Mechatronics

Topic #9

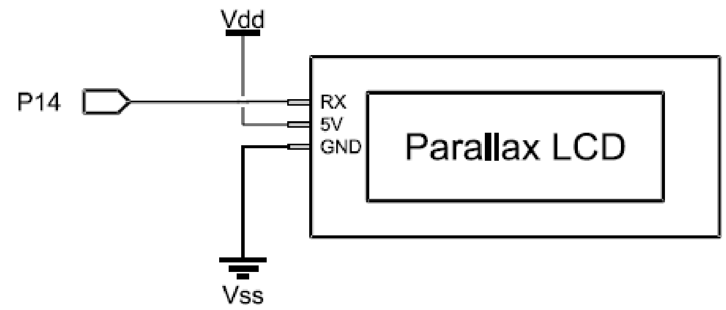
Smart Sensors and sensors for mobile robotics

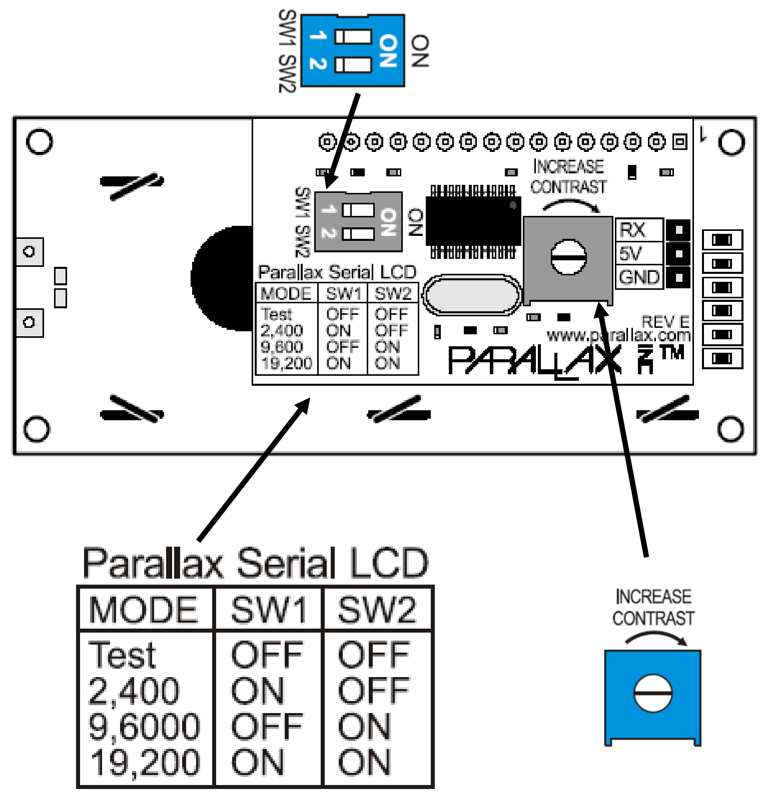
Liquid Crystal Display (LCD)

- Display measurement, status information, etc.
- Field-testing without being tethered to a PC/Laptop
- Parallax 2×16 serial LCD (non-backlit)
- 3-pin connection (V_{ss}, V_{dd}, and V_{sig})
- BS2 commands the LCD serially, using SEROUT

Interfacing LCD to BS2

- Connect BS2's V_{ss} , V_{dd}, and one I/O pin (say P14) to LCD's GND, 5V, and RX pins, respectively
- To test LCD module, on its backside, set switches SW1 and SW2 off
- Turn on power to BS2, LCD should display "Parallax, Inc." on top line and "www.parallax.com" on bottom line
- If display appears dim, adjust the contrast potentiometer
- Turn off power to BS2 and set SW2 ON to allow LCD to receive serial communication from BS2 at 9600 baud rate

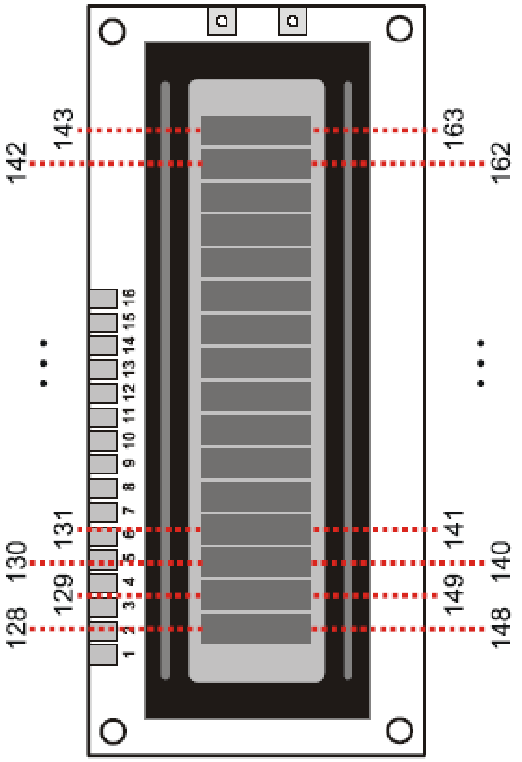




LCD: PBASIC Sample Code I

```
'{$STAMP BS2}
'{$PBASIC 2.5}
SEROUT 14, 84, [22, 12] 'Initialize LCD
PAUSE 5
SEROUT 14, 84, ["Hello World!", 13, "The LCD Works"]
```

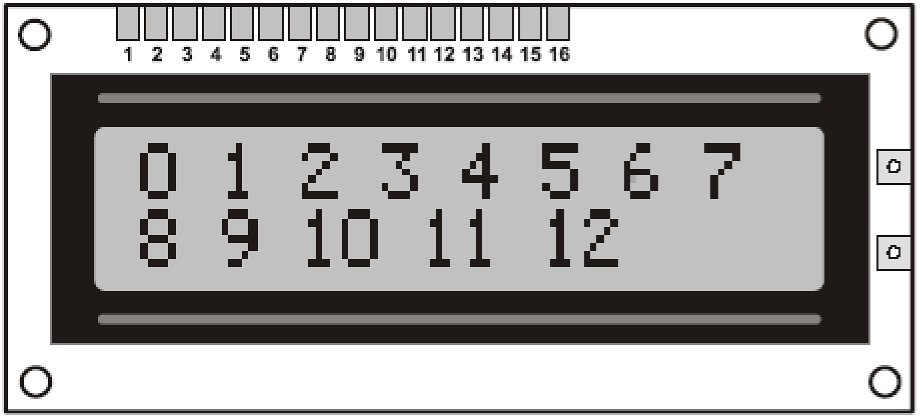
- SEROUT *Pin, BaudMode,* [*DataItem1, DataItem2, ...*]
- BaudMode argument for 9600 bits per second (bps), 8 data bits, no parity, true signal: 84
- DataItems: text to be displayed, control codes, formatters like DEC, BIN, HEX, etc.
- LCD must receive control code 22 from BS2 to turn on
- Control code examples—8: cursor left, 9: cursor right, 12: clear display (follow with PAUSE 5 to allow display to clear), 13: carriage return, 21: LCD off,
- 128 to 143 Position cursor on Line 0, character 0 to 15
- 148 to 163 Position cursor on Line 0, character 1 to 15
- SEROUT 14, 84, [128, "Hello", 148, "World!"]



LCD: PBASIC Sample Code II

```
' {$STAMP BS2}
' {$PBASIC 2.5}
counter VAR Byte            'FOR...NEXT loop index
SEROUT 14, 84, [22, 12]     'Initialize LCD
PAUSE 5                     '5 ms delay for clearing display
FOR counter = 0 TO 12       'Count to 12; increment at 1/2 s
SEROUT 14, 84, [DEC counter, " "]
PAUSE 500
NEXT
END
```

- Display numbers 0 to 12 on LCD
- Each number is followed by a space
- When top line of LCD is filled up by 16 characters
  - text sent by BS2 wraps to the bottom line
  - if the bottom line is filled up by 16 characters then the text wraps again, to top line

