CHAPTER 3. MANIPULATOR KINEMATICS

- Kinematics: Science of motion without regard to the forces and moments that cause it

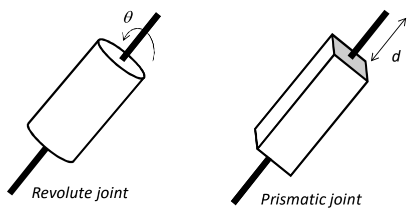
<u>Link Description</u>

- Definitions

  Manipulator: A set of bodies (links) connected in a **chain** by joints
  Links: Bodies of a manipulator or a chain. Mathematical concept relating two neighboring joint axes
  Joints: Connection between a neighboring pair of links

- In robotics, for **modeling**, each joint has one DOF. → One link – one joint – one DOF

  revolute joint vs. prismatic (or sliding) joint



- Note: A (physical) joint with n DOF can be modeled as n joints (revolute and prismatic combined) of one DOF connected with n-1 links of zero and/or non-zero lengths.

- Numbering of links

  Link 0: immobile base of manipulator (e.g., inertial frame, reference frame, ground, etc.)
  Link 1: first moving body
  Link i: ith moving body
  Link n: free end of manipulator

- Joint axis i: vector direction about which link i rotates relative to link i-1

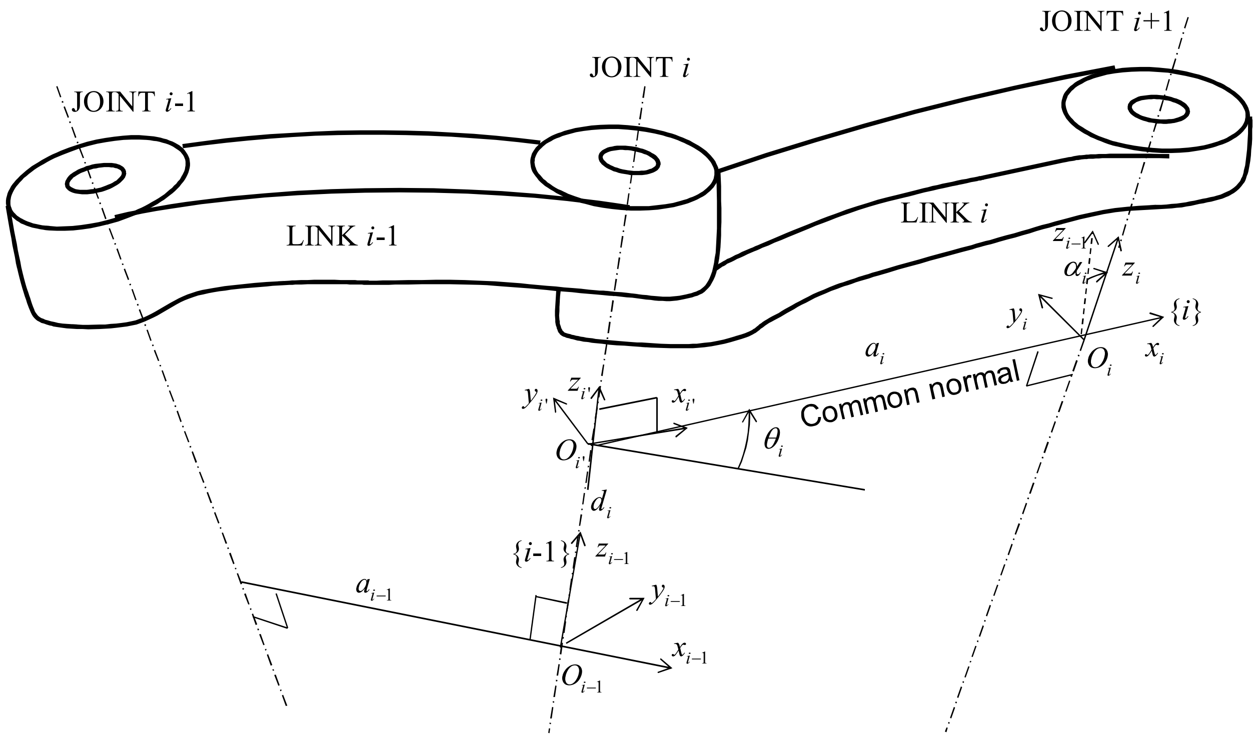
- Recall: Distance between any two axes in 3D is that of the common normal which is perpendicular to both axes.
  (Existence and uniqueness except for parallel axes; parallel axes have infinite number of mutual perpendiculars of equal length.)

