Mathematics for Robotics (ROB-GY 6013 Section A)

- **Week 4:**
  - Basis
  - Dimension
  - More Basis
  - Linear Operators
- **Homework 2 posted**

“Humanity is born free but everywhere is in chains”

L'homme est né libre, et partout il est dans les fers

—Jean-Jacques Rousseau

Freedom to explore

- Vector spaces $(\mathcal{X},\mathcal{F})$ are closed under **vector addition** and **scalar multiplication**
  - Linear combinations of vectors are "trapped" in $\mathcal{X}$.
  - But are they free to explore all of $\mathcal{X}$?

$$(\mathbb{R}^3, \mathbb{R})$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Two is not enough!

Freedom to explore

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$$(\mathbb{R}^3, \mathbb{R})$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Three **linearly independent** vectors can!

Freedom to explore

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$$(\mathbb{R}^3, \mathbb{R})$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.7 \\ 1 \\ 0.3 \end{bmatrix}$$

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$$(\mathbb{R}^3,\mathbb{R})$$

$$\alpha_1\begin{bmatrix}1\\0.5\\2\end{bmatrix}+\alpha_2\begin{bmatrix}0.7\\1\\0.3\end{bmatrix}+\alpha_3\begin{bmatrix}0.41\\0.40\\0.49\end{bmatrix}$$

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$$\alpha_1\begin{bmatrix}1\\0.5\\2\end{bmatrix}+\alpha_2\begin{bmatrix}0.7\\1\\0.3\end{bmatrix}+\alpha_3\begin{bmatrix}0.41\\0.40\\0.49\end{bmatrix}$$

$$\boxed{0.2\begin{bmatrix}1\\0.5\\2\end{bmatrix}+0.3\begin{bmatrix}0.7\\1\\0.3\end{bmatrix}=\begin{bmatrix}0.41\\0.40\\0.49\end{bmatrix}}$$

- **Linearly dependent!**

Freedom to explore

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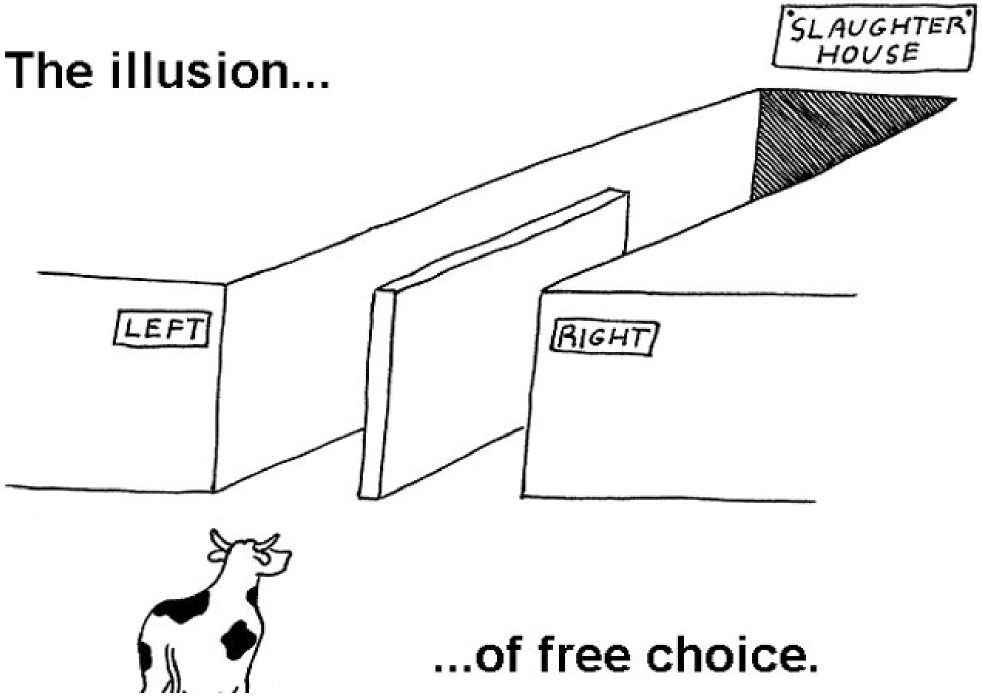


Controllability Theorem (Not quite)

- If there exist n linearly independent **column** vectors of the controllability matrix $\mathcal{C}$, the system is controllable. Otherwise, the system is not controllable.

