1. (Questions on logic and proof methods) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table". **Circle True or False as appropriate for the following statements:**

T **F** (a) There are statements about the natural numbers that can be proved with *Strong Induction* but cannot be proved with *Ordinary Induction*.

T **F** (b) Let n be a natural number. If n^2 is odd then so is n.

T **F** (c) $p \implies q$ is logically equivalent to $(\neg p) \vee q$

T **F** (d) The truth table given below is correct for $\neg q \implies p$

p	q	$\neg q \implies p$
1	1	1
1	0	1
0	1	0
0	0	1

2. (Facts about matrices) For $n, m \geq 1$, let A and B be an $n \times m$ real matrices. For any real matrix M, denote its i-th column by M_i and its ij-element by $[M]_{ij}$. **Circle True or False as appropriate for the following statements:**

T **F** (a) $\text{trace}(AA^\top) = \sum_{i=1}^{n} \left([A]_{ii}\right)^2$

T **F** (b) $[A^\top B]_{ij} = (A_i)^\top B_j$.

T **F** (c) $\text{span}\{A_1, A_2, \cdots, A_m\} = \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^m, \text{ such that } y = Ax\}$.

T **F** (d) Suppose $n = m$ so that A is a square real matrix and let $x \in \mathbb{R}^n$, $x = [x_1, x_2, \ldots, x_n]^\top$. Then

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = x^\top A.$$

3. Let $(\mathcal{X}, \mathbb{R})$ be a finite-dimensional inner product space, let $S_1 \subset \mathcal{X}$ and $S_2 \subset \mathcal{X}$ be nonempty subsets (to be clear, they may or may not be subspaces). **Circle True or False as appropriate for the following statements:**

T **F** (a) If $\text{span}\{S_1\} \subset \text{span}\{S_2\}$, then $S_1 \subset S_2$.

T **F** (b) $\text{span}\{S_1\} \oplus S_1^\perp = \mathcal{X}$.

T **F** (c) $\text{span}\{S_1 \cup S_2\} = \text{span}\{S_1\} + \text{span}\{S_2\}$.

T **F** (d) $S_1^\perp \cap S_2^\perp = \left[\text{span}\{S_1\} \cap \text{span}\{S_2\}\right]^\perp$.

Note: We have not covered normed spaces and inner products in lecture yet. However, you can still complete these problems

4. (Normed spaces, matrices, and inner products) **Circle True or False as appropriate for the following statements:**

T **F** (c) The matrix $M = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ is positive definite.

5. (Vector spaces, representations, and norms) Let $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ be an n-dimensional normed space with $n \geq 4$ and let $L : \mathcal{X} \to \mathcal{X}$ be a linear operator. Let A be the matrix representation of $L : \mathcal{X} \to \mathcal{X}$ when the basis $\{u\} := \{u^1, \ldots, u^n\}$ is used on both copies of $\mathcal{X}$ (i.e., on the domain of L and its range (also called co-domain)). We define a second basis on $\mathcal{X}$ by scaling the first basis:

$$\{\bar{u}\} := \{\bar{u}^1 = u^1, \bar{u}^2 = 2u^2, \ldots, \bar{u}^k = ku^k, \ldots, \bar{u}^n = nu^n\}.$$

Circle True or False as appropriate for the following statements, where u^i and $\bar{u}^j$ always refer to elements of the given bases and the matrix A is as defined in the problem statement.

T **F** (a) The change of basis matrix from $\{u\}$ to $\{\bar{u}\}$ (i.e. $P \in \mathbb{R}^{n\times n}$ s.t. $[x]_{\bar{u}} = P[x]_u$) is $P = \mathrm{diag}\left([1, 2, \ldots, n]\right)$.

T **F** (b) $[L(u^3)]_{\{\bar{u}\}} = \frac{1}{3}A_3$, where A_3 is the third column of A.

T **F** (c) $[L(\bar{u}^3 + \bar{u}^4)]_{\{u\}} = 3A_3 + 4A_4$ where A_3 and A_4 are the corresponding columns of A.

8. (15 points) (Proof Problem) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and v^1, v^2, v^3 vectors in $\mathcal{X}$. Define the following two statements:

- P: each of the sets $\{v^1, v^2\}$, $\{v^2, v^3\}$, and $\{v^3, v^1\}$ is linearly independent.
- Q: the set $\{v^1, v^2, v^3\}$ is linearly independent.

For each of the following statements, decide if it is T or F and then support your conclusion with a proof or a counterexample:

(a) $P \implies Q$

(b) $Q \implies P$

Show your work below. You can use as true anything we have established in ROB 501 lecture or HW. I cannot answer any question of the form: **"do I have to prove this?"** or **"can I assume this?"** or **"have I shown enough?"**.

Remark: If the problem seems completely trivial, that is OK; please write down the few lines it takes to do a (complete) proof or to establish a counterexample. If the problem seems challenging, then maybe you need more than a few lines to work the problem. Both are possible.

(a) $P \implies Q$ is **[T F (circle one)]** and here is my supporting reasoning.

(b) $Q \implies P$ is **[T F (circle one)]** and here is my supporting reasoning.

9. (5 points) A$^+$ Problem: Points earned here will go toward deciding who goes from an A to an A^+ at the end of the term. Recall that for your GPA at Michigan, an A$^+$ counts the same as an A.

Def. Let $(\mathcal{X}, \mathbb{C})$ be a vector space over the complex numbers and $L : \mathcal{X} \to \mathcal{X}$ be a linear operator. $\lambda \in \mathbb{C}$ is an e-value of L if $\exists\ (v \in \mathcal{X}, v \neq 0)$ such that $L(v) = \lambda v$; v is called an e-vector of L.

Prove this: If λ_1, λ_2, and λ_3 are *distinct* e-values of L, then a set of corresponding e-vectors $\{v^1, v^2, v^3\}$ is *linearly independent.*

Note: If you need to use a result from lecture or HW, clearly state the result and note that it is from ROB 501; in that case, you do not need to prove it. Otherwise, any other statements used in your proof should be justified here. **If your proof assumes that $(\mathcal{X}, \mathbb{C})$ is finite dimensional, you will earn at most three (3) points.**