Clarifications

- There are two approaches to deriving the linear acceleration for a prismatic joint:

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\,{}^{i}\dot{v}_i + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}P_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}R_i\,{}^{i}\hat{Z}_i + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1} \times {}^{i+1}R_i\,{}^{i}\hat{Z}_i $$

1. In the lecture notes, we use the following theorem

$$ {}^{A}\dot{\mathbf{V}}_Q = {}^{A}\dot{\mathbf{V}}_{BORG} + {}^{A}R_B\,{}^{B}\dot{\mathbf{V}}_Q + 2\,{}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{V}_Q + {}^{A}\dot{\mathbf{\Omega}}_B \times {}^{A}R_B\,{}^{B}\mathbf{Q} + {}^{A}\mathbf{\Omega}_B \times ({}^{A}\mathbf{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{Q}) $$

2. In the classroom, we instead differentiated the linear velocity of a prismatic joint (below from Chapter 5 lecture notes) directly:

$$ {}^{i+1}v_{i+1} = {}^{i+1}R_i ({}^{i}v_i + {}^{i}\omega_{i+1} \times {}^{i}P_{i+1} + \dot{d}_{i+1}\,{}^{i}\hat{Z}_i) $$

Classroom Approach: Derivative Formula

- Both approaches are fine, but starting from the second approach is a good practice exercise for differentiating vectors based on the first half of lecture.

- The **derivative with respect to** $\{A\}$ of some vector $\mathbf{W}$ (e.g., linear position, linear velocity, angular velocity, etc) can be broken out into two terms in this **formula**:

$$^{A}\dot{\mathbf{W}} = \frac{^{A}d}{dt}\left(^{A}R_{B}\,^{B}\dot{\mathbf{W}}\right) = {}^{A}R_{B}\,^{B}\dot{\mathbf{W}} + {}^{A}\mathbf{\Omega}_{B} \times {}^{A}R_{B}\,^{B}\mathbf{W}$$

  - Note that this notation assumes that the observer and writer frames are the same:

$$^{A}\dot{\mathbf{W}} = {}^{A}\left(^{A}\dot{\mathbf{W}}\right) \qquad {}^{B}\dot{\mathbf{W}} = {}^{B}\left(^{B}\dot{\mathbf{W}}\right) \qquad {}^{B}\mathbf{W} = {}^{B}\left(^{B}\mathbf{W}\right)$$

Linear Velocity to Linear Acceleration

- We would like to derive the **absolute** linear acceleration of a link frame attached to a prismatic joint by differentiating its **absolute** linear velocity:

$$^{i+1}v_{i+1} = {}^{i+1}R_i({}^iv_i + {}^i\omega_{i+1} \times {}^iP_{i+1} + \dot{d}_{i+1}\,{}^i\hat{Z}_i)$$

- Indicate the observer and writer frames for each vector (same for P and Z)

$$^{i+1}({}^0\mathbf{V}_{i+1}) = {}^{i+1}R_i[{}^i({}^0\mathbf{V}_i) + {}^i({}^0\mathbf{\Omega}_{i+1}) \times {}^i({}^iP_{i+1}) + \dot{d}_{i+1}\,{}^i({}^i\hat{Z}_i)]$$

- Multiply both sides by $^0R_{i+1}$ so everything is written in the same frame $\{0\}$

$$^0({}^0\mathbf{V}_{i+1}) = {}^0({}^0\mathbf{V}_i) + {}^0({}^0\mathbf{\Omega}_{i+1}) \times {}^0({}^iP_{i+1}) + \dot{d}_{i+1}\,{}^0({}^i\hat{Z}_i)$$

- There are three terms to differentiate. We will go one by one.

