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Continuous Contact Force Models for Impact Analysis in Multibody Systems

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Abstract. One method for predicting the impact response of a multibody system is based on the assumption that the impacting bodies undergo local deformations and the contact forces are continuous. In a continuous analysis, the integration of the system equations of motion is carried out during the period of contact; therefore, a model for evaluating the contact forces is required. In this paper, two such contact force models are presented, both Hertzian in nature and based upon the direct-central impact of two solid particles.

At low impact velocities, the energy dissipation during impact can be represented by material damping. A model is constructed based on the general trend of the Hertz contact law in conjuction with a hysteresis damping function. The unknown parameters are determined in terms of a given coefficient of restitution and the impact velocity. When local plasticity effects are the dominant factor accounting for the dissipation of energy at high impact velocities, a Hertzian contact force model with permanent indentation is constructed. Utilizing energy and momentum considerations, the unknown parameters in the model are again evaluated. The two particle models are generalized to an impact analysis between two bodies of a multibody system.

Key words: Contact models, impact analysis, multibody systems.

1. Introduction

A collision between two solids is known as an impact during which forces are created that act and disappear over a short period of time. The duration of the contact period governs the choice of the method used to analyze the impact. In general, there are two different approaches for impact analysis. One approach is to assume that impact occurs instantaneously. The analysis is thus divided into two intervals – before and after impact. The interconnection between the two intervals is made by a momentum balance and a parameter representing the amount of energy dissipation in the impact, called the coefficient of restitution. This classical approach has been followed by many for direct-central and oblique impacts of two free solids not connected to other bodies [1–3]. For an impact within a multibody system, a set of momentum balance-impulse equations was developed by Wehage [4]. Khulief and Shabana extended these equations to flexible multibody systems [5]. In references [6] and [7], it was shown that in order to gain numerical efficiency and stability, the equations of motion for a multibody system may be assembled in a canonical form. In this formulation, a reduced set of momentum balance-impulse equations was developed based on the time integration of a canonical form of the equations of motion. In order to perform this so-called "piecewise analysis" the coefficient of restitution between two colliding bodies must be known. Although this method is relatively efficient, the unknown duration of contact limits its applicability. If the duration of contact is large enough that significant changes occur in the configuration of the multibody system, then the assumption of instantaneity of impact is no longer valid.

Another approach for impact analysis is to let the collision forces act in a continuous matter. The impact analysis of a system of colliding bodies is hence performed simply by including the collision of contact forces in the system equations of motion during the contact period. The performance of this so-called "continuous analysis" requires knowledge of the variation of contact force during the contact period. Different models have been postulated to represent this variation. In the most simple one, the contact force is modeled by a parallel linear spring-damper element [8]. This model, known as the Kelvin–Voigt viscoelastic model, has been used for the impact between two bodies within a multibody system [9]. However, this linear model may not be very accurate since it does not represent the overall nonlinear nature of an impact. Furthermore, the half-sine shape solution that it provides for the local deformation of the two bodies in the direction of impact suggests that the two bodies exert tension on each other right before the rebounding stage. A more suitable model of the contact force is the nonlinear Hertz force-displacement law [10]. This model by itself does not represent the energy dissipation process. Although Hertzian theory is based on elasticity, some studies have been performed to extend the theory to include energy dissipation [8, 11–13].

In this paper, two continuous contact force models are presented, for which unknown parameters are analytically evaluated. In the first model, internal damping of the impacting bodies is used to represent the energy dissipation at low impact velocities. A hysteresis damping function of this nature assumes that the loss of energy in impact is all due to the material damping of the colliding solids, which dissipates energy in the form of heat. At fairly moderate or higher velocities of colliding solids, especially metallic solids, permanent indentations are left behind on the colliding surfaces. In the second contact force model, which covers these cases, local plasticity of the surfaces in contact becomes the dominant source of energy dissipation in impact. For both models, the unknown parameters are evaluated by energy and momentum considerations in terms of the velocities of the solids before impact and the coefficient of restitution. The two-particle models are then generalized to impact analysis between two bodies of a multibody mechanical system.

2. A Contact Force Model with Hysteresis Damping

When two solids are in contact, deformation takes place in the local contact zone resulting in a contact force. This suggests that the contact force is directly related to the amount of local deformation or indentation of the two solids. The best-known force model for the contact between two spheres of isotropic material was developed by Hertz based on the theory of elasticity [10]. With radii R_i and R_j of the two spheres i and j, and masses m_i and m_j, the contact force f follows the relation

$$f = K\delta^n, \tag{1}$$

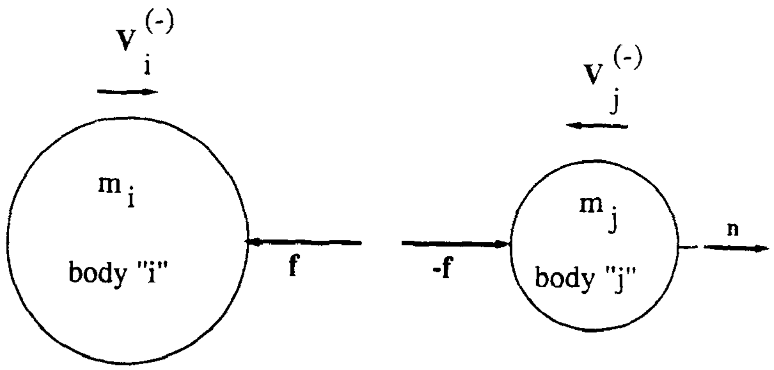


Fig. 1. A direct-central impact of two spheres.

