# CHAPTER 5. JACOBIANS: VELOCITIES AND STATIC FORCES

Part I: Velocities – linear and angular (Sections 5.1 ~ 5.6)
Part II: Jacobians – differential kinematics (Sections 5.7 ~ 5.8)
Part III: Robot statics (Sections 5.9 ~ 5.11)

- Attach a coordinate system (frame) to a body
  → Motion of rigid bodies: motion of **frames** relative to one another

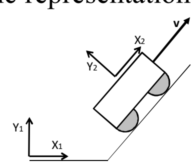
Differentiation of Position Vector (of a Point)

- Derivative of a vector **Q** relative to frame $\{B\}$: ${}^B\mathbf{V}_Q = \frac{d}{dt}{}^B\mathbf{Q} = \lim_{\Delta t \to 0} \frac{{}^B\mathbf{Q}(t+\Delta t) - {}^B\mathbf{Q}(t)}{\Delta t} = {}^B({}^B\mathbf{V}_Q)$
  (Indicate the frame in which the vector is differentiated.)

- A velocity vector is described in terms of a reference frame which is noted with a leading superscript.
  → When expressed in terms of frame $\{A\}$: ${}^A({}^B\mathbf{V}_Q) = \frac{{}^A d}{dt}{}^B\mathbf{Q}$

- Note: Numerical values describing a (linear or translational) velocity vector depend on **two** frames
  Frame (of observer) with respect to which the differentiation is done ($\{B\}$) → vector construction
  Frame (of writer) in which the resulting velocity vector is expressed ($\{A\}$) → vector components

- Dual-superscript notation: **Two** reference frames for description of kinematic vectors (linear position/velocity/acceleration of a point and angular velocity/acceleration of a frame)
  - **Defined** as viewed by an observer fixed in a reference frame: "relative to" or "with respect to" *observer*'s frame → Geometric vector
  - **Resolved** into components with respect to a reference frame: "referred to," "expressed in," or "written in" *writer*'s frame → Algebraic representation of the geometric vector



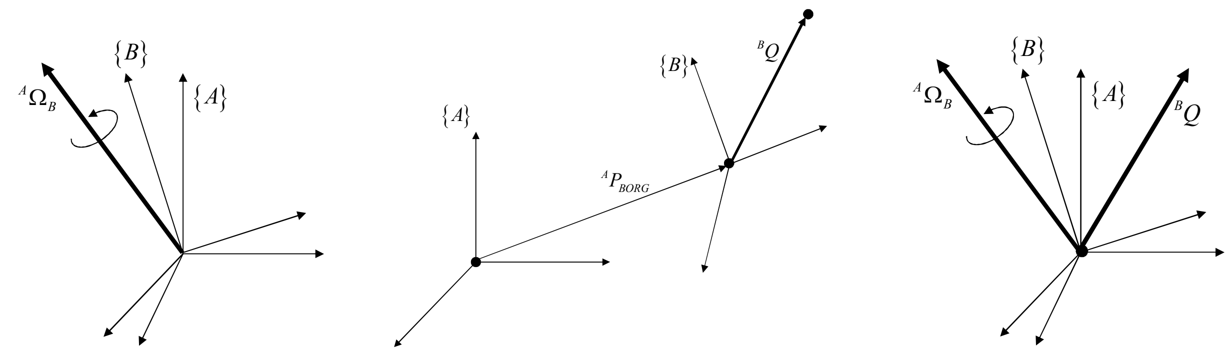

- ${}^B({}^B\mathbf{V}_Q) = {}^B\mathbf{V}_Q$
- $\boxed{{}^A({}^B\mathbf{V}_Q) = {}^A R_B {}^B({}^B\mathbf{V}_Q) = {}^A R_B {}^B\mathbf{V}_Q}$ (use rotation matrix to change the reference frame)
  (Note: ${}^A R_B {}^B\mathbf{V}_Q \neq {}^A\mathbf{V}_Q$)
- Velocity of origin of a frame $\{C\}$ relative to a universe reference frame $\{U\}$: $\boldsymbol{\upsilon}_C = {}^U\mathbf{V}_{CORG}$

- Example 5.1 (Craig's 4th Ed.): (Do it yourself)

Angular Velocity Vector (of a Body)
- Always attach a frame to each rigid body → angular velocity describes rotational motion of a frame

- ${}^A\mathbf{\Omega}_B$ : rotation of frame $\{B\}$ relative to $\{A\}$
  Direction: instantaneous axis of rotation Magnitude: rotation speed

- ${}^{C}({}^{A}\boldsymbol{\Omega}_B)$ : angular velocity of frame $\{B\}$ relative to $\{A\}$ expressed in terms of frame $\{C\}$
- Angular velocity of a frame $\{C\}$ relative to a universe reference frame $\{U\}$: $\boldsymbol{\omega}_C = {}^{U}\boldsymbol{\Omega}_C$




<u>Linear Velocity of Rigid Bodies</u>

- Frame $\{B\}$ attached to a rigid body, and $\{A\}$ is fixed.
- Motion of point $Q$ relative to $\{A\}$: due to ${}^{A}\mathbf{P}_{BORG}$ and ${}^{B}\mathbf{Q}$
- Assume relative orientation of $\{B\}$ and $\{A\}$ is constant.
- Linear velocity (assume constant ${}^{A}R_B$) of point $Q$ in terms of $\{A\}$:
  ${}^{A}({}^{A}\mathbf{V}_Q) = {}^{A}({}^{A}\mathbf{V}_{BORG}) + {}^{A}R_B{}^{B}({}^{B}\mathbf{V}_Q)$ or equivalently, ${}^{A}\mathbf{V}_Q = {}^{A}\mathbf{V}_{BORG} + {}^{A}R_B{}^{B}\mathbf{V}_Q$

<u>Rotational Velocity of Rigid Bodies</u>

- Frames $\{B\}$ and $\{A\}$ with coincident origins (${}^{A}\mathbf{P}_{BORG} = \mathbf{0}$)
- Generally, vector $\mathbf{Q}$ also changes with respect to frame $\{B\}$.
- ${}^{A}\mathbf{V}_Q = \underbrace{{}^{A}({}^{B}\mathbf{V}_Q)}_{wrt\ \{B\}} + \underbrace{{}^{A}\boldsymbol{\Omega}_B \times {}^{A}\mathbf{Q}}_{rotation}$ (from undergraduate dynamics)

  => ${}^{A}\mathbf{V}_Q = {}^{A}R_B{}^{B}\mathbf{V}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}R_B{}^{B}\mathbf{Q}$

  (Note: here and in the textbook, ${}^{A}\mathbf{Q}$ indicates ${}^{A}({}^{B}\mathbf{Q})$, and ${}^{B}\mathbf{Q}$ indicates ${}^{B}({}^{B}\mathbf{Q})$.)

