CHAPTER 4. INVERSE MANIPULATOR KINEMATICS

Inverse kinematics vs. direct (or forward) kinematics

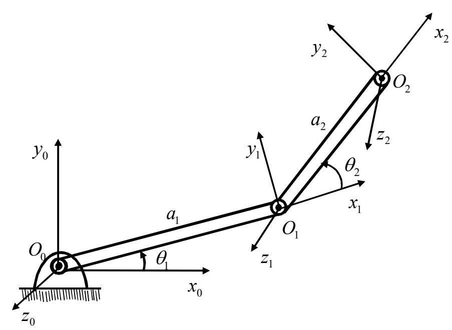
Problem \ Space	Joint Space		Cartesian Space
Forward Kinematics	**q** (known)	→	${}^0T_n(\mathbf{q})$ (unknown)
Inverse Kinematics	**q** (unknown)	←	${}^0T_n(\mathbf{q})$ (known)

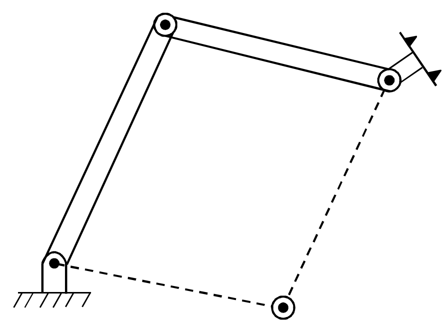
<u>Solvability</u>

- Given ${}^0T_n$ → find $q_1, q_2, \ldots, q_n$ (Cartesian space → joint space)
- $${}^0T_n = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0T_1(q_1)\,{}^1T_2(q_2)\ldots{}^{n-1}T_n(q_n) \;\Rightarrow\; q_i = f_i(r_{11}, \ldots, r_{33}, p_x, p_y, p_z),\; i = 1, \ldots, n$$
- Nonlinear transcendental equations

<u>Existence and Uniqueness</u>

- Workspace: volume of space which the manipulator's end-effector can reach
  - Dexterous workspace: end-effector can reach with all orientations. At each point, end-effector can be arbitrarily oriented.
  - Reachable workspace: end-effector can reach in at least one orientation.
  - {dexterous workspace} $\subset$ {reachable workspace}





- Number of solutions depends on link parameters, joint limits, and number of joints.
  - If [DOF or number of unknowns] = [number of equations] → unique solution
  - If [DOF or number of unknowns] < [number of equations] → solution may not exist; manipulator cannot attain general goal positions and orientations in 3D space.
  - If [DOF or number of unknowns] > [number of equations] → multiple solutions may exist; kinematically redundant (flexible, dexterous, controllable); optimization is required.

(Note: For complete position and orientation of the end-effector, the number of equations is 6. However, in general, the number of equations depends on a given task as well as the manipulator.)

<u>Methods of Solution</u>

- Solvable: all the sets of joint variables can be determined for a given position and orientation.
  - Closed form solutions – analytic expressions or polynomial of degree 4 or less
  - Numerical solutions (e.g., Bisection method, Newton-Raphson method, Secant method, Muller's method, Brent's algorithm, etc.)

- Closed form solution methods of kinematic equations
  1) Algebraic solution: specify end-effector frame relative to base frame → manipulate given equations
  2) Geometric solution: decompose spatial geometry of the manipulator into several plane geometry → use plane geometry to solve for joint angles

  (Note: Frequently, the mix of algebraic and geometric approaches is used.)

- A sufficient condition that a manipulator with 6 revolute joints will have a closed form solution is that three neighboring joint axes intersect at a point.

- Recall: Two-argument arctangent function $\phi = \text{atan2}(y, x)$

  Defined on all four quadrants ($-\pi \le \phi < \pi$)

Case	Quadrants	$\phi = \text{atan2}(y, x)$
$x > 0$	1, 4	$\phi = \arctan(y / x)$
$x = 0$	1, 4	$\phi = \underbrace{\text{sgn}(y)}_{=\pm 1}(\pi / 2)$
$x < 0$	2, 3	$\phi = \arctan(y / x) + \text{sgn}(y) \cdot \pi$

Algebraic Solution by Reduction to Polynomial

- Let $\boxed{u = \tan\frac{\theta}{2}}$ and substitute $\cos\theta = \frac{1-u^2}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$ (Weierstrass Substitution)

  => Transcendental (e.g., trigonometric) equations in θ → polynomial equations in u

  (Note: polynomials up to degree 4 have closed form solutions.)

