Theory

Anticipating the need for delay in achieving a causal linear-phase filter, the desired frequency response is defined as

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\frac{N-1}{2}}, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

The corresponding impulse response is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\frac{N-1}{2})}d\omega \\ &= \frac{\sin(\omega_c(n-\frac{N-1}{2}))}{\pi(n-\frac{N-1}{2})}. \end{aligned}$$

Clearly, $h_d(n)$ has infinite duration. To create a finite-duration linear phase causal filter of length N, we define

$$h(n) = h_d(n)w(n),$$

where $w(n)$ is a window of length N.

Various window functions are given below:

1. **Rectangular Window:**
$$w(n) = 1, \quad 0 \le n \le N-1.$$

2. **Bartlett Window:**
$$w(n) = \begin{cases} \frac{2n}{N-1}, & 0 \le n \le \frac{N-1}{2} \\ 2 - \frac{2n}{N-1}, & \frac{N-1}{2} \le n \le N-1 \end{cases}$$

3. **Hamming Window:**
$$w(n) = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right), 0 \le n \le N-1.$$

4. **Blackman Window:**
$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{2\pi n}{N-1}\right), 0 \le n \le N-1.$$

5. **Kaiser Window:** Kaiser has proposed a flexible family of windows defined by
$$w(n) = \frac{I_0\left[\omega_a\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n-\left(\frac{N-1}{2}\right)\right)^2}\right]}{I_0\left[\omega_a\left(\frac{N-1}{2}\right)\right]}, 0 \le n \le N-1.$$
where $I_0(.)$ is the modified zeroth order Bessel function of the first kind. Kaiser has shown that these windows are nearly optimum in the sense of having the largest energy in the main lobe for a given peak side lobe amplitude. The parameter ω_a can be adjusted so as to tradeoff main-lobe width for side-lobe amplitude. Typical values of $\omega_a\left(\frac{N-1}{2}\right)$ are in the range $4 < \omega_a\left(\frac{N-1}{2}\right) < 9$.

Window Type	Width of Main Lobe	Peak Amplitude of Side Lobe
Rectangular	$\frac{4\pi}{N}$	-13 dB
Bartlett	$\frac{8\pi}{N}$	-25 dB
Hamming	$\frac{8\pi}{N}$	-31 dB
Blackman	$\frac{12\pi}{N}$	-57 dB
Kaiser	read from plot	read from plot

Table 0.1: Characteristics of Various Windows.