<u>General Linear and Rotational Velocity of Rigid Bodies</u>

- Origins are not coincident
- General velocity of a vector in frame $\{B\}$ as seen from $\{A\}$:
  $\boxed{{}^{A}\mathbf{V}_Q = {}^{A}\mathbf{V}_{BORG} + {}^{A}R_B{}^{B}\mathbf{V}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}R_B{}^{B}\mathbf{Q}}$

<u>Rotation Matrix</u> ( = proper orthonormal matrix)

- $RR^T = I_3$ → $\dot{R}R^T + R\dot{R}^T = 0_3$ => $\dot{R}R^T + (\dot{R}R^T)^T = 0_3$
- **Angular velocity matrix**: $\boxed{S = \dot{R}R^T = \dot{R}R^{-1}}$ → $S + S^T = 0_3$ (____________ matrix)

<u>Rotating Reference Frame</u>

- Fixed vector with respect to frame $\{B\}$: ${}^{B}\mathbf{P}$ → With respect to $\{A\}$: ${}^{A}\mathbf{P} = {}^{A}R_B{}^{B}\mathbf{P}$
- If frame $\{B\}$ rotates → ${}^{A}\mathbf{V}_P = {}^{A}\dot{\mathbf{P}} = {}^{A}\dot{R}_B{}^{B}\mathbf{P} = \underbrace{{}^{A}\dot{R}_B{}^{A}R_B^{-1}}_{={}^{A}S_B}{}^{A}\mathbf{P}$ => ${}^{A}\mathbf{V}_P = {}^{A}S_B{}^{A}\mathbf{P}$

- Let $S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$

- **Angular velocity vector**: $\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$ → describes motion of frame $\{B\}$ with respect to $\{A\}$

  => $S\mathbf{P} = \mathbf{\Omega} \times \mathbf{P}$ for any vector $\mathbf{P}$ => $\therefore\ {}^A\mathbf{V}_P = {}^A\mathbf{\Omega}_B \times {}^A\mathbf{P}$

- $\dot{R} = \lim\limits_{\Delta t \to 0} \dfrac{R(t+\Delta t) - R(t)}{\Delta t}$ and let $R(t+\Delta t) = R_K(\Delta\theta)R(t)$ (why?) => $\dot{R} = \left( \lim\limits_{\Delta t \to 0} \dfrac{R_K(\Delta\theta) - I_3}{\Delta t} \right) R(t)$

- Recall: for ${}^A\hat{\mathbf{K}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ → $R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_y k_x v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_z k_x v\theta - k_y s\theta & k_z k_y v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$

  For $\Delta\theta << 1$ → $R_K(\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix}$

  => $\dot{R} = (\lim\limits_{\Delta t \to 0} \dfrac{\begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 0 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 0 \end{bmatrix}}{\Delta t}) \cdot R(t) = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t)$

  $\therefore\ \dot{R}R^{-1} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$

- $\boxed{\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{\mathbf{K}}}$ (← Definition of angular velocity vector)

  : At any instant the change in orientation of rotating frame is a rotation about **instantaneous axis of rotation** $\hat{\mathbf{K}}$ (unit vector). Speed of rotation $(\dot{\theta})$ is the angular velocity vector's magnitude.

Euler Angle Rates

- Rates of *Z*-*Y*-*Z* Euler angles: $\dot{\mathbf{\Theta}}_{Z'Y'Z'} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$

- Recall $S = \dot{R}R^T = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \rightarrow \begin{cases} \Omega_x = \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23} \\ \Omega_y = \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33} \\ \Omega_z = \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13} \end{cases}$

  where entries $r_{ij}$ $(i, j = 1, 2, 3)$ are functions of Euler angles, i.e., $r_{ij} = r_{ij}(\alpha, \beta, \gamma)$

  $\rightarrow \dot{r}_{ij} = \frac{d}{dt} r_{ij}(\alpha, \beta, \gamma) = \dot{\alpha}\frac{\partial r_{ij}}{\partial \alpha} + \dot{\beta}\frac{\partial r_{ij}}{\partial \beta} + \dot{\gamma}\frac{\partial r_{ij}}{\partial \gamma}$ $\quad \therefore\ \Omega_x, \Omega_y, \Omega_z$ are ............................ of $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$

- $\mathbf{\Omega} = E_{Z'Y'Z'}(\mathbf{\Theta}_{Z'Y'Z'})\dot{\mathbf{\Theta}}_{Z'Y'Z'}$

  $E_{Z'Y'Z'}(\mathbf{\Theta}_{Z'Y'Z'})$ : Jacobian matrix relating Euler angle rate vector and angular velocity vector

- Example 5.2 (Craig's 4th Ed.): $E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix}$ (use $R_{Z'Y'Z'}$ for derivation)

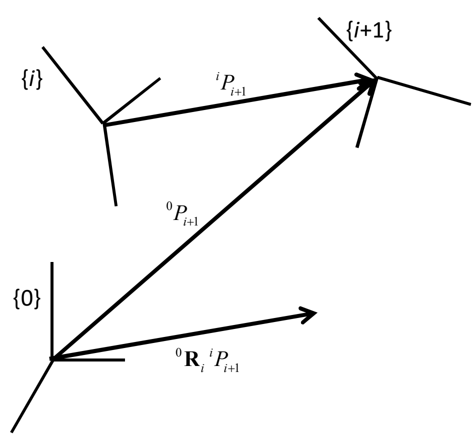
Notation Convention Review

- Generally, three scripts, including two reference frames for dual-superscript notation, are required to describe a kinematic vector **A**: linear position/velocity/acceleration (the subscript indicates a point) or angular velocity/acceleration (the subscript indicates a frame).

$$\boxed{{}^{[\text{expressed in writer's frame}]}\underbrace{\left({}^{[\text{with respect to observer's frame}]}\mathbf{A}_{[\text{describe point or frame of interest}]}\right)}_{\text{a vector is defined}}}$$

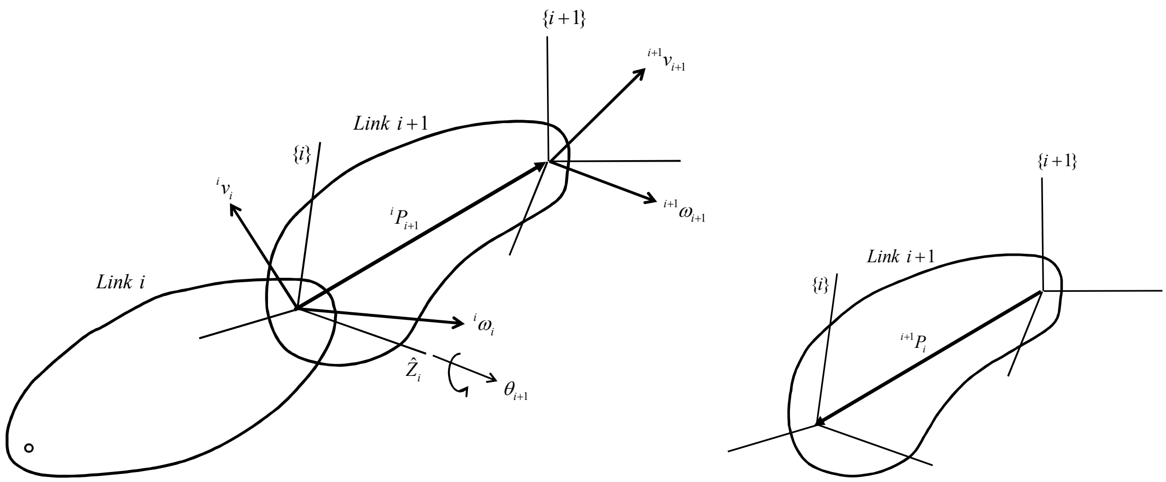
  (Note: The frame of expression for rotation matrix ${}^{[\text{with respect to which frame}]}R_{[\text{describe frame of interest}]}$ is identical to that of the observer.)

- Note: ${}^{0}R_i\,{}^{i}\mathbf{P}_{i+1} \neq {}^{0}\mathbf{P}_{i+1}$ $\quad \therefore$ Even for position vector, all three scripts are required.



