Spong, M.W., Hutchinson, S., and Vidyasagar, M., *Robot Modeling and Control*, Wiley, 2006.

4.6 DERIVATION OF THE JACOBIAN

Consider an n-link manipulator with joint variables $q_1, \ldots, q_n$. Let

$$T_n^0(q) \quad = \quad \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix} \tag{4.40}$$

denote the transformation from the end-effector frame to the base frame, where $q = (q_1, \ldots, q_n)^T$ is the vector of joint variables. As the robot moves about, both the joint variables q_i and the end-effector position o_n^0 and orientation R_n^0 will be functions of time. The objective of this section is to relate the linear and angular velocity of the end-effector to the vector of joint velocities $\dot{q}(t)$. Let

$$S(\omega_n^0) \quad = \quad \dot{R}_n^0 (R_n^0)^T \tag{4.41}$$

define the angular velocity vector ω_n^0 of the end-effector, and let

$$v_n^0 \quad = \quad \dot{o}_n^0 \tag{4.42}$$

denote the linear velocity of the end effector. We seek expressions of the form

$$v_n^0 \quad = \quad J_v \dot{q} \tag{4.43}$$

$$\omega_n^0 \quad = \quad J_\omega \dot{q} \tag{4.44}$$

where J_v and J_ω are $3 \times n$ matrices. We may write Equations (4.43) and (4.44) together as

$$\xi \quad = \quad J \dot{q} \tag{4.45}$$

in which ξ and J are given by

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \tag{4.46}$$

The vector ξ is sometimes called a body velocity. Note that this velocity vector is *not* the derivative of a position variable, since the angular velocity vector is not the derivative of any particular time varying quantity. The matrix J is called the **Manipulator Jacobian** or **Jacobian** for short. Note that J is a $6 \times n$ matrix where n is the number of links. We next derive a simple expression for the Jacobian of any manipulator.

4.6.1 Angular Velocity

Recall from Equation (4.31) that angular velocities can be added as free vectors, provided that they are expressed relative to a common coordinate frame. Thus we can determine the angular velocity of the end-effector relative to the base by expressing the angular velocity contributed by each joint in the orientation of the base frame and then summing these.

If the i-th joint is revolute, then the i-th joint variable q_i equals θ_i and the axis of rotation is z_{i-1}. Following the convention that we introduced above, let ω_i^{i-1} represent the angular velocity of link i that is imparted by the rotation of joint i, expressed relative to frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. This angular velocity is expressed in the frame $i-1$ by

$$\boxed{\omega_i^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i k} \tag{4.47}$$

in which, as above, k is the unit coordinate vector $(0, 0, 1)^T$.

If the i-th joint is prismatic, then the motion of frame i relative to frame $i-1$ is a translation and

$$\boxed{\omega_i^{i-1} = 0} \tag{4.48}$$

Thus, if joint i is prismatic, the angular velocity of the end-effector does not depend on q_i, which now equals d_i.

Therefore, the overall angular velocity of the end-effector, ω_n^0, in the base frame is determined by Equation (4.31) as

$$\begin{aligned} \omega_n^0 &= \rho_1 \dot{q}_1 k + \rho_2 \dot{q}_2 R_1^0 k + \cdots + \rho_n \dot{q}_n R_{n-1}^0 k \\ &= \sum_{i-1}^{n} \rho_i \dot{q}_i z_{i-1}^0 \end{aligned} \tag{4.49}$$

in which ρ_i is equal to 1 if joint i is revolute and 0 if joint i is prismatic, since

$$z_{i-1}^0 = R_{i-1}^0 k \tag{4.50}$$

Of course $z_0^0 = k = (0, 0, 1)^T$.

The lower half of the Jacobian J_ω, in Equation (4.46) is thus given as

$$J_\omega = [\rho_1 z_0 \cdots \rho_n z_{n-1}]. \tag{4.51}$$

Note that in this equation, we have omitted the superscripts for the unit vectors along the z-axes, since these are all referenced to the world frame. In the remainder of the chapter we occasionally will follow this convention when there is no ambiguity concerning the reference frame.