where δ is the relative penetration or identation between the surfaces of the two spheres and $n = 1.5$. The generalized parameter K depends on the material properties and the radii of the spheres:

$$K = \frac{4}{3\pi(h_i + h_j)} \left[\frac{R_i R_j}{R_i + R_j}\right]^{1/2}, \tag{2}$$

where the material parameters h_i and h_j are

$$h_\ell = \frac{1 - \nu_\ell^2}{\pi E_\ell}; \quad \ell = i, j. \tag{3}$$

Variables ν_ℓ and E_ℓ are, respectively, the Poisson's ratio and the Young's modulus associated with each sphere.

Consider now a situation for which the contact between the two spheres is caused by a collision. The two spheres have velocities $V_i^{(-)}$ and $V_j^{(-)}$ right before impact. It is intended to determine the variations of the interaction force between the two spheres during the short period of contact. The normal direction n to the contact surfaces and a pair of forces f and $-f$ are shown in Figure 1. In general, the resulting contact may be considered to occur in two phases – the compression phase and the restitution phase. During the compression phase, the two spheres deform in the normal direction to the impact surface, and the relative velocity of the two spheres is reduced to zero. The end of the compression phase is referred to as the instant of maximum compression, or maximum approach. The restitution phase starts at this point and lasts until the two spheres separate.

Generally, the two spheres will not rebound with the same initial velocities, because part of the initial kinetic energy is dissipated in the form of permanent deformation, heat, etc. It is apparent that the contact force model of equation (1) cannot be used during both phases of contact, since this would suggest that no energy is dissipated in the process of impact. One popular idea is based on the dissipation of energy in the form of internal damping of colliding solids. This assumption is valid for low impact velocities; i.e., those impact situations for which impact velocities are negligible compared to the propagation speed of deformation waves across the solids. The contact force model will then be in terms of a damping coefficient D,

$$f = K\delta^n + D\dot{\delta}, \tag{4}$$

where $\dot{\delta}$ is the relative (or indentation) velocity of the solids. A hysteresis form for the damping coefficient was proposed by Hund and Grossley [11] as

$$D = \mu\delta^n, \tag{5}$$

where the parameter μ is called the "hysteresis damping factor". The contact force model of equation (4) may be used for the entire period of contact. With this model, the energy loss is assumed to be due to the material damping of the bodies, which would dissipate energy in the form of heat. For known parameters K and D (or μ), the shape of the hysteresis loop corresponding to this force model and the solution corresponding to the variations of the indentation with time are shown in Figure 2. In this figure, $t^{(-)}$, $t^{(m)}$, and $t^{(+)}$ denote the time of initial contact, the time of maximum indentation, and the time of separation of the local contact surfaces, respectively. Variables δ_m and f_m refer to the values of indentation and the contact force at time $t^{(m)}$. In the contact force model of equation (4), the damping coefficient D or the hysteresis damping factor μ must be determined. An estimate of the parameter μ based on the classical impulse-momentum equation and the work-energy principal can be determined.

From consideration of the kinetic energies before and after impact, the energy loss ΔT may be expressed in terms of the coefficient of restitution e and the relative approach velocity $\dot{\delta}^{(-)} = V_i^{(-)} - V_j^{(-)}$ as

$$\Delta T = \frac{1}{2} m^{(\text{eff})} \dot{\delta}^{(-)^2} (1 = e^2), \tag{6}$$

where

$$m^{(\text{eff})} = \frac{m_i m_j}{m_I + m_j}. \tag{7}$$

The energy loss may also be expressed by integration of the contact force around the hysteresis loop as [14]

$$\Delta T = \oint D\dot{\delta}\, \mathrm{d}\delta = \oint \mu\delta^n \dot{\delta}\, \mathrm{d}\delta \cong 2 \int_0^{\delta_m} \mu\delta^n \dot{\delta}\, \mathrm{d}\delta = \frac{2}{3}\frac{\mu}{K} m^{(\text{eff})} \dot{\delta}^{(-)^3}, \tag{8}$$

where $\oint$ refers to the integration around a hysteresis loop for a contact force of the form shown in Figure 2(a). The hysteresis damping factor μ may be evaluated by comparing the right sides of equations (6) and (8),