<u>Denavit-Hartenberg (DH) Convention</u>

(References: [1] Sciavicco and Siciliano, *Modeling and Control of Robot Manipulators*, McGraw Hill, 1996; [2] Spong, M.W., Hutchinson, S., and Vidyasagar, M., *Robot Modeling and Control*, Wiley, 2006)

- Overall steps for standard DH Convention

  [DH STEP I] Attach a local frame to each link. Frame $\{i\}$ is attached rigidly to link i.
  [DH STEP II] Assign DH parameters and construct DH table.
  [DH STEP III] Compute homogeneous transformation matrices and forward kinematics.



[DH STEP I] Attach a local frame to each link

- Define and attach link Frame $\{i\}$:
  - **Step I-1)** Let Joint Axis i denote the axis of the joint connecting Link i-1 to Link i.
  - **Step I-2)** Choose axis z_i along the axis of Joint i+1.
  - **Step I-3)** Locate the origin O_i at the intersection of axis z_i with the common normal to axes z_{i-1} and z_i. Also, locate $O_{i'}$ at the intersection of the common normal with axis z_{i-1}.
  - **Step I-4)** Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from Joint i to Joint i+1. ($x_i \perp z_{i-1}$ and points away from z_{i-1})
  - **Step I-5)** Choose axis y_i so as to complete the right-handed frame.

- The DH convention gives a nonunique definition of link frames in the following cases:
  - **Case 1)** For Frame $\{0\}$, only direction of axis z_0 is specified; then O_0 and x_0 can be arbitrarily chosen.
  - **Case 2)** For Frame $\{n\}$, since there is no Joint n+1, z_n is not uniquely defined while x_n has to be normal to z_{n-1}. Typically, Joint n is revolute, and thus z_n is to be aligned with z_{n-1}.
  - **Case 3)** When two consecutive axes are parallel, the common normal between them is not uniquely defined.
  - **Case 4)** When two consecutive axes z_{i-1} and z_i intersect, x_i is chosen normal to the plane formed by z_{i-1} and z_i. The positive direction of x_i is arbitrary. The most natural choice for the origin O_i in this case is at the point of intersection of z_{i-1} and z_i. Note that, in this case, $a_i = 0$. (In general, the line that is normal to the plane formed by two intersecting axes can be viewed as a converging case of the common normal of two non-intersecting axes as they approach to each other and eventually intersect.)
  - **Case 5)** When Joint i is prismatic, the direction sense of z_{i-1} is arbitrary.

- In general, 6 parameters are required for the transformation between two frames. However, the DH convention imposes the following 2 conditions, reducing the required number of parameters to 4:

$$\left.\begin{array}{l}\textbf{DH1)}\ x_i \perp z_{i-1}.\\ \textbf{DH2)}\ x_i \text{ and } z_{i-1} \text{ axes intersect.}\end{array}\right\} \rightarrow x_i \text{ along the common normal to axes } z_{i-1} \text{ and } z_i$$

<u>[DH STEP II]</u> <u>Assign DH parameters and construct DH table</u>

- Once the link frames have been established, the position and orientation of Frame $\{i\}$ with respect to Frame $\{i\text{-}1\}$ are completely specified by the following DH parameters.
  - θ_i : Angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive with counter-clockwise
  - d_i: Coordinate (+/–) of $O_{i'}$ along z_{i-1}
  - a_i: Distance between O_i and $O_{i'}$ (Note: x_i "with direction from Joint i to Joint i+1" or "points away from z_{i-1}" ensures that a_i is positive, thus distance, not coordinate.)
  - α_i : Angle between axes z_{i-1} and z_i about axis x_i to be taken positive with counter-clockwise

