Mathematics for Robotics (ROB-GY 6013 Section A)

- **Week 3:**
  - Subspaces
  - Linear Independence/Dependence
  - Basis

Formal Definitions of Field and Vector Space

- 7 Field Axioms
- 10 Vector Axioms
- Some are more important to remember than others, but it’s nice to review all of them
- No axiom is particularly surprising:
  - **Common themes:** closure, commutativity, associativity, distributivity, identity, inverses

Definition 2.1 *(Chen, 2nd edition, page 8) : A **field** consists of a set, denoted by $\mathcal{F}$, of elements called **scalars** and two operations called addition "+" and multiplication "·"; the two operations are defined over $\mathcal{F}$ such that they satisfy the following conditions:*

1. *To every pair of elements α and β in $\mathcal{F}$, there correspond an element $\alpha + \beta$ in $\mathcal{F}$ called the* sum *of α and β, and an element $\alpha \cdot \beta$ (or simply $\alpha\beta$) in $\mathcal{F}$ called the* product *of α and β.*

2. *Addition and multiplication are respectively commutative: For any α and β in $\mathcal{F}$,*

$$\alpha + \beta = \beta + \alpha \qquad\qquad \alpha \cdot \beta = \beta \cdot \alpha$$

3. *Addition and multiplication are respectively associative: For any α, β, γ in $\mathcal{F}$,*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \qquad\qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

4. *Multiplication is distributive with respect to addition: For any α, β, γ in $\mathcal{F}$,*

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

5. *$\mathcal{F}$ contains an element, denoted by 0, and an element, denoted by 1, such that $\alpha + 0 = \alpha$ and $1 \cdot \alpha = \alpha$ for every α in $\mathcal{F}$.*

6. *To every α in $\mathcal{F}$, there is an element β in $\mathcal{F}$ such that $\alpha + \beta = 0$. The element β is called the* additive inverse.

7. *To every α in $\mathcal{F}$ which is not the element 0, there is an element γ in $\mathcal{F}$ such that $\alpha \cdot \gamma = 1$. The element γ is called the* multiplicative inverse.

Definition 2.2 *(Chen 2nd Edition, page 9) A* **vector space (or, linear space)** *over a field $\mathcal{F}$, denoted by $(\mathcal{X}, \mathcal{F})$, consists of a set, denoted by $\mathcal{X}$, of elements called* **vectors**, *a field $\mathcal{F}$, and two operations called* **vector addition** *and* **scalar multiplication**. *The two operations are defined over $\mathcal{X}$ and $\mathcal{F}$ such that they satisfy all the following conditions:*

1. *To every pair of vectors v^1 and v^2 in $\mathcal{X}$, there corresponds a vector $v^1 + v^2$ in $\mathcal{X}$, called the sum of v^1 and v^2* [1].
2. *Addition is commutative: For any v^1, v^2 in $\mathcal{X}$, $v^1 + v^2 = v^2 + v^1$.*
3. *Addition is associative: For any v^1, v^2, and v^3 in $\mathcal{X}$, $\left(v^1 + v^2\right) + v^3 = v^1 + \left(v^2 + v^3\right)$.*
4. *$\mathcal{X}$ contains a vector, denoted by $\boldsymbol{0}$, such that $\boldsymbol{0}+v = v$ for every v in $\mathcal{X}$. The vector $\boldsymbol{0}$ is called the zero vector or the origin.*
5. *To every v in $\mathcal{X}$, there is a vector $\bar{v}$ in $\mathcal{X}$, such that $v + \bar{v} = 0$.*
6. *To every α in $\mathcal{F}$, and every v in $\mathcal{X}$, there corresponds a vector $\alpha \cdot v$ in $\mathcal{X}$ called the* scalar product *of α and v.*
7. *Scalar multiplication is associative: For any α, β in $\mathcal{F}$ and any v in $\mathcal{X}$, $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot v$.*
8. *Scalar multiplication is distributive with respect to vector addition: For any α in $\mathcal{F}$ and any v^1, v^2 in $\mathcal{X}$, $\alpha \cdot \left(v^1 + v^2\right) = \alpha \cdot v^1 + \alpha \cdot v^2$.*
9. *Scalar multiplication is distributive with respect to scalar addition: For any α, β in $\mathcal{F}$ and any v in $\mathcal{X}$, $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.*
10. *For any v in $\mathcal{X}$, $1 \cdot v = v$, where 1 is the element 1 in $\mathcal{F}$.*

Interesting Vector Spaces $(\mathcal{X}, \mathcal{F})$

- For $\mathcal{F} = \mathbb{R}$
  - $\mathcal{X} = \mathbb{R}$ a scalar can be also be a vector
  - $\mathcal{X} = \mathbb{R}^n$ so can a “list” of scalar
  - $\mathcal{X} = \mathbb{R}^{m \times n}$ or a “table” of scalars
  - $\mathcal{X} = \{f : D \to \mathbb{R}\}$ where $D \subset \mathbb{R}$ a function is like a long “list” of scalars

Vector addition and scalar multiplication are defined the way you’d expect.
Not very surprising either.

