Mathematics for Robotics
ROB-GY 6103
Homework 3 Answers

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Question: 1.(a)

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{S} = \big(p_0 = 1,\ p_1 = x,\ p_2 = x^2\big)$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

$\therefore$ the components of $r(x)$ in basis $\mathcal{S}$ is

$$\mathrm{r}_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{1}$$

Question: 1.(b)

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{Q} = \big(q_0 = 1,\ q_1 = 1 - x,\ q_2 = x + x^2\big)$.

From the definition of the basis $\mathcal{Q}$, we can rewrite it in terms of basis $\mathcal{S}$ as,

$q_{0_{\mathcal{S}}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $q_{1_{\mathcal{S}}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $q_{2_{\mathcal{S}}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Q_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

$$\Rightarrow \mathrm{r}_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{1}$$

Now, we need to find numbers $\tilde{r}_0$, $\tilde{r}_1$ and $\tilde{r}_2$ such that,

$$\mathrm{r}_{\mathcal{S}} = \tilde{r}_0 \mathrm{q}_{0_{\mathcal{S}}} + \tilde{r}_1 \mathrm{q}_{1_{\mathcal{S}}} + \tilde{r}_2 \mathrm{q}_{2_{\mathcal{S}}} \tag{2}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \tilde{r}_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \tilde{r}_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \tilde{r}_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

Eq^n can be rewritten as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{4}$$

Applying the Gauss method,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{5}$$

$R_2 \to (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{6}$$

$R_1 \to R_1 + (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{7}$$

$R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{8}$$

$R_1 \to R_1 + (-1)R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{9}$$

Hence, the solution is,

$$\mathrm{r}_q = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} \tag{10}$$

Question: 2.

Answer: Given the matrix,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \;\Rightarrow (\lambda I - A)v = 0 \;\Leftrightarrow det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 1 & -4 & -10 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \tag{2}$$

$$\Rightarrow det(\lambda I - A) = (\lambda - 1) \cdot (\lambda - 2) \cdot (\lambda - 3) = 0 \tag{3}$$

$$\therefore \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{4}$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 1 - \lambda_i & 4 & 10 \\ 0 & 2 - \lambda_i & 0 \\ 0 & 0 & 3 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

From Eq^n 5, substituting $\lambda_1 = 1$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 4v_2^1 + 10v_3^1 = 0 \\ v_2^1 = 0 \\ 2v_3^1 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{matrix} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -v_1^2 + 4v_2^2 + 10v_3^2 = 0 \\ 0 = 0 \\ v_3^2 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^2 = 4v_2^2 \\ v_2^2 = \text{any value} \\ v_3^2 = 0 \end{matrix} \Rightarrow v^2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

From Eq^n 5, substituting $\lambda_3 = 3$

$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2v_1^3 + 4v_2^3 + 10v_3^3 = 0 \\ -v_2^3 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^3 = 5v_3^3 \\ v_2^3 = 0 \\ v_3^3 = \text{any value} \end{matrix} \Rightarrow v^3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

Now we shall verify that v^1, v^2 and v^3 are *Linearly Independant*,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = 0 \tag{9}$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 0 \tag{10}$$

$$\alpha_1 + 4\alpha_2 + 5\alpha_3 = 0 \quad (11)$$
$$\alpha_2 = 0 \quad (12)$$
$$\alpha_3 = 0 \quad (13)$$

Solving above system of equations gives us $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$.
$\therefore$ v^1, v^2 and v^3 are *Linearly Independant.*

Question: 3.

Answer: Given the matrix,

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (1)$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \Rightarrow (\lambda I - A)v = 0 \Leftrightarrow det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 3 & -1 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \quad (2)$$

$$\Rightarrow det(\lambda I - A) = (\lambda - 3) \cdot (\lambda - 3) \cdot (\lambda - 2) = 0 \quad (3)$$

$$\therefore \lambda_1 = 3, \lambda_2 = 2 \quad (4)$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 3 - \lambda_i & 1 & 0 \\ 0 & 3 - \lambda_i & 0 \\ 0 & 0 & 2 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

From Eq^n 5, substituting $\lambda_1 = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_2^1 = 0 \\ 0 = 0 \\ -v_3^1 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{matrix} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ v_3^2 = \text{any value} \end{matrix} \Rightarrow v^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

Consider the set of eigenvectors $\mathcal{V} = \{v^1, v^2\}$. Firstly,

We know that any set of eigenvectors of a given matrix are Linearly Independent. So, v^1, v^2 are *L.I.* (8)

Secondly, consider a linear combination x such that,

$$\{x \in \mathbb{R} \mid \exists\, \alpha_1, \alpha_2 \in \mathbb{R}, v^1, v^2 \in \mathcal{V}\ ;\ s.t.\ x = \alpha_1 v^1 + \alpha_2 v^2\} \quad (9)$$

By observing above Eq^n 9 we can see that the basis does not span the entire vector space.
$\therefore$ based on the Eq^n 8 and 9 we can say that $\mathcal{V}$ does not form a basis for $\mathbb{R}^3$.

