Foundations of Robotics
ROB-GY 6003
Homework 5 Answers

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Question: 6.15

Answer: Given in below Fig 1 is the manipulator in question -

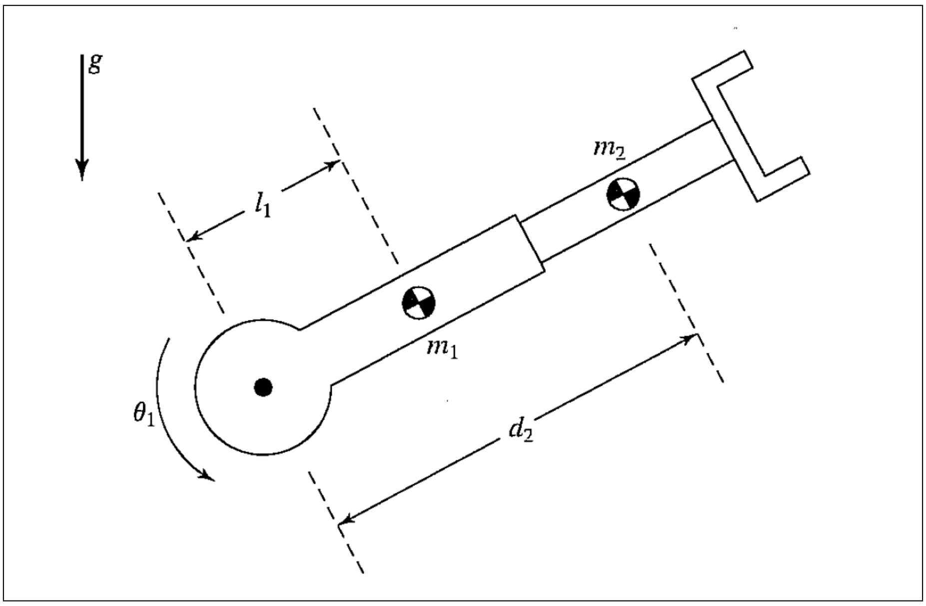

Figure 1: The RP manipulator

The links of the manipulator have the intertia tensors,

$$
\begin{aligned}
{}^{C_1}I_1 &= \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix} \\
{}^{C_2}I_2 &= \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}
\end{aligned} \tag{1}
$$

The links have a mass of m_1 and m_2 respectively. The center of mass of link 1 is located as distance l_1 from joint-1 axis and that of link 2 is at the variable distance d_2, from the joint-1 axis.

We know that,

$$k_i = \frac{1}{2}m_i v_{C_i}^T v_{C_i} + \frac{1}{2}{}^i\omega_i^{T\,C_i}I_i{}^i\omega_i \tag{2}$$

Using above Eq^n (3) we write the kinetic energy for link 1 & 2 $\rightarrow$

$$k_1 = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}I_{zz1}\dot{\theta}_1^2 \tag{3}$$

$$k_2 = \frac{1}{2}m_2(d_2^2\dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2}I_{zz2}\dot{\theta}_1^2 \tag{4}$$

$$\Rightarrow k(\Theta, \dot{\Theta}) = \frac{1}{2}(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1^2 + \frac{1}{2}m_2\dot{d}_2^2 \tag{5}$$

We also know that,

$$u_i = -m_i{}^0g^{T\,0}P_{C_i} + u_{ref_i} \tag{6}$$

Using above Eq^n (7) we write the potential energy for link 1 & 2 $\rightarrow$

$$u_1 = m_1 l_1 g\sin(\theta_1) + m_1 l_1 g \tag{7}$$

$$u_2 = m_2 g d_2\sin(\theta_1) + m_2 g d_{2max} \tag{8}$$

$$\Rightarrow u(\Theta) = g(m_1 l_1 + m_2 d_2)\sin(\theta_1) + m_1 l_1 g + m_2 g d_{2max} \tag{9}$$

Taking partial derivatives,

$$\frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1 \\ m_2 d_2 \end{bmatrix} \tag{10}$$

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} 0 \\ m_2 d_2 \dot{\theta}_1^2 \end{bmatrix} \tag{11}$$

$$\frac{\partial u}{\partial \Theta} = \begin{bmatrix} g(m_1 l_1 + m_2 d_2)\cos(\theta_1) \\ g m_2 \sin(\theta_1) \end{bmatrix} \tag{12}$$

We know that the equation for $n \times 1$ vector of actuator torques is given by,

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau \tag{13}$$

Substituting Eq^n (10), (11) & (12) in Eq^n (11),

$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2 m_2 d_2 \dot{\theta}_1 \dot{d}_2 + (m_1 l_1 + m_2 d_2) g \cos(\theta_1) \tag{14}$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin(\theta_1) \tag{15}$$

Finally,

$$\begin{aligned} M(\Theta) &= \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) & 0 \\ 0 & m_2 \end{bmatrix} \\ V(\Theta, \dot{\Theta}) &= \begin{bmatrix} 2 m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix} \\ G(\Theta) &= \begin{bmatrix} (m_1 l_1 + m_2 d_2) g \cos(\theta_1) \\ m_2 g \sin(\theta_1) \end{bmatrix} \end{aligned} \tag{16}$$

Question: 6.16

Answer: Upon inspection,

$$\tau = \begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix} = M(\theta)\ddot{\theta} + V(\theta_1 \dot{\theta}) + G(\theta) \tag{1}$$

