Lab - 8

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CS-302, Modeling and Simulation

In this lab we numerically and computationally analyze Random-walk models in varying dimensions and try to understand the effects of Monte-Carlo simulations visually as well as analytically.

I. INTRODUCTION

Monte Carlo experiments, are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. The underlying concept is to use randomness to solve problems that might be deterministic in principle. A random walk is a sequence of discrete, fixed-length steps in random directions. Random walks may be 1-dimensional, 2-dimensional, or n-dimensional for any n. Random walk is stochastic process and Monte Carlo simulations can be used to estimate the possible outcomes of this uncertain event. For every type of random walk we shall engage in the average position/distance to average position as well as the average position. In the first method we shall obtain the average position after K simulations for every step and then find the distance from the origin. In the latter case, we first find the distance for every simulation and then take average value for every step.

II. MODEL

A. 1-D Random Walk

The simplest random walk to understand is a 1-dimensional walk. Suppose you start from the origin of a number line. Then, take a step, either forward or backward, with equal probability. Let the 1st step be a1, the second step be a2 and so on. Each step is either +1 or -1. Now we want to know how far you travelled after taking N steps. The distance traveled after N steps will vary each time we repeat the experiment, so what we want to know is, if we repeat the experiment many many times, how far would you have traveled on average. Let's call the displacement that you have traveled, d, which can either be positive or negative, depending on whether you end up to the right or left of 0.
$d = a1 + a2 + a3 + ... + aN$
Next we take average on both sides. But $< a1 >= 0$, because if we repeated the experiment many many times, and a1 has an equal probability of being -1 or +1, we expect the average of a1 to be 0. $<d>$ is the average location after N steps, and since the you are equally likely to move forward or backwards, we expect d to be 0, on average. Even though d can be positive or negative, d^2 is always positive, so it can't average out to 0.
$<d^2> = (<a1^2> + <a2^2> + <a3^2> + ... + <aN^2>) + 2 \cdot (<a1a2> + <a1a3> + ...<a1aN> + <a2a3> + ...<a2aN> + ...)$
a1 can either be +1 or -1. Either way, $a1^2$ =1. The same applies to $<a2^2>$, $<a3^2>$, and all the way up to $<aN^2> >$.
There are 4 possible combinations of a1 and a2, each with equal probability. Since a1a2 is equally likely to be +1 or -1, $< a1a2 > = 0$. The same is true of all of $<a1a3>$, $<a1aN>$, $<a2a3>$, $<a2aN>$ and all of the other terms containing two different steps.
$<d^2> = (1+1+1+...+1)+2(0+0+...+0+0+...) = N$
$\sqrt{<d^2>}$ is average distance away from 0 after N steps, we expect that after N steps, you will be roughly $\sqrt{N}$ steps away from origin.

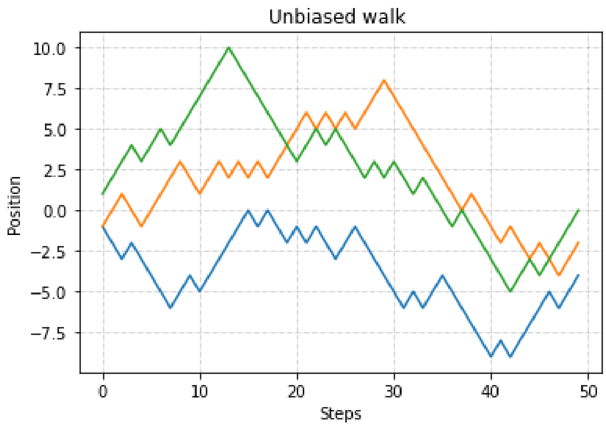

FIG. 1: 3 simulations of unbiased 1-D random walk

So far we have considered you will move with equal probability in any direction. A biased random walk is a random walk that is biased in one direction. Suppose that instead of an equal probability of moving left to right, there was a higher probability of moving to the right, we would expect a net drift to the right. In other words, the random walk is biased toward the right. At each step particle jumps to the left with probability q and to the right with probability 1-q. We calculate $P(x, N)$ which is the probability of finding the random walker at position x after N steps. Probability that particle makes k jumps to the left and N-k jumps to the right obeys the binomial distribution given by