- Reference configuration (= home configuration = zero configuration) of a robot manipulator
  : configuration with respect to which the joint displacements of the manipulator are measured
  - The configuration of a manipulator when all joint variables are equal to zero
  - The location of the end-effector and the locations of the joint axes are known.
  - Can be chosen arbitrarily; usually chosen at the location where the coordinates of all joint axes can be easily identified
  - DH parameters do not represent the angle of rotation or the distance of translation about a joint axis.

- Target configuration (= desired configuration)
  - Manipulator displaced from the reference configuration to the target configuration by a series of joint displacements about all joint axes.
  - To obtain actual joint displacements, subtract joint variables associated with the reference configuration from that of a target configuration.

- Link parameters (design) and joint variables (control) (where $\tilde{\theta}_i$ and $\tilde{d}_i$ are reference configurations)
  - If joint is revolute $\theta_i = \tilde{\theta}_i + q_i$ → joint variable: q_i link parameters: d_i, a_i, α_i
  - If joint is prismatic $d_i = \tilde{d}_i + q_i$ → joint variable: q_i link parameters: θ_i, a_i, α_i

  → Joint variable vector: $\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ (for n-DOF manipulator)

- DH parameter table

  For two types Joint i: revolute and prismatic

Joint i	θ_i	d_i	a_i	α_i	Joint variable $\mathbf{q}$
Revolute	$\theta_i = \tilde{\theta}_i + q_i$	d_i	a_i	α_i	q_i
Prismatic	θ_i	$d_i = \tilde{d}_i + q_i$	a_i	α_i	q_i

For an n-DOF manipulator

Joint #	θ_i	d_i	a_i	α_i	Joint variable $\mathbf{q}$
1 (if revolute)	$\theta_1 = \tilde{\theta}_1 + q_1$	d_1	a_1	α_1	q_1
2 (if prismatic)	θ_2	$d_2 = \tilde{d}_2 + q_2$	a_2	α_2	q_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n (if revolute)	$\theta_n = \tilde{\theta}_n + q_n$	d_n	a_n	α_n	q_n

- Note: "DH parameter = home configuration + joint variable" in DH parameters table
  - Clarifies which DH parameter corresponds to joint degree of freedom (revolute or prismatic)
  - Identifies the home configuration parameter value of the corresponding joint

[DH STEP III] Compute homogeneous transformation matrices and forward kinematics

- Derivation of link transformation ${}^{i-1}T_i$: define Frame $\{i\}$ relative to Frame $\{i-1\}$

  → Four transformations (sub-problems) – each of four transformations will be a function of one DH parameter only

  $Rot(z,\theta_i)$: $T_{z,\theta}$ => $Trans(0,0,d_i)$: $T_{z,d}$ => $Trans(a_i,0,0)$: $T_{x,a}$ => $Rot(x,\alpha_i)$: $T_{x,\alpha}$

  (Note: the rotations in this case are moving frame rotations; Euler angles; .... product)

$$
{}^{i-1}T_i = T_{z,\theta} T_{z,d} T_{x,a} T_{x,\alpha}
$$

$$
= \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\therefore \boxed{ {}^{i-1}T_i = \left[\begin{array}{ccc|c} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] }
$$

- ${}^{i-1}T_i = {}^{i-1}T_i(q_i)$: function of only **one** variable q_i where $\begin{cases} \theta_i = \tilde{\theta}_i + q_i \text{ for revolute joint} \\ d_i = \tilde{d}_i + q_i \text{ for prismatic joint} \end{cases}$

- $Screw_Q(r,\phi)$: translation by distance r along, and rotation by angle ϕ about axis $\hat{Q}$

