CE3007 - Digital Signal Processing

02B_P1: Discrete Time Systems

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Part I

Discrete Time Systems

Discrete Time Systems

References for Discrete Time Systems

1. Reading:
    - Oppenheim, "Signal and Systems" (2nd Edition): Sec 1.5, 1.6, Sec 2.0, 2.1, 2.3 and 2.4
    - Oppenheim, "Discrete Time Signals and System (2nd edition) : Sec 2.2, 2.3, 2.4 and 2.5
2. Video:
    - Oppenheim: MIT Signals and Systems , Lect Video: 4,5,6
      Oppenheim: MIT Digital Signal Processing, Lect Video: 2,3

DT system

1. What is a discrete time system?
2. Types of DT system (memory, memoryless), (causal,not-causal), (finite/infinite impulse response), (stable, non-stable), (linear,not-linear), (time-invariant/time-variant)
3. Focus: linear and time-invariant (LTI) and stable.
4. Representing IO relationship of LTI system:
   - In time domain - impulse response and the convolution.
   - In frequency domain (Fourier Domain) as frequency response.
   - As difference equation.
   - In block diagram.
   - In frequency domain (Z-domain) as transfer function.

DT System: Definition

1. A DT system $f(.)$ processes a DT input sequence $x[n]$ to produce a DT output sequence $y[n]$.

$$y[n] = f(x[n]) \tag{1}$$

$$x[n] = \{\ldots, x[-1], x[0], x[1], x[2], \ldots\} \tag{2}$$

where $n \in \mathbb{Z}$. The output $y[n]$ at index n can depend on all values of $x[n]$.

DT System: Characteristic

1. A DT system can be classify according to the following characteristics
   - (linear,not-linear),
   - (time-invariant vs time-variant)
   - (stable vs non-stable) alternatively BIBO criterion.
   - (causal,non-causal),
   - Others: (memory, memoryless), (finite impulse response, infinite impulse response), ...

Remark: For system deemed to have the above properties, the property must hold for all types of input $x[n]$.

DT System Characteristics: Linearity

The system $f(.)$ is linear if it satisfies the superposition property. That is,the input $x[n]$ can be decomposed to study the output.

$$\begin{aligned} y[n] = f(x[n]) &= f(\alpha x_1[n] + \beta x_2[n]) \\ &= f(\alpha x_1[n]) + f(\beta x_2[n]) \quad (3) \\ &= \alpha f(x_1[n]) + \beta f(x_2[n]) \quad (4) \end{aligned}$$

- where $\alpha, \beta \in \mathbb{R}$.
- Eq 3 is the additive property,
- Eq 4 is the homogeneity or scaling property.
- These two properties together forms the principle of superposition.

Systems which do not possess the above linearity property are called non-linear systems.

Characteristic: Time (shift) invariance

A time invariant DT system $f(.)$ has the property that

- when the input $x[n]$ is delayed by K samples, the output is similarly delayed by K samples.
- Example: time invariant means,

$$\begin{aligned} f(x[n]) &= y[n] \\ \Longrightarrow f(x[n-K]) &= y[n-k] \end{aligned} \tag{5}$$

DT System Characteristics: Stability

We want to design systems to be stable.
If an LTI system is stable, it also means:

1. Its output is bounded if its input is bounded (Bounded input Bounded Output **BIBO**) :
   - By bounded input means $|x[n]| \leq B_x < \infty, \ \forall n$.
2. Its impulse response $h[n]$ is absolute summable: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. See proof pg 25.
3. Its transfer function $H(z)$'s region of convergence (ROC) includes the unit circle. See Z-transform lecture (week 5).

