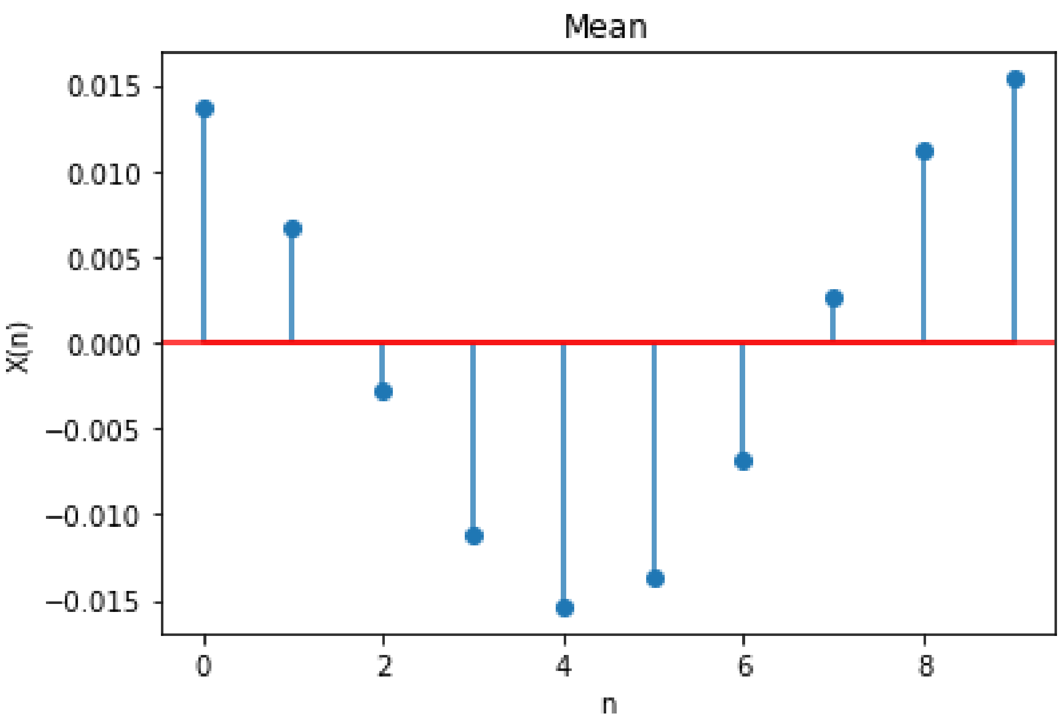
1. (a) $X(n) = cos(0.2\pi n + \theta), \theta \sim U[-1, 1], n \in [0, 9]$

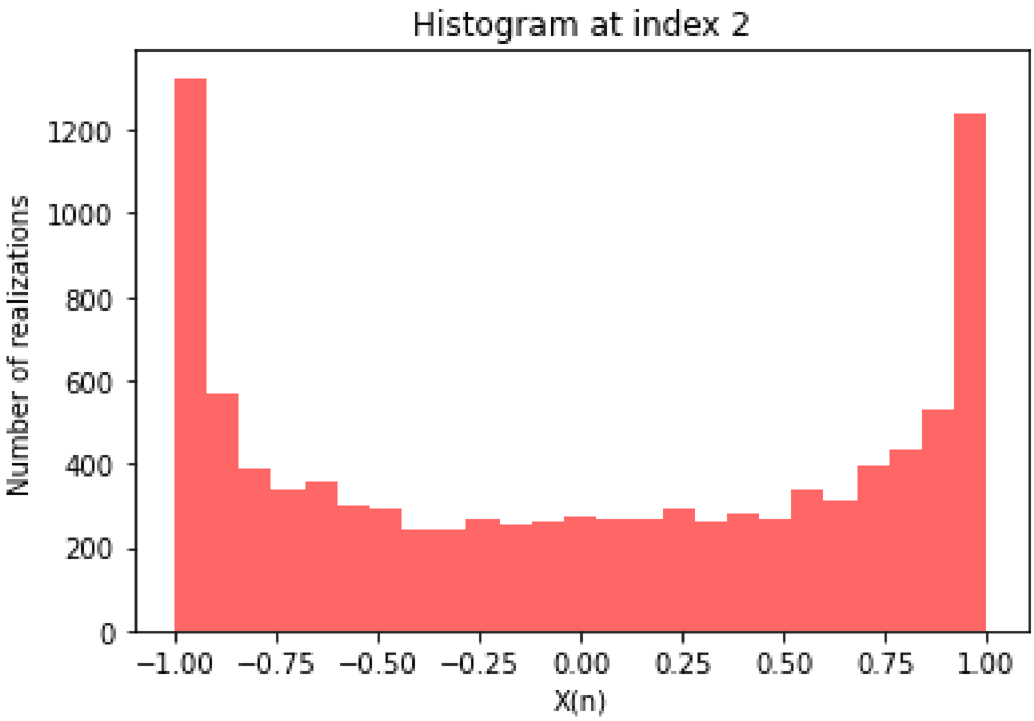
$$
\begin{aligned}
m_x(n_1) &= E[cos(0.2\pi n_1 + \theta] \\
&= E[cos(0.2\pi n_1)cos\theta - sin\theta sin(0.2\pi n)] \\
&= E[cos(0.2\pi n_1)cos\theta] - E[sin\theta sin(0.2\pi n)] \\
&= cos(0.2\pi n_1)E[cos\theta] - sin(0.2\pi n)E[sin\theta]
\end{aligned}
$$