  Examples: $Screw_z(d,\theta) = T_{z,\theta} T_{z,d}$ and $Screw_x(a,\alpha) = T_{x,a} T_{x,\alpha}$

- **Forward kinematics** – concatenating link transformations

  ${}^0T_n = {}^0T_1(q_1)\,{}^1T_2(q_2)\ldots{}^{i-1}T_i(q_i)\ldots{}^{n-1}T_n(q_n)$ (for n-DOF manipulator)

  ${}^0T_i = {}^0T_i(q_1,q_2,\ldots,q_i)$: function of the first i joint variables

  → computes Cartesian position and orientation of the ith link

  ${}^0T_n = {}^0T_n(q_1,q_2,\ldots,q_n)$: function of all n joint variables

  → computes Cartesian position and orientation of the last (nth) link

DH Convention Procedure Summary

1) Find and number consecutively the joint axes; set the directions of axes $z_0, \ldots, z_{n-1}$.

2) Choose Frame {0} by locating the origin on axis z_0; axes x_0 and y_0 are chosen according to right-hand rule. If feasible, it is worth choosing Frame {0} to coincide with the base frame.

Execute steps **3** to **5** for $i = 1, \ldots, n-1$:

3) Locate the origin O_i at the intersection of z_i with the common normal to axes z_{i-1} and z_i. If axes z_{i-1} and z_i are parallel and Joint i is revolute, then locate O_i so that $d_i = 0$; if Joint i is prismatic, locate O_i at a reference position for the joint range, e.g., a mechanical limit.

4) Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from Joint i to Joint i+1.

5) Choose axis y_i according to right-hand rule.

To complete:

6) Choose Frame $\{n\}$; if Joint n is revolute, then align z_n and z_{n-1}; otherwise, if Joint n is prismatic, then choose z_n arbitrarily. Axis x_n is set according to step **4**.

7) For $i = 1, \ldots, n$, construct the table of DH parameters θ_i, d_i, a_i, α_i.

8) On the basis of the DH parameters in **7**, compute the homogenous transformation matrices ${}^{i-1}T_i(q_i)$ for $i = 1, \ldots, n$.

9) Compute the homogenous transformation ${}^0T_n(\mathbf{q}) = {}^0T_1 \ldots {}^{n-1}T_n$ that yields the position and orientation of Frame $\{n\}$ with respect to Frame {0}.

10) Given ${}^bT_0$ (from base to Frame {0}) and ${}^nT_e$ (from Frame $\{n\}$ to end-effector), compute the direct kinematic function as ${}^bT_e(\mathbf{q}) = {}^bT_0{}^0T_n{}^nT_e$ that yields the position and orientation of the end-effector frame with respect to the base frame.

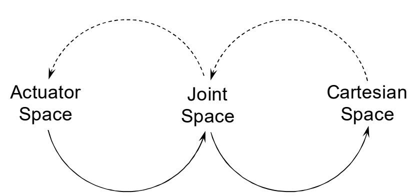
DH Parameters (Quick Summary)

θ_i (joint angle): joint angle from x_{i-1} to x_i about z_{i-1} (revolute joint variable)
d_i (link offset): shortest distance between x_{i-1} to x_i axis (prismatic joint variable)
a_i (link length): shortest distance between z_{i-1} and z_i axis
α_i (link twist): angle from z_{i-1} to z_i about x_i axis

(Note: shortest distance between axes = length of the common normal)

Actuator Space, Joint Space, and Cartesian Space

- nx1 joint vector $\mathbf{q}$: set of n joint variables (generalized coordinates) that specifies the position and orientation of all the links of an n-DOF manipulator
- Joint space: vector space of all joint vectors
- Cartesian space = task-oriented space = operational space
- Actuator vector: actuator positions (determine joint vector) → actuator space
  - Examples: two actuators for a single joint, four-bar linkage (linear → revolute), muscles, etc.
- Mappings between 3 different space representations of manipulator's position and orientation