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$$P(k, N) = \binom{n}{k} \cdot q^k \cdot (1-q)^{N-k}$$

, here,

$$x = 2 \cdot (k - N)$$

assuming unit length steps

$$k = \frac{N + x}{2}$$

As we keep on increasing N, the distribution tends to Gaussian distribution with

$$mean = N \cdot (2 \cdot q - 1)$$

and

$$variance = 4 \cdot N \cdot q \cdot (1 - q)$$

FIG. 2: 3 simulations of biased 1-D random walk

B. 2-D Random Walk

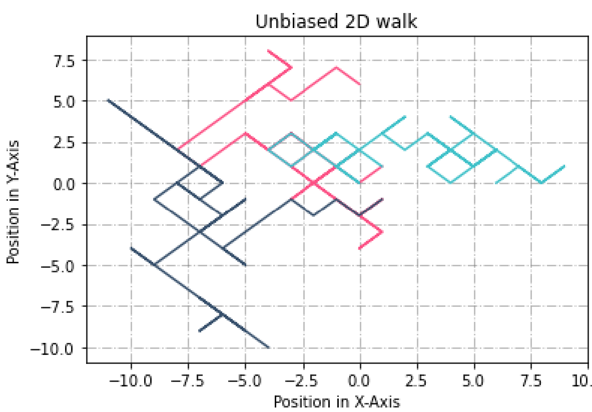

FIG. 3: 3 simulations of unbiased 2-D random walk

2-D random walks are similar to 1-D random walks in a way that they can be decomposed into two equivalent 1-D walks both of which are independent of each other. The most simplistic case is when you are at origin on a 2-D plane and with equal probability, you can either go one step towards North, East, West or South from where you currently are. Going North or South can be considered as a 1-D random walk along y-axis and going East or West can be considered as a 1-D random walk along x-axis. Similar to the calculations we did above, the average displacement expected along x-axis as the simulations increase is 0 and similarly for y-axis it is also expected to be 0, giving us a net average displacement equal to 0. In order to find the average distance travelled from the origin, we take the Pythagorean distance of current location from the origin, i.e. $\sqrt{x^2 + y^2}$.
We can introduce bias in any one or both of the component 1-D random walks to get a biased 2-D random walk.