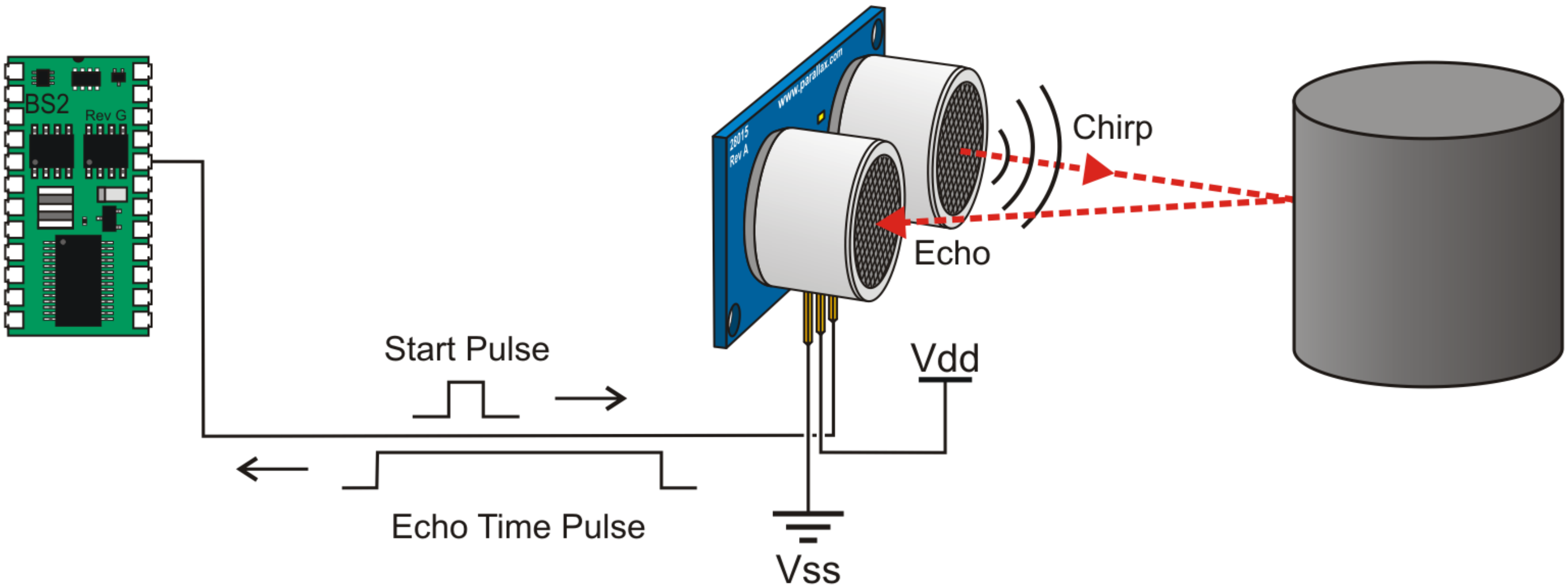


LCD: PBASIC Sample Code III

```
' Display elapsed time with BS2 and Parallax Serial LCD.
' {$STAMP BS2}
' {$PBASIC 2.5}
hours VAR Byte 'hours
minutes VAR Byte 'minutes
seconds VAR Byte 'seconds
SEROUT 14, 84, [22, 12] 'Initialize LCD
PAUSE 5 '5 ms to clear display
SEROUT 14, 84, ["Time Elapsed...", 13] 'Text & carriage return
SEROUT 14, 84, [" h m s"] 'Text on second line
DO 'Main Routine
'Calculate hours, minutes, seconds
IF seconds = 60 THEN seconds = 0: minutes = minutes + 1
IF minutes = 60 THEN minutes = 0: hours = hours + 1
IF hours = 24 THEN hours = 0
'Display digits on LCD on Line 1. The values 148, 153, 158
'place the cursor at character 0, 5, and 10 for the time values.
SEROUT 14, 84, [148, DEC2 hours,
                153, DEC2 minutes,
                158, DEC2 seconds ]
PAUSE 991 'Pause + program overhead ~ 1 second
seconds = seconds + 1 'Increment second counter
LOOP 'Repeat Main Routine
```

Ultrasonic Sensor—PING)))

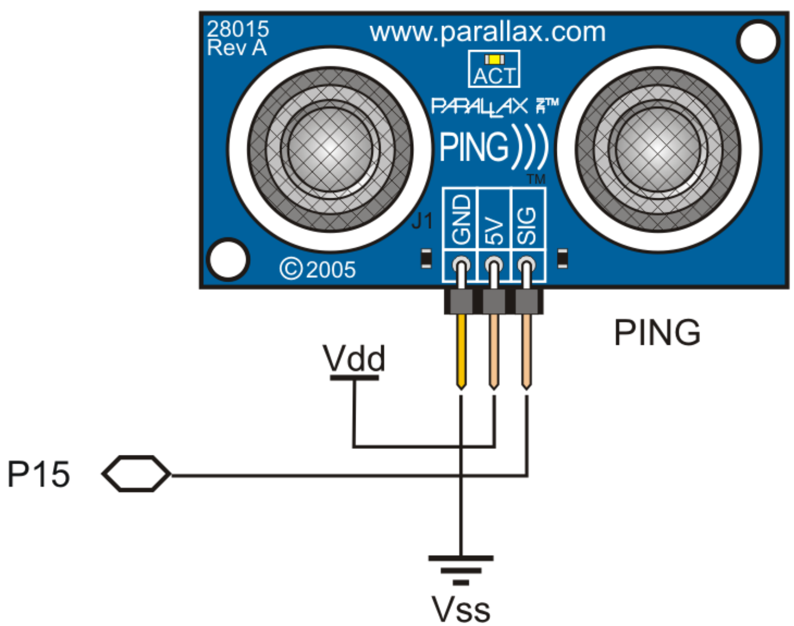
- Time-of-flight distance measurement
- Sensor emits a 40KHz tone and measures time till it receives the echo signal

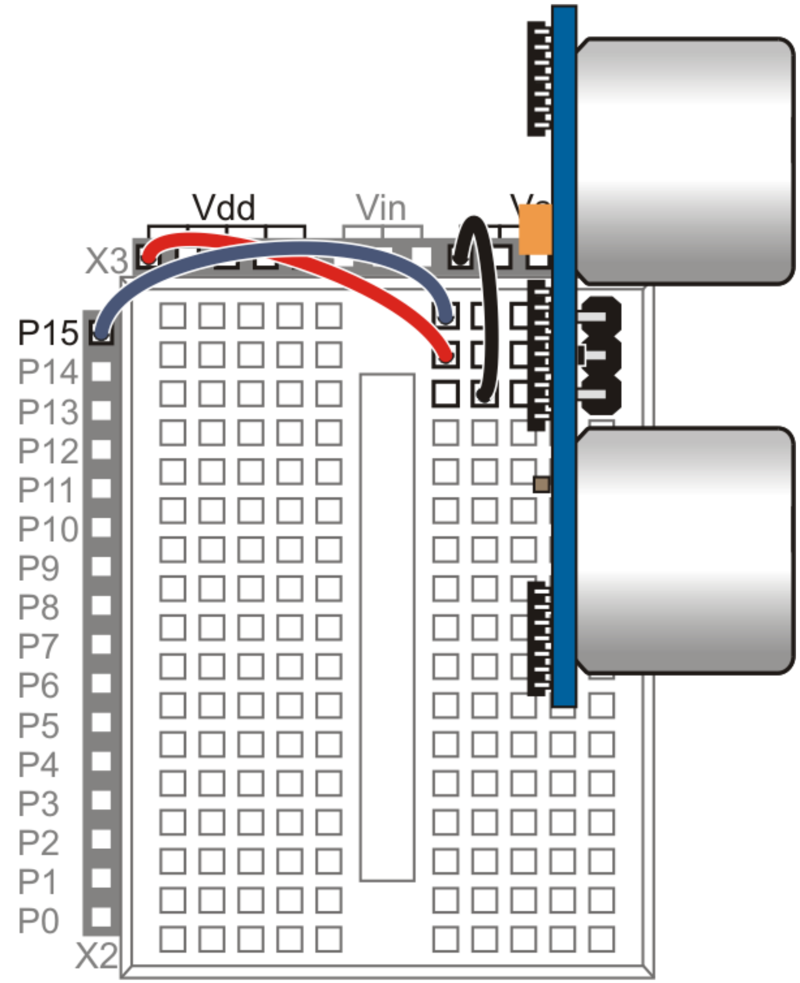


  – Round-trip time-of-flight yields distance measurement: $D=0.5\times C\times T$, D=distance (m), C=speed of sound in air @ 72 ºF (344.8 m/s), T=round trip time (s)
- Range: 3.3 meters

Interfacing PING))) to BS2

- Connect BS2's V_{ss} , V_{dd}, and one I/O pin (say P15) to PING)))'s GND, 5V, and SIG pins, respectively





PING))): PBASIC Sample Code I

```
' {$STAMP BS2}
' {$PBASIC 2.5}
rawtime VAR Word
DO
PULSOUT 15, 5
PULSIN 15, 1, rawtime
DEBUG HOME, “rawtime = ", DEC5 rawtime
PAUSE 100
LOOP
```

- PULSOUT 15, 5: sends a 10μs pulse to P15
- PULSIN 15, 1, time: monitors for the return echo and stores it in the variable time (unit 2μs)

PING))): PBASIC Sample Code II

$$D_{\mathrm{cm}} = \left(\frac{1}{2}\right) \times \overbrace{\left(100 \times 344.8\right)}^{\text{cm/s}} \times \overbrace{\left(T_{\mathrm{raw}} \times 2 \times 10^{-6}\right)}^{\text{seconds}} = T_{\mathrm{raw}} \times 0.03448$$

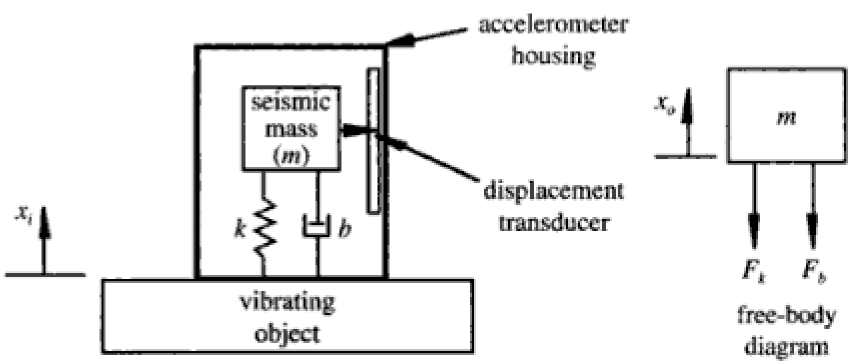
- Let cmConst=0.03448×65536=2260
- Now compute D_{cm} by using T_{raw}**2260

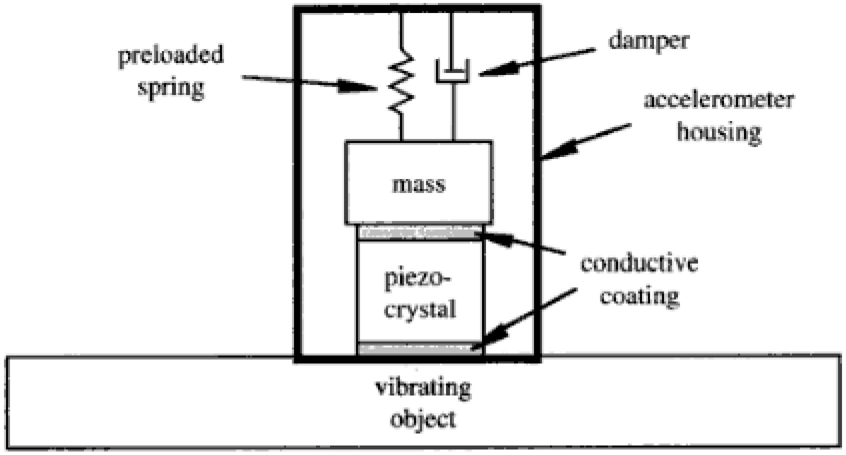
```
' {$STAMP BS2}
' {$PBASIC 2.5}
rawtime VAR Word
cmDist VAR Word
cmConst CON 2260
DO
PULSOUT 15, 5
PULSIN 15, 1, rawtime
cmDist=rawtime**cmConst
DEBUG HOME, "cmDist = ", DEC cmDist
PAUSE 100
LOOP
```

- For D_{inch} let inchConst=(0.03448/2.54)×65536=890
- Now compute D_{inch} by using T_{raw}**890

