Reinforcement Learning and Optimal Control for Robotics ROB-GY 6323 Exercise Series 1

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Links to Jupyter Notebooks -

- Jupyter Notebook for code solutions of Exercises 1, 2 & 3
- Jupyter Notebook for code solution of Exercise 4

Exercise: 1 Find all the minimum(s), if they exist, of the functions below. Characterize the type of minimum (global, local, strict, etc.) and justify your answers (hint: you can plot the functions in Python to get an intuition of their form).
Please see "HW_Series1_Exercise1_2_3.ipynb" for the python code

Ex. 1(a): $f(x) = -e^{-(x-1)^2}$, where $x \in \mathbb{R}$

Ans. 1 (a): The *strict global minimum* of the given function is $x^* = 1$, where $f(x^*) = f(1) = -1$. We can observe this in the plot show below -

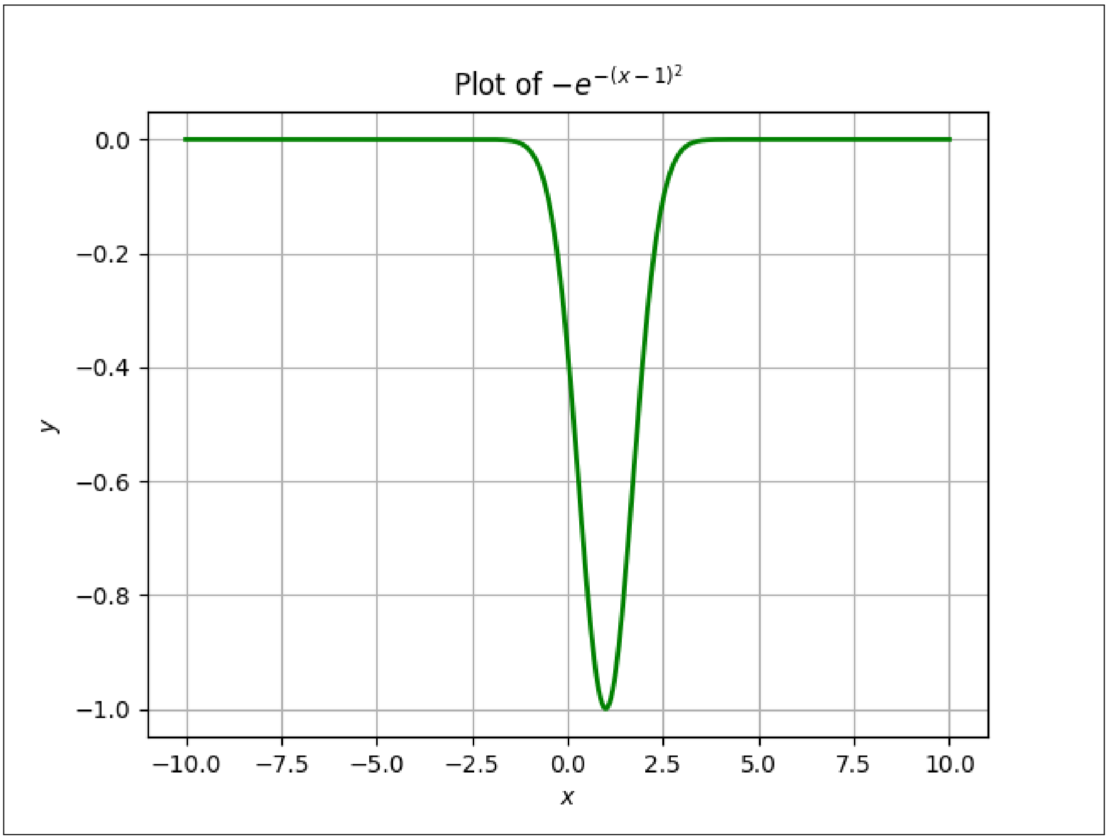

Figure 1: Plot of $f(x) = -e^{-(x-1)^2}$, where $x \in \mathbb{R}$

Looking at the graph, we can see that there are no apparent local minima. Thus, it can be concluded that local minima will be found on the basis of the chosen bounds of x. Finally, when we apply the conditions for recognizing a minimum:

- First order necessary condition,

$$\text{w.k.t, } \nabla f(x^*) = 0 \tag{1}$$
$$\Rightarrow \nabla f(x^*) = -(2 - 2x^*)e^{-(x^*-1)^2} \tag{2}$$
$$= -(2 - 2 \times 1)e^{-(1-1)^2} = 0 \tag{3}$$

- Second order condition,

$$\text{w.k.t, } \nabla^2 f(x^*) \succeq 0 \tag{4}$$
$$\Rightarrow \nabla^2 f(x^*) = -(2 - 2x^*)^2 e^{-(x^*-1)^2} + 2e^{-(x^*-1)^2} \tag{5}$$
$$= -(2 - 2 \times 1)^2 e^{-(1-1)^2} + 2e^{-(1-1)^2} = 2 \succ 0 \tag{6}$$