Term 1: Easy

- We do not need the derivative formula. ($\{A\} = \{B\} = \{0\}$)

$$\frac{d}{dt}\,{}^{0}\left({}^{0}\mathbf{V}_i\right) = {}^{0}\left({}^{0}\dot{\mathbf{V}}_i\right)$$

- **Instructor's Note:** In class, the derivation was derailed because the derivative formula was applied here incorrectly. See below for an example of a similar error

$${}^{i+1}\dot{v}_i = \frac{d}{dt}\left({}^{i+1}v_i\right) = \frac{d}{dt}\left({}^{i+1}R_i\,{}^{i}v_i\right) = {}^{i+1}R_i\,{}^{i}\dot{v}_i + \boxed{{}^{i+1}\omega_i \times {}^{i+1}R_i\,{}^{i}v_i}$$

Terms match

What is this extra term??

Term 2: (Cross) Product Rule and Formula

- We first apply the product rule, which also applies to a cross product.

$$\frac{d}{dt}\,{}^0({}^0\boldsymbol{\Omega}_{i+1}) \times {}^0({}^iP_{i+1}) = {}^0({}^0\dot{\boldsymbol{\Omega}}_{i+1}) \times {}^0({}^iP_{i+1}) + {}^0({}^0\boldsymbol{\Omega}_{i+1}) \times \frac{d}{dt}\,{}^0({}^iP_{i+1})$$

- We need the derivative formula to evaluate $d/dt\left[{}^0({}^iP_{i+1})\right]$ ({A} = 0, {B} = *i*)

$${}^A\dot{\mathbf{W}} = {}^AR_B\,{}^B\dot{\mathbf{W}} + {}^A\boldsymbol{\Omega}_B \times {}^AR_B\,{}^B\mathbf{W}$$

$$\begin{aligned}\frac{d}{dt}\,{}^0({}^0P_{i+1}) &= {}^0R_i\,{}^i({}^i\dot{P}_{i+1}) + {}^0({}^0\boldsymbol{\Omega}_i) \times {}^0R_i\,{}^i({}^iP_{i+1}) \\ &= {}^0R_i\left[\dot{d}_{i+1}\,{}^i({}^i\hat{Z}_i)\right] + {}^0({}^0\boldsymbol{\Omega}_i) \times {}^0R_i\,{}^i({}^iP_{i+1}) \\ &= {}^0R_i\left[\dot{d}_{i+1}\,{}^i({}^i\hat{Z}_i) + {}^i({}^0\boldsymbol{\Omega}_i) \times {}^i({}^iP_{i+1})\right]\end{aligned}$$

For prismatic joint

$${}^i\dot{P}_{i+1} = \dot{d}_{i+1}\,{}^i\hat{Z}_i$$

Term 2: Continued

- Substitution of $d/dt\left[{}^{0}({}^{i}P_{i+1})\right]$ back in

$$
\begin{aligned}
\frac{d}{dt}\,{}^{0}({}^{0}\mathbf{\Omega}_{i+1})\times{}^{0}({}^{i}P_{i+1}) &= {}^{0}({}^{0}\dot{\mathbf{\Omega}}_{i+1})\times{}^{0}({}^{i}P_{i+1})+{}^{0}({}^{0}\mathbf{\Omega}_{i+1})\times\frac{d}{dt}\,{}^{0}({}^{i}P_{i+1}) \\
&= {}^{0}({}^{0}\dot{\mathbf{\Omega}}_{i+1})\times{}^{0}({}^{i}P_{i+1})+{}^{0}({}^{0}\mathbf{\Omega}_{i+1})\times{}^{0}R_{i}\left[\dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i})+{}^{i}({}^{0}\mathbf{\Omega}_{i})\times{}^{i}({}^{i}P_{i+1})\right] \\
&= {}^{0}R_{i}\left[{}^{i}({}^{0}\dot{\mathbf{\Omega}}_{i+1})\times{}^{i}({}^{i}P_{i+1})+{}^{i}({}^{0}\mathbf{\Omega}_{i+1})\times\dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i})+{}^{i}({}^{0}\mathbf{\Omega}_{i+1})\times{}^{i}({}^{0}\mathbf{\Omega}_{i})\times{}^{i}({}^{i}P_{i+1})\right]
\end{aligned}
$$