Accelerometer

- Electromechanical device to measure acceleration forces
  - Static forces like gravity pulling at an object lying at a table
  - Dynamic forces caused by motion or vibration
- How they work
  - Seismic mass accelerometer: a seismic mass is connected to the object undergoing acceleration through a spring and a damper;
  - Piezoelectric accelerometers: a microscopic crystal structure is mounted on a mass undergoing acceleration; the piezo crystal is stressed by acceleration forces thus producing a voltage
  - Capacitive accelerometer: consists of two microstructures (micromachined features) forming a capacitor; acceleration forces move one of the structure causing a capacitance changes.
  - Piezoresistive accelerometer: consists of a beam or micromachined feature whose resistance changes with acceleration
  - Thermal accelerometer: tracks location of a heated mass during acceleration by temperature sensing





Accelerometer Applications

- Automotive: monitor vehicle tilt, roll, skid, impact, vibration, etc., to deploy safety devices (stability control, anti-lock breaking system, airbags, etc.) and to ensure comfortable ride (active suspension)
- Aerospace: inertial navigation, smart munitions, unmanned vehicles
- Sports/Gaming: monitor athlete performance and injury, joystick, tilt
- Personal electronics: cell phones, digital devices
- Security: motion and vibration detection
- Industrial: machinery health monitoring
- Robotics: self-balancing

Helmet: Impact Detection

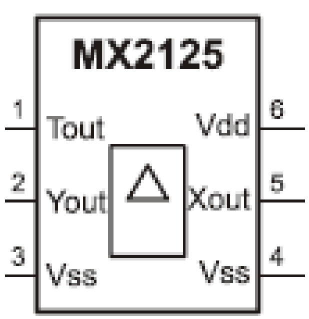
2 axis joystick

WII Nunchuk: 3 axis accelerometer

Segway

Memsic 2125 2-axis Accelerometer

- Measure acceleration, tilt angle, rotation angle
  - G-force measurements for X and Y axis reported in pulse-duration
- Temperature measurement: analog output (T_{out})
- Low current operation: < 4 mA @ 5VDC
- Measures 0 to ±2 g on either axis
- Resolution: <1 mg
- Operating temperature: 0 °C to 70 °C

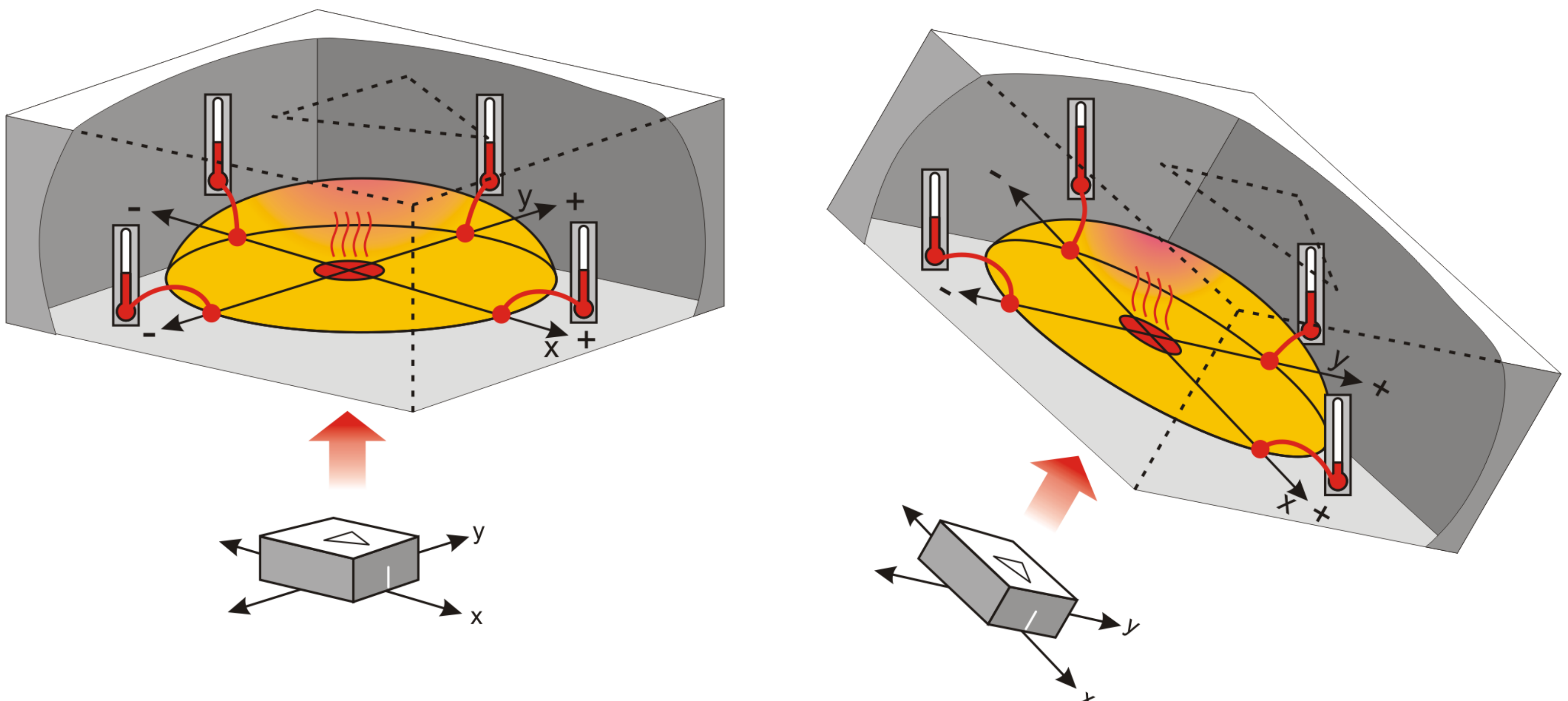


Accelerometer Module

MX2125 Chip

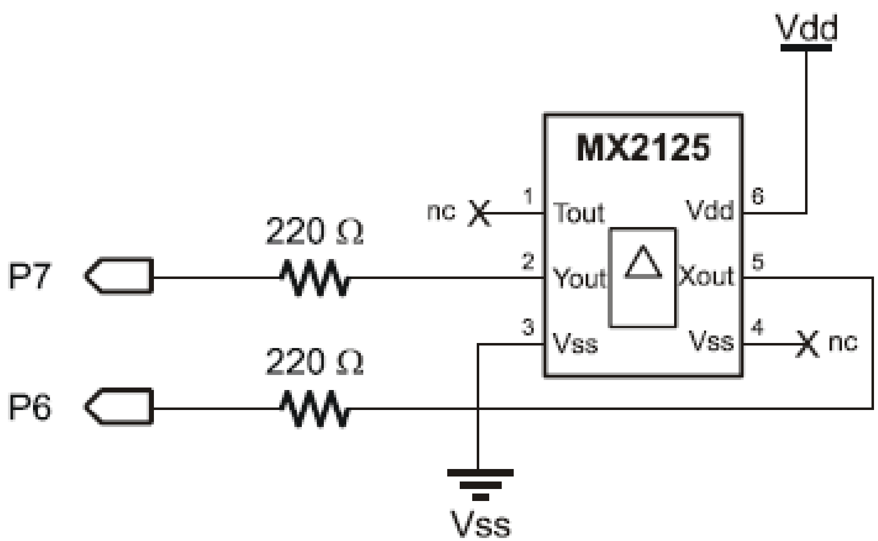
MX2125 Accelerometer: How it Works

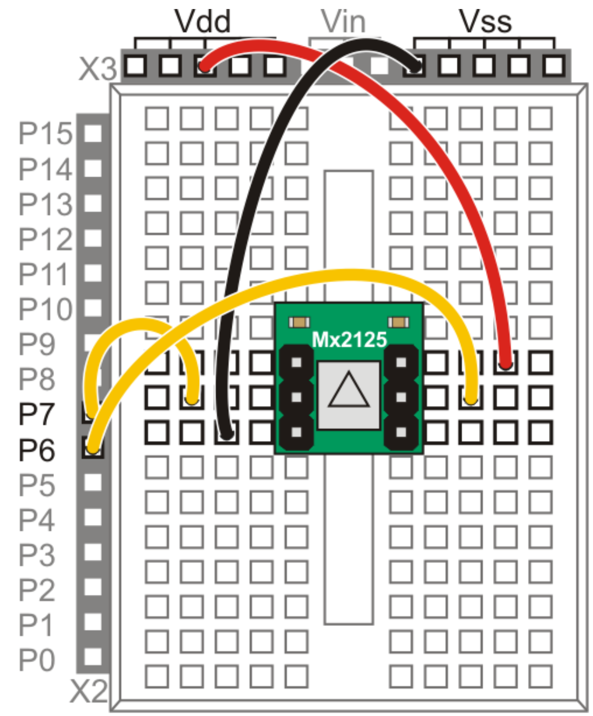
- **A MEMS device consisting of**
  - **a chamber of gas with a heating element in the center**
  - **four temperature sensors around its edge**
- **Hold accelerometer level→hot gas pocket rises to the top-center of the accelerometer's chamber→all sensors measure same temperature**
- **Tilt the accelerometer→hot gas pocket collects closer to one or two temperature sensors→sensors closer to gas pocket measure higher temperature**
- **MX2125 electronics compares temperature measurements and outputs pulses (pulse duration encodes sensor o/p)**



Interfacing Accelerometer to BS2

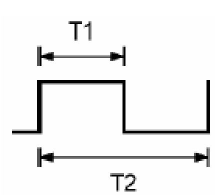
- Connect BS2's V_{ss} , V_{dd}, and two I/O pin (say P6 and P7) to MX2125's pins 3, 6, 5, and 2, respectively





- X_{out} and Y_{out} pulse outputs are set to 50% duty cycle at 0g; the duty cycle changes in proportion to acceleration
- G Force can be computed from the duty cycle as shown below
- T_{out} provides analog output 1.25 volts @25.0°C, output change: 5 mV/°C