Ex. 1(b): $(1-x)^2 + 100(y-x^2)^2$, where $x, y \in \mathbb{R}$

Ans. 1 (b): The *global minimum* of the given function is $x, y = 1, 1$, where $f(x, y) = f(1, 1) = 0$. To compute this, we shall firstly find the gradient and set it to zero,
Partial derivative *w.r.t.* x,

$$\frac{\partial f}{\partial x} = -2(1-x) - 400x(y-x^2) \tag{7}$$

Partial derivative *w.r.t.* y,

$$\frac{\partial f}{\partial y} = 200(y-x^2) \tag{8}$$

As,

$$\frac{\partial f}{\partial y} = 0 \implies 200(y-x^2) = 0 \implies y = x^2 \tag{9}$$

Substitutiting $y = x^2$ into $\frac{\partial f}{\partial x} = 0 \Rightarrow$

$$-2(1-x) - 400x(y-x^2) = -2(1-x) - 400x(0) = -2(1-x) = 0 \implies x = 1 \tag{10}$$

$$\therefore x^* = 1, y^* = x^2 = 1 \Rightarrow f(x^*, y^*) = f(1,1) = (1-1)^2 + 100(1-1^2)^2 + 0 + 100(0)^2 = 0 \tag{11}$$

Now let us compute the Hessian at (x^*, y^*),

$$\nabla^2 f(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 1200\,x^2 - 400\,y + 2 & -400\,x \\ -400\,x & 200 \end{bmatrix} \tag{12}$$

$$\Rightarrow \nabla^2 f(x^*, y^*) = \nabla^2 f(1,1) = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix} \tag{13}$$

The eigenalues of $\nabla^2 f(x^*, y^*)$ were found to be 0.399 and 1001.601 which are both > 0.
$\therefore \nabla^2 f(x^*, y^*) \succ 0$ Since the Hessian is positive definite at $(x^*, y^*) = (1, 1)$, the function has a strict local minimum at this point. Given the nature of the Rosenbrock function, this is also the global minimum. We can observe this in the plot shown below -

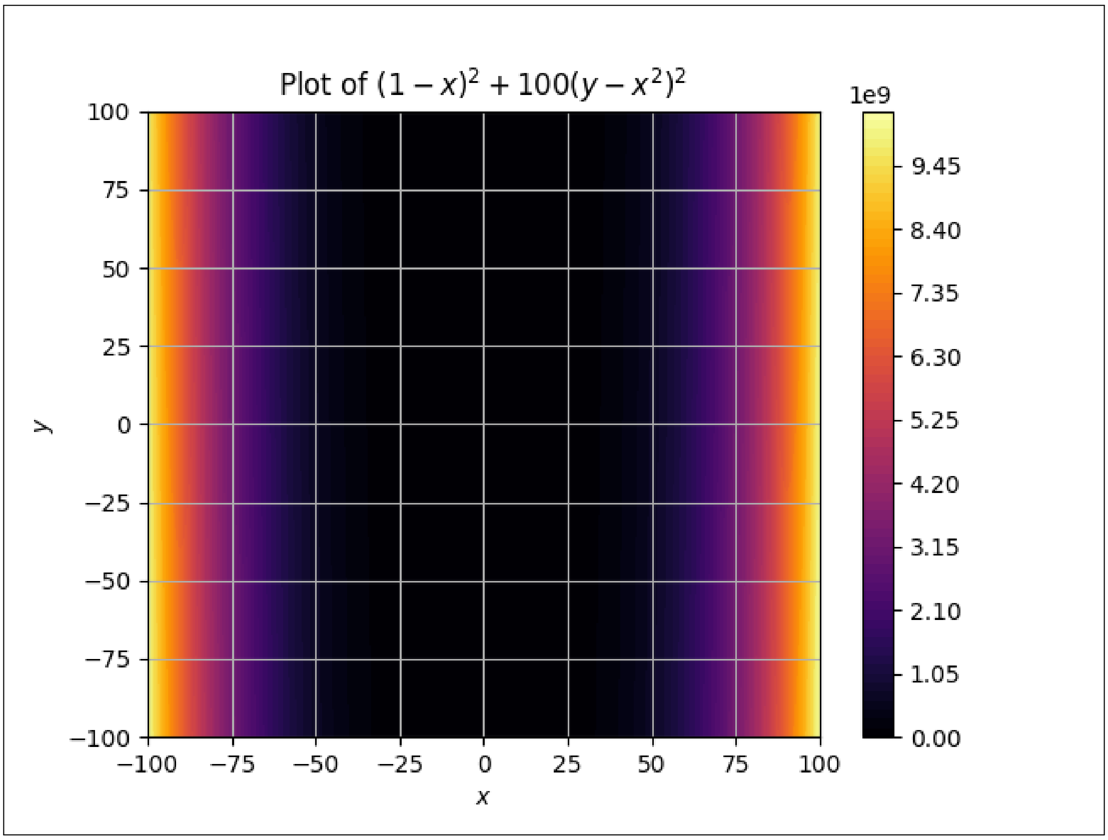

Figure 2: Plot of $f(x, y) = (1-x)^2 + 100(y-x^2)^2$, where $x, y \in \mathbb{R}$