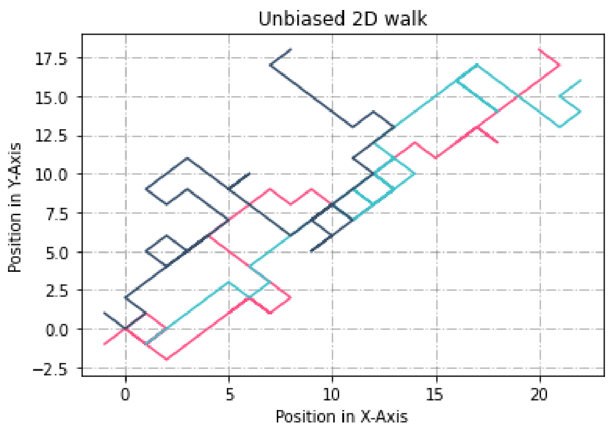

FIG. 4: 3 simulations of biased 2-D random walk

Animation that we made for unbiased 2-D and 3-D random walks can be found here

III. RESULT

A. 1-D Random Walk

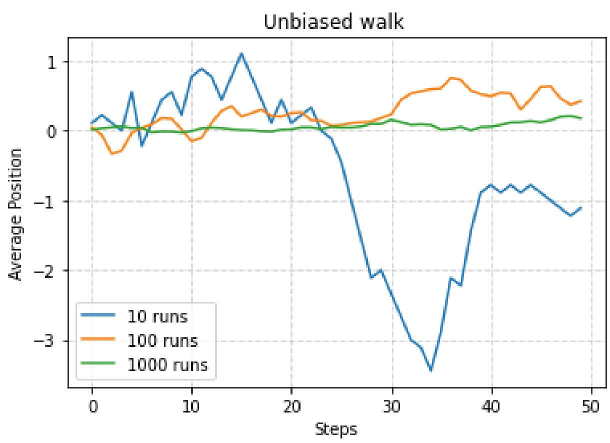

FIG. 5: Average Position after N steps

We observe that for an unbiased random walk, the Average position after N steps, on increasing the number of runs tends to 0. This lies in sync with our actual theoretical conclusion. On further increasing the runs we shall find a flat line on the x-axis.

FIG. 6: Average Distance after N steps

We observe that the average distance increases as a function of $\sqrt{N}$ and can be seen more clearly as we increase the number of runs. This lies in accordance to the theory mentioned above.

FIG. 7: Variance after N steps

For an unbiased walk, we theoretically obtain the variance after N steps to be N. In the above graph, we also observe a similar trend. On increasing the number of runs, the variance plot tends to the y=x line. This can also be understood qualitatively. As we increase the number of steps, the variety of options for the positions that a random walker can be in , also increases. Thus the variance in general increases with N.

FIG. 8: Average Position after N steps with p = 0.3

In the above figure we have considered p =0.3. Thus the walker will prefer to go forward rather than backwards. Thus we see an increasing trend in the average distance. Furthermore, on increasing the number of runs we tend towards the actual plot : $\overline{x}$ = (1-2*0.3)*N = 0.4*N.

FIG. 9: Average Distance after N steps

For a biased random walk since we already tend to move in a particular direction, thus the average distance increases and grows faster than $\sqrt{N}$.

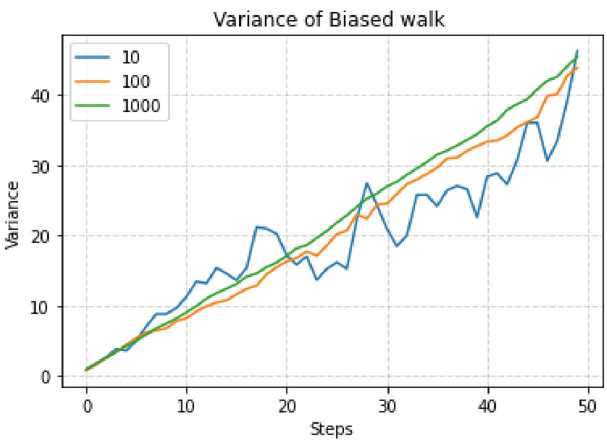

FIG. 10: Variance after N steps with p = 0.3

We observe a behaviour similar to the variance in unbiased walk. However the slope of the line is lower than the previous graph. On increasing the number of runs we obtain a more accurate plot.

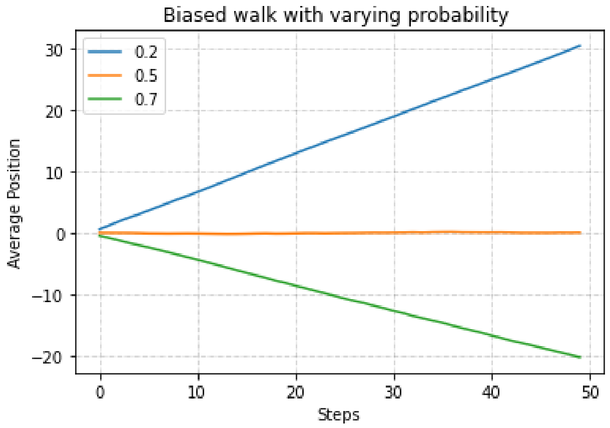

FIG. 11: Average Position after N steps with 1000 runs

The above plot has been generated for 1000 runs. We can clearly see the variation on changing p. On decreasing P from 0.5, we have an increasing line because the walker in general tends towards the forward direction. The exact reverse applies on increasing p from 0.5.

