Physics 101
Spring 2023

Problem Set 1
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1 Jumping off asteroids

Estimate the size of the largest asteroid from whose gravity you could escape by jumping. [Preferably by scaling from jumping on earth—i.e. you should not need to use G, though you may check your answer in any way you like.].

Most people set can jump (on Earth) enough to raise their center of gravity by $d \sim 0.5\,\mathrm{m}$, implying that the speed at the start of the jump is

$$v = \sqrt{2gd} \sim 3\,\mathrm{m\,s^{-1}} \tag{S1}$$

The escape velocity v_{esc} off of a body with mass M and radius R is

$$\frac{1}{2}v_{\mathrm{esc}}^2 - \frac{GM}{R} = 0 \tag{S2}$$

or

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}} \propto M^{1/2}R^{-1/2} \tag{S3}$$

Rocky objects such as the Earth or asteroids are roughly the same density (within a factor of a few), because rock is not particularly compressible. This implies that $M \propto R^3$, or

$$v_{\mathrm{esc}} \propto R \tag{S4}$$

A fun fact is that the Earth's escape velocity is $v_{\mathrm{esc},\oplus} \sim 10\,\mathrm{km\,s^{-1}}$. We can therefore scale by the Earth's values to obtain

$$R = R_\oplus\left(\frac{v}{v_{\mathrm{esc},\oplus}}\right) \sim 2\,\mathrm{km} \tag{S5}$$

It should be possible to jump off of asteroid Bésixdouze (named after B-612, the home of the Little Prince in *the Little Prince*) with some, but not terribly much, effort (it has a diameter $2R \sim 2\,\mathrm{km}$). Of course, the actual asteroid in the Little Prince (which is house-sized) would be very easy to jump off of.

This ignores subtleties about differences between jumping from rest vs running jumps vs pole valuting, and whether in the asteroid's weak gravity you would be able to pre-tension your leg muscles to store and release the same jump energy as you can on earth. It also ignores the mass and any inconvenience of the spacesuit you should probably be wearing on the asteroid.

Landing a spacecraft on an asteroid is also a challenge, because of the tendency to "bounce off": In 2019, the Japanese spacecraft Hayabusa2 collected a sample from the near-earth asteroid Ruygu, which is just about this size (1 km diameter), and returned it to earth in 2020. The NASA mission OSIRIS-REx collected a sample from near-earth asteroid Bennu (0.5 km diameter; OSIRIS-REx also measured its mass, finding that the density is only 1.2g cm^{-3}, hence probably made of rubble, not solid rock), and will return it to earth in 2023.

Figure 1: Like a Caltech student waiting until the last minute to complete their Physics 101 problem set, the Little Prince wonders how it got so bad. Fortunately, as we showed, the option of jumping off of his planet into the great beyond is always open to him.

2 Pizza vs. Rocket fuel

Would the energy of all the calories you have consumed as food be enough to eject you from the solar system?

The author of this solution set is on the order of $\sim 2\times10^1\,\mathrm{yr} \simeq 7\times10^3\,\mathrm{d}$ old. A typical human (which we presume without comment to be relevant in his case) consumes $\sim 2\times10^3\,\mathrm{kcal\,d^{-1}} \simeq 10^7\,\mathrm{J\,d^{-1}}$ (where the conversion ratio to remember is that $1\,\mathrm{kcal} \simeq 4\times10^3\,\mathrm{J}$). This implies a total energy of roughly $E \sim 7\times10^{10}\,\mathrm{J}$ available for this endeavor in extreme cardio.

In order to be ejected from the solar system, one must escape the gravity of both the Earth *and* the Sun, but possibly with an assist from the motion of the Earth—it makes the most sense to jump off the Earth in its direction of motion.

The Earth's velocity in the solar system is

$$v_\oplus = \sqrt{\frac{GM_\odot}{a_\oplus}} \approx 30\,\mathrm{km\,s^{-1}} \tag{S6}$$

where $M_\odot \approx 2\times10^{30}\,\mathrm{kg}$ is the mass of the Sun and $a_\oplus = 1\,\mathrm{AU} \approx 1.5\times10^{11}\,\mathrm{m}$. In the frame of the Earth, the energy E is enough to produce a speed

$$v_\mathrm{rel} = \sqrt{2E/m} \simeq 50\,\mathrm{m\,s^{-1}} \tag{S7}$$

where we have taken $m \simeq 50\,\mathrm{kg}$.

We ask whether

$$3\times10^9\,\mathrm{m^2\,s^{-2}} \simeq \frac{1}{2}(v_\oplus + v_\mathrm{rel})^2 > \frac{GM_\oplus}{R_\oplus} + \frac{GM_\odot}{a_\oplus} \simeq 10^9\,\mathrm{m^2\,s^{-2}} \tag{S8}$$

We see that this criterion is satisfied, and therefore an adult has eaten enough food to launch themselves from the solar system.

3 Tea break energy crisis

In the United Kingdom, at the end of popular BBC television programs, sports events or news events, a large fraction of the country's households turn on their (1 kW) electric kettles to brew a cup of tea. This increases the electrical load on the British power grid over a timescale of just a few minutes. Estimate how much generating power must be put on reserve to meet this demand, in order to prevent a nationwide blackout? Give your answer in gigawatts: a large nuclear or hydroelectric plant can generate about 1 GW, a large gas turbine powerplant about 0.3 GW.

Let f be the fraction of households in the United Kingdom for which a cup of tea is brewed during the end of popular BBC programs (in excess of the background amount). This is likely to be some order-unity quantity less than 1 (although it can be effectively modified by extra water being boiled, etc.). Also let n be the average number of people in each household.

We assume that a kettle runs at a power $P_1 \sim 1\,\mathrm{kW}$. The population of the United Kingdom is $N \sim 7 \times 10^7$ (roughly comparable to France or Germany). The total excess power then is then that of all of the kettles which need to be running to accommodate all of the households

$$P \sim \frac{fN}{n} P_1 \sim 4\,\mathrm{GW}(f/0.2)(n/4)^{-1} \tag{S9}$$

where we have scaled to $f \sim 0.2$ and $n \sim 4$ as a rough guess. This seems to imply the power excess (which must be accounted for via reserve) is on the order of a few large nuclear/hydroelectric plants.

According to Wikipedia[a], this phenomenon (called "TV pickup") typically imposes excess power ≈ 200–$400\,\mathrm{MW}$, but has reached $\approx 2.8\,\mathrm{GW}$ after a penalty shootout in the England–West Germany semifinals round of the 1990 World Cup.

[a] https://en.wikipedia.org/wiki/TV_pickup

4 Order of Magnitude Math I

In Lecture 1, we found the roots of a quadratic equation as an elementary illustration of the simplifying technique of assuming pairs of terms dominate in any multi-term equation. In an equation with n terms, there are ${}_nC_2 = n!/(2!(n-2)!)$ pairs of terms to try. Here is a very slightly less elementary equation for you to try this method on:

$$x^4 - 10^6 x^2 + 10^4 x - 2 = 0 \tag{1}$$

Find good approximations to the roots of this equation by assuming in turn dominance of each of the possible pairs of terms to find the resulting trial values of x, and then checking if the other terms are in fact much smaller than the pair you kept. If they are, you found a good root estimate. If the remaining terms are much larger than the pair you kept, that was a bad choice resulting in a meaningless root estimate. If one or more of the remaining terms are of the same order of magnitude as your chosen pair, you are in the ballpark of a root, but this indicates you may need to keep more than two terms to arrive at better solutions.

Using this method, you should be able to find estimates (good to better than a percent in this case) of the four roots of the quartic, purely by inspection, without need for numerics or using nontrivial formulae.

We only need to check ${}_4C_2 = 6$ combinations of terms to keep. If ignoring them yields an answer self-consistent with the ignored terms being small, the answer is a good approximation for a root. Because this equation is fourth-order, we expect four roots total, which we hope will all be recoverable in this way.

