Mathematical Model for Engineering Drawing Package

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Abstract. In this paper we mathematically modelled the problem presented to us. We first look into the problem of prjecting a 3D object into any projection plane using the elementary concepts of matrices and linear algebra. After dealing with this part we move onto recreating the 3D object, it's isometric views from the 2D othographic views as input with minimal loss of any information. We dealt both the parts with basic knowledge of Matrices and simple operations like Transformation, Rotation and Translation of objects in 3D space. . .

Keywords: Engineering Drawing, Orthographic Projections, Wire-frame, Matrix Representation

1 Introduction

Engineering Drawing is one the most important yet tedious job prone to all kinds of human errors. One of the most effective solution to this problem would be to be mathematically model this problem and let the computer do the job. Since the computing power of the computers has increased tremendously in past years we can save a lot of labour and man power by giving the task of engineering drawing to the computers and saving a lot of time and errors.

2 Projection of 3D objects to 2D planes

The process of converting 3D to 2D is called Projection. There are multiple types of projection but our concern remains on parallel and orthographic projection of 3D objects. Parallel projection are used to create working drawings of objects which preserves the shape and scale of objects which is very important in terms of minimizing any loss of important piece of information.In parallel projection, image points are found at the intersection of the view plane with a ray drawn from the object point and having a fixed direction. The direction of projection is the same for all rays (all rays are parallel). A parallel projection is described by prescribing a direction of projection vector v and a viewplane. The object point P is located at (x, y, z) and we need to determine the image point coordinates $P(x, y, z)$. If the projection vector v has the same direction as the view plane normal, then the projection is said to be orthogonal, otherwise the projection is oblique.

2.1 Mathematical formulation of Parallel Projection using Matrices

We can represent a trasformation matrix as 4×4 homogenous co-ordinates as

$$[T] = \begin{pmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{pmatrix} \tag{1}$$

where l, m, n are translations along x, y, z axis, s is overall scaling , p, q, r are perspective transformations and others are Linear Transformations- local scaling, shear, rotation reflection.

Orthographic projection matrices would be

$$[T_z] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; [T_y] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; [T_x] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Since the co-ordinate axes can be translated and rotated arbitrarily, the easiest way to solve the problem would be a attaching the co-ordinate axes to the object.The computation would be a lot easier than choosing any other position of the axes in space.

Another possibility of paralell projection is projection of the object on to an oblique/ cutting plane. In this case, we can project the points on to the plane using matrix mentioned in equation (1). The last step in the process of projection of 3D object would be the joining of the projected points onto the projection plane. One possible solution would be labelling the face in the direction of ray of the projection. So, the points facing the projection plane would be transformed to 2D co-ordinates and connected respectively in the projection plane with solid lines. Now if any other point lies outside the closed curve formed by projected point would be the point on the back face and would be connected with solid line. All other projected point would be connected with dashed lines as they would be hidden in the view. So,using this algorithm we can create the projection of the given polyhederal solid on any given plane and also create their orthographic projection.

In the next section we deal with the opposite problem i.e recreating the 3D object using 2D orthographic drawing of the given object in the input.

3 2D to 3D orthographic projections

In this section we will consider the case of 3D object reconstruction from given projections. The first thing that comes to anybody's mind would be how do we do it? How many views are necessary? What is the most effecient approach to reconstruct the 3D model/isometric view with minimal loss of information.

All of these concerns are addressed here. We shall first address the question of how many views are necessary to create 3D model or the isometric view from the 2D orthographic views for the simplicity.

A number of techniques have been proposed for the reconstruction of the 3D objects among them B-rep (Boundary Representation) methods are the most studied till now. It consists of following steps:

1. Generate 3D vertices from the 2D projections.
2. Generate possible 3D edges from 3D vertices.
3. Construct possible 3D faces from possible 3D vertices obtained.
4. Construct 3D object from the possible/candidate faces

We will use a bit of different approach to solve the problem and the steps would be really simple.

1. The first step would be the processing of the Engineering drawing given in the input. Processing would involve checking the data for the validity, each view is identified and extracted from the input provided.
2. The next step would be to create 3D candidate edges and vertices from the three orthographic views. We will be analysing the relationship between 3D edges and their projections using the theory of the matrices and would then try to construct a wire frame model.
3. The last step would be an attempt to reconstruct the 3D object using the wire frame model. The information about depth would be used to remove false edges and faces that we would extract from the input engineering drawing. And then finally we will assemble the all the true faces to form the 3D object.

Matrix Representation of Curved edges which can be approximated by Conics in 2D orthographic view We will develop a relationship between a conic section and matrices, in particular a symmetric matrix.
A general equation of a conic can be represented by

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \tag{3}$$

Using matrix notation we can represent the equation as

$$f(u) = u^T P u \tag{4}$$

where u=$[x, y, 1]^T$=0, u^T is Transpose of u, P is a 3×3 symmetric matrix given by

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{5}$$

The matrix P is non-singular matrix for $f(u)$ to be a non-degenerate conic i.e it can't be factorised over complex numbers or is irreducible.