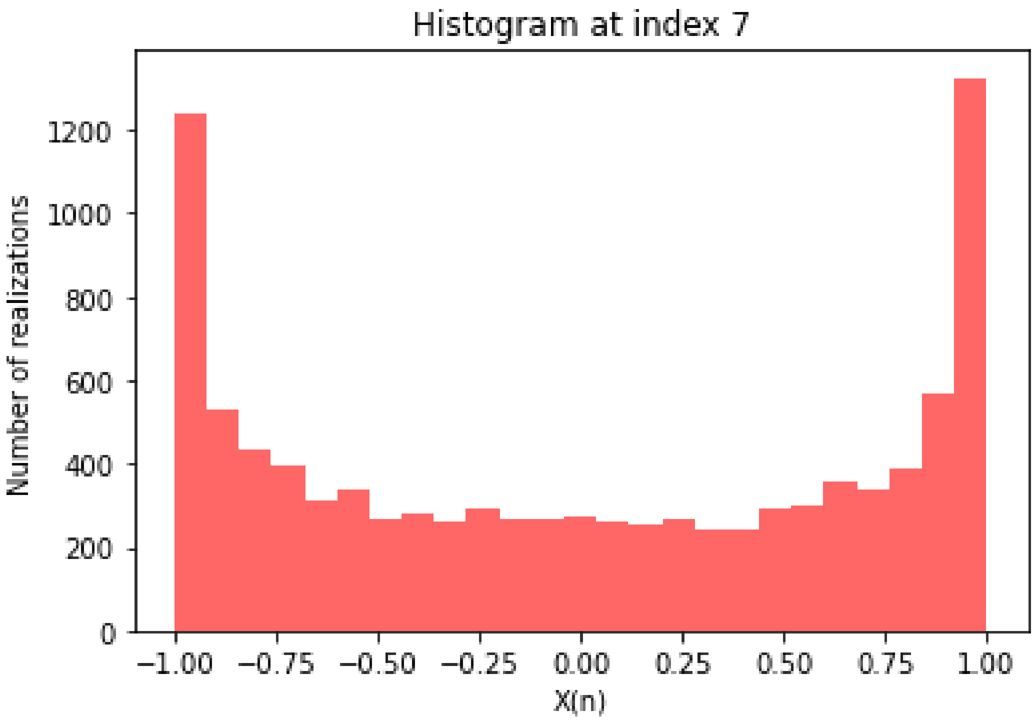
since, $E[cos\theta] = \int_{-\pi}^{\pi} cos\theta \cdot \frac{1}{2\pi} d\theta = 0 = E[sin\theta]$
so, $m_x(n_1) = 0$

$$
\begin{aligned}
R_x(n_1, n_2) &= E[cos(0.2\pi n_1 + \theta) \cdot cos(0.2\pi n_2 + \theta)] \\
&= E[0.5 \cdot cos(0.2\pi(n_1 - n_2) + 0.5 \cdot cos(0.2\pi(n_1 + n_2) + 2 \cdot \theta)] \\
&= E[0.5 \cdot cos(0.2\pi(n_1 - n_2))] + E[0.5 \cdot cos(0.2\pi(n_1 + n_2) + 2 \cdot \theta)] \\
&= 0.5 \cdot cos(0.2\pi(n_1 - n_2)) + 0 \\
&= 0.5 \cdot cos(0.2\pi(n_1 - n_2))
\end{aligned}
$$

From analytical solution we see that mean is 0, and autocorrelation depends only on difference of n1 and n2, therefore this equation is wide-sense stationary.
We can see the same by computing them. The results are shown below.
Mean is 0 as we can see from the graph just below.
The other graphs show the density functions for random variable X(2) and X(7).