Interesting Vector Spaces $(\mathcal{X}, \mathcal{F})$

- $\mathcal{X} = \{f : D \to \mathbb{R}\}$ where $D \subset \mathbb{R}$, and $\mathcal{F} = \mathbb{R}$
  The **set of all real-valued functions on *D***, where vector addition and scalar multiplication are defined as:

$$(a)\ \forall\, f, g \in \mathcal{X},\ define\ f + g \in \mathcal{X}\ by\ \forall t \in D,\ (f+g)(t) := f(t) + g(t);$$

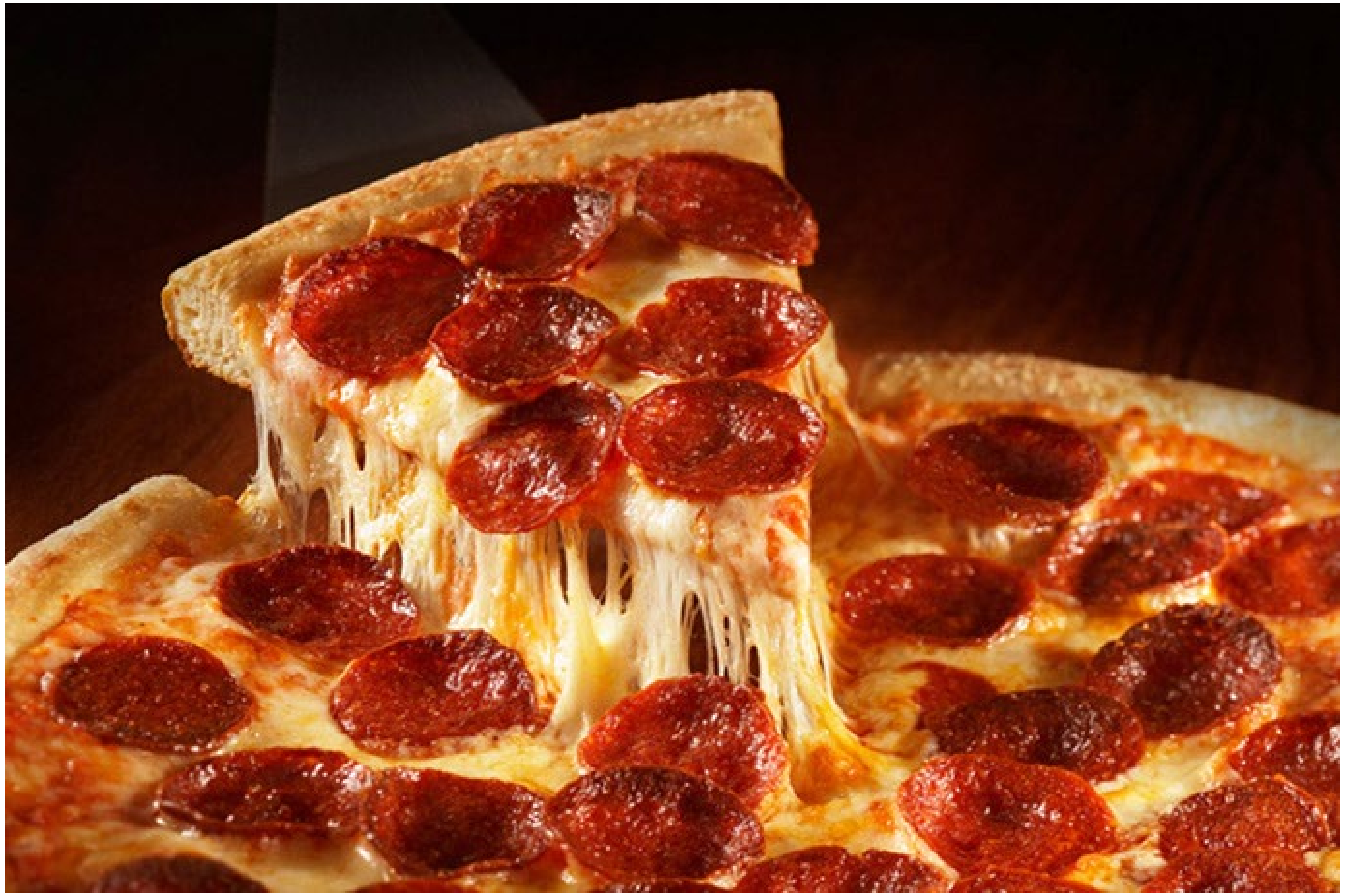
$$(b)\ \forall\, f \in \mathcal{X}\ and\ \alpha \in \mathbb{R},\ define\ \alpha \cdot f \in \mathcal{X}\ by\ \forall t \in D, (\alpha \cdot f)(t) := \alpha \cdot f(t).$$

- Checking **Axioms 4 and 10**

4. $\mathcal{X}$ contains a vector, denoted by $\mathbf{0}$, such that $\mathbf{0}+v = v$ for every v in $\mathcal{X}$. The vector $\mathbf{0}$ is called the zero vector or the origin.

10. For any v in $\mathcal{X}$, $1 \cdot v = v$, where 1 is the element 1 in $\mathcal{F}$.

Parts and the Whole



Set Notation $=, \subset, \not\subset, \subseteq, \subsetneq$

- Let A and B be sets. Then:
  - $(A \subset B) \iff (a \in A \implies a \in B)$
  - $(A = B) \iff (A \subset B \text{ } and \text{ } B \subset A)$

- **Following the book:** We do not use the notion of A being a strict subset of B, which in some books is denoted as $A \subsetneq B$ or $A \subsetneqq B$.

