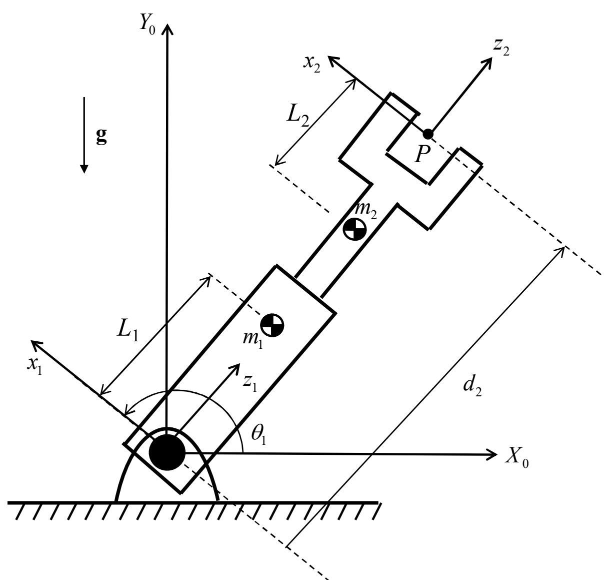
HW CH6 Solution

Craig 4th ed. Prob.: 6.15, 6.16*, 6.20*
*6.16 and 6.20: Derive using two different methods: (1) Lagrangian and (2) Newton-Euler, all in standard DH convention, and compare the results. Show that the two resulting dynamic equations are identical. Also, identify any Coriolis term(s), if exist(s), in the results.

6.15) The overall framework of the solution procedure is similar to that given in Craig's Section 6.7, except that joint 2 is now prismatic. Also, the frame numbers and dimensions should be consistent with the standard DH convention as given in the lecture note for Example 6.5. Use the accelerations derived in the lecture notes. Check your results with the answers from Example 6.5.



Inertia tensors:

$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix} \text{ and } {}^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

The center of mass of Link 1 and 2 is located at a distance L_1 and L_2 from the origin of link frame {1} and {2}, respectively.

Recall that Frame $\{C_i\}$ has its origin at the link mass center, and has the same orientation as the link frame $\{i\}$. Thus, I_{yyi} (i = 1, 2) should be used for the axis that is perpendicular to the plane, as seen from the attached link frames. (Note: the textbook uses I_{zzi}, which is not consistent with the standard DH convention.)

- Angular acceleration of link i+1 with respect to frame {i+1}
  - For revolute joint i+1: $^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i({}^i\dot{\omega}_i + \dot{\theta}_{i+1}\,{}^i\omega_i \times {}^i\hat{Z}_i + \ddot{\theta}_{i+1}\,{}^i\hat{Z}_i)$
  - For prismatic joint i+1: $^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\,{}^i\dot{\omega}_i$

For Link 1 (i = 0 in the above equation):

$${}^{1}\dot{\omega}_1 = {}^{1}R_0(\ddot{\theta}_1\,{}^{0}\hat{Z}_0)$$

$${}^{1}R_0 = {}^{0}R_1^T = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix} \Rightarrow {}^{1}\dot{\omega}_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix}$$

For Link 2 (i = 1 in the above equation):

$${}^{2}\dot{\omega}_2 = {}^{2}R_1\,{}^{1}\dot{\omega}_1$$

$${}^{2}R_1 = {}^{1}R_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{2}\dot{\omega}_2 = \begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix}$$

- Linear acceleration of origin of frame $\{i+1\}$ with respect to frame $\{i+1\}$

  For prismatic joint $i+1$:

  $${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^{i}\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}\hat{Z}_i$$

  For revolute joint $i+1$: ${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1})$