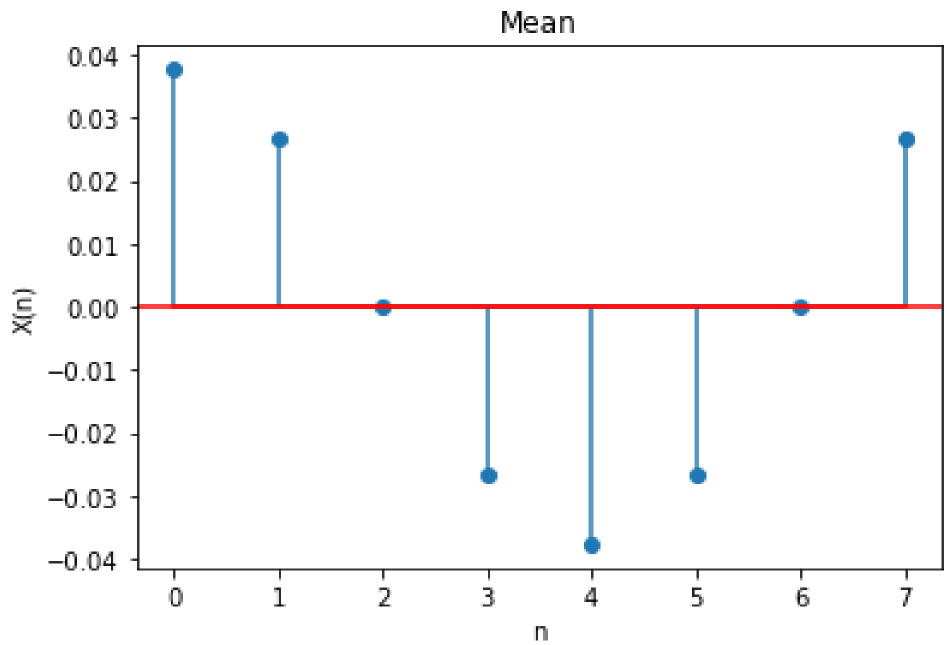


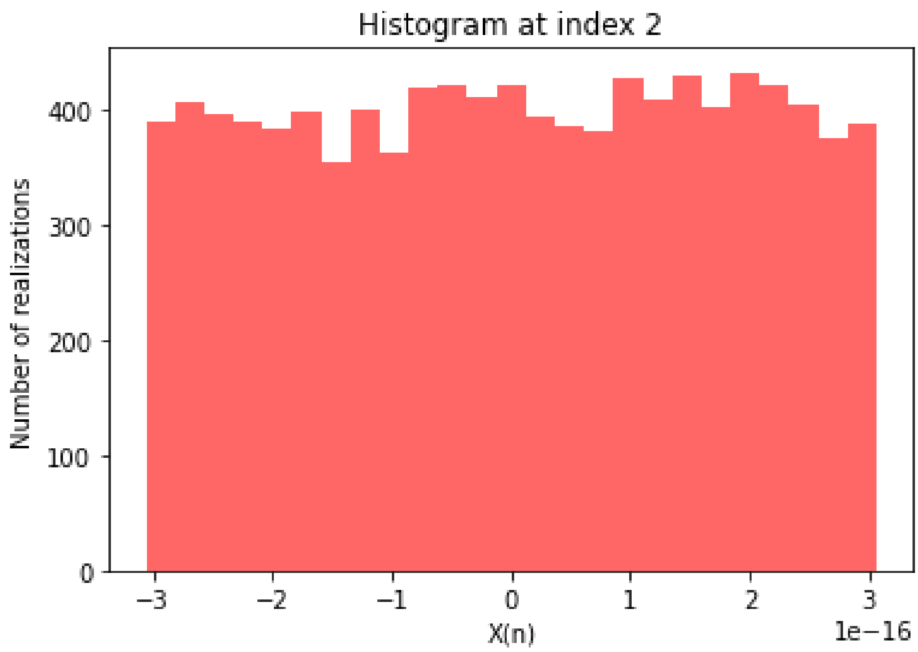
(b) $X(n) = A \cdot cos(0.25\pi n), A \sim U[-5,5], n \in [0,7]$

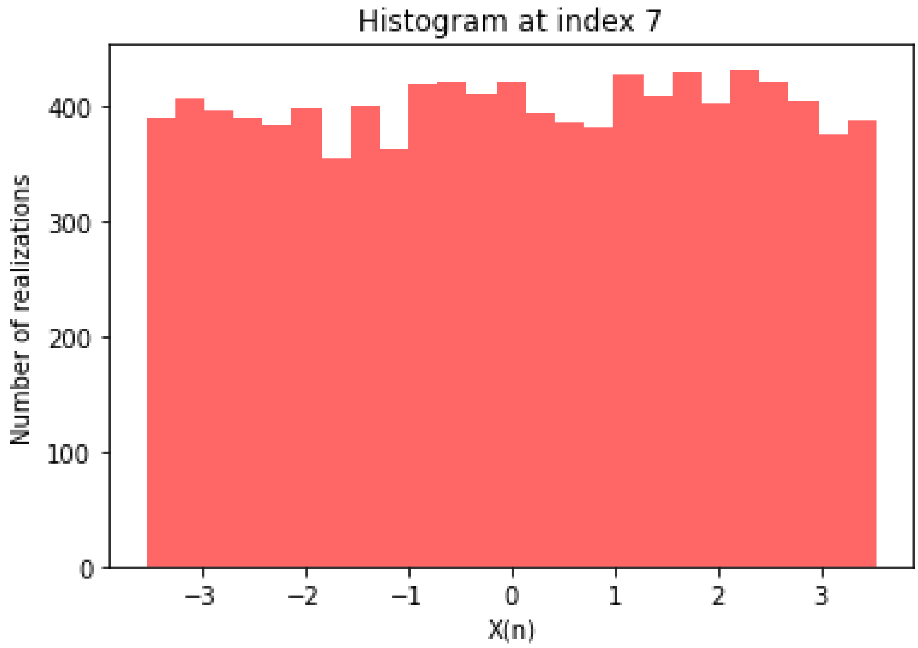
$$\begin{aligned} m_x(n_1) &= E[A \cdot cos(0.25\pi n)] \\ &= cos(0.25\pi n)E[A] \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
R_x(n_1, n_2) &= E[A \cdot cos(0.25\pi n_1) \cdot A \cdot cos(0.25\pi n_2)] \\
&= E[A^2] \cdot cos(0.25\pi n_1) \cdot cos(0.25\pi n_2) \\
&= \frac{25}{3} \cdot cos(0.25\pi n_1) \cdot cos(0.25\pi n_2)
\end{aligned}
$$

From analytical solution we see that mean is 0, but auto-correlation does not depend only on difference of n1 and n2, therefore this equation is not wide-sense stationary. We can see the same by computing them. The results are shown below.
Mean is 0 as we can see from the graph just below.
The other graphs show the density functions for random variable X(2) and X(7).







(c) $X(n) = A(n), A(n) \sim \mathcal{N}[0,1]$

$$\begin{aligned} m_x(n_1) &= E[A(n_1)] \\ &= E[\mathcal{N}[0,1]] \\ &= 0 \end{aligned}$$

