Foundations of Robotics
ROB-GY 6003
Homework 1 Answers

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Question: 2.1

Answer: It is given that a vector ${}^{\mathrm{A}}\mathrm{P}$ is rotated about $\hat{Z}_{\mathrm{A}}$ by $\theta°$, the roation matrix is given by -

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The vector is subsequently rotated about $\hat{X}_{\mathrm{A}}$ by $\phi°$, then the rotation matrix is given by -

$$R_{ZX}(\theta,\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$R_{ZX}(\theta,\phi) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{bmatrix} \tag{3}$$

Question: 2.3

Answer: It is given that a frame {B} is rotated about $\hat{Z}_{\mathrm{B}}$ by $\theta°$, the roation matrix is given by -

$${}^{A}_{B}R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The frame B is subsequently rotated about $\hat{X}_{\mathrm{B}}$ by $\phi°$, then the rotation matrix is given by -

$${}^{A}_{B}R_{ZX}(\theta,\phi) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{2}$$

$${}^{A}_{B}R_{ZX}(\theta,\phi) = \begin{bmatrix} \cos\theta & -\sin\theta\cos\phi & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{3}$$

Question: 2.12

Answer: Given :

A velocity vector

$${}^{B}V = \begin{bmatrix} 10.0 \\ 20.0 \\ 30.0 \end{bmatrix} \tag{1}$$

And a Transformation Operator,

$${}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 11.0 \\ 0.500 & 0.866 & 0.000 & -3.0 \\ 0.000 & 0.000 & 1.000 & 9.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

As Velocity is a free vector, we have the relation (Eq$^{\mathrm{n}}$ 2.94),

$${}^{A}V = {}^{A}_{B}R\,{}^{B}V \tag{3}$$

$${}^{A}V = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 10.0 \\ 20.0 \\ 30.0 \end{bmatrix} \tag{4}$$

Finally,

$${}^{A}V = \begin{bmatrix} -1.34 \\ 22.32 \\ 30 \end{bmatrix} \tag{5}$$

Question: 2.14

Answer: Consider below Figure 1.

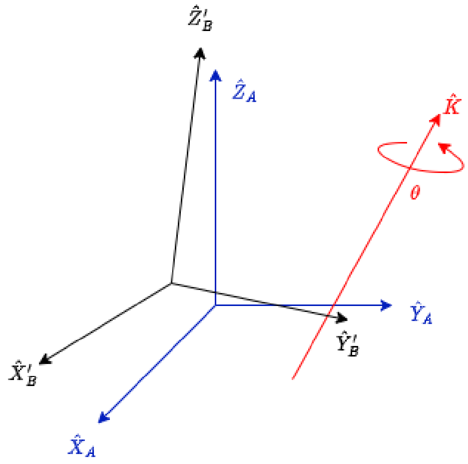

Figure 1: Frame $\{B\}$ rotated about a general axis $\hat{K}$ that does not pass through the origin of the original frame

In order for us to obtain a generalised formula, we shall use the one which we already know as Rodriques' Formula,

$$R_K(\theta) = \begin{bmatrix} k_xk_xv\theta + c\theta & k_xk_yv\theta - k_zs\theta & k_xk_zv\theta + k_ys\theta \\ k_xk_yv\theta + k_zs\theta & k_yk_yv\theta + c\theta & k_yk_zv\theta - k_xs\theta \\ k_xk_zv\theta - k_ys\theta & k_yk_zv\theta + k_xs\theta & k_zk_zv\theta + c\theta \end{bmatrix} \tag{1}$$

But the above formula is for a general axis that passes through the origin of the initial coincidence. In order for us to use $Eq^n 1$ we will first have to translate the initial coincidence to a location that allows the general axis $\hat{K}$ to pass through its origin, then rotate the frame by the angle θ and then finally, undo the original translation, $i.e.$ bring it back to its original location.
So, this transformation can be written as,

$$^A_BT = D_Q(q)R_K(\theta)D_Q(-q) \tag{2}$$

Where,

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

And,

$$D_Q(-q) = \begin{bmatrix} 1 & 0 & 0 & -q_x \\ 0 & 1 & 0 & -q_y \\ 0 & 0 & 1 & -q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Substituting $Eq^n 1$, $Eq^n 3$ and $Eq^n 4$ in $Eq^n 2$, we get,

$$^A_BT = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{5}$$

Where, r_{ij} is given by $Eq^n 1$.
And,

$$p_x = -k_xq_x - k_yq_y - k_zq_z + q_x \tag{6}$$
$$p_y = -k_xq_x - k_yq_y - k_zq_z + q_y \tag{7}$$
$$p_z = -k_xq_x - k_yq_y - k_zq_z + q_z \tag{8}$$

Question: 2.19

Answer: Consider the below Figure 2.