Using the procedure outlined, we find four self-consistent roots (Table 1):

$$x_1 = -10^3 \tag{S10a}$$
$$x_2 = 2 \times 10^{-4} \tag{S10b}$$
$$x_3 = 10^{-2} \tag{S10c}$$
$$x_4 = 10^3 \tag{S10d}$$

Maple and Wolfram Alpha confirm this, finding numerical roots

$$x_1 \approx -1000.005 \tag{S11a}$$
$$x_2 \approx 2.04 \times 10^{-4} \tag{S11b}$$
$$x_3 \approx 0.98 \times 10^{-2} \tag{S11c}$$
$$x_4 \approx 0.999995 \times 10^3 \tag{S11d}$$

terms kept	equation	root(s)	self-consistent root(s)
1 & 2	$x^4 - 10^6 x^2 = 0$	$x = 0, \pm 10^3$	$x = \pm 10^3$; terms 3,4 smaller than 1,2: ok
1 & 3	$x^4 - 10^4 x = 0$	$x = 0, 10^{4/3}$	none: term 2 larger than 1,3
1 & 4	$x^4 - 2 = 0$	$x = 2^{1/4}$	none: terms 2,3 larger than 1,4
2 & 3	$-10^6 x^2 + 10^4 x = 0$	$x = 0, 10^{-2}$	$x = 10^{-2}$: terms 1,4 smaller than 2,3: ok
2 & 4	$-10^6 x^2 - 2 = 0$	$x = \pm 0.0014i$	none: term 3 is $14i$, larger mag than 2 and 4
3 & 4	$10^4 x - 2 = 0$	$x = 2 \times 10^{-4}$	$x = 2 \times 10^{-4}$: terms 1,2 smaller than 3,4: ok

Table 1: Searches for approximations which yield self-consistent roots

5 Order of Magnitude Math vs Sophisticated Black Box Software

Rudolf Peierls in 1993 gave wise advice to his new young colleague and housemate at the University of Manchester, the future Nobel Laureate Hans Bethe: *Erst kommt das Denken, dann das Integral*. Bethe truly mastered this, and so should you.

(a) Consider the integral

$$I_1 \equiv \int_0^\infty \exp(-[x \sin x]^2)\,\mathrm{d}x \tag{2}$$

Neither Mathematica nor Maple find any analytic expression for this, but will happily give a numerical evaluation of I_1, using sophisticated adaptive numerical quadrature methods combined with some automatic symbolic changes of variable in the integral to transform the range to a finite interval.

- Mathematica gives $I_1 = 2.37231$.
- Maple gives $I_1 = 2.835068335$.

Sketch the integrand, and make rough estimates of the areas of any spikes that you find. Based on that, decide whether or not Mathematica or Maple's values of I_1 are correct to order of magnitude. You need not, and should not actually do any numerical or analytical integration to answer this question.

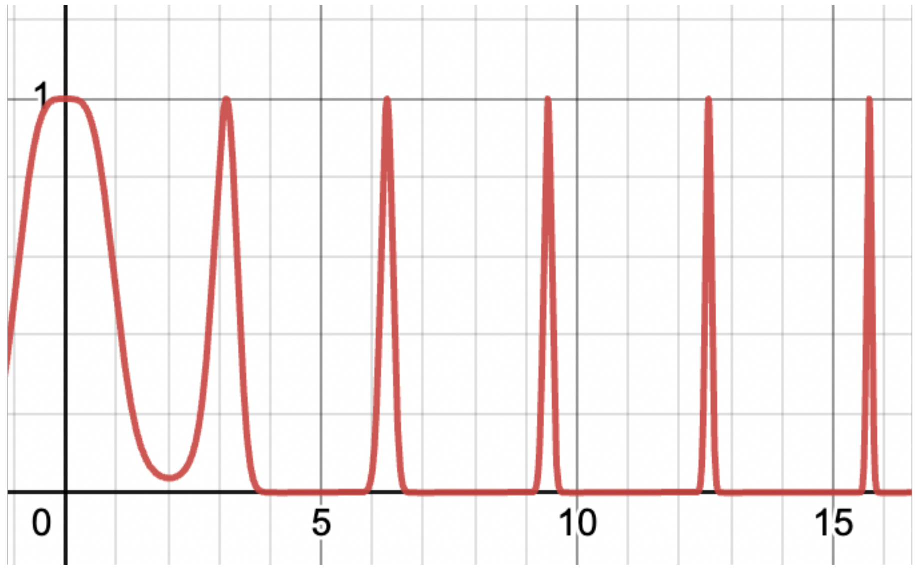

Figure 2: A sketch of the function $x^{-(x \sin x)^2}$, which comprises a series of progressively narrowing spikes.

Looking at the graph (Figure 2), we see that the integral from $x = 0$ to $x = 5$ is roughly two bumps (around $n = 0$ and $n = 1$) of height 1 and width about 1, so the contribution to I_1 from $x = 0$ to $x = 5$ is about 2, as suggested by the Maple and Mathematica results.

Farther from the origin, exponent only becomes relatively small when in the vicinity of a root of $\sin x$, i.e., $x = n\pi + \epsilon$, with $|\epsilon| \ll 1$. Expanding around such a point yields

$$(x \sin x)^2 \simeq n^2\pi^2\epsilon^2 \tag{S12}$$

Thus, in the vicinity of the nth spike at $x = n\pi + \epsilon$, the integrand is

$$\exp(-n^2\pi^2\epsilon^2 + \mathcal{O}(n\pi\epsilon^3)) \ . \tag{S13}$$

This is exponentially small except for $\epsilon < 1/(n\pi)$, where it is of order unity. Thus a crude estimate of the area of each spike is (height)$\times$(width)$=1 \times (1/n\pi) = 1/(n\pi)$.

A more accurate estimate is $\int_{-1}^{1} \exp(-n^2\pi^2\epsilon^2)\, d\epsilon \to \int_{-\infty}^{\infty} \exp(-n^2\pi^2\epsilon^2)\, d\epsilon = \sqrt{\pi}/(n\pi)$, larger than our crude estimate by a constant factor of $\sqrt{\pi} = 1.77$. This better estimate becomes asymptotically perfectly accurate as $n \to \infty$, and the width of the spikes become s very narrow.

Thus we have a good estimate for I_1:

$$I_1 \simeq 2 + \frac{1}{\sqrt{\pi}} \sum_{2}^{\infty} \frac{1}{n} = \infty \tag{S14}$$

since the harmonic series $\sum^N (1/n) \sim \int^N (1/n) dn$ diverges as $\ln(N)$ as $N \to \infty$. Note that the conclusion does not depend on whether we estimated the area of the spikes crudely (which would replace the $1/\sqrt{\pi}$ in equation S14 by $1/\pi$) or not, since $\infty/\sqrt{\pi} = \infty$.

$\boxed{\text{Hence } I_1 \text{ is } \textit{infinite}}$. The fancy numerical integrators in Mathematica (claiming $I_1 = 2.37231$) and Maple (claiming $I_1 = 2.835068335$) clearly started missing the narrow spikes for $x > 20$ and $x > 40$ respectively, and incorrectly concluded that their numerical integrals had converged.

(b) Now generalize this integral to

$$I_s \equiv \int_0^\infty \exp(-[x^s \sin x]^2)\, \mathrm{d}x \tag{3}$$

As in part (a), use only order-of-magnitude estimates to answer: for what real values of $s > 0$ is I_s infinite, and for what values is I_s finite, with a value within an order of magnitude of 1?

Using the same logic before, we attempt to assess when the expression in the exponent is close to zero, i.e., the integrand is close to 1. Around $x = 0$, we have

$$(x^s \sin x)^2 \simeq x^{2s+2} \tag{S15}$$

For $s > 0$, this quantity will be close to zero, and the integrand of order unity, for $x \lesssim 1$, so the integral of the broad peak around the origin will be of order unity.

For the spikes at larger $x = n\pi + \epsilon$, we have

$$(x \sin x)^2 \simeq (n\pi)^{2s}\epsilon^2 \tag{S16}$$

and the integrand $\exp(-((n\pi)^s\epsilon)^2$ is exponentially small except for when $\epsilon < 1/(n\pi)^s$, where it is of order unity.

The integral is therefore roughly

$$I_s \simeq \mathcal{O}(1) + \frac{\mathcal{O}(1)}{\pi^s}\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{S17}$$

This sum diverges if the terms in the series fall off slower than (or as slow as) the harmonic series, i.e., it only converges when $s > 1$. $\boxed{\text{Hence } I_s \text{ is infinite for } s \leq 1\text{, and finite for } s > 1.}$

To estimate when this integral is $\mathcal{O}(1)$, we can roughly approximate this sum as an integral:

$$\sum_{n=1}^{\infty}\frac{1}{n^s} \simeq \int_1^{\infty}\frac{1}{x^s}\,\mathrm{d}x = \frac{1}{s-1} \tag{S18}$$

To keep $I_s < 10$, it is required that roughly

$$\frac{1}{\pi^s}\sum_{n=1}^{\infty}\frac{1}{n^s} \sim \frac{1}{\pi^s}\frac{1}{s-1} \lesssim 10 \tag{S19}$$

This yields that $I_s < 10$ when $s \gtrsim 1.03$.