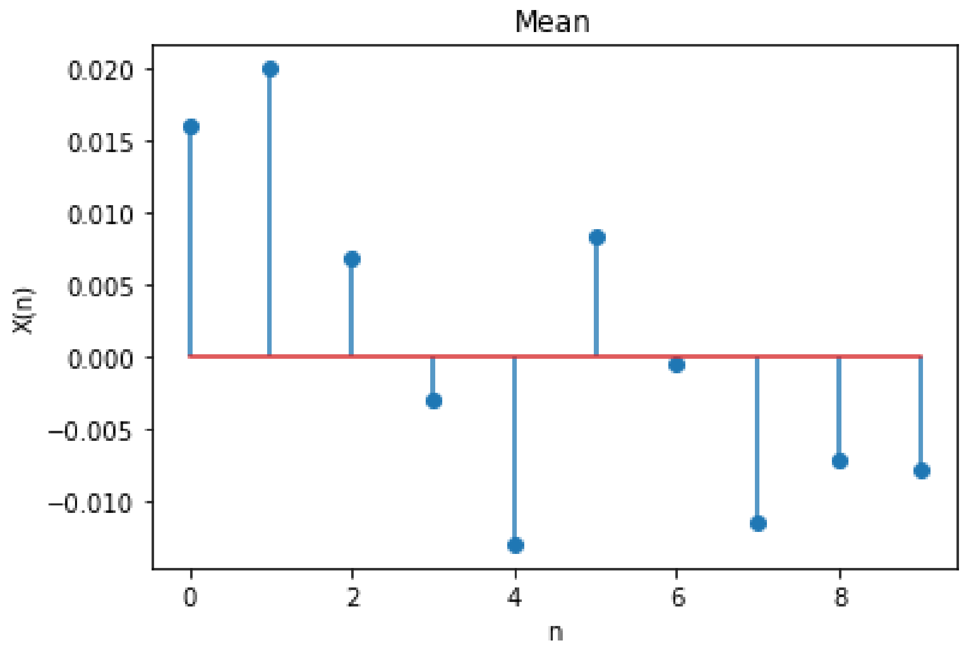
$$\begin{aligned} R_x(n_1, n_2) &= E[A(n_1) \cdot A(n_2)] \\ &= E[A(n_1)] \cdot E[A(n_2)] \\ &= 0 \end{aligned}$$

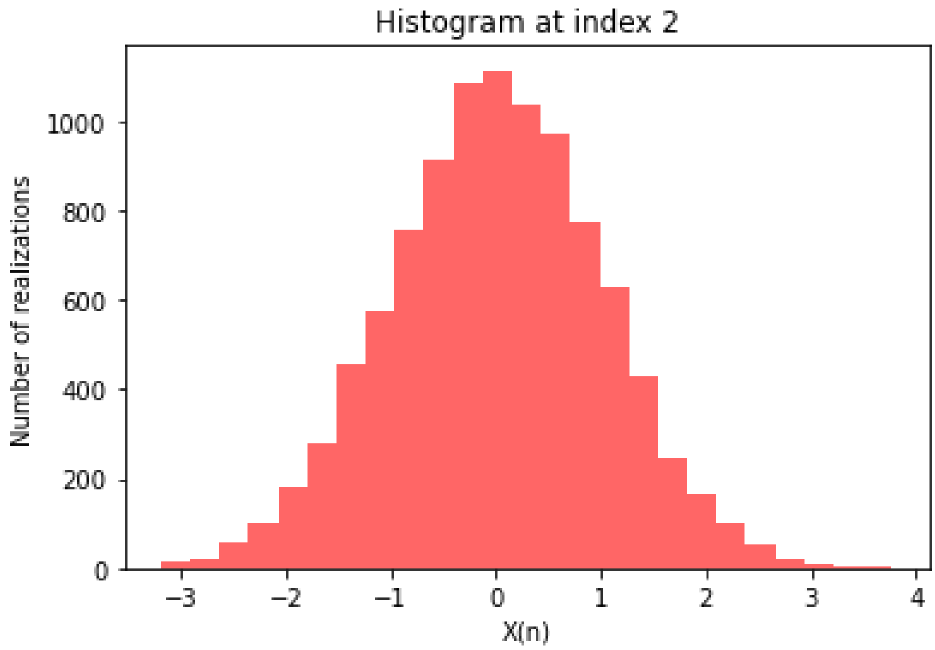
Every time instance n has a different Random Variable and each such variable has the same distribution and are independent of each other. Let X contain a family of Independent and Identically Distributed Random Variables. We get,

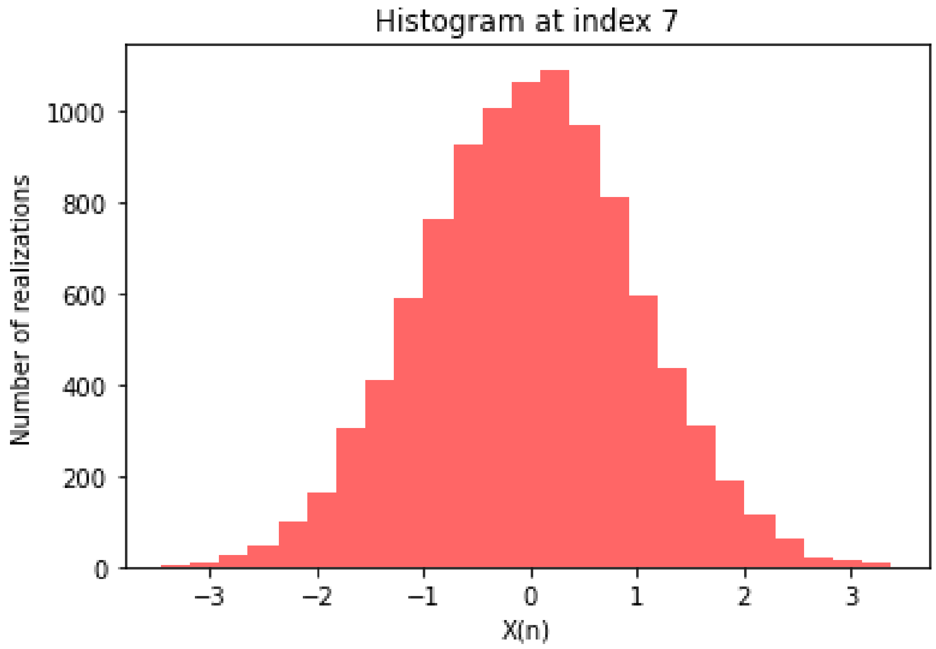
$$\begin{aligned} p(X_{t1+T}, X_{t2+T}, ...X_{tn+T}) = \prod_{i=1}^{i=n} p(X_{ti+T}) \\ = \prod_{i=1}^{i=n} p(X_{ti}) \\ = p(X_{t1}, X_{t2}, ...X_{tn}) \end{aligned}$$

From analytical solution we see that mean is 0 and auto-correlation is 0, and hence it depends on nothing other than difference of n1 and n2, therefore this equation is wide-sense stationary. It also satisfies the condition of being strict sense stationary. Since strict sense also implies wide sense stationary, this process is **strict sense stationary** process.