For Link 1 (i = 0 in the above equation):

$${}^{1}\dot{v}_1 = {}^{1}\dot{\omega}_1 \times {}^{1}R_0\,{}^{0}P_1 + {}^{1}\omega_1 \times ({}^{1}\omega_1 \times {}^{1}R_0\,{}^{0}P_1)$$

$${}^{0}P_1 = \mathbf{0} \Rightarrow {}^{1}\dot{v}_1 = \mathbf{0}$$

For Link 2 (i = 1 in the above equation):

$${}^{2}\dot{v}_2 = {}^{2}R_1\,{}^{1}\dot{v}_1 + {}^{2}\dot{\omega}_2 \times {}^{2}R_1\,{}^{1}P_2 + {}^{2}\omega_2 \times ({}^{2}\omega_2 \times {}^{2}R_1\,{}^{1}P_2) + \ddot{d}_2\,{}^{2}R_1\,{}^{1}\hat{Z}_1 + 2\dot{d}_2\,{}^{2}\omega_2 \times {}^{2}R_1\,{}^{1}\hat{Z}_1$$

$${}^{2}R_1\,{}^{1}P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}$$

$${}^{2}\dot{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times (\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}) + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_2 \end{bmatrix} + 2\dot{d}_2 \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow {}^{2}\dot{v}_2 = \begin{bmatrix} d_2\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 \\ 0 \\ -d_2\dot{\theta}_1^2 + \ddot{d}_2 \end{bmatrix}$$

- Linear acceleration of COM frame $\{C_i\}$ origin of link i (for both revolute and prismatic joint $i+1$):

  $${}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{C_i}) + {}^{i}\dot{v}_i$$

For Link 1 (i = 1 in the above equation):

$${}^{1}\dot{v}_{C_1} = {}^{1}\dot{\omega}_1 \times {}^{1}P_{C_1} + {}^{1}\omega_1 \times ({}^{1}\omega_1 \times {}^{1}P_{C_1}) + {}^{1}\dot{v}_1$$

$${}^{1}\dot{v}_{C_1} = \begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times (\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1\ddot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow {}^{1}\dot{v}_{C_1} = \begin{bmatrix} L_1\ddot{\theta}_1 \\ 0 \\ -L_1\dot{\theta}_1^2 \end{bmatrix}$$

For Link 2 (i = 2 in the above equation):

$$ {}^2\dot v_{C_2} = {}^2\dot\omega_2 \times {}^2P_{C_2} + {}^2\omega_2 \times ({}^2\omega_2 \times {}^2P_{C_2}) + {}^2\dot v_2 $$

$$ {}^2\dot v_{C_2} = \begin{bmatrix} 0 \\ \ddot\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -L_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} \times (\begin{bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -L_2 \end{bmatrix}) + \begin{bmatrix} d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1 \\ 0 \\ -d_2\dot\theta_1^2 + \ddot d_2 \end{bmatrix} \quad => \quad {}^2\dot v_{C_2} = \begin{bmatrix} -L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1 \\ 0 \\ L_2\dot\theta_1^2 - d_2\dot\theta_1^2 + \ddot d_2 \end{bmatrix} $$

<u>Newton-Euler Equations at Link i</u>

| Frame $\{C_i\}$ attached to link i with its origin at the link COM and has same orientation as $\{i\}$

$$ \sum f = F_i = m\dot v_{C_i} \quad \& \quad \sum n = N_i = {}^{C_i}I\dot\omega_i + \omega_i \times {}^{C_i}I\omega_i $$

- From higher to lower numbered:

$$ {}^if_i = {}^iR_{i+1}\,{}^{i+1}f_{i+1} - m_i\,{}^iR_0\,{}^0\mathbf{g} - \sum_j {}^iR_0\,{}^0f_j^{ext} + {}^iF_i $$

$$ {}^in_i = {}^iR_{i+1}\,{}^{i+1}n_{i+1} - ({}^iP_{i-1} - {}^iP_{c_i}) \times {}^iF_i - {}^iP_{i-1} \times {}^iR_{i+1}\,{}^{i+1}f_{i+1} + ({}^iP_{i-1} - {}^iP_{c_i}) \times m_i\,{}^iR_0\,{}^0\mathbf{g} $$
$$ - \sum_j [({}^iP_j - {}^iP_{i-1}) \times {}^iR_0\,{}^0f_j^{ext}] - \sum_k {}^iR_0\,{}^0n_k^{ext} + {}^iN_{i,C_i} $$

