Laboratory Exercises

- **Set 1 (24-01-2020): Euler method and Taylor methods of the second and third order**

  1. Refer to the *Mathematics Theory* exercises on *Numerical integration of differential equations*, and consider the first-order differential equations in *Questions 2, 3, 4, 7, 8, 9, 10*. Thereafter, perform the following tasks:
     (a) Find the analytical solutions of all the differential equations by integrating.
     (b) Decrease the step-size in each problem to $\Delta t = 0.01$. Then write a computer code, applying the Euler method over the full range in time.
     (c) Repeat the same procedure applying the second-order Taylor method.
     (d) For the autonomous equations, apply the third-order Taylor method.
     (e) For every differential equation, in a single plot, show the exact integral solution, the solutions of the Euler method and the Taylor methods.
     (f) Separately plot the difference between the exact result and the approximate results to compare the accuracy in every case.

- **Set 2 (24-01-2020): Basic physical systems of growth and decay**

  1. The position $x(t)$ of an object moving with a constant horizontal velocity v is given by $\dot{x} = v$. With $v = 40\,\mathrm{ms}^{-1}$ and $x(0) = 0$, find the exact integral of $x \equiv x(t)$. Then by Euler's method, numerically solve for $x(t)$ from $t = 0\,\mathrm{s}$ to $t = 10\,\mathrm{s}$. Perform this exercise for two different time steps, a small one and a large one, both of your choice. Compare the analytical result with the numerical result, and comment on the accuracy of the Euler method in this problem.
  2. Free-fall velocity $v(t)$ is given by $\dot{v} = -g$, where $g = 9.8\,\mathrm{ms}^{-2}$. Taking the initial condition $v(0) = 0$, carry out the same exercise as in the previous problem.
  3. The decay rate of a radioactive element, whose amount is $x(t)$, is governed by $\dot{x} = -\lambda x$ ($\lambda > 0$). Using the initial condition $x(0) = x_0$ write the analytical solution. Now for the following three radioactive isotopes of uranium, radium and lead,

     A. U-238, $T_{1/2} = 4.5 \times 10^9\,\mathrm{yrs}$ B. Ra-226, $T_{1/2} = 1600\,\mathrm{yrs}$ C. Pb-210, $T_{1/2} = 22\,\mathrm{yrs}$

     choose an appropriate time step Δt. Then setting $x_0 = 1\,\mathrm{unit}$, numerically integrate from $t = 0$ to $t = 20\Delta t$, by Euler's method, and the Taylor methods of the second and third orders. For each element, plot the numerical solutions along with the analytical solution to compare for accuracy.

- **Set 3 (07-02-2020): Growth and its saturation**

  1. Radioactive lead-210 (Pb-210), with a half-life of 22 years, decays to non-radioactive lead-206 (Pb-206). On the other hand, Pb-210 is replenished by the radioactive decay of radium-226 (Ra-226), which has a half-life of 1600 years.
     (a) For time scales that are much less than 1600 years, and comparable to the half-life of Pb-210, show how the decay rate of Pb-210 is approximated by $\dot{x} = r - \lambda x$, in which $x(t)$ is the amount of Pb-210, with r and λ being constant. (Hint: In the decay equation of Ra-226, expand the exponential function in a series and retain only the linear order for small time.)
     (b) Taking the initial amount of Ra-226 to be $1\,\mathrm{unit}$, estimate r and λ. With the initial value of $x = 0$, plot the analytical solution of $x(t)$.
     (c) Choosing a suitable time step Δt, numerically integrate the decay-rate equation by Euler's method, and the second and third-orders Taylor methods. Plot the numerical solutions along with the analytical solution to compare for accuracy. Separately plot the error between both.

     Note: The foregoing analysis is relevant in detecting art forgery.

