Unsteady Aerodynamics Assignment

Construction of Non-simple region of an expansion wave in a shock tube using MOC

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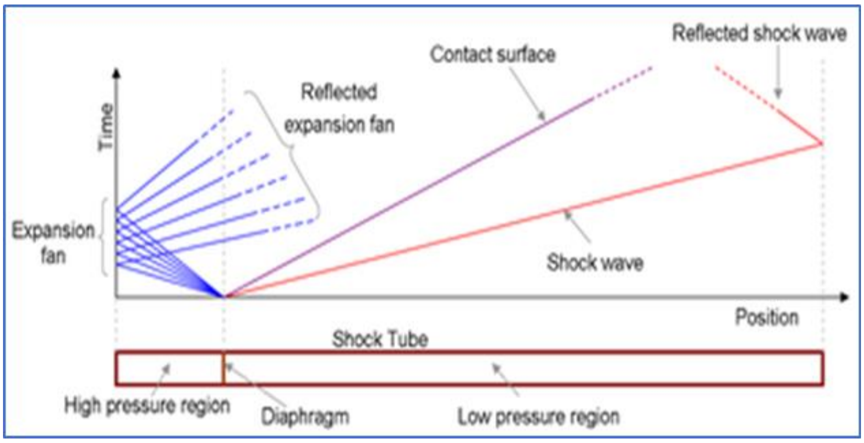
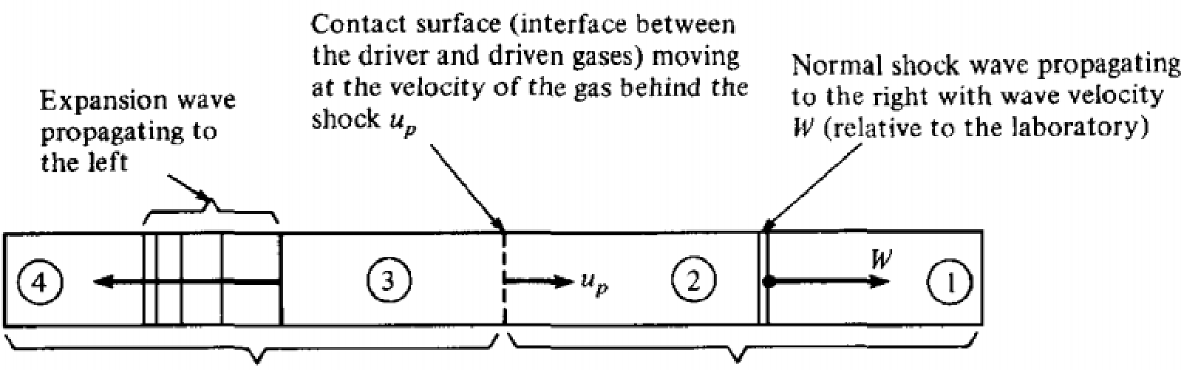
A simple shock tube is a tube, rectangular or circular in cross-section, usually constructed of metal, in which a gas at low pressure and a gas at high pressure are separated using some form of diaphragm.

The diaphragm suddenly bursts open under predetermined conditions to produce a wave propagating through the low-pressure section. The shock that eventually forms increases the temperature and pressure of the test gas and induces a flow in the direction of the shock wave. Simultaneously, a rarefaction wave, often referred to as the Prandtl-Meyer wave, travels back in to the driver gas.

The following figure shows different waves which are formed in the tube once the diaphragm is ruptured.



In this report, we have constructed the non-simple region of a reflected expansion wave using MATLAB. The entire algorithm has been described in the following sections. Through the report, we use the numbering of regions as shown in the figure below.



After the algorithm, the results have been discussed. Also, our code has been included as an appendix to the report.

Defining Initial conditions

$P_4/P_1 = 5$

$T_1 = T_4 = 300$ K

Gas → air ⇒ $R = 287$ and $\gamma = 1.4$

Resolution of plot $(n) = 4$ (number of characteristic lines to use for calculation)

Length of driving section $(l) = 3$ m

Shock tube relations

In order to construct the simple and non-simple regions of the expansion wave, we require a_4, a_3 and u_3. We used the following procedure to determine them using the initial conditions:

(i) $a_1 = \sqrt{\gamma R T_1} = 347.1887$ m/s and $a_4 = \sqrt{\gamma R T_4} = \boxed{347.1887 \text{ m/s}}$

(ii) $$P_4/P_1 = P_2/P_1 \left[1 - \frac{(\gamma_4 - 1)(a_1/a_4)(P_2/P_1 - 1)}{\sqrt{2\gamma_1 [2\gamma_1 + (\gamma_1+1)(P_2/P_1 - 1)]}} \right]^{\frac{-2\gamma_4}{\gamma_4 - 1}}$$

P_2/P_1 can be obtained by iteratively solving the above equation.