For Link 2 ($i = 2$ in the above equation):

${}^2f_2 = -m_2\,{}^2R_0\,{}^0\mathbf{g} + {}^2F_2$

$$ {}^2f_2 = -m_2 \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} -L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1 \\ 0 \\ L_2\dot\theta_1^2 - d_2\dot\theta_1^2 + \ddot d_2 \end{bmatrix} $$

$$ => \quad {}^2f_2 = m_2 \begin{bmatrix} g\sin\theta_1 - L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1 \\ 0 \\ -g\cos\theta_1 + L_2\dot\theta_1^2 - d_2\dot\theta_1^2 + \ddot d_2 \end{bmatrix} $$

$$ {}^2n_2 = -({}^2P_1 - {}^2P_{c_2}) \times {}^2F_2 + ({}^2P_1 - {}^2P_{c_2}) \times m_2\,{}^2R_0\,{}^0\mathbf{g} + {}^2N_{2,C_2} $$

$$ {}^2n_2 = -(\begin{bmatrix} 0 \\ 0 \\ -d_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -L_2 \end{bmatrix}) \times m_2 \begin{bmatrix} -L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1 \\ 0 \\ L_2\dot\theta_1^2 - d_2\dot\theta_1^2 + \ddot d_2 \end{bmatrix} + (\begin{bmatrix} 0 \\ 0 \\ -d_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -L_2 \end{bmatrix}) \times m_2 \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} $$
$$ + \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix} \begin{bmatrix} 0 \\ \ddot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix} \begin{bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} $$
$$ = -m_2 \begin{bmatrix} 0 \\ (L_2 - d_2)(-L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1) \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ 0 \\ L_2 - d_2 \end{bmatrix} \times \begin{bmatrix} -g\sin\theta_1 \\ 0 \\ g\cos\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{yy2}\ddot\theta_1 \\ 0 \end{bmatrix} $$

$$ => \quad {}^2n_2 = \begin{bmatrix} 0 \\ -m_2(L_2 - d_2)(-L_2\ddot\theta_1 + d_2\ddot\theta_1 + 2\dot d_2\dot\theta_1) - m_2g(L_2 - d_2)\sin\theta_1 + I_{yy2}\ddot\theta_1 \\ 0 \end{bmatrix} $$

For Link 1 (i = 1 in the above equation):
${}^1f_1 = {}^1R_2{}^2f_2 - m_1{}^1R_0{}^0\mathbf{g} + {}^1F_1$

$$ {}^1f_1 = m_2\begin{bmatrix} g\sin\theta_1 - L_2\ddot{\theta}_1 + d_2\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 \\ 0 \\ -g\cos\theta_1 + L_2\dot{\theta}_1^2 - d_2\dot{\theta}_1^2 + \ddot{d}_2 \end{bmatrix} - m_1\begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + m_1\begin{bmatrix} L_1\ddot{\theta}_1 \\ 0 \\ -L_1\dot{\theta}_1^2 \end{bmatrix} $$

$$ \Rightarrow \quad {}^1f_1 = \begin{bmatrix} m_2(g\sin\theta_1 - L_2\ddot{\theta}_1 + d_2\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1) + m_1(g\sin\theta_1 + L_1\ddot{\theta}_1) \\ 0 \\ m_2(-g\cos\theta_1 + L_2\dot{\theta}_1^2 - d_2\dot{\theta}_1^2 + \ddot{d}_2) + m_1(-g\cos\theta_1 - L_1\dot{\theta}_1^2) \end{bmatrix} $$