This representation gives us a simple characterisation of it's orthographic projection. And any linear transformation of the form $u = Au$ is visible from the change in the matrix from P to P_a which is given by

$$P_a = A^T P A \tag{6}$$

Hence the relationship between a conic and its orthographic projection in this representation is linear.

4 Minimum number of views for reconstruction

Here we will discuss the minimum number of views required for the reconstruction of 3D object from the 2D orthographic views limiting to 3D conics.

Definition 1. *Under a parallel projection, if the plane containing the space conic is not perpendicular to the projection plane, then paralell projection is irreducible(non-degenerate).*

Under non-degenerate projection all conics are mapped to conics. Also, the class of the conics does not change under such projection.It follows that, under non-degenerate paralell projections ellipses, hyperebolas, parabolas in line drawings are projections of ellipses, hyperbolas and parabolas respectievely. Therefore, a conclusion that can be drawn from this is that if any one of the projection of the planar curve is a conic, then the planar curve must be a conic.

Suppose a space conic lies on a plane p and the object co-ordinate system that it follows is c_p such that x_p and y_p lies on the plane p. Let c be the standard(global) co-ordinate system. Then the standard representaion $x = [x, y, z, 1]^T$ in c of a point $x_p = [x_p, y_p, z_p, 1]^T$ in c_p can be obtained by applying the transformation of the form

$$x = Rx + t \tag{7}$$

where R and t are rotation and translation respectievely. Now, for any point on the plane p has its $z_p = 0$, so we obtain the following matrix

$$x = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{00} & r_{01} & t_0 \\ r_{10} & r_{11} & t_1 \\ r_{20} & r_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} = P u_p \tag{8}$$

where u_p=$[x_p, y_p, 1]^T$ and P is a matrix 4×3 given in the above equation. The space geometry of conic is represented in P.

We consider the relationship between a space conic and it's orthographic projections onto some projection planes. Let c_i represent the co-ordinate system on the ith projection plane, and $u_i = [x_i, y_i, 1]^T$ denote the co-ordinte of any arbitrary point on the the plane. If C_i is a 3×4 matrix in which 3 of the columns form an orthogonal basis for the projection, then the transformation from point x in 3D to point u_i in the projection plane is given by the relation

$$u_i = C_i x \tag{9}$$

The matrix which relates u_i and u_p can be obtained from the equation

$$u_i = C_i P u_p = H_i u_p \tag{10}$$

where $H_i = C_i P_i$

Theorem 1. *Three distinct orthographic views sufficient to uniquely recover a space curve.*

Proof. Let us suppose that non of the three projection are irreducible.
Let M be a space conic that lies on plane P

$$u_p^T M u_p = 0 \tag{11}$$

and its projection curves $M_i arerepresentedby$

$$u_i^T M_i u_i = 0 \quad i = 0, 1, 2, ..., q \tag{12}$$

Substitute the linear transformation of the form $u_i = G_i u_p$ into equation (10), we get

$$u_p^T G_i^T M_i C_i P_i = M \quad i = 1, 2, ..., q \tag{13}$$

where M and P are unknown matrices. This implies that three orthographic projections yeilds 18 equations in 15 unknowns. Using Bernstein's seminal theorem, we can derive that the system of equation (10) is solvable. Hence, three distinct views are sufficient to uniquely reconstruct a curve(conic curve) if none of its projection is irreducible.

We now consider the case where we have at least one of the orthographic view which is reducible or degenerate. So, the projection of conic onto the projection plane is a straight line. By definiton of orthographic projection we can determine the plane on which the given conic curve lies, which can be obtained by extruding the straight line along the reducible projection direction. Since, at least one of the projection is a conic we can determine its centre point(vertex) of space conic by finding its position in the other views. Accordingly the matrix P is obtained.Let us now consider the projection in the front view be a conic, without any loss of generality. To reconstruct the conic we need to solve the equation $P^T C_t^T M_f C_t P = M$ for M, where t indicates the top view. Hence if at least one of the orthographic projection is irreducible/degenerate, three distinct orthogonal projections are also sufficient to identity the conic. Our proof is complete.

Reconstruction Now we represent the 3D wire-frame model in preparation for automatic interpretation of faces and solid and uses Boundary Representation model to represent the reconstructed solid.

The input consists of three orthographic views i.e Front view, Top view and side view. Every visible feature is drawn with solid line while those which are invisible are drawn with dotted/dashed line.Views are separated by the fact that the projection of three views on co-ordinate axes are two separated segements respectively.So, we project each curve onto the two co-ordinate axes and four different segments.Two on x-axis and two on y-axis.We also draw a vertical line between two segments on x-axis and horizontal lines between segments on y-axis.