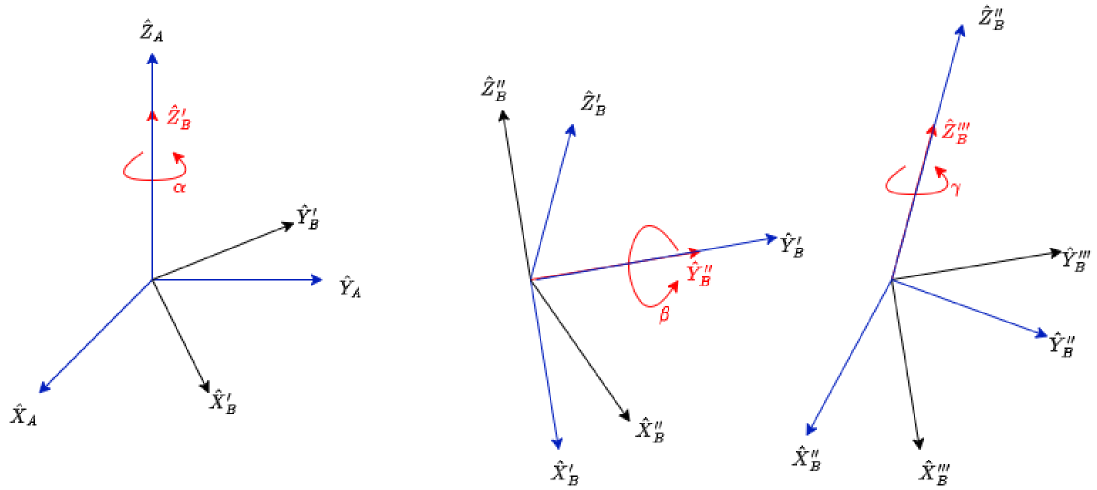

Figure 2: Z-Y-Z Euler Angles

Start with the frame coincident with a known frame A. Rotate B first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and, finally, about $\hat{Z}_B$ by an angle γ . (See above Figure 1)

A rotation matrix parameterised by Z-Y-Z Euler angles is indicated by the notation ${}^A_BR_{Z'Y'Z'}(\alpha, \beta, \gamma)$ With reference to *Figure 1*, we can ues the intermediate frames $\{B'\}$, $\{B''\}$ and $\{B'''\}$ in order to give an expression for the rotation matrix ${}^A_BR_{Z'Y'Z'}(\alpha, \beta, \gamma)$.

The first rotation, from $\{A\}$ to $\{B'\}$ is given by,

$$ {}^A_{B'}R = R_Z(\alpha) \tag{1} $$

The second rotation, now expressed in fixed coordinates is,

$$ {}^{B'}_{B''}R = R_Z(\alpha)R_Y(\beta)R_Z^{-1}(\alpha) \tag{2} $$

And the third rotation is given by,

$$ {}^{B''}_{B'''}R = (R_Z(\alpha)R_Y(\beta)R_Z^{-1}(\alpha))R_Z(\gamma)(R_Z(\alpha)R_Y(\beta)R_Z^{-1}(\alpha))^{-1} \tag{3} $$

The result,

$$ {}^A_BR = {}^{B''}_{B}R \, {}^{B'}_{B''}R \, {}^A_{B'}R \tag{4} $$

Upon substituting $Eq^n 1$, $Eq^n 2$ and $Eq^n 3$ in $Eq^n 4$, we get,

$$ \Rightarrow {}^A_BR_{Z'Y'Z'}(\alpha, \beta, \gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) \tag{5} $$

$$ = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6} $$

$$ \therefore {}^A_BR_{Z'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix} \tag{7} $$

Question: 2.20

Answer: It is given that a vector Q rotates about a vect $\hat{K}$ by $\theta°$to form a new vector Q', given by the equation,

$$ Q' = R_K(\theta)Q \tag{1} $$

Let us start with Equation (2.80),

$$R_K(\theta) = \begin{bmatrix} k_xk_xv\theta + c\theta & k_xk_yv\theta - k_zs\theta & k_xk_zv\theta + k_ys\theta \\ k_xk_yv\theta + k_zs\theta & k_yk_yv\theta + c\theta & k_yk_zv\theta - k_xs\theta \\ k_xk_zv\theta - k_ys\theta & k_yk_zv\theta + k_xs\theta & k_zk_zv\theta + c\theta \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} k_xk_xv\theta & k_xk_yv\theta & k_xk_zv\theta \\ k_xk_yv\theta & k_yk_yv\theta & k_yk_zv\theta \\ k_xk_zv\theta & k_yk_zv\theta & k_zk_zv\theta \end{bmatrix} + \begin{bmatrix} c\theta & -k_zs\theta & k_ys\theta \\ k_zs\theta & c\theta & -k_xs\theta \\ -k_ys\theta & k_xs\theta & c\theta \end{bmatrix} \tag{3}$$