In a bit more detail: The sum $\sum_1^\infty n^{-s} = \zeta(s)$ where $\zeta(s)$ is the Riemann zeta function. For s close to 1, the sum converges very slowly, and equation S18 shows that $\zeta(s) \simeq 1/(s-1)$.

However we expect the sum of the narrow spikes in equation S17, and hence $\boxed{I_s \text{ to be larger than 10 for } 1 < s \lesssim 1.03.}$ So we estimate that $\boxed{I_s \text{ is within an order of magnitude of unity only for } s > 1.03.}$ Note also that for $s \gtrsim 3$, the $n \geq 2$ spikes no longer dominate the integral, as they do for small s.

Solutions to Problem Set 2

Homework Problems:

1. **Spinning neutron stars**. [8 pts] The cosmic radio sources known as pulsars are believed to be spinning, magnetic neutron stars. Let the star's radius be R, its surface magnetic field (at $r = R$) be B_s (most conveniently measured in Gaussian EM units of Gauss, but if you prefer a bit of extra suffering, you can also use SI electromagnetic units), the star's radius R and the spin rotation frequency Ω. We would like to know the total electromagnetic power radiated by this spinning magnet. Write down all the relevant variables in your preferred system of units. Use the Buckingham Pi theorem to find an expression for the radiated power in terms of those variables. You should find it involves an unknown function. However many papers refer to this as "magnetic dipole radiation". That bit of information suggests that the power should depend on B only via the magnetic dipole moment. Using only that, together with your knowledge of the external magnetic field of a magnetic dipole, eliminate the unknown function, and find a complete expression for the radiated power, up to a numerical factor of order unity.

Solution

In Gaussian CGS units (simpler): There are 5 variables: power P, surface field B_s, radius R, angular spin frequency Ω, speed of light c (since EM power). 5 variables in 3 units (mass, length, time), so Buckingham Pi theorem says there must be $5 - 3 = 2$ Π's. By inspection, two simple choices are $\Pi_1 = P/(B_s^2 R^2 c)$, $\Pi_2 = \Omega R/c$.

Buckingham tells us that$f(\Pi_1, \Pi_2) = 0$, i.e., a root is given by $\Pi_2 = g(\Pi_1)$, or $P = B_s^2 R^2 c g(\Omega R/c)$. Since a magnetic dipole has $\mu \sim B_s R^3$, if the power is to depend just on the magnetic moment, i.e. on $B_s R^3$, we must have $g(\Pi_2) \sim \Pi_2^4$. Hence $P = B_s^2 R^2 c(\Omega R/c)^4 = \Pi B^2 R^6 \Omega^4/c$. This is the correct expression for vacuum magnetic dipole radiation, up to the numerical factor $\Pi = (2/3)\sin^2\alpha$, where α is the angle between the spin axis and the magnetic dipole axis. In real pulsars, the rotation-induced electric fields are so strong that they create particles from the vacuum, and this modifies the vacuum result, making $\Pi \sim 1$, independent of α. So the OoM result is actually better than the "exact" vacuum solution!

You can also do the problem in SI units. Physical quantities involved in this problem are R, B_s, Ω, c (speed of light, which often occurs in electromagnetic problems), μ_0 (magnetic permittivity in the vacuum), and P (total electromagnetic power). There are now 6 variables in 4 fundamental units: length, time, mass, and current. As a result, Buckingham gives $6 - 4 = 2$ dimensionless quantities (or two Π's), same as in Gaussian units, but with an extra variable and an extra unit.

Note that P is in the unit of "energy per unit time", while $B_s^2/2\mu_0$ is the magnetic energy density. With additional volume (R^3) and time (Ω^{-1}), we may quickly find

$$\Pi_1 = \frac{P}{(B_s^2/\mu_0)R^3\Omega}. \tag{1}$$

The second one must include c, which is in the unit of "length per unit time", so

$$\Pi_2 = \frac{\Omega R}{c}. \tag{2}$$

With the Pi theorem, we should expect $\Pi_1 = g(\Pi_2)$, where g is an unknown function, so

$$P = \frac{B_s^2}{\mu_0} R^3 \Omega \cdot g\left(\frac{\Omega R}{c}\right). \tag{3}$$

We also note that this is also magnetic dipole radiation. With some external knowledge of the magnetic dipole field, we know $B_s \sim \mu_0 m/R^3$ (where m is the magnetic dipole moment) and $m \sim B_s R^3/\mu_0$. If the radiation power is only dependent on m, by comparing powers of B, we find that g must be a simple cubic function. In summary,

$$P = \alpha \frac{B_s^2 R^6 \Omega^4}{\mu_0 c^3}, \tag{4}$$

where α is a numerical factor.

Note that there are degrees of freedom in choosing Π's. Another possible form of Π_1 is to use c instead of Ω, which implies $\Pi_1 = P/(B_s^2 R^2 c/\mu_0)$. In this case, the function g is a biquadratic function.

2. **Thermodynamics** [2+4+3+3=12pts] [Note: to answer this question, you don't have to know any thermodynamics, though you'll appreciate it more if you do. Enthalpy has units of energy.] The enthalpy per particle $h = H/N$ of a relativistic gas containing N particles in a volume V depends on:

$s = S/N$,	the entropy per particle,
p,	the pressure,
$\hbar$,	Planck's constant,
c,	the speed of light,
m,	the rest mass of the particles in the gas,
k,	Boltzmann's constant.

a) How many independent dimensionless quantities Π_i can be formed from these 7 variables?

Solution

These variables involve four fundamental units: mass, length, time, and temperature. Therefore, we can form $7 - 4 = 3$ dimensionless quantities from them (the seventh variable is h).

b) One of these is $\Pi_1 = s/k$. Find all the others, and give an expression for H in the form $H = N v_1^{\alpha_1} v_2^{\alpha_2} \phi(\Pi_1, \Pi_2, \ldots)$, where v_1 and v_2 are two of the variables listed above. The function ϕ cannot be determined from dimensional analysis alone (and in fact also depends on the particles' dimensionless spin).

Solution

Noting that h has units of energy, one obvious choice is $\Pi_2 = h/mc^2$. The other dimensionless combination is less obvious but after some playing around, one can show that

$$\Pi_3 = \frac{p\hbar^3}{m^4 c^5} \tag{5}$$

is dimensionless. Note that any combinations of these quantities would be just as good. The general solution, according to the Buckingham theorem, can be written

$$\Pi_2 = \phi(\Pi_1, \Pi_3) \tag{6}$$

or equivalently

$$h = mc^2\phi(\Pi_1, \Pi_3). \tag{7}$$

and thus

$$H = Nmc^2\phi(\Pi_1, \Pi_3). \tag{8}$$

c) If the gas is nonrelativistic, the speed of light is no longer a relevant variable, so c must cancel out of the equation in (b). Give the resulting equation for H, and using the thermodynamic relation $\partial H/\partial p|_s = V$, use that equation to prove that for any nonrelativistic gas $H = (5/2)pV$.

Solution

Since Π_1 does not contain any factors of c, the only way for c to cancel out would be if we have

$$\phi(\Pi_1, \Pi_3) = \psi(\Pi_1)\Pi_3^{2/5} \tag{9}$$

for some new function ψ. In that case, we find

$$V \equiv \left.\frac{\partial H}{\partial p}\right|_s = Nmc^2\psi(\Pi_1)\left(\frac{2}{5}\Pi_3^{-3/5}\right)\frac{\partial \Pi_3}{\partial p} \tag{10}$$

$$= \frac{2}{5}\frac{H}{\Pi_3}\frac{\Pi_3}{p} = \frac{2}{5}\frac{H}{p}, \tag{11}$$

from which the result $H = (5/2)pV$ follows.

d) If the gas is ultrarelativistic, the speed of light c will be relevant, but the rest mass of the particles shouldn't matter. Give the resulting equation for H, and prove (as in (c)) that $H = 4pV$ for any gas of ultrarelativistic particles.

Solution

Again since Π_1 does not contain any factors of m, the only way for m to cancel out would be if we have

$$\phi(\Pi_1, \Pi_3) = \psi(\Pi_1)\Pi_3^{1/4} \tag{12}$$

for some new function ψ. Following the same steps as in (c), we obtain the result $H = 4pV$.

Note that since $H = E + pV$, this corresponds to $E/V = 3p$, which may be more familiar to you as the relation between energy density and pressure for a photon gas, and also in the early universe when all particles are very relativistic.