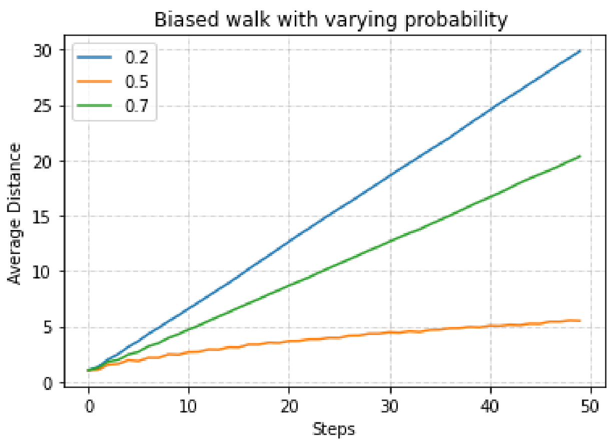

FIG. 12: Average Distance after N steps with 1000 runs

As we diverge more from p=0.5 the plot for average distance grows faster. This is because we tend to move farther away in a quicker time span.

The lines below tends toward their theoretical counterparts. One can observe that as the probability shifts farther away from 0.5, the slope decreases even further. This is because as we tend towards an extreme probability (0 or 1), the walker tends to move in that particular direction only. As a result of which it doesn't utilize the full set of positions that it can avail in the original conditions (p=0.5). Thus on account of the implicit lack of more positions, the variance is lower as we diverge more from 0.5. That is why p =0.7 has a higher slope than p = 0.2.

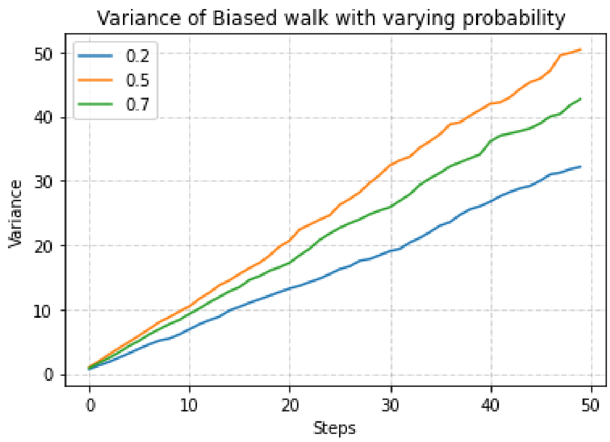

FIG. 13: Variance after N steps with 1000 runs

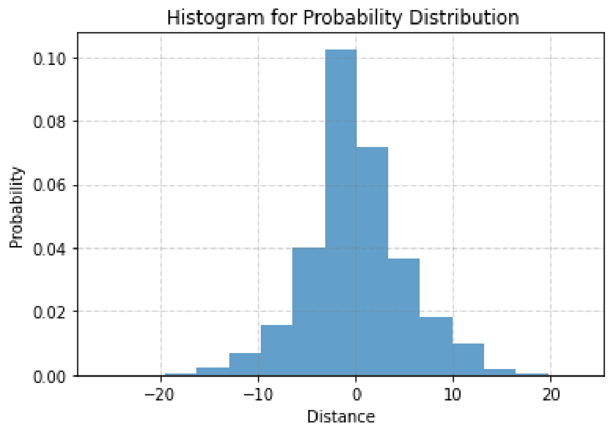

FIG. 14: Unbiased walk Histogram for Position after N steps

Although the actual distribution is a binomial distribution, yet by the Central Limit theorem for large value of N, we obtain a normal distribution. In this case we have N=50 and we observe a behaviour somewhat similar to a normal distribution with mean and variance mentioned above. Here, we can find the histogram centered around the mean value (0).