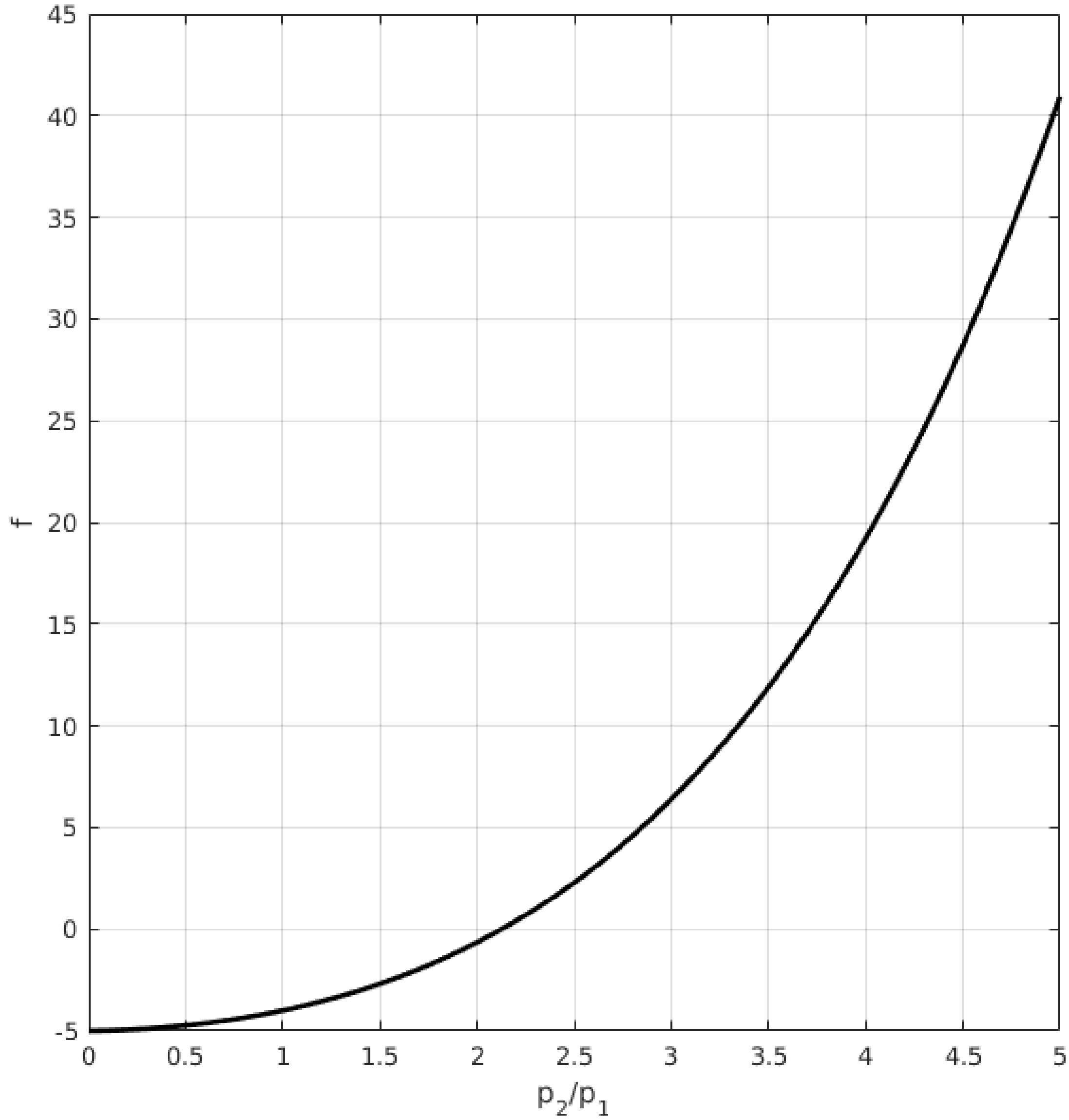
For this, we define

$$f = [RHS] - P_4/P_1$$

Hence, we will get the correct P_2/P_1 when $f = 0$.

If we plot f against P_2/P_1, we will observe that f monotonically increases with P_2/P_1, as shown below. Hence, we don't need to implement Newton-Rapson method by deriving f'.