4.6.2 Linear Velocity

The linear velocity of the end-effector is just $\dot{o}_n^0$. By the chain rule for differentiation

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i \tag{4.52}$$

Thus we see that the i-th column of J_v, which we denote as J_{v_i} is given by

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i} \tag{4.53}$$

Furthermore this expression is just the linear velocity of the end-effector that would result if $\dot{q}_i$ were equal to one and the other $\dot{q}_j$ were zero. In other words, the i-th column of the Jacobian can be generated by holding all joints fixed but the i-th and actuating the i-th at unit velocity. We now consider the two cases (prismatic and revolute joints) separately.

(i) Case 1: Prismatic Joints

If joint i is prismatic, then it imparts a pure translation to the end-effector. From our study of the DH convention in Chapter 3, we can write the T_n^0 as the product of three transformations as follows

$$\begin{aligned}
\begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix} &= T_n^0 && (4.54)\\
&= T_{i-1}^0 T_i^{i-1} T_n^i && (4.55)\\
&= \begin{bmatrix} R_{i-1}^0 & o_{i-1}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_n^i & o_n^i \\ 0 & 1 \end{bmatrix} && (4.56)\\
&= \begin{bmatrix} R_n^0 & R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \\ 0 & 1 \end{bmatrix}, && (4.57)
\end{aligned}$$

which gives

$$o_n^0 = R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \tag{4.58}$$

If only joint i is allowed to move, then both of o_n^i and o_{i-1}^0 are constant. Furthermore, if joint i is prismatic, then the rotation matrix R_{i-1}^0 is also constant (again, assuming that only joint i is allowed to move). Finally, recall from Chapter 3 that, by the DH convention, $o_i^{i-1} = (a_i c_i, a_i s_i, d_i)^T$. Thus, differentiation of o_n^0 gives

$$\boxed{\frac{\partial o_n^0}{\partial q_i}} = \frac{\partial}{\partial d_i} R_{i-1}^0 o_i^{i-1} \tag{4.59}$$

$$= R_{i-1}^0 \frac{\partial}{\partial d_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} \tag{4.60}$$

$$= \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.61}$$

$$\boxed{= \dot{d}_i z_{i-1}^0,} \tag{4.62}$$

in which d_i is the joint variable for prismatic joint i. Thus, (again, dropping the zero superscript on the z-axis) for the case of prismatic joints we have

$$J_{v_i} = z_{i-1} \tag{4.63}$$

(ii) Case 2: Revolute Joints

If joint i is revolute, then we have $q_i = \theta_i$. Starting with Equation (4.58), and letting $q_i = \theta_i$, since R_i^0 is not constant with respect to θ_i, we obtain

$$\boxed{\frac{\partial}{\partial \theta_i} o_n^0} = \frac{\partial}{\partial \theta_i} \left[R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} \right] \tag{4.64}$$

$$= \frac{\partial}{\partial \theta_i} R_i^0 o_n^i + R_{i-1}^0 \frac{\partial}{\partial \theta_i} o_i^{i-1} \tag{4.65}$$

$$= \dot{\theta}_i S(z_{i-1}^0) R_i^0 o_n^i + \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 o_i^{i-1} \tag{4.66}$$

$$= \dot{\theta}_i S(z_{i-1}^0) \left[R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} \right] \tag{4.67}$$

$$= \dot{\theta}_i S(z_{i-1}^0)(o_n^0 - o_{i-1}^0) \tag{4.68}$$

$$\boxed{= \dot{\theta}_i z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)} \tag{4.69}$$

The second term in Equation (4.66) is derived as follows:

$$R_{i-1}^0 \frac{\partial}{\partial \theta_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} = R_{i-1}^0 \begin{bmatrix} -a_i s_i \\ a_i c_i \\ 0 \end{bmatrix} \dot{\theta}_i \tag{4.70}$$

$$= R_{i-1}^0 S(k\dot{\theta}_i) o_i^{i-1} \tag{4.71}$$

$$= R_{i-1}^0 S(k\dot{\theta}_i) \left(R_{i-1}^0\right)^T R_{i-1}^0 o_i^{i-1} \tag{4.72}$$

$$= S(R_{i-1}^0 k\dot{\theta}_i) R_{i-1}^0 o_i^{i-1} \tag{4.73}$$

$$= \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 o_i^{i-1} \tag{4.74}$$

Equation (4.71) follows by straightforward computation. Thus for a revolute joint

$$J_{v_i} = z_{i-1} \times (o_n - o_{i-1}) \tag{4.75}$$

in which we have, following our convention, omitted the zero superscripts. Figure 4.1 illustrates a second interpretation of Equation (4.75). As can be seen in the figure, $o_n - o_{i-1} = r$ and $z_{i-1} = \omega$ in the familiar expression $v = \omega \times r$.