DT System Characteristics: Memory, Causal

1. **memory**: A system has memory if $y[n]$ depends not only on $x[n]$ but past values of $x[n]$, e.g, $x[n-1], x[n-2], ..$ etc.
   - Example: $y[n] = x[n]^2$ has no memory.
   - $y[n] = 2.5x[n] + 3.4x[n-1]$ has memory
2. **causalilty**: A system is a causal system if its output depends only on past values of $x[n]$,
   - Example: $y[n] = f(x[n], x[n-1], x[n-2], ..)$ is causal.
   - $y[n] = f(.., x[n+2], x[n+1], x[n], x[n-1], ..)$ is not causal.
   - For causal LTI system, its impulse response h[n] is causal.

DT System Characteristics: Impulse response (finite/infinite)

The response $y[n]$ of a system to a DT impulse $\delta[n]$ input sequence is called its impulse response.

1. Finite impulse response: The output $y[n]$ becomes 0 after $n > N$, where $N < \infty$.
    - A finite impulse response system is ALWAYS stable.
2. Infinite impulse response: the output $y[n]$ does not become 0 as $n \to \infty$.
    - An infinite impulse response system may be either stable or unstable!
    - Check stability by examining if the impulse response is absolutely summable.

LTI System

In this course,

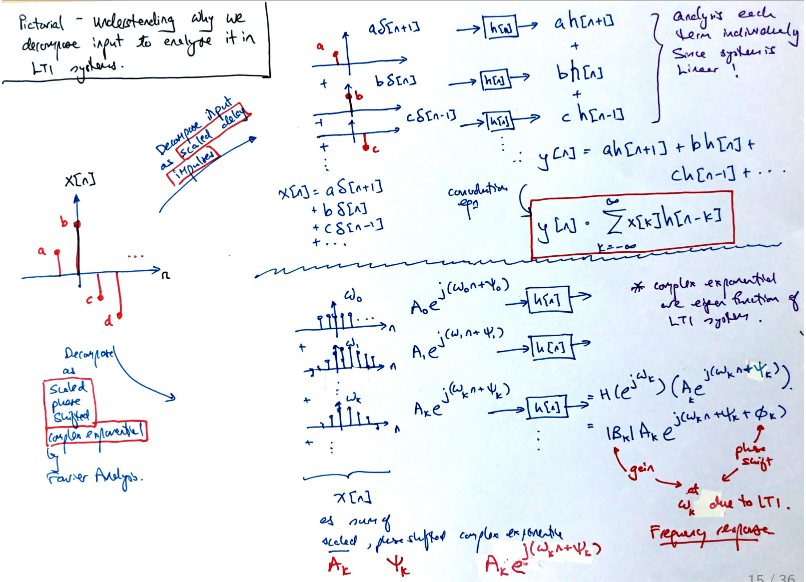
- Focus only on Linear Time-invariant (LTI) system.
- Design stable LTI system.
- If require real time processing, i.e, cannot use future values as input, system must be causal.

LTI System: Examining IO relationship

Because the system is linear and time invariant, the input/output relationship can be examined by decomposing the input $x[n]$ in two ways (see Fig Page 15):

1. Express $x[n]$ as a sum of scaled delayed DT impulses.
   - Using linearity and time invariance condition, we derive convolution equation
   - - Convolution equation shows that the output is a summation of scaled delayed impulse responses.
2. Express $x[n]$ as sum of scaled and phase shifted complex exponentials by Fourier Analysis.
   - complex exponentials are eigen functions of LTI system.
   - For each discrete frequency, we find the frequency response (gain and phase shift at that frequency).
   - the output is represented as the summation of scaled and phase shifted complex exponentials (decomposed $x[n]$) modified by the frequency response.

Decomposing input: Two ways

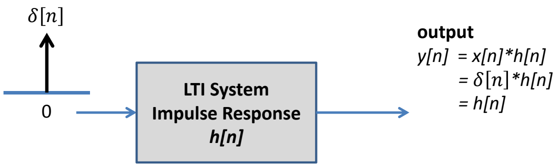


LTI system: Convolution Eq

Output of an LTI system for input $x[n]$ is given by the convolution equation

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{6}$$

where $h[n]$ is its impulse response.



