HW CH2 Solution

Craig 4th ed. Prob.: 2.1, 2.3, 2.12, 2.14, 2.19, 2.20, 2.21, 2.22, 2.27, 2.37, 2.38

2.1) Fixed frame rotation: apply rotations "from right to left."

$$R = rot(\hat{x},\phi)rot(\hat{z},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix}\begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta & -S\theta & 0 \\ C\phi S\theta & C\phi C\theta & -S\phi \\ S\phi S\theta & S\phi C\theta & C\phi \end{bmatrix}$$

2.3) Since rotations are performed about axes of the frame being rotated, these are Euler-Angle rotations: apply rotations "from left to right."

$R = rot(\hat{z},\theta)rot(\hat{x},\phi)$

We might also use the following reasoning:

$${}^{A}R_{B}(\theta,\phi) = {}^{B}R_{A}^{-1}(\theta,\phi) = [rot(\hat{x},-\phi)rot(\hat{z},-\theta)]^{-1} = rot^{-1}(\hat{z},-\theta)rot^{-1}(\hat{x},-\phi) = rot(\hat{z},\theta)rot(\hat{x},\phi)$$

Another way of viewing the same operation:

1st rotate by $rot(\hat{z},\theta)$; 2nd rotate by $rot(\hat{z},\theta)rot(\hat{x},\phi)rot^{-1}(\hat{z},\theta)$

(See similarity transform in Problem 2.19.)

2.12) Velocity is a "free vector" and only will be affected by rotation, and not by translation:

$${}^{A}V = {}^{A}R_{B}\,{}^{B}V = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} -1.34 \\ 22.32 \\ 30.0 \end{bmatrix}$$

2.14)

[Method 1] This rotation can be written as: ${}^{A}T_{B} = \text{trans}({}^{A}\hat{P}, |{}^{A}P|) \cdot \text{rot}(\hat{K},\theta) \cdot \text{trans}(-{}^{A}\hat{P}, |{}^{A}P|)$

where $\text{rot}\left(\hat{K},\theta\right)$ is written as in eq. (2.77),

$$\text{trans}\left({}^{A}\hat{P}, \left|{}^{A}P\right|\right) = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \text{trans}\left(-{}^{A}\hat{P}, \left|{}^{A}P\right|\right) = \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying out we get:

$${}^{A}T_{B} = \left[\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & Q_x \\ R_{21} & R_{22} & R_{23} & Q_y \\ R_{31} & R_{32} & R_{33} & Q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where the R_{ij} are given by eq. (2.77), and:

$$Q_x = P_x - P_x\left(K_x^2 V\theta + C\theta\right) - P_y\left(K_x K_y V\theta - K_z S\theta\right) - P_z\left(K_x K_z V\theta + K_y S\theta\right)$$

$$Q_y = P_y - P_x\left(K_x K_y V\theta + K_z S\theta\right) - P_y\left(K_y^2 V\theta + C\theta\right) - P_z\left(K_y K_z V\theta + K_x S\theta\right)$$
$$Q_z = P_z - P_x\left(K_x K_z V\theta - K_y S\theta\right) - P_y\left(K_y K_z V\theta + K_x S\theta\right) - P_z\left(K_z^2 V\theta + C\theta\right)$$

[Method 2] See Fig. 2.20. ${}^{A}T_{B} = {}^{A}T_{A'}\,{}^{A'}T_{B'}\,{}^{B'}T_{B}$

(i) Rotation

$${}^{A'}\hat{\mathbf{X}}_B = R_K(\theta)\,{}^{A'}\mathbf{P}_{A\hat{\mathbf{X}}} - R_K(\theta)\,{}^{A'}\mathbf{P}_{AORG} = R_K(\theta)({}^{A'}\mathbf{P}_{A\hat{\mathbf{X}}} - {}^{A'}\mathbf{P}_{AORG}) = R_K(\theta)\,{}^{A'}\hat{\mathbf{X}}_A = R_K(\theta)\,{}^{A}\hat{\mathbf{X}}_A$$

Similarly for **Y** and **Z**.

$${}^{A'}R_B = \left[{}^{A'}\hat{\mathbf{X}}_B \quad {}^{A'}\hat{\mathbf{Y}}_B \quad {}^{A'}\hat{\mathbf{Z}}_B\right] = \left[R_K(\theta)\,{}^{A}\hat{\mathbf{X}}_A \quad R_K(\theta)\,{}^{A}\hat{\mathbf{Y}}_A \quad R_K(\theta)\,{}^{A}\hat{\mathbf{Z}}_A\right] = R_K(\theta) I_d = R_K(\theta) = {}^{A'}R_{B'}$$

Therefore, the rotation portion of Frame $\{B\}$ is same as the rotation portion of Frame $\{B'\}$.