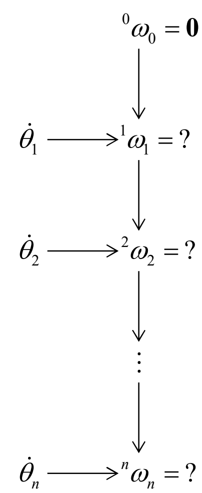

Velocities (absolute) of Links

Reference (or global) frame - link Frame {0}
$v_i$: linear velocity of origin of link Frame $\{i\}$ with respect to (or observer is at) the global frame
$\omega_i$ : angular velocity of link Frame $\{i\}$ with respect to (or observer is at) the global frame




- Each link as a rigid body → linear and angular velocity vectors written in its own link frame (rather than the global frame); note the notations here that ${}^{i}\upsilon_i = {}^{i}({}^{0}\upsilon_i)$ and ${}^{i}\omega_i = {}^{i}({}^{0}\omega_i)$
- [Link $i$+1 velocity] = [Link $i$ velocity] + [relative velocity added by Joint $i$+1]
- Compute the velocities of each link starting from the base (**outward**) → Apply successively from link 0 to link $n$. → ${}^{n}\omega_n$ and ${}^{n}\upsilon_n$.
- Multiply by ${}^{0}R_n$ → expressed in global frame

| Joint \ Velocity | Linear | Angular |
|---|---|---|
| Revolute | ✓ | ✓ |
| Prismatic | ✓ | ✓ |



Link Velocities for Revolute Joint $i$+1

- $\dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i = {}^{i}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$
- Angular velocity of Link $i$+1
  with respect to Frame $\{i\}$: ${}^{i}\omega_{i+1} = {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i$
  with respect to Frame $\{i+1\}$: $\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}R_i({}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i)}$
- Proofs
  1) ${}^{k}({}^{k}\omega_{i+1}) = {}^{k}({}^{k}\omega_i) + \dot{\theta}_{i+1}\,{}^{k}\hat{Z}_i$ => ${}^{i}R_k \times [{}^{k}({}^{k}\omega_{i+1}) = {}^{k}({}^{k}\omega_i) + \dot{\theta}_{i+1}\,{}^{k}\hat{Z}_i]$
  => ${}^{i}({}^{k}\omega_{i+1}) = {}^{i}({}^{k}\omega_i) + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i \quad \therefore\ {}^{i}\omega_{i+1} = {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i$
  2) ${}^{i+1}R_i \times [{}^{i}({}^{k}\omega_{i+1}) = {}^{i}({}^{k}\omega_i) + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i]$ => ${}^{i+1}({}^{k}\omega_{i+1}) = {}^{i+1}R_i[{}^{i}({}^{k}\omega_i) + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i]$
  $\therefore\ {}^{i+1}\omega_{i+1} = {}^{i+1}R_i({}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i}\hat{Z}_i)$
- Linear velocity of origin of Frame $\{i+1\}$
  with respect to Frame $\{i\}$: ${}^{i}\upsilon_{i+1} = {}^{i}\upsilon_i + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1}$
  with respect to Frame $\{i+1\}$: $\boxed{{}^{i+1}\upsilon_{i+1} = {}^{i+1}R_i({}^{i}\upsilon_i + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1})}$

- Proofs

1) In ${}^{A}\mathbf{V}_Q = {}^{A}\mathbf{V}_{BORG} + {}^{A}R_B\,{}^{B}\mathbf{V}_Q + {}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{Q}$ (textbook equation (5.13)), let $\{A\} = \{K\}$, $\{B\} = \{i+1\}$, $\mathbf{Q}$ = origin of $\{i\}$, ${}^{A}P_{BORG} = {}^{K}P_{i+1}$, and ${}^{B}Q = {}^{i+1}P_i$.

$${}^{K}({}^{K}v_i) = {}^{K}({}^{K}v_{i+1}) + {}^{K}R_{i+1}\,\cancel{{}^{i+1}({}^{i+1}v_i)} + {}^{K}({}^{K}\omega_{i+1}) \times {}^{K}R_{i+1}\,{}^{i+1}P_i$$

$${}^{i}R_K \times [{}^{K}({}^{K}v_i) = {}^{K}({}^{K}v_{i+1}) + {}^{K}({}^{K}\omega_{i+1}) \times {}^{K}R_{i+1}\,{}^{i+1}P_i] \Rightarrow {}^{i}({}^{K}v_{i+1}) = {}^{i}({}^{K}v_i) + {}^{i}({}^{K}\omega_{i+1}) \times \underbrace{\left(-{}^{i}R_{i+1}\,{}^{i+1}P_i\right)}_{= {}^{i}P_{i+1} \text{ from (2.44)}}$$

$$\therefore\ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1}$$

2) ${}^{i+1}R_i \times [{}^{i}({}^{K}v_{i+1}) = {}^{i}({}^{K}v_i) + {}^{i}({}^{K}\omega_{i+1}) \times {}^{i}P_{i+1}] \Rightarrow \therefore\ {}^{i+1}v_{i+1} = {}^{i+1}R_i\,{}^{i}v_i + {}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1}$

<u>Link Velocities for Prismatic Joint *i*+1</u>

- Angular velocity of Link $i$+1 with respect to Frame $\{i+1\}$: $\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i}$
- Linear velocity of Frame $\{i+1\}$ origin with respect to Frame $\{i+1\}$:

$$\boxed{{}^{i+1}\upsilon_{i+1} = {}^{i+1}R_i({}^{i}\upsilon_i + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1} + \dot{d}_{i+1}\,{}^{i}\hat{Z}_i)}$$

<u>Link Velocities for Joint *i* (Unified Form)</u>

- In general, if Joint $i$ is:

revolute $\theta_i = \tilde{\theta}_i + q_i \rightarrow \dot{\theta}_i = \dot{q}_i$ and $\dot{d}_i = 0$

prismatic $d_i = \tilde{d}_i + q_i \rightarrow \dot{d}_i = \dot{q}_i$ and $\dot{\theta}_i = 0$

Therefore, regardless of the joint type (revolute or prismatic),

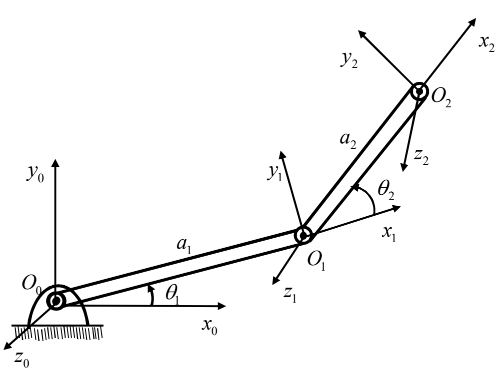
angular velocity of Link $i$: $\omega_i = \omega_{i-1} + \dot{\theta}_i \hat{Z}_{i-1}$

linear velocity of origin of Frame $\{i\}$: $\upsilon_i = \upsilon_{i-1} + \omega_i \times {}^{i-1}P_i + \dot{d}_i \hat{Z}_{i-1}$

(For simplicity, the frames of expression are omitted in the notations.)

<u>Example 5.3</u> (with standard DH convention)

A two-link manipulator with rotational joints is shown in the figure below. Calculate the (absolute linear) velocity of the tip (i.e., the origin of Frame {2}) of the arm as a function of joint rates (i.e., joint velocities). Give the answer in two forms—in terms of (i.e., written in) Frame {2} and Frame {0}.




Solution) Two different methods—with and without using the iterative formulas—are available.