LTI System: Convolution Eq

To proof the convolution relationship,

1. Decompose input $x[n]$ into scaled and delayed DT impulses,

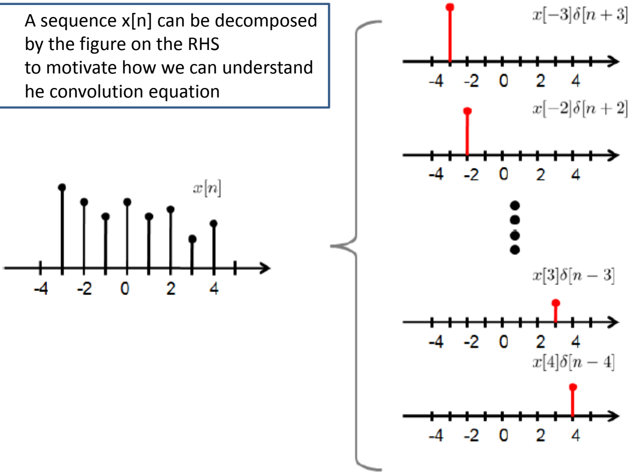
$$\begin{aligned} x[n] &= \{\ldots, x[-1], x[0], x[1], x[2], \ldots\} \\ &= \ldots + x[-1]\delta[n+1] + x[0]\delta[n] \\ &\quad + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots \end{aligned} \tag{7}$$

2. The LTI system $f(.)$ impulse response is $h[n] = f(\delta[n])$.
3. Using linearity and time invariance property, examine $f(.)$ for each $x[k]\delta[n-k]$ (Eq 7) separately.
4. The output for each term $x[k]\delta[n-k]$ is

$$\begin{aligned} y_k[n] &= f(x[k]\delta[n-k]) \\ &= x[k]f(\delta[n-k]) \\ &= x[k]h[n-k] \end{aligned} \tag{8}$$

Convolution Eq: Decomposing x[n] to scaled and delayed DT impulses

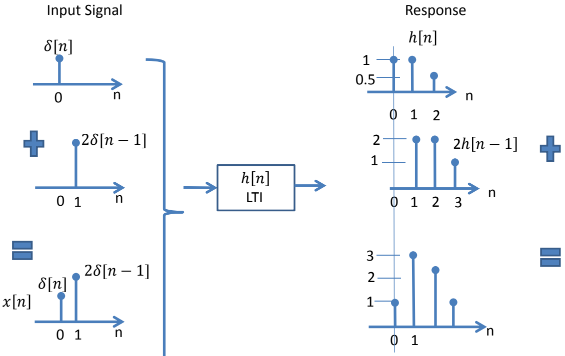
A sequence x[n] can be decomposed by the figure on the RHS
to motivate how we can understand
he convolution equation



Convolution Equation

Therefore by considering all $x[k]\delta[n-k]$ terms that makes up $x[n]$, we have the convolution equation,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{9}$$



Decompose x[n] into scaled delayed impulse

The output $y[n]$ is the sum of individual response of scaled and delayed impulse response

Summary: Impulse Response and Convolution Equation

The impulse response is the most important response to an LTI system. This is because an LTI system is fully characterise by $h[n]$,

1. Given $h[n]$ and $x[n]$, the output $y[n]$ is described by the convolution equation.
2. If $h[n]$ is absolutely summable, the system is stable. Proof in page 25.

Frequency Response: Decomposing $x[n]$ to complex exponential

For periodic DT signal, using Fourier Analysis, we can decompose $x[n]$ into sum of scaled and phased shifted complex exponentials

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn} \\ &= A_0 e^{j(\frac{2\pi}{N}0n+\phi_0)} + \ldots + A_k e^{j(\frac{2\pi}{N}kn+\phi_k)} + \\ &\qquad \ldots + A_{N-1} e^{j(\frac{2\pi}{N}(N-1)n+\phi_{(N-1)})} \end{aligned} \tag{10}$$

where $c_k = A_k e^{j\phi_k}$. If the signal is not periodic, the DT Fourier Transform is used.