Ex. 1(c): $20x + 2x^2 + 4y - 2y^2$, where $x, y \in \mathbb{R}$

Ans. 1 (c): Firstly, let us rewrite the equation in matrix form,

$$\Rightarrow f(\mathbf{x}) = \min_{\mathbf{x}} \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 20 & 4 \end{bmatrix} \mathbf{x} \tag{14}$$

Let,

$$P = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \qquad q = \begin{bmatrix} 20 \\ 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Using the shortcut, we can easily find the gradient to be,

$$\begin{aligned} \nabla f(\mathbf{x}) &= 2P\mathbf{x} + q \quad (15) \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 20 \\ 4 \end{bmatrix} \\ \Rightarrow \frac{\partial f}{\partial x} &= 20 + 4x \quad (16) \\ \frac{\partial f}{\partial y} &= -4y + 4 \quad (17) \end{aligned}$$

Equating $\nabla f(\mathbf{x}) = 0$ gives us $\mathbf{x}^* = (-5, 1)$. Now, we can compute the hessian to be as,

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \tag{18}$$

The hessian has eigenvalues 4 and -4. As one of the eigenvalues are < 0 it is not positive definite. Hence, the function has no global negative. It shall have only local negatives, that depend on the bounds or constraints that have been set on the function. In our case, the domain is $\mathbb{R}$.
$\therefore$ The minimum value of this function tends to $-\infty$.
The below figure shows the plot of this function.

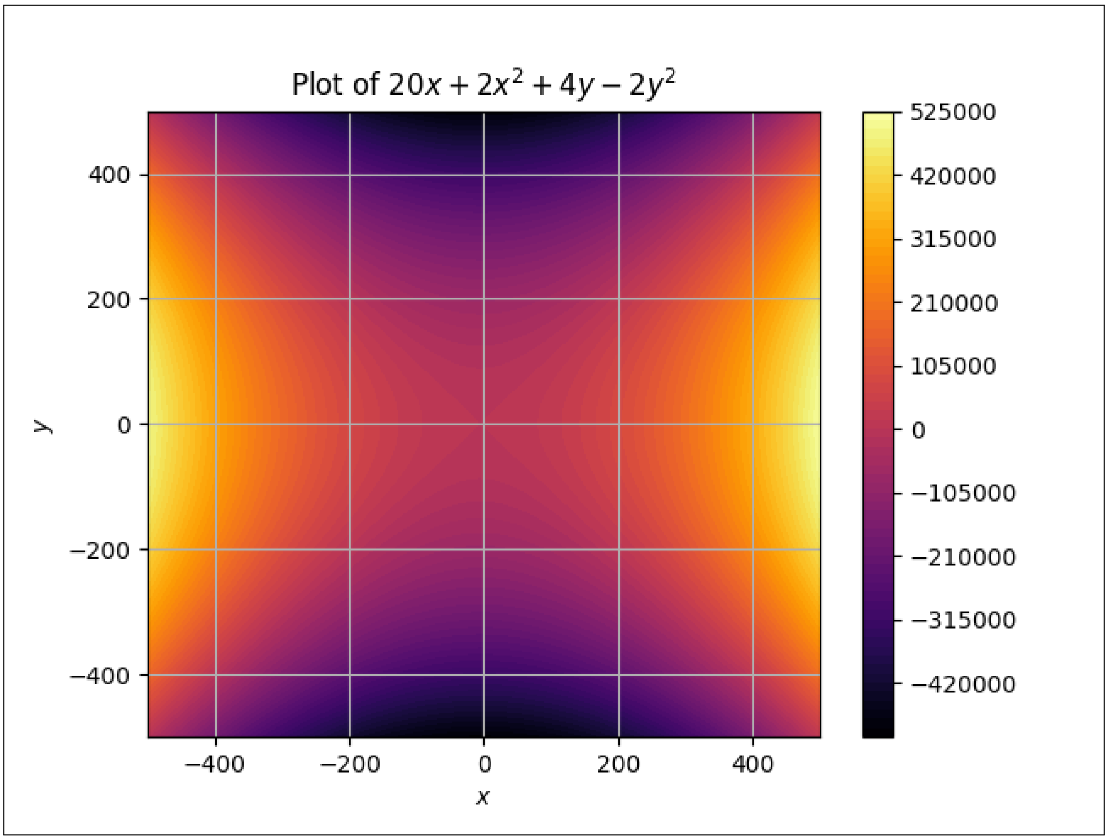

Figure 3: Plot of $20x + 2x^2 + 4y - 2y^2$, where $x, y \in \mathbb{R}$

Ex. 1(d): $\mathbf{x}^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^2$

Ans. 1 (d): For the given function $f(\mathbf{x})$ we can use the same shortcut method to find the gradient and the hessian. Let,

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad q = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{19}$$