This simplefyes the iterations to: (i) initialize $P_2/P_1 = P_4/P_1$
(ii) if $f > 0$ then reduce P_2/P_1
(iii) if $f < 0$ then increase P_2/P_1
(tolerance of $f = 0 \pm 0.1$)



Plot of *f* vs p_2/p_1

As seen in the above figure, we got:

$$p_2/p_1 = 2.13 \quad \text{when} \quad p_4/p_1 = 5$$

(iii) We know, $p_2 = p_3 \Rightarrow p_3/p_1 = p_2/p_1$

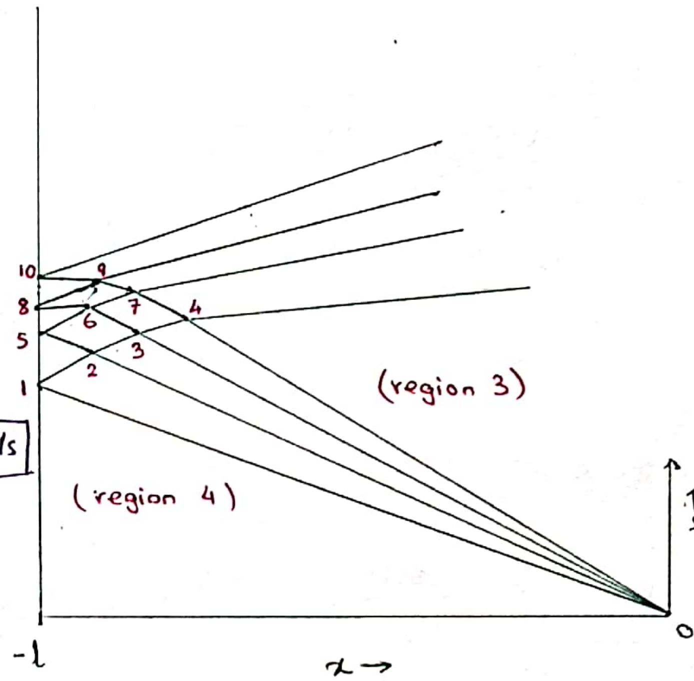
$$\Rightarrow p_3/p_4 = p_2/p_1 \div p_4/p_1 = 0.4260$$

$$\Rightarrow T_3/T_4 = (p_3/p_4)^{\gamma-1/\gamma}$$

$$\Rightarrow T_3 = T_4 \times (T_3/T_4) = 235.0922 \text{ K}$$

Hence, $a_3 = \sqrt{\gamma R T_3} = \boxed{307.3436 \text{ m/s}}$

(iv) For the rest of this report, we have followed the numbering of pts. as shown in the adjecent figure.



We know, $(J_+)_1 = (J_+)_4$

$$\Rightarrow \cancel{u_4}^{\,0} + \frac{2}{\gamma-1} a_4 = u_3 + \frac{2}{\gamma-1} a_3$$

$$\Rightarrow u_3 = \frac{2}{\gamma-1}(a_4 - a_3) = \boxed{199.2258 \text{ m/s}}$$

Now that we have determined a_4, a_3 and u_3, we can proceed towards constructing the non-simple region of the expansion wave using method to characteristics.

We know, $$u = \frac{2}{\gamma+1}(a_4 + x/t) \quad \text{[at pts. 1, 2, 3, 4]} \quad -(1)$$

To generate c^- lines of the required $n=4$ between regions 4 and 3, we need to vary the value of x/t from $(-a_4)$ to $(u_3 - a_3)$.

We got, $\boxed{x/t = \{-347.1887, -267.4984, -187.8081, -108.1173\}}$

We know, $J_+ = u + \frac{2}{\gamma-1} a$ and $(J_+)_1 = (J_+)_2 = (J_+)_3 = (J_+)_4 = \frac{2}{\gamma-1} a_4$

$\Rightarrow \frac{2a}{\gamma-1} = \frac{2}{\gamma-1} a_4 - u$ — ② [at pts. 1, 2, 3, 4] — ②

Also, $J_- = u - \frac{2}{\gamma-1} a$

$\Rightarrow J_- = \frac{4}{\gamma+1} (a_4 + x/t) - \frac{2}{\gamma-1} a_4$ [at pts. 1, 2, 3, 4]

Thus, for i^{th} incident line : $(J_-)_i = \frac{4}{\gamma+1} (a_4 + (x/t)_i) - \frac{2}{\gamma-1} a_4$

Thus we got $\boxed{J_- = \{-1735.9,\ -1603.1,\ -1470.3,\ -1337.5\}}$

Further, at wall pts. (1, 5, 8, 10), $u = 0$

Hence, $J_+ = \frac{2}{\gamma-1} a$ and $J_- = -\frac{2}{\gamma-1} a$

So, we can say that $\underbrace{(J_+)_i}_{i^{th}\text{ reflected line}} = - \underbrace{(J_-)_i}_{i^{th}\text{ incident line}}$

Therefore, we have determined J_+ and J_- for all pts. (1 - 10).

