CHAPTER 2. SPATIAL DESCRIPTIONS AND TRANSFORMATIONS

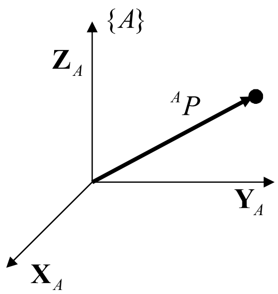
- Global (= universe = inertial = Newtonian = world) coordinate system

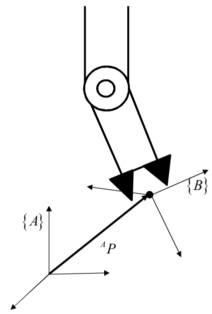
Position (of a point)

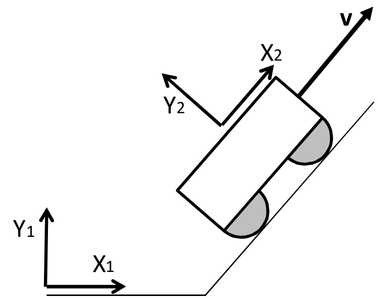
- 3x1 position vector (e.g., ${}^{A}\mathbf{P}$) → identify the coordinate system $\{A\}$ of description

$$ {}^{A}\mathbf{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$

  - Components of ${}^{A}\mathbf{P}$: distances along axes of $\{A\}$







Orientation (of a rigid body)

Attach a coordinate system to a body → describe this frame relative to the reference frame
$\{B\}$ relative to $\{A\}$ → orientation of the body
Write unit vectors of principal axes of $\{B\}$ in terms of $\{A\}$.

- Dual-superscript notation: **Two** reference frames for description of kinematic vectors (linear position/velocity/acceleration of a point and angular velocity/acceleration of a frame)
  - **Defined** as viewed by an observer fixed in a reference frame: "relative to" or "with respect to" *observer*'s frame → Geometric vector
  - **Resolved** into components with respect to a reference frame: "referred to," "expressed in," or "written in" *writer*'s frame → Algebraic representation of the geometric vector

- $\begin{bmatrix} \hat{\mathbf{X}}_B \\ \hat{\mathbf{Y}}_B \\ \hat{\mathbf{Z}}_B \end{bmatrix}$: Principal unit vectors of $\{B\}$ written in terms of $\{B\}$

- $\begin{bmatrix} {}^{A}\hat{\mathbf{X}}_B \\ {}^{A}\hat{\mathbf{Y}}_B \\ {}^{A}\hat{\mathbf{Z}}_B \end{bmatrix}$: Principal unit vectors of $\{B\}$ written in terms of $\{A\}$

  → Columns of 3x3 **rotation matrix** (= direction cosine matrix) of $\{B\}$ relative to $\{A\}$: ${}^{A}_{B}R$ or ${}^{A}R_B$

- $\boxed{ {}^{A}R_B = [\, {}^{A}\hat{\mathbf{X}}_B \;\vdots\; {}^{A}\hat{\mathbf{Y}}_B \;\vdots\; {}^{A}\hat{\mathbf{Z}}_B \,] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} }$ (orientation)

- Note
  - Position of a point → vector (position vector)
  - Orientation of a body → matrix (rotation matrix)

- Note

	Configuration	Motion
Linear	....	....
Angular	....	....

- $$\boxed{{}^{A}R_{B} = [\,{}^{A}\hat{\mathbf{X}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Y}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Z}}_{B}\,] = \begin{bmatrix} \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{X}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{X}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{X}}_{A} \\ \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{Y}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{Y}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{Y}}_{A} \\ \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{Z}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{Z}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{Z}}_{A} \end{bmatrix}}$$ (arbitrary choice of frame for description)
  - Elements are the **direction cosines**.
  - ${}^{A}\hat{\mathbf{X}}_{B}$, ${}^{A}\hat{\mathbf{Y}}_{B}$, ${}^{A}\hat{\mathbf{Z}}_{B}$: unit orthogonal vectors

- Note: rotation matrix ${}^{\text{[with respect to which frame]}}R_{\text{[describe frame of interest]}}$ does not require the frame of expression