Method 1 (using the iterative formulas):

$$ {}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad {}^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad {}^0T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Use ${}^{i+1}\omega_{i+1} = {}^{i+1}R_i({}^i\omega_i + \dot{\theta}_{i+1}\,{}^i\hat{Z}_i)$ and ${}^{i+1}\upsilon_{i+1} = {}^{i+1}R_i({}^i\upsilon_i + {}^i\omega_{i+1} \times {}^iP_{i+1})$ sequentially from link to link to compute the velocity of the origin of each frame, starting from the base frame {0}, which has zero velocity:

$$ {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; {}^1\omega_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \; {}^2\omega_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and since } {}^0\omega_1 = {}^0R_1\,{}^1\omega_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \text{ and } {}^0P_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix}, $$

$$ {}^1v_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ a_1\dot{\theta}_1 \\ 0 \end{bmatrix}. $$

$$ \text{Likewise, since } {}^1\omega_2 = {}^1R_2\,{}^2\omega_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \text{ and } {}^1P_2 = \begin{bmatrix} a_2c_2 \\ a_2s_2 \\ 0 \end{bmatrix}, $$

$$ {}^2v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 0 \\ a_1\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} a_2c_2 \\ a_2s_2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} a_1\dot{\theta}_1s_2 \\ a_1\dot{\theta}_1c_2 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}. \text{ (Ans.)} $$

To find these velocities with respect to the nonmoving base frame, we rotate them with the rotation matrix as follows:

$$ {}^0v_2 = {}^0R_2\,{}^2v_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1\dot{\theta}_1s_2 \\ a_1\dot{\theta}_1c_2 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} = \begin{bmatrix} -a_1\dot{\theta}_1s_1 - a_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} \\ a_1\dot{\theta}_1c_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \\ 0 \end{bmatrix} \text{ (Ans.)} $$

Method 2 (without using the iterative formulas):

$$ {}^0P_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix} \rightarrow {}^0v_2 = {}^0\dot{P}_2 = \begin{bmatrix} -a_1\dot{\theta}_1s_1 - a_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} \\ a_1\dot{\theta}_1c_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \\ 0 \end{bmatrix} \text{ (Ans.)} $$

$$ {}^2v_2 = {}^2R_0\,{}^0v_2 = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a_1\dot{\theta}_1s_1 - a_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} \\ a_1\dot{\theta}_1c_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} a_1\dot{\theta}_1s_2 \\ a_1\dot{\theta}_1c_2 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} \text{ (Ans.)} $$

<u>End-Effector Velocities</u>

Angular velocity of end-effector: $\boxed{\omega_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1}}$

Linear velocity of end-effector frame's origin: $\boxed{\upsilon_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1} \times (P_n - P_{i-1}) + \dot{d}_i \hat{Z}_{i-1}}$

- Proofs

<u>Angular velocity</u>. Sum up $\omega_i - \omega_{i-1} = \dot{\theta}_i \hat{Z}_{i-1}$ where $\omega_0 = 0$:

$$
\begin{aligned}
&\cancel{\omega_1} - \omega_0 = \dot{\theta}_1 \hat{Z}_0 \\
&\cancel{\omega_2} - \cancel{\omega_1} = \dot{\theta}_2 \hat{Z}_1 \\
&\quad\vdots \\
&\cancel{\omega_{n-1}} - \cancel{\omega_{n-2}} = \dot{\theta}_{n-1} \hat{Z}_{n-2} \\
+)\ &\omega_n - \cancel{\omega_{n-1}} = \dot{\theta}_n \hat{Z}_{n-1} \\
&\qquad\Downarrow \\
\therefore\ &\omega_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1}
\end{aligned}
$$

<u>Linear velocity</u>. Sum up $\upsilon_i - \upsilon_{i-1} = \omega_i \times {}^{i-1}P_i + \dot{d}_i \hat{Z}_{i-1}$ where $v_0 = 0$:

$$
\begin{aligned}
&\cancel{\upsilon_1} - \upsilon_0 = \omega_1 \times {}^{0}P_1 + \dot{d}_1 \hat{Z}_0 \\
&\cancel{\upsilon_2} - \cancel{\upsilon_1} = \omega_2 \times {}^{1}P_2 + \dot{d}_2 \hat{Z}_1 \\
&\quad\vdots \\
&\cancel{\upsilon_{n-1}} - \cancel{\upsilon_{i-2}} = \omega_{n-1} \times {}^{n-2}P_{n-1} + \dot{d}_{n-1} \hat{Z}_{n-2} \\
+)\ &\upsilon_n - \cancel{\upsilon_{n-1}} = \omega_n \times {}^{n-1}P_n + \dot{d}_n \hat{Z}_{n-1} \\
&\qquad\Downarrow \\
&\upsilon_n = \sum_{i=1}^{n} (\omega_i \times {}^{i-1}P_i + \dot{d}_i \hat{Z}_{i-1})
\end{aligned}
$$

From $\omega_i = \sum_{j=1}^{i} \dot{\theta}_j \hat{Z}_{j-1}$, and a double summation identity $\sum_{i=1}^{n}\sum_{j=1}^{i} a_{i,j} = \sum_{j=1}^{n}\sum_{i=j}^{n} a_{i,j}$, the first term is:

$$
\begin{aligned}
\sum_{i=1}^{n} \omega_i \times {}^{i-1}P_i &= \sum_{i=1}^{n} [\sum_{j=1}^{i} (\dot{\theta}_j \hat{Z}_{j-1}) \times {}^{i-1}P_i] = \sum_{i=1}^{n}\sum_{j=1}^{i} [\dot{\theta}_j \hat{Z}_{j-1} \times {}^{i-1}P_i] \\
&= \sum_{j=1}^{n}\sum_{i=j}^{n} [\dot{\theta}_j \hat{Z}_{j-1} \times {}^{i-1}P_i] = \sum_{j=1}^{n} [\dot{\theta}_j \hat{Z}_{j-1} \times \underbrace{\sum_{i=j}^{n} {}^{i-1}P_i}_{= {}^{j-1}P_n = P_n - P_{j-1}}] = \sum_{j=1}^{n} \dot{\theta}_j \hat{Z}_{j-1} \times (P_n - P_{j-1})
\end{aligned}
$$

$$
\therefore\ \upsilon_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1} \times (P_n - P_{i-1}) + \dot{d}_i \hat{Z}_{i-1}
$$

Jacobian (in general; analytical method)

**Derivative** in multidimensional (vector) space (vs. derivative with respect to scalar variable(s))
**Mapping** (linear) in tangential (velocity) space

- Given $m$ functions with $n$ independent variables

$$y_1 = f_1(x_1, \ldots, x_n),\quad y_2 = f_2(x_1, \ldots, x_n),\quad \vdots \quad y_m = f_m(x_1, \ldots, x_n).$$

or, $\mathbf{Y} = \mathbf{F}(\mathbf{X})$

- Differentials of $y_i$ with respect to $x_j$ (linear combinations)

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_n}\delta x_n,$$
$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_n}\delta x_n,$$
$$\vdots$$
$$\delta y_m = \frac{\partial f_m}{\partial x_1}\delta x_1 + \frac{\partial f_m}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_m}{\partial x_n}\delta x_n.$$

or, $\delta\mathbf{Y} = \dfrac{\partial\mathbf{F}}{\partial\mathbf{X}}\delta\mathbf{X} = J(\mathbf{X})\delta\mathbf{X}$

- $J(\mathbf{X}) = \dfrac{\partial\mathbf{F}}{\partial\mathbf{X}}$: $m$ x $n$ **Jacobian matrix**; time-varying linear transformation

$$\boxed{J(\mathbf{X}) = \frac{\partial\mathbf{F}_{(m\times1)}}{\partial\mathbf{X}_{(n\times1)}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{(m\times n)}}$$

(derivative of a vector with respect to another vector)

- Det($J$): **Jacobian**
- In kinematics, $\delta\mathbf{Y} = J(\mathbf{X})\delta\mathbf{X}$ : infinitesimal (or differential) motion
- In kinematics, $\boxed{\dot{\mathbf{Y}} = J(\mathbf{X})\dot{\mathbf{X}}}$: mapping velocities in **X** to **Y**
- Note: Jacobian for angular velocity cannot be derived directly from analytical method.