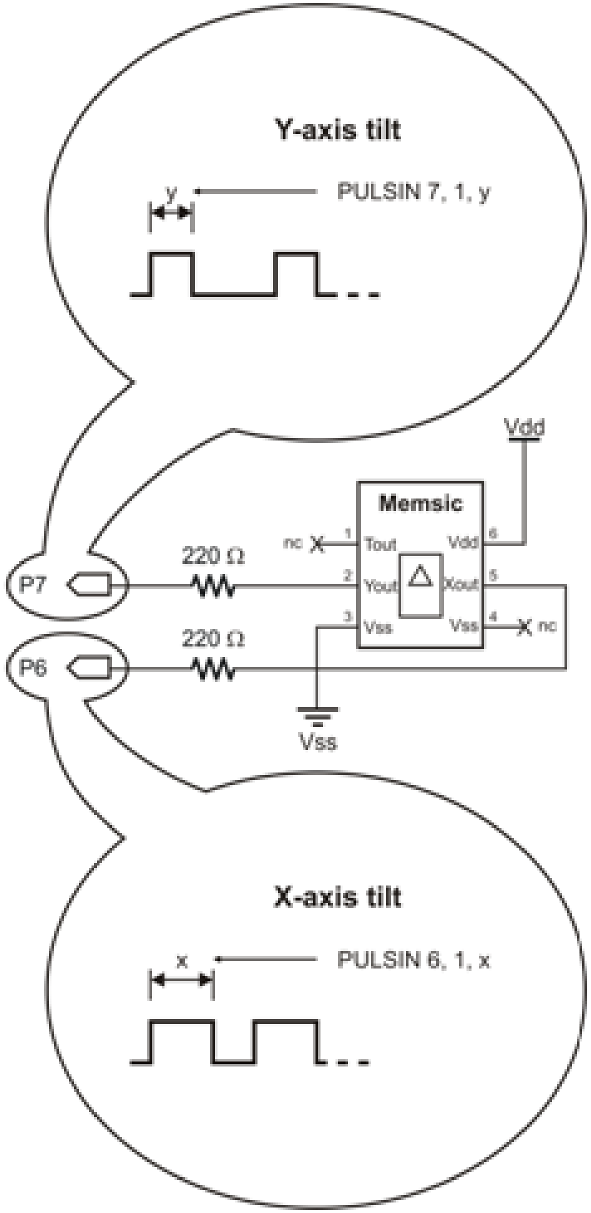
Memsic 2125 Pulse Output



$$A(g) = ((T1 / T2) - 0.5) / 12.5\%$$

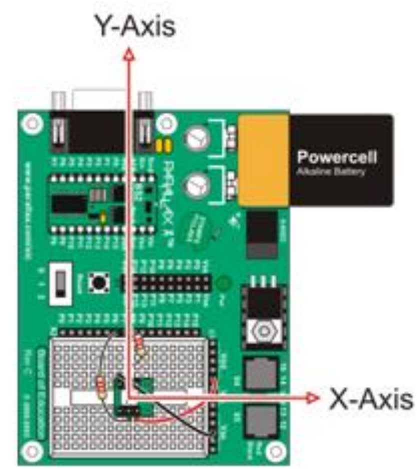
T2 duration is calibrated to 10 milliseconds at 25° C (room temperature)

Accelerometer Axis Pulse Measurements



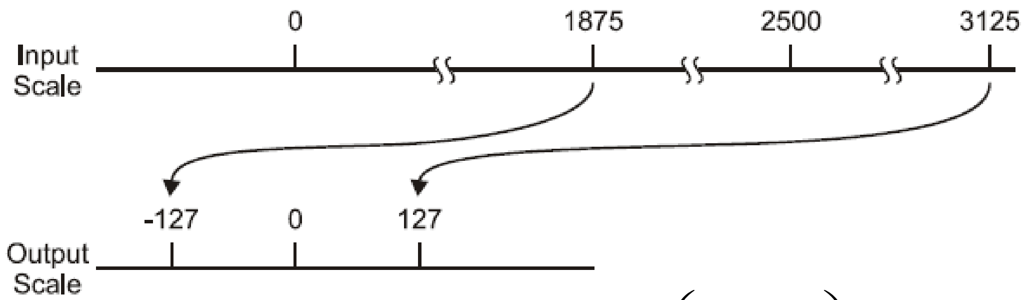

```
'{$STAMP BS2}
'{$PBASIC 2.5}
x VAR Word
y VAR Word
DEBUG CLS
DO
PULSIN 6, 1, x
PULSIN 7, 1, y
DEBUG HOME, DEC4 ? x, DEC4 ? y
PAUSE 100
LOOP
```

Pulsin o/p range: 1875 to 3125
When level: o/p=2500



Pulse Measurements: Offset and Scaling

- **Let X_{raw}= Pulsin output**
- **$X_{raw} \in$ {1875, 3125} and when level X_{raw}=2500**
- **We wish X_{out}: $X_{raw} \rightarrow X_{out} \in$ {-127, 127}, and X_{out}=0 when level**



$$X_{\text{out}} = \left(X_{\text{raw}} - 2500\right) \times \left(\frac{254}{1250}\right)$$

$$= X_{\text{raw}} \times \left(\frac{254}{1250}\right) - 508$$

- **Let Scale=INT((254/1250) ×65536)=13316**
- **Now compute X_{out} by using X_{raw}**13316-508**

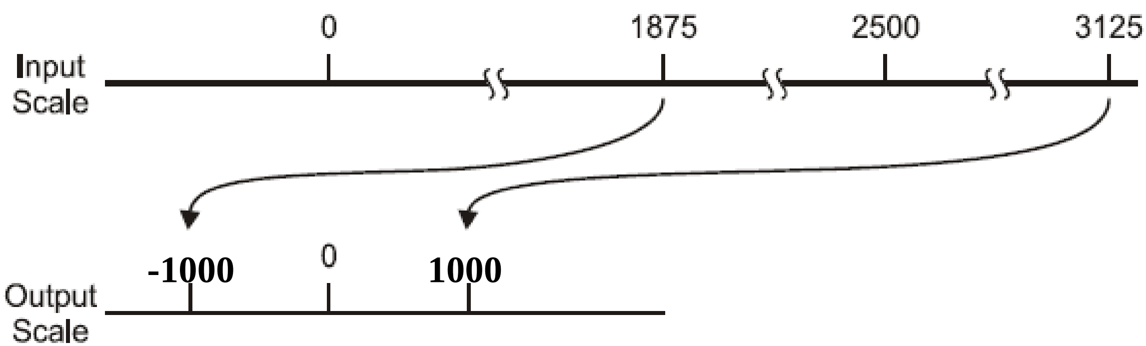
```
'{$STAMP BS2}
'{$PBASIC 2.5}
scalecon CON 13316
xraw VAR Word
yraw VAR Word
Xo VAR Word
Yo VAR Word
DEBUG CLS
DO
PULSIN 6, 1, xraw
PULSIN 7, 1, yraw
Xo=xraw**scalecon-508
Yo=yraw**scalecon-508
DEBUG HOME, SDEC Xo, SDEC Yo
PAUSE 100
LOOP
```

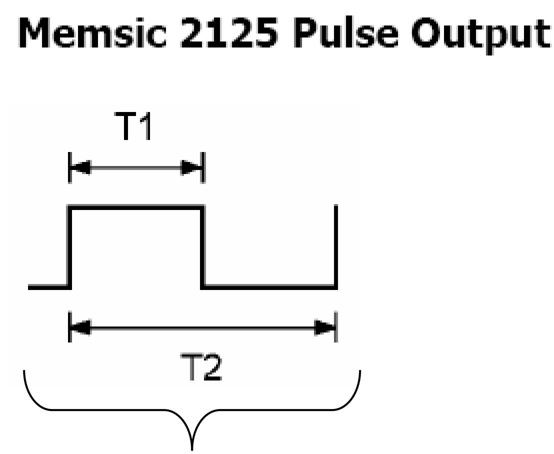
Clamp input range to {1875, 3125} using the following:

```
xout=(xraw Min 1875 Max 3125) **scalecon-508
yout=(yraw Min 1875 Max 3125) **scalecon-508
```

g-Force Measurements in mili-g—I

- **Let T_{raw}= Pulsin output (2μs units)**
- **$T_{raw} \in$ {1875, 3125} and when level T_{raw}=2500**
- **T_{raw}=1875→-g (-1000 milli-g) and T_{raw}=3125→g (-1000 mili-g)**
- **So, we wish T_{out}: $T_{raw} \rightarrow T_{out} \in$ {-1000, 1000}, and T_{out}=0 when level**





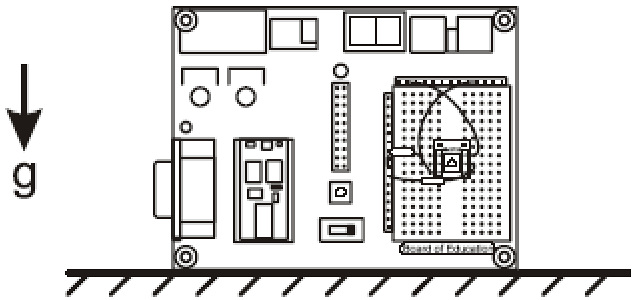
T_1: Pulsin output returns T_{raw}
T_2: 10milli-seconds @ 25ºC

