2. The symmetric matrix below has repeated e-values $2, 2, -1$. In lecture, we stated that even with repeated e-values, we can still diagonalize a symmetric matrix using orthogonal matrices. The objective of the problem is to see why this is true by working a numerical example. We will follow the proof attached at the end of the HW set and factor $A$ as a product $O\Lambda O^\top$, where $O$ is an orthogonal matrix.

$$A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}.$$

Each of the steps below is motivated by a step in the proof. **Suggestion:** Open a script file in MATLAB and execute each step of the problem. It will save time.

(a) Verify that $v^1 = [0, 1, 0]^\top$ satisfies $Av^1 = 2v^1$, and thus $v^1$ is an e-vector corresponding to $\lambda = 2$.

(b) Choose $v^2$ and $v^3$ such that $\{v^1, v^2, v^3\}$ is orthonormal, and verify that $V = [v^1 \, |v^2 \, |v^3]$ is an orthogonal matrix. In general, you would accomplish this by completing $\{v^1\}$ to a basis of $\mathbb{R}^n$ and applying Gram Schmidt. Here, you can do it by inspection.

4. Load DataHW5_Prob4.mat (see MATLAB folder on NYU Brightspace) into your MATLAB workspace (See also the ReadMe.txt file). The data file provides "perturbed or noisy" data for the model $y_i = C_i x + e_i$, $1 \leq i \leq N$, where $N = 500$, $x \in \mathbb{R}^{100}$ and $y_i \in \mathbb{R}^3$. The data set contains the measured values $y_i$, the model matrices $C_i$, and the true value of $x$. The true value is given so that you can compare your estimated values to the true value. Of course, in real life, we would not have $x$ available to us.

For $1 \leq k \leq N$, define $S_k = I_{3\times 3}$ and as in the lecture on Recursive Least Squares,

$$Y_k = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}, \quad A_k = \begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix}, \quad R_k = \mathrm{diag}[S_1, \cdots, S_k] = I.$$

(a) Find $n$ such that $A_k$ has at least $\dim(x) = 100$ independent columns for $k \geq n$. For each $n \leq k \leq N$, define

$$\hat{x}_k := \arg\min ||Y_k - A_k x|| = \arg\min \sqrt{(Y_k - A_k x)^\top R_k (Y_k - A_k x)}$$

but do not compute anything except $n$ at this step.

(b) For each $n \leq k \leq N$, compute $\hat{x}_k$ in a batch process, that is,

$$\hat{x}_k = (A_k^\top R_k A_k)^{-1} A_k^\top R_k Y_k,$$

and, using the standard Euclidean norm, compute

$$E_k := ||\hat{x}_k - x||.$$

Make a plot of $E_k$ versus $k$ and turn it in. Put a clear title on your plot, such as "Norm error in x-hat using Batch Process". Implementing the "for each $n \leq k \leq N$" will require a `for loop or while loop`. Use the `tic` and `toc` commands to determine how long it takes to compute your *entire set of estimates* and report this value. Either write it on your error plot by hand or place it there with a MATLAB command.

(c) For each $n \leq k \leq N$, compute $\hat{x}_k$ using the RLS (Recursive Least Squares) Algorithm. First implement it without using the Matrix Inversion Lemma. Turn in a plot of $E_k$ versus $k$, and record on your plot the amount of time it takes to do your computations.

(d) For each $n \leq k \leq N$, compute $\hat{x}_k$ once again using the RLS (Recursive Least Squares) Algorithm, but this time, implement it using the Matrix Inversion Lemma. Turn in a plot of $E_k$ versus $k$, and record on your plot the amount of time it takes to do your computations. Note that this time you are numerically inverting a $3 \times 3$ matrix and then computing the inverse of the $100 \times 100$ matrix $Q_k$ with the Matrix Inversion Lemma. This is the main point of the Matrix Inversion Lemma.

5. Load DataHW5_Prob5.mat (see MATLAB folder on NYU Brightspace) into your MATLAB workspace. It provides "perturbed or noisy" data for the model $y_i = C_i x_i + e_i$, $1 \le i \le N$, where this time the "state" or "parameter" $x$ that we are estimating is slowly "drifting" (means that it is slowly varying with time), which is why it has an index $x_i$. We will see that basic least squares does not work very well when $x$ can drift. We will learn a way to fix it.

   In this problem, $N = 500$, $x \in \mathbb{R}^{20}$ and $y_i \in \mathbb{R}^3$. The data set contains the measured values $y_i$, the model matrices $C_i$, and the true value of $x_i$. The true value is given so that you can compare your estimated values to the true value. As you know very well, in real life, we would not have $x_i$ available to us.

