Reading Assignment: Chapters 6 and 7 of Nagy.

1. Nagy, Page 136, Prob. 4.4.3 (Nagy's "ordered bases" are simply called bases in lecture. Finding the components is what we call finding the representation). In addition, find the change of basis matrix P from the basis $\mathcal{S}$ to the basis $\mathcal{Q}$.

2. Calculate by hand a set of e-vectors of the matrix A_3 below (It is noted that the e-values are obvious since the matrix is upper triangular.)

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   Verify that the e-vectors you computed are linearly independent.

   Remark: Check you answers in MATLAB. We noted in lecture that e-vectors are not unique. Do your e-vectors agree with those computed by MATLAB using the `eig` command?

3. For the matrix A_4 below, compute its e-values and e-vectors.

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

   Can you find a basis for $\mathbb{R}^3$ consisting of e-vectors?

4. Two square matrices A and B are said to be **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$. Show that if A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues. **Note:** The characteristic polynomial of a square matrix A is $\det(\lambda I - A)$; the characteristic equation is $\det(\lambda I - A) = 0$.

5. For the matrix A_3 in Prob. 2 above, show that A is similar to a diagonal matrix. You can use results from lecture.

6. Let $\mathcal{F} = \mathbb{R}$ and let $\mathcal{X}$ be the set of 2×2 matrices with real coefficients. Define $L : \mathcal{X} \to \mathcal{X}$ by

$$L(M) = \frac{1}{2}(M + M^\top),$$

   where $M \in \mathcal{X}$ is a 2×2 real matrix.

   (a) Show that L is a linear operator.

   (b) Compute A, the matrix representation of L, when the basis

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   is used on both copies of $\mathcal{X}$ (i.e., on both the domain and codomain of L).

7. Let A be an $n \times n$ matrix with possibly complex coefficients. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ by $L(x) = Ax$. Note that the field is $\mathcal{F} = \mathbb{C}$.

   (a) Compute the matrix representation of L when the "natural" (also called canonical) basis is used in $\mathbb{C}^n$. Call your representation $\hat{A}$ and find its relation to the original matrix A.

   (b) Suppose that the e-values of A are distinct. Compute the matrix representation L with respect to a basis constructed from the e-vectors of A. Call your representation $\hat{A}$.

Hints

Hints: Prob. 1 Recall that the change of basis matrix **from** the basis $\{v^1, \ldots, v^n\}$ **to** the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$ is the $n \times n$ matrix P with i-th column given by

$$P_i = [v^i]_{\tilde{v}},$$

the representation of v^i in the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$, and the change of basis matrix **from** the basis $\{\tilde{v}^1, \ldots, \tilde{v}^n\}$ **to** the basis $\{v^1, \ldots, v^n\}$ is the $n \times n$ matrix $\tilde{P}$ with i-th column given by

$$\tilde{P}_i = [\tilde{v}^i]_v.$$

Moreover, $P\tilde{P} = \tilde{P}P = I$, the $n \times n$ identity matrix. Consequently, you should always compute whichever of the two matrices is easier, and then get the other by matrix inversion in MATLAB.

Hints: Prob. 3 The answer is NO, you cannot find such a basis The matrix is "defective" (check definition of defective matrix on the web), meaning it does not have a full set of e-vectors. To treat such matrices, one has to learn about the Jordan canonical form, a subject we will not cover in ROB 501. The point of the problem is simply that when e-values are not distinct, you may not have enough e-vectors to build a basis. Jordan canonical forms are treated in EECS 560 = ME 564 = AERO 550.

Hints: Prob. 4 Recall that for compatible square matrices A and B, $\det(AB) = \det(A)\det(B)$.

Hints: Prob. 5 Translate the equation $Av^i = \lambda_i v^i$ into an equation involving matrices

$$P = [v^1 \mid \cdots \mid v^n] \quad \text{and} \quad \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$$

where in this problem, $n = 3$. Questions to ask yourself: Is P invertible? And which is correct, $AP = \Lambda P$ or $AP = P\Lambda$? Perhaps neither?

Hints: Prob. 6 A will be 4×4. From lecture, we have a formula for the columns of A.

Hints: Prob. 7 In both parts, you are using the same basis on the domain and the co-domain of L. The point of (a) is that when you view a matrix as linear operator, and write $L(x) = Ax$, you are using the natural basis on $\mathcal{F}^n$. The answer for (b) will show that A is similar to a diagonal matrix, viz Prob. 5, though the perspectivis different.