Mathematics for Robotics (ROB-GY 6013) **Section A**

Formula Sheet

$q \to p$ (converse of $p \to q$) $\sim q \to \sim p$ (contrapositive of $p \to q$)

A **field** consists of a set, denoted by $\mathcal{F}$, of elements called **scalars** and two operations called addition “+” and multiplication “·”; the two operations are defined over $\mathcal{F}$ such that they satisfy the following conditions:

1. To every pair of elements α and β in $\mathcal{F}$, there correspond an element $\alpha + \beta$ in $\mathcal{F}$, called the sum of α and β, and an element $\alpha \cdot \beta$ (or simply $\alpha\beta$) in $\mathcal{F}$ called the product of α and β.
2. Addition and multiplication are respectively commutative: For any α and β in $\mathcal{F}$,

   $$\alpha + \beta = \beta + \alpha \qquad\qquad \alpha \cdot \beta = \beta \cdot \alpha$$

3. Addition and multiplication are respectively associative: For any α, β, γ in $\mathcal{F}$,

   $$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \qquad\qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

4. Multiplication is distributive with respect to addition: For any α, β, γ in $\mathcal{F}$,

   $$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

5. $\mathcal{F}$ contains an element, denoted by 0, and an element, denoted by 1, such that $\alpha + 0 = \alpha$ and $1 \cdot \alpha = \alpha$ for every α in $\mathcal{F}$.
6. To every α in $\mathcal{F}$, there is an element β in $\mathcal{F}$ such that $\alpha + \beta = 0$. The element β is called the *additive inverse*.
7. To every α in $\mathcal{F}$ which is not the element 0, there is an element γ in $\mathcal{F}$ such that $\alpha \cdot \gamma = 1$. The element γ is called the *multiplicative inverse*.

A **vector space** (or, **linear space**) over a field $\mathcal{F}$, denoted by $(\mathcal{X}, \mathcal{F})$, consists of a set, denoted by $\mathcal{X}$, of elements called **vectors**, a field $\mathcal{F}$, and two operations called **vector addition** and **scalar multiplication**. The two operations are defined over $\mathcal{X}$ and $\mathcal{F}$ such that they satisfy all the following conditions:

1. To every pair of vectors v^1 and v^2 in $\mathcal{X}$, there corresponds a vector $v^1 + v^2$ in $\mathcal{X}$, called the sum of v^1 and v^2.
2. Addition is commutative: For any v^1, v^2 in $\mathcal{X}$, $v^1 + v^2 = v^2 + v^1$.
3. Addition is associative: For any v^1, v^2, and v^3 in $\mathcal{X}$, $(v^1 + v^2) + v^3 = v^1 + (v^2 + v^3)$
4. $\mathcal{X}$ contains a vector, denoted by $\mathbf{0}$, such that $\mathbf{0} + v = v$ for every v in $\mathcal{X}$. The vector $\mathbf{0}$ is called the zero vector or the origin.
5. To every v in $\mathcal{X}$, there is a vector $\overline{v}$ in $\mathcal{X}$, such that $v + \overline{v} = 0$.
6. To every α in $\mathcal{F}$, and every v in $\mathcal{X}$, there corresponds a vector $\alpha \cdot v$ in $\mathcal{X}$ called the scalar product of α and v.
7. Scalar multiplication is associative: For any α, β in $\mathcal{F}$ and any v in $\mathcal{X}$, $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$.
8. Scalar multiplication is distributive with respect to vector addition: For any α in $\mathcal{F}$ and any v^1, v^2 in $\mathcal{X}$, $\alpha \cdot (v^1 + v^2) = \alpha \cdot v^1 + \alpha \cdot v^2$.
9. Scalar multiplication is distributive with respect to scalar addition: For any α, β in $\mathcal{F}$ and any v in $\mathcal{X}$, $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.
10. For any v in $\mathcal{X}$, $1 \cdot v = v$, where 1 is the element 1 in $\mathcal{F}$.

Let $(\mathcal{X},\mathcal{F})$ be a vector space, and let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Then $\mathcal{Y}$ is a **subspace** if using the rules of vector addition and scalar multiplication defined in $(\mathcal{X},\mathcal{F})$, we have that $(\mathcal{Y},\mathcal{F})$ is a vector space.

The following are equivalent:

a) $(\mathcal{Y},\mathcal{F})$ is a subspace of $(\mathcal{X},\mathcal{F})$.
b) $\forall v^1, v^2 \in \mathcal{Y}$, $v^1 + v^2 \in \mathcal{Y}$ (closed under vector addition) and
   $\forall y \in \mathcal{Y}$, $\forall \alpha \in \mathcal{F}$, $\alpha y \in \mathcal{Y}$ (closed under scalar multiplication)
c) $\forall v^1, v^2 \in \mathcal{Y}$, $\forall \alpha \in \mathcal{F}$, $\alpha \cdot v^1 + v^2 \in \mathcal{Y}$
d) $\forall v^1, v^2 \in \mathcal{Y}$, $\forall \alpha_1, \alpha_2 \in \mathcal{F}$, $\alpha_1 \cdot v^1 + \alpha_2 \cdot v^2 \in \mathcal{Y}$

A **linear combination** is a finite sum of the form:

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n \text{ where } n \geq 1,\ \alpha_1 \in \mathcal{F},\ v^i \in \mathcal{X},\ v^i \in \mathcal{X},\ 1 \leq i \leq n$$

$$[x]_v := \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{bmatrix}^T \in \mathcal{F}^n$$

$$[x]_{\bar{u}} = P[x]_u$$

$$[\mathcal{L}(x)]_v = A[x]_u \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathcal{L}} & \mathcal{Y} \\ [x]_u \downarrow & & \downarrow [y]_v \\ \mathcal{F}^m & \xrightarrow{\mathcal{A}} & \mathcal{F}^n \end{array}$$

$$Av = \lambda v \qquad A = M \Lambda M^{-1}$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$