State vector $\mathbf{x} \in \mathbb{R}^n$

Input vector $\mathbf{u} \in \mathbb{R}^m$

State matrix $A \in \mathbb{R}^{n \times n}$

Input matrix $B \in \mathbb{R}^{n \times m}$

State-Space Representation of a System

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

Review

- **Span** of a non-empty (infinite or finite) set of vectors forms a **subspace** that represent how much of the vector space you can "explore" by taking **linear combinations** of vectors

- The "smallest" set of vectors that can "explore" the entire vector space is the **basis**.

  a) $\mathcal{B}$ is linearly independent.

  b) $\mathrm{span}\{\mathcal{B}\} = \mathcal{X}$

- For a finite vector space, its **dimension** is the number of linearly independent vectors can you fit inside of it.

  a) there exists a set with n linearly independent vectors,

  b) and any set with $n+1$ vectors is linearly dependent.

Definition: Basis

- Let $(\mathcal{X},\mathcal{F})$ be a vector space.

  A set of vectors $\mathcal{B}$ in $(\mathcal{X},\mathcal{F})$ is a basis for $\mathcal{X}$ if

  a) $\mathcal{B}$ is linearly independent.

  b) $\text{span}\{\mathcal{B}\} = \mathcal{X}$

Just enough to span the entire vector space without anything extra

Definition: Dimension

- Let n be a natural number. The vector space $(\mathcal{X}, \mathcal{F})$ has **finite dimension** n if
  a) there exists a set with n linearly independent vectors,
  b) and any set with $n + 1$ vectors is linearly dependent.

- The vector space $(\mathcal{X}, \mathcal{F})$ is **infinite-dimensional** if for every n

  there exists a set with n linearly independent vectors

Linking Basis and Dimension: Theorems

- Let $(\mathcal{X}, \mathcal{F})$ be an n-**dimensional** vector space (always means n is finite). Then, any set of n linearly independent vectors is a **basis**.

Linking Basis and Dimension: Theorems

- Let $(\mathcal{X},\mathcal{F})$ be an **n-dimensional** vector space (always means n is finite). Then, any set of n linearly independent vectors is a **basis**.
    - Sound obvious but still requires proof. Does not immediately follow from definitions of dimension and basis.
    - Already know they are linearly independent.
    - Just need to show their span is all of $\mathcal{X}$.

Completing the basis

- See slides 55-57 in last week's lecture
- Given 2 linearly independent vectors

$$(\mathbb{R}^5, \mathbb{R})$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Completing the basis

- See slides 55-57 in last week's lecture
- Given 2 linearly independent vectors, you can always add 3 more to complete the basis

$$(\mathbb{R}^5, \mathbb{R})$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Linking Basis and Dimension: Theorems

- Let $(\mathcal{X},\mathcal{F})$ be an n**-dimensional** vector space with a **basis** $\{v^1, \ldots, v^n\}$ and let $x \in \mathcal{X}$. Then, there exist ***unique*** coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

Linking Basis and Dimension: Theorems

- Let $(\mathcal{X}, \mathcal{F})$ be an n-**dimensional** vector space with a **basis** $\{v^1, \ldots, v^n\}$ and let $x \in \mathcal{X}$. Then, there exist ***unique*** coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

- Foreshadowing that **representations** are unique
- How do you prove uniqueness?
  - Assume $\exists x, y \in S$ such that $P(x) \wedge P(y)$ is true and show $x = y$.
  - Argue by assuming that $\exists x, y \in S$ are distinct such that $P(x) \wedge P(y)$, then derive a contradiction.

Definition: Representation

- Let $(\mathcal{X}, \mathcal{F})$ be an n-**dimensional** vector space with a **basis** $v := \{v^1, \ldots, v^n\}$ and write $x \in \mathcal{X}$ as a unique linear combination of the basis vectors:

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

Then $[x]_v := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$

is the **representation** of x with respect to the basis v.