$$\mu = \frac{3K(1 - e^2)}{4\dot{\delta}^{(-)}}, \tag{9}$$

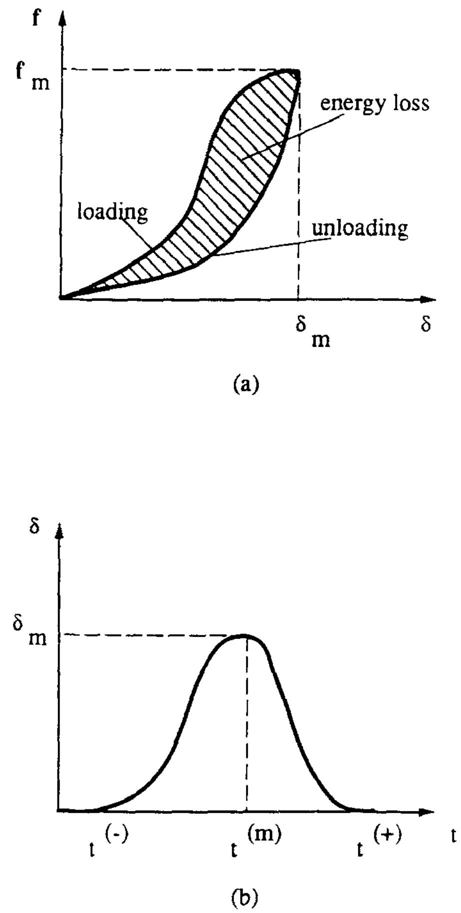


Fig. 2. Hertz contact force model with hysteresis damping: (a) contact force versus indentation, and (b) indentation versus time.

which shows a direct relationship between the coefficient of restitution and an equivalent damping factor. The contact force in conjuntion with the damping representation may then be written in an alternative form as

$$f = K\delta^n \left[1 + \frac{3(1-e^2)}{4}\frac{\dot{\delta}}{\dot{\delta}^{(-)}}\right], \tag{10}$$

which shows a direct relationship between the coefficient of restitution and the contact force.

The clear advantage of the Hertz contact force model, $f = K\delta^n$, with its damping representation in equation (10) over the Kelvin–Voigt viscoelastic model is its nonlinearity. The overall pattern of impact is far from linear, while the Kelvin–Voigt model and its damping representation are linear models. The solution for indentation corresponding to the linear models is a half-damped harmonic. This indicates that the bodies in impact must exert tension on each other right before separation. On the other hand, the Hertzian contact force model predicts no tension on the bodies before separation, as observed from the solution for its corresponding indentation of Figure 2(b).

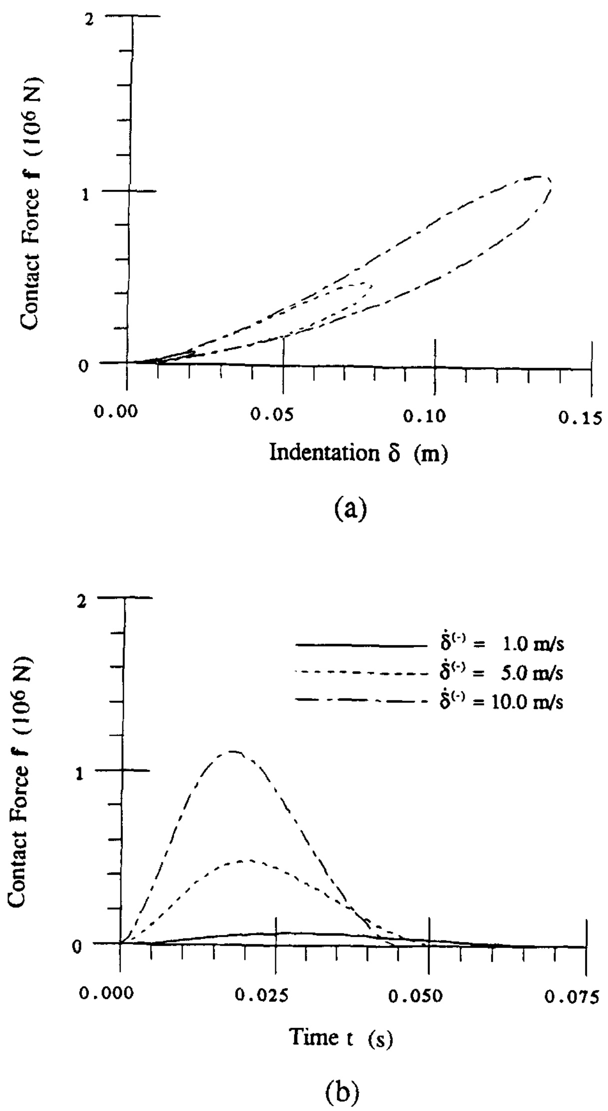
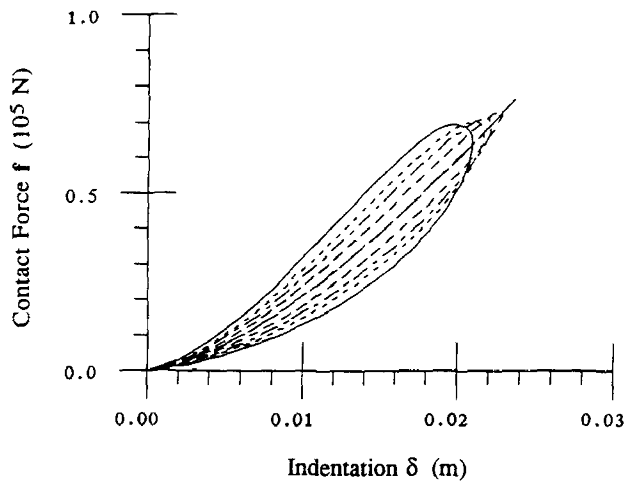
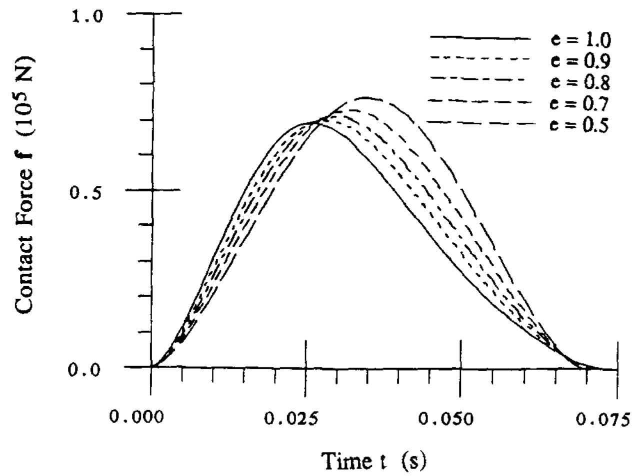