The gradient is found as,

$$\begin{aligned} \nabla f(\mathbf{x}) &= 2P\mathbf{x} + q \qquad (20) \\ &= 2 \times \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \Rightarrow \frac{\partial f}{\partial x} &= 6\,x + 2\,y - 1 \qquad (21) \\ \frac{\partial f}{\partial y} &= 2\,x + 6\,y + 1 \qquad (22) \end{aligned}$$

Equating $\nabla f(\mathbf{x}) = 0$ gives us $\mathbf{x}^* = (0.25, -0.25)$. Now, we can compute the hessian to be as,

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} \tag{23}$$

The hessian has eigenvalues 4 and 8 $\Rightarrow \nabla^2 f(\mathbf{x}) \succ 0$. Thus the $\mathbf{x}^* = (0.25, -0.25)$ gives us the minimum value $f(\mathbf{x}^*) = -0.25$, which is a *strict global minimum*. The plot for this function is shown below -

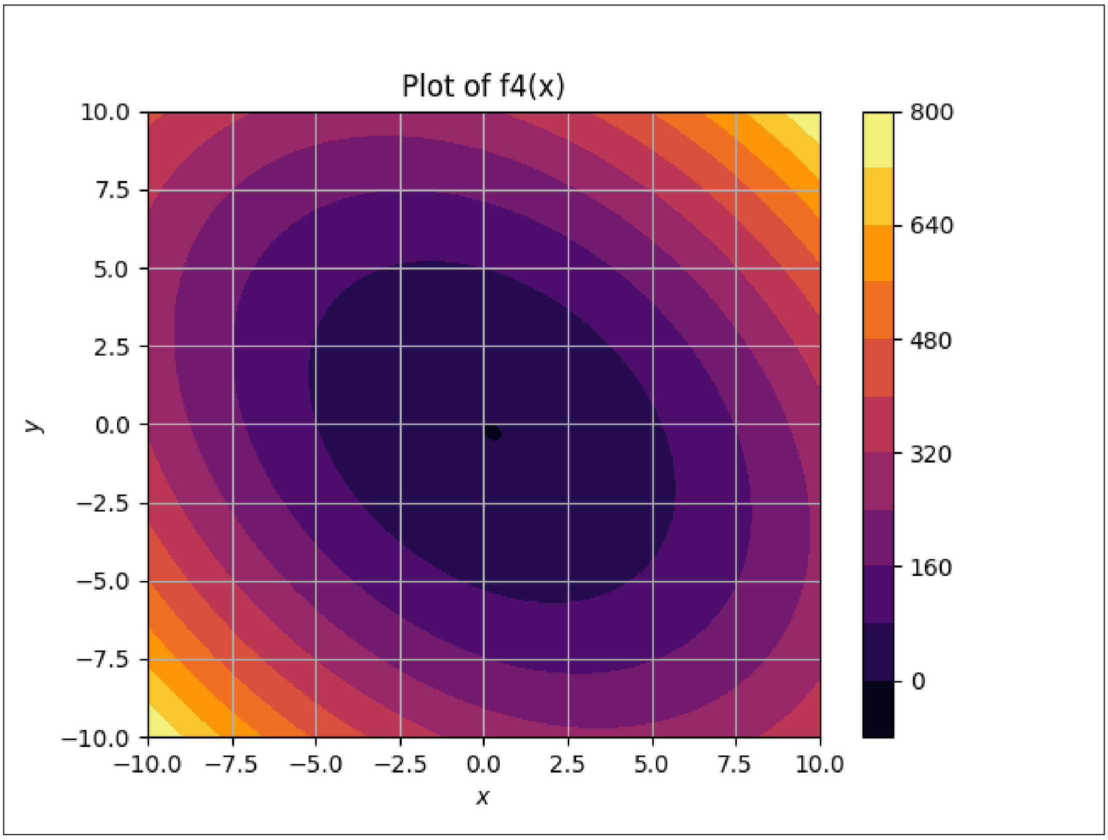

Figure 4: Plot of $\mathbf{x}^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^2$

Ex. 1(e): $x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 10 \end{bmatrix} x, \text{ where } x \in \mathbb{R}^2$

Ans. 1 (e): For the given function $f(\mathbf{x})$ we can use the same shortcut method to find the gradient and the hessian. Let,

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad q = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{24}$$

The gradient is found as,

$$\nabla f(\mathbf{x}) = 2P\mathbf{x} + q \tag{25}$$
$$= 2 \times \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$
$$\Rightarrow \frac{\partial f}{\partial x} = 2\,x + 4\,y + 1 \tag{26}$$
$$\frac{\partial f}{\partial y} = 4\,x + 2\,y + 10 \tag{27}$$

Equating $\nabla f(\mathbf{x}) = 0$ gives us $\mathbf{x}^* = (-3.1667, 1.3333)$. Now, we can compute the hessian to be as,

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \tag{28}$$

The hessian has eigenvalues -2 and 6. As one of the eigenvalues is negative, the function is not convex and there is no global minimum. As there are no bounds, the minimum values tends to $-\infty$. Shown below is the plot of this function -