Example: 2-Link 2R Planar Manipulator

- DH parameters table

i	θ	d	a	α	Joint variable q_i
1	$\theta_1 = 0° + q_1$	0	a_1	0	q_1
2	$\theta_2 = 0° + q_2$	0	a_2	0	q_2

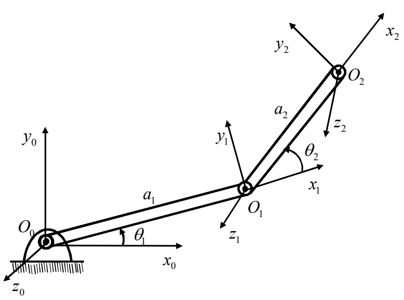
- Homogeneous transformation matrix in joint space

$$
{}^0T_2 = {}^0T_1\,{}^1T_2 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 & a_1c_1 + a_2c_1c_2 - a_2s_1s_2 \\ c_1s_2 + s_1c_2 & c_1c_2 - s_1s_2 & 0 & a_1s_1 + a_2s_1c_2 + a_2c_1s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

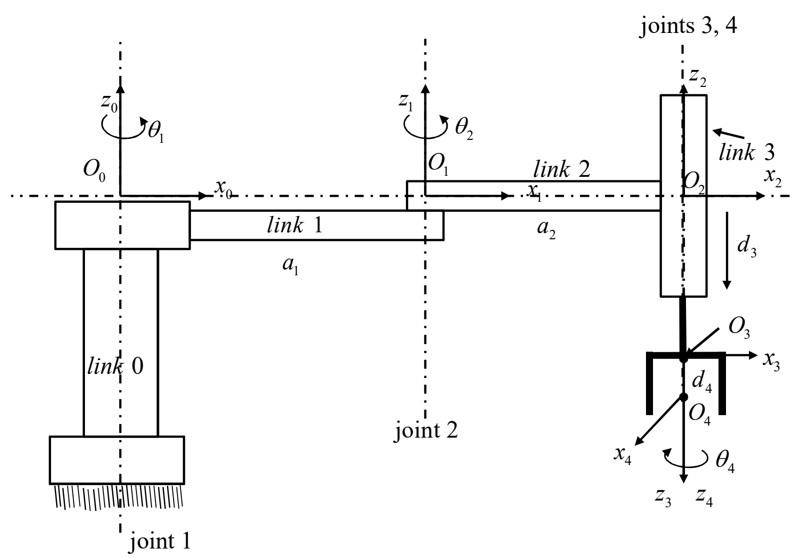
$$
\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \Rightarrow c_{12} = c_1c_2 - s_1s_2
$$

$$
\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \Rightarrow s_{12} = s_1c_2 + c_1s_2
$$

$$
\therefore\ {}^0T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



2R Planar Manipulator



SCARA Robot

Example: SCARA Robot

- Local frame: since all joint axes are parallel, the locations of the origins are not unique. In this case, the origins are located at each joint.
- DH parameters table

i	θ_i	d_i	a_i	α_i	Joint variable q_i
1	$\theta_1 = 0 + q_1$	0	a_1	0	q_1

2	$\theta_2 = 0 + q_2$	0	a_2	0	q_2
3	0	$d_3 = 0 + q_3$	0	π	q_3
4	$\theta_4 = 0 + q_4$	d_4	0	0	q_4

- Homogeneous transformation matrices

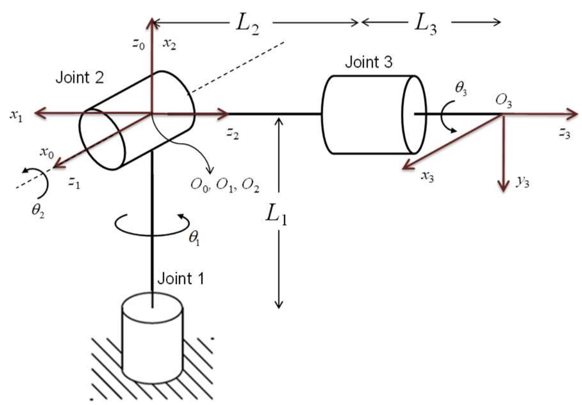
$$
{}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^2T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^3T_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Forward kinematics equation