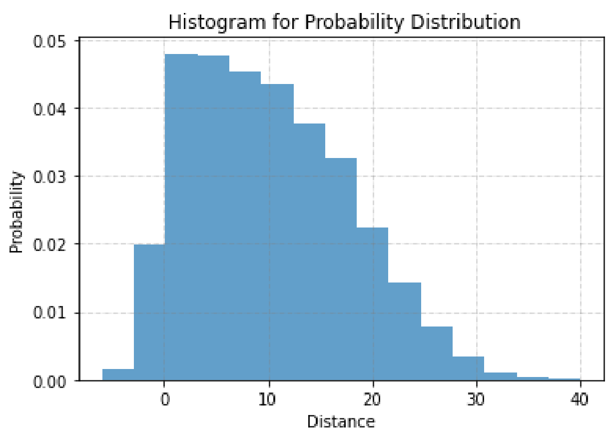

FIG. 15: Biased walk (P=0.3) Histogram for Position after N steps

Here we obtain a shifted version of the previous diagram. All of the above principle apply here as well. Since the walker prefers going forward, thus it is centered around a positive mean value.

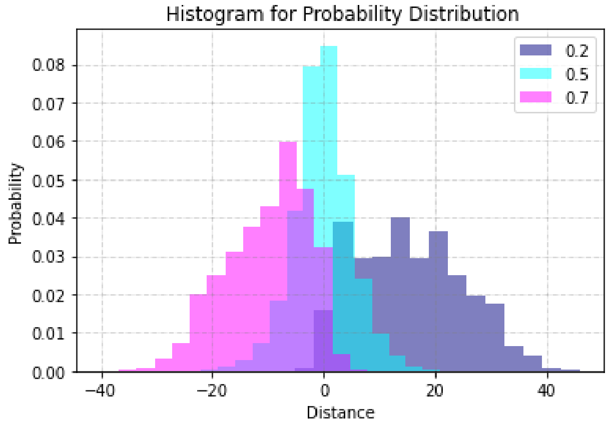

FIG. 16: Histogram for Position after N steps

A comparative analysis for multiple values of p has been done. We observe shifted distributions as we change the value of p. If we increase N to extremely high values we shall obtain perfect normal distributions for all the 3 histograms. In this case, we also observe that the more we diverge from p=0.5, the peaks of the histogram tend to flatten out. This is a direct consequence of the binomial distribution coefficients.

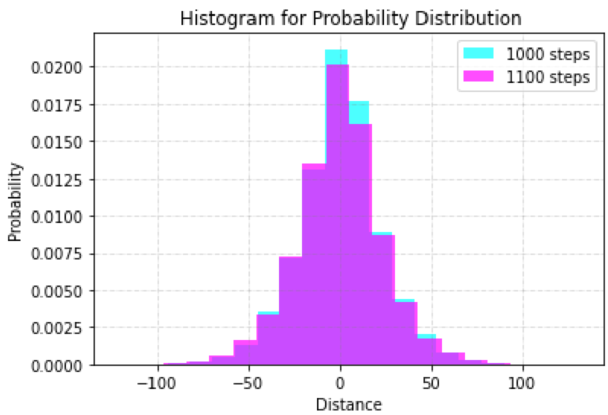

FIG. 17: Checking stationarity by increasing steps

Random walk is in essence not a stationary distribution/process. This is because we have a changing mean and variance for varied values of N. For an unbiased walk even though the mean is 0 but the variance still changes. However by central limit theorem, if we sufficiently increase N, it tends to a Normal distribution with the given mean and variance. For high values of N, on perturbing it there is no significant change in the distribution and we can say it tends to a stationary distribution under those particular set of conditions. In our case on taking 1000 steps for 10,000 runs we obtain a distribution which doesn't change much on perturbing N.