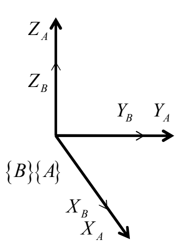
- Rows are unit vectors of $\{A\}$ expressed in $\{B\}$: $${}^{A}R_{B} = [\,{}^{A}\hat{\mathbf{X}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Y}}_{B} \;\vdots\; {}^{A}\hat{\mathbf{Z}}_{B}\,] = \begin{bmatrix} {}^{B}\hat{\mathbf{X}}_{A}^{T} \\ \hline {}^{B}\hat{\mathbf{Y}}_{A}^{T} \\ \hline {}^{B}\hat{\mathbf{Z}}_{A}^{T} \end{bmatrix}$$

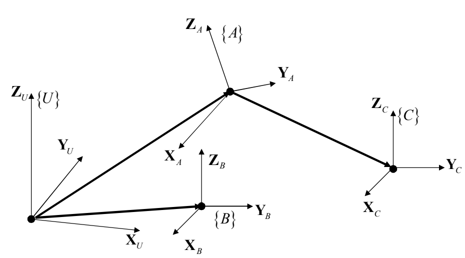
  → ${}^{A}R_{B} = {}^{B}R_{A}^{T}$ and ${}^{A}R_{B} = {}^{B}R_{A}^{-1}$

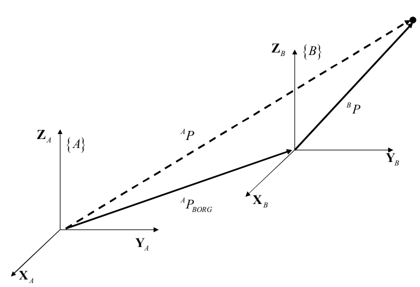
  $\therefore\ {}^{A}R_{B} = {}^{B}R_{A}^{-1} = {}^{B}R_{A}^{T}$ → Rotation matrix is .... matrix (i.e., $RR^{T} = I_3$).

- Example

$${}^{A}R_{B} = \begin{bmatrix} \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{X}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{X}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{X}}_{A} \\ \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{Y}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{Y}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{Y}}_{A} \\ \hat{\mathbf{X}}_{B}\bullet\hat{\mathbf{Z}}_{A} & \hat{\mathbf{Y}}_{B}\bullet\hat{\mathbf{Z}}_{A} & \hat{\mathbf{Z}}_{B}\bullet\hat{\mathbf{Z}}_{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$







<u>Frame</u>

- Describes one coordinate system with respect to another.
- Represents both position and orientation.
- A set of four vectors – position vector and rotation matrix

- Position description – in general, choose the origin of the body-attached (= local) frame

- $\{B\} = \{{}^A R_B, {}^A\mathbf{P}_{BORG}\}$

Mapping

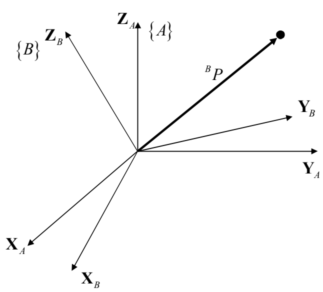
Changing **descriptions** (only!) from frame to frame
Original vector is not changed in space
Computes new description of the vector relative to another frame

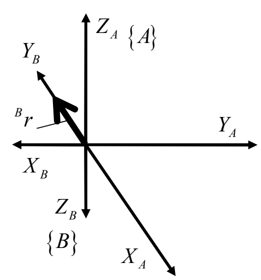
- Mapping of translation (**same orientations**): ${}^A\mathbf{P} = {}^B\mathbf{P} + {}^A\mathbf{P}_{BORG}$
  (Note: vector additions in terms of different frames can be calculated only when their orientations are equivalent!)