Jacobian (in robotics; geometric method)

Directly mapping joint velocities to Cartesian (angular, as well as linear) velocities of end-effector

- Let **q**: $n$-DOF joint variables vector; ${}^0\mathbf{V} = \begin{bmatrix} {}^0\mathbf{v}_{(3\times1)} \\ {}^0\boldsymbol{\omega}_{(3\times1)} \end{bmatrix}_{(6\times1)}$ : Cartesian linear and angular velocity vector

$\boxed{{}^0\mathbf{V}_{(6\times1)} = {}^0J(\mathbf{q})_{(6\times n)}\dot{\mathbf{q}}_{(n\times1)}}$ (differential kinematics)

- Changing Jacobian's frame of reference from $\{B\}$ to $\{A\}$

Given $\begin{bmatrix} {}^B\mathbf{v} \\ {}^B\boldsymbol{\omega} \end{bmatrix} = {}^B\mathbf{V} = {}^BJ(\mathbf{q})\dot{\mathbf{q}}$ ; use $\begin{bmatrix} {}^A\mathbf{v} \\ {}^A\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} {}^AR_B & 0 \\ 0 & {}^AR_B \end{bmatrix} \begin{bmatrix} {}^B\mathbf{v} \\ {}^B\boldsymbol{\omega} \end{bmatrix}$

$\therefore\ {}^AJ(\mathbf{q}) = \left[\begin{array}{c|c} {}^AR_B & 0 \\ \hline 0 & {}^AR_B \end{array}\right] {}^BJ(\mathbf{q})$

- Example: 2-link arm linear Jacobian ${}^0J(\mathbf{q}) = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{bmatrix}$, ${}^2J(\mathbf{q}) = \begin{bmatrix} l_1s_2 & 0 \\ l_1c_2 + l_2 & l_2 \end{bmatrix}$

<u>Jacobian Matrix Computation using Geometric Method</u>

- Partition into 3x1 column vectors: $J_{P,i}(\mathbf{q})$ for position and $J_{O,i}(\mathbf{q})$ for orientation

$$J(\mathbf{q})_{(6\times n)} = [J_1(\mathbf{q})_{(6\times 1)} \mid \cdots \mid J_i(\mathbf{q})_{(6\times 1)} \mid \cdots \mid J_n(\mathbf{q})_{(6\times 1)}] = \left[\begin{array}{c|c|c|c|c} J_{P,1}(\mathbf{q})_{(3\times 1)} & \cdots & J_{P,i}(\mathbf{q})_{(3\times 1)} & \cdots & J_{P,n}(\mathbf{q})_{(3\times 1)} \\ J_{O,1}(\mathbf{q})_{(3\times 1)} & \cdots & J_{O,i}(\mathbf{q})_{(3\times 1)} & \cdots & J_{O,n}(\mathbf{q})_{(3\times 1)} \end{array}\right]$$

$$\mathbf{V}_{(6\times 1)} = J(\mathbf{q})_{(6\times n)}\dot{\mathbf{q}}_{(n\times 1)} \rightarrow \upsilon_n = \sum_{i=1}^{n} \dot{q}_i J_{P,i}(\mathbf{q}) \ \ \& \ \ \omega_n = \sum_{i=1}^{n} \dot{q}_i J_{O,i}(\mathbf{q})$$

$\dot{q}_i J_{P,i}(\mathbf{q})$: contribution of single Joint $i$ velocity to the end-effector frame origin's linear velocity

$\dot{q}_i J_{O,i}(\mathbf{q})$ : contribution of single Joint $i$ velocity to the end-effector frame's angular velocity

$$\boxed{J_i(\mathbf{q})_{(6\times 1)} = \left[\begin{array}{c} J_{P,i}(\mathbf{q})_{(3\times 1)} \\ \hline J_{O,i}(\mathbf{q})_{(3\times 1)} \end{array}\right] = \begin{cases} \left[\begin{array}{c} \hat{Z}_{i-1} \\ \hline \mathbf{0} \end{array}\right] & \leftarrow \text{Prismatic joint } i \\ \left[\begin{array}{c} \hat{Z}_{i-1} \times (P_n - P_{i-1}) \\ \hline \hat{Z}_{i-1} \end{array}\right] & \leftarrow \text{Revolute joint } i \end{cases}}$$

The vectors $\hat{Z}_{i-1}$, $P_n$, and $P_{i-1}$ are all functions of the joint variables. If written in Frame {0}:

$\hat{Z}_{i-1}$: obtained from the third column of ${}^0R_{i-1}(q_1,\ldots,q_{i-1})$ or ${}^0T_{i-1}(q_1,\ldots,q_{i-1})$.

$\rightarrow\ \hat{Z}_{i-1} = {}^0R_{i-1}[0 \ \ 0 \ \ 1]^T$ OR $\left[\begin{array}{c} \hat{Z}_{i-1} \\ \hline 0 \end{array}\right] = {}^0T_{i-1}[0 \ \ 0 \ \ 1 \ \ 0]^T$

$P_n$: obtained from the fourth column of ${}^0T_n(q_1,\ldots,q_n)$. $\rightarrow \left[\begin{array}{c} P_n \\ \hline 1 \end{array}\right] = {}^0T_n[0 \ \ 0 \ \ 0 \ \ 1]^T$

$P_{i-1}$: obtained from the fourth column of ${}^0T_{i-1}(q_1,\ldots,q_{i-1})$. $\rightarrow \left[\begin{array}{c} P_{i-1} \\ \hline 1 \end{array}\right] = {}^0T_{i-1}[0 \ \ 0 \ \ 0 \ \ 1]^T$

- Proofs

(a) $J_{O,i}(\mathbf{q})$: Since $\omega_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1}$ and $\omega_n = \sum_{i=1}^{n} \dot{q}_i J_{O,i}(\mathbf{q})$, $\sum_{i=1}^{n} \dot{q}_i J_{O,i}(\mathbf{q}) = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1}$.

$\therefore\ J_{O,i}(\mathbf{q}) = \hat{Z}_{i-1}$ for revolute joint and $J_{O,i}(\mathbf{q}) = \mathbf{0}$ for prismatic joint.

(b) $J_{P,i}(\mathbf{q})$: Since $\upsilon_n = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1} \times (P_n - P_{i-1}) + \sum_{i=1}^{n} \dot{d}_i \hat{Z}_{i-1}$ and $\upsilon_n = \sum_{i=1}^{n} \dot{q}_i J_{P,i}(\mathbf{q})$,

$$\sum_{i=1}^{n} \dot{q}_i J_{P,i}(\mathbf{q}) = \sum_{i=1}^{n} \dot{\theta}_i \hat{Z}_{i-1} \times (P_n - P_{i-1}) + \sum_{i=1}^{n} \dot{d}_i \hat{Z}_{i-1} .$$

$\therefore$ $J_{P,i}(\mathbf{q}) = \hat{Z}_{i-1} \times (P_n - P_{i-1})$ for revolute joint and $J_{P,i}(\mathbf{q}) = \hat{Z}_{i-1}$ for prismatic joint.

- Kinematic interpretations
  - Assume that all joints, other than Joint $i$, are instantaneously fixed, and thus all the links from Link $i$ to the end-effector can be regarded as a single rigid body.

The contribution of prismatic (allows pure translation) joint velocity to the end-effector frame's

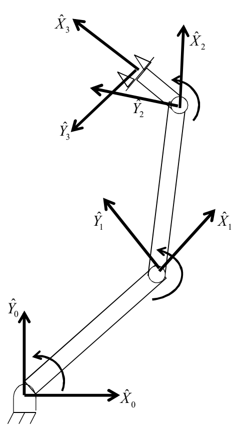
angular velocity: None. $\because$ A rigid body in translation has zero angular velocity.