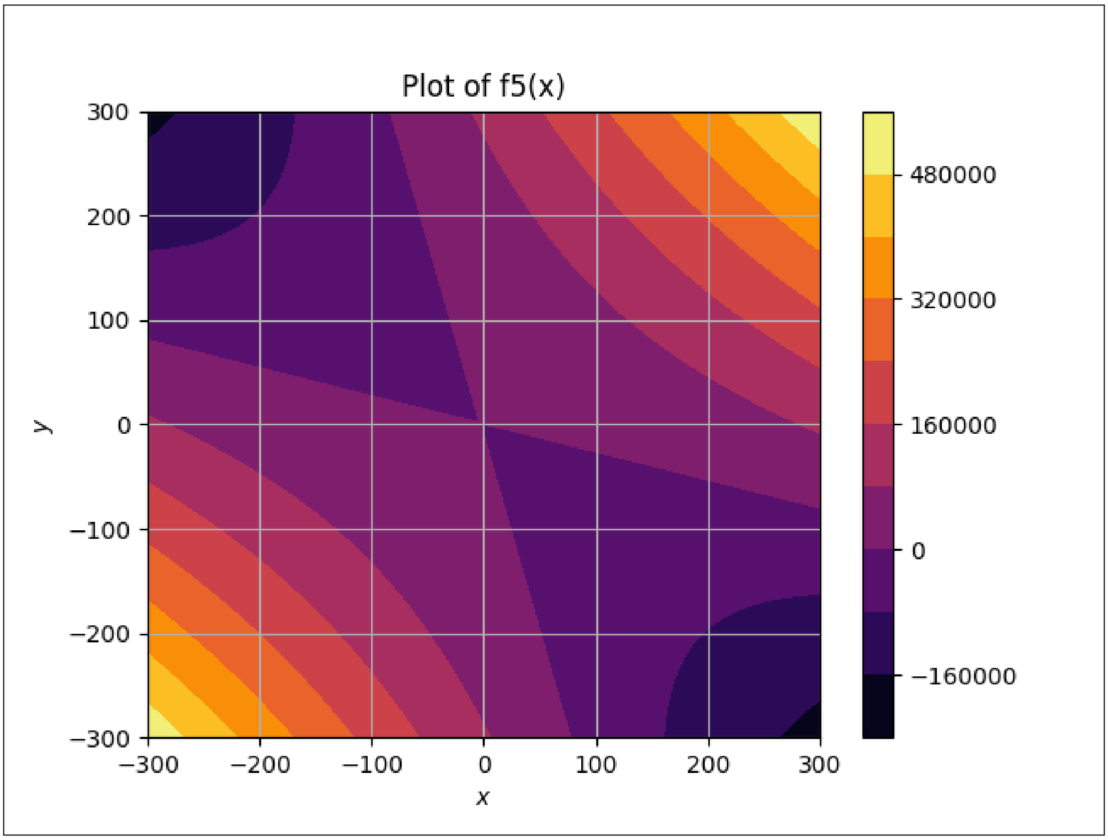

Figure 5: Plot of $x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 10 \end{bmatrix} x, \text{ where } x \in \mathbb{R}^2$

Ex. 1(f): $\frac{1}{2}x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x,$ where $x \in \mathbb{R}^3$

Ans. 1 (f): For the given function $f(\mathbf{x})$ we can use the same shortcut method to find the gradient and the hessian. Let,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad q = -\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{29}$$

The gradient is found as,

$$\nabla f(\mathbf{x}) = 2P\mathbf{x} + q \tag{30}$$

$$= 2 \times \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2\,x + 2\,y \quad \frac{\partial f}{\partial y} = 2\,x + 2\,y \quad \frac{\partial f}{\partial z} = 8\,z + 2 \tag{31}$$

Equating $\nabla f(\mathbf{x}) = 0$ gives us $\mathbf{x}^* = (x, -x, 0.5)$. Now, we can compute the hessian to be as,

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \tag{32}$$

The hessian has eigenvalues 0, 2 and 4 $\Rightarrow \nabla^2 f(\mathbf{x}^*) \succeq 0$. Thus, the *global minima* is found at all $\mathbf{x}^* = (x, -x, 0.5)$ but this is not strict as the hessian is not positive definite. The plot is shown below,