- Mapping of rotation (**same origins**)

$$\rightarrow \text{Components of } {}^A\mathbf{P}: \begin{cases} {}^A p_x = {}^B\hat{\mathbf{X}}_A \bullet {}^B\mathbf{P} = {}^B\hat{\mathbf{X}}_A^T\, {}^B\mathbf{P} \\ {}^A p_y = {}^B\hat{\mathbf{Y}}_A \bullet {}^B\mathbf{P} = {}^B\hat{\mathbf{Y}}_A^T\, {}^B\mathbf{P} \\ {}^A p_z = {}^B\hat{\mathbf{Z}}_A \bullet {}^B\mathbf{P} = {}^B\hat{\mathbf{Z}}_A^T\, {}^B\mathbf{P} \end{cases} \Rightarrow {}^A\mathbf{P} = \begin{bmatrix} {}^B\hat{\mathbf{X}}_A^T \\ \hdashline {}^B\hat{\mathbf{Y}}_A^T \\ \hdashline {}^B\hat{\mathbf{Z}}_A^T \end{bmatrix} {}^B\mathbf{P}$$

$\therefore\ {}^A\mathbf{P} = {}^A R_B\, {}^B\mathbf{P}$: mapping of a same vector's description from $\{B\}$ to $\{A\}$.

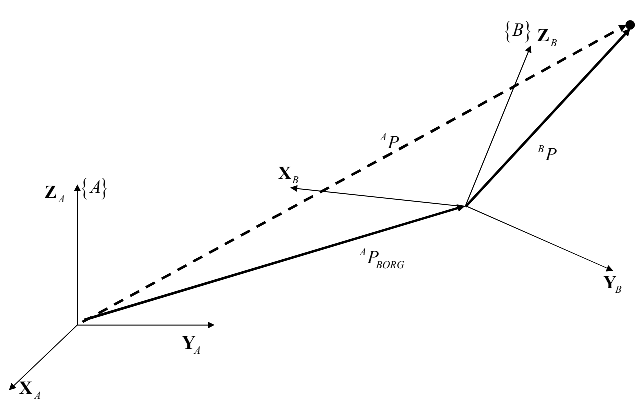


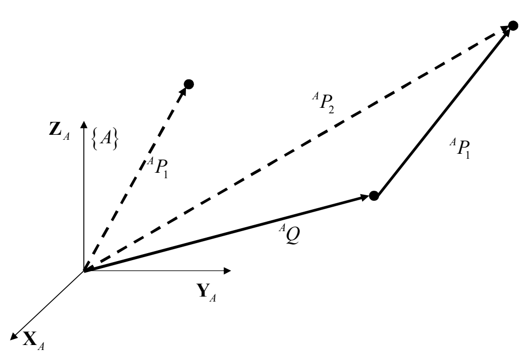


- Example

$${}^A R_B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix};\ {}^B\mathbf{r} = [0\ \ 2\ \ 0]^T;\ {}^A\mathbf{r} = {}^A R_B\, {}^B\mathbf{r} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

- Mapping of general frames' transformation: translation + rotation (affine transformation)
  $\boxed{{}^A\mathbf{P} = {}^A R_B\, {}^B\mathbf{P} + {}^A\mathbf{P}_{BORG}}$ (Note 1: nonhomogeneous; Note 2: ${}^A R_{\cancel{B}}\, {}^{\cancel{B}}\mathbf{P} \neq {}^A\mathbf{P}$)





- Construct a 4x4 "augmented" matrix operator T using 4x1 "augmented" position vectors

$$\begin{bmatrix} {}^{A}\mathbf{P} \\ \hline 1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \underbrace{{}^{A}R_{B}}_{orientation} & \underbrace{{}^{A}\mathbf{P}_{BORG}}_{position} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right]}_{={}^{A}T_{B}} \begin{bmatrix} {}^{B}\mathbf{P} \\ \hline 1 \end{bmatrix} \Rightarrow {}^{A}\mathbf{P} = {}^{A}T_{B}\,{}^{B}\mathbf{P}$$

${}^{A}T_{B} = \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}\mathbf{P}_{BORG} \\ \hline \mathbf{0}^{T} & 1 \end{array}\right]$: **Homogeneous transform** – describes $\{B\}$ relative to $\{A\}$; mapping ${}^{B}\mathbf{P} \mapsto {}^{A}\mathbf{P}$

- Example

<u>Operators</u>

→ Transform points and/or vectors in a given frame (**only one coordinate system** is involved)
  - Use the mapping transform

- Translational operators: moves a point in space by a vector

  → ${}^{A}\mathbf{P}_1$ translated by ${}^{A}\mathbf{Q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$: ${}^{A}\mathbf{P}_2 = {}^{A}\mathbf{P}_1 + {}^{A}\mathbf{Q}$