Term 3: Product Rule

- Product Rule: $$\frac{d}{dt}\dot{d}_{i+1}\,{}^{0}({}^{i}\hat{Z}_i) = \ddot{d}_{i+1}\,{}^{0}({}^{i}\hat{Z}_i) + \dot{d}_{i+1}\frac{d}{dt}\,{}^{0}({}^{i}\hat{Z}_i)$$

- We need the derivative formula to evaluate $\frac{d}{dt}\,{}^{0}({}^{i}\hat{Z}_i)$ ({A} = 0, {B} = *i*)

$${}^{A}\dot{\mathbf{W}} = {}^{A}R_B\,{}^{B}\dot{\mathbf{W}} + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}R_B\,{}^{B}\mathbf{W}$$

$$\begin{aligned}
\frac{d}{dt}\,{}^{0}({}^{i}\hat{Z}_i) &= {}^{0}R_i\left[\frac{d}{dt}\,{}^{i}({}^{i}\hat{Z}_i)\right] + {}^{0}({}^{0}\boldsymbol{\Omega}_i) \times {}^{0}R_i\,{}^{i}({}^{i}\hat{Z}_i) \\
&= \qquad 0 \qquad + {}^{0}({}^{0}\boldsymbol{\Omega}_i) \times {}^{0}R_i\,{}^{i}({}^{i}\hat{Z}_i) \\
&= {}^{0}R_i\left[{}^{i}({}^{0}\boldsymbol{\Omega}_i) \times {}^{i}({}^{i}\hat{Z}_i)\right]
\end{aligned}$$

Term 3: Continued

- Substitution of $\frac{d}{dt}\,{}^{0}({}^{i}\hat{Z}_i)$ back in

$$\frac{d}{dt}\dot{d}_{i+1}\,{}^{0}({}^{i}\hat{Z}_i) = \ddot{d}_{i+1}\,{}^{0}({}^{i}\hat{Z}_i) + \dot{d}_{i+1}\frac{d}{dt}\,{}^{0}({}^{i}\hat{Z}_i)$$

$$= \ddot{d}_{i+1}\,{}^{0}({}^{i}\hat{Z}_i) + \dot{d}_{i+1}\,{}^{0}R_i\left[{}^{i}({}^{0}\mathbf{\Omega}_i)\times{}^{i}({}^{i}\hat{Z}_i)\right]$$

$$= {}^{0}R_i\left[\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_i) + \dot{d}_{i+1}\,{}^{i}({}^{0}\mathbf{\Omega}_i)\times{}^{i}({}^{i}\hat{Z}_i)\right]$$

Sum Terms 1-3

$$^{0}({}^{0}\dot{v}_{i+1}) = {}^{0}({}^{0}\dot{\mathbf{V}}_{i}) + {}^{0}R_{i}\Big[{}^{i}({}^{0}\dot{\mathbf{\Omega}}_{i+1}) \times {}^{i}({}^{i}P_{i+1}) + {}^{i}({}^{0}\mathbf{\Omega}_{i+1}) \times \dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + {}^{i}({}^{0}\mathbf{\Omega}_{i+1}) \times {}^{i}({}^{0}\mathbf{\Omega}_{i}) \times {}^{i}({}^{i}P_{i+1})\Big]$$
$$+ {}^{0}R_{i}\Big[\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + \dot{d}_{i+1}\,{}^{i}({}^{0}\mathbf{\Omega}_{i}) \times {}^{i}({}^{i}\hat{Z}_{i})\Big]$$