Subspaces

- **Definition:** Let $(\mathcal{X}, \mathcal{F})$ be a vector space, and let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Then $\mathcal{Y}$ is a **subspace** if using the rules of vector addition and scalar multiplication **defined in** $(\mathcal{X}, \mathcal{F})$, we have that $(\mathcal{Y}, \mathcal{F})$ is a vector space.

- The definition builds on definition of a vector space
- How do you check if something is a subspace?

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- The definition builds on definition of a vector space
- How do you check if something is a subspace?
- If you're lazy, try this one first

4. $\mathcal{X}$ contains a vector, denoted by $\boldsymbol{0}$, such that $\boldsymbol{0}+v = v$ for every v in $\mathcal{X}$. The vector $\boldsymbol{0}$ is called the zero vector or the origin.

Subspaces

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- (**Proposition 2.8 in the book**) The following are equivalent (TFAE)

  (a) $(\mathcal{Y},\mathcal{F})$ *is a subspace of* $(\mathcal{X},\mathcal{F})$.

  (b) $\forall v^1, v^2 \in \mathcal{Y}, v^1 + v^2 \in \mathcal{Y}$ *(closed under vector addition),*
  and $\forall y \in \mathcal{Y}$ *and* $\alpha \in \mathcal{F}, \alpha y \in \mathcal{Y}$ *(closed under scalar multiplication).*

  (c) $\forall v^1, v^2 \in \mathcal{Y}, \forall \alpha \in \mathcal{F}, \alpha \cdot v^1 + v^2 \in \mathcal{Y}$.

  (d) $\forall v^1, v^2 \in \mathcal{Y}, \forall \alpha_1, \alpha_2 \in \mathcal{F}, \alpha_1 \cdot v^1 + \alpha_2 \cdot v^2 \in \mathcal{Y}$.

Interesting (or Obvious) Subspaces?

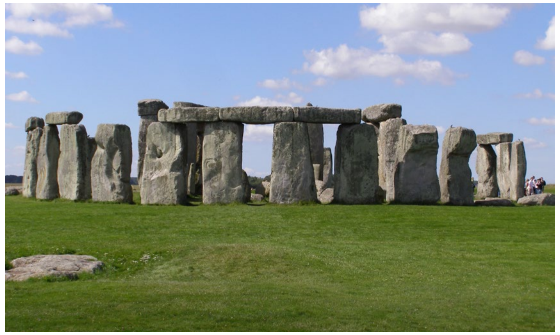
- Think about it

1. *To every pair of vectors* v^1 *and* v^2 *in* $\mathcal{X}$*, there corresponds a vector* $v^1 + v^2$ *in* $\mathcal{X}$*, called the sum of* v^1 *and* v^2 [1].
2. *Addition is commutative: For any* v^1, v^2 *in* $\mathcal{X}$*,* $v^1 + v^2 = v^2 + v^1$.
3. *Addition is associative: For any* v^1, v^2*, and* v^3 *in* $\mathcal{X}$*,* $(v^1 + v^2) + v^3 = v^1 + (v^2 + v^3)$.
4. $\mathcal{X}$ *contains a vector, denoted by* $\mathbf{0}$*, such that* $\mathbf{0} + v = v$ *for every* v *in* $\mathcal{X}$*. The vector* $\mathbf{0}$ *is called the zero vector or the origin.*
5. *To every* v *in* $\mathcal{X}$*, there is a vector* $\bar{v}$ *in* $\mathcal{X}$*, such that* $v + \bar{v} = 0$.
6. *To every* α *in* $\mathcal{F}$*, and every* v *in* $\mathcal{X}$*, there corresponds a vector* $\alpha \cdot v$ *in* $\mathcal{X}$ *called the* scalar product *of* α *and* v.
7. *Scalar multiplication is associative: For any* α, β *in* $\mathcal{F}$ *and any* v *in* $\mathcal{X}$*,* $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot v$.
8. *Scalar multiplication is distributive with respect to vector addition: For any* α *in* $\mathcal{F}$ *and any* v^1, v^2 *in* $\mathcal{X}$*,* $\alpha \cdot (v^1 + v^2) = \alpha \cdot v^1 + \alpha \cdot v^2$.
9. *Scalar multiplication is distributive with respect to scalar addition: For any* α, β *in* $\mathcal{F}$ *and any* v *in* $\mathcal{X}$*,* $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.
10. *For any* v *in* $\mathcal{X}$*,* $1 \cdot v = v$*, where* 1 *is the element* 1 *in* $\mathcal{F}$.

Interesting (or Obvious) Subspaces?