Next, we define $u_{ij} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ & u_5 & u_6 & u_7 \\ & & u_8 & u_9 \\ & & & u_{10} \end{bmatrix}$ and similarly a_{ij}

Here, u_{ij} and a_{ij} are the local u and a at the intersection of the j^{th} incident and i^{th} reflected line.

Now, we can easily obtain the u and a at all pts. (1 - 10).

$$u_{ij} = \frac{(J_+)_i + (J_-)_j}{2} \quad \text{and} \quad a_{ij} = \frac{\gamma-1}{4}\left[(J_+)_i - (J_-)_j\right]$$

We got,

$$u_{ij} = \begin{bmatrix} 0 & 66.4086 & 132.8172 & 199.2258 \\ & 0 & 66.4086 & 132.8172 \\ & & 0 & 66.4086 \\ & & & 0 \end{bmatrix}$$

and $\quad a_{ij} = \begin{bmatrix} 347.1887 & 333.9070 & 320.6253 & 307.3436 \\ & 320.6253 & 307.3436 & 294.0618 \\ & & 294.0618 & 280.7801 \\ & & & 267.4984 \end{bmatrix}$

Therefore, the u and a has been determined at all points in the non-simple region. We can now proceed towards generating the char. lines of the expansion wave on the x-t plane.

Plotting the results

To generate the plot, we need to determine the locations of all points (1 – 10) in the t-x plane.

So, we define $(x,t)\big|_{i,j}$ → intersection pt. of i^{th} reflected and j^{th} incident characteristic line.

We already know u_{ij} and a_{ij}. So we can find the slopes of the C^+ and C^- lines at any Location using:

$$mp\big|_{i,j} = \frac{1}{u_{ij} + a_{ij}} \quad \text{and} \quad mn\big|_{i,j} = \frac{1}{u_{ij} - a_{ij}}$$

(C^+ line) (C^- line)

We got,

$$mp_{ij} = \begin{bmatrix} 0.0029 & 0.0025 & 0.0022 & 0.0020 \\ & 0.0031 & 0.0027 & 0.0023 \\ & & 0.0034 & 0.0029 \\ & & & 0.0037 \end{bmatrix}$$

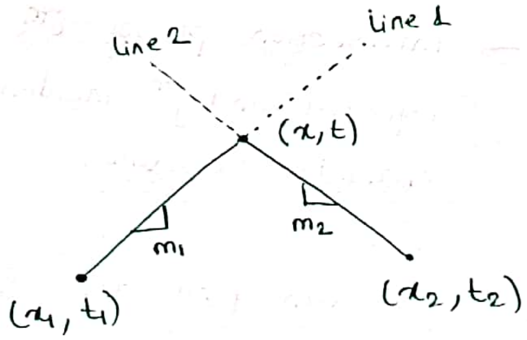
and

$$mn_{ij} = \begin{bmatrix} -0.0029 & -0.0037 & -0.0053 & -0.0092 \\ & -0.0031 & -0.0042 & -0.0062 \\ & & -0.0034 & -0.0047 \\ & & & -0.0037 \end{bmatrix}$$

To find the slope of interpolated line segment between two pts., we used Heun's Method, given by:

$$m_{12} = \frac{m_1 + m_2}{2}$$

To find the intersection pts. 1 to 10, we used the formulation described below. Note that, through this formulation, the pts. (1 - 10) are determined in the order of numbering.



Here (x,t) is the pt. we need to find.

We assume,

$(x_2,t_2) \rightarrow$ pt. lying on same incident and previous reflected line.