3. **Ultracentrifuges.** [6+6=12pts] Biologists use ultracentrifuges to separate proteins and other components of cells by molecular weight and size (which can vary with solvent: e.g. swelling in salt water vs fresh water vs alcohol, linkage with other molecules, etc). Assume that the centrifuge is a swinging-bucket model (where the test tubes swing freely towards the horizontal as the rotor spins up). Standard test tubes for centrifuges are 7.5 cm long, and the swing mechanism is such that their ends, when spinning, are 9 cm from the axis of rotation.

a) Show that the time to settle a particular particle into the bottom (outer when spinning) half of the test tube scales as Ω^{-2}, where Ω is the angular rotation velocity of the centrifuge, and is almost *independent* of the length of the test tube.

Solution

A particle's free-fall settling velocity v is set by balancing the gravitational force (minus buoyancy) by the draft force on it, and for small particles $\mathrm{Re} \ll 1$, the drag force $\propto v$, so $v \propto g \propto g = \Omega^2 r$. In a centrifuge the effective gravitational acceleration $g = \Omega^2 r$. Thus, $dr/dt \propto \Omega^2 r$. Solving for the time t to move from $r_{\min}$ to $r_{\max}$, t, we find

$$\boxed{t \propto \Omega^{-2} \ln \frac{r_{\max}}{r_{\min}}.} \tag{13}$$

Thus the time depends on the angular rotation velocity Ω, but is almost independent of the length of the tube.

b) Proteins like albumin ("egg white, blood plasma") and collagen ("glue, jello") have atomic weight $A \sim 10^5$, and density $\rho_s \sim 1.3\text{g cm}^{-3}$. Approximating these protein molecules as spheres, use the "marble in corn syrup" equations derived in class to estimate how long it would take to settle the proteins to the bottom of the test tube in a Beckman-Coulter model SW 65 titanium-rotor ultracentrifuge at its maximum speed of 65,000 rpm, and compare to the time it would take to settle under earth gravity.

Solution

The expression given in class for the low Re terminal velocity $v = 2gR^2(\rho_s - \rho_l)/(9\rho_l\nu)$ can be rewritten as $v = gm(1 - \rho_l/\rho_s)/(6\pi\rho_l\nu R)$. We can speed up this settling by increasing g in a centrifuge, where the effective $g = \Omega^2 r$. The sedimentation coefficient s is defined as v/g (units of seconds, or more commonly "Svedbergs", 1 Svedberg= $1\text{S} = 10^{-13}\text{s}$). Thus, $s = m(1 - \rho_l/\rho_s)/(6\pi\rho_l\nu R)$ for a spherical body of density ρ_s in water at room temperature $\rho_l\nu = 10^{-2}\text{g cm}^{-1}\text{s}^{-1}$, and mass A in atomic mass units m_u. Combining these, we find

$$s = m_u A \frac{1}{6\pi\rho_l\nu[Am_u 3/4\pi\rho_s]^{1/3}}(1 - \rho_l/\rho_s) \sim 10^{-15}\text{s} \times A^{2/3}\rho_s^{1/3}(1 - \rho_l/\rho_s). \tag{14}$$

Note that $10^{-15}\text{s} = 10^{-2}\text{S}$. For $A \sim 10^5$ and $\rho_s \sim 1.3$, $s \sim 5\text{S} = 5 \times 10^{-13}\text{s}$. As in 3-a), $dr/dt = sg = s\Omega^2 r$. Solving the differential equation gives

$$\boxed{t \sim \Omega^{-2}s^{-1} \sim [(65000/60)2\pi\text{rad s}^{-1}]^{-2}s^{-1} \sim 4 \times 10^4 \text{ sec} \sim 12\text{hr}.} \tag{15}$$

To settle 5 cm in a test tube in normal gravity would take

$$\boxed{5\text{cm}/(sg) = 10^{10}\text{s} \sim 300\text{yr!}} \tag{16}$$

4. **Blown off your feet.** [8pts] Estimate: into how a fast a wind can you walk without being blown backwards? Lifted off your feet?

Solution

When the drag force is about the same as friction, people would be blown backwards. The drag force is given by $\frac{1}{2}c_D\rho v^2 A$, where c_D is the drag coefficient, ρ is the air density, A is the frontal area of the human, and v is the wind speed. The friction is αmg where $\alpha \sim 1$ is coefficient of friction of feet on sidewalk (assumed dry, $\alpha < 1$ if wet or icy). Using an area $A \sim .3\text{m} \times 2\text{m}$ and $c_D \sim 1$ yields

$$\boxed{v \sim \sqrt{\frac{2\alpha mg}{c_D \rho A}} \sim \sqrt{\frac{2 \times 65\text{kg} \times 9.8\text{m s}^{-2}}{(1.2\text{kg m}^{-3} \times 0.6\text{m}^2)}} \sim 40\text{m s}^{-1} \simeq 90\text{mph.}} \tag{17}$$

To understand why you lean into the wind (and why lift matters), you also have to consider torque: at this speed, the wind is applying a force about equal to mg at a height about your belly button. So to avoid being flipped onto your back, you have to lean into forward into the wind at about a 45 degree angle, so the gravitational torque will balance the wind torque. This slightly reduces your frontal area, but also introduces lift, which reduces your mg and the friction force, and can eventually lift you off your feet:

http://www.aerospaceweb.org/question/airfoils/q0150b.shtml shows that coefficient of lift at 45 degree angle of attack for NACA airfoil section 0015 is about 1. This is also true for a flat plate, and probably about right for a human too. So already at $\sim 40\text{m s}^{-1}$ wind speed, the lift force $F_L \sim mg$. Thus your effective weight mg is significantly reduced ($mg = F_L \ll mg$), so walking into the wind requires $\alpha(mg - F_L) > \frac{1}{2}c_D\rho v^2 A$, and will only be possible for speeds even less than our rough estimate of equation 17. And above about $\sim 40\text{m s}^{-1}$, leaning forward at about 45 degrees into the wind, you will will start to be lifted off your feet.

For demonstration: https://www.youtube.com/watch?v=rKUipxR3bDc#t=1 Published on Dec 13, 2013 "People Christmas shopping in downtown Aalesund, Norway had trouble crossing the street today. The storm "Ivar" was blowing things and people around and police had to assist people in moving around. One man blew off the sidewalk and into the street in an intersection downtown Aalesund earlier. An ambulance was required and the police is now present in the streets, to follow the development and help out where needed." Ivar had sustained winds in Norway of 35m/s=78mph, agrees with our estimation.

5. **Lead poisoning.** [5+5=10pts] First some facts: Tetraethyl lead was used as an octane booster in gasoline between 1929 and 1986 (when it was banned in the US for automobile use), at a concentration of about 5 grams of lead per gallon of gas. Most of that lead was exhausted in small soot particles which settled near roads (directly onto the ground and onto shrubbery and the walls of buildings, where rains washed it to the ground). Also, exterior house paint used before 1978 in the US had lead concentrations of 5-10% by weight. Lead is a potent neurotoxin which binds to hemoglobin in the blood. It takes about a month for lead levels in the blood to change (about equally by excretion and by storage in the bones and teeth). Clinical symptoms (fatigue, muscle pain and reduction of IQ) appear when the lead levels in blood reach 100μg/liter; coma and death occur at ten times that level. Now the problems:

   a) Estimate the total mass of lead spread over the city of Pasadena. Assume that this is mixed into the top 2 cm of soil in Pasadena. How much Pasadena dirt can a toddler safely suck off his or her fingers per day and still avoid lead poisoning?

   b) A typical Pasadena-area house was constructed in 1920. Assuming sanded paint is also concentrated in the top 2 cm of soil around the house, estimate the lead concentration in the dirt there due to accumulated house sandings and scrapings, and compare to (a).