- Just the zero vector* (over the field $\mathcal{F}$)
- $\mathcal{X}$ is a subspace of $\mathcal{X}$ (over the field $\mathcal{F}$)

- Footnote: {0} is a valid vector space. However, mathematicians explicitly exclude it from being a field by having a field axiom that the 0 and 1 element in a field cannot be the same (also known as **zero-one law.)**



Subspace Examples

$$(\mathcal{X}, \mathcal{F}) := (\mathbb{R}^2, \mathbb{R})$$

- $$\mathcal{Y} := \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} \mid \beta \in \mathbb{R} \right\} \subset \mathcal{X}$$
- $$\mathcal{Y} := \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \beta \in \mathbb{R} \right\} \subset \mathcal{X}$$

Check (b) closure under **vector addition** and **scalar multiplication**

Subspace Examples

$$(\mathcal{X}, \mathcal{F}) := (\mathbb{R}^2, \mathbb{R}) \; and \; \mathcal{Y} := \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} \mid \beta \in \mathbb{R} \right\} \subset \mathcal{X}$$

- Check (b) closure under **vector addition** and **scalar multiplication**

$$v^1, v^2 \in \mathcal{Y}, \qquad \underbrace{\begin{bmatrix} \beta_1 \\ 2\beta_1 \end{bmatrix}}_{v^1} + \underbrace{\begin{bmatrix} \beta_2 \\ 2\beta_2 \end{bmatrix}}_{v^2} = \begin{bmatrix} \beta_1 + \beta_2 \\ 2(\beta_1 + \beta_2) \end{bmatrix} \in \mathcal{Y}$$

$$v \in \mathcal{Y} \; and \; \alpha \in \mathbb{R}, \qquad \alpha \underbrace{\begin{bmatrix} \beta \\ 2\beta \end{bmatrix}}_{v} = \begin{bmatrix} \alpha\beta \\ 2(\alpha\beta) \end{bmatrix} \in \mathcal{Y}.$$

Subspace Examples

$$(\mathcal{X}, \mathcal{F}), \mathcal{F} = \mathbb{R}, \textit{where } \mathcal{X} = \{f : \mathbb{R} \to \mathbb{R}\}$$

- $\mathcal{Y} := \mathcal{P}(t) := \{\textit{polynomials in } t \textit{ with real coefficients}\}$
- $\widetilde{\mathcal{Y}} := \{f : \mathbb{R} \to \mathbb{R} \mid f \textit{ is differentiable }, \frac{d}{dt} f \equiv 0\}$
- $\widehat{\mathcal{Y}} := \{f : \mathbb{R} \to \mathbb{R} \mid f(2) = 1.0\}$

Subspace Examples

$$(\mathcal{X}, \mathcal{F}), \mathcal{F} = \mathbb{R}, \textit{ where } \mathcal{X} = \{f : \mathbb{R} \to \mathbb{R}\}$$

- $\mathcal{Y} := \mathcal{P}(t) := \{\textit{polynomials in } t \textit{ with real coefficients}\}$ **Yes**
- $\widetilde{\mathcal{Y}} := \{f : \mathbb{R} \to \mathbb{R} \mid f \textit{ is differentiable }, \frac{d}{dt} f \equiv 0\}$ **Yes**
- $\widehat{\mathcal{Y}} := \{f : \mathbb{R} \to \mathbb{R} \mid f(2) = 1.0\}$ **No, missing zero vector**

Definition: Linear Combination

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space. A **linear combination** is a **finite** sum of the form

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

where $n \geq 1,\ \alpha_1 \in \mathcal{F},\ v^i \in \mathcal{X},\ v^i \in \mathcal{X},\ 1 \leq i \leq n$

To be extra clear, a sum of the form $\sum_{i=1}^{\infty} \alpha_i v^i$ is not a linear combination because it is not finite.

Needs more structure to develop notion of convergence/limits. We will come back to this point later in this lesson!

Motivation: Matrix Equations

- Think about column vectors of a matrix
  - Forget for a moment that matrices can also be thought of as vectors too
- A n $\times$ n matrix A is invertible if and only if the solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$

Definition: Linear (In)dependence

- Let $(\mathcal{X},\mathcal{F})$ be a vector space.

  A finite set of vectors $\{v^1,\ldots,v^k\}$ is **linearly dependent** if there exists $\alpha_1, \ldots \alpha_k \in \mathcal{F}$ **not all zero** such that such that $\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0$.

  Otherwise, the set is **linearly independent**.

Linear Dependence

Suppose $\{v^1, \ldots, v^k\}$ *is a linearly dependent set. Then,* $\exists \alpha_1, \ldots, \alpha_k$ *are not all zero such that*

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0.$$

- We would like to say more interesting than *there exists something…*

Linear Dependence

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$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0.$$

- We would like to say more interesting than *there exists something…*
- **Without loss of generality (WLOG),** suppose $\alpha_k \neq 0$:

$$\alpha_k v^k = -\alpha_1 v^1 - \alpha_2 v^2 - \cdots - \alpha_{k-1} v^{k-1}$$

$$\boxed{v^k = -\frac{\alpha_1}{\alpha_k} v^1 - \frac{\alpha_2}{\alpha_k} v^2 - \cdots - \frac{\alpha_{k-1}}{\alpha_1} v^{k-1}}$$

- v^k is a **linear combination** of the set $\{v^1, \ldots, v^{k-1}\}$

Linear Independence

- For a finite set of vectors, $S := \{v^1, \ldots, v^k\}$, TFAE
    a) The set S is linearly independent.
    b) $\{v^1, \ldots, v^{k-1}\}$ is linearly independent and v^k cannot be written as a linear combination of $\{v^1, \ldots, v^{k-1}\}$.
    c) Every **finite** subset of S is linearly independent.

