Reminder

- Invitation to complete course feedback by end of Saturday

Mathematics for Robotics (ROB-GY 6013 Section A)

- **Exam Format**
- **Mental Map**
- **Things to expect/not expect**
- **Extra Study Material Posted**
- **Recorded session link:** https://nyu.zoom.us/rec/share/ZgyIF5gTGop6TkLgRc6bdkHKN72Z7P3vW9gyaCXXgad6whzJv6TY0oRu3fttI0XI.th903pisLGRm5BLX

Exam Format

- **Not cumulative.**
- **No proofs.**
- The exam will be closed-book and closed notes. A formula sheet will be provided to you (shared on NYU Brightspace).
- There will be both short answer and long answer questions.
    - Short answers can be true/false, multiple choice, or fill-in-the-blank.
    - For long answer questions, you must show all your work to obtain full credit.

Mental Map (How to think about the last 8 weeks)

- **Lengths, Distance, and Orthogonality**
  - Norms and Inner Products (Inner Product Spaces)
  - Gram-Schmidt Process
- **Road to Least Squares**
  - (Pre-)Projection Theorem, Normal Equations
  - Weighted Least Squares, Recursive Least Squares
- **Symmetric Matrices**
  - Positive-(semi)definiteness, Quadratic Form, Schur's Complement, other properties
- **Road to Kalman Filter**
  - Probability, BLUE, MVE, Kalman Filter
- **Computations**
  - QR factorization, SVD, LU Factorization, Euler's Method, Newton's Method

Things to Expect/Not Expect

- **Week 7–8 content:**
  - There will be a problem that applies Gram-Schmidt process

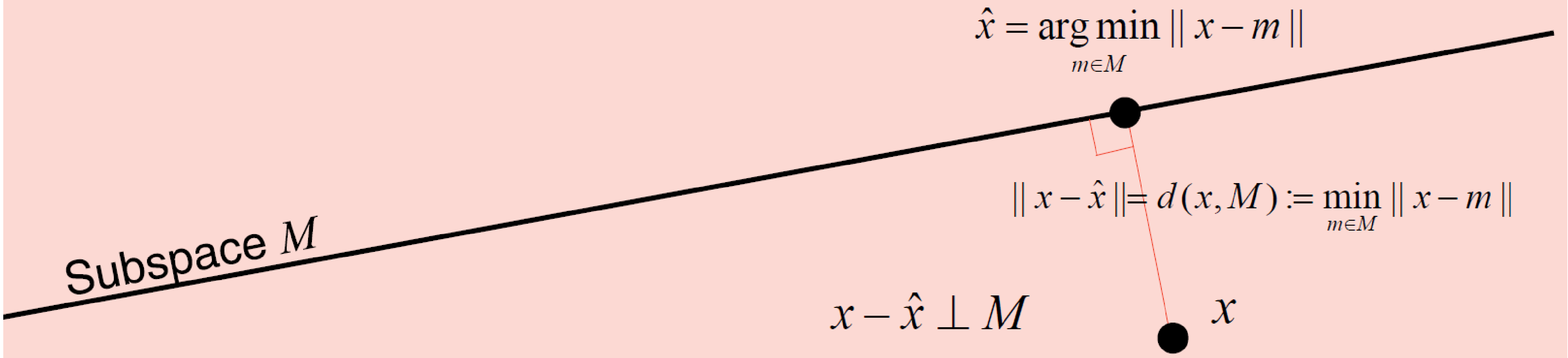
  - Reminder that you can take the norms and inner products of functions
    - Reminder that many things that don't look like vectors are vectors and have norms/inner products. Be prepared.

Things to Expect/Not Expect

- **Week 9 content:**
  - There may be a problem that requires you to creatively apply the normal equations (see next slide)
  - Use this big picture to guide you

Let $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$ be a <u>**finite-dimensional**</u> (real) **inner product space**, M be a **subspace** of $\mathcal{X}$, and x be an arbitrary **point** in $\mathcal{X}$.