$(x_1,t_1) \rightarrow$ pt. lying on same reflected and previous incident line

Hence, $(x,t)_{1-10}$ can be determined (in order) through the following general case:

Eqn. of line 1 $\rightarrow$ $\dfrac{t-t_1}{x-x_1} = m_1$ — ①

Eqn. of line 2 $\rightarrow$ $\dfrac{t-t_2}{x-x_2} = m_2$ — ②

Solving ① and ②, we get:

$$x = \frac{m_1 x_1 - m_2 x_2 + t_2 - t_1}{m_1 - m_2} \quad \text{and} \quad t = m_1(x - x_1) + t_1$$

The above has 2 special cases:

(i) Pts. lying on 1st reflected line $(i=1)$ $\rightarrow$ (1, 2, 3, 4)

Here, $(x_2, t_2) \rightarrow$ origin $\Rightarrow x_2 = t_2 = 0$

and $m_2 = mn_{ij}$ (without averaging, since it lyes in the simple region)

(ii) Pts. lying on the wall $(i=j)$ $\rightarrow$ (1, 5, 8, 10)

Here, $m_1 = \infty$ since line 1: $x = -l$

So, we directly get $x = -l$ and $t = m_2(-l - x_2) + t_2$

Note that pt. 1 satisfies both case (i) and case (ii)

We obtained,

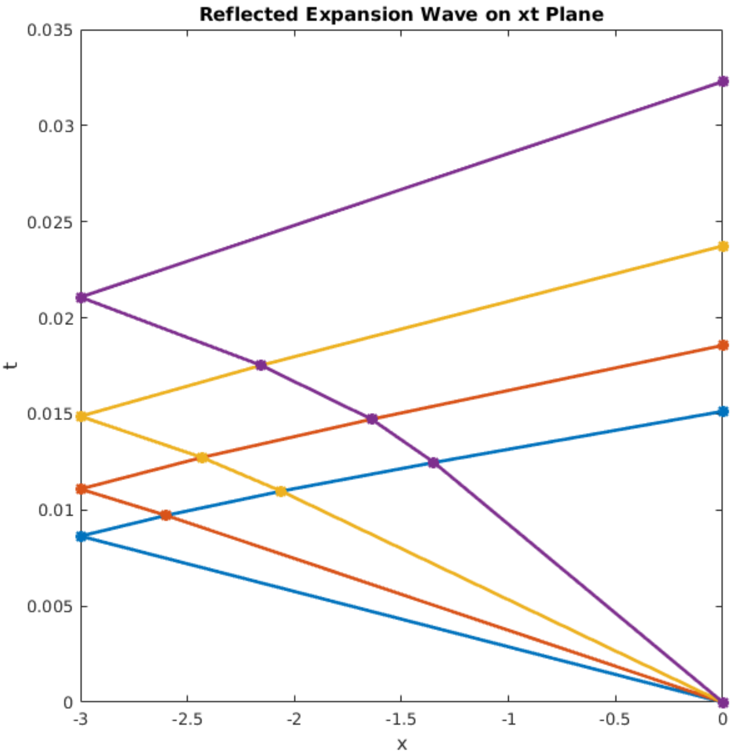
$$x_{ij} = \begin{bmatrix} -3 & -2.5995 & -2.0623 & -1.3485 \\ & -3 & -2.6326 & -1.6399 \\ & & -3 & -2.1560 \\ & & & -3 \end{bmatrix}$$

and

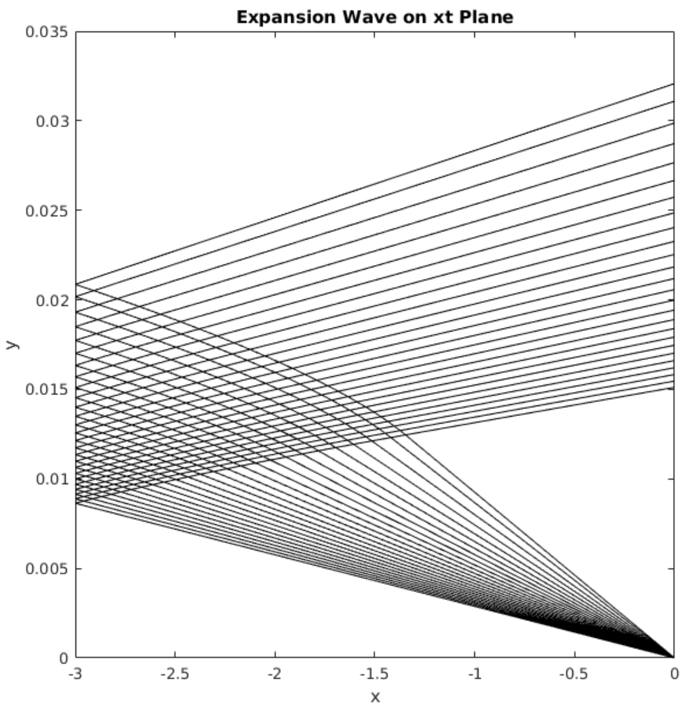
$$t_{ij} = \begin{bmatrix} 0.0086 & 0.0097 & 0.0110 & 0.0125 \\ & 0.0111 & 0.0127 & 0.0147 \\ & & 0.0149 & 0.0175 \\ & & & 0.0211 \end{bmatrix}$$

So far, all the quantities abtained through the MATLAB programe and through the hand-calculations matched well. Hence, we could proceed fowards generating the plots.