Definition: Linear Independence *for Infinite Sets*

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space.

  An arbitrary set of vectors $S \subset \mathcal{X}$ is **linearly independent** if every finite subset is **linearly independent**.

Definition: Linear Independence *for Infinite Sets*

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space.

  An arbitrary set of vectors $S \subset \mathcal{X}$ is **linearly independent** if every finite subset is **linearly independent**.

- Let $(\mathcal{X}, \mathcal{F})$ be the following vector space.

  $$\mathcal{F} = \mathbb{R} \textit{ and } \mathcal{X} = \mathbb{P}(t) = \{ \textit{ set of polynomials with real coefficients } \}$$

  The monomials are **linearly independent**. That is, for each $n \geq 0$, the set $\{1, t, \ldots, t_n\}$ is linearly independent. We will show this very soon.

Definition: Linear (In)dependence

- How to prove something is **linearly dependent**.
  - Straightforward to use the definition.
    - Just find a linear combination that adds up to zero other than the trivial solution ($\alpha_1 = \ldots = \alpha_k = 0$).

- How to prove something is **linearly independent** (NOT **linearly dependent**).
  - Start by writing down $\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0$.
  - Do some work…
  - Show from your work that $\alpha_1 = 0$, $\alpha_2 = 0$, … , $\alpha_k = 0$ **MUST** be true.

Linear Independence Examples

- Let $\mathcal{F} = \mathbb{R}$ *and* $\mathcal{X} = \mathbb{R}^{2\times 3}$, the set of 2 × 3 matrices with real coefficients

$$v^1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, v^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; v^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, v^4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Is the set $\{v^1, v^2, v^3, v^4\}$ **linearly independent**?

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Is the set $\{v^1, v^2, v^3, v^4\}$ **linearly independent**?

- What is my zero vector?

Solution: To show independence, we must show that the only linear combination resulting in the zero vector in $\mathcal{X}$ is a trivial linear combination. Hence, we check

$$\alpha_1 v^1 + \alpha_2 v^2 = 0_{2\times 3} \iff \begin{bmatrix} \alpha_1 & 0 & 0 \\ 2\alpha_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \iff \alpha_1 = \alpha_2 = 0.$$

Hence, $\{v^1, v^2\}$ is a linearly independent set.

For $\{v^1, v^2, v^4\}$, we form a linear combination and seek coefficients resulting in the zero vector,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_4 v^4 = 0_{2\times 3} \iff \begin{bmatrix} \alpha_1 & 0 & 0 \\ 2\alpha_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \alpha_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 & 0 & 0 \\ 2\alpha_1 + \alpha_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One then checks that $\alpha_1 = 1, \alpha_2 = -1, \alpha_4 = -2$ is a nontrivial solution. There are infinite number of other nontrivial solutions. We only needed to find one to show that $\{v^1, v^2, v^4\}$ is a linearly dependent set of vectors.

$\blacksquare$

Linear Independence Examples

- Let $\mathcal{F} = \mathbb{R}$ *and* $\mathcal{X} = \mathbb{R}^{2\times 3}$, the set of 2×3 matrices with real coefficients

$$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

Is the set $\{A_1, A_2\}$ **linearly independent**?

Linear Independence Examples

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$$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

  Is the set $\{A_1, A_2\}$ **linearly independent**?

- What is the maximum number of linear independent vectors I can have?

Solution: We form a linear combination of A_1 and A_2 and check for a nontrivial solution.

$$\alpha_1 A_1 + \alpha_2 A_2 = \begin{bmatrix} \alpha_1 + 4\alpha_2 & \alpha_2 & 4\alpha_1 \\ 3\alpha_1 + 6\alpha_2 & -\alpha_1 & 2\alpha_1 + 6\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \iff \begin{cases} \alpha_1 + 4\alpha_2 & = 0 \\ \alpha_2 & = 0 \\ 4\alpha_1 & = 0 \\ 3\alpha_1 + 6\alpha_2 & = 0 \\ -\alpha_1 & = 0 \\ 2\alpha_1 + 6\alpha_2 & = 0 \end{cases}$$

We wrote out all of the equations to emphasize that when you set a 2×3 matrix equal to the zero matrix, each of its entries must be zero. We only have two unknowns, so we could have gotten by with just noting that the second and fifth equations together imply that $\alpha_1 = \alpha_2 = 0$. We conclude that the set $\{A_1, A_2\}$ is linearly independent. ■

Linear Independence Examples

- Let $(\mathcal{X},\mathcal{F})$ be the following vector space.

$$\mathcal{F} = \mathbb{R} \textit{ and } \mathcal{X} = \mathbb{P}(t) = \{ \textit{ set of polynomials with real coefficients } \}$$

The monomials are **linearly independent**. That is, for each $n \geq 0$, the set $\{1, t, \ldots, t_n\}$ is linearly independent.