*Pay attention to the notation of brackets [ ] and subscript

Finding a representation: Examples

$$(\mathbb{R}^{2\times 2}, \mathbb{R}) \qquad x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$$

Basis 1: $v^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis 2: $w^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, w^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, w^3 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, w^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis 1: This one can be done by inspection because the basis is so simple:

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5v^1 + 3v^2 + 1v^3 + 4v^4 \iff [x]_w = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^4.$$

Basis 2: We'll work this one out

$$\alpha_1 w^1 + \alpha_2 w^2 + \alpha_3 w^3 + \alpha_4 w^4 = \begin{bmatrix} \alpha_1 & \alpha_2+\alpha_3 \\ \alpha_2-\alpha_3 & \alpha_3+\alpha_4 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}.$$

This gives us four equations in four unknowns, which we express in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix}.$$

The solution is, $\alpha_1 = 5, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 3$. Therefore,

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5w^1 + 2w^2 + 1w^3 + 3w^4 \iff [x]_w = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix} \in \mathbb{R}^4.$$

Representations

- Changing the basis also changes the representation
  - Sets don't care about order, but the order of basis vectors matters to us!
  - All bases used in this class are **ordered bases**
- Addition and scalar multiplication of **representations:**

$$[x + y]_v = [x]_v + [y]_v \qquad\qquad [\alpha x]_v = \alpha [x]_v$$

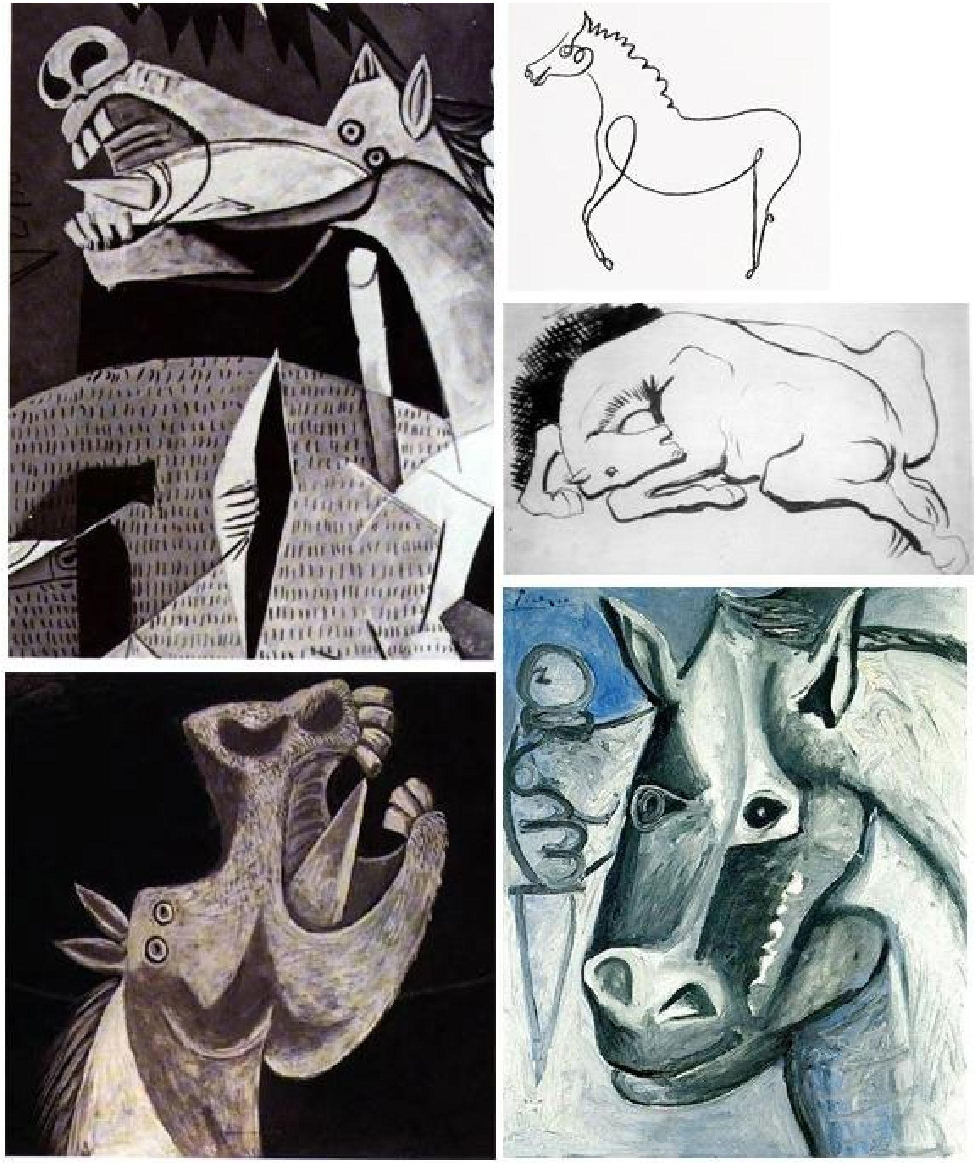
- Once a basis $v := \{v^1, \ldots, v^n\}$ is chosen, an n-dimensional vector space "looks like":

$$(\mathcal{X}, \mathcal{F}) \overset{v}{\longleftrightarrow} (\mathcal{F}^n, \mathcal{F})$$