2. The motion of an object falling through a liquid is described by $\dot{v} = A - Bv$, with A and B as constants. Consider now a steel ball of radius $r = 2\,\mathrm{mm}$ falling through a column of oil. The viscosity of the oil is $\eta = 2.42\,\mathrm{N\,s\,m^{-2}}$, and its density is $940\,\mathrm{kg\,m^{-3}}$. The density of steel is $7800\,\mathrm{kg\,m^{-3}}$, and $g = 9.8\,\mathrm{m\,s^{-2}}$.
   (a) If the drag coefficient for a spherical object is $k = 6\pi\eta r$, estimate A and B.
   (b) Calculate the terminal velocity and the approximate time after which the velocity starts to approach its terminal value.
   (c) Plot the velocity-time graph using the analytical solution of $v(t)$.
   (d) Choosing an appropriate time step Δt, numerically integrate by Euler's method, and the Taylor methods of the second and third orders. Plot the numerical solutions along with the analytical solution to compare for accuracy. Separately plot the error between both.
3. Worldwide, the human population $x(t)$ grows according to the logistic equation $\dot{x} = ax - bx^2$. Take $a \simeq 0.03$ per annum and $b \simeq 3 \times 10^{-12}$ as per the global census data of 1961.
   (a) Estimate the limiting value of the world population. Compare this value with the present global population (to be found by a Google search).
   (b) Plot $x(t)$ using its analytical solution starting with an initial value of $x_0 \simeq 3 \times 10^9$.
   (c) Choosing an appropriate time step Δt, numerically integrate by Euler's method, and the Taylor methods of the second and third orders. Plot the numerical solutions along with the analytical solution to compare for accuracy. Separately plot the error between both.

- **Set 4 (14-02-2020): Phase plot of first-order autonomous systems**

1. Refer to the *Mathematics Theory* exercises on *Phase plots and linear stability of first-order autonomous systems*, and consider all the first-order differential equations in *Question 1*. For every equation perform the following tasks:
   (a) Plot x–$\dot{x}$ using a plotting software. Observe the fixed point(s) (if any) and their stability.
   (b) Starting from $t = 0$ and with a suitable value of Δt, apply the Euler method to obtain $x(t)$. The initial value of x should be chosen near a fixed point, preferably a stable one, to show a convergence for $t \longrightarrow \infty$. If there are multiple fixed points, choose a stable one that is nearest to $x = 0$ (guided by the x–$\dot{x}$ plot). If there is no fixed point, then apply your judgement about the initial value of x. Clearly state the initial value of x and Δt in all cases.
   (c) Repeat the same procedure, applying the second and third-order Taylor methods.
   (d) In a single plot, show the solutions of the Euler method and the Taylor methods.
   (e) Check the correspondence between the $x(t)$ graph and the x–$\dot{x}$ graph. Especially observe the behaviour of $x(t)$ when $\dot{x} = 0$ and $t \longrightarrow \infty$.

- **Set 5 (14-02-2020): Air resistance**

1. The vertical fall of a skydiver is given by $m\dot{v} = mg - kv^2$. The total mass of the skydiver and his equipment is $110\,\mathrm{kg}$. The air drag coefficient is $k = 0.18\,\mathrm{kg\,m^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$.
   (a) Calculate the terminal velocity and the approximate time after which the velocity starts to approach its terminal value.
   (b) Plot the velocity-time graph using the analytical solution of $v(t)$.
   (c) Choosing an appropriate time step Δt, numerically integrate by Euler's method, and the Taylor methods of the second and third orders. Plot the numerical solutions along with the analytical solution to compare for accuracy.
2. The motion of a bicycle is driven by the power of its rider, and is resisted by the air drag. The relevant data for this system are power $P = 400\,\mathrm{W}$, the drag coefficient $b \simeq 0.5$, the total mass of the bicycle and rider $m = 70\,\mathrm{kg}$, the frontal area $A = 0.33\,\mathrm{m^2}$, the air density $\rho = 1.25\,\mathrm{kg\,m^{-3}}$ and the initial velocity $v_0 = 4\,\mathrm{m\,s^{-1}}$.

(a) Ignoring air drag, plot the velocity-time graph using the analytical solution of $v(t)$. Then, with a time step $\Delta t = 0.1\,\mathrm{s}$, numerically integrate by Euler's method, and the second and third-order Taylor methods. Plot the numerical solutions along with the analytical solution to compare the accuracy of the numerical methods.

(b) Estimate the limiting velocity when air drag is present. Then perform the same numerical integrations as was done without air drag. Plot the three numerical solutions with air drag, along with the analytical solution without air drag.