To do this in MATLAB, we started with the 1st incident line from origin and moved along the same until we reached the wall. Then, we followed the 1st reflected line until we reached $x = 0$. We repeated the same for all 4 lines.

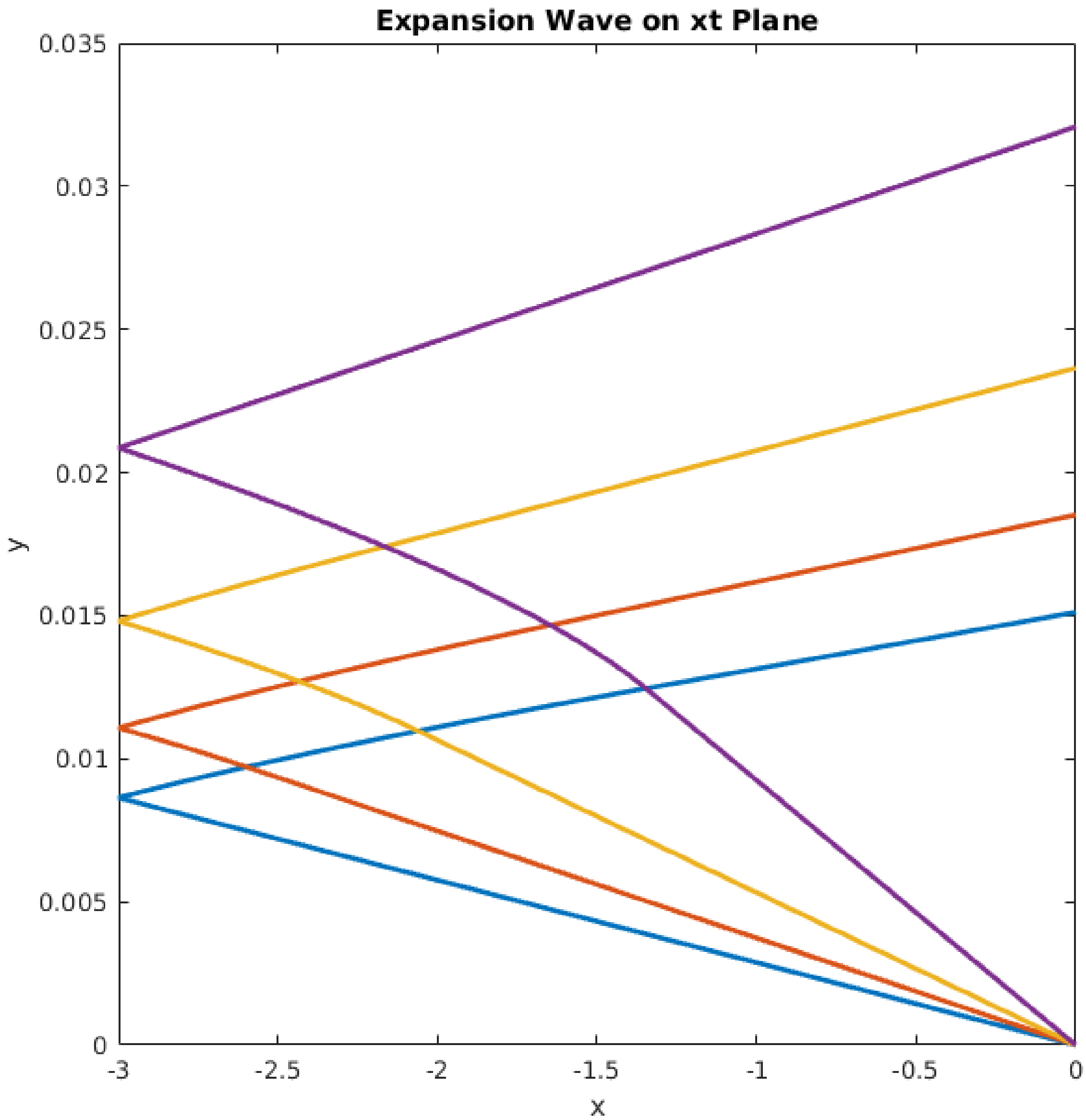


Since the program worked as expected for 4 characteristic lines, we can now generate the same plot using more number of characteristic lines. For $n = 25$, we got the following:



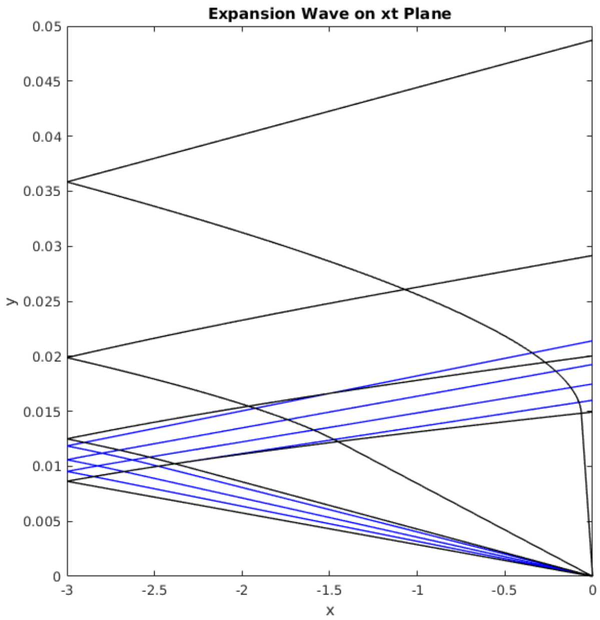
To increase the resolution of the plot, we need to increase *n* to a larger number like *100*. But then, with 100 lines in the graph, it would be difficult to comprehend. Thus we defined another parameter *n_plot* to choose the number of lines to plot.

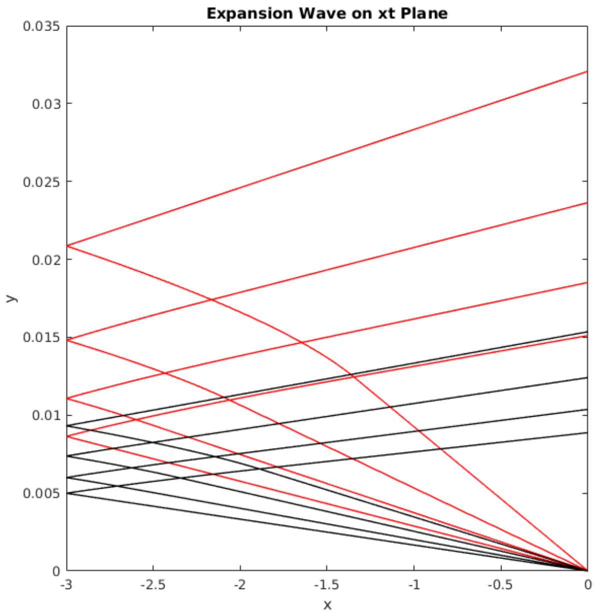
Hence, for $n = 100$ and $n\_plot = 4$, we get:



Thus, this graph is much smoother than the one obtained with just 4 lines.

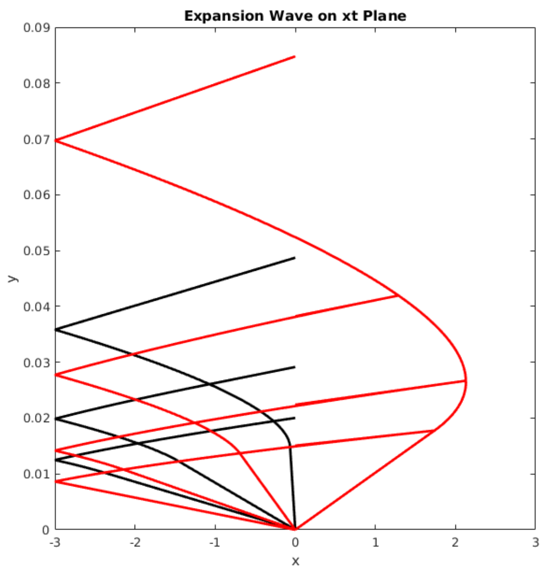
We made some observations when we varied the pressure ratio and the initial temperature of the gases. As we increased the pressure ratio or the temperature of the driving gas, the expansion wave travelled slower in the *x-t* plane. This is intuitively correct because a larger pressure ratio or a hotter driving gase, both increase the strength of the travelling shock wave. As a result the velocity of mass motion, i.e. the contact surface velocity u_3 increases. Due to which, the tail of the expansion wave travels slower (at $u_3 - a_3$).