Question: 4.

Answer: We are given two similar square matrices A and B such that,

$$B = P^{-1}AP \tag{1}$$

Consider the characteristic relation of matrix B,

$$det(\lambda I - B) \tag{2}$$

Apply Eq^n 1 in Eq^n **??**,

$$\Rightarrow det(\lambda I - B) = det(\lambda I - P^{-1}AP) \tag{3}$$

The identity matrix I can also be written as $P^{-1}IP$. Substituting in Eq^n 3,

$$\Rightarrow det(\lambda I - B) = det(\lambda P^{-1}IP - P^{-1}AP) \tag{4}$$

Now we shall take P^{-1} and P out common,

$$\Rightarrow det(\lambda I - B) = det(P^{-1}(\lambda I - A)P) \tag{5}$$

We know that for compatible square matrices A and B, $det(AB) = det(A)det(B)$. So Eq^n 5 becomes,

$$\Rightarrow det(\lambda I - B) = det(P^{-1})det(\lambda I - A)det(P) \tag{6}$$

Cancelling out $det(P^{-1})$ by $det(P)$

$$\Rightarrow det(\lambda I - B) = det(\lambda I - A) \tag{7}$$

When the relation in Eq^n 8 is equated to zero, we prove that the two matrices have the same eigenvalues as well as the characteristic equations. I.E.,

$$\Rightarrow det(\lambda I - B) = det(\lambda I - A) = 0 \tag{8}$$

Q.E.D.

Question: 5.

Answer: Given a matrix A_3 such that,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

From Question: 2. We found the eigenvalues to be $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.
Using these eigenvalues, we can create the diagonal martix, $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{2}$$

To show similarity between A and Λ, we need to find a matrix P such that $\Lambda = PAP^{-1}$
Consider a matrix $P = \begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix}$ where v^1, v^2 and v^3 are the eigenvectors of A.

$$P = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Testing for invertibility, $det(P) = 1$. As the determinant is non-zero, we can conclude that P is invertible.

In Question: 2. we already proved that v^1, v^2 and v^3 are *Linearly Independant*.

As these two conditions are satisfied, we can therefore test for similarity,

$$A_3 = P\Lambda P^{-1} \tag{4}$$

$$A_3 = \begin{bmatrix}1 & 4 & 5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 4 & 5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{-1} \tag{5}$$

$$= \begin{bmatrix}1 & 4 & 5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0.5 & 0\\0 & 0 & 0.333\end{bmatrix} \tag{6}$$

$$= \begin{bmatrix}1 & 8 & 15\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0.5 & 0\\0 & 0 & 0.333\end{bmatrix} \tag{7}$$

$$= \begin{bmatrix}1 & 4 & 10\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix} \tag{8}$$

$\therefore$ **LHS = RHS**

Thus, we can say that matrix A is similar to a diagonal matrix Λ. **Q.E.D.**

Question: 6.(a)

Answer: Given, a vector space $(\mathcal{X}, \mathbb{R})$ where $\mathcal{X}$ is a set of 2 x 2 matrices with real coefficients.

An operation is defined $L : \mathcal{X} \to \mathcal{X}$ by

$$L(M) = \frac{1}{2}(M + M^T) \tag{1}$$

Where $M \in \mathcal{X}$ is a 2 x 2 real matrix

The operator L will be considered a *Linear Operator* if,

$$\forall x, y \in \mathcal{X}\ ,\ \alpha, \beta \in \mathbb{R} \mid L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \tag{2}$$

Applying above Statement 2 to Eq^n 1,

$$L(\alpha x + \beta y) = \frac{1}{2}((\alpha x + \beta y) + (\alpha x + \beta y)^T) \tag{3}$$

Applying the property of sum of transpose of two matrices
(i.e. for two matrices A and $B \to (A + B)^T = A^T + B^T$)

$$L(\alpha x + \beta y) = \frac{1}{2}(\alpha x + \beta y + \alpha x^T + \beta y^T) \tag{4}$$

$$= \frac{1}{2}(\alpha x + \alpha x^T + \beta y + \beta y^T) \tag{5}$$

$$= \alpha(\frac{1}{2}(x + x^T)) + \beta(\frac{1}{2}(y + y^T)) \tag{6}$$

$$= \alpha L(x) + \beta L(y) \tag{7}$$

Thus, Eq^n 7 proves Statement 2. $\therefore$ the given operator L is actually a *Linear Operator*.
Q.E.D.