(c) With air drag present, plot the behaviour of v–$\dot{v}$. Note the correspondence between the $v(t)$ graph and the v–$\dot{v}$ graph, especially when $\dot{v} = 0$.

3. A projectile is fired with an initial velocity of $700\,\mathrm{m\,s^{-1}}$. If air drag is considered, then $(B/m) = 4\times10^{-5}\,\mathrm{m^{-1}}$. Start with a projection angle of $30°$. Then in steps of $5°$ go up to $55°$. $g = 9.8\,\mathrm{m\,s^{-2}}$.
   (a) Ignoring air drag, plot the x–y graph using the analytical solution of $y(x)$, which is the trajectory of the projectile. Then use the Euler method to plot the projectile trajectories alongside the analytical solution for every projection angle. Show all plots together.
   (b) Now accounting for air drag, carry out the same numerical treatment as above. Show all the plots together. Note how the range of the projectile is reduced due to air drag. Carefully record the angle (correct within the $5°$ step), for which the range is maximum.

- **Set 6 (28-02-2020): Second-order systems**

1. Consider radioactive decay involving two types of nuclei, A and B, with amounts $x_A(t)$ and $x_B(t)$, respectively. Type-A nuclei decay according to $\dot{x}_A = -x_A/\tau_A$, forming Type-B nuclei, which in turn decay according to $\dot{x}_B = -\dot{x}_A - (x_B/\tau_B)$. Both τ_A and τ_B are the characteristic decay time scales of the two types of nuclei. Initially, $x_A(0) = 1\,\mathrm{unit}$, and $x_B(0) = 0\,\mathrm{unit}$.
   (a) By Euler's method integrate the foregoing system for the three cases of $\tau_A = \tau_B$, $\tau_A \simeq 10^4\tau_B$ and $\tau_A \simeq 10^{-4}\tau_B$, keeping $\tau_B = 1\,\mathrm{unit}$. Carefully choose Δt in each case.
   (b) For each case of τ_A and τ_B, plot $x_A(t)$ and $x_B(t)$ versus t, first separately, and then together in the same graph for comparison.
   (c) For each case also plot the x_A–x_B graph.

2. Consider radioactive decay with two types of nuclei, A and B, in which Type-A nuclei decay into ones of type B, while Type-B nuclei decay into ones of type A. The corresponding decay rate equations are then $\dot{x}_A = (x_B - x_A)/\tau$ and $\dot{x}_B = -\dot{x}_A$. The decay time scale is $\tau = 1\,\mathrm{unit}$, which is the same for both nuclei. Initially, $x_A(0) = 100\,\mathrm{unit}$, and $x_B(0) = 0\,\mathrm{unit}$.
   (a) By Euler's method integrate the foregoing system, with an appropriate choice of Δt.
   (b) Plot $x_A(t)$ and $x_B(t)$ versus t, first separately, and then together in the same graph.
   (c) Also plot the x_A–x_B graph.

3. The conserved harmonic oscillator equation is described by $\ddot{x} = -\omega^2 x$, in which $\omega = 2\pi/T$, with T being the natural time period of the oscillator. Decompose this second-order differential equation into two coupled first-order equations, $\dot{x} = v$ and $\dot{v} = -\omega^2 x$. For each of the two sets of initial conditions, $x(0) = 0$, $v(0) = 1$ and $x(0) = 1$, $v(0) = 0$, perform the following tasks:
   (a) Integrate the coupled first-order equations by Euler's method, for $T = 0.1, 1, 10$ (in some unit of time). Be mindful about the choice of Δt for a given T.
   (b) For each T, plot $x(t)$ and $v(t)$ versus t in the same graph.
   (c) Also plot the x–v graph using the data generated by the Euler method. It should be an ellipse. If not, give a reason for it. To check how much the result of the numerical integration deviates from the analytical theory, in the same graph also plot the analytical function of energy conservation $(v^2/2) + (\omega^2 x^2/2) = E$, without using the numerical data. The total energy E can be fixed from the initial values of $x(0)$ and $v(0)$.
   (d) The total energy E should be constant according to the analytical result. To verify it plot E–t as generated by the Euler method.