Blue: $p_4/p_1 = 2$; Black: $p_4/p_1 = 10$ **Black: $T4 = 300\ K$; Red: $T4 = 900\ K$**

Finally, we observed another interesting thing. When the pressure ratio became too high, the tail of the expansion actually travelled towards the positive *x* direction, and then eventually travelled back towards region 4.



$p_4/p_1 = 20$

Appendix

MATLAB Code

MOC for reflected expansion wave in shock-tube

This script plots the x-t graph for the reflection of the expansion wave in a 1-D shock-tube using method of characteristics (MOC).

Contents

Initialisation

This section defines the properties at the driving the driven sections.

```
clear

n = 100;    % solver resolution
n_plot = 4;   % number of lines to plot

gamma = 1.4;    % adiabatic constant of air
R = 287;    % gas constant of air

PR41 = 5;    % pressure ratio = P4/P1
T1 = 300;   % driving section temperature
T4 = 300;    % driven section temperature
l = 3;    % length of driving section
```

Shock tube relations

This section uses the shocktube relations to evaluate flow quantities in regions 3 and 4.

```
a1 = (gamma*R*T1)^0.5;
a4 = (gamma*R*T4)^0.5;

% iteratively solving for PR21
PR21 = PR41;
f = 1;
while abs(f)>0.1
    f = -PR41 + PR21*(1 - ((gamma-1)*(a1/a4)*(PR21-1)/...
        ((2*gamma*(2*gamma + (gamma+1)*(PR21-1)))^0.5)))^(-2*gamma/(gamma-1));
    if f>0
        PR21 = PR21 - 0.01;
    else
        PR21 = PR21 + 0.01;
    end
end

PR34 = PR21/PR41;

T2 = T1*PR21*((gamma+1)/(gamma-1) + PR21)/(1 + PR21*(gamma+1)/(gamma-1));
a2 = (gamma*R*T2)^0.5;
T3 = T4*(PR34^((gamma-1)/gamma));
a3 = (gamma*R*T3)^0.5;
u3 = 2*(a4 - a3)/(gamma-1);
```

MOC solver routine

This section uses the method-of-characteristics to evaluate behaviour of the reflected expansion wave in the x-t plane.

```
xbyt = -a4:((u3 - a3 + a4)/(n-1)):(u3 - a3);    % equally spacing (x/t) between (-a4) and (u3-a3)
jn = (a4 + xbyt)*4/(gamma+1) - 2*a4/(gamma-1);  % vector containing j-minus invariants
jp = -jn;   % vector containing j-plus invariants
u = zeros(n);   % (n X n) matrix containing local u at points of intersection
a = u; x = u; t = u; mp = u; mn = u;    % local a, c-minus slope, c-plus slope and (x,t) coordinates

for i = 1:n
    for j = i:n
        u(i,j) = (jp(i) + jn(j))/2;
        a(i,j) = (jp(i) - jn(j))*(gamma-1)/4;
        mp(i,j) = 1/(u(i,j) + a(i,j));
        mn(i,j) = 1/(u(i,j) - a(i,j));

        if i==1 % for points lying on 1st reflected line
            x2 = 0; t2 = 0; m2 = mn(i,j);
        else
            x2 = x(i-1,j); t2 = t(i-1,j);
            m2 = (mn(i-1,j) + mn(i,j))/2;
        end

        if j==i % for points lying at wall
            x(i,j) = -l;
            t(i,j) = m2*(x(i,j) - x2) + t2;
        else
            x1 = x(i,j-1); t1 = t(i,j-1);
            m1 = (mp(i,j-1) + mp(i,j))/2;

            x(i,j) = (m1*x1 - m2*x2 + t2 - t1)/(m1 - m2);
            t(i,j) = m1*(x(i,j) - x1) + t1;
        end
    end
end
```

Plots

This section generates the plots of the characteristic lines on the x-t plane.

```
cx = zeros(1,n+2); ct = cx;     % characteristic lines to plot in x-t plane

for k = [(1:round((n-1)/(n_plot-1)):(n-1)),n]
    cx(1) = 0; ct(1) = 0;
    for j = 1:k
        cx(j+1) = x(j,k);
        ct(j+1) = t(j,k);
    end
    for j = (k+1):n
        cx(j+1) = x(k,j);
        ct(j+1) = t(k,j);
    end
    cx(n+2) = 0;
    ct(n+2) = -mp(k,n)*cx(n+1) + ct(n+1);
    plot(cx,ct,'LineWidth',2,'Color','k');
    xlabel('x'); ylabel('y'); title('Expansion Wave on xt Plane')
    hold on
end

% end of script
```