Direct Proof: Suppose that $p(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n = 0$ is the zero polynomial. We need to show that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. From Calculus, we know that a polynomial of degree n is identically zero if, and only, if, $p(0) = 0$ and $\frac{d^k p(t)}{dt^k}|_{t=0} = 0$ for $k = 1, 2, \ldots, n$. Armed with this notion, we check that

$$\begin{aligned}
0 = p(0) &\implies \alpha_0 = 0 \\
0 = \frac{dp(t)}{dt}|_{t=0} = (\alpha_1 + 2\alpha_2 t + \cdots + n\alpha_n t^{n-1})|_{t=0} &\implies \alpha_1 = 0 \\
0 = \frac{d^2 p(t)}{dt^2}|_{t=0} = (2\alpha_2 + 6\alpha_3 t + \cdots + n(n-1)\alpha_n t^{n-2})|_{t=0} &\implies \alpha_2 = 0 \\
&\vdots \\
0 = \frac{d^n p(t)}{dt^n}|_{t=0} = n!\alpha_n &\implies \alpha_n = 0.
\end{aligned}$$

Linear Independence Examples

$$(\mathbb{C}, \mathbb{R}) \quad (\mathbb{C}, \mathbb{C})$$

- A set with just one vector

Linear Independence Examples

$$(\mathbb{C}, \mathbb{R}) \quad (\mathbb{C}, \mathbb{C})$$

Remark 2.19 *The field $\mathcal{F}$ is important when determining whether a set is linearly independent or not. For example, let $\mathcal{X} = \mathbb{C}$ and $v^1 = 1$, $v^2 = j := \sqrt{-1}$. v^1 and v^2 are linearly independent when $\mathcal{F} = \mathbb{R}$. However, they are linearly dependent when $\mathcal{F} = \mathbb{C}$.*

- A set with just one vector
    - The set {0} containing only the zero vector is ***not*** linearly independent! Any scalar multiplied by {0} will give {0}, which is a non-trivial solution!

Definition: Span

- Let S be a subset of a vector space $(\mathcal{X},\mathcal{F})$.

  The **span** of S, denoted $\mathrm{span}\{S\}$, is the set of all linear combinations of elements of S.

$$\mathrm{span}\{\mathcal{S}\} := \{x \in \mathcal{X} \mid \exists n \geq 1, \alpha_1, \ldots, \alpha_n \in \mathcal{F}, v^1, \ldots, v^n \in \mathcal{S},\ s.t.\ x = \alpha_1 v^1 + \cdots + \alpha_n v^n\}$$

- By construction, $\mathrm{span}\{S\}$, is ***closed*** under linear combinations. Hence, it is a **subspace** of X.

Span Example

- What is the span of $S = \{1, t, t^2, ...\}$

- and is $e^t \in \mathrm{span}\{S\}$?

Span Example

- What is the span of $S = \{1, t, t^2, \ldots\}$

  $\text{span}\{S\} = \text{span}\{t^k, k \geq 0\}$, the set of polynomials in t with real coefficients

- and is $e^t \in \text{span}\{S\}$?

  No. No infinite sums/Taylor series allowed! The following is NOT a finite sum of monomials. It is NOT a linear combination of monomials!

$$e^t = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Basis

- Let $(\mathcal{X}, \mathcal{F})$ be a vector space.

  A set of vectors $\mathcal{B}$ in $(\mathcal{X}, \mathcal{F})$ is a basis for $\mathcal{X}$ if

  a) $\mathcal{B}$ is linearly independent.

  b) $\operatorname{span}\{\mathcal{B}\} = \mathcal{X}$

Just enough to span the entire vector space without anything extra

Natural Basis

- **Natural basis** $\{e^1, e^2, \ldots, e^n\}$

$$(\mathcal{F}^n, \mathcal{F})$$

$$(\mathbb{R}^n, \mathbb{R})$$

$$(\mathbb{C}^n, \mathbb{C})$$

$$(\mathbb{Q}^n, \mathbb{Q})$$

$$e^1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e^2 := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e^n := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Natural Basis

- **Natural basis** $\{e^1, e^2, \ldots, e^n\}$

$(\mathcal{F}^n, \mathcal{F})$

$(\mathbb{R}^n, \mathbb{R})$

$(\mathbb{C}^n, \mathbb{C})$

$(\mathbb{Q}^n, \mathbb{Q})$

$$e^1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e^2 := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e^n := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Linear combinations of $\{e^1, e^2, \ldots, e^n\}$ $\Rightarrow$ $\alpha_1 e^1 + \alpha_2 e^2 + \cdots + \alpha_n e^n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ $\Rightarrow$ $\mathcal{F}^n$

More Bases

- **Another basis** $\{v^1, v^2, \ldots, v^n\}$

$$(\mathcal{F}^n, \mathcal{F})$$

$$(\mathbb{R}^n, \mathbb{R})$$

$$(\mathbb{C}^n, \mathbb{C})$$

$$(\mathbb{Q}^n, \mathbb{Q})$$

$$v^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v^2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, v^n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

More Bases

- $\{je^1, je^2, \ldots, je^n\}$ *is also a basis for* $(\mathbb{C}^n, \mathbb{C})$
- $\{e^1, e^2, \ldots, e^n, je^1, je^2, \ldots, je^n\}$ *is a basis for* $(\mathbb{C}^n, \mathbb{R})$
- *The infinite set* $\{1, t, \ldots, t^n, \ldots\}$ *is a basis for* $(\mathbb{P}(t), \mathbb{R})$

- $\{e^1, e^2, \ldots, e^n\}$ *is NOT a basis for* $(\mathbb{C}^n, \mathbb{R})$
- $\{e^1, e^2, \ldots, e^n, je^1, je^2, \ldots, je^n\}$ *is not a basis of* $(\mathbb{C}^n, \mathbb{C})$

Definition: Dimension

- Let n be a natural number. The vector space $(\mathcal{X}, \mathcal{F})$ has **finite dimension** n if
  a) there exists a set with n linearly independent vectors,
  b) and any set with $n + 1$ vectors is linearly dependent.