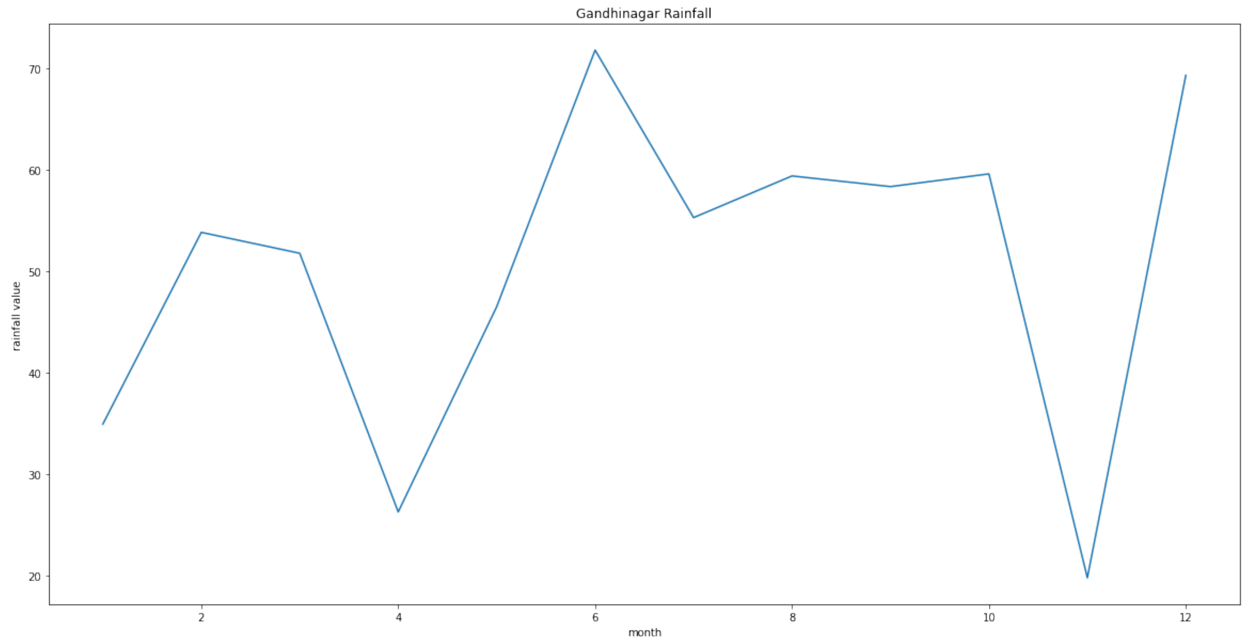
We can see the same by computing them. The results are shown below.
Mean is 0 as we can see from the graph just below.
The other graphs show the density functions for random variable X(2) and X(7).



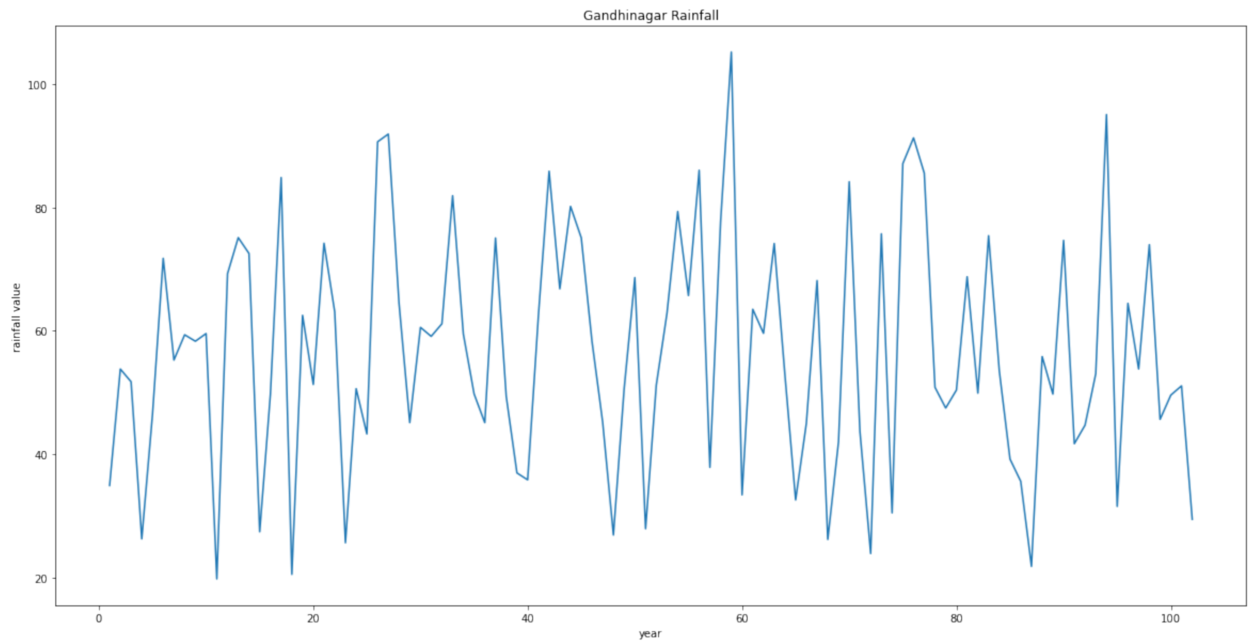



Histogram at index 7
1000
800
600
400
200
0
Number of realizations
−3
−2
−1
0
1
2
3
X(n)