- **Moreover, recall g force is given by**

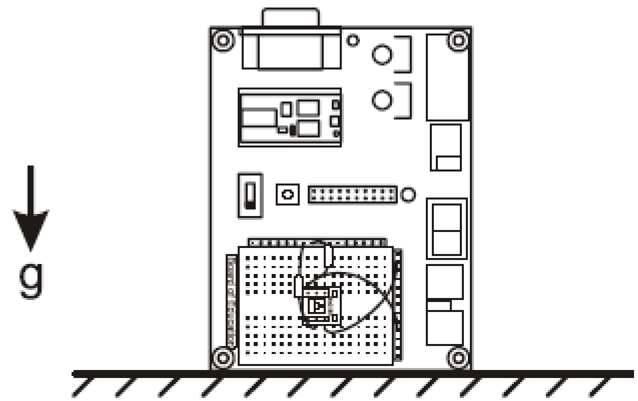
$$g_{\text{Force}} = \left(\frac{T_1}{T_2} - 0.5 \right) \times \left(\frac{1}{12.5\%} \right) \quad (\text{units}: \text{g})$$

g-Force Measurements in mili-g—II

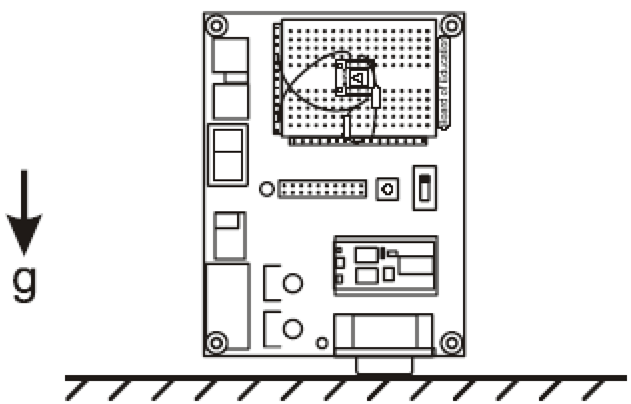
a. x=1000/1000, y=0/1000



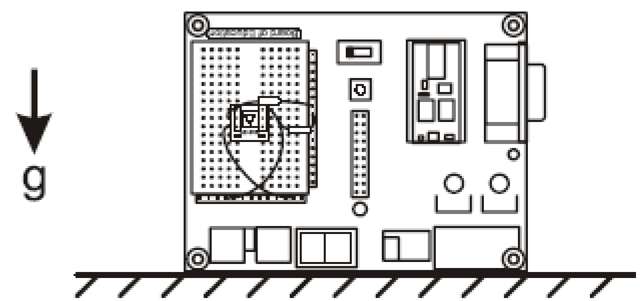
b. x=0/1000, y=1000/1000



d. x=0/1000, y=-1000/1000

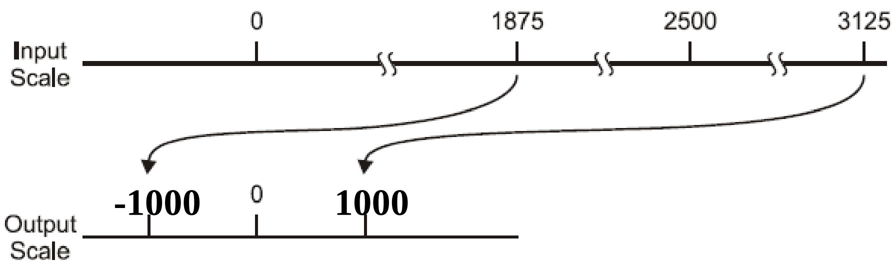


c. x=-1000/1000, y=0/1000



Sample Readings at Various Orientations (start at top left, rotate clockwise)

g-Force Measurements in mili-g—III



- T_1: Pulsin output returns T_{raw} in 2μs units
- T_2: 10mili-seconds @ 25ºC
- Thus,
  $T_1 = 2\times10^{-6}\times T_{raw}$ seconds $= 2\times10^{-3}\times T_{raw}$ mili-seconds

$$
\begin{aligned}
T_{\text{out}} &= \left(T_{\text{raw}} - 2500\right)\times\left(\frac{2000}{1250}\right) \\
&= \left(\frac{2\times T_{\text{raw}}}{10}\right)\times\left(\frac{1000}{125}\right) - 2500\times\left(\frac{2000}{1250}\right) \\
&= \left(\frac{2\times T_{\text{raw}}}{10}\right)\times 8 - 4000 \\
&= \left(\left(\frac{2\times T_{\text{raw}}}{10}\right) - 500\right)\times 8
\end{aligned}
$$

$$
\begin{aligned}
g_{\text{Force}} &= \left(\frac{T_1}{T_2} - 0.5\right)\times\left(\frac{1}{12.5\%}\right),\ \text{(units: g)} \\
&= \left(\frac{T_1}{T_2} - 0.5\right)\times\left(\frac{1}{12.5\%}\right)\times 10^3,\ \text{(units: milli-g)} \\
&= \left(\frac{T_{\text{raw}}\times 2\times 10^{-3}}{10} - 0.5\right)\times\left(\frac{100}{12.5}\right)\times 10^3 \\
&= \left(\frac{T_{\text{raw}}\times 2}{10} - 500\right)\times 8
\end{aligned}
$$

MX2125 Angle of Rotation in Vertical Plane—I

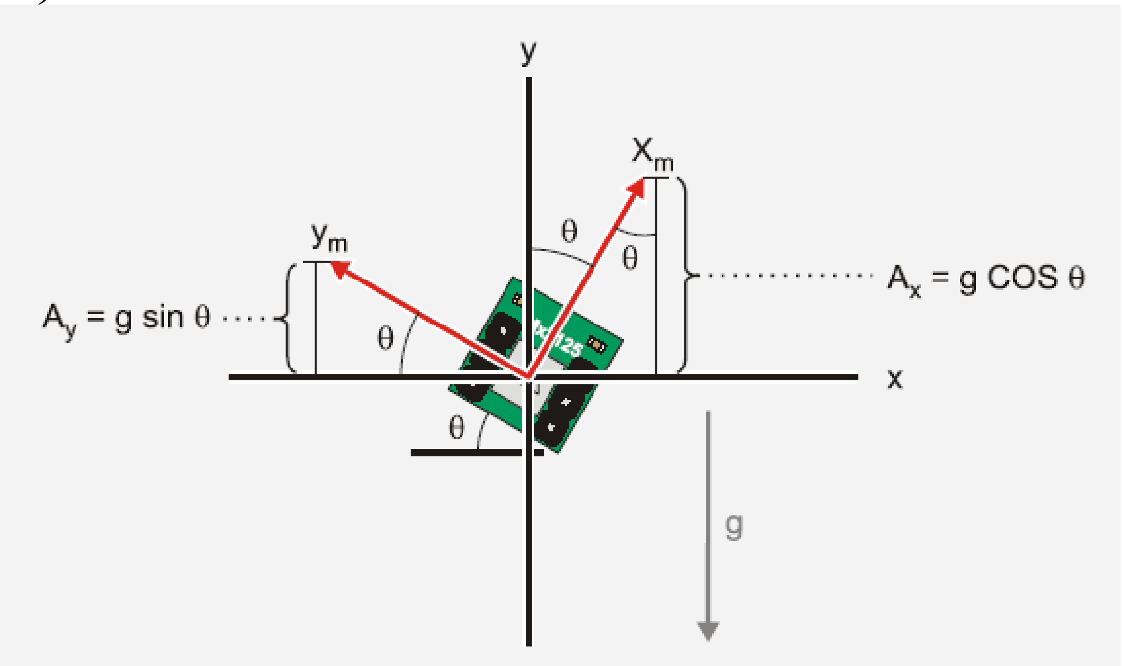
- MX2125's angle of rotation in the vertical plane:

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right), \text{ BS2 returns } A_x, A_y \in \{1875, 3125\}$$

- To compute tan^{-1}(Y/X) use PBASIC ATAN command: X ATN Y; ATN requires X, Y$\in${-127, 127} which is accomplished using

$$X = (A_x - 2500) \times \left(\frac{254}{1250}\right)$$
$$= A_x \times \left(\frac{254}{1250}\right) - 508$$

- **Let INT((254/1250) ×65536)=13316**
- **Now compute X by using Ax**13316-508**

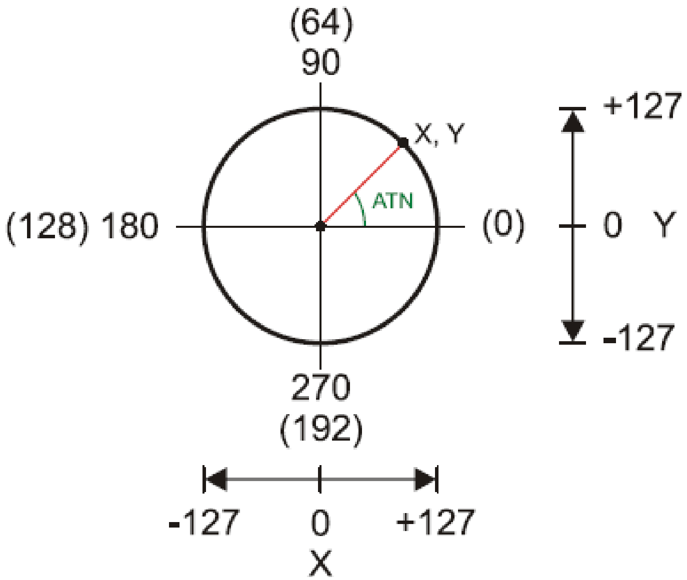


MX2125 Angle of Rotation in Vertical Plane—II

- ATN returns its output in binary radians (i.e., a circle is split up into 256 segments instead of 360 segments as in degrees)
- Convert ATN output from brad to degrees as follows:

$$\theta_{\text{Deg}} = \theta_{\text{BRad}} \times \left(\frac{360}{256}\right)$$

- **Let INT((360/256)×256)=360**
- **Now compute θ_{Deg} by using θ_{BRad}*/360**



Unit circle in degrees and binary radians

MX2125 Angle of Rotation in Vertical Plane: Sample Code

```
'{$STAMP BS2}
'{$PBASIC 2.5}
scale1 CON 13316
scale2 CON 360
Ax VAR Word
Ay VAR Word
angle VAR Word
DEBUG CLS
DO
PULSIN 6, 1, Ax
PULSIN 7, 1, Ay
Ax=(Ax MIN 1875 MAX 3125)**scale1-508
Ay=(Ay MIN 1875 MAX 3125)**scale1-508
angle=Ax ATN Ay
angle=angle*/scale2
DEBUG HOME, " Ax =", SDEC Ax, " Ay=", SDEC Ay, " angle=", SDEC3 angle, 176, "   "
PAUSE 300
LOOP
```

Dead Reckoning

- Dead Reckoning: derived from deduced reckoning of sailing days
  - Establish present location by advancing over a previous known position through known course and velocity information over a given length of time
- Measure vehicle displacement:
  - Wheel rotation (odometry using pot, encoder, magnetic/inductive proximity sensor, etc.).
  - Doppler navigation (motion relative to ground)
  - Inertial navigation (accelerometers)
- Measure vehicle heading:
  - Onboard steering
  - Magnetic compass
  - Rate gyro
  - Differential odometry