Solution

a) The area of Pasadena is approximately 10km × 10km = 100km^2 = 10^{12}cm^2. The population is about 2×10^5 people so there are probably about 10^5 cars in Pasadena. The average annual mileage for a car is about 10^4 miles. Of course a lot of Pasadena residents drive their cars out of Pasadena but a lot of nonresidents drive their cars into Pasadena so we can assume that the number of cars in the city at any one time is always about 10^5 (this certainly isn't true on January 1st, but never mind!). However, the number of cars in Pasadena between 1929 and 1986 obviously isn't constant but we can approximate what is probably a close to exponential increase over that time by a uniform number of cars over, say, the period 1966 to 1986. Also, over this time we assume the cars in the city of Pasadena were doing about 10 miles to the gallon. Let's say that the mean number of cars in Pasadena from 1966 to 1986 is half of the current number. Thus, the number of gallons of gasoline consumed is

$$N_{cars} = \frac{1}{2}\frac{(10^5\text{cars})(10^4\text{miles/car/year})(20\text{years})}{(10\text{miles/gallon})} = 10^9\text{gallons}. \tag{18}$$

Since there is 5g of lead per gallon of gasoline, the mass of lead deposited in Pasadena is

$$M_{Pb} = (5\text{g/gallon})(10^9\text{gallons}) = 10^{10}\text{g}. \tag{19}$$

So the mass of lead deposited per unit area is

$$\frac{10^{10}\text{g}}{10^{12}\text{cm}^2} = 0.005\text{gcm}^{-2}. \tag{20}$$

Lead deposited on roads probably gets flushed into storm drains but we assume that lead deposited on soil gets mixed into the top 2 cm of soil giving a lead concentration of

$$\frac{0.005\text{gcm}^{-2}}{2\text{cm}} = 0.0025\text{gcm}^{-3}. \tag{21}$$

The critical level of lead in the bloodstream is 100μg/liter. About 10% of body mass is blood, so a toddler has about 1 liter of blood. Therefore the toddler shouldn't eat more that 100μg of lead per month; that's about 3μg of lead per day. So the daily intake of soil should not exceed

$$\frac{3 \times 10^{-6}\text{g}}{2.5 \times 10^{-3}\text{gcm}^{-3}} = 3 \times 10^{-4}\text{cm}^3. \tag{22}$$

That's not a whole lot! Soil consists of a mixture of weathered rock ($\rho \approx 3$gcm^{-3}) and organic material ($\rho \approx 1$gcm^{-3}). So the density of soil is probably about 2gcm^{-3}. This means that the abundance by mass of lead in soil is about 1000 ppm. The measured mass abundances of lead in soil range from 20 ppm to 800 ppm (the 800 ppm figure is for New Orleans) (American Scientist, Vol 87, 62–73 1999). We have assumed that no lead is removed from the soil once it is deposited, and that no soil is removed either. The fact that our answer is within a factor of 2 of the measurements suggests that this is not a big effect.

b) Suppose that the typical house is repainted every 10 years, with two coats of paint each time. Thus, over the period 1920–1978, there will be about 12 coats of paint applied. The typical house has walls about 10 feet tall; since one gallon of paint covers about 400 square feet, it will be spread across 40 linear feet of wall (or 1200 linear centimeters). Now, suppose that when paint is removed from the house, all of it ends up in the soil.

Suppose that 90% of the scrapings are picked up, but that the remaining 10% are left in the soil. Suppose further that the paint is spread over a distance of about 3 feet (90 cm) from the wall. I estimate a density of paint of about 1gcm^{-3}. Since $1\,\text{gallon} \approx 3800\text{cm}^3$, one gallon of paint weighs about 3800g. So we have 12 gallons spread out over an area of $1200\text{cm} \times 90\text{cm}$, with a depth of 2cm. This gives a density in the soil of

$$(12\,\text{coats}) \cdot \frac{(3800\text{g})(10\%)}{(1200\text{cm})(90\text{cm})(2\text{cm})} = 0.02\text{gcm}^{-3}. \qquad (23)$$

Since paint is 5–10% lead by weight, the density of lead in the soil is

$$(0.02\text{gcm}^{-3})(7.5\%) = 0.002\text{gcm}^{-3}. \qquad (24)$$

This is similar to the amount of lead in the soil due to leaded gasoline.

Solutions to Problem Set 3

Homework Problems:

1. **Fitting teeth. (8 pts: 6+2)** After a dentist installs a crown, cap, or onlay on a tooth, or installs a dental implant, much time is spent having the patient clench and chew with their teeth. After each such action by the patient, the dentist grinds down parts of the new tooth surface. The process is repeated until the clenching and chewing feels comfortable, and dyed sheets placed between the teeth show a broad contact area.

   a) Estimate how much higher a "badly fitting tooth" would need to be to be noticeably different in "feel" than a properly fitting one.

   Solution:
   Biting my kitchen scale with my front teeth, I can easily reach its 5kg limit (i.e. 50 N force). Biting the edge of my bathroom scale hard enough to cause some tooth pain, again with the front teeth, it reads 8kg (i.e. 80 N force), though I am unsure if it is accurate with such an off-axis strain. [1] Since the jaw is a lever with mechanical advantage considerably less than one (the jaw muscles are attached near the fulcrum of the jaw lever), this suggests that the bite force at the rear molars, several times nearer the fulcrum, could be ~ 30 kg (i.e. 300 N force), i.e. about 3×10^7dyn. Molars have about 1mm wide "mountains", so if a single one of these were too high (so all the chewing force were applied to it), the stress on it during chewing would be about 3×10^9dyn/cm^2 $\sim$ 300MPa. That exceeds the 60 Mpa breaking stress. Indeed it, people often do crack a tooth by biting down on a large grain of sand or small piece of nut shell accidentally left in with their food.

   In normal chewing, the biting force is not so hard as to cause pain, and the force is spread over several "molar mountains" on several teeth which are simultaneously in contact. Perhaps 5 kg (30 N) force spread over 10mm^2, or a stress of 5×10^7dyn cm^{-2} = 5Mpa. The elastic compression in that normal stress would be

   $$\Delta h \sim (5\text{MPa}/1.5\text{GPa})h \sim 0.0007(h/0.2\text{cm})\text{cm}$$

   or around 7μm across an $h \sim 0.2$cm high molar "mountain" (the stress is spread across a wider area lower in the tooth, so the compression there will be less).

   If a bad crown left a single mountain much higher than that $\boxed{7\mu\text{m}}$, the stress on it would be much higher, threatening tooth breakage, getting painful, and not be great for

[1] In dentistry, devices for measuring bite force are called *gnathodynamometers* (from the Greek $\gamma\nu\alpha\theta o\varsigma$: jaw). They are used to diagnose conditions of the teeth and jaw, and the effectiveness of false teeth, implants, etc. Their history starts with Borelli in 1681 (who hung weights on a strap passing over his molars), through various spring and lever devices, through modern piezoelectric strain gauge devices. There is a nice review "Bite Force and Influential Factors on Bite Force Measurements: A Literature Review" by Koc, Dogan and Bek 2010, European Journal of Dentistry, 4, 223. A historical review with pictures and results from early devices can be found in "The Gnathodynamometer and Its Use in Dentistry" by A. E. Rowlett, 1933, Proceedings of the Royal Society of Medicine, 26 (4), 464.

masticating your food. Thus the dentist should grind the new tooth surfaces to at least that height accuracy.

Sterl's dentist said that the dye films he uses to show contact areas have a thickness of 8 microns, so the accuracy needed must a few microns in height, consistent with the estimate above (made by Sterl while having his new crown fitted).

b) Would there be any danger of breakage of the "badly fitting tooth" or its counterpart if it were not adjusted?

Solution:
Yes. As estimated above, biting hard on a single 1mm^2 molar "mountain" could produce 300 MPa stress, exceeding the breaking stress of 60MPa. Don't crack nuts with your teeth, keep sand out of your food, and don't clench/grind your teeth at night. And if you get a crown or filling, make sure the dentist adjusts till you feel a good contact.

Useful information: the elastic modulus of the two main components of teeth, enamel and dentin are respectively about 1.3GPa and 1.6GPa (1GPa= 10^{10}dyne cm^{-2}), while that of the gold alloy commonly used for tooth crowns is 2.3GPa. The breaking stress of tooth enamel in compression is about 60MPa. You are responsible for investigating the size and shape of your own teeth, and estimating your typical chewing forces.

2. **Fermion tea. 10 points (5+5)** My teacup is impervious to nucleons. At absolute zero,

a) (5pts) How many neutrons can I put in the cup before my cup runneth over?

Solution:
My cup runneth over when the uncertainty in velocity of the degenerate neutrons is enough to escape the gravitational well (the height of the cup). If the number density of neutrons is n, then each neutron is 'confined' to a cube of side $\Delta x \sim n^{-1/3}$; from the uncertainty principle, $\Delta p \sim \hbar/\Delta x \sim \hbar n^{1/3}$, and $\Delta v \sim \hbar n^{1/3}/m_{\text{n}}$. Confined is in quotes because the neutron wave functions actually extend over the whole cup. You solve for the energy levels in a three-dimensional box, and put neutrons in starting from the ground state, *i.e.*, from longer to shorter wavelengths (2 neutrons in each state, spin up and spin down, as allowed by the Pauli principle). The lowest energy neutrons will have wavefunctions with wavelengths comparable to the box size; the highest energy neutrons will have wavefunctions with wavelengths comparable to $n^{-1/3}$ (see any solid state text for the honest derivation of the Fermi energy or Fermi velocity). These highest energy neutrons will jump out of the cup first, as we increase the number density, and it is these whose energy we are estimating with our 'uncertainty velocity' method for finding the number density threshold.

