Mathematics for Robotics
ROB-GY 6103
Homework 1 Answers

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Question: 1. (a) Negate $(P \wedge Q)$

Answer: We must find the negation of the given expression $(P \wedge Q)$

According to De Morgan's Laws, $\sim(P \wedge Q) = \sim P \vee \sim Q$

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

Question: 1. (b) Negate $(P \vee Q)$

Answer: We must find the negation of the given expression $(P \vee Q)$

According to De Morgan's Laws, $\sim(P \vee Q) = \sim P \wedge \sim Q$

P	Q	$P \vee Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F

Question: 2. (a) Negate P : "For every integer n, 2n + 1 is odd."

Answer: $\sim P$: "For some integer n, 2n + 1 is not odd"

Question: 2. (b) Negate P : "For some integer n, 2n + 1 is prime."

Answer: $\sim P$: "For every integer n, 2n + 1 is not prime"

Question: 2. (c) Let A be an $n \times n$ real matrix and $\lambda \in \mathbb{R}$. P:" $\exists\, v \in \mathbb{R}^n, v \neq 0, such\ that\ Av = \lambda v$."

Answer: $\sim P$: "$\forall\, v \in \mathbb{R}^n, v \neq 0, such\ that\ Av \neq \lambda v$"

Question: 2. (d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. P: "$\forall\, \eta > 0, \exists\, \delta > 0\ such\ that\ |x| \leq \delta \Rightarrow |f(x)| \leq \eta|x|$"

Answer: $\sim P$: "$\exists\, \eta > 0, \forall\, \delta > 0\ such\ that\ |x| \leq \delta \Rightarrow |f(x)| > \eta|x|$"

Question: 3. Prove that $\sqrt{7}$ is irrational.
Assume that "Let m be an integer. If 7 divides m^2, then 7 also divides m" is *true*

Answer: We will prove this by contradiction. Suppose that $\sqrt{7}$ is a rational number. Then there exist two integer p and q without common factors such that,

$$\sqrt{7} = \frac{p}{q} \tag{1}$$

Squaring both sides,

$$7 = \frac{p^2}{q^2} \tag{2}$$

Rearranging $Eq^n(2)$,

$$7q^2 = p^2 \tag{3}$$

$Eq^n(3) \Rightarrow p^2$ is a multiple of 7 $\Rightarrow$ 7 is a factor of p (Using given assumption)$\rightarrow$ (3.1)

Consider, $\exists\, r \in \mathbb{Q} \mid p = 7r \leftarrow$ substitute in $Eq^n(3)$

$$7q^2 = 49r^2 \tag{4}$$

$$q^2 = 7r^2 \tag{5}$$

$Eq^n(5) \Rightarrow q^2$ is a multiple of 7 $\Rightarrow$ 7 is a factor of q (Using given assumption)$\rightarrow$ (5.1)

From statements (3.1) & (5.1) we can deduce that p and q do have the common factor of 7, which goes against the definition of rational numbers.
$\therefore$ by Contradiction, $\sqrt{7}$ is irrational. **QED**.

Question: 4. Let A be a square matrix. Prove: if $det(A) = 0$, then A is not invertible.

Answer: We shall prove this by contradiction.

Consider a square matrix A such that $det(A) = 0$ and assume that A is invertible.
$\Rightarrow$ there exists a matrix, A', such that,

$$A \times A' = I \tag{1}$$

Now, take the determinant of both sides,

$$det(A \times A') = det(I) \tag{2}$$

It is given that $det(AB) = det(A)det(B)$; Applying this relation, we get,

$$det(A) \times det(A') = det(I) \tag{3}$$

We know that, det(I) = 1 and it is given that det(A) = 0

$$0 \times det(A') = 1 \tag{4}$$

$$\Rightarrow 0 = 1 \tag{5}$$

$Eq^n(5)$ states an impossibilty $\Rightarrow$ our initial assumption that A is invertible was *wrong*.
$\therefore$ a materix A having $det(A) = 0$ is $Non - Invertible$. **QED**

Question: 5. Prove that, for all integers,

$$n \geq 1, \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

Answer: We shall prove the given statement using standard induction.