Fig. 3. Contact force model with hysteresis damping for one coefficient of restitution ($e = 0.7$) and different impact velocities: (a) force-indentation relationship, and (b) variations of contact force with time.

EXAMPLE 1. To illustrate the shape of the contact force law, the impact between two spheres, each of mass of 1 kg, was considered. The results from a series of continuous analyses are summarized in Figures 3 and 4. The equations of motion for the two spheres in conjunction with the nonlinear contact force model of equation (10) are numerically integrated forward in time. Figure 3(a) shows the contact force for one restitution coefficient and different initial indentation velocities. The variations of the contact force with time, for the same parameters, are shown in Figure 3(b). The hysteresis loops shown in Figure 4(a) correspond to one initial



(a)



(b)

Fig. 4. Contact force model with hysteresis damping for one impact velocity ($\dot{\delta}^{(-)} = 1$ m/s) and different coefficients of restitution: (a) force-indentation relationship, and (b) variations of the contact force with time.

indentation velocity and different coefficients of restitution. The corresponding time variations of the contact force are shown in Figure 4(b).

3. A Contact Force Model with Permanent Indentation

At higher impact velocities, the dissipation of energy is mostly in the form of local plasticity. In other words, there is some permanent indentation that is left behind on the surfaces of the two spheres after separation, and this indentation accounts for the energy loss in impact. This is not an unreasonable assumption for impact problems in which two metallic bodies with

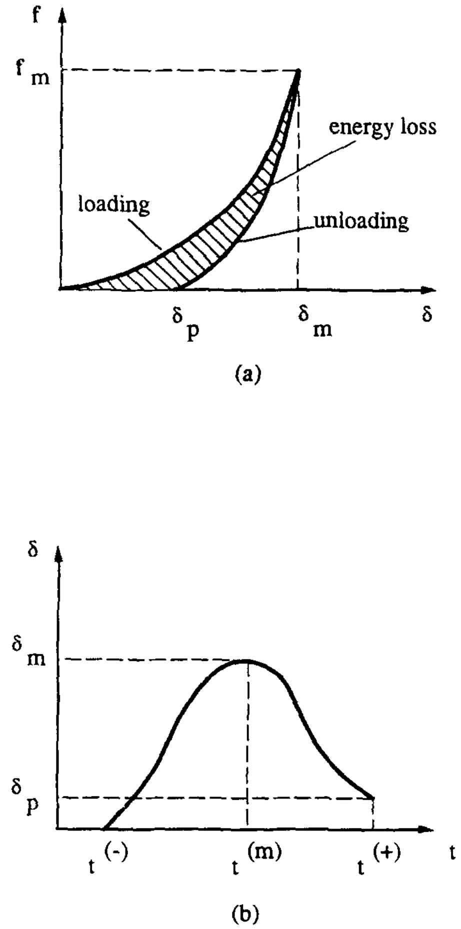


Fig. 5. Hertz contact force model with permanent indentation: (a) contact force versus time, and (b) indentation versus time.

initial relative velocity larger than $10^{-5}\sqrt{E/\rho}$ collide [15], where ρ is the mass density and the quality $\sqrt{E/\rho}$ is the larger of two propagation speeds of the elastic deformation waves in the colliding solids. With this condition, the contact force changes according to equation (1) during the compression period, and its variation during the restitution follows

$$f = f_m \left[\frac{\delta - \delta_p}{\delta_m - \delta_p} \right]^n , \tag{11}$$

where variable δ_p is the permanent indentation of the two spheres after separation. The shape of the hysteresis loop corresponding to this contact force model and the solution corresponding to the variation of the indentation with time are shown in Figure 5.