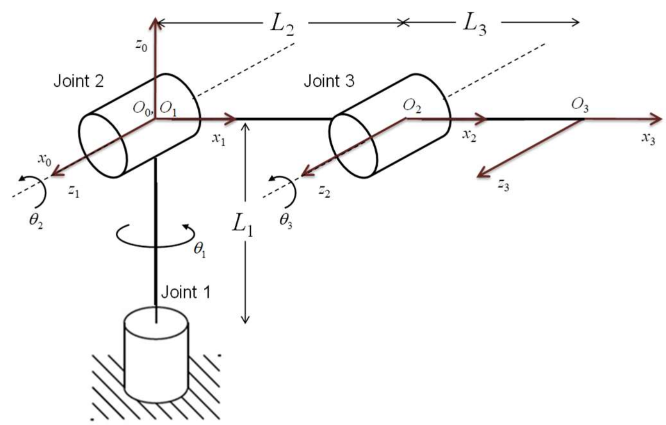
linear velocity (of origin): Vector addition of the prismatic joint velocity. $\because$ All points in the rigid body have same linear velocities (translation).

The contribution of revolute (allows pure rotation) joint velocity to the end-effector frame's

angular velocity: Vector addition of the revolute joint velocity. $\because$ An angular velocity vector (free vector) due to the revolute joint's rotation can be transported to the end-effector frame.

linear velocity (of origin): Rotation of the position vector of the end-effector frame's origin relative to the origin of Joint $i$ axis frame. Note that, unlike the other three cases, this is the only quantity that depends on the end-effector's (relative) position.


Three-link planar arm



Anthropomorphic arm

Example: Three-link Planar Arm

- The position vectors and the joint axes' unit vectors, all written in Frame {0}, are:

$$P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ P_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix},\ P_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix},\ P_3 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}, \text{and } \hat{Z}_0 = \hat{Z}_1 = \hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

where $c_{12} = \cos(\theta_1 + \theta_2)$, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, $s_{12} = \sin(\theta_1 + \theta_2)$, $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$, etc.

$$\therefore\ J = \begin{bmatrix} \hat{Z}_0 \times (P_3 - P_0) & \hat{Z}_1 \times (P_3 - P_1) & \hat{Z}_2 \times (P_3 - P_2) \\ \hat{Z}_0 & \hat{Z}_1 & \hat{Z}_2 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

<u>Example: Anthropomorphic Arm</u> (shown in its home configuration)

| $i$ | $\theta_i$ | $d_i$ | $a_i$ | $\alpha_i$ | Variable |
|---|---|---|---|---|---|
| 1 | $\theta_1 = 90^o + q_1$ | 0 | 0 | 90° | $q_1$ |
| 2 | $\theta_2 = 0 + q_2$ | 0 | $L_2$ | 0 | $q_2$ |
| 3 | $\theta_3 = 0 + q_3$ | 0 | $L_3$ | 0 | $q_3$ |

$${}^0T_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & L_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$${}^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & L_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ {}^0T_2 = {}^0T_1\,{}^1T_2 = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & L_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & L_2s_1c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^0T_3 = {}^0T_2\,{}^2T_3 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & c_1(L_2c_2 + L_3c_{23}) \\ s_1c_{23} & -s_1s_{23} & -c_1 & s_1(L_2c_2 + L_3c_{23}) \\ s_{23} & c_{23} & 0 & L_2s_2 + L_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The position vectors and the joint axes' unit vectors, all written in Frame $\{0\}$, are:

$$P_0 = P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ P_2 = \begin{bmatrix} L_2c_1c_2 \\ L_2s_1c_2 \\ L_2s_2 \end{bmatrix}, \text{ and } P_3 = \begin{bmatrix} c_1(L_2c_2 + L_3c_{23}) \\ s_1(L_2c_2 + L_3c_{23}) \\ L_2s_2 + L_3s_{23} \end{bmatrix};\ \hat{Z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \hat{Z}_1 = \hat{Z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}.$$

$$\therefore\ J = \begin{bmatrix} \hat{Z}_0 \times (P_3 - P_0) & \hat{Z}_1 \times (P_3 - P_1) & \hat{Z}_2 \times (P_3 - P_2) \\ \hat{Z}_0 & \hat{Z}_1 & \hat{Z}_2 \end{bmatrix} = \begin{bmatrix} -s_1(L_2c_2 + L_3c_{23}) & -c_1(L_2s_2 + L_3s_{23}) & -L_3c_1s_{23} \\ c_1(L_2c_2 + L_3c_{23}) & -s_1(L_2s_2 + L_3s_{23}) & -L_3s_1s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(Note: If the global origin is on the ground, the solution will be different and will include $L_1$.)

Singularities

Determinant = 0
In robotics: $Det(J) = 0$ → $J$ loses full rank

- If $J$ is nonsingular, i.e., $Det(J) \neq 0$ → $\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\mathbf{V}$ (differential kinematics → inverse kinematics)
- If $J$ is singular → manipulator loses one or more DOFs in Cartesian space (from implicit function theorem and/or differential geometry theory); it cannot move along some direction(s).
  - Joint rates approach infinity (why?)

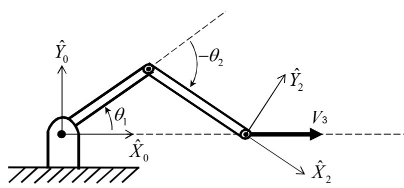
- Workspace singularities
  Workspace boundary singularities: links are fully stretched out or folded back
  Workspace interior singularities: when two or more joint axes are aligned
  → Use to construct workspaces

- Example: 2-link arm

$$Det[{}^0J(\mathbf{q})] = \begin{vmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{vmatrix} = l_1 l_2 s_2 = 0$$

→ singular when $\theta_2$ = 0, 180° (stretched out or folded back) → workspace boundary singularities

- Example 5.5 (Craig's 4th Ed.): Consider a two-link robot moving its end-effector along the $\hat{X}$ axis at 1.0 m/s. Show that as a singularity is approached at $\theta_2 = 0$, joint rates tend to infinity.




Sol) The inverse of the Jacobian written in Frame {0} is ${}^0J^{-1}(\mathbf{q}) = \dfrac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}$.

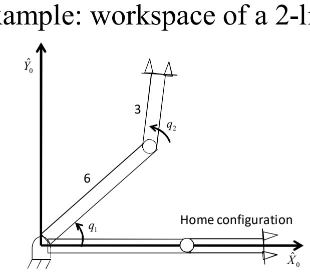
Then using $\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\mathbf{V}$ with $\mathbf{V} = [1, 0]^T$, the joint rates as a function of manipulator configuration is:

$\dot{\theta}_1 = \dfrac{c_{12}}{l_1 s_2}$ and $\dot{\theta}_2 = -\dfrac{c_1}{l_2 s_2} - \dfrac{c_{12}}{l_1 s_2}$ $\therefore$ As $\theta_2 \to 0$ (arm stretches out), $\dot{\theta}_1 \to \infty$ and $\dot{\theta}_2 \to \infty$.

Robot Workspace

The (continuum) set of points in space that can be reached by a point on end-effector.