Doppler Navigation—I

- Stationary Observer, moving source

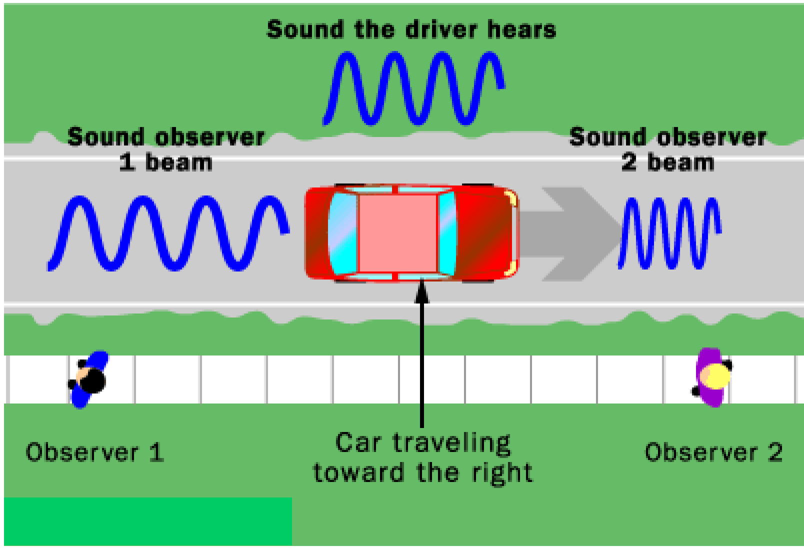
$$f_{\text{rec}} = f\left(\frac{s}{s \pm v_{\text{s}}}\right)$$

  - f: source frequency, f_{rec}: frequency @ observer (Doppler frequency), s: speed of sound in air, v_{s}: velocity of source
  - +/- sign: source moving away from/toward observer

- Moving observer, stationary source

$$f_{\text{rec}} = f\left(\frac{s \pm v_{\text{o}}}{s}\right)$$

  - v_{o}: velocity of observer



- For reflected wave, instead of Doppler frequency, we consider the change in frequency (Doppler shift)

$$\Delta f = f - f_{rec} = \frac{2 f v \cos\theta}{s}$$

  - f: source frequency, f_{rec}: frequency received, s: speed of sound in air, v: velocity of target object, θ: relative angle between direction of motion and beam axis

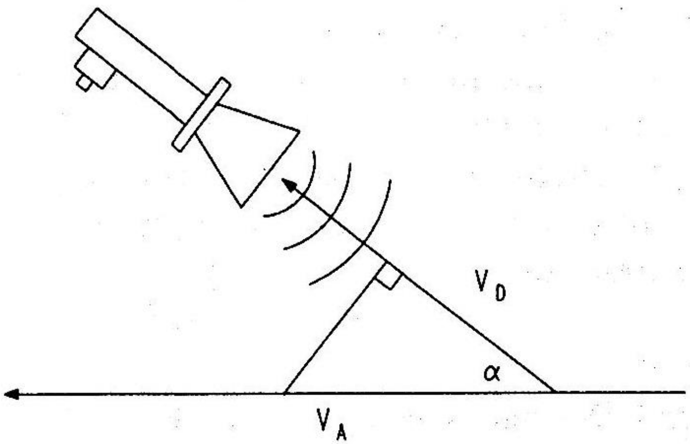
Doppler Navigation—II

- Use ultrasonic sensor aimed downward at a prescribed angle to sense ground movement
- Use Doppler shift equation to determine the ground speed V_A of the vehicle as follows:

$$\Delta f = f - f_{rec} = \frac{2 f V_D}{s}$$

$$\Rightarrow V_D = \frac{s \Delta f}{2 f}$$

$$\Rightarrow V_A = \frac{V_D}{\cos \alpha} = \frac{s \Delta f}{2 f \cos \alpha}$$



Doppler ground speed sensor

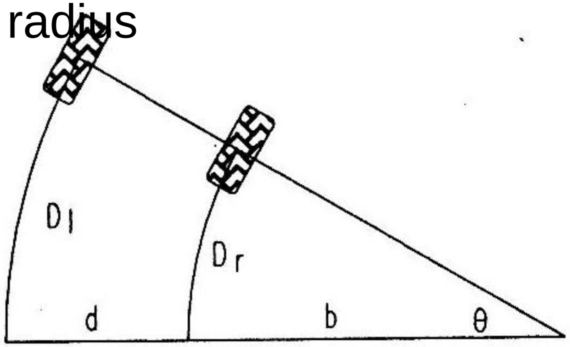
  – f: transmitted frequency, Δf: Doppler shift, s: speed of sound in air, V_D: measured velocity, α: declination angle

Vehicle heading via Differential Odometry

- Displacement *D* of a differential-drive robot platform:

$$D = \frac{D_{\mathrm{L}} - D_{\mathrm{R}}}{2}$$

- D_L and D_R: displacements of left and right wheels, respectively
- D_L: portion of the circumference of a circle with radius $d+b$, $C_L = 2\pi(d+b)$
- D_R: portion of the circumference of a circle with radius b, $C_R = 2\pi b$
  - d: distance between left and right wheels, b: inner turn radius



- Moreover:

$$D_{\mathrm{L}} = \left(\frac{C_{\mathrm{L}}}{2\pi}\right)\theta \;\rightarrow\; C_{\mathrm{L}} = \frac{2\pi D_{\mathrm{L}}}{\theta} \;\rightarrow\; \theta = \frac{D_{\mathrm{L}}}{d+b}$$

- Similarly,

$$D_{\mathrm{R}} = \left(\frac{C_{\mathrm{R}}}{2\pi}\right)\theta \;\rightarrow\; C_{\mathrm{R}} = \frac{2\pi D_{\mathrm{R}}}{\theta} \;\rightarrow\; \theta = \frac{D_{\mathrm{R}}}{b} \;\rightarrow\; b = \frac{D_{\mathrm{R}}}{\theta}$$

- Finally,

$$\theta = \frac{D_{\mathrm{L}} - D_{\mathrm{R}}}{d}$$

Situational Awareness

- To move about intelligently in its environment, a robot must have situational awareness

- Situational awareness necessitates knowing ranges and bearings of nearby objects

- Tactile sensor: direct physical contact between an on-board sensor and an object indicates collision with the object (tactile feelers, tactile bumpers, micro-switches, etc.)

- Proximity sensor: a non-contact sensor provides advance warning on the presence of an object in close vicinity of the sensor (magnetic, inductive, capacitive, ultrasonic, optical, etc.)

  – While tactile sensors indicate presence of object after physical contact with it, proximity sensors do not quanitfy the range to the object

- Range senor: provides actual distance to a target of interest without physical contact (triangulation, time-of-flight, phase-shift measurement, frequency modulation, interferometry, return signal intensity); broadly classified as *active* and *passive*

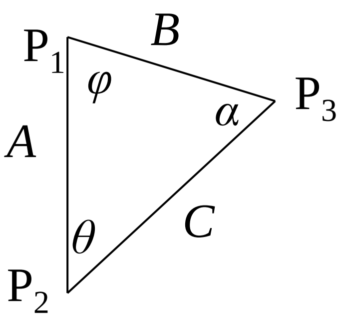
  – Radar (radio direction and ranging): typically uses, time-of-flight, phase-shift measurements, or frequency modulation

  – Sonar (sound navigation and ranging): typically uses, time-of-flight since speed of sound is slow enough to be measured with inexpensive electronic

  – Lidar (light direction and ranging): laser-based schemes that typically use, time-of-flight or phase-shift measurements,

Triangulation Ranging

- Basis (Law of sines): If the sides of a triangle are a, b, and c and the angles opposite to those sides are *α, θ,* and *φ*, then



$$\frac{A}{\sin\alpha} = \frac{B}{\sin\theta} = \frac{C}{\sin\varphi}$$

$$\rightarrow \quad B = A\frac{\sin\theta}{\sin\alpha} = A\frac{\sin\theta}{\sin(180-\theta-\varphi)} = A\frac{\sin\theta}{\sin(\theta+\varphi)}$$

- Therefore, given the length of a side and two angles of a triangle, the length of the other two sides and the third angle can be determined.

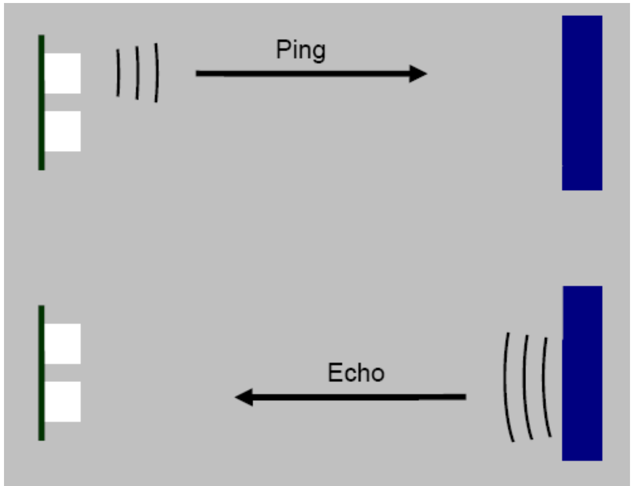
- In ranging applications, length *B* represents the distance to the object of interest at point P_3.

- In a passive ranging system, directional detectors can be placed at P_1 and P_2 to view the object point P_3, forming an imaginary triangle.

- Measurement of angles *θ* and *φ* along with the known orientation and lateral separation of the detectors allows the calculation of range to the object at P_3.

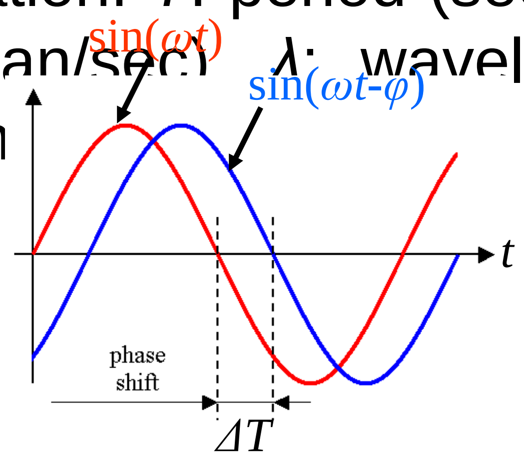
Time-of-Flight Ranging

- Measure the round-trip time required for a pulse (burst) of emitted energy (acoustic, radio, or optical) to travel to an object and then reflect/echo back to a receiver.
- Range to the object: $d=v(T/2)$, where v is the speed of the propagated wave, T=round-trip time of travel
- Ultrasonic emitter/detector pairs (transceivers) are commonly used
  - Common ultrasonic transducers: capacitive, electrostatic, and piezoelectric
- Laser-based time-of-flight systems
- Speed of sound ≈ 0.3m/ms, speed of light ≈ 0.3m/ns
  - Time of flight for 3 meters: ultrasonic system: 10ms; laser system: 10ns
  - → sophisticated timing circuitry necessitated in laser-based time-of-flight ranging instruments.