Where, $\theta = \begin{bmatrix} d_1 \\ \theta_2 \end{bmatrix}$. This implies that,

$$\begin{aligned} M(\theta) &= \begin{bmatrix} M_1 + M_2 & 0 \\ 0 & I_{zz2} \end{bmatrix} \\ V(\theta_1 \dot{\theta}) &= 0 \\ G(\theta) &= 0 \end{aligned}$$

Question: 6.20

Answer: Given in below Fig 2 is the manipulator in question -

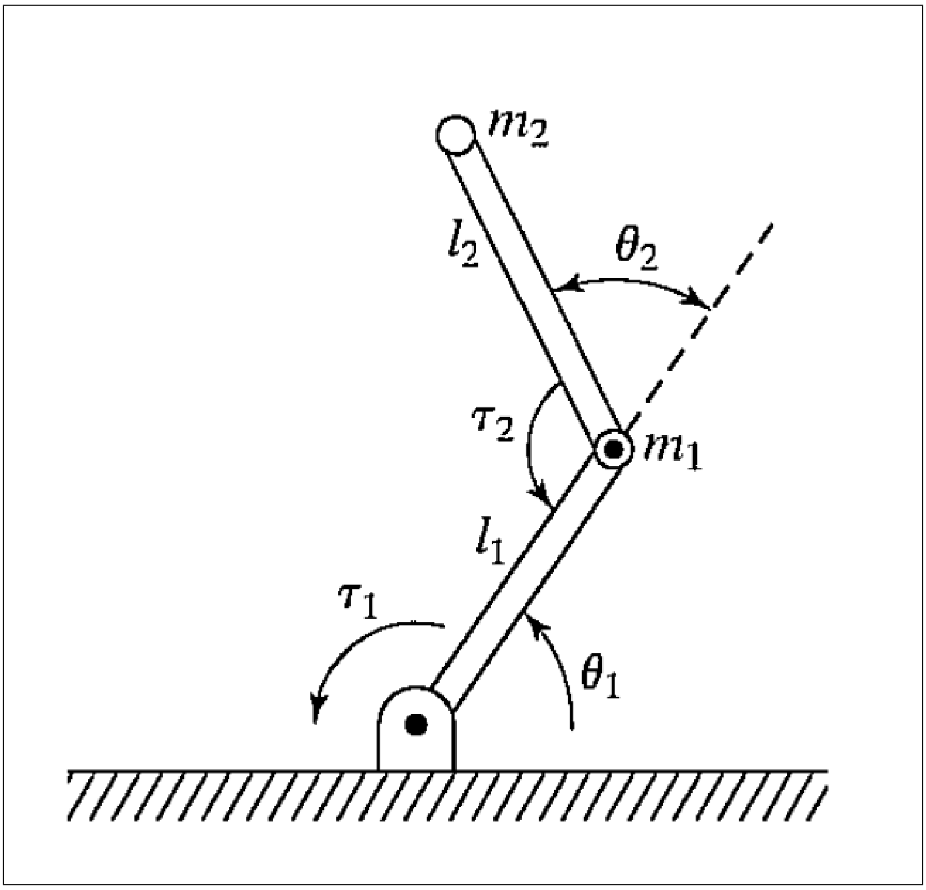

Figure 2: Two-link planar manipulator with point masses at distal ends of links

Because of the point-mass assumption,

$$\begin{aligned} {}^{C_1}I_1 &= 0 \\ {}^{C_2}I_2 &= 0 \end{aligned} \tag{1}$$

Also, as the base of the robot is not rotating,

$$\begin{aligned} \omega_0 &= 0 \\ \dot{\omega_0} &= 0 \end{aligned} \tag{2}$$

We know that,

$$k_i = \frac{1}{2} m_i v_{C_i}^T v_{C_i} + \frac{1}{2} {}^i\omega_i^T {}^{C_i}I_i {}^i\omega_i \tag{3}$$

Using above Eq^n (3) we write the kinetic energy for link 1 & 2 $\rightarrow$

$$k_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \tag{4}$$

$$k_2 = \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 \tag{5}$$

$$\Rightarrow k(\Theta, \dot{\Theta}) = \frac{1}{2}(m_1 l_1^2 \dot{\theta}_1^2 + m_2 l_2^2 \dot{\theta}_2^2) \tag{6}$$

We also know that,
$$u_i = -m_i {}^0g^{T\,0}P_{C_i} + u_{ref_i} \tag{7}$$

Using above Eq^n (7) we write the potential energy for link 1 & 2 $\rightarrow$

$$u_1 = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g \tag{8}$$
$$u_2 = m_2 l_2 g \sin(\theta_2) + m_2 l_2 g \tag{9}$$
$$\Rightarrow u(\Theta) = g(m_1 l_1 \sin(\theta_1) + m_2 l_2 \sin(\theta_2)) + m_1 l_1 g + m_2 l_2 g \tag{10}$$

We know that the equation for $n \times 1$ vector of actuator torques is given by,

$$\tau = \frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} \tag{11}$$
$$\begin{aligned}\Rightarrow \tau_1 = {} & m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 \\ & - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1\end{aligned} \tag{12}$$
&
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{13}$$

Therefore,

$$M(\Theta) = \begin{bmatrix} m_2 + \frac{m_1}{s_2^2} & 0 \\ 0 & m_2 \end{bmatrix} \tag{14}$$
$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -(m_2 l_2 c_2 + m_2 l_2)\dot{\theta}_1^2 - m_2 l_2 \dot{\theta}_2^2 - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{s_2^2})\dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1^2 + l_1 m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix} \tag{15}$$
$$G(\Theta) = \begin{bmatrix} m_1 g \frac{c_1}{s_2} + m_2 g s_{12} \\ m_2 g c_{12} \end{bmatrix} \tag{16}$$