Inner product space $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$



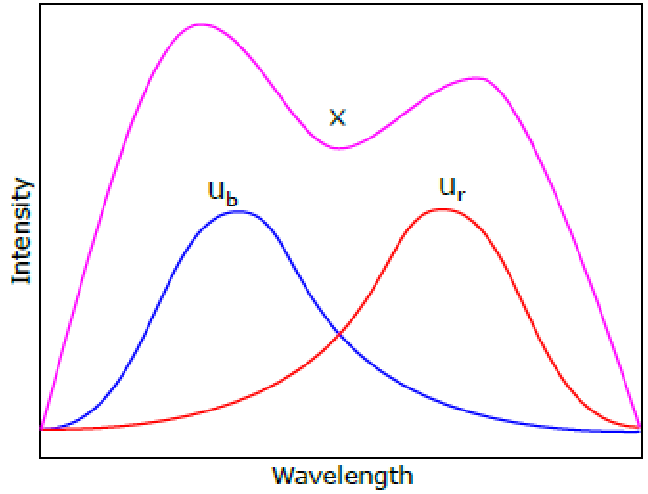


Figure 1: Diagram of our LED color profiles u_b, and u_r, and our target color x.

7. (15 points) The space of all color profiles can be specified by functions $I : [\Lambda_{min}, \Lambda_{max}] \to \mathbb{R}$, where $I(\lambda)$ is the intensity of the color at each wavelength λ. We can represent this space of colors as an inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ where $\mathcal{X} := \{I : [\Lambda_{min}, \Lambda_{max}] \to \mathbb{R}\}$ is a finite-dimensional vector space of continuous functions and the inner product is defined as

$$< f, g > := \int_{\Lambda_{min}}^{\Lambda_{max}} f(\lambda) g(\lambda) d\lambda.$$

Suppose we have two LEDs, one red, one blue, with corresponding intensities $\{u_b, u_r\}$ that can be combined and scaled linearly to form different colors (see Figure. 1). Our goal is to best approximate a desired color with intensity function x using these two LEDs.

We are given the following properties:

$$< u_b, u_b >= 1 \qquad < u_b, u_r >= 0.5 \qquad < u_r, u_r >= 1 \qquad < x, u_b >= 2.5 \qquad < x, u_r >= 1.5$$

Find a linear combination of the colors blue and red that best approximates our desired color profile x. That is, find:

$$\alpha^* = \begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \arg \min_{\alpha_b, \alpha_r \in \mathbb{R}} ||x - (\alpha_b u_b + \alpha_r u_r)||^2$$

using the norm induced by the inner product. Record your answer in the box.

Problem 7:

The set of linear combinations of $\{u_b, u_r\}$ is a subspace of $\mathcal{X}$, so we use the normal equations to find α^*:

$$\begin{bmatrix} < u_b, u_b > & < u_b, u_r > \\ < u_b, u_r > & < u_r, u_r > \end{bmatrix} \begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \begin{bmatrix} < x, u_b > \\ < x, u_r > \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_b^* \\ \alpha_r^* \end{bmatrix} = \begin{bmatrix} 7/3 \\ 1/3 \end{bmatrix}$$

(a) The answer is unique, since the Gram matrix: $\begin{bmatrix} < u_b, u_b > & < u_b, u_r > \\ < u_b, u_r > & < u_r, u_r > \end{bmatrix}$ is full rank and invertible.

Things to Expect/Not Expect

- **Weeks 10–13 content:**
  - Only ***conceptual*** and ***short answer*** questions on probability
    - If X_1 and X_2 are Gaussian random vectors, is their linear combination also a Gaussian random vector?
      - Answer: No
    - What does $E\{\hat{x}-x\}=0$ mean in the context of our derivations?
      - Answer: the estimation algorithm is **unbiased**.

  - You should review the derivations of BLUE, MVE, and Kalman filter from the perspective of understanding the major steps rather than memorizing anything.

Things to Expect/Not Expect

- **Week 14 content:**
    - There will be a problem requiring computations for each of the following:
        - QR factorization (Largest will be 3 x 3 matrix)
        - SVD (Largest will be 3 x 3 matrix)
        - LU factorization (Largest will be 3 x 3 matrix)
        - Euler's Method
        - **Formula sheet will not include anything on Week 14. You must memorize!**
        - **For the factorizations, you may use any method you like.**
        - **For Euler's Method, you must use Euler's method.**
    - **Newton-Raphson method is not on the exam**

Extra Study Material

- Formula sheet
- Handout for Euler's Method posted on NYU Brightspace
- Will post select solutions to homework along with code

- WolframAlpha and MATLAB are very convenient for creating matrices and practicing QR factorization, SVD, and LU Factorization.