When the velocity of the highest energy neutrons is comparable to the escape velocity, $v_{\text{e}} \sim \sqrt{gL}$, where L is the side length of the cup, the neutrons will get out. So $\Delta v \sim \sqrt{gL}$. Substituting for Δv, we get

$$n^{1/3} \sim \frac{m_{\text{n}}}{\hbar}\sqrt{gL}.$$

Now we use a useful trick, based on $\hbar c \simeq 200\,\text{MeV fm}$. You can shift powers of ten from the electron-volts to the meters, to use the most convenient units for the problem; here we want centimeters because the cup has sides of a few centimeters, so we use $\hbar c \simeq 2 \times 10^{-5}\,\text{eV cm}$. But there's no c, so we fix that, by multiplying by c/c, to get an $\hbar c$ in the denominator. Then we multiply by c/c again, to get $m_{\text{n}}c^2$ (which you know is

$\simeq 1\,\mathrm{GeV}$—this is another trick, to avoid looking up particle masses). We find

$$n^{1/3} \sim \frac{m_{\mathrm{n}}c^2}{\hbar c}\frac{\sqrt{gL}}{c}.$$

Putting in the numbers (taking $L \sim 10\,\mathrm{cm}$),

$$\begin{aligned} n^{1/3} &\sim \tfrac{10^9\,\mathrm{eV}}{2\times10^{-5}\,\mathrm{eV\,cm}} \times \tfrac{\sqrt{1000\,\mathrm{cm/\,sec^2}\times 10\,\mathrm{cm}}}{3\times10^{10}\,\mathrm{cm/\,sec}} \\ &\sim 2\times10^5\,\mathrm{cm}^{-1}. \end{aligned} \tag{1}$$

With this number density, the number of neutrons in the cup is

$$nL^3 = (n^{1/3}L)^3 \sim \left(2\times10^5\,\mathrm{cm}^{-1}\times 10\,\mathrm{cm}\right)^3 \sim 10^{19}.$$

Note that if neutrons were bosons, then we could pack them all into the same state: they would each have wavefunctions with wavelength comparable to L and their uncertainty energy would be negligible.

b) (5pts) How many protons?

Solution:

Protons, like neutrons, obey Fermi statistics, but protons also have charge, and most likely the Coulomb repulsion will kick protons out before the Pauli repulsion will. If there are N protons spread around the cup, a single proton sees an electrostatic repulsive potential $U \sim Ne^2/L$. When $U \sim m_{\mathrm{p}}gL$, protons will jump out. So, $Ne^2 \sim m_{\mathrm{p}}gL^2$. To avoid remembering e in esu, or any other system, we use the $\hbar c$ trick again, because $e^2/\hbar c$ is defined to be the fine structure constant, which in this class is 0.01. So we divide both sides by $\hbar c$, and also multiply the right side by c^2/c^2 to get a $m_{\mathrm{p}}c^2$. These manipulations give

$$\frac{Ne^2}{\hbar c} \equiv N\alpha \sim \frac{m_p c^2}{\hbar c} g\left(\frac{L}{c}\right)^2.$$

Putting in numbers, and moving the α to the other side,

$$\begin{aligned} N &\sim \alpha^{-1}\tfrac{10^9\,\mathrm{eV}}{2\times10^{-5}\,\mathrm{eV\,cm}} \times 1000\,\mathrm{cm/\,sec^2} \times \left(\tfrac{10\,\mathrm{cm}}{3\times10^{10}\,\mathrm{cm/\,sec}}\right)^2 \\ &\sim 100\times5\times10^{13}\,\mathrm{cm}^{-1}\times1000\,\mathrm{cm/\,sec^2}\times10^{-19}\,\mathrm{sec}^2 \sim 0.5. \end{aligned}$$

So one proton is about the limit; putting in another will push the first out of the cup. Moral of the story: gravity is weak.

3. **Violin strings. 10 points (5+5)** The E-string on the violin is made of high-quality steel, stretched almost to the breaking point, and tuned to 659Hz.

a) Using *only* the information given above, estimate the length of the E-string.

Solution:

From Ph 12 we know the transverse wave speed in a string is, $c_T = \sqrt{T/\mu}$ where T is the tension and μ is the mass per unit length of the string. Using this we find $c_T = \sqrt{(T/A)/(\mu/A)} = \sqrt{\sigma/\rho}$. We know from the lectures that the yield strain of good steel is $\sigma_\gamma \sim 10^{-2}B \sim 10^{10}\mathrm{dyn/cm}^2$ and that the density of steel is $7.8\mathrm{g/cm}^3$. Using these gives $c_T = 3.5\times10^4\mathrm{cm/s}$ ($\sim$speed of sound in air!) and the size of the violin is $L = c_T/2f = 27\mathrm{cm}$. An actual violin from bridge to end is 33 cm so we are pretty close!

b) Beginning violinists have trouble keeping their bow perpendicular to the string. Their bowing then excites longitudinal vibrations along the string, not just the desired transverse vibrations. Estimate the fundamental frequency of the resulting cringe-inducing "beginner's squawk" on the E-string.

Solution:

In this case the wave speed will just be the speed of sound in our steel wire: $c_L = \sqrt{B/\rho}$. Comparing this to the expression in a gives $c_T \sim \sqrt{10^{-2}B/\rho} = 10^{-1}c_L$. So we see the longitudinal wave speed is around $1\sqrt{\epsilon_Y} \sim 10$ times higher than the transverse one. Hence the frequency of the "squawk" will be about ten times higher than that of the more desirable transverse waves: $\sim 10 \times 659\text{Hz} = \boxed{6.6 \text{ kHz}}$.

4. **Bathyspheres. (10 pts: 7+3)** Bathyspheres are spherical vessels designed to withstand the pressure at great depths in the ocean.

a) How thick must the wall of a 2 meter radius steel bathysphere be in order for it to safely travel to the deepest parts of the ocean, depth 10 kilometers? (7pts)

Solution

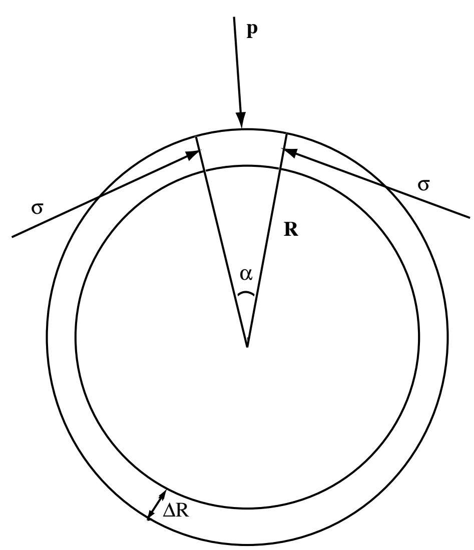

Figure 1: Estimating strain in a bathysphere.

The minimum thickness is determined by requiring the stress induced in the vessel be less than the yield stress. The stress, σ, in a spherical shell of radius R and thickness ΔR is related to the external water pressure, p, by

$$\sigma = \frac{pR}{2\Delta R}.$$

To see this, we can examine a small circular piece of the bathysphere's hull as shown in figure 1. Let this be a cone of opening angle α. So, the radius of our small piece of bathysphere hull that is exposed to the pressure of the ocean is given by

$$r = R \cdot \sin\frac{\alpha}{2} \approx \frac{R\alpha}{2}$$

Then, the area exposed to the ocean's pressure is $A = \pi r^2$, meaning that the force on the piece of the hull is given by

$$F_{pressure} = Ap = \pi r^2 p \approx \pi R^2 \frac{\alpha^2}{4} p$$

Now, this force from the pressure must be equalized by the stress, σ, in the bathysphere's hull. The cone intersects the hull in an nearly cylindrical cut, of area

$$A \approx 2\pi r \Delta R \approx \pi \alpha R \Delta R,$$

through which the stress is transmitted to the rest of the bathysphere. This stress σ is normal to the sides of the cut (i.e., tangential to the surface of the bathysphere), but has a small radial component (see figure 1), $\sigma\alpha/2$.
Thus, the force from metal stress on our hull element is given by

$$F_{stress} \approx A\sigma \sin\frac{\alpha}{2} \approx \frac{1}{2}\pi\alpha^2 \sigma R \Delta R$$