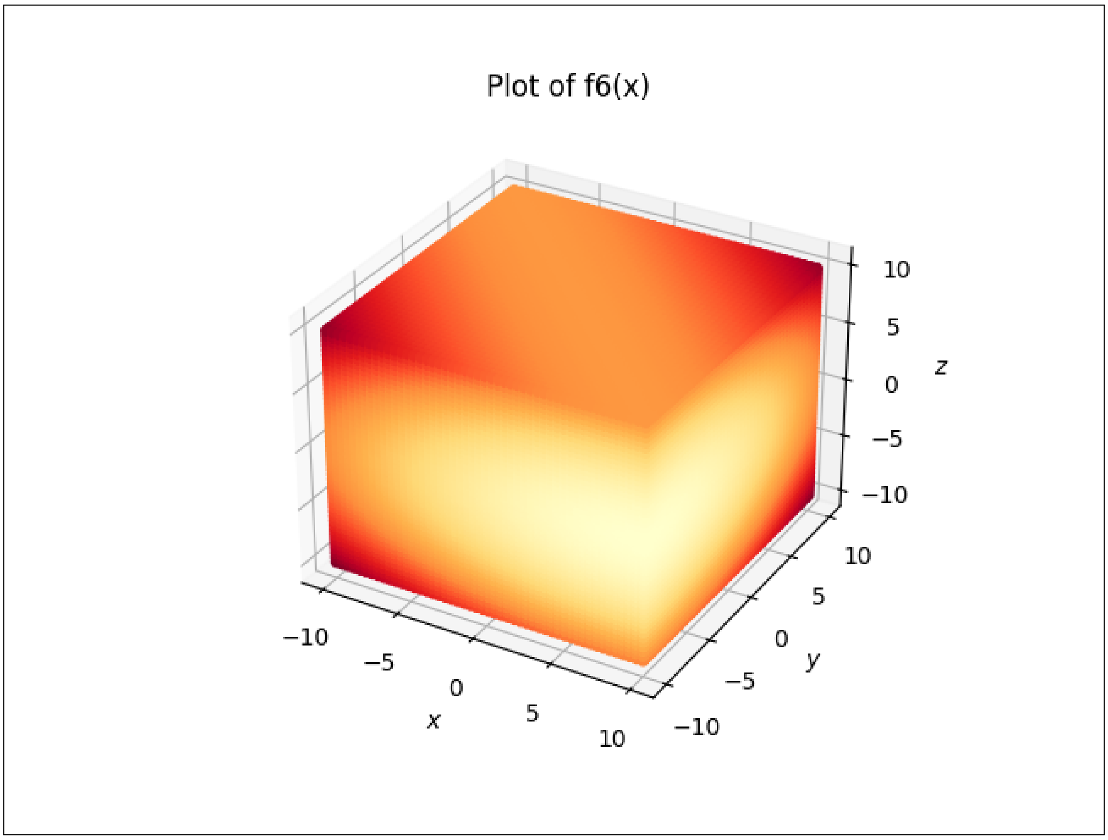

Figure 6: Plot of $\frac{1}{2}x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x,$ where $x \in \mathbb{R}^3$

Exercise: 2 We would like to find a 2D point (x, y) as close as possible to the point $(1, 1)$ under the constraints that the sum $x + y$ is lower than 1 and that the differences $y - x$, $x - y$ and $-y - x$ are lower than 1.

Please see "HW_Series1_Exercise1_2_3.ipynb" for the python code

Ex. 2(a) Write the problem above as a minimization problem with constraints (hint: use a quadratic cost)

Ans. 2(a) We know that the distance between two points in 2D Cartesian Space is written as,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

Where (x_1, y_1) and (x_2, y_2) are the two points, the distance between which needs to be computed.

In our case, it is given $(x_1, y_1) = (1, 1)$. We now need to find a point (x, y) such that the distance between the two points is minimized. Thus, we can form a quadratic function of the form,

$$f(x, y) = d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \tag{2}$$

$$\Rightarrow \min_{x,y} f(x, y) = \min_{x,y} (x - 1)^2 + (y - 1)^2 \tag{3}$$

Subject to the given constraints $h(x, y)$,

$$\begin{aligned} x + y &\leq 1 & \quad y - x &\leq 1 \\ x - y &\leq 1 & \quad -y - x &\leq 1 \end{aligned}$$

Finally, we can rewrite $f(x, y)$ in matrix form as, $\begin{bmatrix} x \\ y \end{bmatrix}$

$$f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 2 \begin{bmatrix} x \\ y \end{bmatrix} + 2 \tag{4}$$

And subject to the conditions,

$$h(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix} - B < 0 \tag{5}$$

$$\text{Where, } A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This builds $f(x, y)$ and $h(x, y)$ to be substituted in the formula for KKT conditions of optimality.

Ex. 2(b) Draw a geometric sketch of the problem showing the level sets of the function to minimize and the constraints