The proposed contact force model can be used for the impact between two spheres, if the parameters in the model are known. The generalized parameter K may be evaluated from the radii and the material properties of the two spheres using equation (2). The remaining parameters to be determined are δ_m, f_m, and δ_p.

Consider first the compression phase of contact, during which the equations of motion of the two spheres are

$$m_i \ddot{\delta}_i = -K\delta^n, \quad m_j \ddot{\delta}_j = K\delta^n, \tag{12}$$

where $\ddot{\delta}_i$ and $\ddot{\delta}_j$ are the indentation accelerations of the two spheres. Equation (12) can be written as

$$m^{(\text{eff})}\ddot{\delta} = -K\delta^n, \tag{13}$$

where $\ddot{\delta} = \ddot{\delta}_i - \ddot{\delta}_j$ is the relative indentation acceleration. The relative indentation equation of motion expressed by equation (13) is a second-order ordinary differential equation. A proper set of initial conditions at the initial time of contact, $t^{(-)}$, contains

$$\delta^{(-)} = 0; \quad \dot{\delta}^{(-)} = V_i^{(-)} - V_j^{(-)}.$$

Equation (13) can be integrated analytically with respect to time in order to obtain the time variation of the indentation velocity during the compression phase of contact. This results in

$$\frac{1}{2}m^{(\text{eff})}[\dot{\delta}^2 - \dot{\delta}^{(-)^2}] = -K\frac{\delta^{n+1}}{n+1}. \tag{14}$$

At the instant of maximum compression, $t^{(m)}$, the indentation velocity $\dot{\delta}$ is zero; thus,

$$-\frac{1}{2}m^{(\text{eff})}[\dot{\delta}^{(-)}]^2 = -K\frac{\delta_m^{n+1}}{n+1}, \tag{15}$$

which can be solved for the maximum indentation, δ_m, as

$$\delta_m = \left[\frac{n+1}{2K}m^{(\text{eff})}\dot{\delta}^{(-)^2}\right]^{1/(n+1)}. \tag{16}$$

This equation states that the maximum indentation of the two spheres depends on the material properties, masses, radii, and velocities of the two spheres right before impact. The maximum contact force, using equation (1), may also be evaluated as

$$f_m = K\delta_m^n. \tag{17}$$

The remaining parameter in equation (11) to be determined is the permanent indentation, δ_p, which is a measure of the dissipated energy. It is obvious that there must be a relationship between the permanent indentation and the coefficient of restitution. The dissipated energy of the impact can be evaluated by integration of the contact force around the hysteresis loop of Figure 5(a), in the form of

$$\Delta T = \int_0^{\delta_m} K\delta^n \, \mathrm{d}\delta + \int_{\delta_m}^{\delta_p} f_m \left[\frac{\delta - \delta_p}{\delta_m - \delta_p}\right]^n \mathrm{d}\delta = \frac{f_m \delta_p}{n+1}. \tag{18}$$

An expression is thus obtained to relate the permanent indentation to the total dissipated energy. Equating the right sides of the energy balance relations in equations (6) and (18), one obtains

$$\delta_p = \frac{(n+1)m^{(\text{eff})}\dot{\delta}^{(-)^2}}{2f_m}(1 - e^2). \tag{19}$$

Hence, the permanent indentation is evaluated from the initial approach velocities of the spheres and a known coefficient of restitution between the spheres.

With the preceding parameter evaluation process, the variations of the contact force during the entire period of contact are thus obtained. A continuous analysis may then be performed by numerically integrating the equations of motion of the two spheres forward in time in conjunction with the developed contact force model. A solution is thus obtained in the form of positions, velocities, and accelerations of the spheres at any instant in time during the contact period. As a by-product of the preceding parameter indentification process, one can approximate the duration of the contact period between the two spheres. If $\dot{\delta}$ in equation (14) is integrated with respect to time between times $t^{(-)}$ and $t^{(m)}$, and assuming that the durations of the compression phase and the restitution phase are almost the same, then an approximate formula for the period of contact is obtained as [6]

$$\Delta t \cong 2.94 \frac{\delta_m}{\dot{\delta}^{(-)}}. \tag{20}$$

EXAMPLE 2. To illustrate the parameter estimation process, a numerical example is considered here. For a direct-central impact of two identical aluminum spheres, a coefficient of restitution of 0.7 and a duration of contact of 135 μs were recorded from experiment [8]. Both spheres had equal and opposite impact velocities of 0.15 m/s, and each had a radius of 0.02 m.