Complex Exponentials are eigenfunction of LTI

Key idea:

- Represent $x[n]$ into sum of scaled and phase delayed version of $A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$
- Study how the LTI system will modify each $A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$

Proof: Consider $x[n] = e^{j\omega n}$, and $h[n]$ impulse response, the output of LTI system by convolution is:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\ &= \left(\sum_{k=-\infty}^{\infty} h[k]e^{j\omega(-k)}\right) e^{j\omega n} \\ &= H(e^{j\omega})e^{j\omega n} = \left(|H(e^{j\omega})|e^{j\angle H(e^{j\omega})}\right) e^{j(\omega n)} \end{aligned} \tag{11}$$

Frequency Response of LTI

- Above eqn shows that for input $e^{j\omega n}$, the output remains as $e^{j\omega n}$ but modified by the first term.
- The term $H(e^{j\omega})$ is a complex number, and it is also known as the (Frequency Response) of the system:

$$
\begin{aligned}
H(e^{j\omega}) &= \left(\sum_{k=-\infty}^{\infty} h[k] e^{j\omega(-k)} \right) && (12) \\
&= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\
&= |H(e^{j\omega})| e^{j\angle H(e^{j\omega})} && (13)
\end{aligned}
$$

- The components of $H(e^{j\omega})$ in polar form shows amplitude and phase component. These values introduce scaling and phase shift to the input $e^{j\omega n}$.
- Equation 12 is the well known Fourier Transform of the impulse response $h[k]$ to get the Frequency Response of the system.

Summary: Eigenfunction and Frequency Response of LTI

1. The LTI system will only modify the a single pure complex exponential by changing its amplitude and phase. No new signal is produced.
   - In other words, complex exponential are Eigen function of LTI systems.
2. Frequency response of a system $H(e^{j\omega})$ is produced by Fourier transforming $\mathcal{F}(h[n])$ the impulse response ,
   - Frequency response shows how each complex exponential (at different frequency) is modified (gain and phase shift) when it passes through an LTI.
   - The gain and phase change imposed by the system on the complex exponential at frequency ω is the magnitude and phase (polar form) of the complex value at $H(\omega)$.

Proof- Stability LTI systems

Showing BIBO stability.
Since $y[n] = x[n] * h[n]$, and $|x[n]| < B_x < \infty$ for all n (bounded input), then:

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]| \, |x[n-k]| \leq \sum_{k=-\infty}^{\infty} |h[k]| B_x \end{aligned} \tag{14}$$

- When $h[n]$ is absolutely summable, then $y[n]$ is bounded by B_x multiply with $\sum_{k=-\infty}^{\infty} |h[k]|$.
- When the output is finite, the system is BIBO.
- Eq 14 is a sufficient condition to guarantee stability.

Properties of LTI System: Commutative

1. Convolution is Commutative

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad (15)$$

$$= h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad (16)$$

The above can be shown by substituting $m = n - k$ into Eq 16, and $k = n - m$, then we have $y[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m]$ as in Eq 15.

2. Order of the convolution is un-important! See illustration in Pg 29.

Properties of LTI System: Associative

Given two LTI connected in series with impulse response $h_1[n]$ and $h_2[n]$, and input $x[n]$,the output $y[n]$ is

1. By associative property of convolution,

$$\begin{aligned} y[n] &= x[n] * h_1[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n] \\ &= x[n] * (h_1[n] * h_2[n]) \\ &= x[n] * h[n] \end{aligned} \tag{17}$$

2. From a system point of view, two LTI systems cascaded can be interpreted as one equivalent system $h[n]$ that is the convolution of the individual two systems. See illustration in Pg 29.