Ans. 2(b) The geometric plot is shown below -

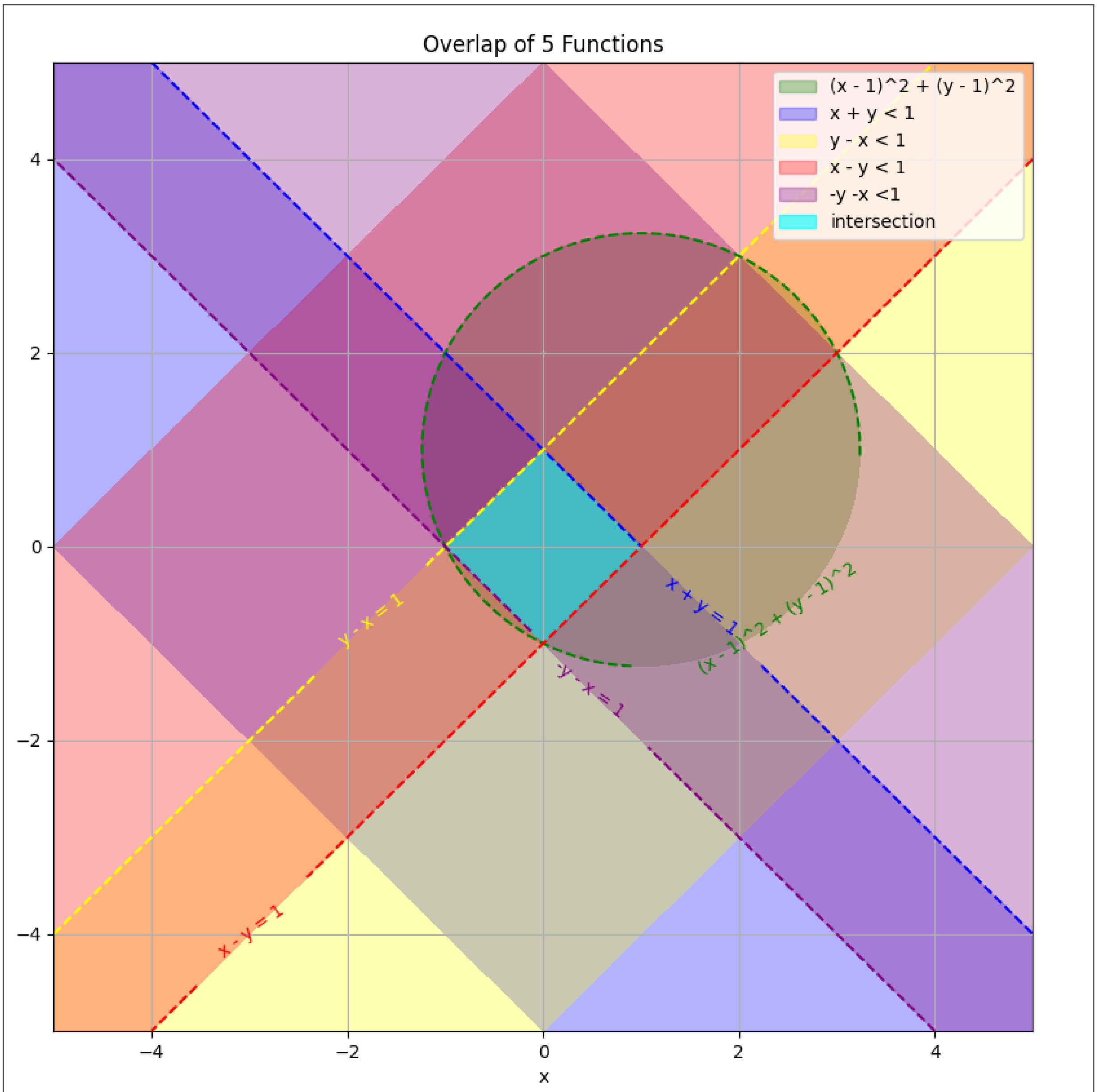

Figure 7: Plot of $f(x, y)$ and all $h(x, y)$

The cyan region highlights the intersection between $f(x, y)$ and all the $h(x, y)$.

Ex. 2(c) Write the Lagrangian of the optimization problem

Ans. 2(c) The Lagrangian of the problem can be formulated as,

$$L(x, y, \mu) = f(x, y) + \mu^T h(x, y) \tag{6}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 2 \begin{bmatrix} x \\ y \end{bmatrix} + 2 + \mu^T \left(A \begin{bmatrix} x \\ y \end{bmatrix} - B \right) \tag{7}$$

Substituting for A, B and $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$

$$\therefore L(x, y, \mu) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 2 \begin{bmatrix} x \\ y \end{bmatrix} + 2 + \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \tag{8}$$

Ex. 2(d) Write the KKT necessary conditions for a point x^* to be optimal

Ans. 2(d) Firstly, we know that, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ & $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$. Also, let $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Based on this, we can rewrite the Lagrangian as,

$$L(X, \mu) = X^T X - 2X + 2 + \mu^T (AX - B) \tag{9}$$

Now, we shall find the gradient of L with respect to X and μ,

$$\nabla_X L = 2X - 2 + A^T \mu = 0 \tag{10}$$

$$\nabla_\mu L = AX - B = 0 \tag{11}$$

This is equivalent to,

$$\begin{bmatrix} 2 \times I & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} X \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ \mu \end{bmatrix} = \begin{bmatrix} 2 \times I & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{12}$$

Where I is the 2×2 Identity matrix.

Ex. 2(e) Find the minimum and the values of x and y that reach this minimum

Ans. 2(e) To find the minimum values of x and y, the following python code can be employed -

```python
import numpy as np
from qpsolvers import solve_qp

# Quadratic term matrix (P)
P = np.array([[2, 0], [0, 2]])

# Linear term vector (q) reshaped as a column vector
q = np.array([-2, -2])

# Constraint matrix (G) and RHS vector (h)
G = np.array([[1, 1], [-1, 1], [1, -1], [-1, -1]])
h = np.array([1, 1, 1, 1])

# Solve the QP problem using qpsolver
x = solve_qp(P=P, q=q, G=G, h=h, solver="clarabel")
print(f"QP solution: {x = }")
```

The code provided to solution to be $\begin{bmatrix} x & y \end{bmatrix}^T = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$. The Clarabel (Interior Point Conic Optimization for Rust and Python) was chosen because it showed the highest accuracy of all available solvers [1].

