**Problem 1:** Let $X = \text{span}\{S_1 \cup S_2\}$ and $Y = \text{span}\{S_1\} + \text{span}\{S_2\}$. We need to show that $X \subset Y$ and $Y \subset X$, because this is equivalent to $X = Y$. We denote the field by $\mathcal{F}$.

Claim 1: $X \subset Y$.

Proof: Let $x \in \text{span}\{S_1 \cup S_2\}$. Then $\exists v_1, ..., v_k \in S_1$ and $w_1, ..., w_m \in S_2$, as well as $\alpha_1, ..., \alpha_k \in \mathcal{F}$ and $\beta_1, ..., \beta_m \in \mathcal{F}$ such that

$$X = \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=1}^{m} \beta_i w_i.$$

We observe, by definition of $\text{span}\{S_1\}$ and $\text{span}\{S_2\}$, that $\sum_{i=1}^{k} \alpha_i v_i \in \text{span}\{S_1\}$ and $\sum_{i=1}^{m} \beta_i w_i \in \text{span}\{S_2\}$.

Therefore, by the definition of the sum of two subspaces,

$$X = \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=1}^{m} \beta_i w_i \in \text{span}\{S_1\} + \text{span}\{S_2\} = Y.$$

Claim 2: $Y \subset X$.

Proof: $y \in Y$ means there exists $y_1 \in \text{span}\{S_1\}$ and $y_2 \in \text{span}\{S_2\}$ such that $y = y_1 + y_2$. $y_1 \in \text{span}\{S_1\}$ implies there exist $\bar{\alpha}_1, ..., \bar{\alpha}_{\bar{k}} \in \mathcal{F}$ and $\bar{v}_1, ..., \bar{v}_{\bar{k}} \in S_1$, such that

$$y_1 = \sum_{i=1}^{\bar{k}} \bar{\alpha}_i \bar{v}_i.$$

Similarly, there exist $\bar{\beta}_1, ..., \bar{\beta}_{\bar{m}} \in \mathcal{F}$ and $\bar{w}_1, ..., \bar{w}_{\bar{m}} \in S_2$, such that

$$y_2 = \sum_{i=1}^{\bar{m}} \bar{\beta}_i \bar{w}_i.$$

Therefore, $y = y_1 + y_2$ is a linear combination of elements of $S_1 \cup S_2$ and thus $y \in \text{span}\{S_1 \cup S_2\} = X$. $\square$

**Problem 4:**

Claim: Let $Y \subset X$ be a subspace and $S \subset Y$. Then $\text{span}\{S\} \subset Y$.

Proof: From the definition of span, if $S_1 \subset S_2$, then $\text{span}\{S_1\} \subset \text{span}\{S_2\}$. Also, if $Y$ is a subspace, then $\text{span}\{Y\} = Y$. Putting these two observations together we have

$$S \subset Y,\ Y \text{ a subspace} \implies \text{span}\{S\} \subset Y. \qquad \square$$

**Problem 5:**

$(a) \implies (b)$:
Suppose $x \in V + W$ is written as

$$x = v_1 + w_1,$$

and as

$$x = v_2 + w_2,$$

where $v_1,\ v_2 \in V$ and $w_1,\ w_2 \in W$.
To Show: $v_1 = v_2$ and $w_1 = w_2$.

Proof:

$$0 = x - x = (v_1 - v_2) + (w_1 - w_2).$$

Because $V$ and $W$ are subspaces,

$$v_1 - v_2 \in V \quad \text{and} \quad w_1 - w_2 \in W.$$

Because $v_1 - v_2 = w_2 - w_1 = -(w_1 - w_2)$, we have

$$v_1 - v_2 \in W \quad \text{and} \quad w_1 - w_2 \in V.$$

Hence,

$$(v_1 - v_2) \in V \cap W \quad \text{and} \quad (w_1 - w_2) \in V \cap W.$$

However, $V \cap W = \{0\}$, and thus $v_1 - v_2 = 0$ and $w_1 - w_2 = 0$, which is what we needed to show. $\square$

$(b) \implies (a)$:
In order to prove $(b) \implies (a)$, we will prove that $\sim (a) \implies \sim (b)$.

$$\begin{aligned} \sim (a) &\Leftrightarrow \exists u \in V \cap W \ \ \text{with} \ \ u \neq 0. \\ \sim (b) &\Leftrightarrow \exists x \in V + W \ \ \text{that can be expressed in at least two} \\ & \qquad\qquad\qquad\ \ \text{different ways (i.e., NOT UNIQUE).} \end{aligned}$$

Because $x \in V + W$, $\exists v \in V$ and $w \in W$ such that $x = v + w$. We have that

$$x = v + w + u - u = (v + u) + (w - u)$$

also holds. Moreover, because $u \in V \cap W$, and because $V$ and $W$ are subspaces, we have that $v + u \in V$ and $w - u \in W$. Because $u \neq 0$, we have that

$$v \neq v + u \quad \text{and} \quad w \neq w - u.$$

$\therefore$ $x = v + w$ and $x = (v + w) + (w - u)$ are distinct ways of expressing $x$ as a sum of elements from $V$ and $W$, and thus the representation is not unique. $\square$

**Problem 10:**

(a) We need to show that $\{M_1,\ M_2,\ M_3,\ M_4\}$ is linearly independent. Hence, consider

$$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{bmatrix} + \begin{bmatrix} \alpha_4 & 0 \\ 0 & -\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned} \alpha_3 + \alpha_4 &= 0\,, \\ \alpha_1 - \alpha_2 &= 0\,, \\ \alpha_1 + \alpha_2 &= 0\,, \\ \alpha_3 - \alpha_4 &= 0\,. \end{aligned}$$

That is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because both matrices are non-singular, we deduce that the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, and thus the set is linearly independent.

(b) We write

$$A = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4$$

Using our work from part (a), we deduce that

$$\begin{aligned} \alpha_3 + \alpha_4 &= 1\,, \\ \alpha_1 - \alpha_2 &= 2\,, \\ \alpha_1 + \alpha_2 &= 3\,, \\ \alpha_3 - \alpha_4 &= 4\,. \end{aligned}$$

Hence,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Which yields

$$
\begin{aligned}
\alpha_1 &= 2\frac{1}{2}\,, \\
\alpha_2 &= \frac{1}{2}\,, \\
\alpha_3 &= 2\frac{1}{2}\,, \\
\alpha_4 &= -1\frac{1}{2}\,.
\end{aligned}
$$

That is,

$$
[A]_M = \begin{bmatrix} 2.5 \\ 0.5 \\ 2.5 \\ -1.5 \end{bmatrix}.
$$