  → Matrix operator: ${}^{A}\mathbf{P}_2 = D_Q(q)\,{}^{A}\mathbf{P}_1$ ($q = \left\|\hat{Q}\right\| = \sqrt{q_x^2 + q_y^2 + q_z^2}$)

$$D_Q(q) = \begin{bmatrix} I_3 & \hat{Q} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotational operators: rotates ${}^{A}\mathbf{P}_1$ to become ${}^{A}\mathbf{P}_2$ by means of R

  → ${}^{A}\mathbf{P}_2 = R\,{}^{A}\mathbf{P}_1$ or ${}^{A}\mathbf{P}_2 = R_K(\theta)\,{}^{A}\mathbf{P}_1$ ($\hat{K}$: axis direction, θ : angle)

  Example: $R_z(\theta) = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$ (3x3 or 4x4)

- General transformation operator: Frame

→ ${}^{A}\mathbf{P}_2 = T\,{}^{A}\mathbf{P}_1$: T operates on (i.e., rotates and translates) ${}^{A}\mathbf{P}_1$ to compute ${}^{A}\mathbf{P}_2$

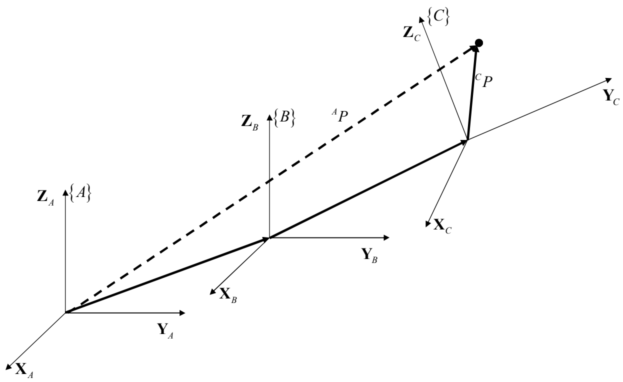
<u>Transformation Arithmetic</u>

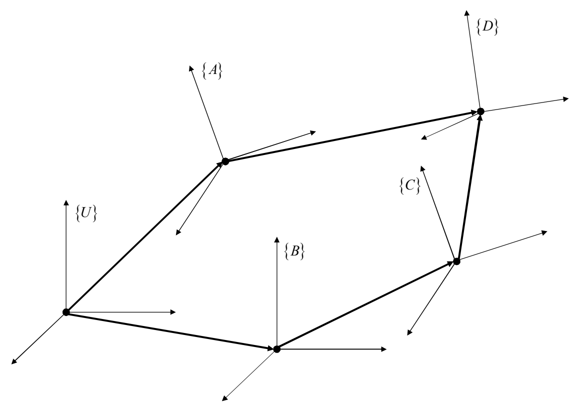
- Compound: ${}^{A}T_C = {}^{A}T_B\,{}^{B}T_C \Rightarrow {}^{A}T_C = \left[\begin{array}{c|c} {}^{A}R_B\,{}^{B}R_C & {}^{A}R_B\,{}^{B}\mathbf{P}_{CORG} + {}^{A}\mathbf{P}_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]$

- Inversion: ${}^{B}\mathbf{P}_{BORG} = {}^{B}R_{A}\,{}^{A}\mathbf{P}_{BORG} + {}^{B}\mathbf{P}_{AORG} = \mathbf{0}$
  (Note: The point of interest is $\mathbf{P}_{BORG}$. Thus the notation ${}^{B}({}^{A}\mathbf{P}_{BORG})$ in the textbook is not proper.)
  $$\Rightarrow\ {}^{B}\mathbf{P}_{AORG} = -{}^{B}R_{A}\,{}^{A}\mathbf{P}_{BORG} = -{}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG} \ \Rightarrow\ {}^{A}T_{B}^{-1} = {}^{B}T_{A} = \left[\begin{array}{c|c} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