   (a) Find $n$ such that $A_k$ has at least $\dim(x) = 20$ independent columns for $k \ge n$. For each $n \le k \le N$, define
   $$\hat{x}_k := \arg\min ||Y_k - A_k x||,$$
   but do not compute anything except $n$ at this step. You should find $n = 7$.

   (b) As in Prob. 4, use constant weights, with $S_k = I_{3\times 3}$. For each $n \le k \le N$, compute $\hat{x}_k$ (any method you wish) and compute $E_k := ||\hat{x}_k - x_k||$. It does not matter how fast your MATLAB code is for the computation of $\hat{x}_k$ because in this problem we will not record the time. Make a plot of $E_k$ versus $k$ and turn it in. Put a clear title on your plot. Note that the error gets pretty bad.

   (c) **The forgetting factor:** Let $0 < \lambda < 1$ (some number strictly between zero and one). A typical value for the *forgetting factor* might be $\lambda = 0.98$. The idea is to discount old measurements when we do the least squares problem. This is done by selecting at time $k$ the weight matrices for $1 \le i \le k$ to be
   $$S_i = \lambda^{(k-i)} I_{3\times 3}.$$
   With this choice, the $3k \times 3k$ weighting matrix $R_k$ is given by
   $$R_k = \text{diag}(\lambda^{k-1} I_3, \lambda^{k-2} I_3, \cdots, \lambda I_3, I_3).$$
   We see that the errors in older measurements are "discounted" by higher powers of $\lambda$, and thus the estimation process " exponentially forgets" them and "focuses" on the more recent measurements. It is important to note that at each step $k$, we are redefining the weights $R_k$ so that errors in the newest measurements are penalized the most. This can be done recursively in our `for loop`, by
   $$R_{k+1} = \begin{bmatrix} \lambda R_k & 0_{3k\times 3} \\ 0_{3\times 3k} & I_{3\times 3} \end{bmatrix}.$$
   For each $n \le k \le N$, compute $\hat{x}_k$ using the Batch Method. Turn in a plot of $E_k$ versus $k$, and label your plot appropriately. You can use $\lambda = 0.98$ or you can tune the forgetting factor to see what works best. How can you resist playing with it once you have your code working? :)

   (d) For each $n \le k \le N$, compute $\hat{x}_k$ now using the RLS (Recursive Least Squares) Algorithm, with forgetting factor. The algorithm (without using the Matrix Inversion Lemma) becomes

   - **Initialization Step:** Set
   $$\begin{aligned} Q_n :=& \sum_{i=1}^{n} C_i^\top \lambda^{n-i} C_i \\ \Gamma_n :=& \sum_{i=1}^{n} C_i^\top \lambda^{n-i} y_i \\ \hat{x}_n :=& (Q_n)^{-1} \Gamma_n \end{aligned}$$
   - **Recursion:** For $n \le k < N$
   $$\begin{aligned} Q_{k+1} :=& \lambda Q_k + C_{k+1}^\top C_{k+1} \\ K_{k+1} :=& (Q_{k+1})^{-1} C_{k+1}^\top \\ \hat{x}_{k+1} :=& \hat{x}_k + K_{k+1} \left( y_{k+1} - C_{k+1} \hat{x}_k \right) \end{aligned}$$

- If you want the version with the Matrix Inversion Lemma, see the hints!
- Turn in a plot of $E_k := ||\hat{x}_k - x_k||$ versus $k$, and label your plot appropriately. To be clear, there are no $\lambda$'s in the computation of $E_k$; we are just using the standard Euclidean norm to see how well we are doing in tracking $x$ as it slowly drifts.

## Hints

**Hints: Prob. 2** The important point here is that when a matrix is symmetric, repeated e-values do not pose a problem as they do for a general square matrix. The last page of the HW gives a proof by induction.

(a) **Base Step:** The first thing to note is that a $1 \times 1$ matrix can always be factored.

(b) **Inductive Step:** The induction hypothesis is to assume that $(n-1) \times (n-1)$ symmetric matrices can be factored as $O\Lambda O^\top$ where $O$ is an orthogonal matrix and $\Lambda$ is diagonal.

(c) **To show:** Next, you must show that the same is true for $n \times n$ symmetric matrices. The key step in the proof is to show that if $A$ is symmetric and $\lambda$ is an e-value, then there exists an orthogonal matrix $P$ such that

$$P^\top AP = \begin{bmatrix} \lambda & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & B \end{bmatrix},$$

where $B$ is symmetric and $(n-1)\times(n-1)$. The orthogonal matrix $P$ is produced by using an e-vector associated with $\lambda$ and the Gram-Schmidt process. Hence, if you care to understand the proof, it is within your means to do so.