- Rewrite in $\{i + 1\}$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}R_{i}\,{}^{i}\dot{v}_{i} + {}^{i+1}R_{i}\Big[{}^{i}\dot{\omega}_{i+1} \times {}^{i}({}^{i}P_{i+1}) + {}^{i}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + {}^{i}\omega_{i+1} \times {}^{i}\omega_{i+1} \times {}^{i}({}^{i}P_{i+1})\Big]$$
$$+ {}^{i+1}R_{i}\Big[\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + \dot{d}_{i+1}\,{}^{i}\omega_{i+1} \times {}^{i}({}^{i}\hat{Z}_{i})\Big]$$

- Note for prismatic joint: ${}^{i}({}^{0}\mathbf{\Omega}_{i}) = {}^{i}({}^{0}\mathbf{\Omega}_{i+1})$
- Notation reminder: ${}^{A}v_{B} = {}^{A}({}^{0}\mathbf{V}_{B});\ {}^{A}\omega_{B} = {}^{A}({}^{0}\mathbf{\Omega}_{B})$

Check with equation from lecture notes (Boxed)

$$^{0}({}^{0}\dot{v}_{i+1}) = {}^{0}({}^{0}\dot{\mathbf{V}}_{i}) + {}^{0}R_{i}\left[{}^{i}({}^{0}\dot{\mathbf{\Omega}}_{i+1})\times{}^{i}({}^{i}P_{i+1}) + {}^{i}({}^{0}\mathbf{\Omega}_{i+1})\times\dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + {}^{i}({}^{0}\mathbf{\Omega}_{i+1})\times{}^{i}({}^{0}\mathbf{\Omega}_{i})\times{}^{i}({}^{i}P_{i+1})\right]$$
$$+{}^{0}R_{i}\left[\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + \dot{d}_{i+1}\,{}^{i}({}^{0}\mathbf{\Omega}_{i})\times{}^{i}({}^{i}\hat{Z}_{i})\right]$$

- Rewrite in $\{i+1\}$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}R_{i}\,{}^{i}\dot{v}_{i} + {}^{i+1}R_{i}\left[{}^{i}\dot{\omega}_{i+1}\times{}^{i}({}^{i}P_{i+1}) + {}^{i}\omega_{i+1}\times\dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + {}^{i}\omega_{i+1}\times{}^{i}\omega_{i+1}\times{}^{i}({}^{i}P_{i+1})\right]$$
$$+{}^{i+1}R_{i}\left[\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i}) + \dot{d}_{i+1}\,{}^{i}\omega_{i+1}\times{}^{i}({}^{i}\hat{Z}_{i})\right]$$

$$\underline{^{i+1}\dot{v}_{i+1}} = \underline{{}^{i+1}R_{i}\,{}^{i}\dot{v}_{i}} + {}^{i+1}R_{i}\left[\underline{{}^{i}\dot{\omega}_{i+1}\times{}^{i}({}^{i}P_{i+1})} + \underline{{}^{i}\omega_{i+1}\times\dot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i})} + \underline{{}^{i}\omega_{i+1}\times{}^{i}\omega_{i+1}\times{}^{i}({}^{i}P_{i+1})}\right] + {}^{i+1}R_{i}\left[\underline{\ddot{d}_{i+1}\,{}^{i}({}^{i}\hat{Z}_{i})} + \underline{\dot{d}_{i+1}\,{}^{i}\omega_{i+1}\times{}^{i}({}^{i}\hat{Z}_{i})}\right]$$

$$\boxed{^{i+1}\dot{v}_{i+1} = {}^{i+1}R_{i}\,{}^{i}\dot{v}_{i} + {}^{i+1}\dot{\omega}_{i+1}\times{}^{i+1}R_{i}\,{}^{i}P_{i+1} + {}^{i+1}\omega_{i+1}\times({}^{i+1}\omega_{i+1}\times{}^{i+1}R_{i}\,{}^{i}P_{i+1}) + \ddot{d}_{i+1}\,{}^{i+1}R_{i}\,{}^{i}\hat{Z}_{i} + 2\dot{d}_{i+1}\,{}^{i+1}\omega_{i+1}\times{}^{i+1}R_{i}\,{}^{i}\hat{Z}_{i}}$$