- Example: workspace of a 2-link arm




| Joint | $\theta$ | $d$ | $a$ | $\alpha$ |
|---|---|---|---|---|
| 1 | $0 + q_1$ | 0 | 6 | 0 |
| 2 | $0 + q_2$ | 0 | 3 | 0 |

${}^0T_n$ → $x = 6\cos q_1 + 3\cos(q_1+q_2)$, $y = 6\sin q_1 + 3\sin(q_1+q_2)$

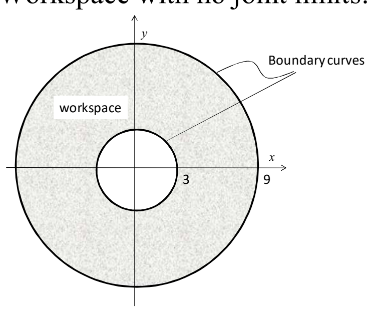
Jacobian $J = \begin{bmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2} \\ \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2} \end{bmatrix}$; $Det(J) = 18\sin q_2 = 0$ → $q_2 = 0, \pi$

Find set(s) of $\mathbf{q}$ that make $J$ singular:

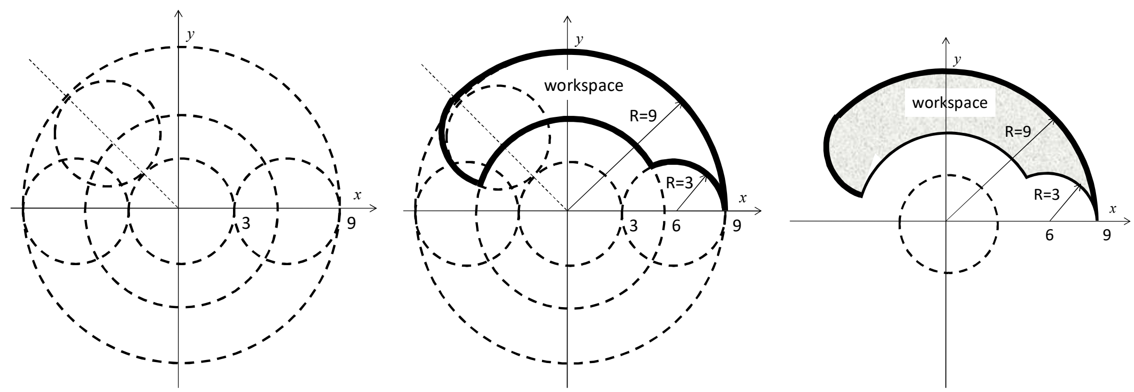
$P_{singular}(q_1, q_2 = 0) = \begin{bmatrix} 6\cos q_1 + 3\cos(q_1 + 0) \\ 6\sin q_1 + 3\sin(q_1 + 0) \end{bmatrix} = \begin{bmatrix} 9\cos q_1 \\ 9\sin q_1 \end{bmatrix}$ → circle of radius 9 and center at the origin

$P_{singular}(q_1, q_2 = \pi) = \begin{bmatrix} 6\cos q_1 + 3\cos(q_1 + \pi) \\ 6\sin q_1 + 3\sin(q_1 + \pi) \end{bmatrix} = \begin{bmatrix} 3\cos q_1 \\ 3\sin q_1 \end{bmatrix}$ → circle of radius 3 and center at the origin

(a) Workspace with no joint limits:



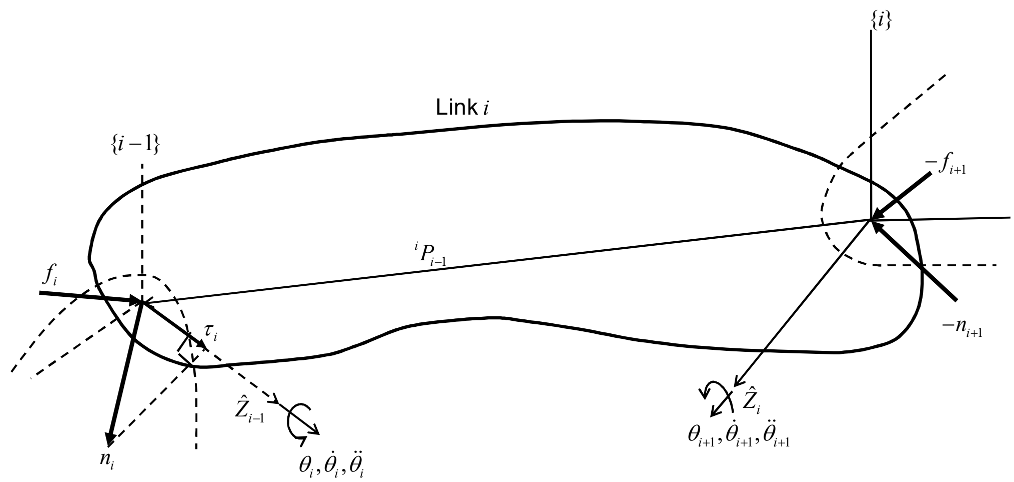

(b) Workspace with joint limits $0 < q_1 < 135$ and $0 < q_2 < 120$:




Static Forces

$f_i$ = force exerted.....Link $i$.....Link $i$-1
$n_i$ = moment exerted.....Link $i$.....Link $i$-1

- Note: In general, a FBD should include all forces/moments exerted "on" the system of interest "by" the environment.




- Static equilibrium
  - Force: $\sum f = 0$ => ${}^{i}f_i - {}^{i}f_{i+1} = 0$ => ${}^{i}f_i = {}^{i}f_{i+1}$
  - Moment **about** origin of Frame $\{i\}$: $\sum n = 0$ => ${}^{i}n_i - {}^{i}n_{i+1} + {}^{i}P_{i-1} \times {}^{i}f_i = 0$ => ${}^{i}n_i = {}^{i}n_{i+1} - {}^{i}P_{i-1} \times {}^{i}f_i$

- Start with a description of the forces and moments applied at the end-effector (Link $n$)
  → calculate from Link $n$ to Link 0 (inward)

- Static force/moment propagation from link to link expressed in each link frame:
  $\boxed{{}^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1},\ {}^{i}n_i = {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} - {}^{i}P_{i-1} \times {}^{i}f_i}$

- All components of the force and moment vectors are resisted by the reaction from the structure of the mechanism itself, except for the force/moment component (actuation) along the joint axis.
- Actuation required to maintain static equilibrium
  - Joint actuator torque (revolute joint $i$): $\tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_{i-1}$
  - Joint actuator force (prismatic joint $i$): $\tau_i = {}^{i}f_i^T\,{}^{i}\hat{Z}_{i-1}$

Jacobians in the Force Domain

- Let
  - $\mathbf{F}$: 6x1 Cartesian force-moment vector applied on the end-effector
  - $\delta\mathbf{X}$: 6x1 infinitesimal Cartesian displacement of the end-effector
  - $\boldsymbol{\tau}$: $n$x1 joint actuator torque vector
  - $\delta\mathbf{q}$: $n$x1 infinitesimal joint variables vector

- **Principle of virtual work** (static equilibrium)
  $\mathbf{F}\bullet\delta\mathbf{X} - \boldsymbol{\tau}\bullet\delta\mathbf{q} = 0$ => $\mathbf{F}^T\delta\mathbf{X} = \boldsymbol{\tau}^T\delta\mathbf{q}$
  → [work done in Cartesian terms] = [work done in joint space terms]
  : Work is the same measured in any set of generalized coordinates

- Recall: Jacobian $\delta\mathbf{X} = J\delta\mathbf{q}$ → $\mathbf{F}^T J\delta\mathbf{q} = \boldsymbol{\tau}^T\delta\mathbf{q}$ ($\forall\ \delta\mathbf{q}$)
  $\therefore$ $\boxed{\boldsymbol{\tau} = J^T\mathbf{F}}$
  : Jacobian transpose maps Cartesian forces/moments into equivalent joint torques

(Note: In the above equation, $\boldsymbol{\tau}$ are the joint torques producing effects that are "equivalent" to those of **F**; on the other hand, the joint torques that are in static "equilibrium" with **F** is $\boldsymbol{\tau} = -J^T\mathbf{F}$.)