B. 2-D Random Walk

FIG. 18: Simulation of Unbiased 2D Random walk

We observe that as we increase the runs, the walker tends to move more around the origin itself with low variance. On increasing the number of runs, the distance to the average position at all steps tends to 0.

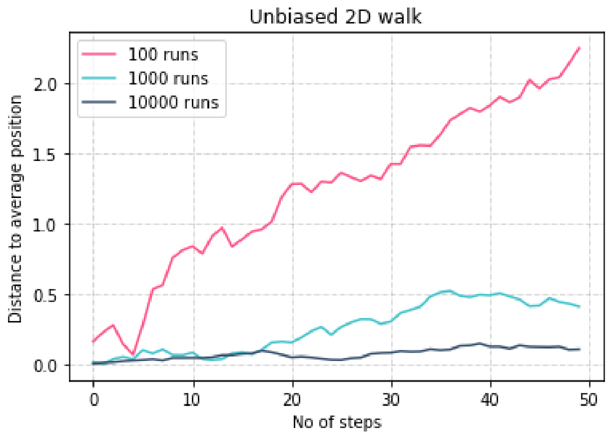

FIG. 19: Distance to Average position after N steps

FIG. 20: Simulation of Unbiased 2D Random walk

We furthermore, observe that the average distance increases as a function of $\sqrt{N}$ and can be seen more clearly as we increase the number of runs.

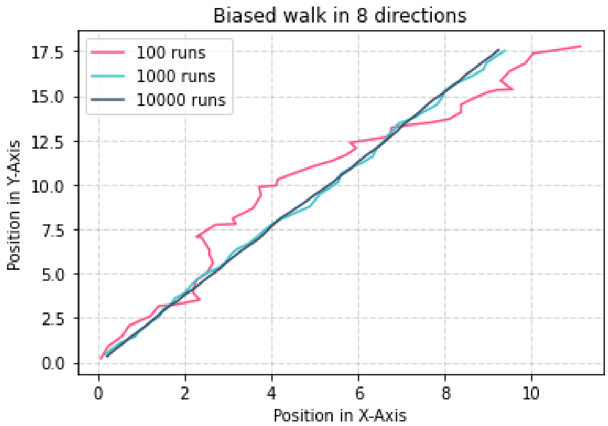

FIG. 21: Simulation of Biased 2D Random walk

At each step probabilities of going in the indicated directions are as follows, N : 19%, NE : 24%, E : 17%, SE : 10%, S : 2%, SW : 3%, W : 10%, NW : 15% We can observe that the P(N) + P(E) + P(NE) = 0.6. Thus the walker primarily tends to move in a direction pointing to the weighted vector sum of the 3 directions. As we increase the number of runs we observe a better and smooth plot.

FIG. 22: Distance to Average position after N steps

Since the position is increasing in the first quadrant, thus the average distance also tends to increase after every consecutive step as can be seen in the graph.

FIG. 23: Average distance of biased 2D Random walk

For a biased random walk since we already tend to move in a particular direction, thus the average distance increases and grows faster than $\sqrt{N}$.

IV. CONCLUSIONS

From the above graphs we observed various aspects of Random Walks. For an unbiased walk, the average position is 0, the variance increases linearly with the number of steps and the average distance increase as function of $\sqrt{N}$ as well. For a biased walk, the average position/displacement is no longer 0.The slope of the variance also decreases as well. Furthermore, we concluded for an unbiased random walk, it is not a stationary distribution because the variance changes with N. But if we sufficiently increase N, we observe that the plot doesn't change significantly as it tends to a normal distribution (By central limit theorem), and hence it tends to a stationary distribution. For a 2D walk, similar observations were made as well. We observed the distance to average position to be 0 and the average distance to be increasing. For a biased walk, we observe the results mentioned beforehand.

[1] Module 9.5, A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).