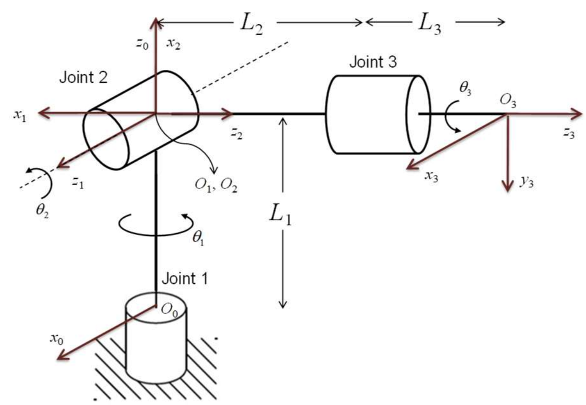
$$
{}^0T_4 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

<u>Example: Spherical Manipulator</u>

- Three orthogonal revolute joints, shown in its home configuration in the figures.
- The positive directions for the rotations of Joints 1, 2, and 3 are given as upward, out of the plane, and to the right, respectively, in the figures.
- Indeterminacies on: the global frame's origin O_0 and its x_0 axis; the local frame {3}; and the positive directions of x_1 and x_2 along their lines of axes (since z_0, z_1, and z_2 intersect). Here, they are given as in the figures, with two options for O_0 and x_0.



Option 1



Option 2

<u>Option 1</u>

- DH parameters table

i	θ_i	d_i	a_i	α_i	Joint variable q_i
1	$\theta_1 = -\pi/2 + q_1$	0	0	$-\pi/2$	q_1
2	$\theta_2 = -\pi/2 + q_2$	0	0	$\pi/2$	q_2
3	$\theta_3 = \pi/2 + q_3$	L_2+L_3	0	0	q_3

- Homogeneous transformation matrices ($c_1 = \cos\theta_1$, $s_1 = \sin\theta_1$, etc.)

$$
{}^0T_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^1T_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Forward kinematics equation

$$
{}^0T_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & c_1s_2 & (L_2 + L_3)c_1s_2 \\ s_1c_2c_3 + c_1s_3 & -s_1c_2s_3 + c_1c_3 & s_1s_2 & (L_2 + L_3)s_1s_2 \\ -s_2c_3 & s_2s_3 & c_2 & (L_2 + L_3)c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

<u>Option 2</u>

- DH parameters table

i	θ_i	d_i	a_i	α_i	Joint variable q_i
1	$\theta_1 = -\pi/2 + q_1$	L_1	0	$-\pi/2$	q_1
2	$\theta_2 = -\pi/2 + q_2$	0	0	$\pi/2$	q_2
3	$\theta_3 = \pi/2 + q_3$	L_2+L_3	0	0	q_3

- Homogeneous transformation matrices

$$
{}^0T_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^1T_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; {}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Forward kinematics equation

$$
{}^0T_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & c_1s_2 & (L_2 + L_3)c_1s_2 \\ s_1c_2c_3 + c_1s_3 & -s_1c_2s_3 + c_1c_3 & s_1s_2 & (L_2 + L_3)s_1s_2 \\ -s_2c_3 & s_2s_3 & c_2 & (L_2 + L_3)c_2 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

<u>Remarks</u>

- If $L_1 = L_2 = 0$, the first three frame origins intersect at a single point for both options in the figures, representing a spherical manipulator.
- As a consequence of the choice made for the coordinate frames, the block matrix ${}^0R_3$ that can be extracted from ${}^0T_3$ coincides with the rotation matrix of *ZYZ* Euler angles for $\theta_1, \theta_2, \theta_3$ with respect to the reference frame O_0-$x_0y_0z_0$.