(ii) Translation

$${}^{A'}\mathbf{P}_{AORG} = -{}^{A}P_{A'ORG} = -{}^{A}\mathbf{P}$$
$${}^{A'}\mathbf{P}_{BORG} = R_K(\theta)\,{}^{A'}\mathbf{P}_{AORG} = -R_K(\theta)\,{}^{A}\mathbf{P}$$
$${}^{B'}\mathbf{P}_{BORG} = {}^{B'}R_{A'}\,{}^{A'}\mathbf{P}_{BORG} = {}^{B'}R_{A'}\left(-R_K(\theta)\,{}^{A}\mathbf{P}\right) = -{}^{A}\mathbf{P} \quad (\because\ {}^{B'}R_{A'} = R_K(\theta)^{-1})$$

$$\therefore\ {}^{A}T_{B} = {}^{A}T_{A'}\,{}^{A'}T_{B'}\,{}^{B'}T_{B} = \begin{bmatrix} I_d & {}^{A}\mathbf{P} \\ 000 & 1 \end{bmatrix}\begin{bmatrix} R_K(\theta) & 0 \\ 000 & 1 \end{bmatrix}\begin{bmatrix} I_d & -{}^{A}\mathbf{P} \\ 000 & 1 \end{bmatrix} = \begin{bmatrix} R_K(\theta) & {}^{A}\mathbf{P} - R_K(\theta)\,{}^{A}\mathbf{P} \\ 000 & 1 \end{bmatrix}$$

2.19) In the *Z*-*Y*-*Z* Euler Angle set, the first rotation is: $R_1 = rot(\hat{z},\alpha)$

The second rotation expressed in fixed coordinates is: $R_2 = rot(\hat{z},\alpha)rot(\hat{y},\beta)rot^{-1}(\hat{z},\alpha)$

The third is: $R_3 = (R_2R_1)rot(\hat{z},\gamma)(R_2R_1)^{-1}$

The result is: $R = R_3R_2R_1 = rot(\hat{z},\alpha)rot(\hat{y},\beta)rot(\hat{z},\gamma)$, which gives the result of (2.72).

<u>Additional explanation</u> about the description given in the problem statement: In order to perform the rotation about the fixed frame's *y* axis, the *y* axes of the fixed and moving frames need to be made coincident. Therefore, first, bring back the previous rotation about the fixed frame's *x* axis, perform the rotation about the fixed frame's *y* axis, and then re-perform the first rotation about the fixed frame's *x* axis.

2.20)

[Method 1] Transform matrix operations into vector operations.

$$R_K(\theta) = \begin{bmatrix} k_xk_xv\theta + c\theta & k_xk_yv\theta - k_zs\theta & k_xk_zv\theta + k_ys\theta \\ k_yk_xv\theta + k_zs\theta & k_yk_yv\theta + c\theta & k_yk_zv\theta - k_xs\theta \\ k_zk_xv\theta - k_ys\theta & k_zk_yv\theta + k_xs\theta & k_zk_zv\theta + c\theta \end{bmatrix}$$
$$= c\theta \cdot I_d + s\theta\begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} + v\theta\begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_yk_x & k_yk_y & k_yk_z \\ k_zk_x & k_zk_y & k_zk_z \end{bmatrix}$$

$$Q' = R_K(\theta)Q = c\theta \cdot Q + s\theta \underbrace{\begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} Q}_{=\hat{K}\times Q} + v\theta \begin{bmatrix} k_x k_x & k_x k_y & k_x k_z \\ k_y k_x & k_y k_y & k_y k_z \\ k_z k_x & k_z k_y & k_z k_z \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$= Q \cdot c\theta + s\theta(\hat{K} \times Q) + v\theta \underbrace{\begin{bmatrix} k_x(k_x Q_x + k_y Q_y + k_z Q_z) \\ k_y(k_x Q_x + k_y Q_y + k_z Q_z) \\ k_z(k_x Q_x + k_y Q_y + k_z Q_z) \end{bmatrix}}_{=\hat{K}(\hat{K}\bullet Q)} = Q \cdot c\theta + s\theta(\hat{K} \times Q) + (1 - c\theta)(\hat{K} \bullet Q)\hat{K}$$