**Hints: Prob. 4** Recursive Least Squares (RLS)

(a) **Basic Version:**

- Initialization Step: Choose $n$ such that $Q_n$ is invertible (full rank)

$$\begin{aligned} Q_n &:= \sum_{i=1}^{n} C_i^\top S_i C_i \\ \Gamma_n &:= \sum_{i=1}^{n} C_i^\top S_i y_i \\ \hat{x}_n &:= (Q_n)^{-1}\Gamma_n \end{aligned}$$

- Recursion: For $n \le k < N$

$$\begin{aligned} Q_{k+1} &:= Q_k + C_{k+1}^\top S_{k+1} C_{k+1} \\ K_{k+1} &:= (Q_{k+1})^{-1} C_{k+1}^\top S_{k+1} \\ \hat{x}_{k+1} &:= \hat{x}_k + K_{k+1}\left(y_{k+1} - C_{k+1}\hat{x}_k\right) \end{aligned}$$

(b) **Improved Version Using the Matrix Inversion Lemma:**

- Initialization Step: Choose $n$ such that $Q_n$ is invertible (full rank)

$$\begin{aligned} Q_n &:= \sum_{i=1}^{n} C_i^\top S_i C_i \\ P_n &:= (Q_n)^{-1} \\ \Gamma_n &:= \sum_{i=1}^{n} C_i^\top S_i y_i \\ \hat{x}_n &:= P_n \Gamma_n \end{aligned}$$

- Recursion: For $n \le k < N$

$$\begin{aligned} P_{k+1} =& P_k - P_k C_{k+1}^\top [S_{k+1}^{-1} + C_{k+1} P_k C_{k+1}^\top]^{-1} C_{k+1} P_k. \\ K_{k+1} :=& P_{k+1} C_{k+1}^\top S_{k+1} \\ \hat{x}_{k+1} :=& \hat{x}_k + K_{k+1} \left( y_{k+1} - C_{k+1} \hat{x}_k \right) \end{aligned}$$

- How to Derive the Riccati Equation? It comes from the Matrix Inversion Lemma

$$\begin{aligned} Q_{k+1} =& Q_k + C_{k+1}^\top S_{k+1} C_{k+1} \\ Q_{k+1}^{-1} =& \left( Q_k + C_{k+1}^\top S_{k+1} C_{k+1} \right)^{-1} \\ =& Q_k^{-1} - Q_k^{-1} C_{k+1}^\top \left[ S_{k+1}^{-1} + C_{k+1} Q_k^{-1} C_{k+1}^\top \right]^{-1} C_{k+1} Q_k^{-1} \\ P_k :=& Q_k^{-1} \\ P_{k+1} =& P_k - P_k C_{k+1}^\top \left[ S_{k+1}^{-1} + C_{k+1} P_k C_{k+1}^\top \right]^{-1} C_{k+1} P_k. \end{aligned}$$

- Jacopo Francesco Riccati (1676-1754) `http://en.wikipedia.org/wiki/Jacopo_Riccati`

**Hints: Prob. 5**

(a) Rewrite $Q_{k+1} = \lambda Q_k + C_{k+1}^\top C_{k+1}$ as

$$\frac{1}{\lambda} Q_{k+1} = Q_k + C_{k+1}^\top \frac{1}{\lambda} C_{k+1}$$

Therefore

$$\lambda Q_{k+1}^{-1} = [Q_k + C_{k+1}^\top \frac{1}{\lambda} C_{k+1}]^{-1} \quad (*)$$

Using the Matrix Inversion Lemma, we have

$$\lambda Q_{k+1}^{-1} = Q_k^{-1} - Q_k^{-1} C_{k+1}^\top [\lambda I + C_{k+1} Q_k^{-1} C_{k+1}^\top]^{-1} C_{k+1} Q_k^{-1}.$$

and thus

$$Q_{k+1}^{-1} = \frac{1}{\lambda} Q_k^{-1} - \frac{1}{\lambda} Q_k^{-1} C_{k+1}^\top [\lambda I + C_{k+1} Q_k^{-1} C_{k+1}^\top]^{-1} C_{k+1} Q_k^{-1}.$$

If we define $P_k := Q_k^{-1}$, we obtain

$$P_{k+1} = \frac{1}{\lambda} P_k - \frac{1}{\lambda} P_k C_{k+1}^\top [\lambda I + C_{k+1} P_k C_{k+1}^\top]^{-1} C_{k+1} P_k.$$

(b) If you are using your m-file for the Matrix Inversion Lemma, you can stop at (*), apply your function to get the inverse of $[Q_k + C_{k+1}^\top \frac{1}{\lambda} C_{k+1}]$, and then divide by the forgetting factor.