Each view may be considered as a graph of 2D curves and node placed on the plane of drawing. A node is a place where two curves meet i.e. no crossovers are allowed if graph is planar.If two 2D curves intersect,a new node must be created and the curves must be subdivided at that node. When two faces of the 3D object meet tangentially, the crresponding tanget line must be added to the views. Also, centre lines and axes are added to the orthographic views to reconstruct conic edges.

Now we start with reconstruction of wire-frame. Now, we construct all possible 3D vertices and 3D edges that constitutes the wire-frame model by eastibilishing the correspondence among the orthographic views. We can reconstruct the 3D vertices from the following idea. A candidate vertex is created from the 2D vertices present in the three orthographic views. Let $N_f = (N_f(x), N_f(z))$, $N_t = (N_t(x), N_t(y))$ and $N_s = (N_s(x), N_s(y))$ be 2D vertices in the front,top and side views respectively. If
$|N_f(x) - N_t(x)| < \epsilon|$, $|N_t(x) - N_s(y) < \epsilon|$, $|N_s(x) - N_s(y) < \epsilon|$
then we know that N_f,N_t and N_s are corresponding projections in each view of a 3D vertex. And ϵ is the tolrance for inexact matches. To determine possible edges we need do determie the matrix M from the equation given below

$$G_i^T M_i G_i = P^T C_i^T A_i C_i P = A \quad i = 1, 2, 3 \tag{14}$$

Without loss of generality, suppose the three orthographic views are front,top and side views. Thus, the projection matrices are

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} C_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

And from $G_i = C_i P$, we obtain

$$G_1 = \begin{pmatrix} r_{00} & r_{01} & t_0 \\ r_{10} & r_{11} & t_1 \\ 0 & 0 & 1 \end{pmatrix} G_2 = \begin{pmatrix} r_{00} & r_{01} & t_0 \\ r_{20} & r_{21} & t_2 \\ 0 & 0 & 1 \end{pmatrix} G_3 = \begin{pmatrix} r_{10} & r_{11} & t_1 \\ r_{20} & r_{21} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

In particular the relationship between M_i and M is linear in the element of M. This transformation provides us a simple way to characterise the solvability of inverse of problem of determining M from eqauation (12). This system of equation can be solved using various methods. Hence, a conic M can be represented by its projection $A_i (i = 1, 2, 3)$ can be described by an algabraic expression

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{13}x + a_{23}y + a_{33} = 0$$

Since conic curves are invariant under parallel projection, they can be determined by the type of its projection curves.

Now we come to the reconstruction part. We begin with the psudo wire-frame model generated in the last steps. Firstly, all possible faces are found within wire-frame, using a *maximum turning angle* method. Next step would be the detection and deletion of psuedo faces formed from back projection using definition of manifold and Moebius rule. At last, all the remaining true faces are assembled to form the 3D object.

We now come the part in which we try to generate surfaces and faces. We use the term surface to refer to an unbounded face and the term face to refer to the region bounded by an edge loop. All possible surfaces can be constructed very easily from the information of two edges sharing a vertex.The type of those two edges mentioned above and the relationship between them determine the type and features of generated surface.Once a surface is identified we will search all the edges on the surface for . For each vertex on the surface, all edges connected to the vertex are checked. A possibile algorithm for searching can be a depth-first traversal on the vertices.
For each face found in the previous step, we will find all its edge loops from the wire-frame model to form the corresponding faces. We will use a method called maximum turning method but first we should know what turning angle is. It is the angle by which the tangent vector of a curved edge is to be rotated about the common vertex to go from tangent vector of current face to tangent vector of its adjacent curved edge. COunterclockwise rotation is assumed to be positive.
We find all edge loops by sorting the edges at each vectex in descending the order with respect to the turning angle from selected edge, then traversing the resulting graph of oriented edges in this order to find all the cicuits in the face.

The last step is to find all possible solids that can be constructed from the set of faces and to finally compare them with the original given projections. Among all the possible faces that are generated int he previous steps, some really lie at the boundary of the 3D model, they are the real faces, and the others are caled psuedo faces.
Determination of these faces is done using Moebius rule and the definition of Manifolds. The Moebius rule says that each edge in a manifold solid belongs to two faces and the orientation of the edge is inverted by each face. We can impliment this idea by traversing the whole graph of possible face configurations, backtracking when necessary, to find all possible solids.// Initailly, we use depth information including solid curves and dashed lines to removes psuedo-faces.Faces that obstruct solid edges must be removed. On the other hand, if projection of a 3D candidate edge is a dashed curve in one view, there should exist at least one face that obstructs the edges in projecting direction. Otherwise

3D candidate edge is psuedo and must be deleted. The final solid is made by removing all the left psuedo faces using Divide and conquer approach based on Moebius Rule and the defintion of Manifold. The final solid consists of set of real faces oreinted with respected to one another according to the Moebius rule. On last thing that is left is that in the searching process, we compute an integer for each 2D curve views, which is the nukmber of 3D edges that project to that 2D curve. When solid is built, some edges maybe removed but that may cause decrese in number of edges. In order to match the original views , we must keep an integer associated with each 2D curve positive.

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