- Different ***representations*** of the same underlying ***thing*** (the **vector**)

Switching Bases

- Different ***representations*** of the same underlying ***thing*** (the **vector**)
  - Can we switch between representations?
  - How do we switch?
  - **Hint:** Most of you do this every Thursday 6:00-8:00 PM



Rotation Matrix (Excuse the notation here)

- The rotation matrix ${}^{A}R_{B}$ switches from each **representation** *or linear combination of the basis vectors* in $\{\mathrm{B}\}$ **TO** its **representation** or *linear combination of the basis vectors* in $\{\mathrm{A}\}$.
- Underlying vector is the **same** (**has not moved**)

$$ {}^{A}R_{B} = [\,{}^{A}\hat{\mathbf{X}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Y}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Z}}_{B}\,] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ {}^{B}\mathbf{P} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\;{}^{A}\mathbf{P} = {}^{A}R_{B}\,{}^{B}\mathbf{P}\;} {}^{A}\mathbf{P} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{22} \end{bmatrix} = {}^{A}\hat{\mathbf{X}}_{B} $$

First column of rotation matrix

$$ {}^{B}\mathbf{P} = 1\hat{\mathbf{X}}_{B} + 0\hat{\mathbf{Y}}_{B} + 0\hat{\mathbf{Z}}_{B} $$

$$ {}^{A}\mathbf{P} = r_{11}\hat{\mathbf{X}}_{A} + r_{21}\hat{\mathbf{Y}}_{A} + r_{21}\hat{\mathbf{Z}}_{A} $$

Change of Basis Matrix (Excuse the notation here)

- We suspect that most if not all change of bases can be expressed with a matrix
- If that is the case, how can we “discover” this matrix? **Throw unit vectors at it.**

$$^{A}R_{B} = [\,{}^{A}\hat{\mathbf{X}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Y}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Z}}_{B}\,] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

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Change of Basis Matrix: Example

- Two bases:

$$u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\overline{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- Look for matrix P to switch from u to $\overline{u}$: $[x]_{\overline{u}} = P[x]_u$

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- Look for matrix P to switch from u to $\overline{u}$: $[x]_{\overline{u}} = P[x]_u$

- Work column by column: $P = \left[P_1 \,\vdots\, P_2 \,\vdots\, P_3 \,\vdots\, P_4 \right]$

- What should the first column P_1 be?

Change of Basis Matrix: Example

- Two bases:

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$$\overline{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- Look for matrix P to switch from u to $\overline{u}$: $\quad [x]_{\overline{u}} = P[x]_u$

- Work column by column: $\quad P = \left[P_1 \;\vdots\; P_2 \;\vdots\; P_3 \;\vdots\; P_4 \right]$

- What should the first column P_1 be? $\quad P_1 = P[u^1]_u = P[1 \quad 0 \quad 0 \quad 0]^T = [u^1]_{\overline{u}}$

Column by column

$$P_1 = [u^1]_{\overline{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_2 = [u^2]_{\overline{u}} = \begin{bmatrix} 0 \\ .5 \\ .5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = [u^3]_{\overline{u}} = \begin{bmatrix} 0 \\ .5 \\ -.5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_4 = [u^4]_{\overline{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Change of Basis Matrix: Example

- How about the other way?

$$u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\overline{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

$$[x]_{\overline{u}} = P[x]_u$$

$$[x]_u = \overline{P}[x]_{\overline{u}}$$

Change of Basis Matrix: Example

$$\overline{P}_1 = [\overline{u}^1]_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \overline{P}_2 = [\overline{u}^2]_u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overline{P}_3 = [\overline{u}^3]_u = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad \overline{P}_4 = [\overline{u}^4]_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Change of Basis Matrix: Example

- How about the other way?
- They are **inverses**!