Based on the presented theory, it is intended to construct the contact force model between the two spheres. The initial indentation velocity between the two spheres is $\dot{\delta}^{(-)} = 0.3$ m/s. The limiting value based on the speed of deformation waves is about 0.03 m/s. Hence the Hertz contact force model with permanent indentation is a valid one. The generalized parameter K is calculated from equation (2), with $\nu = 0.33$, to be equal to $5.50(10^9)$ N/m$^{1.5}$. The equivalent mass of the two spheres is obtained from equation (7) as $m^{(\text{eff})} = 0.046$ kg. From equations (16), (17), and (19), the unknown parameters in the contact force model are evaluated as

$$\begin{aligned} \delta_m &= 1.55(10^{-5})m, \\ f_m &= 336N, \\ \delta_p &= 7.92(10^{-6})m. \end{aligned}$$

With all the parameters determined, the contact force model of the two spheres is constructed as shown in Figure 6. Note that the permanent indentation is slightly larger than half of the maximum relative indentation for this impact situation, which is a significant amount. This suggests that the local plasticity of the contact surfaces during impact cannot be neglected, and it again justifies the validity of the Hertzian contact force model with permanent indentation. A continuous analysis may be performed on the two spheres with the constructed contact force model. The relative indentation equation of motion, $m^{(\text{eff})}\ddot{\delta} = -f$, with its corresponding initial conditions, and the developed contact force model of Figure 6 are integrated forward in time until the separation time. This results in a relative departing velocity, $\dot{\delta}^{(+)}$, of 0.21 m/s and a duration contact of 130 μs, which is very close to the experimental value of 135 μs. The solution for variations of indentation, indentation velocity, and indentation acceleration versus time is shown in Figure 7. If a piecewise analysis with the coefficient of restitution 0.7 is performed, the departing velocities of the two spheres is obtained as

$$\dot{\delta}^{(+)} = -e\dot{\delta}^{(-)}.$$

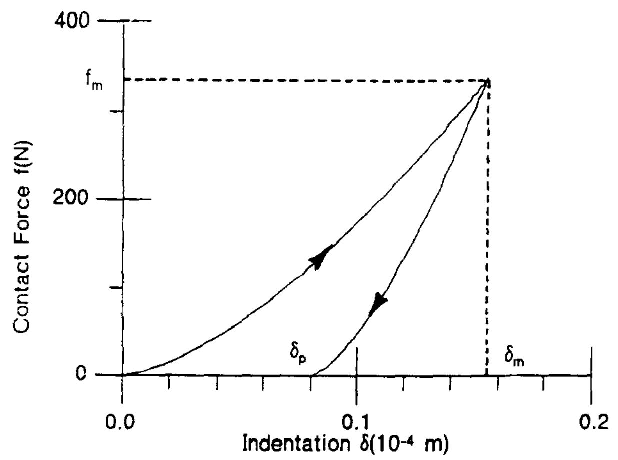


Fig. 6. A Hertzian contact force model with permanent indentation for the impact of two identical aluminum spheres with impact velocities of 0.15 m/s each.

The relative indentation velocity after impact is thus determined to be $\dot{\delta}^{(+)} = 0.21$ m/s. Hence, the results from a continuous analysis and a piecewise analysis match closely. The energy dissipated in impact can also be evaluated from equation (18) as $\Delta T = 1.06(10^{-3})J$. Also, the approximate expression of equation (20) for the duration of contact results in a value of $\Delta t = t^{(+)} - t^{(-)} \cong 152\ \mu$s. The apparent difference between this approximate value and the experimental value of 135 μs is due to the fact that the restitution period is relatively shorter than the compression period.

4. Generalization to Multibody Impact

We now generalize the preceding discussion to an impact situation within a multibody system. The objective is to determine the unknown parameters in the contact force models for the impact of two bodies in the system. Let the two colliding bodies be "i" and "j" in the system of Figure 8. The points of contact on the two bodies are P_i and P_j, and n is a unit vector in the normal direction to the contact surfaces of the two bodies. No matter which type of coordinates are used to assemble the equations of motion for the multibody system, the coordinates and the velocities of the bodies can be calculated, at any instant of time, from the solution of the equations of motion. For a known system configuration at the initial time of contact, the location of the contact points $\mathbf{r}_i^{P(-)}$ and $\mathbf{r}_j^{P(-)}$ and the components of the algebraic unit vector $\mathbf{n}$, with respect to a non-moving xyz coordinate system, may be calculated. From the known system velocities, the velocities of the contact points in the xyz coordinate system, $\dot{\mathbf{r}}_i^{P(-)}$ and $\dot{\mathbf{r}}_j^{P(-)}$, may also be calculated at that time. Hence, the indentation and the indentation velocity at the initial time of contact are

$$\delta^{(-)} = 0 \tag{21}$$

$$\dot{\delta}^{(-)} = \mathbf{n}^T[\dot{\mathbf{r}}_i^{P(-)} - \dot{\mathbf{r}}_j^{P(-)}], \tag{22}$$