  - Alternative derivation: ${}^{A}\mathbf{P} = {}^{A}R_{B}\,{}^{B}\mathbf{P} + {}^{A}\mathbf{P}_{BORG}$
    $$\Rightarrow\ {}^{A}R_{B}^{T}\,{}^{A}\mathbf{P} = {}^{A}R_{B}^{T}\,{}^{A}R_{B}\,{}^{B}\mathbf{P} + {}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG} = {}^{B}\mathbf{P} + {}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG} \ \Rightarrow\ {}^{B}\mathbf{P} = {}^{A}R_{B}^{T}\,{}^{A}\mathbf{P} - {}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG}$$
    $$\Rightarrow \left[\begin{array}{c} {}^{B}\mathbf{P} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}\mathbf{P}_{BORG} \\ \hline \mathbf{0}^{T} & 1 \end{array}\right] \left[\begin{array}{c} {}^{A}\mathbf{P} \\ \hline 1 \end{array}\right]$$

- Transform equation: ${}^{U}T_{A}\,{}^{A}T_{D} = {}^{U}T_{B}\,{}^{B}T_{C}\,{}^{C}T_{D}$





Orientation

- Rotation matrix $R = [\qquad\qquad]$ → $Det(R) = 1$ (i.e., Proper orthonormal matrix)
  Recall: ${}^{A}R_{B}\,{}^{B}R_{C} \neq {}^{B}R_{C}\,{}^{A}R_{B}$ (not commutative)

- Cayley's formula: $R = (I_3 - S)^{-1}\,(I_3 + S)$ (where S is a skew-symmetric matrix; ______)

- $S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$ → $\therefore$ R: 3 independent parameters

- $\|\hat{\mathbf{X}}\| = \|\hat{\mathbf{Y}}\| = \|\hat{\mathbf{Z}}\| = 1$ and $\hat{\mathbf{X}}\bullet\hat{\mathbf{Y}} = \hat{\mathbf{X}}\bullet\hat{\mathbf{Z}} = \hat{\mathbf{Y}}\bullet\hat{\mathbf{Z}} = 0$ → 9 elements and 6 equations → $\therefore$ ___ unknowns

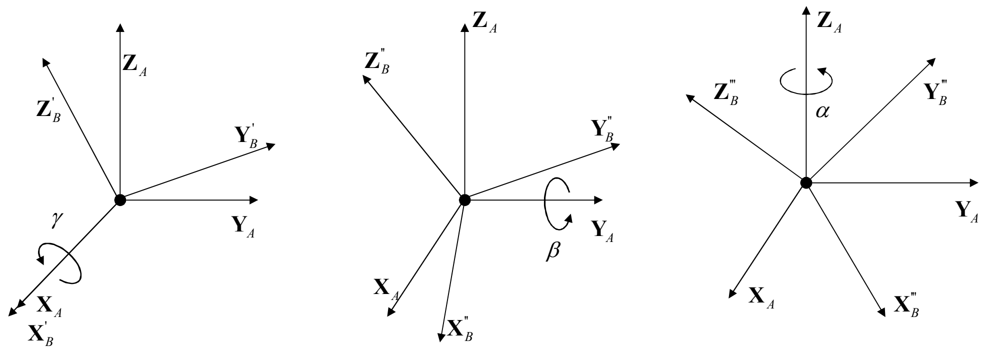
Rotation of Frames

Fixed angle rotation (absolute transform)
Moving (i.e., current) frame rotation (relative transform)

Fixed Angle Rotation

Rotations are specified about the fixed frame.
Each of three rotations takes place about an axis in the fixed frame (e.g., $\{A\}$).

- X-Y-Z fixed angles (roll-pitch-yaw): initially $\{B\}$ coincides with $\{A\}$
  → (1) rotate $\{B\}$ about $\hat{\mathbf{X}}_A$ by γ → (2) rotate $\{B\}$ about $\hat{Y}_A$ by β → (3) rotate $\{B\}$ about $\hat{Z}_A$ by α



$$\boxed{{}^{A}R_{BXYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)}$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

- "Multiply rotation matrices **from right to left**; premultiplying" (rotations as operators)

- Let ${}^{A}R_{BXYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$. If $c\beta \neq 0$, then $\begin{cases} \beta = \text{Atan2}(-r_{31}, \sqrt{r^2_{11} + r^2_{21}}) \\ \alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta) \\ \gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta) \end{cases}$.

