Foundations of Robotics
ROB-GY 6003
Homework 4 Answers

November 20, 2023

Shantanu Ghodgaonkar
Univ ID: N11344563
Net ID: sng8399
Ph.No.: +1 (929) 922-0614

Question: 5.4

Answer: In the *position domain*, the Jacobian maps the velocities in X to Y,

$$\dot{Y} = J(X)\dot{X} \tag{1}$$

Similarly in the *force domain*, the Jacobian maps the force $\mathcal{F}$ to the torque τ,

$$\tau = J^T \mathcal{F} \tag{2}$$

And by definition of a Singularity, $Det(J) = 0 \rightarrow J$ *loses full rank*. But as J is common in Eq^n (1) and (2), we can say that *singularities in the force domain exist at the same configurations as singularities in the position domain.*

Question: 5.8

Answer: We know that the Jacobian of the given two-link planar manipulator is given by,

$$J = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \tag{1}$$

For above $Eq^n(1)$ to be isotropic (*i.e. columns are orthogonal and of equal magnitude*),

$$J' = \begin{bmatrix} l_2 & 0 \\ 0 & l_2 \end{bmatrix} \tag{2}$$

Equating J and J', we get,

$$l_1 s_2 = l_2 \tag{3}$$

$$l_1 c_2 + l_2 = 0 \tag{4}$$

Simplifying above $Eq^n s$ we get,

$$s_2 = \frac{l_2}{l_1} \tag{5}$$

$$c_2 = -\frac{l_2}{l_1} \tag{6}$$

But W.K.T., $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow {s_2}^2 + {c_2}^2 = 1 \tag{7}$$

$$\left(\frac{l_2}{l_1}\right)^2 + \left(-\frac{l_2}{l_1}\right)^2 = 1 \tag{8}$$

$$2(l_2)^2 = (l_1)^2 \tag{9}$$

$$l_1 = \pm\sqrt{2}\ l_2 \tag{10}$$

$$\Rightarrow s_2 = \pm\frac{1}{\sqrt{2}} \tag{11}$$

$\therefore$ We can say that an isotropic point exists when $l_1 = \sqrt{2}\ l_2$ and $\theta_2 = \pm 0.7854$ rad.

Question: 5.11

Answer: Given,

$$ {}^{A}T_{B} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ {}^{A}V_{A} = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1.414 \\ 1.414 \\ 0 \end{bmatrix} \tag{1} $$

$$ \Rightarrow {}^{A}R_{B} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ {}^{A}P_{BORG} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix} \tag{2} $$

$$ \Rightarrow {}^{B}R_{A} = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ {}^{A}P_{BORG}\times = \begin{bmatrix} 0 & -5 & 0 \\ 5 & 0 & -10 \\ 0 & 10 & 0 \end{bmatrix} \tag{3} $$

$$ \Rightarrow {}^{B}T_{vA} = \left[\begin{array}{c:c} {}^{B}R_{A} & -{}^{B}R_{A}{}^{A}P_{BORG}\times \\ \hdashline 0 & {}^{B}R_{A} \end{array}\right] = \begin{bmatrix} 0.866 & 0.5 & 0 & -2.5001 & 4.3302 & 5.0002 \\ -0.5 & 0.866 & 0 & -4.3302 & -2.5001 & 8.6604 \\ 0 & 0 & 1 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4} $$

$$ \text{W.K.T.,} {}^{B}V_{B} = {}^{B}T_{vA}{}^{A}V_{A} \tag{5} $$

$$ \Rightarrow {}^{B}V_{B} = \begin{bmatrix} 0.866 & 0.5 & 0 & -2.5001 & 4.3302 & 5.0002 \\ -0.5 & 0.866 & 0 & -4.3302 & -2.5001 & 8.6604 \\ 0 & 0 & 1 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1.414 \\ 1.414 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5878 \\ -7.9260 \\ -17.1400 \\ 1.9316 \\ 0.5175 \\ 0 \end{bmatrix} \tag{6} $$

Question: 5.13

Answer: Given,

$$ {}^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}, \ {}^{0}\mathcal{F} = 10\hat{X}_0 \tag{1} $$

W.K.T.,

$$ \tau = J^T \mathcal{F} \tag{2} $$

Substituting Eq^n (1) in (2)

$$ \tau = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} \tag{3} $$

$$ = \begin{bmatrix} -10 l_1 s_1 - 10 l_2 s_{12} \\ -10 l_2 s_{12} \end{bmatrix} \tag{4} $$

Question: 5.16

Answer: We know that for Z-Y-Z Euler angles, the rotation matrix is given by -

$$ {}^{A}_{B}R_{Z'Y'Z'}(\theta_1, \theta_3, \theta_3) = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & c_1s_2 \\ s_1c_2c_3 - c_1s_3 & -s_1c_2s_3 - c_1c_3 & s_1s_2 \\ -s_2c_3 & s_2s_3 & c_2 \end{bmatrix} \tag{1} $$

And we also know that,

$$ \begin{aligned} \Omega_x &= \dot{r_{31}}r_{21} + \dot{r_{32}}r_{22} + \dot{r_{33}}r_{23} \\ \Omega_y &= \dot{r_{11}}r_{31} + \dot{r_{12}}r_{32} + \dot{r_{13}}r_{33} \\ \Omega_z &= \dot{r_{12}}r_{11} + \dot{r_{22}}r_{12} + \dot{r_{23}}r_{13} \end{aligned} \tag{2} $$

Solving for $Eq^n s$ (1) and (2), we can find $E_{Z'Y'Z'}$ to be -

$$ E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s_1 & c_1s_2 \\ 0 & c_1 & s_1s_2 \\ 1 & 0 & c_2 \end{bmatrix} \tag{3} $$

Question: 5.20

Answer: A *singularity* is a special condition wherein the determinant of the Jacobian equals zero. It implies that the Jacobian is no longer invertible. In the real world, it just means that the joint corresponding to the Jacobian is stuck in its current orientation. So, we might as well assume that it behaves like a link.

Generalising the above explanation, if we have a n-DOF and one of the joints reaches a singularity, it has lost one joint, *i.e. it can be treated as a* $n-1$*-DOF manipulator.*