Properties of LTI System: Distributive

1. Convolution is distributive, specifically, convolution distributes over addition. Given two LTI connected in parallel with impulse response $h_1[n]$ and $h_2[n]$, and input $x[n]$,then the output

$$\begin{aligned} y[n] &= x[n] * (h_1[n] + h_2[n]) \\ &= (h_1[n] + h_2[n]) * x[n] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}$$

2. From a systems point of view, the equivalent impulse response of a system that has two LTI connected in parallel, is one that impulse response is the sum of the two. See illustration in Pg 29.

$$h[n] = (h_1[n] + h_2[n])$$

Properties of LTI System

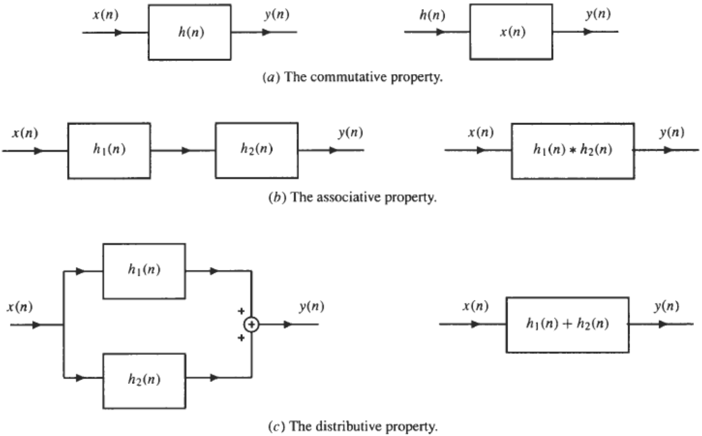


Figure: Illustration of Distributive property of convolution

Describing IO of LTI

1. Convolution Equation: $y[n] = x[n] * h[n]$
2. In Frequency domain (Fourier): $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, the frequency domain output is the multiplication of the Fourier transform of the input and impulse response.
3. By Constant Coefficient Difference equation

$$\begin{aligned} y[n] &= b[0]x[n] + b[1]x[n-1] + \dots \\ &\quad - a[1]y[n-1] - a[2]y[n-2] + \dots \end{aligned} \tag{18}$$

4. By Block diagram
5. In Frequency domain (z-transform): $Y(z) = X(z)H(z)$, the z domain output is the multiplication of the Z transform of the input and impulse response.

Linear Constant Coefficient Difference Equation

An important sub-class of LTI system are the N-order linear constant coefficient difference equation.

$$
\begin{aligned}
\sum_{k=0}^{N} a[k]y[n-k] &= \sum_{m=0}^{M} b[m]x[n-m] \\
a[0]y[n] &= \sum_{m=0}^{M} b[m]x[n-m] - \sum_{k=1}^{N} a[k]y[n-k]
\end{aligned}
$$

Usually have $a[0] = 1$, and hence

$$
\begin{aligned}
y[n] &= \sum_{m=0}^{M} b[m]x[n-m] - \sum_{k=1}^{N} a[k]y[n-k] \\
&= \quad \text{feedforward} \qquad\qquad \text{feedback}
\end{aligned}
$$

Linear Constant Coefficient Difference Equation (LCCDE)

A causal LTI system has the property that when the input is zero for all time, the output will also be zero for all time. Hence for LCCDE system's memory $\{x[n-m], y[n-k]\}$ must be zero - initial rest condition.

1. Finite Impulse Response (FIR): when $N = 0$, there is no feedback terms of past values of $y[n]$ in the eqn.
   - The output is fully characterise by the M terms of $b[m]$.
   - If input $x[n] = \delta[n]$, the impulse response is simply the coefficients of $b[m]$.
2. Infinite impulse response (IIR): when $N > 0$, then the feedback from past values of $y[n]$ will usually cause $y[n]$ not to become 0 even as $n \to \infty$, i.e, infinite impulse response.

see Oppenheim's notes in link `http://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring-2011/lecture-notes/MITRES_6_007S11_lec06.pdf`