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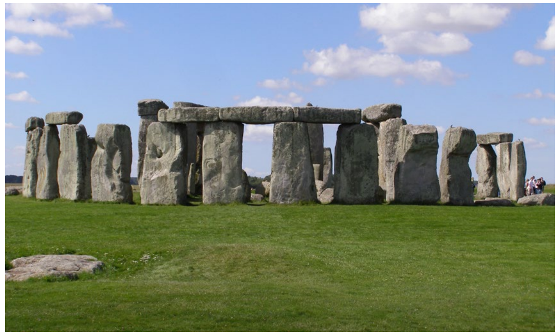
- Isn't this a waste of time? Why not build on the definition of **basis**?
  - Takes extra work to show that a basis exists, especially for infinite-dimensional spaces (see **Hamel basis**)

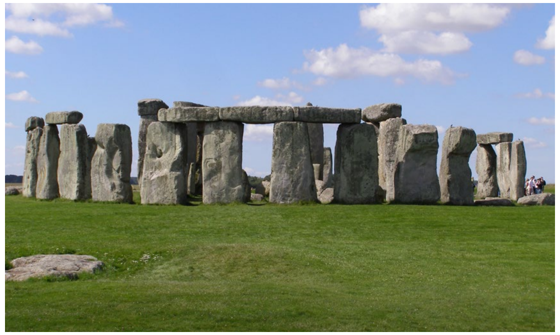


Dimension Examples

- $\dim(\mathcal{F}^n, \mathcal{F}) =$
- $\dim(\mathbb{C}^n, \mathbb{R}) =$
- $\dim(\mathbb{P}(t), \mathbb{R}) =$
- $\dim(\mathbb{R}, \mathbb{Q}) =$

Dimension Examples

- $\dim(\mathcal{F}^n, \mathcal{F}) = n.$
- $\dim(\mathbb{C}^n, \mathbb{R}) = 2n.$
- $\dim(\mathbb{P}(t), \mathbb{R}) = \infty.$
- $\dim(\mathbb{R}, \mathbb{Q}) = \infty.$



See Problem 6:
http://www2.math.ou.edu/~aroche/courses/LinAlg-Fall2011/solutions1.pdf

Back to Basi(c)s

- Let $(\mathcal{X}, \mathcal{F})$ be an n-**dimensional** vector space (always means n is finite). Then, any set of n linearly independent vectors is a **basis**.

Theorem 2.31 *Let $(\mathcal{X}, \mathcal{F})$ be an n-dimensional vector space (*always means n is finite*). Then, any set of n linearly independent vectors is a basis.*

Proof: Let $\{v^1, \ldots, v^n\}$ be a linearly independent set in $(\mathcal{X}, \mathcal{F})$. To show it is a basis, we need to show the "span" property, namely,

$$\forall x \in \mathcal{X}, \exists\, \alpha_1, \ldots, \alpha_n \in \mathcal{F} \text{ such that } x = \alpha_1 v^1 + \cdots + \alpha_n v^n.$$

Let $x \in \mathcal{X}$ be given. Because $(\mathcal{X}, \mathcal{F})$ is n-dimensional, the set $\{x, v^1, \ldots, v^n\}$ is a linearly dependent set, because, otherwise, $\dim(\mathcal{X}) > n$. Hence, $\exists\, \beta_0, \beta_1, \ldots, \beta_n \in \mathcal{F}$, NOT ALL ZERO, such that $\beta_0 x + \beta_1 v^1 + \cdots + \beta_n v^n = 0$.

Claim 2.32 $\beta_0 \neq 0$.

Proof: We do a proof by contradiction. Suppose that $\beta_0 = 0$. Then,

(a) At least one of $\beta_1, \ \ldots, \ \beta_n$ is non-zero, and

(b) $\beta_1 v^1 + \cdots + \beta_n v^n = 0$.

(a) and (b) imply that $\{v^1, \ldots, v^n\}$ is a linearly dependent set. However, by assumption, $\{v^1, \ldots, v^n\}$ is a basis and hence linearly independent. This is a contradiction. Hence, $\beta_0 = 0$ cannot hold. $\square$

Because $\beta_0 \neq 0$, we complete the proof by

$$\beta_0 x = -\beta_1 x^1 - \ldots - \beta_n v^n$$
$$\Downarrow$$
$$x = \left(\frac{-\beta_1}{\beta_0}\right) v^1 + \ldots + \left(\frac{-\beta_n}{\beta_0}\right) v^n$$

and therefore, $\alpha_1 := \frac{-\beta_1}{\beta_0}, \ldots, \alpha_n := \frac{-\beta_n}{\beta_0}$ are the required coefficients in $\mathcal{F}$. $\blacksquare$

Back to Basi(c)s

- Let $(\mathcal{X},\mathcal{F})$ be an n-**dimensional** vector space (always means n is finite). Then, any set of n linearly independent vectors is a **basis**.