Ranging by Phase Shift Measurement

- A continuous-wave (e.g., amplitude-modulated laser, RF, or acoustic) energy source is directed towards a target.
- The reflected signal that strikes back at the detector is compared to a reference signal (tapped off from the transmitted signal).
- The relative phase shift between the reference and reflected signal is measured to determine the round-trip distance from the object.
- Notation: T: period (sec), f: frequency (Hz), ω: radial frequency (radian/sec), λ: wavelength (m), $\omega=2\pi f$, $\lambda=s/f$, s: speed of soun



$$\phi = \omega \Delta T = 2\pi f \Delta T = 2\pi \left(\frac{s}{\lambda} \right) \Delta T$$

$$= \frac{2\pi}{\lambda} s \Delta T = \frac{2\pi (2d)}{\lambda} = \frac{4\pi d}{\lambda}$$

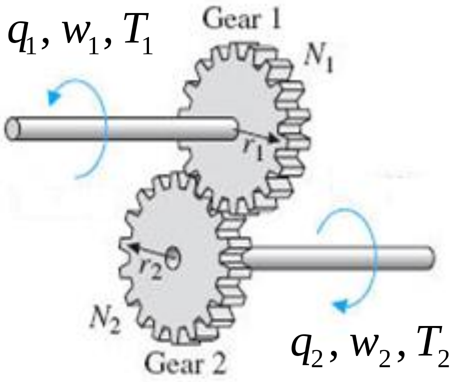
$$\Rightarrow d = \frac{\phi \lambda}{4\pi}$$

Motion Transmission—I

- Motion transmission:
  - Mechanism used to transmit mechanical motion produced by an actuator (electric motor, hydraulic/pneumatic cylinder, etc.) to a device
  - Manipulate torque-speed between input-output shafts
  - Types: rotary-rotary, rotary-translation, etc.
- Efficiency:
  - Ratio between output and input power $\eta \cong \frac{P_{\text{out}}}{P_{\text{in}}}$
  - System with perfect efficiency (100%, no mechanical losses): $P_{\text{out}}=P_{\text{in}}$
- Gear Ratio:
  - For a system of input-output gears, assuming no slip condition, linear distance traveled by each gear at contact point is same, i.e.,

$$s_1 = s_2 \Rightarrow \theta_1 r_1 = \theta_2 r_2 \Rightarrow \frac{\theta_1}{\theta_2} = \frac{r_2}{r_1}$$

$$N \cong \frac{\theta_1}{\theta_2} = \underbrace{\frac{r_2}{r_1} = \frac{N_2}{N_1}}_{\text{Gears of same pitch}}$$



Motion Transmission—II

- Effect of less than 100% efficiency:
  - Loss of a % of transmitted force/torque (no influence on effective gear ratio)

$$P_2 = \eta P_1 \Rightarrow T_2\dot{\theta}_2 = \eta T_1\dot{\theta}_1 \Rightarrow T_2 = \eta\left(\frac{\dot{\theta}_1}{\dot{\theta}_2}\right)T_1 = \eta N T_1$$

- Torque to be produced at the i/p shaft for a required torque at the o/p shaft

$$T_2 = \eta N T_1 \Rightarrow T_{2\cong 1} = \frac{1}{\eta N}T_2 \overset{\eta=1}{\Rightarrow} T_{2\cong 1} = \frac{1}{N}T_2$$

- How to reflect inertia of o/p shaft to the i/p shaft
  - Kinetic energy of the o/p shaft (the input shaft must supply this energy and losses!)

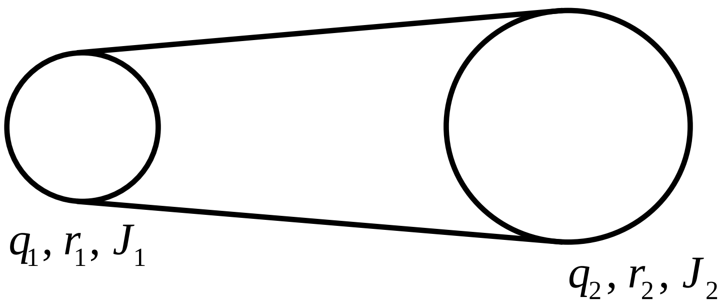
$$\mathrm{KE}_2 = \eta\,\mathrm{KE}_1$$

$$\frac{1}{2}J_2\dot{\theta}_2 = \eta\frac{1}{2}J_{2\cong 1}\dot{\theta}_1$$

$$J_{2\cong 1} = \frac{1}{\eta}\left(\frac{\dot{\theta}_2}{\dot{\theta}_1}\right)^2 J_2 = \frac{1}{\eta N^2}J_2 \overset{\eta=1}{\Rightarrow} J_{2\cong 1} = \frac{1}{N^2}J_2$$

Motion Transmission—III

- Belt-pulley system:
  - Let η=1



$$s_1 = s_2 \text{ Þ } q_1 r_1 = q_2 r_2 \text{ Þ } \frac{q_1}{q_2} = \frac{r_2}{r_1}$$

$$N @ \frac{q_1}{q_2} = \frac{r_2}{r_1} = \frac{N_2}{N_1}$$

$$T_{2@1} = \frac{1}{N} T_2$$

$$J_{2@1} = \frac{1}{N^2} J_2$$

Motion Transmission—III

- Rotary to translational motion:
  - Let $\eta=1$

$$\Delta x = r\Delta\theta \Rightarrow v = r\dot{\theta}$$

$$N \cong \frac{\theta_1}{\theta_2} = \frac{\Delta x}{\Delta \theta_1} = \frac{1}{r}$$

  - Mass *m* reflected at the pinion

$$\mathrm{KE}_2 = \mathrm{KE}_1$$

$$\frac{1}{2} m v^2 = \frac{1}{2} J_{\mathrm{eq}} \dot{\theta}^2$$

$$J_{\mathrm{eq}} = m\left(\frac{v}{\dot{\theta}}\right)^2$$

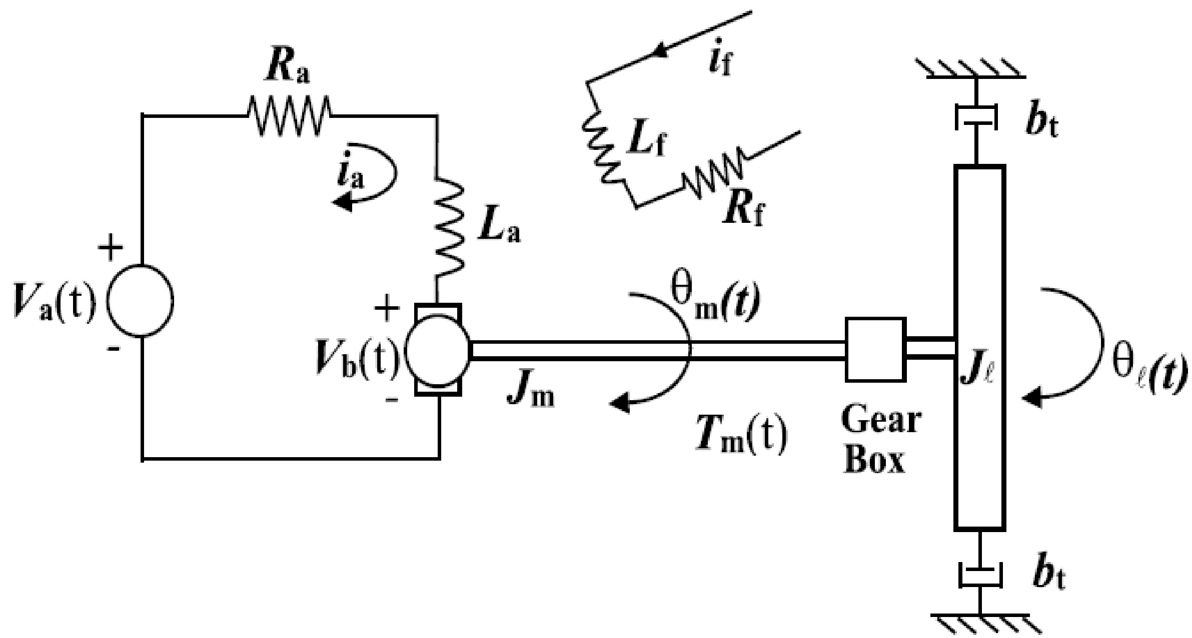
$$J_{\mathrm{eq}} = \frac{m}{r^2}$$

DC Motor Modeling—I

- Electrical subsystem:
  - Motor torque ∝ armature current → $T_m = K_T i_a$
  - Armature back e.m.f. ∝ armature angular velocity → $V_b = K_b \omega_m$
  - Apply K.V.L. to the armature circuit

$$L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + R_a i_a + V_b = V_a$$

$$L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + R_a i_a + K_b \frac{\mathrm{d}q_m}{\mathrm{d}t} = V_a$$

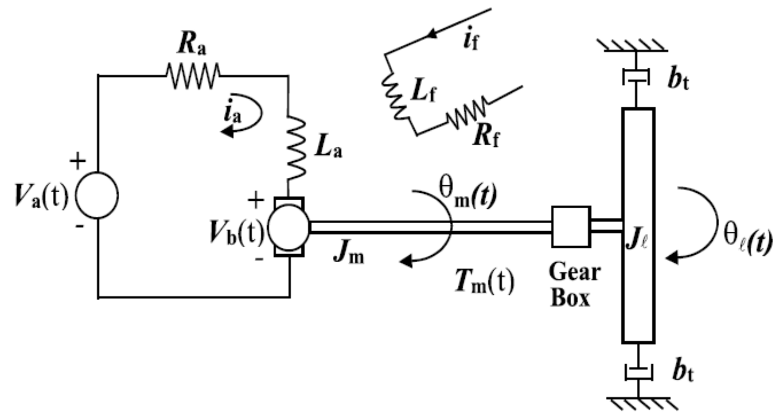


DC Motor Modeling—II

- Mechanical subsystem: Torque balance for the motor shaft

$$J_m \ddot{\theta}_m = -\left(T_l + T_f\right)\frac{1}{K_g} + T_m$$

- Load torque and friction torque:



$$T_l = J_l \ddot{\theta}_l = J_l \frac{\ddot{\theta}_m}{K_g} = \left(\frac{J_l}{K_g}\right)\ddot{\theta}_m$$

$$T_f = b_t \dot{\theta}_l = b_t \frac{\dot{\theta}_m}{K_g} = \left(\frac{b_t}{K_g}\right)\dot{\theta}_m$$