Basic Building Blocks of LTI systems

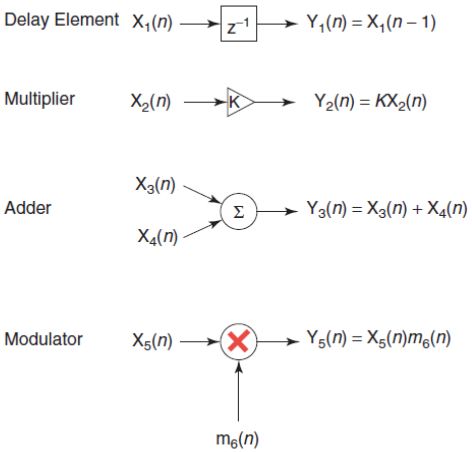


Figure: Basic Building Blocks of LTI systems

Example: A simple IIR Filter

$$y[n] = \sum_{m=0}^{2} b[m]x[n-m] - \sum_{k=1}^{2} a[k]y[n-k]$$

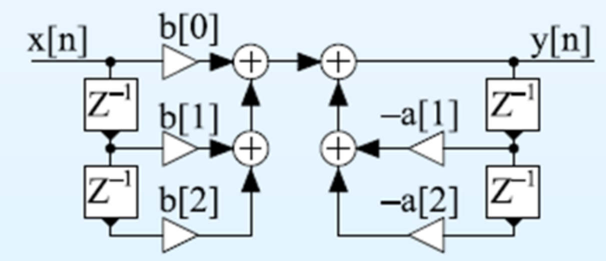


Figure: Building a IIR Filter

Frequency Domain

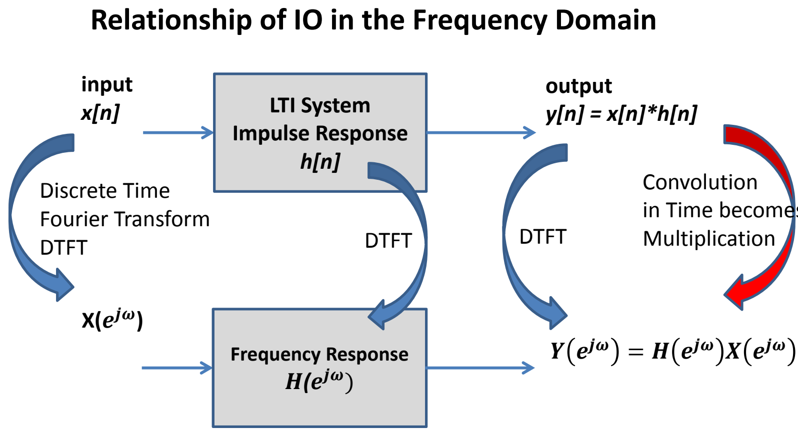


Figure: IO relationship in the Fourier Domain

Frequency Z-Domain

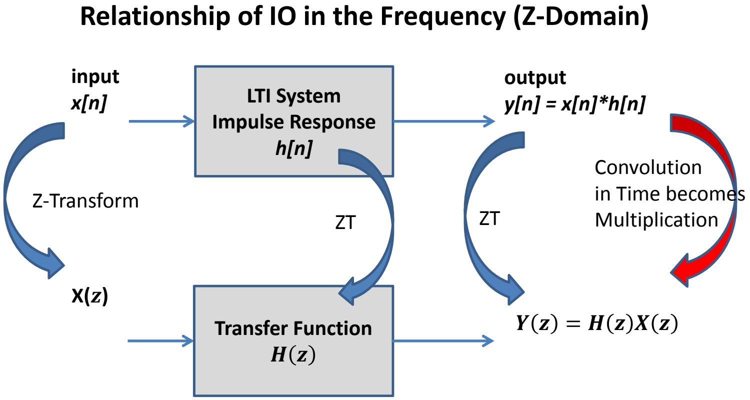


Figure: IO relationship in the Z-domain