Equating F_{stress} and $F_{pressure}$ and solving for σ yields our original equation for the stress as a function of pressure:

$$\sigma = pR/(2\Delta R)$$

At 10 km depth, the water pressure is $p = \rho g h \sim 10^9$ dyne/cm^2. For the best quality Cr-Mo steel, the yield stress $\sigma_Y \sim 10^{10}$ dyne/cm^2. Trusting one's life to a sphere in corrosive salt water with no margin of safety at all seem's pretty unwise. Engineers usually allow a safety factor of two or more. For a $\boxed{\text{safety factor } f_s \sim 2}$ (i.e., at maximum depth, $\sigma = \sigma_Y/f_s$),

$$\Delta R/R = \frac{f_s p}{2\sigma_Y} = 0.1(f_s/2),$$

and thus $\boxed{\Delta R = 20(f_s/2)\text{cm}}$. The record setting vessels actually had $\Delta R/R \approx 0.15$, i.e. $f_s \sim 3$. If I were going down in one, I'd also like a big margin of safety!

b) Would an empty bathysphere float? (3pts)

Solution

For small ΔR, the volume of the steel hull is $\approx 4\pi R^2 \Delta R$. Assuming interior is filled with air (negligible mass compared to the steel hull), the mean density is the mass of the hull divided by the volume of the entire sphere.

$$\rho \approx \frac{4\pi R^2 \Delta R \rho_{\text{steel}}}{\frac{4}{3}\pi R^3} = 3\frac{\Delta R}{R}\rho_{\text{steel}}$$

So, with $\rho_{\text{steel}} \sim 8\text{g cm}^{-3}$, and our result from (a) with a safety factor $f_s \sim 2$, $3\Delta R/R \times \rho_{\text{steel}} \sim 2.4\text{g cm}^{-3}(f_s/2)$ Since this is a higher mean density than water (1g cm^{-3}), our bathysphere would sink. And sink even faster if we included a safety factor $f_s > 2$ in the hull thickness.

5. **Flying safety. 12 points (7+5)**

a) In the US, the rate of passenger car fatalities in recent years has been about 13 per billion vehicle miles driven, while the rate of fatalities on commercial airlines has been 0.02 per billion passenger miles. What is the largest number of airplanes that could be in blindly random flight over the continental U.S., and between 25,000 and 35,000 feet altitude, without risking more than one fatality per billion passenger miles as a result of midair collisions?

Solution

Let A be the area of the continental US, H be the altitude range in the flight layer, and N be the total number of planes in the sky at once. Let h, l, w be the height, length and wingspan of the plane, respectively. For a medium-sized plane like the Boeing 757-200, $h \sim 15$m (engine to top of tail), $l \sim 50$m, $w \sim 40$m. If any part of two planes overlap (e.g. in a typical T-bone collision the tail of one plane might slice through the fuselage or wing of another). So we take the collision cross-section $\sigma \sim hl$.

The number density of planes per volume of airspace is $n = N/(AH)$. The range of altitudes given $H = 10,000\text{feet} \sim 3\text{km}$, and inspection of a map shows that the continental US has area $A \sim 3500\text{km} \times 2000\text{km} \sim 7 \times 10^6\text{km}^3$. This gives a mean free path of

$$\lambda \sim \frac{1}{n\sigma} \sim \frac{AH}{hl} \times \frac{1}{N}.$$

The (small) probability that a given plane, flying at a mean speed v relative to other planes, has a collision in time t is $n\sigma vt$. Since there are N planes in the air at any time, the expectation value for the total number of collisions in time t is

$$N_{\text{coll}} = \frac{N}{2} n\sigma vt$$

(the factor of 1/2 is to avoid double-counting planes, since each collision involves two planes). If each plane has N_p passengers, each collision will kill $2N_p$ passengers, so the expected number of deaths is

$$\langle N_{\text{deaths}} \rangle = 2N_p N \frac{N}{2AH} \sigma vt.$$

The total number of passenger miles flown in time t is

$$PM = N_p N vt$$

Thus the expected number of deaths per passenger mile

$$dpm \equiv \frac{\text{deaths}}{\text{passenger} - \text{mile}} = \frac{N_p N N \sigma vt}{AHN_p Nvt} = \frac{N\sigma}{AH}.$$

Notice that t, v and N_p all cancel out: more passengers per plane means more passenger miles per plane, but more deaths per collision by the same factor.

We were asked to keep the deaths per passenger mile $dpm < 10^{-9}\text{mile}^{-1} = 6\times10^{-10}\text{km}^{-1}$ (less than 1/10 the rate for cars), so, using the numerical values given above, including the Boeing 757's cross-section $\sigma \sim hl \sim 15\text{m} \times 50\text{m} \sim 8 \times 10^{-4}\text{km}^2$,

$$N < dpm\frac{AH}{\sigma} \sim 6 \times 10^{-10}\text{km}^{-1} \frac{7 \times 10^6\text{km}^2 \times 3\text{km}}{8 \times 10^{-4}\text{km}^2} \sim 15$$

This remarkably small number is about the number the planes in close approach of a single large airport at any time. With a total US fleet of commercial planes of about 6000, each of which spends a significant part of the day aloft, the number of planes actually flying over the US at any time must be in the thousands. FAA rules and traffic control are important!

Notice again that the answer does *not* depend on the number of passengers per plane (fatalities per crash scales with passenger miles per flight) or the speed of the plane (the collision frequency and number of miles travelled both scale with speed). If the planes were free to move only in two dimensions, one can show that the limit on N would the same.

b) At 5:21 PDT, Apr 19, 2023, NASA's 300kg RHESSI (Reuven Ramaty High Energy Solar Spectroscopic Imager) satellite NORAD # 27370, launched 5 Feb 2002, re-entered the earth's atmosphere over the desert in northern Egypt. It was expected to break up, with several components reaching the ground. In 2022, there were 180 rocket launches worldwide, with pieces of boosters rentering after launch (mostly controlled to land in unpopulated areas, but some, notably Chinese Long March 5B rocket boosters (21 tons empty) are not controlled: in May 11, 2020, a 12m long piece of a Long March 5B crashed into a village in Ivory Coast, in July 24, 2022, 5m by 1.5m piece of another landed in Mindoro Province, the Philippines, and in Nov 4, 2022, Spain and France closed their airspace as a Long March 5B re-entered). There are currently about 2,000 satellites in orbits similar to RHESSI's (altitudes below 550km, with lifetimes against atmospheric drag of a few decades or less).

Estimate the probability per year that someone on earth will be killed or injured by a piece of a falling rocket or satellite.

Solution

There are about 10^{10} people on earth. For each person, the projected area in the view of a falling satellite piece is 0.3m $\times$ 1m (1/2 time lying, 1/2 time standing). We also note that satellites mainly in orbits of inclination less than 40 degrees, so area of ground tracks is $2\pi R_\oplus^2 \sim 2.6 \times 10^{14}\mathrm{m}^2$ (not 4π, since not many polar orbits). Most satellite pieces are tiny and smaller than the projected area of a person, so we can safely ignore their surface when estimating the cross section (though not the case for some large rocket debris). The possibility that a satellite piece hits a person outdoors is about $0.3\mathrm{m}^2/(3 \times 10^{14}\mathrm{m}^2) = 10^{-15}$. Considering 10^{10} people, the possibility of a fatal hit is about 10^{-5} per piece. Taking 200 rockets plus satellites re-entering each year, each in 5 pieces, that is 0.01 hits per year.

There are also cases that we could consider larger areas being fatal, e.g. if a piece hit an airplane, but or train, could kill hundreds of people at once.

Problem Set 4

Due in Canvas 11pm Wednesday, 3 May 2023

Homework Problems (point values indicated; 50 points total):

1. **Rapunzel physics. (10pts: 5+5)** (**+2 bonus** points for including your own measurements of the tensile stress of your hair with a statement of the hair color/ethnicity; **+2 extra bonus** points if you also include a laser-pointer diffraction measurement of your hair's diameter. The n'th minimum of the diffraction pattern is at angle $\theta_n = n\lambda/d$, where d the diameter of the hair and λ is the wavelength of the laser pointer: for red pointers, either ~ 650nm for old cheap dim ones, or ~ 635nm for newer brighter-looking ones).