- Step 0 : For $n \in \mathbb{Q} | n \geq 1$, $P(n) = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$
- Step 1 : For the base case, n = 1,

$$P(1) = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1+1}$$

$$\frac{1}{1(1+1)} = \frac{1}{2}$$

$$\therefore \frac{1}{2} = \frac{1}{2}$$

- Step 2 : Now, we must show that the induction hypothesis is true. Using the fact that for $1 \geq j \geq k$, show that $P(k+1)$ is true. So for, $n = k+1$

$$\Rightarrow P(k+1) = \sum_{k=1}^{k+1} \frac{1}{k(k+1)} = \frac{k+1}{(k+1)+1}$$

$$= \sum_{k=1}^{k} \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Finding the LCM of the LHS,

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Cancelling out $(k+2)$ on the denominator for both sides,

$$= \frac{k(k+2)+1}{(k+1)} = k+1$$

Mulitplying the numerator on both sides by $(k+1)$,

$$= k(k+2)+1 = (k+1)(k+1)$$

$$= k^2+2k+1 = (k+1)^2$$

Rewriting LHS,

$$= (k+1)^2 = (k+1)^2$$

Dividing both sides by $(k+1)$,

$$= k+1 = k+1$$

$$\therefore LHS = RHS$$

Hence, $P(k+1)$ is true. Because we have shown that $P(1)$ is true and for all $n \geq 1$,$P(n) \Rightarrow P(n+1)$, by the Principle of Induction, we conclude that,

$$\forall n \in \mathbb{Q} | n \geq 1 \ , \ P(n) = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

QED.

Question: 6. (a) Prove that, for all integers, $n \geq 12$, there exist non-negative integers k_1 and k_2 such that $n = k_1 4 + k_2 5$. Is the same statement true for $n \geq 8$?

Answer: We shall prove the given statement using strong induction.

- Step 0 : For $n \geq 12$, $P(n) : \exists\, k_1, k_2 \in \mathbb{Q}\ and\ k_1, k_2 \geq 0 \mid n = k_1 4 + k_2 5$
- Step 1 : For the base case, n = 12,

$$P(12) : 12 = k_1 4 + k_2 5$$

Substituting $k_1 = 3$ and $k_2 = 0$,

$$P(12) : 12 = 3 \cdot 4 + 0 \cdot 5$$

$$P(12) : 12 = 12$$

- Step 2 : Now, we must show that the induction hypothesis is true. Using the fact that for $12 \geq j \geq k$, show that $P(k+1)$ is true.
So for, $n = k+1$

$$\Rightarrow k+1 \geq 12$$

$$k \geq 13$$

Condsider the base case here, n = 13

$$P(13) : 13 = k_1 4 + k_2 5$$

Substituting $k_1 = 2$ and $k_2 = 1$,

$$P(13) : 13 = 2 \cdot 4 + 1 \cdot 5$$

$$P(13) : 13 = 13$$

$\therefore$ We can see that this satisfies the original statement and has already been proven by the induction hypothesis. **QED**.

- Is it true for $n \geq 8$?
  Consider the case $n = 11$,
  $$\Rightarrow P(11) : 11 = k_1 4 + k_2 5$$
  Upon observing the above equation, we can deduce that there is no possible combination of non-negative integers k_1, k_2 that can satisfy the relation.
  $\therefore$ the given equation is false for $n \geq 8$. **QED**.

Question: 6. (b) Prove that, for all even integers, $n \geq 6$, there exist non-negative integers k_1 and k_2 such that $n = k_1 3 + k_2 5$.

Answer: We shall prove the given statement using strong induction.

- Step 0 : For $n \geq 6 | n = 2x$ where $x \in \mathbb{N}$, $P(n) : \exists\, k_1, k_2 \in \mathbb{Q}$ and $k_1, k_2 \geq 0 \mid n = k_1 3 + k_2 5$
- Step 1 : For the base case, n = 6,
  $$P(6) : 6 = k_1 3 + k_2 5$$
  Substituting $k_1 = 2$ and $k_2 = 0$,
  $$P(6) : 6 = 2 \cdot 3 + 0 \cdot 5$$
  $$P(6) : 6 = 6$$
- Step 2 : Now, we must show that the induction hypothesis is true. Using the fact that for $6 \geq j \geq k$, show that $P(k + 1)$ is true.
  So for, $n = k + 1$
  $$\Rightarrow k + 1 \geq 6$$
  $$k \geq 7$$
  Condsider the base case here, n = 8 (as 7 is an odd number)
  $$P(8) : 8 = k_1 3 + k_2 5$$
  Substituting $k_1 = 1$ and $k_2 = 1$,
  $$P(8) : 8 = 1 \cdot 3 + 1 \cdot 5$$
  $$P(8) : 8 = 8$$
  $\therefore$ We can see that this satisfies the original statement and has already been proven by the induction hypothesis. **QED**.