$$u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\overline{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[x]_{\overline{u}} = P[x]_u$$

$$[x]_u = \overline{P}[x]_{\overline{u}}$$

Change of Basis Matrix: Theorem (2.40)

- **There exists an invertible matrix** P, with coefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X},\mathcal{F})$,

$$[x]_{\overline{u}} = P[x]_u$$

where, $P = \left[P_1 \mid P_2 \mid \dots \mid P_n\right]$ and its i^{th} column is given by $P_i := [u^i]_{\overline{u}} \in \mathcal{F}^n$

and $[u^i]_{\overline{u}}$ is the **representation** of u^i with respect to $\overline{u}$.

Change of Basis Matrix: Theorem (2.40)

- **There exists an invertible matrix** P, with coefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X},\mathcal{F})$,

$$[x]_{\overline{u}} = P[x]_u$$

where, $P = \left[P_1 \mid P_2 \mid \ldots \mid P_n\right]$ and its i^{th} column is given by $P_i := [u^i]_{\overline{u}} \in \mathcal{F}^n$

and $[u^i]_{\overline{u}}$ is the **representation** of u^i with respect to $\overline{u}$.

- **Similarly, there exists an invertible matrix** $\overline{P} = \left[\overline{P}_1 \mid \overline{P}_2 \mid \ldots \mid \overline{P}_n\right]$

with $\overline{P}_i = [\overline{u}^i]_u$ the representation of $\overline{u}^i$ with respect to u,

- **and** $P\overline{P} = \overline{P}P = I$

Theorem 2.40 *There exists an invertible matrix P, with coefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X}, \mathcal{F})$, $[x]_{\overline{u}} = P[x]_u$, where, $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}$ and its i^{th} column is given by $P_i := [u^i]_{\overline{u}} \in \mathcal{F}^n$, and $[u^i]_{\overline{u}}$ is the representation of u^i with respect to $\overline{u}$. Similarly, there exists an invertible matrix $\overline{P} = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 & \cdots & \overline{P}_n \end{bmatrix}$ with $\overline{P}_i = [\overline{u}^i]_u$, the representation of $\overline{u}^i$ with respect to u, and $P \cdot \overline{P} = \overline{P} \cdot P = I$.*

Proof: We can express $x \in \mathcal{X}$ in terms of both bases, $x = \alpha_1 u^1 + \cdots + \alpha_n u^n = \overline{\alpha}_1 \overline{u}^1 + \cdots + \overline{\alpha}_n \overline{u}^n$, so that

$$\alpha := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} := [x]_u \text{ and } \overline{\alpha} := \begin{bmatrix} \overline{\alpha}_1 \\ \overline{\alpha}_2 \\ \vdots \\ \overline{\alpha}_n \end{bmatrix} = [x]_{\overline{u}}$$

From the linearity of the representation operation,

$$\overline{\alpha} := [x]_{\overline{u}} = \left[\sum_{i=1}^{n} \alpha_i u^i \right]_{\overline{u}} = \sum_{i=1}^{n} \alpha_i [u^i]_{\overline{u}} = \sum_{i=1}^{n} \alpha_i P_i = P\alpha. \tag{2.1}$$

