Question: 1.(a) Nagy, Page 136, Prob. 4.4.3

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{S} = \big(p_0 = 1,\ p_1 = x,\ p_2 = x^2\big)$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

$\therefore$ the components of $r(x)$ in basis $\mathcal{S}$ is

$$\mathrm{r}_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{1}$$

Question: 1.(b)

Answer: Given, $V = \mathbb{P}_2$ with the ordered basis $\mathcal{Q} = \big(q_0 = 1,\ q_1 = 1 - x,\ q_2 = x + x^2\big)$.

From the definition of the basis $\mathcal{Q}$, we can rewrite it in terms of basis $\mathcal{S}$ as,

$q_{0_{\mathcal{S}}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $q_{1_{\mathcal{S}}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $q_{2_{\mathcal{S}}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Q_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

And the given polynomial is $r(x) = 2 + 3x - x^2$.

$$\Rightarrow \mathrm{r}_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{1}$$

Now, we need to find numbers $\tilde{r}_0$, $\tilde{r}_1$ and $\tilde{r}_2$ such that,

$$\mathrm{r}_{\mathcal{S}} = \tilde{r}_0 \mathrm{q}_{0_{\mathcal{S}}} + \tilde{r}_1 \mathrm{q}_{1_{\mathcal{S}}} + \tilde{r}_2 \mathrm{q}_{2_{\mathcal{S}}} \tag{2}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \tilde{r}_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \tilde{r}_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \tilde{r}_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

Eq^n can be rewritten as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{4}$$

Applying the Gauss method,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{5}$$

$R_2 \to (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{6}$$

$R_1 \to R_1 + (-1)R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{7}$$

$R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{8}$$

$R_1 \to R_1 + (-1)R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{9}$$

Hence, the solution is,

$$\mathrm{r}_q = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} \tag{10}$$

Question: 2.

Answer: Given the matrix,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \;\Rightarrow (\lambda I - A)v = 0 \;\Leftrightarrow det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 1 & -4 & -10 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \tag{2}$$

$$\Rightarrow det(\lambda I - A) = (\lambda - 1) \cdot (\lambda - 2) \cdot (\lambda - 3) = 0 \tag{3}$$

$$\therefore \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \tag{4}$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 1 - \lambda_i & 4 & 10 \\ 0 & 2 - \lambda_i & 0 \\ 0 & 0 & 3 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

From Eq^n 5, substituting $\lambda_1 = 1$

$$\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 4v_2^1 + 10v_3^1 = 0 \\ v_2^1 = 0 \\ 2v_3^1 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{matrix} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -v_1^2 + 4v_2^2 + 10v_3^2 = 0 \\ 0 = 0 \\ v_3^2 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^2 = 4v_2^2 \\ v_2^2 = \text{any value} \\ v_3^2 = 0 \end{matrix} \Rightarrow v^2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

From Eq^n 5, substituting $\lambda_3 = 3$

$$\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2v_1^3 + 4v_2^3 + 10v_3^3 = 0 \\ -v_2^3 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^3 = 5v_3^3 \\ v_2^3 = 0 \\ v_3^3 = \text{any value} \end{matrix} \Rightarrow v^3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

Now we shall verify that v^1, v^2 and v^3 are *Linearly Independant*,

$$\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = 0 \tag{9}$$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 0 \tag{10}$$

$$\alpha_1 + 4\alpha_2 + 5\alpha_3 = 0 \tag{11}$$
$$\alpha_2 = 0 \tag{12}$$
$$\alpha_3 = 0 \tag{13}$$

Solving above system of equations gives us $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$.
$\therefore$ v^1, v^2 and v^3 are *Linearly Independant.*

Question: 3.

Answer: Given the matrix,

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1}$$

By definition, we know that, $\exists v \neq 0$ s.t. $Av = \lambda v \ \Rightarrow (\lambda I - A)v = 0 \ \Leftrightarrow det(\lambda I - A) = 0$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda - 3 & -1 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \tag{2}$$

$$\Rightarrow det(\lambda I - A) = (\lambda - 3) \cdot (\lambda - 3) \cdot (\lambda - 2) = 0 \tag{3}$$

$$\therefore \lambda_1 = 3, \lambda_2 = 2 \tag{4}$$

Now, we shall apply the known relation $Av^i = \lambda_i v^i \Rightarrow (A - \lambda_i I)v^i = 0$

$$\begin{bmatrix} 3 - \lambda_i & 1 & 0 \\ 0 & 3 - \lambda_i & 0 \\ 0 & 0 & 2 - \lambda_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

From Eq^n 5, substituting $\lambda_1 = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_2^1 = 0 \\ 0 = 0 \\ -v_3^1 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^1 = \text{any value} \\ v_2^1 = 0 \\ v_3^1 = 0 \end{matrix} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

From Eq^n 5, substituting $\lambda_2 = 2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1^2 = 0 \\ v_2^2 = 0 \\ v_3^2 = \text{any value} \end{matrix} \Rightarrow v^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

Consider the set of eigenvectors $\mathcal{V} = \{v^1, v^2\}$. Firstly,

We know that any set of eigenvectors of a given matrix are Linearly Independent. So, v^1, v^2 are *L.I.* (8)

Secondly, consider a linear combination x such that,

$$\left\{x \in \mathbb{R} \mid \exists\ \alpha_1, \alpha_2 \in \mathbb{R}, v^1, v^2 \in \mathcal{V}\ ;\ s.t.\ x = \alpha_1 v^1 + \alpha_2 v^2\right\} \tag{9}$$

By observing above Eq^n 9 we can see that it holds true for all cases.
$\therefore$ based on the Eq^n 8 and 9 we can say that $\mathcal{V}$ forms a basis for $\mathbb{R}^3$.