[1] As per the project description of qpsolvers 4.3.3 on pypi.org

Ex. 2(f) At the minimum, which constraints are active (if any) and what are their associated Lagrange multipliers?

Ans. 2(f) At the minimum, the only constraint that is active is,

$$h_1(x, y) = x + y - 1 \rightarrow h(x^*, y^*) = x^* + y^* - 1 = 0.5 + 0.5 - 1 = 0 \tag{13}$$

To find the value of μ_1 we can simply go back and substitute the values of x^*, y^* in $\nabla_X L$,

$$\nabla_X L(X^*) = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 2 + \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = 0 \tag{14}$$

$$= -1 - 1 + \mu_1 + \mu_1 = 0 \tag{15}$$

$$= 2\mu_1 = 2 \tag{16}$$

$$\Rightarrow \mu_1 = 1 \tag{17}$$

When it comes to the other constraints, for them to satisfy the condition $\mu_i h_i(x^*, y^*) = 0 \ \ \forall i = 2, 3, 4$,

$$h_2(x^*, y^*) = -x^* + y^* - 1 = -0.5 + 0.5 - 1 = -1 \qquad \Rightarrow \mu_2 = 0 \tag{18}$$

$$h_3(x^*, y^*) = x^* - y^* - 1 = 0.5 - 0.5 - 1 = -1 \qquad \Rightarrow \mu_3 = 0 \tag{19}$$

$$h_4(x^*, y^*) = -x^* - y^* - 1 = -0.5 - 0.5 - 1 = -2 \qquad \Rightarrow \mu_4 = 0 \tag{20}$$

Exercise: 3 Consider the following optimization problem,

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{1}$$

Where, $\mathbf{Q} \in \mathbb{R}^{n\times n} > 0$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$ is full rank with $m < n$ and $\mathbf{b} \in \mathbb{R}^m$ is an arbitrary vector.
Please see "HW_Series1_Exercise1_2_3.ipynb" for the python code

Ex. 3(a) Write the Lagrangian of the optimization problem as well as the KKT conditions for optimality

Ans. 3(a) It is given that,

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{2}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \Rightarrow g(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = 0 \tag{3}$$

Based on above equations, we can write the Lagrangian as,

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda^T g(\mathbf{x}) \tag{4}$$

$$= \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \tag{5}$$

Based on the above formulated Lagrangian, the KKT Conditions can be formulated as,

$$\nabla_{\mathbf{x}} L = \mathbf{Q}\mathbf{x} + \mathbf{A}^T\lambda = 0 \tag{6}$$

$$\nabla_{\lambda} L = \mathbf{A}\mathbf{x} - \mathbf{b} = 0 \tag{7}$$

Ex. 3(b) Solve the KKT system and find the optimal Lagrange multipliers as a function of $\mathbf{Q}$, $\mathbf{A}$ and $\mathbf{b}$.

Ans. 3(b) Based on the Lagrangian and KKT Conditions formulated above, we can find $\mathbf{x}$ and λ as shown below,

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \tag{8}$$

$$\Rightarrow \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \tag{9}$$

Ex. 3(c) Use the above results to compute the minimum of the function below (and the value of $\mathbf{x}$ and of the Lagrange multipliers)

$$\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} \mathbf{x} \tag{10}$$

under the constraint that the sum of the components of the vector $\mathbf{x} \in \mathbb{R}^3$ should be equal to 1. Verify that the constraint is indeed satisfied for your result. (Hint: use python for all your numerical computation.)

Ans. 3(c) We now have,

$$\mathbf{Q} = \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} \tag{11}$$

And a constraint is given as,

$$x + y + z = 1 \tag{12}$$

$$\Rightarrow x + y + z - 1 = 0 \tag{13}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 1 = 0 \tag{14}$$

$$\text{Where, } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;,\; \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \;,\; \mathbf{b} = 1$$

The problem can be solved using the below python code,

```
import numpy as np

# Quadratic term matrix (Q)
Q = np.array([[100, 2, 1], [2, 10, 3], [1, 3, 1]])

# Constraint matrix (A) and RHS term (b)
A = np.ones((1, 3))
b = np.array([1])

# Stacking the LHS matrices
lhs_a = np.vstack((np.hstack((Q, A.T)), np.hstack((A, [[0]]))))

# stacking the RHS matrices
rhs_b = np.vstack((np.zeros((3, 1)), [1]))

# Solve the QP problem using numpy.linalg.solve
x = np.linalg.solve(lhs_a, rhs_b)
print(f"QP solution: {x = }")
```

The solution that the code provides is,

$$\begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} -0.00404858 \\ -0.40080972 \\ 1.4048583 \\ -0.19838057 \end{bmatrix} \tag{15}$$