Therefore, $\overline{\alpha} := P\alpha = P[x]_u$. Similarly,

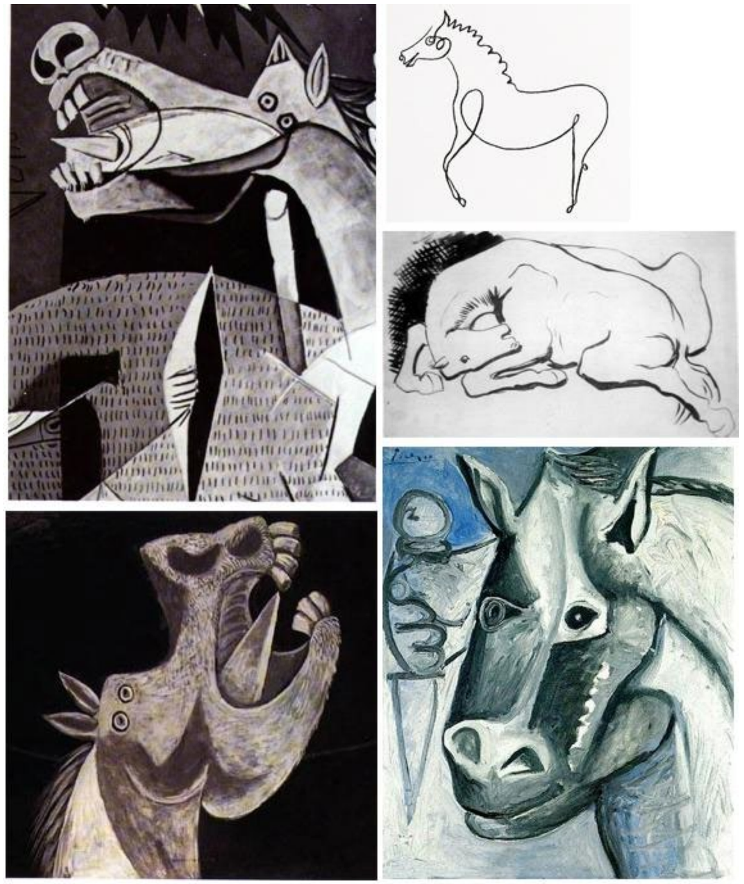
$$\alpha = [x]_u = \left[\sum_{i=1}^{n} \overline{\alpha}_i \overline{u}^i \right]_u = \sum_{i=1}^{n} \overline{\alpha}_i [\overline{u}^i]_u = \sum_{i=1}^{n} \overline{\alpha}_i \overline{P}_i = \overline{P}\overline{\alpha}, \tag{2.2}$$

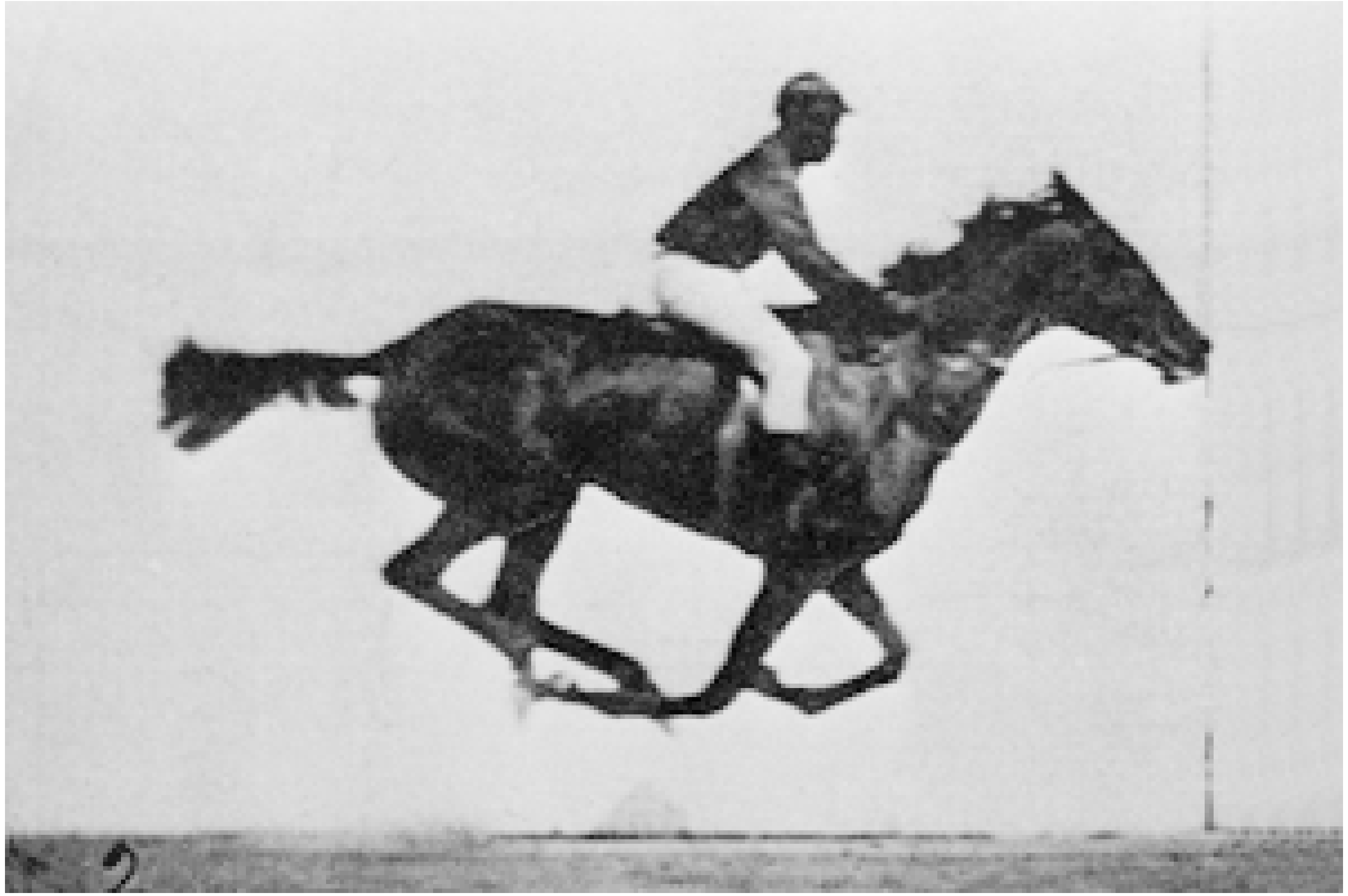
yielding $\alpha = \overline{P}\overline{\alpha}$. Combining (2.1) and (2.2) gives $\alpha = \overline{P}P\alpha$ and $\overline{\alpha} = P\overline{P}\overline{\alpha}$. Because this holds for all x, and hence for all $\alpha =$ and $\overline{\alpha}$, we deduce $P\overline{P} = \overline{P}P = I$.

