 1

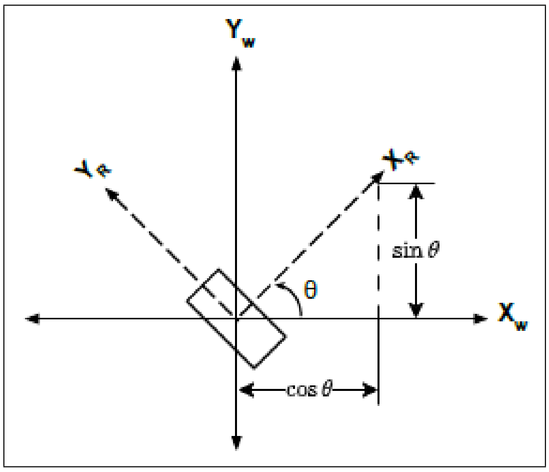

Figure 1: World coordinate system and Robot coordinate system

And the standard basis for the world frame,

$$[x]_W = \left(X_W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Y_W = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \tag{1}$$

It is given that the rotated by an angle θ as shown in Figure 1.
So, we get the new basis by applying trogonometric relations as,

$$[x]_R = \left(X_R = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, Y_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right) \tag{2}$$

Now, we need to find a matrix P such that, $[x]_R = P[x]_W$
But as it is easier to find $\bar{P}$ first, we shall do it by working column by column,

$$\bar{P} = \left[\begin{array}{c:c} \bar{P}_1 & \bar{P}_2 \end{array} \right] \tag{3}$$

$$\bar{P}_1 = [X_R]_W = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \bar{P}_2 = [Y_R]_W = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \Rightarrow \bar{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

As, $\bar{P} = P^{-1} \Rightarrow P = \bar{P}^{-1}$,

$$\Rightarrow P = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{6}$$