Question: 4.

Answer: We are given two similar square matrices A and B such that,

$$B = P^{-1}AP \tag{1}$$

Consider the characteristic relation of matrix B,

$$det(\lambda I - B) \tag{2}$$

Apply Eq^n 1 in Eq^n **??**,

$$\Rightarrow det(\lambda I - B) = det(\lambda I - P^{-1}AP) \tag{3}$$

The identity matrix I can also be written as $P^{-1}IP$. Substituting in Eq^n 3,

$$\Rightarrow det(\lambda I - B) = det(\lambda P^{-1}IP - P^{-1}AP) \tag{4}$$

Now we shall take P^{-1} and P out common,

$$\Rightarrow det(\lambda I - B) = det(P^{-1}(\lambda I - A)P) \tag{5}$$

We know that for compatible square matrices A and B, $det(AB) = det(A)det(B)$. So Eq^n 5 becomes,

$$\Rightarrow det(\lambda I - B) = det(P^{-1})det(\lambda I - A)det(P) \tag{6}$$

Cancelling out $det(P^{-1})$ by $det(P)$

$$\Rightarrow det(\lambda I - B) = det(\lambda I - A) \tag{7}$$

When the relation in Eq^n 8 is equated to zero, we prove that the two matrices have the same eigenvalues as well as the characteristic equations. I.E.,

$$\Rightarrow det(\lambda I - B) = det(\lambda I - A) = 0 \tag{8}$$

Q.E.D.

Question: 5. **P is the eigenvector matrix**

Answer: Given a matrix A_3 such that,

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

From Question: 2. We found the eigenvalues to be $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.
Using these eigenvalues, we can create the diagonal martix, $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{2}$$

To show similarity between A and Λ, we need to find a matrix P such that $\Lambda = PAP^{-1}$
Consider a matrix $P =][$ such

Question: 6.(a)

Answer: Given, a vector space $(\mathcal{X}, \mathbb{R})$ where $\mathcal{X}$ is a set of 2 x 2 matrices with real coefficients.
An operation is defined $L : \mathcal{X} \to \mathcal{X}$ by

$$L(M) = \frac{1}{2}(M + M^T) \tag{1}$$

Where $M \in \mathcal{X}$ is a 2 x 2 real matrix

The operator L will be considered a *Linear Operator* if,

$$\forall x, y \in \mathcal{X}\ ,\ \alpha, \beta \in \mathbb{R} \mid L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \tag{2}$$

Applying above Statement 2 to Eq^n 1,

$$L(\alpha x + \beta y) = \frac{1}{2}((\alpha x + \beta y) + (\alpha x + \beta y)^T) \tag{3}$$

Applying the property of sum of transpose of two matrices
(i.e. for two matrices A and $B \rightarrow (A+B)^T = A^T + B^T$)

$$L(\alpha x + \beta y) = \frac{1}{2}(\alpha x + \beta y + \alpha x^T + \beta y^T) \tag{4}$$

$$= \frac{1}{2}(\alpha x + \alpha x^T + \beta y + \beta y^T) \tag{5}$$

$$= \alpha(\frac{1}{2}(x + x^T)) + \beta(\frac{1}{2}(y + y^T)) \tag{6}$$

$$= \alpha L(x) + \beta L(y) \tag{7}$$

Thus, Eq^n 7 proves Statement 2. $\therefore$ the given operator L is actually a *Linear Operator*.
Q.E.D.

Question: 6.(b)

Answer: Given a linear transformation $L : \mathcal{X} \rightarrow \mathcal{X}$ and a basis E given by $E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ used on both copies of $\mathcal{X}$.

By the theorem, we know that $A_i = [L(u^i)]_v$. But in our case, $u = v = E$. So, we can rewrite it as $A_i = [L(E^{ij})]_E$ where $\forall i, j \in \mathbb{N} \mid 1 \leq i, j \leq 2$

But as it is given that $L : \mathcal{X} \rightarrow \mathcal{X} \Rightarrow L$ is the Identity Operation Id. Thus, $[L(E^{ij})]_E = [E^{ij}]_E$.

So, now we can form the columns of $A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix}$ to be as,

$$A = \begin{bmatrix} [E^{11}]_E & [E^{12}]_E & [E^{21}]_E & [E^{22}]_E \end{bmatrix} \tag{1}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Question: 7.(a)

Answer:

Question: 7.(b)

Answer: