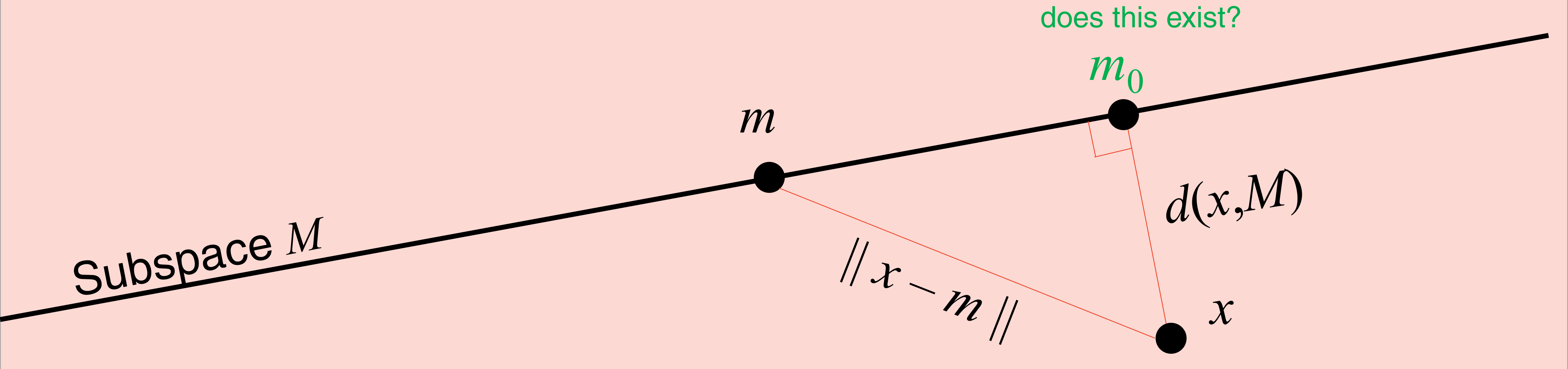
Mathematics for Robotics (ROB-GY 6013 Section A)

- **Week 9:**
  - Theme: Least Squares
    - Using the Projection Theorem: Normal Equations
    - Quadratic Forms

Recap: Without Classical Projection Theorem

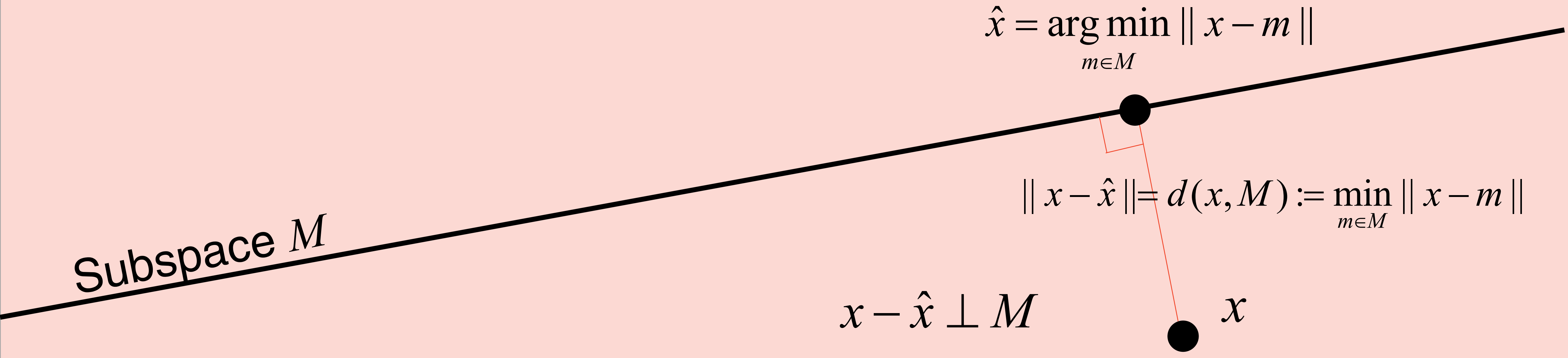
Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be a **inner product space**, M be a **subspace** of $\mathcal{X}$, and x be an arbitrary **point** in $\mathcal{X}$.



Recap: Classical Projection Theorem

Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be a **finite-dimensional** (real) **inner product space**, M be a **subspace** of $\mathcal{X}$, and x be an arbitrary **point** in $\mathcal{X}$.