Question: 6.(b)

Answer: Given a linear transformation $L : \mathcal{X} \to \mathcal{X}$ and a basis E given by $E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ used on both copies of $\mathcal{X}$.

By the theorem, we know that $A_i = [L(u^i)]_v$. But in our case, $u = v = E$. So, we can rewrite it as $A_i = [L(E^{ij})]_E$ where $\forall i, j \in \mathbb{N} \mid 1 \leq i, j \leq 2$

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix} \tag{1}$$

$$A_1 = [L(E^{11})]_E = \left[\frac{1}{2}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})\right]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$A_2 = [L(E^{12})]_E = \left[\frac{1}{2}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})\right]_E = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{3}$$

$$A_3 = [L(E^{21})]_E = \left[\frac{1}{2}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})\right]_E = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{4}$$

$$A_4 = [L(E^{22})]_E = \left[\frac{1}{2}(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})\right]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

$$\tag{6}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Question: 7.(a)

Answer: Given a linear operator $L : \mathbb{C}^n \to \mathbb{C}^n$ defined by $L(x) = Ax$ with the natural basis $(u = \{u^1 \ldots u^n\})$ applied on both, the domain and codomain.

Consider $x \in \mathbb{C}^n$, such that $x = \alpha_1 u^1 + \ldots + \alpha_n u^n$. It can be represented as,

$$[x]_u = \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix} \in \mathbb{C}^n \tag{1}$$

Now let us compute the matrix representation,

$$[L(x)]_u = [L(\alpha_1 u^1 + \ldots + \alpha_n u^n)]_u \tag{2}$$

$$[L(x)]_u = [L(\alpha_1 u^1) + \ldots + L(\alpha_n u^n)]_u \tag{3}$$

$$[L(x)]_u = [\alpha_1 L(u^1) + \ldots + \alpha_n L(u^n)]_u \tag{4}$$

But from the theorem on matrix representation, we know that $[L(x)]_v = A[x]_u$. But in our case $u = v$. So, above equation can be rewitten as,

$$\hat{A}[x]_u = [\alpha_1 L(u^1) + \ldots + \alpha_n L(u^n)]_u \tag{5}$$

Where, $\hat{A}$ is the Matrix Representation of L.

$$\hat{A}[x]_u = [\alpha_1 Au^1 + \ldots + \alpha_n Au^n]_u \tag{6}$$

$$\hat{A}[x]_u = \begin{bmatrix} Au^1 & \ldots & Au^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{7}$$

We know that for the natural basis u, $A \times u = A$.

$$\hat{A}[x]_u = A[x]_u \tag{8}$$

$\therefore \hat{A} = A$. **Q.E.D.**

Question: 7.(b)

Answer: Given a linear operator $L : \mathbb{C}^n \to \mathbb{C}^n$ defined by $L(x) = Ax$ with the basis created of eigenvectors of A ($M = \{v^1, v^2, \ldots v^n\}$) applied on the domain.

And the eigenvalues $\Lambda \in \mathbb{C}^n$, such that

$$[\Lambda]_M = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{C}^n \tag{1}$$

Now let us compute the matrix representation,

$$[L(M\Lambda)]_M = [L(\lambda_1 v^1 + \ldots + \lambda_n v^n)]_M \tag{2}$$

$$[L(M\Lambda)]_M = [\lambda_1 L(v^1)]_M + \ldots + [\lambda_n L(v^n)]_M \tag{3}$$

But from the theorem on matrix representation, we know that $[L(x)]_v = A[x]_u$,

$$\hat{A}[M\Lambda]_M = \begin{bmatrix} L(v^1) & \ldots & L(v^n) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \tag{4}$$

$$\hat{A}[M\Lambda]_M = \begin{bmatrix} Av^1 & \ldots & Av^n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \tag{5}$$

But by Diagonalisation, $AM = M\Lambda$

$$\hat{A}[M\Lambda]_M = \Lambda[M\Lambda]_M \Rightarrow \hat{A} = \Lambda \tag{6}$$

Now, it is given that $L(x) = Ax$. Substituting x as M,

$$L(M) = AM \tag{7}$$

Substituting the matrix representation of L,

$$\hat{A}[M]_M = AM \tag{8}$$

But from Eq^n 6, $\hat{A} = \Lambda$ and $[M]_M = M$

$$\Lambda M = AM \tag{9}$$

$$\Rightarrow \Lambda = M^{-1}AM \tag{10}$$

$\therefore$ the given matrix A is similar to a diagonal matrix Λ.