Problem 2:

(a) Each of these is correct:

- $\forall\, v \in \mathbb{R}^n$ with $v \neq 0$, it follows that $Av \neq \lambda v$.
- $\forall\, v \in \mathbb{R}^n$, either $v = 0$ or $Av \neq \lambda v$.
- $\forall\, v \in \mathbb{R}^n$, $v \neq 0 \Longrightarrow Av \neq \lambda v$.

We can also rewrite (c) as: For at least one $v \in \mathbb{R}^n$, $v \neq 0$, it is true that $Av = \lambda v$.

We can negate this statement as

- For all $v \in \mathbb{R}^n$ with $v \neq 0$, $Av \neq \lambda v$.
- For all nonzero $v \in \mathbb{R}^n$, $Av \neq \lambda v$.

(b) Each is correct:

- $\exists\, \eta > 0$ such that $\forall\, \delta > 0$, $\exists\, x$ such that $|x| \leq \delta$, while $|f(x)| > \eta|x|$.
- There is at least one $\eta > 0$ such that, for all $\delta > 0$, there is at least one x satisfying $|x| \leq \delta$ and $|f(x)| > \eta|x|$.

Problem 4: Hint: this is a proof by contrapositive.

Problem 5: Proof by (ordinary) induction

$$P(n) : \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

We check $P(1)$: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{1+1}$ and hence $P(1)$ holds.

We assume that $P(k)$: $\sum_{j=1}^{k} \frac{j}{j(j+1)} = \frac{k}{k+1}$ and seek to show that

$$P(k+1) = \frac{k+1}{(k+1)+1}$$

We can write $P(k+1)$ as

$$\begin{aligned}
P(k+1) &= \sum_{j=1}^{k+1} \frac{1}{j(j+1)} \\
&= \left(\sum_{j=1}^{k} \frac{1}{j(j+1)}\right) + \left(\frac{1}{(k+1)(k+2)}\right) \\
&= P(k) + \frac{1}{(k+1)(k+2)}
\end{aligned}$$

By the induction hypothesis, we have that

$$\begin{aligned}
P(k+1) &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \frac{k+1}{(k+1)+1}
\end{aligned}$$

which is what we wanted to show. $\square$