$$ {}^1n_1 = {}^1R_2\,{}^2n_2 - ({}^1P_0 - {}^1P_{c_1}) \times {}^1F_1 - {}^1P_0 \times {}^1R_2\,{}^2f_2 + ({}^1P_0 - {}^1P_{c_1}) \times m_1\,{}^1R_0\,{}^0\mathbf{g} + {}^1N_{1,C_1} $$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ -m_2(L_2 - d_2)(-L_2\ddot{\theta}_1 + d_2\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1) - m_2g(L_2 - d_2)\sin\theta_1 + I_{yy2}\ddot{\theta}_1 \\ 0 \end{bmatrix} - (\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}) \times m_1\begin{bmatrix} L_1\ddot{\theta}_1 \\ 0 \\ -L_1\dot{\theta}_1^2 \end{bmatrix} $$

$$ -\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times m_2\begin{bmatrix} g\sin\theta_1 - L_2\ddot{\theta}_1 + d_2\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 \\ 0 \\ -g\cos\theta_1 + L_2\dot{\theta}_1^2 - d_2\dot{\theta}_1^2 + \ddot{d}_2 \end{bmatrix} + (\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}) \times m_1\begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} $$

$$ +\begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}\begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ m_2(L_2 - d_2)(L_2\ddot{\theta}_1 - d_2\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1) - m_2g(L_2 - d_2)\sin\theta_1 + I_{yy2}\ddot{\theta}_1 \\ 0 \end{bmatrix} + m_1\begin{bmatrix} 0 \\ L_1^2\ddot{\theta}_1 \\ 0 \end{bmatrix} + m_1\begin{bmatrix} 0 \\ L_1g\sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{yy1}\ddot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ \Rightarrow \quad {}^1n_1 = \begin{bmatrix} 0 \\ \{m_1L_1^2 + I_{yy1} + I_{yy2} + m_2(d_2 - L_2)^2\}\ddot{\theta}_1 + 2m_2(d_2 - L_2)\dot{d}_2\dot{\theta}_1 + \{m_1L_1 + m_2(d_2 - L_2)\}g\sin\theta_1 \\ 0 \end{bmatrix} $$

- Required actuation: $\tau_i = {}^in_i^T\,{}^i\hat{Z}_{i-1}$ (for revolute joint i) or $\tau_i = {}^if_i^T\,{}^i\hat{Z}_{i-1}$ (for prismatic joint i)

$$ {}^2\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } {}^1\hat{Z}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

$\tau_2 = {}^2f_2^T\,{}^2\hat{Z}_1 \quad \Rightarrow \quad \tau_2 = m_2\ddot{d}_2 - m_2(d_2 - L_2)\dot{\theta}_1^2 - m_2g\cos\theta_1$

$\tau_1 = {}^1n_1^T\,{}^1\hat{Z}_0 \quad \Rightarrow \quad \tau_1 = \{m_1L_1^2 + I_{yy1} + I_{yy2} + m_2(d_2 - L_2)^2\}\ddot{\theta}_1 + 2m_2(d_2 - L_2)\dot{d}_2\dot{\theta}_1 + \{m_1L_1 + m_2(d_2 - L_2)\}g\sin\theta_1$

6.16)
(1) Using Lagrangian formulation:
$\| v_{C_1} \| = \dot{d}_1$; $\| \omega_1 \| = 0$; $\| v_{C_2} \| = \dot{d}_1$; $\| \omega_2 \| = \dot{\theta}_2$

$$k_1 = \frac{1}{2}m_1(\dot{d}_1)^2;\ k_2 = \frac{1}{2}m_2(\dot{d}_1)^2 + \frac{1}{2}I_{zz2}(\dot{\theta}_2)^2$$

$u_1 = u_2 = 0$

$L = k_1 + k_2 - u_1 - u_2$

$$\frac{\partial L}{\partial \dot{d}_1} = (m_1 + m_2)\dot{d}_1 \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right) = (m_1 + m_2)\ddot{d}_1;\ \frac{\partial L}{\partial d_1} = 0$$

$$f_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right) - \frac{\partial L}{\partial d_1} \quad => \quad \underline{f_1 = (m_1 + m_2)\ddot{d}_1}$$ (Coriolis term does not exist in this equation.)