2. Non-stationary.

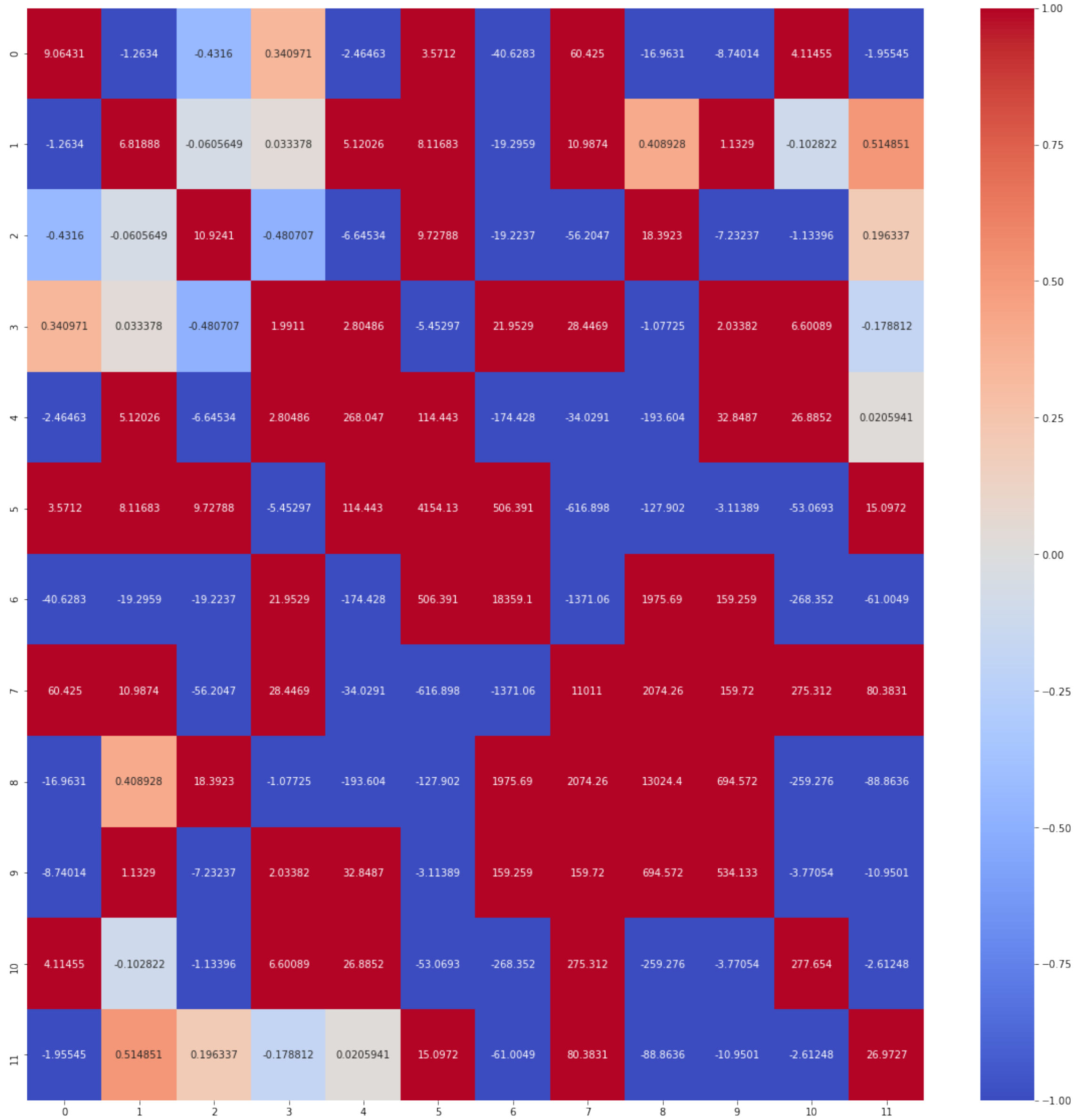


This graph shows the mean value of rainfall for all months over 102 years. These values vary largely with the rainfall season months and the mean is not constant. This is the correct representation since the 102 years are the different sample functions and months are the random variables. We have taken the mean of each random variable over all the sample functions.



This graph shows the mean value of rainfall for all years over 12 months. These values vary around the overall mean which is around 56.4214cm of rainfall. This interpretation of the random process is not correct and we will go with the previous graph.

Now we plot the heat map for the co-variance of the entire data set to get the 12 x 12 co-variance matrix. We can see that it's a non stationary process.



3. (a)

$$R = E[x \cdot x^T]$$

$$Corr(X_j, X_i) = \frac{Cov(X_j, X_i)}{\sigma_{X_j} \cdot \sigma_{X_i}}$$

$$Corr(X_j, X_i) = \frac{Cov(X_i, X_j)}{\sigma_{X_i} \cdot \sigma_{X_j}}$$

$$Cov(X_i, X_j) = Cov(X_j, X_i)$$

Hence, we can say that:

$$R = R^T$$

Auto correlation matrix $R = E[x \cdot x^T], \forall y \in \mathbb{R}^n$

$$\begin{aligned} y^T R y &= y^T E[x x^T] y \\ &= y^T \|x\|^2 y \\ &= \|x\|^2 y^T y \\ &= \|x\|^2 \cdot \|y\|^2 \geq 0 \end{aligned}$$