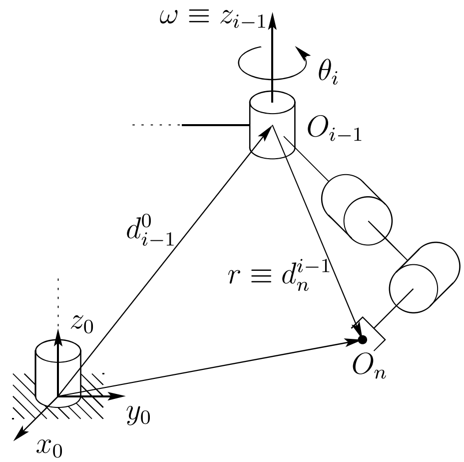

Fig. 4.1 Motion of the end-effector due to link i.

4.6.3 Combining the Angular and Linear Jacobians

As we have seen in the preceding section, the upper half of the Jacobian J_v is given as

$$J_v = [J_{v_1} \cdots J_{v_n}] \tag{4.76}$$

where the i-th column J_{v_i} is

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases} \tag{4.77}$$

The lower half of the Jacobian is given as

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}] \tag{4.78}$$

where the i-th column J_{ω_i} is

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases} \tag{4.79}$$

Putting the upper and lower halves of the Jacobian together, we the Jacobian for an n-link manipulator is of the form

$$\boxed{J = [J_1 J_2 \cdots J_n]} \tag{4.80}$$

where the i-th column J_i is given by

$$\boxed{J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}} \tag{4.81}$$

if joint i is revolute and

$$\boxed{J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}} \tag{4.82}$$

if joint i is prismatic.

The above formulas make the determination of the Jacobian of any manipulator simple since all of the quantities needed are available once the forward kinematics are worked out. Indeed the only quantities needed to compute the Jacobian are the unit vectors z_i and the coordinates of the origins $o_1, \ldots, o_n$. A moment's reflection shows that the coordinates for z_i with respect to the base frame are given by the first three elements in the third column of T_i^0 while o_i is given by the first three elements of the fourth column of T_i^0. Thus only the third and fourth columns of the T matrices are needed in order to evaluate the Jacobian according to the above formulas.

The above procedure works not only for computing the velocity of the end-effector but also for computing the velocity of any point on the manipulator. This will be important in Chapter 6 when we will need to compute the velocity of the center of mass of the various links in order to derive the dynamic equations of motion.

4.7 EXAMPLES

We now provide a few examples to illustrate the derivation of the manipulator Jacobian.

Example 4.5 Two-Link Planar Manipulator

Consider the two-link planar manipulator of Example 3.1. Since both joints are revolute the Jacobian matrix, which in this case is 6×2, *is of the form*

$$J(q) = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix} \tag{4.83}$$

The various quantities above are easily seen to be

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \tag{4.84}$$

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.85}$$

Performing the required calculations then yields

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{4.86}$$

It is easy to see how the above Jacobian compares with Equation (1.1) derived in Chapter 1. The first two rows of Equation (4.85) are exactly the 2×2 *Jacobian of Chapter 1 and give the linear velocity of the origin* o_2 *relative to the base. The third row in Equation (4.86) is the linear velocity in the direction of* z_0*, which is of course always zero in this case. The last three rows represent the angular velocity of the final frame, which is simply a rotation about the vertical axis at the rate* $\dot{\theta}_1 + \dot{\theta}_2$*.*

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Example 4.6 Jacobian for an Arbitrary Point