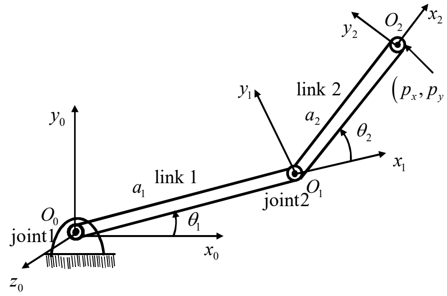
- Closed form solvable manipulators: Manipulators which are sufficiently simple to be solved by algebraic equations of up to degree 4.

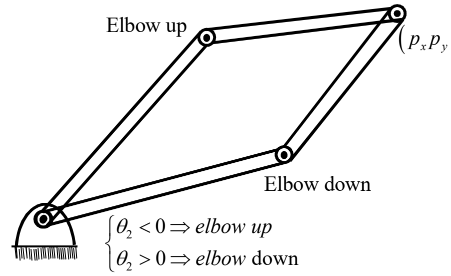
Repeatability and Accuracy

- Taught point: point that the manipulator is moved to physically, and then the joint position sensors are read, and the joint angles are stored; teach and playback
- Repeatability of manipulator: specification of how precisely a manipulator can return to a taught point
- Computed point: point in a manipulator's workspace which was never taught; if a goal position and orientation are specified in Cartesian space, required joint variables must be solved for by computing inverse kinematics.
- Accuracy: Precision with which a computed point can be attained. Accuracy of a manipulator is bounded by the repeatability.

<u>Example: 2R Planar Manipulator</u>

Determine θ_1 and θ_2 in terms of p_x and p_y.





Kinematic equations

Homogeneous transformation between links: $${}^0T_2 = {}^0T_1\,{}^1T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

End-effector's position and orientation: $${}^RT_H = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & p_x \\ \sin\phi & \cos\phi & 0 & p_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1) Calculation of θ_2

$p_x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) = a_1c_1 + a_2c_{12}$ and $p_y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) = a_1s_1 + a_2s_{12}$

$\Rightarrow\ p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1a_2(c_1c_{12} + s_1s_{12}) = a_1^2 + a_2^2 + 2a_1a_2c_2$ (also from law of cosines)

$\Rightarrow\ c_2 = \cos\theta_2 = \dfrac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2}$ → The solution exists only if $-1 \le \dfrac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2} \le 1$. If not, the target point is outside of the reachable workspace.

$s_2 = \sin\theta_2 = \pm\sqrt{1-c_2^2}$; $\theta_2 = \text{atan2}(s_2, c_2)$ → redundancy – elbow-up vs. elbow-down

2) Calculation of θ_1

$p_x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) = a_1c_1 + a_2c_{12} = a_1c_1 + a_2(c_1c_2 - s_1s_2) = (a_1 + a_2c_2)c_1 - a_2s_2s_1$

$p_y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) = a_1s_1 + a_2s_{12} = a_1c_1 + a_2(c_1s_2 + s_1c_2) = (a_1 + a_2c_2)s_1 + a_2s_2c_1$

Method 1 for θ_1:

$(a_1 + a_2c_2)c_1 - (a_2s_2)s_1 = p_x$ and $(a_2s_2)c_1 + (a_1 + a_2c_2)s_1 = p_y$

Cramer's formula $\Rightarrow$ $c_1 = \dfrac{(a_1 + a_2c_2)p_x + a_2s_2p_y}{(a_1 + a_2c_2)^2 + (a_2s_2)^2}$ and $s_1 = \dfrac{-a_2s_2p_x + (a_1 + a_2c_2)p_y}{(a_1 + a_2c_2)^2 + (a_2s_2)^2}$

$\theta_1 = \text{atan2}(s_1, c_1) = \text{atan2}(-a_2s_2p_x + (a_1 + a_2c_2)p_y, (a_1 + a_2c_2)p_x + a_2s_2p_y)$

Method 2 for θ_1:

$$\frac{p_y}{p_x} = \frac{(a_1 + a_2 c_2)s_1 + a_2 s_2 c_1}{(a_1 + a_2 c_2)c_1 - a_2 s_2 s_1} = \frac{\dfrac{s_1}{c_1} + \dfrac{a_2 s_2}{a_1 + a_2 c_2}}{1 - \dfrac{s_1}{c_1}\dfrac{a_2 s_2}{a_1 + a_2 c_2}}$$

Let $\tan\gamma = \dfrac{a_2 s_2}{a_1 + a_2 c_2}$ or $\gamma = \text{atan2}(a_2 s_2, a_1 + a_2 c_2)$

$$\frac{p_y}{p_x} = \frac{\tan\theta_1 + \tan\gamma}{1 - \tan\theta_1 \tan\gamma} = \tan(\theta_1 + \gamma) \;\Rightarrow\; \theta_1 = \text{atan2}(p_y, p_x) - \gamma$$

$$\Rightarrow\; \theta_1 = \text{atan2}(p_y, p_x) - \text{atan2}(a_2 s_2, a_1 + a_2 c_2)$$