- Kineto-statics duality: $\delta\mathbf{X} = J\delta\mathbf{q}$ vs. $\boldsymbol{\tau} = J^T\mathbf{F}$
- If $J$ is singular (i.e., loses full rank) or near singular: **F** can be increased or decreased in null-space basis directions without changes in $\boldsymbol{\tau}$.
  → mechanical advantage goes infinity; small $\boldsymbol{\tau}$ required to generate large forces at the end-effector
- Singular configuration → singularity in both position and force domains

<u>Cartesian Transformation of Velocities and Static Forces</u>

- 6x1 general velocity of a body: $\mathbf{V} = \begin{bmatrix} \mathbf{v}_{(3\times1)} \\ \boldsymbol{\omega}_{(3\times1)} \end{bmatrix}$
- 6x1 general force vector: $\mathbf{F} = \begin{bmatrix} f_{(3\times1)} \\ n_{(3\times1)} \end{bmatrix}$ ($f$: 3x1 force vector; $n$: 3x1 moment vector)
- 6x6 transformations to map from Frame $\{A\}$ to $\{B\}$ at each time instant
- Velocity transformation
  Recall: ${}^{i+1}\omega_{i+1} = {}^{i+1}R_i({}^i\omega_i + \dot{\theta}_{i+1}\,{}^i\hat{Z}_i)$ and ${}^{i+1}\upsilon_{i+1} = {}^{i+1}R_i({}^i\upsilon_i + {}^i\omega_{i+1} \times {}^iP_{i+1})$
  with $\dot{\theta}_{i+1} = 0$ ($\because$ the two frames are rigidly connected) and $\{i\} = \{A\}$, $\{i+1\} = \{B\}$
  → matrix form: $\begin{bmatrix} {}^B\mathbf{v}_B \\ {}^B\boldsymbol{\omega}_B \end{bmatrix} = \left[\begin{array}{c:c} {}^BR_A & -{}^BR_A\,{}^AP_{BORG}\times \\ \hdashline 0 & {}^BR_A \end{array}\right] \begin{bmatrix} {}^A\mathbf{v}_A \\ {}^A\boldsymbol{\omega}_A \end{bmatrix}$ or ${}^B\mathbf{V}_B = {}^BT_{vA}{}^A\mathbf{V}_A$ (6x6 operator)

  where $P\times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$ (Note: recall similar formula for angular velocity matrix!)

- Inversion: $\begin{bmatrix} {}^A\mathbf{v}_A \\ {}^A\boldsymbol{\omega}_A \end{bmatrix} = \left[\begin{array}{c:c} {}^AR_B & ({}^AP_{BORG}\times)\cdot{}^AR_B \\ \hdashline 0 & {}^AR_B \end{array}\right] \begin{bmatrix} {}^B\mathbf{v}_B \\ {}^B\boldsymbol{\omega}_B \end{bmatrix}$ or ${}^A\mathbf{V}_A = {}^AT_{vB}{}^B\mathbf{V}_B$
- Force-moment transformation
  Recall: ${}^if_i = {}^iR_{i+1}{}^{i+1}f_{i+1}$ and ${}^in_i = {}^iR_{i+1}{}^{i+1}n_{i+1} - {}^iP_{i-1} \times {}^if_i$
  → matrix form: $\begin{bmatrix} {}^Af_A \\ {}^An_A \end{bmatrix} = \left[\begin{array}{c:c} {}^AR_B & 0 \\ \hdashline -({}^AP_{\text{joint}A}\times)\cdot{}^AR_B & {}^AR_B \end{array}\right] \begin{bmatrix} {}^Bf_B \\ {}^Bn_B \end{bmatrix}$ or ${}^A\mathbf{F}_A = {}^A\mathbf{T}_{fB}{}^B\mathbf{F}_B$
- ${}^AT_{fB} = {}^AT_{vB}^T$
- Example 5.8 (Craig's 4th Ed.): (Do it yourself)

<u>Redundancy Resolution</u>

- Given $m$ function equations with $n$-DOF joint variables → $J$: $m$x$n$ Jacobian matrix
- If $m < n$ (i.e., redundant), infinite solutions of $\dot{\mathbf{q}}$ exist for $\mathbf{V}_{(m\times1)} = J(\mathbf{q})_{(m\times n)}\dot{\mathbf{q}}_{(n\times1)}$.

- Solution methods: Formulate as a constrained optimization problem.
  - Jacobian pseudo-inverse
  - Numerical trajectory optimization (e.g., collocation method, single/multiple shooting methods, etc.)

Jacobian Pseudo-Inverse

- Let the end-effector velocity is $\mathbf{V}$, Jacobian $J$ (for given $\mathbf{q}$) has full rank, and $W$ is a suitable ($n$x$n$) symmetric positive definite weight matrix. Then the optimal solution $\dot{\mathbf{q}}^*$ that satisfies $\mathbf{V} = J\dot{\mathbf{q}}$ and minimizes the quadratic cost functional $g(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T W \dot{\mathbf{q}}$ is $\dot{\mathbf{q}}^* = J^+\mathbf{V}$, where $J^+ = W^{-1}J^T(JW^{-1}J^T)^{-1}$ is the weighted right pseudo-inverse of $J$, i.e., $JJ^+ = I_n$.

- Proof (Use Method of Lagrange multipliers)
  Minimize $g(\dot{\mathbf{q}}, \boldsymbol{\lambda}) = \frac{1}{2}\dot{\mathbf{q}}^T W \dot{\mathbf{q}} + \boldsymbol{\lambda}^T(\mathbf{V} - J\dot{\mathbf{q}})$, where $\boldsymbol{\lambda}$ is a ($m$x1) vector of unknown Lagrange multipliers. Since $\partial^2 g / \partial \dot{\mathbf{q}}^2 = W$ is positive definite, the necessary conditions for minimum are:
  $\frac{\partial g}{\partial \dot{\mathbf{q}}} = \mathbf{0}^T$ → $\dot{\mathbf{q}} = W^{-1}J^T\boldsymbol{\lambda}$ (where $W^{-1}$ exists); and $\frac{\partial g}{\partial \boldsymbol{\lambda}} = \mathbf{0}^T$ → $\mathbf{V} = J\dot{\mathbf{q}}$
  => $\mathbf{V} = JW^{-1}J^T\boldsymbol{\lambda}$ → $\boldsymbol{\lambda} = (JW^{-1}J^T)^{-1}\mathbf{V}$ ($\because$ $JW^{-1}J^T$: ($m$x$m$) square matrix of rank $m$ and invertible)
  → $\dot{\mathbf{q}}^* = W^{-1}J^T(JW^{-1}J^T)^{-1}\mathbf{V}$

- If $W = I_n$ → $\boxed{J^+ = J^T(JJ^T)^{-1}}$: right pseudo-inverse of $J$ → minimizes $\|\dot{\mathbf{q}}\|$

- If the cost functional is $g'(\dot{\mathbf{q}}) = \frac{1}{2}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_0)^T(\dot{\mathbf{q}} - \dot{\mathbf{q}}_0)$, where $\dot{\mathbf{q}}_0$ is a vector of arbitrary joint velocities
  → $\dot{\mathbf{q}}^* = J^+\mathbf{V} + (I_n - J^+J)\dot{\mathbf{q}}_0$ (from the Method of Lagrange multipliers)
  - $J^+\mathbf{V}$: minimizes $\|\dot{\mathbf{q}}\|$
  - $(I_n - J^+J)\dot{\mathbf{q}}_0$: homogeneous solution; attempts to satisfy additional constraints to specify via $\dot{\mathbf{q}}_0$.

  Remark: $J(I_n - J^+J)\dot{\mathbf{q}}_0 = \mathbf{0}$, i.e., $I_n - J^+J$ projects $\dot{\mathbf{q}}_0$ in the null space of $J$, and $\dot{\mathbf{q}}_0$ generates internal motions of $(I_n - J^+J)\dot{\mathbf{q}}_0$ without violating the end-effector's $\mathbf{V} = J\dot{\mathbf{q}}$.

- Remark: If $m > n$ (i.e., over-constrained), no solution of $\dot{\mathbf{q}}$ exists for $\mathbf{V}_{(m\times 1)} = J(\mathbf{q})_{(m\times n)}\dot{\mathbf{q}}_{(n\times 1)}$.
  → (weighted) left pseudo-inverse of $J$ ($J^+J = I_n$) → approximate solution to minimize $\|\mathbf{V} - J\dot{\mathbf{q}}\|$