- Thus, governing equation for mechanical subsystem

$$J_m \ddot{\theta}_m = -\left(\frac{J_l}{K_g}\ddot{\theta}_m + \frac{b_t}{K_g}\dot{\theta}_m\right)\frac{1}{K_g} + K_T i_a \quad \Rightarrow \quad \left(J_m + \frac{J_l}{K_g^2}\right)\ddot{\theta}_m + \frac{b_t}{K_g^2}\dot{\theta}_m = K_T i_a$$

$$\left(J_m + \frac{J_l}{K_g^2}\right)\left(K_g \ddot{\theta}_l\right) + \frac{b_t}{K_g^2}\left(K_g \dot{\theta}_l\right) = K_T i_a \quad \Rightarrow \quad J_{\text{eq}}\ddot{\theta}_l + b_t \dot{\theta}_l = K_g K_T i_a$$

$$J_{\text{eq}} \cong J_m K_g^2 + J_l$$

DC Motor Modeling—III

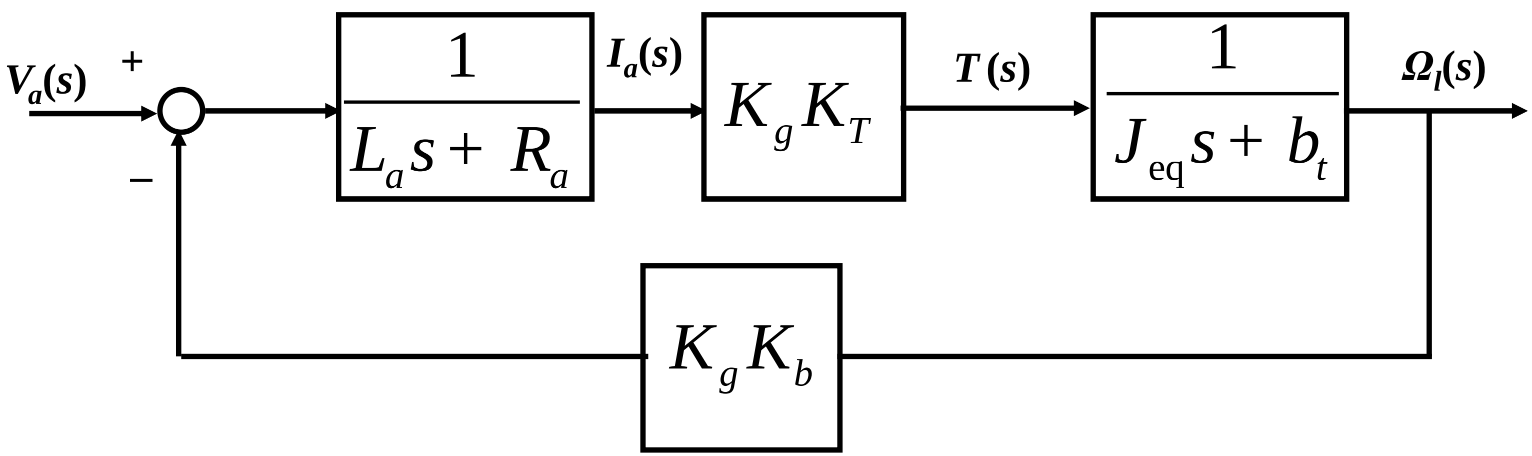
- Electrical subsystem:

$$L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + R_a i_a + K_b \frac{\mathrm{d}\theta_m}{\mathrm{d}t} = V_a \rightarrow L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + R_a i_a + K_g K_b \frac{\mathrm{d}\theta_l}{\mathrm{d}t} = V_a$$

- Mechanical subsystem:

$$J_{\mathrm{eq}} \ddot{\theta}_l + b_t \dot{\theta}_l = K_g K_T i_a$$

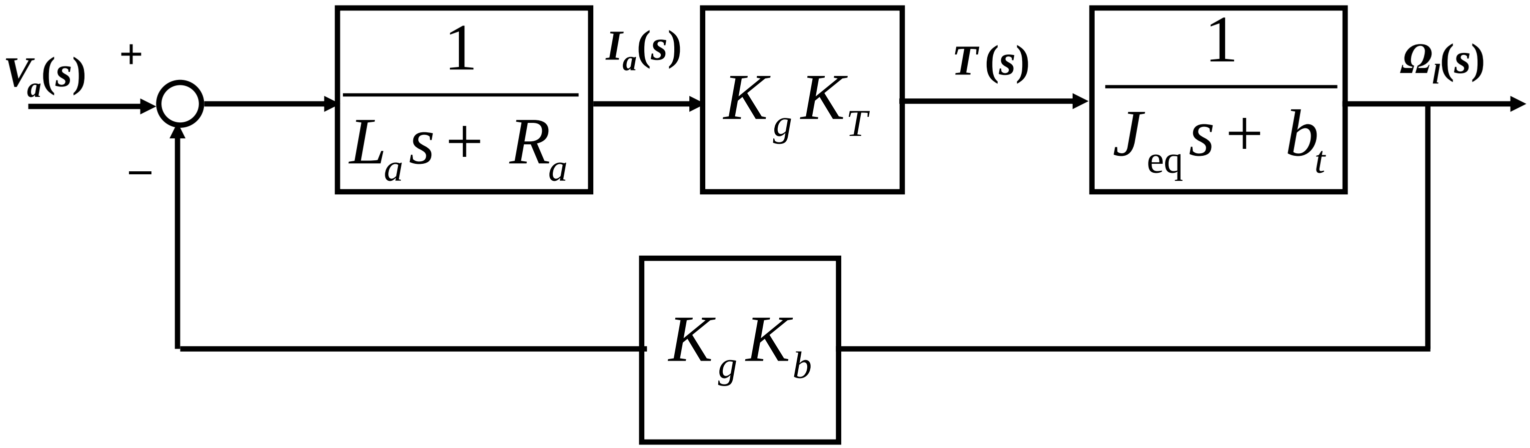
- Use Laplace transforms to produce:



$$\omega_l \triangleq \dot{\theta}_l \quad \Omega_l \triangleq \mathcal{L}(\omega_l)$$

DC Motor Modeling—IV

- Closed-loop TF: (use $L_a \approx 0$)



$$\frac{W_l(s)}{V_a(s)} = \frac{\dfrac{K_g K_T}{(L_a s + R_a)(J_{\text{eq}} s + b_t)}}{1 + \dfrac{K_g^2 K_T K_b}{(L_a s + R_a)(J_{\text{eq}} s + b_t)}} = \frac{K_g K_T}{(L_a s + R_a)(J_{\text{eq}} s + b_t) + K_g^2 K_T K_b}$$

$$\frac{W_l(s)}{V_a(s)} = \frac{K_g K_T}{R_a J_{\text{eq}} s + R_a b_t + K_g^2 K_T K_b} = \frac{K}{t\, s + 1}$$

DC Motor Torque/Speed & Power/Speed Curves—I

- Electrical and Mechanical subsystems:

$$L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + R_a i_a + K_g K_b \frac{\mathrm{d}\theta_l}{\mathrm{d}t} = V_a \quad \& \quad J_{\mathrm{eq}} \ddot{\theta}_l + b_t \dot{\theta}_l = K_g K_T i_a$$

- In steady-state, current is constant, motor runs at constant angular velocity, and armature voltage is constant, so:

$$R_a i_a + K_g K_b \dot{\theta}_l = V_a \Rightarrow R_a \left(\frac{T_m}{K_T} \right) + K_g K_b \dot{\theta}_l = V_a \Rightarrow$$

$$T_m = \frac{K_T}{R_a} V_a - \frac{K_T K_g K_b}{R_a} \dot{\theta}_l \Rightarrow T_m \text{ v/s } \dot{\theta}_l \text{: linear relation}$$

- Starting torque (as motor is starting up, $\omega_l = 0$)

$$T_s = \frac{K_T}{R_a} V_a$$

DC Motor Torque/Speed & Power/Speed Curves—II

- No-load speed $\omega_{\max}$ (maximum speed of motor): note that in this case $T_\mathrm{m} \to 0$

$$0 = \frac{K_T}{R_a} V_a - \frac{K_T K_g K_b}{R_a} \dot{\theta}_l \Rightarrow \dot{\theta}_l = \frac{1}{K_g K_b} V_a \Rightarrow \dot{\theta}_m = \frac{1}{K_b} V_a \Rightarrow \dot{\theta}_{\max} = \frac{1}{K_b} V_a$$

- Power delivered by the motor:

$$P = T_m \dot{\theta}_m$$

  – Note that the torque is a function of angular velocity!

$$T_m = \frac{K_T}{R_a} V_a - \frac{K_T K_g K_b}{R_a} \dot{\theta}_l = T_s - \frac{K_T K_b}{R_a} \times \frac{T_s}{T_s} \times \left(K_g \dot{\theta}_l\right)$$

$$T_m = T_s - \frac{K_T K_b}{R_a T_s} T_s \dot{\theta}_m = T_s - \frac{K_b}{V_a} T_s \dot{\theta}_m = T_s \left(1 - \frac{\dot{\theta}_m}{\dot{\theta}_{\max}}\right)$$

$$T_m = T_s \left(1 - \frac{\omega_m}{\omega_{\max}}\right)$$

DC Motor Torque/Speed & Power/Speed Curves—III

- Power and maximum power delivered by motor:

$$P = T_m \dot{\theta}_m = T_s \left(1 - \frac{\omega_m}{\omega_{\max}}\right) \omega_m$$

  - To obtain maximum power, evaluate

$$\frac{dP}{d\omega_m} = \frac{d}{d\omega_m}\left[T_s \left(1 - \frac{\omega_m}{\omega_{\max}}\right) \omega_m \right] = 0$$

$$1 - 2\frac{\omega_m}{\omega_{\max}} = 0 \Rightarrow$$

$$\text{Max power @ } \omega_m = \frac{\omega_{\max}}{2}$$