  (Atan2(y, x): two-argument arc tangent function or four-quadrant arc tangent)

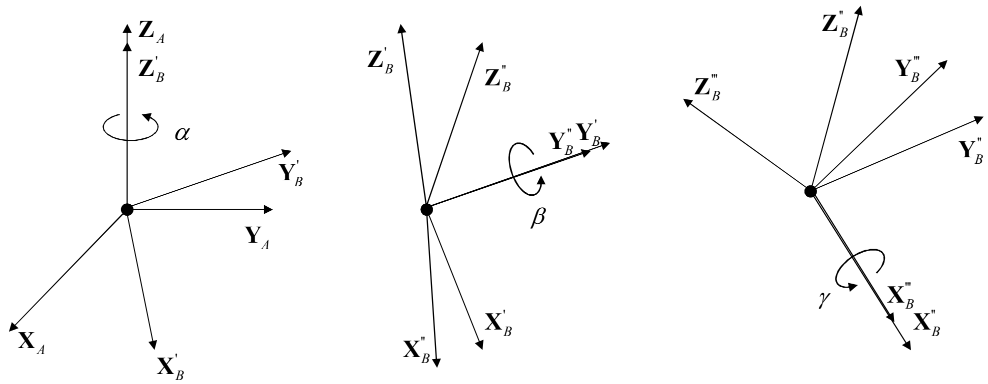
$$\theta = \text{Atan2}(y,x) = \begin{cases} 0 \le \theta \le 90 & +x & +y \\ 90 \le \theta \le 180 & -x & +y \\ -180 \le \theta \le -90 & -x & -y \\ -90 \le \theta \le 0 & +x & -y \end{cases}$$

- For one-to-one function, assume $-90.0^o \le \beta \le 90.0^o$.
- If $\beta = \pm 90.0^o$ (i.e., $\cos\beta = 0$): singular → only sum or difference of α and γ available. Choose arbitrary α or γ (e.g., $\alpha = 0.0$). (Read textbook for further development.)

Moving Frame Rotation

Each rotation is performed about an axis of the moving system (e.g., $\{B\}$).
Euler angles

- Z-Y-X Euler angles: initially $\{B\}$ coincides with $\{A\}$
  → (1) rotate $\{B\}$ about $\hat{Z}_B$ by α → (2) rotate $\{B\}$ about $\hat{Y}_B$ by β → (3) rotate $\{B\}$ about $\hat{X}_B$ by γ



- ${}^A R_B = {}^A R_{B'}\,{}^{B'}R_{B''}\,{}^{B''}R_B$ ($\because$ for a given vector $\mathbf{P}$, ${}^A\mathbf{P} = {}^A R_{B'}\,{}^{B'}\mathbf{P}$, ${}^{B'}\mathbf{P} = {}^{B'}R_{B''}\,{}^{B''}\mathbf{P}$, and ${}^{B''}\mathbf{P} = {}^{B''}R_B\,{}^B\mathbf{P}$)

$$\boxed{{}^A R_{BZ'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)}$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

- "Multiply rotation matrices **from left to right**; postmultiplying" (rotations as mapping)
- Note: Same final orientation as the fixed axes rotation in **opposite** order.

- *Z-Y-Z* Euler angles: (Read textbook)

- 24 Angle set conventions (12 fixed angles + 12 Euler angles)

Equivalent Angle-Axis

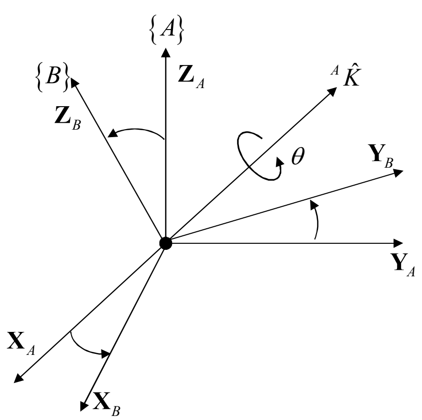
If the axis is a general direction, any orientation may be obtained through proper axis and angle selection.