$$= v\theta \begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_xk_y & k_yk_y & k_yk_z \\ k_xk_z & k_yk_z & k_zk_z \end{bmatrix} + \begin{bmatrix} c\theta & -k_zs\theta & k_ys\theta \\ k_zs\theta & c\theta & -k_xs\theta \\ -k_ys\theta & k_xs\theta & c\theta \end{bmatrix} \tag{4}$$

$$= v\theta \begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_xk_y & k_yk_y & k_yk_z \\ k_xk_z & k_yk_z & k_zk_z \end{bmatrix} + \begin{bmatrix} 0 & -k_zs\theta & k_ys\theta \\ k_zs\theta & 0 & -k_xs\theta \\ -k_ys\theta & k_xs\theta & 0 \end{bmatrix} + \begin{bmatrix} c\theta & 0 & 0 \\ 0 & c\theta & 0 \\ 0 & 0 & c\theta \end{bmatrix} \tag{5}$$

$$= v\theta \begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_xk_y & k_yk_y & k_yk_z \\ k_xk_z & k_yk_z & k_zk_z \end{bmatrix} + s\theta \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} + c\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Substituting above equation in equation (1),

$$Q' = \left(v\theta \begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_xk_y & k_yk_y & k_yk_z \\ k_xk_z & k_yk_z & k_zk_z \end{bmatrix} + s\theta \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} + c\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) Q \tag{7}$$

$$= Qv\theta \begin{bmatrix} k_xk_x & k_xk_y & k_xk_z \\ k_xk_y & k_yk_y & k_yk_z \\ k_xk_z & k_yk_z & k_zk_z \end{bmatrix} + Qs\theta \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} + Qc\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Converting the matrices to their vector notation and substituting for
$v\theta = 1 - \cos\theta$, we get,

$$= (1 - \cos\theta)\hat{K} \times \hat{K} \times Q + \sin\theta(\hat{K} \times Q) + Q\cos\theta \tag{9}$$

Rearranging equation (9)

$$Q' = Q\cos\theta + \sin\theta(\hat{K} \times Q) + (1 - \cos\theta)(\hat{K} \cdot Q)\hat{K} \tag{10}$$

Equation (9) represents Rodriques's Formula, with equation (2) being the matrix representation of the same.
QED

Question: 2.21

Answer: Given,

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix} \tag{1}$$

And the applying the approximations: $\sin\theta = \theta$, $\cos\theta = 1$, $\theta^2 = 0$, $v\theta = 1 - \cos\theta = 1 - 1 = 0$ to the Equation (1), we get,

$$R_K(\theta) = \begin{bmatrix} k_x k_x(0) + 1 & k_x k_y(0) - k_z\theta & k_x k_z(0) + k_y\theta \\ k_x k_y(0) + k_z\theta & k_y k_y(0) + 1 & k_y k_z(0) - k_x\theta \\ k_x k_z(0) - k_y\theta & k_y k_z(0) + k_x\theta & k_z k_z(0) + 1 \end{bmatrix} \tag{2}$$

$$R_K(\theta) = \begin{bmatrix} 1 & -k_z\theta & k_y\theta \\ k_z\theta & 1 & -k_x\theta \\ -k_y\theta & k_x\theta & 1 \end{bmatrix} \tag{3}$$

Question: 2.22

Answer: Consider the answer of the previous question,

$$R_K(\theta) = \begin{bmatrix} 1 & -k_z\theta & k_y\theta \\ k_z\theta & 1 & k_x\theta \\ -k_y\theta & k_x\theta & 1 \end{bmatrix} \tag{1}$$

Now, consider two rotations $\mathrm{R}_1 = R_A(\alpha)$ and $\mathrm{R}_2 = R_B(\beta)$.
Such that, $\alpha << 1$ and $\beta << 1$ and $\alpha\beta \approx 0$

If we form the product of the two rotations, *i.e*, perform R_1 and then R_2

$$R_1 R_2 = \begin{bmatrix} 1 & -a_z\alpha & a_y\alpha \\ a_z\alpha & 1 & a_x\alpha \\ -a_y\alpha & a_x\alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & -b_z\beta & b_y\beta \\ b_z\beta & 1 & b_x\beta \\ -b_y\beta & b_x\beta & 1 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 1 & -a_z\alpha - b_z\beta & a_y\alpha + b_y\beta \\ a_z\alpha + b_z\beta & 1 & -a_x\alpha - b_x\beta \\ -a_y\alpha - b_y\beta & a_x\alpha + b_x\beta & 1 \end{bmatrix} \tag{3}$$