Inner product space $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$



Recap: Classical Projection Theorem

- Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be a finite-dimensional (real) **inner product space** and M a **subspace** of $\mathcal{X}$. Then, $\forall x \in \mathcal{X}$, $\exists$ **unique** $\hat{x} \in M$ such that

$$\|x-\hat{x}\| = d(x,M) := \inf_{m\in M} \|x-m\| = \min_{m\in M} \|x-m\|$$

  where we can write **min**imum instead of **inf**imum, because the infimum is achieved. Moreover $\hat{x} \in M$ is characterized by $x-\hat{x} \perp M$

- Use of $\hat{x}$ (best approximation) instead of m_0 because this is the standard notation for estimation problems.

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$$
\begin{gathered}
\langle \widehat{x}, y^i \rangle = \langle x, y^i \rangle \; 1 \leq i \leq k \\
\Updownarrow \\
\langle \alpha_1 y^1 + \alpha_2 y^2 + \cdots + \alpha_k y^k, y^i \rangle = \langle x, y^i \rangle \; 1 \leq i \leq k \\
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Normal Equations

$$G(y^1,\cdots,y^k) := \begin{bmatrix} \langle y^1,y^1\rangle & \langle y^1,y^2\rangle & \cdots & \langle y^1,y^k\rangle \\ \langle y^2,y^1\rangle & \langle y^2,y^2\rangle & \cdots & \langle y^2,y^k\rangle \\ \vdots & \vdots & & \vdots \\ \langle y^k,y^1\rangle & \langle y^k,y^2\rangle & \cdots & \langle y^k,y^k\rangle \end{bmatrix} \quad \alpha := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \ \beta := \begin{bmatrix} \langle x,y^1\rangle \\ \langle x,y^2\rangle \\ \vdots \\ \langle x,y^k\rangle \end{bmatrix} =: \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}$$

Gram Matrix

$G_{ij} := \langle y^i, y^j \rangle$

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Is this invertible?

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Proposition 3.41 (Invertibility of the Gram Matrix)

- Let $g(y^1, \dots, y^k) := \det G(y^1, \dots, y^k)$ be the **determinant** of the **Gram Matrix**.

  Then $g(y^1, \dots, y^k) \neq 0$ if and only if $\{y^1, \dots, y^k\}$ is **linearly independent**.

Normal Equations

- Seek $\hat{x} = \arg\min_{m \in M} \|x - m\|$
- Pick basis
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Wealth (USD)	Sleep (Avg Hrs/day)	Happiness (0, 100)
1	1	0
100	2	10
25	5	50
50	8	100

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$$A\hat{\alpha} \approx b$$

$$\begin{bmatrix} 1 & 1 \\ 100 & 2 \\ 25 & 5 \\ 50 & 8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 10 \\ 50 \\ 100 \end{bmatrix}$$

Linear Regression (Overdetermined Equations)

- **Seek** $\hat{x} = \arg\min_{m \in M} \|x - m\| = \arg\min_{A\hat{\alpha} \in M} \|b - A\hat{\alpha}\|$

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Remarks

- No vector calculus!

$$\widehat{\alpha} = \arg\min_{\alpha \in \mathbb{R}^m} \sqrt{(A\alpha - b)^\top (A\alpha - b)} = \arg\min_{\alpha \in \mathbb{R}^m} (A\alpha - b)^\top (A\alpha - b)$$

$$\frac{\partial}{\partial \alpha} \left((A\alpha - b)^\top (A\alpha - b)\right) = A^\top \cdot (A\alpha - b)$$

$$A^\top \cdot (A\widehat{\alpha} - b) = 0 \iff (A^\top A)\widehat{\alpha} = A^\top b \iff \widehat{\alpha} = (A^\top A)^{-1} A^\top b$$

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$$(a)\ \ x - \widehat{x} \perp M.$$

$$(b)\ \ \exists m^{\perp} \in M^{\perp} \textit{ such that } x = \widehat{x} + m^{\perp}.$$

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- A function $P\colon \mathcal{X} \to M$ by $P(x) = \hat{x}$, where $\hat{x}$ satisfies any one of (a), (b), or (c), is called the **orthogonal projection** of $\mathcal{X}$ onto M.

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  - P is a linear operator
  - Let $\{v^1, \ldots, v^k\}$ be an **orthonormal basis** for M. Then $P(x) = \sum_{i=1}^{k} \langle x, v^i \rangle v^i.$

Many Claims about Symmetric and Orthogonal Matrices

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$$\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$$

- $\overline{A \cdot v} = \overline{A} \cdot \overline{v} = A \cdot \overline{v}$ (because A is real)
- $\overline{A \cdot v} = \overline{\lambda \cdot v} = \overline{\lambda} \cdot \overline{v}$ (because $Av = \lambda v$)