- Euler's theorem on rotation: initially $\{B\}$ coincides with $\{A\}$

  → rotate $\{B\}$ about ${}^A\hat{K}$ by θ (according to right hand rule)

  ${}^A\hat{K}$: Equivalent axis of finite rotation; unit vector

  $K = \theta \cdot {}^A\hat{K}$: 3x1 orientation vector



- Equivalent rotation matrix for ${}^{A}\hat{K}=[k_x\ k_y\ k_z]^T$

$$R_K(\theta)={}^{A}R_B(\hat{K},\theta)=\begin{bmatrix} k_xk_xv\theta+c\theta & k_xk_yv\theta-k_zs\theta & k_xk_zv\theta+k_ys\theta \\ k_yk_xv\theta+k_zs\theta & k_yk_yv\theta+c\theta & k_yk_zv\theta-k_xs\theta \\ k_zk_xv\theta-k_ys\theta & k_zk_yv\theta+k_xs\theta & k_zk_zv\theta+c\theta \end{bmatrix}$$

(versed sine: $\text{versine}(\theta)=\text{vers}(\theta)=v\theta=1-c\theta$)

Examples: $R_X(\theta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $R_Y(\theta)=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, $R_Z(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Rotate a vector Q about a vector $\hat{K}$ by θ → a new vector Q'

  Rodriques' formula: $Q'=R_K(\theta)Q=Q\cos\theta+\sin\theta\left(\hat{K}\times Q\right)+(1-\cos\theta)\left(\hat{K}\bullet Q\right)\hat{K}$

- Let ${}^{A}R_{BK}(\theta)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ => $\theta=\text{Acos}\left(\dfrac{r_{11}+r_{22}+r_{33}-1}{2}\right)$; $\hat{K}=\dfrac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$ ($0^\circ<\theta<180^\circ$)
- $({}^{A}\hat{K},\theta)\equiv(-{}^{A}\hat{K},-\theta)$
- Small angular rotation: $\theta\to 0$ => ill-defined rotation axis ($\theta=0$ or $\theta=\pi$)
- Two special cases

  i) $\theta=0$: No rotation; R is identity; any nonzero $\hat{K}$ is suitable

  ii) $\theta=\pi$: Half turn; sense of axis vector is arbitrary; $R(\hat{K},\pi)=R(-\hat{K},\pi)$

  To find $\hat{K}$, set $\sin\theta=0$, $\cos\theta=-1$, and $v\theta=1-\cos\theta=2$, and use the first row of R

  => $2K_x^2-1=r_{11}$, $2K_xK_y=r_{12}$, $2K_yK_z=r_{13}$

  => $\therefore\ K_x=\sqrt{(1+r_{11})/2}$, $K_y=\dfrac{r_{12}}{2K_x}=\dfrac{r_{12}+r_{21}}{4K_x}$, $K_z=\dfrac{r_{13}+r_{31}}{4K_x}$

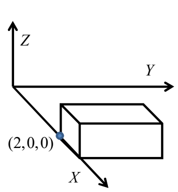
- Rotation about $\hat{K}$ which does not pass through the origin

  : [position change] + [same final orientation as if $\hat{K}$ had passed through the origin]

- Example: Rotate about Z-axis

  $K=[0\ \ 0\ \ 1]^T$; $\phi=90^o$; $\mathbf{r}=[2\ \ 0\ \ 0]^T$

  => $\mathbf{r}'=[0\ \ 2\ \ 0]^T$



- Example 2.9 (Craig's 4th Ed.): A frame $\{B\}$ is described as initially coincident with $\{A\}$. We then rotate $\{B\}$ about the vector ${}^{A}\hat{K} = [0.707 \;\; 0.707 \;\; 0.0]^{T}$ (passing through point ${}^{A}P$ = [1.0 2.0 3.0]) by an amount $\theta = 30$ degrees. Give the frame description of $\{B\}$.
  (Do it yourself)

- Exercise 2.14 (Craig's 4th Ed.): (Do it yourself)

Euler Parameters (= Unit Quaternion)
(Skip)

Transformation of Free Vectors

Equal vectors: same magnitude and direction
Equivalent vectors: produce same effect in a certain capacity

- Vector quantities
  - Free vector: may be positioned anywhere in space (e.g., couple vector on a rigid body, translational velocity of a nonrotating body)
  - Sliding (or line) vector: effects depend on specified line of action (e.g., force applied on a rigid body)
  - Bound (or fixed) vector: effects depend on point of application (e.g., force applied on a deformable body, force applied on a particle)