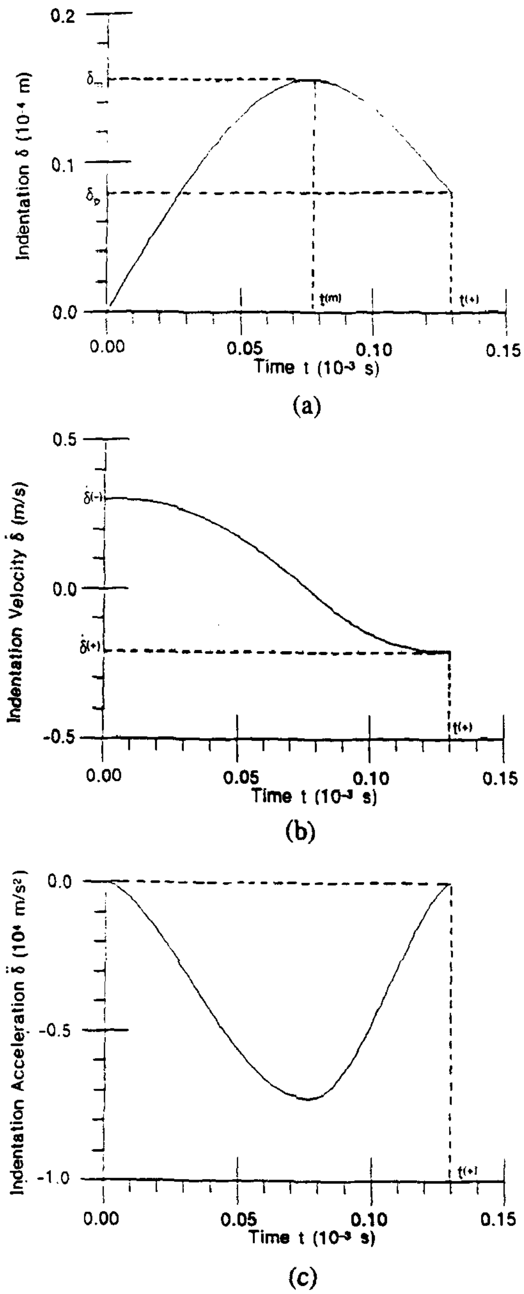


Fig. 7. Results from a continuous analysis of the impact of two aluminum spheres: (a) relative indentation versus time, (b) relative indentation velocity versus time, and (c) relative indentation acceleration versus time.

in which the symbol T performs the transpose operation and velocities of the contact points are projected in the normal direction to the contact surface. The expression in equation (2) for the parameter K can be used for the contact between any two bodies if the local surfaces of contact are both spherically shaped. Similar expressions have been obtained by Hertz [10] and others [6, 8] for other shapes of the local contact surfaces such as sphere on plane, parallel

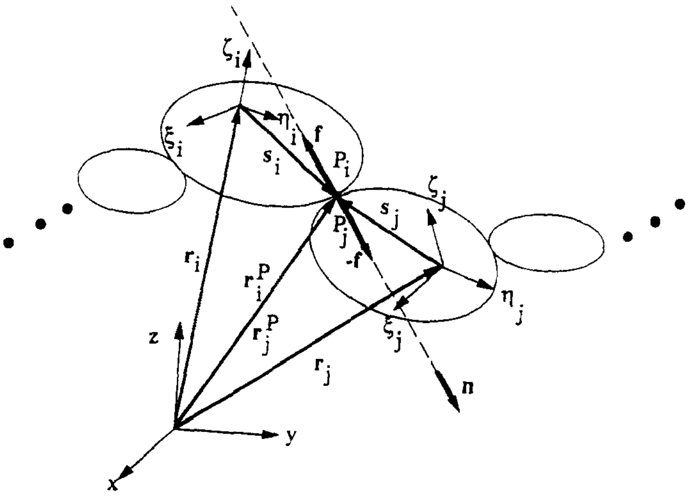


Fig. 8. Impact within a multibody system.

cylinders, and plane on plane. Once the generalized parameter K is calculated, with a given coefficient of restitution e and the initial approach velocity $\dot{\delta}^{(-)}$, all other parameters in the contact force model can be determined.

With known variations of the contact force during the contact period, a continuous analysis of the system can be performed simply by adding these forces to the multibody system equations of motion. This analysis method provides accurate results, since all of the equations of motion are integrated over the period of contact. It thus accounts for the changes in the configuration and the velocities of the system during that period.

EXAMPLE 3. To illustrate an impact analysis in a multibody system, the impact between the slider of a slider-crank mechanism and another sliding block [9] is considered, as shown in Figure 9. The system is kinematically constrained due to the existence of the prismatic and the pin joints interconnecting different components. The slider-crank is driven by a restoring torque with maximum value of 200 Nm on the crank such that the crank maintains an almost constant angular velocity of 150 rad/s. At some instant, the slider impacts a free block, which is inertially driven at a constant velocity of 15 m/s to the left.