In conclusion, $\overline{P}$ is the inverse of P ($\overline{P} = P^{-1}$). ∎

Change of ~~Basis~~ Vector?

- What if the underlying thing (the **vector**) is changing? Can we describe the **“action”**?





Operator

- What if the underlying thing (the **vector**) is changing? Can we describe the **"action"**?
- Yes. If the underlying thing is some **"action,"** the mathematical object we use to describe it is an **operator**.
    - **Operators *do*** things. An **operator** is a function that maps from one vector space to another vector space.
        - A rotation operator maps a vector in 3-D space ($\mathbb{R}^3$) to another vector in 3-D space ($\mathbb{R}^3$).
        - A derivative operator (i.e., taking the derivative) maps a function to another function (e.g., $\sin(2x)$ to $2\cos(2x)$).

Definition: Linear Operator

- Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be vector spaces.

  $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is a **linear operator** if for all $x, z \in \mathcal{X}$, $\alpha, \beta \in \mathcal{F}$,

$$\mathcal{L}(\alpha x + \beta z) = \alpha \mathcal{L}(x) + \beta \mathcal{L}(z).$$

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- Note that it is always over the same field $\mathcal{F}$.

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- Let A be an $n \times m$ matrix with coefficients in $\mathcal{F}$.

  Show that $\mathcal{L}: \mathcal{F}^m \to \mathcal{F}^n$ by $\mathcal{L}(x) = Ax$ is a linear operator.

- Let $\mathcal{X} = \{polynomials\ of\ degree \leq 3\}$, $\mathcal{F} = \mathbb{R}$, $\mathcal{Y} = \mathcal{X}$.

  Show that $\mathcal{L}: \mathcal{X} \to \mathcal{Y}$ by $p \in \mathcal{X}$, $\mathcal{L}(p) := d/dt\{p(t)\}$ is a linear operator.

Matrix Representation

- Can **all** linear operators be written down as a matrix multiplication?
  - Rotation operations can be written with rotation matrixes

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Let's try this one

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Matrix Representation for Differentiation

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- Think of polynomials as vectors that are going to be multiplied by some matrix A

  (choose a **basis**)

- **Once again, throw unit vectors at the problem! (Or rather, vectors whose representations are the unit vectors)**

Matrix Representation for Differentiation

- After picking the basis (e.g., monomials)
- Write the representations of the basis vectors. Unsurprisingly, the representations look like the **natural basis**. One 1 and all the rest zeros.

$$[1]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [t]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[t^2]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [t^3]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix Representation for Differentiation

- Take the linear operator of each vector.
- Write out the representations of the resulting vectors.
- These representations form column vectors of the desired matrix!

$$A_1 = [\mathcal{L}(1)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad A_2 = [\mathcal{L}(t)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = [\mathcal{L}(t^2)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad A_4 = [\mathcal{L}(t^3)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Choose some other basis for other functions

- Makes numerical differentiation very easy.
- Fit function to a finite basis made of monomials, sines/cosines, exponentials, etc.
- Taking the derivative is a matrix product!