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- **Proof:**

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Note $<v, v> \neq 0$

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- **Definition:** A matrix A is **symmetric** if $A^T = A$
- **Proposition:** If A is real and symmetric, then $\forall\, x, y \in \mathbb{C}^n,\ \langle Ax, y\rangle = \langle x, Ay\rangle$
  - Choose this inner product: $< x, y >= x^T \overline{y}$

$$\langle Ax, y\rangle = x^\top A^\top \overline{y} = x^\top A \overline{y} \qquad \langle x, Ay\rangle = x^\top \overline{Ay} = x^\top \overline{A}\overline{y} = x^\top A \overline{y}$$

- **Proof:**

$$\langle Av, v\rangle = \langle v, Av\rangle$$
$$\Updownarrow$$
$$\langle \lambda v, v\rangle = \langle v, \lambda v\rangle$$
$$\Updownarrow$$
$$\lambda \langle v, v\rangle = \overline{\lambda} \langle v, v\rangle$$

Then the corresponding eigenvector must be also real!

Note $<v, v> \neq 0$

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- **Proof:** Show that for the corresponding eigenvectors $\langle v^1, v^2 \rangle = 0$

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$$\langle \lambda_1 v^1, v^2 \rangle = \langle v^1, \lambda_2 v^2 \rangle$$
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$$\lambda_1 \langle v^1, v^2 \rangle = \lambda_2 \langle v^1, v^2 \rangle$$
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$$0 = (\lambda_1 - \lambda_2) \langle v^1, v^2 \rangle$$

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Note $\lambda_1 - \lambda_2 \neq 0$

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- The eigenvectors of an $n \times n$ real symmetric matrix can always be chosen to form an orthonormal basis of $\mathbb{R}^n$
- Repeated eigenvalues require special consideration (deflected to homework exercise)
- Consider a real but not symmetric matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Definition: Orthogonal Matrix

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- Column vectors of an orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$

Diagonalization with Orthogonal Matrix

- Suppose A is an $n \times n$ real symmetric matrix. Then there exists an orthogonal matrix Q such that

$$Q^T AQ = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$$

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- Very useful property for later (e.g., **singular value decomposition**)

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- Rotations and reflections preserve length

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  - Show: $\lambda \geq 0$

Eigenvalues of $A^T A$ and AA^T are non-negative (real numbers)

- **Proof for $A^T A$:**
  - Let $A^T A v = \lambda v$ where $v \in \mathbb{R}^n$, $v \neq 0$, $\lambda \in \mathbb{R}$.

$$v^\top A^\top A v = v^\top \lambda v$$
$$\Updownarrow$$
$$\langle Av, Av \rangle = \lambda \langle v, v \rangle$$
$$\Updownarrow$$
$$\|Av\|^2 = \lambda \|v\|^2.$$

  - Show: $\lambda \geq 0$

Quadratic Forms and Positive (Semi)Definiteness

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Symmetric Part of Quadratic Form

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- For the quadratic form $x^T M x$ only the symmetric part matters. Thus, all theorems about quadratic forms consider only symmetric M.

$$x^\top M x = x^\top \left(\frac{M+M^\top}{2}\right) x$$

Connections with Eigenvalues and Quadratic Forms

- For a real symmetric matrix form M. Then $\forall\, x \in \mathbb{R}^n$

$$\lambda_{\min}\, x^\top x \leq x^\top M x \leq \lambda_{\max}\, x^\top x$$

- Bounds are “tight”

Theorem: Positive Definite

- **Definition:** A real symmetric matrix form P is **positive definite** if

$$\forall x \in \mathbb{R}^n, x \neq 0 \quad \Rightarrow \quad x^T P x > 0$$

- **Theorem:** A symmetric real matrix P is **positive definite** if and only if all of its eigenvalues are **strictly greater than** zero.

- **Notation:** $P > 0$

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- **Proof Outline:**

$$P = O^T \Lambda O$$

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$$\Lambda^{1/2} := \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}),$$

Square root $N = \Lambda^{1/2} O$

$$(\Lambda^{1/2})^\top \Lambda^{1/2} = \Lambda^{1/2}\Lambda^{1/2} = \Lambda.$$

$$N^\top N = O^\top \left(\Lambda^{1/2}\right)^\top \Lambda^{1/2} O = O^\top \Lambda O = P.$$

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  - Deep significance in relation to quadratic forms
- Check whether a certain matrix is positive definite:
  - Find eigenvalues. **HARD**
  - **You should only look for things that you have to**
  - **For example, you don’t have to invert a matrix to check if the column vectors are linearly independent**

Theorem: Schur's Complements

- Suppose that A, B and C are real matrices, $A = n \times n$ is symmetric, $B = n \times m$, and $C = m \times m$ is symmetric.

Then for the symmetric matrix $$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

The following are equivalent

(a) $M > 0.$

(b) $A > 0,$ *and* $C - B^\top A^{-1} B > 0.$

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