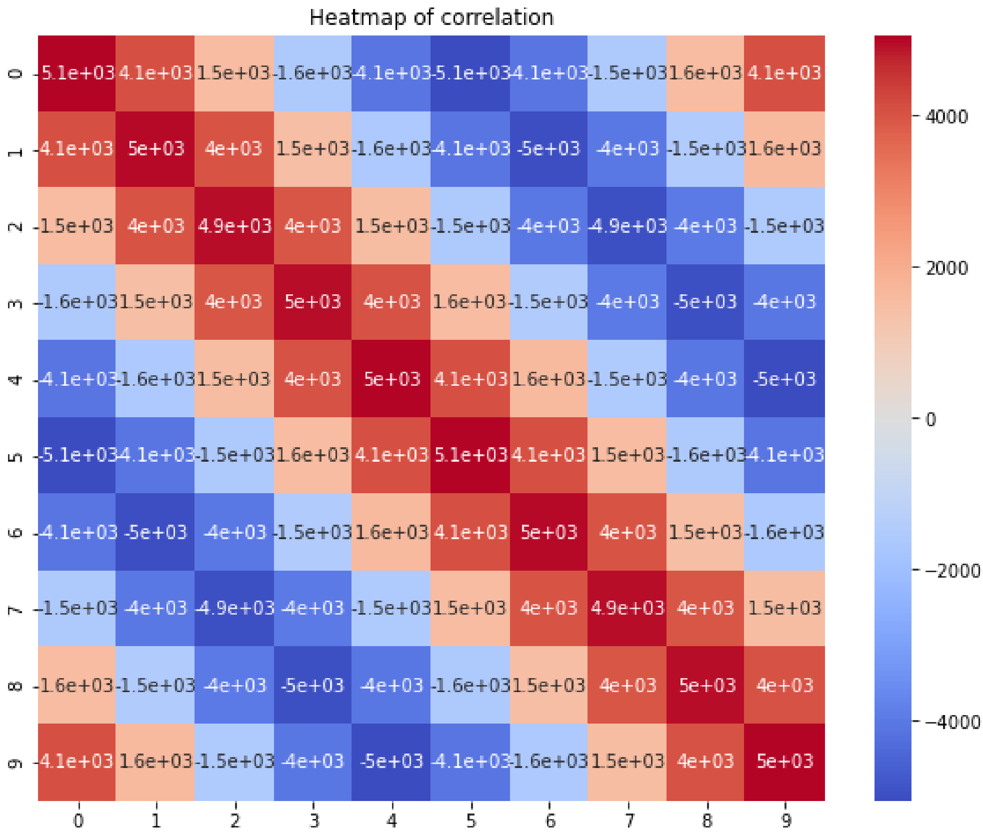
So, we got $A = A^T$ and $x^T R x \geq 0$ Hence, R is a symmetric positive semi-definite matrix.
Similarly, we have to follow the same steps for Co-variance matrix C
We just need to put $x = x - m$ and in doing so Correlation(R) becomes equal to Co-variance(C).

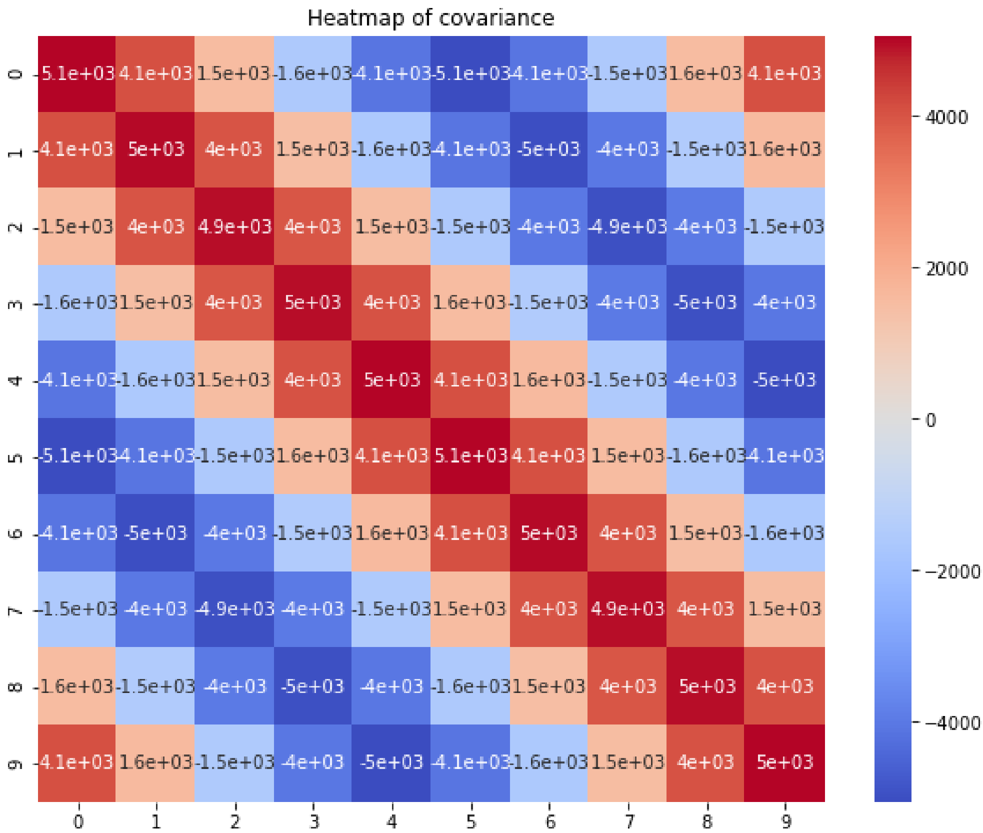
$$Cov = E[(x - m)(x - m)^T]$$

Also, we get $(x - m)^T C (x - m) \geq 0$ so we conclude that C is also symmetric positive semi-definite matrix.