The occurrence of contact between the two blocks is determined by evaluating variable δ at any time during the numerical integration of the system equations of motion as

$$\delta = x_5 - x_4 - 2a. \tag{23}$$

At the initial time of contact, the relative indentation and its velocity are

$$\delta^{(-)} = 0 \tag{24}$$

$$\dot{\delta}^{(-)} = \dot{x}_5 - \dot{x}_4. \tag{25}$$

If the two blocks are made from steel and are spherically shaped near the local contact surfaces, the generalized parameter K can be determined from equation (2) to be 9.5×10^9 N/m$^{1.5}$.

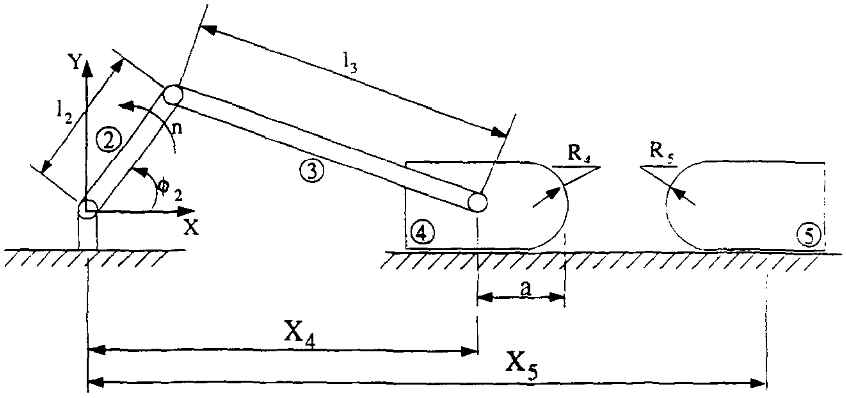


Fig. 9. Impact of a slider-crank mechanism with a free block.

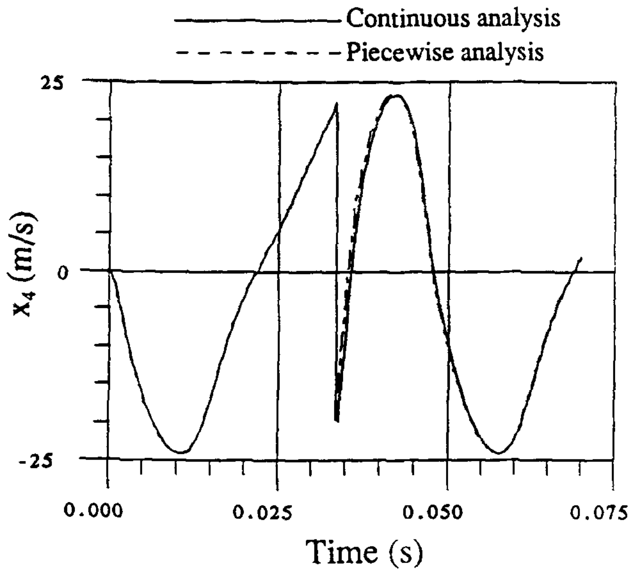


Fig. 10. Slider velocity versus time for multibody continuous and piecewise analysis.

For the piecewise analysis of this multibody system, a coefficient of restitution of 0.83 has been used [6]. A continuous analysis was then performed. The contact force model of equation (10) was directly used in the multibody system equations of motion. These equations were numerically integrated forward in time over the period of contact.

A plot of slider velocity versus time for the piecewise analysis [6] and continuous analysis is shown in Figure 10. The period of contact resulting from the continuous analysis is clearly shown by the time delay in the system response. As observed, the results agree closely. This is due to the fact that the duration of contact is small enough so that no significant changes occur in the system configuration before and after the impact. For larger durations of contact, the continuous analysis must be performed by direct incorporation of the contact force model in the multibody system equations of motion. The piecewise analysis method would then not be suitable for such a case.

5. Summary

A continuous analysis method has been presented for the direct-central impact of two solids. To represent the variation of the contact force during the contact period, two Hertzian models are used, one with hysteresis damping and one with local plasticity effects. At low impact velocities, energy is dissipated in the form of internal damping or heat. If the initial indentation velocity is not negligible compared with the propagation speed of deformation waves across the solid, then permanent indentation is the dominant factor accounting for energy dissipation. For both models, based on energy and momentum considerations, the unknown parameters were evaluated in terms of the geometrical and material properties of the contact surfaces, velocities of the solids before impact, and the coefficient of restitution. A procedure for impact analysis of multibody systems was then developed. The Hertzian contact force models with the hysteresis damping or permanent indentation representations provide means for performing continuous analysis with either multibody system equations of motion or a single relative indentation equation of motion.

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