(c) The recursion on $\hat{x}_k$ is unchanged from the RLS algorithm without the forgetting factor. In case you want to see the derivation, the key formulas are:

$$
\begin{aligned}
Q_k &:= \sum_{i=1}^{k} C_i^\top \lambda^{k-i} C_i \\
Q_k \hat{x}_k &:= \sum_{i=1}^{k} C_i^\top \lambda^{k-i} y_i \\
Q_{k+1} &= \sum_{i=1}^{k+1} C_i^\top \lambda^{k+1-i} C_i \\
&= \lambda Q_k + C_{k+1}^\top C_{k+1} \\
Q_{k+1} \hat{x}_{k+1} &= \sum_{i=1}^{k+1} C_i^\top \lambda^{k+1-i} y_i \\
&= \lambda \sum_{i=1}^{k} C_i^\top \lambda^{k-i} y_i + C_{k+1}^\top y_{k+1} \\
&= \lambda Q_k \hat{x}_k + C_{k+1}^\top y_{k+1} \\
\lambda Q_k &= Q_{k+1} - C_{k+1}^\top C_{k+1}
\end{aligned}
$$

and thus, putting all of this together

$$
\begin{aligned}
\hat{x}_{k+1} &= Q_{k+1}^{-1} \left[ \left( Q_{k+1} - C_{k+1}^\top C_{k+1} \right) \hat{x}_k + C_{k+1}^\top y_{k+1} \right] \\
&= \hat{x}_k + Q_{k+1}^{-1} C_{k+1}^\top \left( y_{k+1} - C_{k+1} \hat{x}_k \right)
\end{aligned}
$$

Hence, the only change is to the update formula for $Q_{k+1}$.

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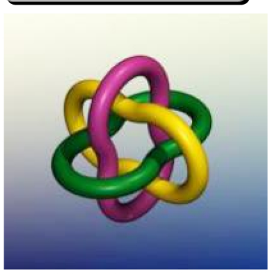

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**DECEMBER 23RD, 2012, 12:21** **#3**

> Originally Posted by **matqkks**
>
> I have been trying to prove the following result:
> If A is real symmetric matrix with an eigenvalue lambda of multiplicity m then lambda has m linearly independent e.vectors.
> Is there a simple proof of this result?

This is a slight variation of Deveno's argument. I will assume you already know that the eigenvalues of a real symmetric matrix are all real.

Let $A$ be an $n\times n$ real symmetric matrix, and assume as an inductive hypothesis that all $(n-1)\times(n-1)$ real symmetric matrices are diagonalisable. Let $\lambda$ be an eigenvalue of $A$, with a normalised eigenvector $x_1$. Using the Gram–Schmidt process, form an orthonormal basis $\{x_1,x_2,\ldots,x_n\}$ with that eigenvector as its first element.

Let $P$ be the $n\times n$ matrix whose columns are $x_1,x_2,\ldots,x_n$, and denote by $T:\mathbb{R}^n\to\mathbb{R}^n$ the linear transformation whose matrix with respect to the standard basis is $A$. Then $P$ is an orthogonal matrix ($P^{\mathrm{T}}=P^{-1}$), and the matrix of $T$ with respect to the basis $\{x_1,x_2,\ldots,x_n\}$ is $P^{\mathrm{T}}AP$. The $(i,j)$-element of that matrix is $(P^{\mathrm{T}}AP)_{ij}=\langle Ax_j,x_i\rangle$. In particular, the elements in the first column are

$$(P^{\mathrm{T}}AP)_{i1}=\langle Ax_1,x_i\rangle=\langle\lambda x_1,x_i\rangle=\begin{cases}\lambda & (i=1)\\ 0 & (i>1)\end{cases}$$

(because the vectors $x_i$ are orthonormal). Thus the first column of $P^{\mathrm{T}}AP$ has $\lambda$ as its top element , and $0$ for each of the other elements. Since $P^{\mathrm{T}}AP$ is symmetric, the top row also consists of a $\lambda$ followed by all zeros. Hence the matrix $P^{\mathrm{T}}AP$ looks like this:

$$\begin{bmatrix}\lambda & 0 & \ldots & 0\\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{bmatrix},$$

where $B$ is an $(n-1)\times(n-1)$ real symmetric matrix. By the inductive hypothesis, $B$ is diagonalisable, so there is an orthogonal $(n-1)\times(n-1)$ matrix $Q$ such that $Q^{\mathrm{T}}BQ$ is diagonal. Let

$$R = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{bmatrix}.$$

Then $R^{\mathrm{T}}P^{\mathrm{T}}APR$ is diagonal, as required.

**Was sich überhaupt sagen lässt, lässt sich klar sagen; und wovon man nicht reden kann, darüber muss man schweigen.**

(Anything that can be said at all, can be said clearly; and whereof one cannot speak, thereon one must be silent.)

– Ludwig Wittgenstein's good advice for forum contributors, in *Tractatus Logico-Philosophicus*.