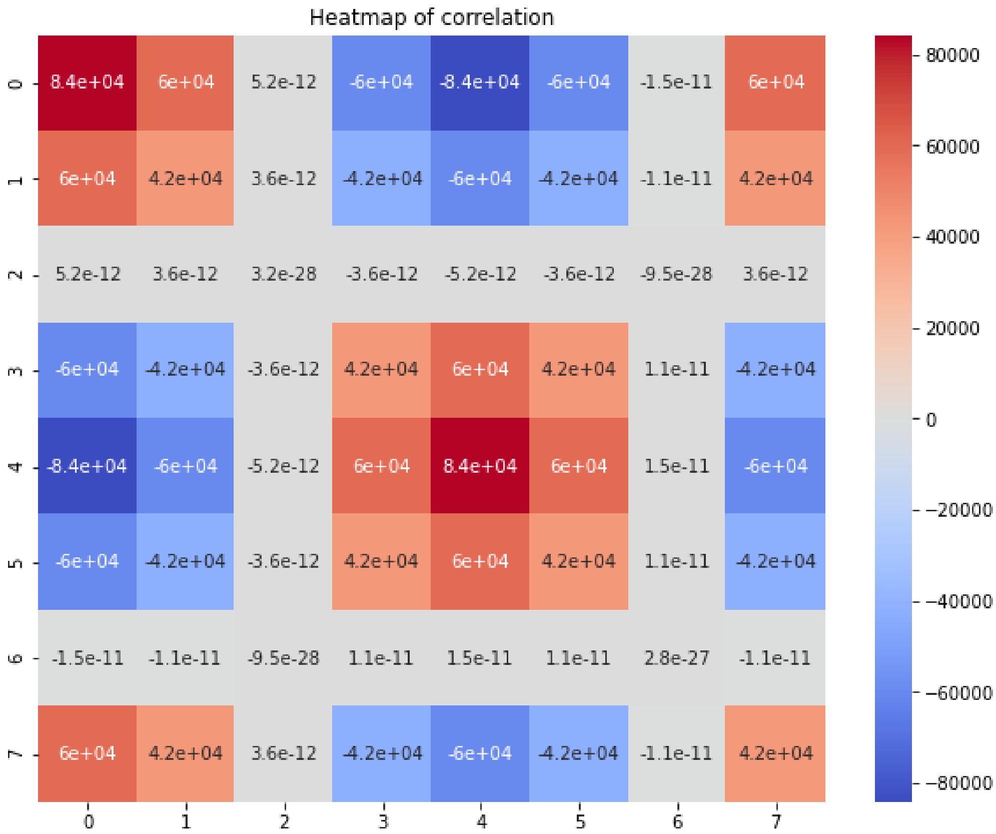
In the code, for $y^T A y \geq 0$ we check the whether value of eigenvalues is greater than zero. This can be understood from the following steps:

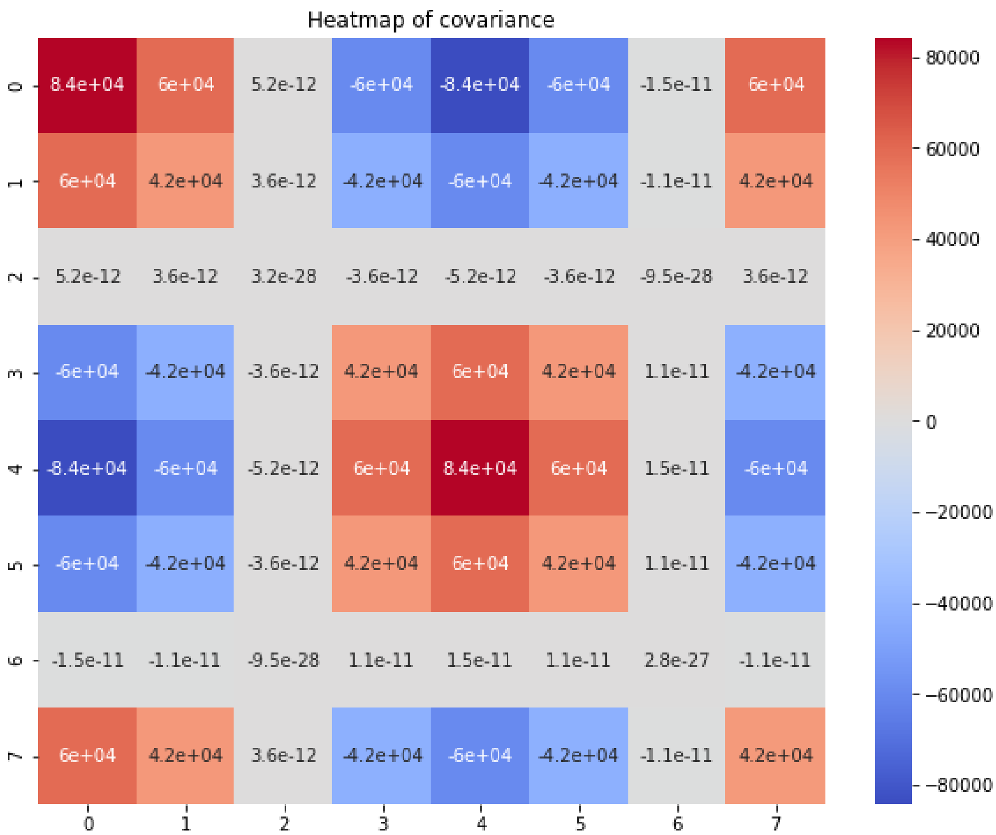
- Consider that λ is eigenvalue of A
- $\exists$ y, such that $Ay = \lambda y$
- So, $y^T A y = \lambda y^T y$
- Since, $y^T y \geq 0$ for all y $\implies$ λ is non-negative





The above two images are correlation matrix and co-variance matrix, respectively for Q1 (a).
We can see that both are symmetric and their eigenvalues are positive therefore, both of them are symmetric positive-semi definite.





The above two images are correlation matrix and co-variance matrix, respectively for Q1 (b).
We can see that both are symmetric and their eigenvalues are positive therefore, both of them are symmetric positive-semi definite.

(b) Such type of matrices are called **constant diagonal** matrices or Toeplitz matrices.
A diagonal-constant matrix is a matrix in which each descending diagonal from left to right is a constant.
We can observe that from in Q1(a)'s heat-maps shown above.