Now let's perform the two roations in the opposite order,

$$R_2 R_1 = \begin{bmatrix} 1 & -b_z\beta & b_y\beta \\ b_z\beta & 1 & b_x\beta \\ -b_y\beta & b_x\beta & 1 \end{bmatrix} \begin{bmatrix} 1 & -a_z\alpha & a_y\alpha \\ a_z\alpha & 1 & a_x\alpha \\ -a_y\alpha & a_x\alpha & 1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & -a_z\alpha - b_z\beta & a_y\alpha + b_y\beta \\ a_z\alpha + b_z\beta & 1 & -a_x\alpha - b_x\beta \\ -a_y\alpha - b_y\beta & a_x\alpha + b_x\beta & 1 \end{bmatrix} \tag{5}$$

Thus, upon comparing equations (3) and (5) we see that,

$$R_1 R_2 = R_2 R_1 \tag{6}$$

Therefore, we can say that two infinitesimal rotations commute.
QED

Question: 2.27

Answer: We are given the following Figure 2.25 -

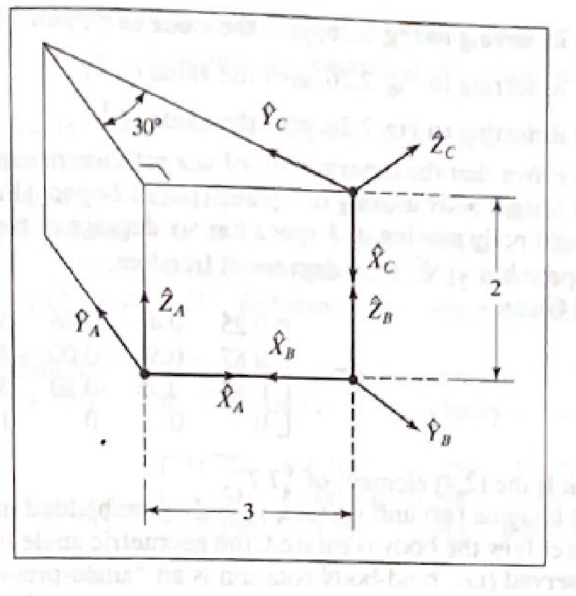

FIGURE 2.25: Frames at the corners of a wedge.

Referring to the given figure, we can see that the distance between frame {A} and {B} is that of 3
And that with respect to frame {A}, frame {B} has rotated about $\hat{Z}$ by 180°

And using Homogenous Transform Equation,

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}.$$

We get the matrix

$$ {}^{A}_{B}T = \begin{bmatrix} \cos(180) & -\sin(180) & 0 & 3 \\ \sin(180) & \cos(180) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$ {}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Question: 2.37

Answer: We know that a homogenous transform given by,

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

And that the inverse of the transformation is given by,

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Applying Equation (2) on the given homogenous transform,

$$ {}^{A}_{B}T = \begin{bmatrix} 0.25 & 0.43 & 0.86 & 5.0 \\ 0.87 & -0.5 & 0.00 & -4.0 \\ 0.43 & 0.75 & -0.5 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3} $$

We get,

$$ {}^{B}_{A}T = \begin{bmatrix} 0.25 & 0.87 & 0.43 & 0.94 \\ 0.43 & -0.5 & 0.75 & -6.4 \\ 0.86 & 0.00 & -0.5 & -2.8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4} $$

Therefore, the (2,4) element of ${}^{B}_{A}T =$ -6.4

Question: 2.38

Answer: We are given two vectors v_1 and v_2 embedded in a rigid body such that the geometric angle between the two is constant, immaterial of the number and order of rotations that the body undergoes.
So, we can deduce that the scalar product of the two vectors remains constant.

$$ v_1 \cdot v_2 = \cos\theta \tag{1} $$

Where θ is the angle between $v_1 \& v_2$.
$Eq^n(1)$ can be rewritten in matrix form as,

$$ v_1^T v_2 = \cos\theta \tag{2} $$

Now, let us rotate the given rigid body. $Eq^n(2)$ can now be written as,

$$ (Rv_1)^T (Rv_2) = \cos\theta \tag{3} $$

But we know that $\cos\theta = v_1^T v_2$ (From $Eq^n(1)$),

$$ R^T v_1^T R v_2 = v_1^T v_2 \tag{4} $$

Cancelling out $v_1^T v_2$ from both sides, we get,

$$ R^T R = I \tag{5} $$

Where, I represents the Identity Matrix.
But we also know that by definition, the inverse of a matrix can be expressed as,

$$ R^{-1} R = I \tag{6} $$

By comparing $Eq^n(5)$ and $Eq^n(6)$, we can see that,

$$ R^T = R^{-1} \tag{7} $$

QED.