

MODULE 7.1

Radioactive Chains—Never the Same Again

Prerequisite: Module 2.2, “Unconstrained Growth and Decay.”

Introduction

The mass $Q(t)$ of a radioactive substance decays at a rate proportional to the mass of the substance (see the section “Unconstrained Decay” in Module 2.2, “Unconstrained Growth and Decay”). Thus, for positive **disintegration constant**, or **decay constant**, r , we have the following differential equation:

$$dQ/dt = -rQ(t)$$

and its difference equation counterpart:

$$\Delta Q = -rQ(t - \Delta t)\Delta t$$

In this module, we model the situation where one radioactive substance decays into another radioactive substance, forming a chain of such substances. For example, radioactive bismuth-210 decays to radioactive polonium-210, which in turn decays to lead-206. We consider the amounts of each substance as time progresses.

Modeling the Radioactive Chain

If a radioactive substance, $substanceA$, decays into substance $substanceB$, we say that $substanceA$ is the **parent** of $substanceB$ and that $substanceB$ is the **child** of $substanceA$. If $substanceB$ is also radioactive, $substanceB$ is the parent of another substance, $substanceC$, and we have a **chain** of substances. Figure 7.1.1 depicts the situation where A , B , and C are the masses of radioactive substances, $substanceA$, $substanceB$, and $substanceC$, respectively; and different disintegration constants, $decay_rate_of_A$ (a) and $decay_rate_of_B$ (b), exist for each decay.

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decay A to B decay B to C

decay rate of A decay rate of B

Figure 7.1.1 Chain of decays

Quick Review Question 1

Suppose A and B are the masses of *substanceA* and *substanceB*, respectively, at time t ; ΔA and ΔB are the changes in these masses; and a and b are the positive disintegration constants.

- Using these constants and variables along with arithmetic operators, such as minus and plus, give the difference equation for the change in the mass of *substanceA*, ΔA .
- Through disintegration of *substanceA*, *substanceB*'s mass increases, while some of *substanceB* decays to *substanceC*. Give the difference equation for the change in the mass of *substanceB*, ΔB .
- In Figure 7.1.1, where A , B , and C are the masses of three radioactive substances, give the formula as it appears in a system dynamics tool's equation for the flow *decay_A_to_B*.
- Give the formula as it appears in a systems dynamics tool's equation for the flow *decay_B_to_C*.

The mass of *substanceA* that decays to *substanceB* is aA . Thus, in Figure 7.1.1, the flow *decay_A_to_B* contains the formula *decay_rate_of_A* * A . What *substanceA* loses, *substanceB* gains. However, *substanceB* decays to *substanceC* at a rate proportional to the mass of *substanceB*, bB . Consequently, in Figure 7.1.1, the flow *decay_B_to_C* contains the mass that flows from one stock to another, *decay_rate_of_B* * B . The total change in the mass of *substanceB* consists of the gain from *substanceA* minus the loss to *substanceC* with the result multiplied by the change in time, Δt :

$$\Delta B = (aA - bB) \Delta t$$

We consider the initial amounts of *substanceB* and *substanceC* to be zero.

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Projects

- With a system dynamics tool or a computer program, develop a model for a radioactive chain of three elements, from *substanceA* to *substanceB* to *substanceC*. Allow the user to designate constants. Generate a graph and a table for the amounts of *substanceA*, *substanceB*, and *substanceC* versus time. Answer the following questions using this model.

- b. Explain the shapes of the graphs.
- c. As the decay rate of A, a , increases from 0.1 to 1, describe how the time of the maximum total radioactivity changes. The total radioactivity is the sum of the change from *substanceA* to *substanceB* and the change from *substanceB* to *substanceC*, or the total number of disintegrations. Why?
- d. (The verification in Part d requires calculus.) With b being the decay rate of B, in several cases where $a < b$, observe that eventually we have the following approximation:

$$\frac{B}{A} \approx \frac{a}{b-a}$$

With the ratio of the mass of *substanceB* (B) to the mass of *substanceA* (A) being almost constant, $a/(b-a)$, we say the system is in **transient equilibrium**. Eventually, *substanceA* and *substanceB* appear to decay at the same rate. Using the following material, verify this approximation:

Find the exact solution to the differential equation for the rate of change of A with respect to time, $dA/dt = -aA$ (see the section "Analytic Solution" in Module 2.2, "Unconstrained Growth and Decay"). Verify

that $B = \frac{aA}{b-a} + A_0 e^{-at}$, where A_0 is the initial mass of *substanceA*, is a solution to the differential equation for the rate of change of

$$B = \frac{aA}{b-a} + A_0 e^{-at}$$

B with respect to time (see the difference equation for ΔB). What number does e^{-at} approach as t goes to infinity? For $a < b$, which is smaller, e^{-at} or e^{-bt} ? Thus, for large t , B is approximately equal to what?

- e. Using your model from Part a, observe in several cases where $a > b$ that the ratio of the mass of *substanceB* to the mass of *substanceA* does not approach a number. Thus, transient equilibrium (see Part d) does not occur in this case.
- f. (Requires calculus) Verify the observation from Part e analytically using work similar to that in Part d.
- g. If a is much smaller than b , we have $A \approx A_0$ and $B \approx \frac{aA}{b-a}$

. With the two

$$\frac{b}{a}$$

amounts being almost constant, we have a situation called **secular equilibrium**.

librium. Observe this phenomenon for the radioactive chain from radium-226 to radon-222 to polonium-218: $\text{Ra}^{226} \rightarrow \text{Rn}^{222} \rightarrow \text{Po}^{218}$, where the decay rate of Ra^{226} , a , is 0.00000117/da and the decay rate of Rn^{222} , b , is 0.181/da. Using your work from Part a, run the simulation for at least one year.

- h. (Requires calculus) Show analytically that the approximations from Part g hold.

i. In the radioactive chain $\text{Bi}^{210} \rightarrow \text{Po}^{210} \rightarrow \text{Pb}^{206}$ (bismuth-210 to polonium-210 to lead-206), the decay rate of Bi^{210} , a , is 0.0137/day and the decay rate of Po^{210} , b , is 0.0051/day. Assuming the initial mass of Bi^{210} is 10^{-8} g and using your model from Part a, find, approximately, the maximum mass of Po^{210} and when the maximum occurs.

j. (Requires calculus) In Part d, we verified that $B = aA$

$$b a e^{b t} - a e^{b t}$$

Using

this result, find analytically the maximum of mass of substance B and when this maximum occurs.

k. Check your approximations of Part i using your solution to Part j. l. For the chain in Part g, use your solution to Part j to find when the largest mass of Rn^{222} occurs.

m. For the chain in Part g, use your simulation of Part a to approximate the time when the largest mass of Rn^{222} occurs. How does your approximation compare with the analytical solution of Part l?

2. Develop a model for a chain of four elements. Perform simulations, observations, and analyses similar to those before. Discuss your results

Answers to Quick Review Question

1. a. $\Delta A = -aA \Delta t$

b. $\Delta B = (aA - bB)\Delta t$

c. $\text{decay_A_to_B} = \text{decay_rate_of_A} * A$

d. $\text{decay_B_to_C} = \text{decay_rate_of_B} * B$

Reference

Horelick, Brindell, and Sinan Koont. 1979, 1989. "Radioactive Chains: Parents and Children." *UMAP Module 234*. COMAP, Inc.