Strategy

- Write polynomials in monomial basis

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\mathcal{L}"] \arrow[d, "{[x]_u}"'] & \mathcal{Y} \arrow[d, "{[y]_v}"] \\
\mathcal{F}^m \arrow[r, "A"] & \mathcal{F}^n
\end{tikzcd}$$

- Write polynomials in monomial basis

Commutative Diagram

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Definition: Matrix Representation

- Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be **finite-dimensional** vector spaces and $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ be a **linear operator.**

  A **matrix representation** of $\mathcal{L}$ with respect to a **basis** $u := \{u^1, \ldots, u^m\}$ for $\mathcal{X}$ and **basis** $v := \{v^1, \ldots, v^n\}$ for $\mathcal{Y}$ is an $n \times m$ **matrix** A, with coefficients in $\mathcal{F}$, such that

$$\forall x \in \mathcal{X}, \left[\mathcal{L}(x)\right]_v = A \left[x\right]_u$$

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- Next, we will show that we can always find a **matrix representation** and how.

Theorem: Matrix Representation

- Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be **finite-dimensional** vector spaces, $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ a **linear operator**, $u := \{u^1, \ldots, u^m\}$ a **basis** for $\mathcal{X}$, and $v := \{v^1, \ldots, v^n\}$ a **basis** for $\mathcal{Y}$,

  then $\mathcal{L}$ has a **matrix representation** $A = [\ A_1 \ \ldots \ A_m \ ]$, where the i^{th} column of A is given by

$$A_i := \left[\mathcal{L}(u^i)\right]_v, \quad 1 \leq i \leq m$$

Theorem 2.45 *Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces, $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ a linear operator, $u := \{u^1, \ldots, u^m\}$ a basis for $\mathcal{X}$ and $v := \{v^1, \ldots, v^n\}$ a basis for $\mathcal{Y}$, then $\mathcal{L}$ has a matrix representation $A = \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}$, where the i^{th} column of A is given by*

$$A_i := \left[\mathcal{L}(u^i)\right]_v, \;\; 1 \leq i \leq m.$$

Proof: $x \in \mathcal{X}$, we write $x = \alpha_1 u^1 + \cdots + \alpha_m u^m$ so that its representation is

$$[x]_u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \in \mathcal{F}^m.$$

As in the theorem, we define

$$A_i = \left[\mathcal{L}(u^i)\right]_v, \;\; 1 \leq i \leq m.$$

Using linearity

$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}(\alpha_1 u^1 + \cdots + \alpha_m u^m) \\ &= \alpha_1 \mathcal{L}(u^1) + \cdots + \alpha_m \mathcal{L}(u^m). \end{aligned}$$

Hence, computing representations, we have

$$\begin{aligned} [\mathcal{L}(x)]_v &= [\alpha_1 \mathcal{L}(u^1) + \cdots + \alpha_m \mathcal{L}(u^m)]_v \\ &= \alpha_1 [\mathcal{L}(u^1)]_v + \cdots + \alpha_m [\mathcal{L}(u^m)]_v \\ &= \alpha_1 A_1 + \cdots + \alpha_m A_m \end{aligned}$$

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&= \alpha_1 [\mathcal{L}(u^1)]_v + \cdots + \alpha_m [\mathcal{L}(u^m)]_v \\
&= \alpha_1 A_1 + \cdots + \alpha_m A_m \\
&= \begin{bmatrix} A_1 & A_2 & \cdots & A_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \\
&= A \left[x\right]_u .
\end{aligned}$$

Hence, $\left[\mathcal{L}(x)\right]_v = A \left[x\right]_u .$ ■

One last question

- What do you call an operator that does nothing?

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- What do you call an operator that does nothing?
  - **Identity Operation**

$$\mathcal{L} : X \to X \quad \textbf{OR} \quad Id$$

Commutative Diagram

- What is A?

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "Id"] \arrow[d, "{[x]_u}"'] & \mathcal{X} \arrow[d, "{[x]_{\bar{u}}}"] \\
\mathcal{F}^n \arrow[r, "A"] & \mathcal{F}^n
\end{tikzcd}$$

Commutative Diagram

- What is A? The **change of basis matrix**!

$$
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\end{tikzcd}
$$