4. The linear damped oscillator is given by $\ddot{x} + 2b\dot{x} + \omega^2 x = 0$. As before, decompose it into a coupled first-order system in x and v. Also consider the initial conditions of the previous question, and the same three values of T. Now for the overdamped oscillator ($b > \omega$, for which set two different values of b, one slightly greater than ω and the other much greater than ω), the underdamped oscillator ($b < \omega$, for which set two different values of b, one slightly less than ω and the other much less than ω), and the critically-damped oscillator ($b = \omega$), separately perform the following tasks:
   (a) Integrate the coupled first-order equations by Euler's method,
   (b) For each T (which determines ω) and b, plot $x(t)$ and $v(t)$ versus t in the same graph.
   (c) Also plot the x–v graph.
   (d) Plot E–t.

- **Lockdown assignment (11-05-2020): Model of epidemic spread**

1. During the spread of an epidemic, the affected population is divided into three classes, namely, the infected, x, the susceptible, y, and the removed class, z. The last class can consist of both the recovered patients and those who have died because of the epidemic. For simplicity, assume that the disease is non-fatal and, therefore, the removed class is made only of recovered patients, who acquire permanent immunity against the disease. Further, assume that the disease has a short incubation period. Under all these simplifying assumptions, we can write $x(t)+y(t)+z(t) = N$, in which N is the conserved (constant) total number of the population, and $x, y, z > 0$.

   The initial condition is that at $t = 0$, $x(0) = x_0$ and $z(0) = 0$. Hence, $y(0) = y_0 = N - x_0$. The value of x_0 is small, i.e. initially a small number of infected persons introduce the disease in a population of total number, N ($x_0 \ll N$). Thereafter, the growth rate of x is $\dot{x} = Axy - Bx$, the growth rate of y is $\dot{y} = -Axy$, and the growth rate of z is $\dot{z} = Bx$. In these coupled first-order equations, A is the infection rate and B is the removal rate ($A, B > 0$). Now define $\rho = B/A$. According to the *Threshold Theorem of Epidemiology*, an epidemic will break out in a population if $y_0 > \rho$. Equipped with this basic knowledge, perform the following tasks:
   (a) First take $N = 10^5$, $x_0 = 10$, $A = 10$ and $B = 1$. These numbers ought to match realistic conditions. The typical population of a small Indian town like Gandhinagar is $N \sim 10^5$ (in lakhs). Large cities like Bangalore, Chennai, Pune and Ahmedabad have populations of $N \sim 10^6$ (in tens of lakhs). Mega cities like Delhi and Mumbai have $N \sim 10^7$ (in crores). Plot y–x (x along the *vertical* axis and y along the *horizontal* axis). For the values of the parameters and initial conditions provided here, it is clear that an epidemic should break out. This will be more so true when N is large. Which fairly well explains why the large cities of India (especially Mumbai, Delhi, Ahmedabad and Chennai) are most affected by the current *COVID-19* pandemic. You are advised to experiment with the parameter values (A, B and N) and the initial condition (x_0) to test for many possible outcomes.
   (b) By Euler's method, numerically integrate the coupled equations of $\dot{x}$, $\dot{y}$ and $\dot{z}$ to obtain $x(t)$, $y(t)$ and $z(t)$. The realistic scale of time is one day. This is the time scale on which the Government of India publishes all data related to the spread of the *COVID-19* epidemic in India. Hence, choose Δt carefully by applying your judgement.
   (c) Plot x, y and z (along the vertical axis) versus t (along the horizontal axis). Plot them in separate graphs and also in the same graph. You should find that x and z grow in time, while y decays in time. These trends are visible in the three first-order equations of $\dot{x}$, $\dot{y}$ and $\dot{z}$, which show $\dot{x} > 0$ (at least in the early stages), $\dot{y} < 0$ and $\dot{z} > 0$.
   (d) In a separate graph, plot $\ln x$ and $\ln z$ along the vertical axis, and t along the horizontal axis. If the early growth in this linear–log graph is a straight line, then that confirms an exponential growth for x and z in the early stages. Compare your results with the plots in `https://en.wikipedia.org/wiki/COVID-19_pandemic_in_India`. Especially be attentive to the linear–log graph at the end of this website, which shows that currently in India the recovery rate has become higher than the growth rate of active cases.