Consider the three-link planar manipulator of Figure 4.2. Suppose we wish

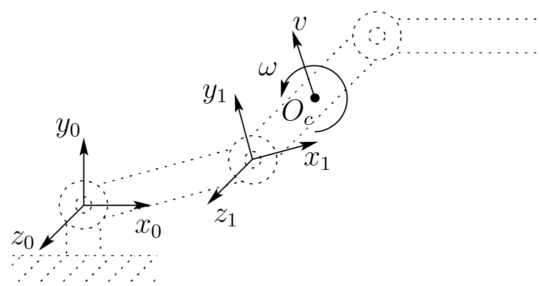

Fig. 4.2 Finding the velocity of link 2 of a 3-link planar robot.

to compute the linear velocity v *and the angular velocity* ω *of the center of link 2 as shown. In this case we have that*

$$J(q) = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix} \tag{4.87}$$

which is merely the usual the Jacobian with o_c *in place of* o_n*. Note that the third column of the Jacobin is zero, since the velocity of the second link is unaffected by motion of the third link*[2]*. Note that in this case the vector* o_c *must be computed as it is not given directly by the* T *matrices (Problem 4-13).*

[2]Note that we are treating only kinematic effects here. Reaction forces on link 2 due to the motion of link 3 will influence the motion of link 2. These dynamic effects are treated by the methods of Chapter 6.

◇

Example 4.7 Stanford Manipulator

Consider the Stanford manipulator of Example 3.5 with its associated Denavit-Hartenberg coordinate frames. Note that joint 3 is prismatic and that $o_3 = o_4 = o_5$ *as a consequence of the spherical wrist and the frame assignment. Denoting this common origin by* o *we see that the columns of the Jacobian have the form*

$$\begin{aligned} J_i &= \begin{bmatrix} z_{i-1} \times (o_6 - o_{i-1}) \\ z_{i-1} \end{bmatrix} \quad i = 1, 2 \\ J_3 &= \begin{bmatrix} z_2 \\ 0 \end{bmatrix} \\ J_i &= \begin{bmatrix} z_{i-1} \times (o_6 - o) \\ z_{i-1} \end{bmatrix} \quad i = 4, 5, 6 \end{aligned}$$

Now, using the A-matrices given by Equations (3.18)-(3.23) and the T-matrices formed as products of the A-matrices, these quantities are easily computed as follows: First, o_j *is given by the first three entries of the last column of* $T_j^0 = A_1 \cdots A_j$*, with* $o_0 = (0, 0, 0)^T = o_1$*. The vector* z_j *is given as*

$$z_j = R_j^0 k \tag{4.88}$$

where R_j^0 *is the rotational part of* T_j^0*. Thus it is only necessary to compute the matrices* T_j^0 *to calculate the Jacobian. Carrying out these calculations one obtains the following expressions for the Stanford manipulator:*

$$o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 - c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix} \tag{4.89}$$

$$o_3 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix} \tag{4.90}$$

The z_i are given as

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \tag{4.91}$$

$$z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \qquad z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \tag{4.92}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \tag{4.93}$$

$$z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}. \tag{4.94}$$

The Jacobian of the Stanford Manipulator is now given by combining these expressions according to the given formulae (Problem 4-19).
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Example 4.8 SCARA Manipulator

We will now derive the Jacobian of the SCARA manipulator of Example 3.6. This Jacobian is a 6×4 matrix since the SCARA has only four degrees-of-freedom. As before we need only compute the matrices $T_j^0 = A_1 \ldots A_j$, where the A-matrices are given by Equations (3.26)-(3.29).

Since joints 1,2, and 4 are revolute and joint 3 is prismatic, and since $o_4 - o_3$ is parallel to z_3 (and thus, $z_3 \times (o_4 - o_3) = 0$), the Jacobian is of the form

$$J = \begin{bmatrix} z_0 \times (o_4 - o_0) & z_1 \times (o_4 - o_1) & z_2 & 0 \\ z_0 & z_1 & 0 & z_3 \end{bmatrix} \tag{4.95}$$

Performing the indicated calculations, one obtains

$$o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \tag{4.96}$$

$$o_4 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_2 + a_2 s_{12} \\ d_3 - d_4 \end{bmatrix} \tag{4.97}$$

Similarly $z_0 = z_1 = k$, and $z_2 = z_3 = -k$. Therefore the Jacobian of the SCARA Manipulator is

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \tag{4.98}$$