- Let $(\mathcal{X},\mathcal{F})$ be an n-**dimensional** vector space with a **basis** $\{v^1, \ldots, v^n\}$ and let $x \in \mathcal{X}$. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

Proposition 2.33 *Let $(\mathcal{X}, \mathcal{F})$ be a vector space with basis $\{v^1, \ldots, v^n\}$ and let $x \in \mathcal{X}$. Then, there exist* unique *coefficients $\alpha_1, \ldots, \alpha_n$ such that*

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n.$$

Proof: Suppose x can also be written as $x = \beta_1 v^1 + \beta_2 v^2 + \ldots + \beta_n v^n$. We need to show: $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$. To do so, write

$$0 = x - x = (\alpha_1 - \beta_1)v^1 + \ldots + (\alpha_n - \beta_n)v^n.$$

By the linear independence of $\{v^1, \ldots, v^n\}$, we obtain that

$$\alpha_1 - \beta_1 = 0, \ldots, \alpha_n - \beta_n = 0.$$

Hence, $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$, that is, the coefficients are unique. ■

Back to Basi(c)s

- Let $(\mathcal{X},\mathcal{F})$ be an **n-dimensional** vector space (always means n is finite). Then, any set of n linearly independent vectors is a **basis**. (See Theorem 2.31)

- Let $(\mathcal{X},\mathcal{F})$ be an **n-dimensional** vector space with a **basis** $\{v^1, \ldots, v^n\}$ and let $x \in \mathcal{X}$. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n$$

- Let $(\mathcal{X},\mathcal{F})$ be an **n-dimensional** vector space and let $\{v^1, \ldots, v^k\}$ be a **linearly independent** set with k strictly less than n. Then, there exists $v^{k+1} \in \mathcal{X}$ such that $\{v^1, \ldots, v^k, v^{k+1}\}$ is **linearly independent**
  - Keep completing a basis if $k < n$!

Proposition 2.34 *Let* $(\mathcal{X}, \mathcal{F})$ *be an* n*-dimensional vector space and let* $\{v^1, \cdots, v^k\}$ *be a linearly independent set with* k *strictly less than* n*. Then,* $\exists\, v^{k+1} \in \mathcal{X}$ *such that* $\{v^1, \cdots, v^k, v^{k+1}\}$ *is linearly independent.*

Proof: We use proof by contradiction. Suppose that $\{v^1, \cdots, v^k\}$ is a linearly independent and $k < n$, but no such v^{k+1} exists. Then, $\forall x \in \mathcal{X}$, $x \in \operatorname{span}\{v^1, \cdots, v^k\}$, and therefore, $\mathcal{X} \subset \operatorname{span}\{v^1, \ldots, v^k\}$. This in turn implies that $n = \dim(\mathcal{X}) \leq \dim(\operatorname{span}\{v^1, \ldots, v^k\}) = k$, which contradicts $k < n$. Hence, there must exist $v^{k+1} \in \mathcal{X}$ such that $\{v^1, \ldots, v^k, v^{k+1}\}$ is linearly independent. $\blacksquare$

Completing the basis

- In a finite dimensional vector space, any linearly independent set can be completed to a basis.

  More precisely, let $\{v^1, \ldots, v^k\}$ be **linearly independent**, $n = \dim(\mathcal{X})$ and $k < n$. Then, there exists $v^{k+1}, \ldots, v^n$ such that $\{v^1, \ldots, v^k, v^{k+1}, \ldots, v^n\}$ is a **basis** for $\mathcal{X}$.

Proposition 2.34 *Let $(\mathcal{X}, \mathcal{F})$ be an n-dimensional vector space and let $\{v^1, \cdots, v^k\}$ be a linearly independent set with k strictly less than n. Then, $\exists\, v^{k+1} \in \mathcal{X}$ such that $\{v^1, \cdots, v^k, v^{k+1}\}$ is linearly independent.*

Proof: We use proof by contradiction. Suppose that $\{v^1, \cdots, v^k\}$ is a linearly independent and $k < n$, but no such v^{k+1} exists. Then, $\forall x \in \mathcal{X}$, $x \in \operatorname{span}\{v^1, \cdots, v^k\}$, and therefore, $\mathcal{X} \subset \operatorname{span}\{v^1, \ldots, v^k\}$. This in turn implies that $n = \dim(\mathcal{X}) \leq \dim(\operatorname{span}\{v^1, \ldots, v^k\}) = k$, which contradicts $k < n$. Hence, there must exist $v^{k+1} \in \mathcal{X}$ such that $\{v^1, \ldots, v^k, v^{k+1}\}$ is linearly independent. ■

Corollary 2.35 *In a finite dimensional vector space, any linearly independent set can be completed to a basis. More precisely, let $\{v^1, \cdots, v^k\}$ be linearly independent, $n = \dim(\mathcal{X})$ and $k < n$. Then, $\exists v^{k+1}, \ldots, v^n$ such that $\{v^1, \cdots, v^k, v^{k+1}, \ldots, v^n\}$ is a basis for $\mathcal{X}$.*

Proof: Previous Proposition plus induction. ■

Back to Basics

$$\mathbf{p} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$