$$\frac{dL}{d\dot{\theta}_2} = I_{zz2}\dot{\theta}_2 \rightarrow \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) = I_{zz2}\ddot{\theta}_2;\ \frac{dL}{d\theta_2} = 0$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} \quad => \quad \underline{\tau_2 = I_{zz2}\ddot{\theta}_2}$$ (Coriolis term does not exist in this equation.)

(2) Using Newton-Euler formulation:
Similar to the solution of Problem 6.15 (above), except for the order of the joints. The final results for f_1 and τ_2 should be identical to the results from Lagrangian formulation as given above.

6.20)
(1) Using Lagrangian formulation:

$$\mathbf{v}_1 = l_1\dot{\theta}_1\begin{bmatrix}-\sin\theta_1\\ \cos\theta_1\end{bmatrix} \text{ and } \mathbf{v}_2 = \mathbf{v}_1 + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\begin{bmatrix}-\sin(\theta_1+\theta_2)\\ \cos(\theta_1+\theta_2)\end{bmatrix} = \begin{bmatrix}-l_1\dot{\theta}_1\sin\theta_1 - l_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\sin(\theta_1+\theta_2)\\ l_1\dot{\theta}_1\cos\theta_1 + l_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\cos(\theta_1+\theta_2)\end{bmatrix}$$

$$k_1 = \frac{1}{2}m_1\|\mathbf{v}_1\|^2 = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 \text{ and } k_2 = \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + 2l_1l_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\cos\theta_2\right]$$

$$u_1 = m_1gl_1\sin\theta_1 \text{ and } u_2 = m_2g\left(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2)\right)$$

$L = k_1 + k_2 - u_1 - u_2$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1l_1^2\dot{\theta}_1 + m_2l_1^2\dot{\theta}_1 + m_2l_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right) + m_2l_1l_2\left(2\dot{\theta}_1+\dot{\theta}_2\right)\cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1l_1^2\ddot{\theta}_1 + m_2l_1^2\ddot{\theta}_1 + m_2l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2l_1l_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right)\cos\theta_2 - m_2l_1l_2\left(2\dot{\theta}_1+\dot{\theta}_2\right)\dot{\theta}_2\sin\theta_2$$

$$\frac{\partial L}{\partial \theta_1} = -m_1gl_1\cos\theta_1 - m_2g\left(l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)\right)$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \quad => \quad \begin{aligned}&\tau_1 = m_2l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2l_1l_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right)\cos\theta_2 + (m_1+m_2)l_1^2\ddot{\theta}_1 - m_2l_1l_2\dot{\theta}_2^2\sin\theta_2\\ &\underline{-2m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2 + m_2l_2g\cos(\theta_1+\theta_2) + (m_1+m_2)l_1g\cos\theta_1}\end{aligned}$$

(Coriolis term: $\underline{-2m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2}$)

$$\frac{dL}{d\dot{\theta}_2} = m_2 l_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2 l_1 l_2 \dot{\theta}_1 \cos\theta_2$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) = m_2 l_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

$$\frac{dL}{d\theta_2} = -m_2 l_1 l_2 \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\theta_2 - m_2 l_2 g\cos\left(\theta_1 + \theta_2\right)$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} \quad => \quad \underline{\tau_2 = m_2 l_1 l_2 \ddot{\theta}_1 \cos\theta_2 + m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 + m_2 l_2 g\cos(\theta_1 + \theta_2) + m_2 l_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)}$$

(Coriolis term does not exist in this equation.)

(2) Using Newton-Euler formulation:
Similar to the solution of Problem 6.15 (above), except that now both joints are revolute joints. The final results for τ_1 and τ_2 should be identical to the results from Lagrangian formulation as given above.