[Method 2] Derive backwards, i.e., expand the right-hand side of Rodriques' formula.

$$\begin{bmatrix} Q_x \cos\theta \\ Q_y \cos\theta \\ Q_z \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} k_y Q_z - k_z Q_y \\ k_z Q_x - k_x Q_z \\ k_x Q_y - k_y Q_x \end{bmatrix} + (1-\cos\theta)(k_x Q_x + k_y Q_y + k_z Q_z) \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$

$$= \begin{bmatrix} Q_x \cos\theta + \sin\theta(k_y Q_z - k_z Q_y) + (1-\cos\theta)(k_x Q_x + k_y Q_y + k_z Q_z)k_x \\ Q_y \cos\theta + \sin\theta(k_z Q_x - k_x Q_z) + (1-\cos\theta)(k_x Q_x + k_y Q_y + k_z Q_z)k_y \\ Q_z \cos\theta + \sin\theta(k_x Q_y - k_y Q_x) + (1-\cos\theta)(k_x Q_x + k_y Q_y + k_z Q_z)k_z \end{bmatrix}$$

$$= \begin{bmatrix} [k_x k_x(1-\cos\theta) + \cos\theta]Q_x + [k_x k_y(1-\cos\theta) - k_z \sin\theta]Q_y + [k_x k_z(1-\cos\theta) + k_y \sin\theta]Q_z \\ [k_y k_x(1-\cos\theta) + k_z \sin\theta]Q_x + [k_y k_y(1-\cos\theta) + \cos\theta]Q_y + [k_y k_z(1-\cos\theta) - k_x \sin\theta]Q_z \\ [k_z k_x(1-\cos\theta) - k_y \sin\theta]Q_x + [k_z k_y(1-\cos\theta) + k_x \sin\theta]Q_y + [k_z k_z(1-\cos\theta) + \cos\theta]Q_z \end{bmatrix}$$

$$= \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_y k_x v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_z k_x v\theta - k_y s\theta & k_z k_y v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

2.21) Just use the given approximations in (2.80) to obtain:

$$R_K(\delta\theta) = \begin{bmatrix} 1 & -K_Z\delta\theta & K_Y\delta\theta \\ K_Z\delta\theta & 1 & -K_X\delta\theta \\ -K_Y\delta\theta & K_X\delta\theta & 1 \end{bmatrix}$$

More on this is in Chapter 5.

2.22) So, given $R_1 = R_J(\alpha)$ and $R_2 = R_K(\beta)$ with $\alpha << 1$ and $\beta << 1$; show $R_1R_2 = R_2R_1$. If we form the product R_1R_2 and use $\alpha\beta \cong 0$ we have:

$$R_1R_2 = \begin{bmatrix} 1 & -J_Z\alpha - K_Z\beta & J_Y\alpha + K_Y\beta \\ J_Z\alpha + K_Z\beta & 1 & -J_X\alpha - K_X\beta \\ -J_Y\alpha - K_Y\beta & J_X\alpha + K_X\beta & 1 \end{bmatrix}$$

We see that j and k, as well as α and β, appear symmetrically, so $R_1R_2 = R_2R_1$.

2.27) For rotation part, use the definition of the rotation matrix in Equation (2.2). For translation part, write the position vector of the origin of Frame $\{B\}$ with respect to Frame $\{A\}$.

$$\therefore \; {}^{A}T_{B} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.37) Form (2, 4) element of $-{}^{A}R_{B}^{T}\,{}^{A}P_{BORG}$ → to get: –6.4
(See Equation (2.45).)

2.38) $v_1 \bullet v_2 = v_1^T v_2 = \cos\theta$, R preserves angles, so, $(Rv_1)^T(Rv_2) = v_1^T v_2$
$v_1^T R^T R v_2 = v_1^T v_2 \quad \therefore\; R^T R = Id \;\Rightarrow\; R^T = R^{-1}$