   First remind yourself of the famous Brother's Grimm fairytale Rapunzel. If your German is rusty, I recommend Andrew Lang's legendary 1890 English translation: `https://www.worldoftales.com/fairy_tales/Andrew_Lang_fairy_books/Red_fairy_book/Rapunzel.html`. You should read this (it'll take a couple of minutes) to gather data and set the mood for this problem. The reference to "golden hair" allows us to deduce that Rapunzel is a blonde caucasian. Normal blonde hair has a diameter of about 70μm, and mechanical properties that you can deduce from the appropriate stress-strain curve in figure 1, though you are encouraged to experiment on your own hair and report the results, which may differ very substantially from those in the figure. Notice that hair has mechanical properties rather similar to those of the copper and chromel wires that were demonstrated in class: it stretches elastically for $\epsilon \lesssim 0.02$ ("yield"); for slightly higher stresses, it stretches dramatically in a ductile and increasingly inelastic fashion to $\epsilon \sim 0.3$ and then breaks suddenly (the curves in figure 1 end at the breaking point). All human head hair has density $\rho \sim 1.3\text{g cm}^{-3}$. and is of ellipsoidal cross-section with major and minor axes $2a$, $2b$ listed in table 1 below, along with its elastic properties, compiled from various fascinating sources[1].

	Caucasian	Asian	African	Caucasian (wet)
Major axes 2a/2b (μm)	74/47	92/71	89/44	
Elastic Modulus (GPa)	3.3	4.7	2.5	0.9
Yield Strength (Mpa)	67	100	58	47
Breaking Strength (Mpa)	117	139	101	82
Strain at breaking	0.35	0.32	0.20	0.39

Table 1: Mechanical properties of human head hair

   a) The hairs in a crewcut or cowlick stand up straight. Long hair falls down. Estimate the length l of a single hair held vertically at its base which would bend in gravity over 90

[1] Seshadri & Bhushan 2008 "In-situ tensile deformation characterization of human hair with atomic force microscopy", *Acta Materialia* **56**, 774, Reutsch & Weigmann 1996 "Mechanism of tensile stress release in the keratin fiber", *J. Soc. Cosmetic Chemists* **47**, 13, and "The mechanics of fracture of human hair", *Int. J. Cosmetic Science*, **21**, 227, and Bharat Bhushan's book *Biophysics of Human Hair* (2010, Springer). Bonus points for actually measuring properties of your own or friends' hair. Some students have found hair to be considerably stronger than this "reference hair", which may not all have been from healthy young college students!

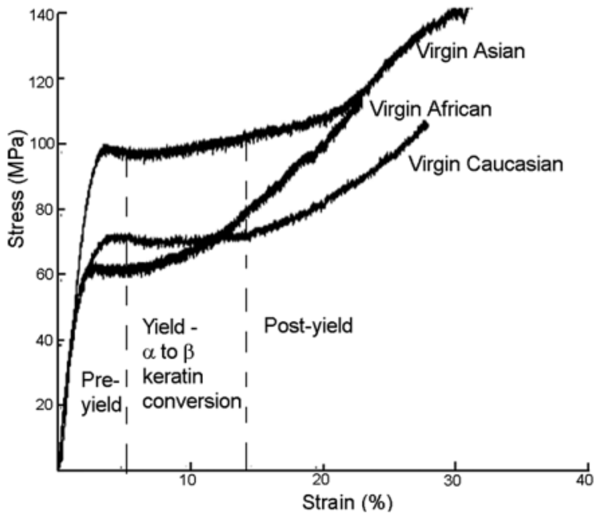

Figure 1: Mechanical properties of head human hair. Note: experiments by Sterl on his family, and by students, get answers rather different than these graphs. You are encouraged to make your own measurements. Figure is from Fig 2.6 of I.P. Seshadri 2008 MS Thesis, OSU, which is in turn reproduced as Fig 4.16 in B. Bhushan's book *Biophysics of Human Hair* (2010, Springer). Note: 'Virgin' does not refer to sexual experience, but to hair not bleached, dyed, straightened or subject to other chemical, heat or mechanical treatments, which can considerably weaken it.

degrees, in terms of its elastic modulus (E), density ρ, diameter d and g, and evaluate this length numerically in centimeters.

Solution

This is a buckling strut. Equate elastic energy stored in semi-circular bend $\sim E(d/l)^2 d^2 l$ to the gravitational potential energy released $\sim g(\rho d^2 l)l$, and get

$$l \sim \left(\frac{E}{\rho}\frac{d^2}{g}\right)^{1/3} \sim 10\text{cm}$$

Actually the quantity in parenthesis should be multiplied by a factor of (1/16), so the actual answer is about 4 cm for buckling hair.

b) For caucasian hair, the average circumference of a compressed ponytail is 3 inches, and the hair grows about 5 inches per year. For most healthy young people, each hair follicle shuts off the growth of its hair after 5-7 years, and the strand subsequently falls out, to be replaced by a new growth, limiting the length even of never-cut hair to a length that would not be very useful for climbing tall towers. Let us make the story more realistic (and more comfortable for Rapunzel's scalp) by assuming that the witch, in anticipation of locking Rapunzel up in the tower at age 12, started saving all the hair from Rapunzel's comb and shower drain from birth, and constructed the rope from this hair. Let's also assume one end is tied round a firm anchor (iron candle sconce?) at the tower window. Assuming a slender and handsome young prince as the climber, how long and thin a climbing rope could have been made from Rapunzel's hair? The story mentions a 20-

yard (18m) tower. Is that too high, just right, or could it have been even higher?

Solution

If the witch cut off all Rapunzel's hair when she locked her up at age 12, 12 years of hair growth= two 30-inch (75 cm) lengths, of cross-sectional area $(3\text{inch}\times 2.5\text{cm inch}^{-1}/(2\pi))^2 \times \pi = 5.5\text{cm}^2$, for a total volume of hair of $V \sim 5.5 \times 2 \times 75 \sim 800\text{cm}^3$. Since the witch was heavier than the prince, let's give the prince a bit of safety margin by keeping stress σ within the elastic regime, so $\sigma < \sigma_Y$ the yield stress, which from the figure is about 70 Mpa, or $\sigma_Y = 7 \times 10^8 \text{dyn cm}^{-2}$. The mass of a handsome young prince we take to be $m = 65$ kg. So the rope's cross-sectional area must be $A > mg/\sigma_Y = 0.09\text{cm}^2$. Thus the length of rope can be up to $\boxed{L = V/A = 800/0.09 \sim 9000\text{cm} = 90\text{m}}$, about 30 stories, and the roof height of New York's famous Flatiron Building skyscraper. For the $\sim$ 10m tower shown in the illustration, the witch wouldn't have had to shave Rapunzel at all, just collect strands off her comb for a few years.[2]

2. **All steamed up. (10 pts)** When you boil a pot of water on a range burning natural gas (CH_4), the kitchen steams up. What fraction of this water vapor is from the pot, and what fraction is water produced in the combustion of the natural gas?

Solution

Quick answer: the Purcell sheet gives the heat of vaporization of materials as 10^4cal mol^{-1}, or about 500cal per gram of water evaporated. Purcell also gives heats of combustion as 10^4cal g^{-1}. Combusting a methane molecule ($A = 16$) produces two water molecules ($A = 2 \times 18 = 36$), so that is $10^4\text{cal}(16/36) \sim 4000\text{cal}$ per gram of water produced. But gas stoves have an efficiency $\epsilon < 1$: a lot of the flame of the heat goes into the air, not the pot. Then about $4000\epsilon/500 = 8\epsilon$ grams of water is evaporated for each gram released in combustion. If we guess that $\epsilon \sim 1/4$, that would imply about 2/3 water from the pot and 1/3 from combustion.

More detailed answer with efficiency estimate: Natural gas is composed mostly of methane. Methane combusts in the reaction

$$\mathrm{CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O}. \tag{1}$$

We could guess that this releases a few eV's per CH_4. According to Purcell's sheet, the combustion heat should be $\Delta H_{\mathrm{CH_4}} \sim 10^4\text{cal/g} = 7\text{eV/CH}_4$. The actual combustion heat of methane is 9.3eV/CH_4

To boil water, we need to break hydrogen bonds. The hydrogen bond is an order of magnitude weaker than covalent bonds, so we could guess a few tenths of an eV per water molecule is required to boil water. According to Purcell's sheet, the heat of vaporization of water is $\Delta H_{\mathrm{H_2O}} \sim 10^4\text{cal/mole} \sim 0.4\text{eV/H}_2\text{O}$. The actual heat of vaporization is $0.42\text{eV/H}_2\text{O}$. Our guess that hydrogen bonds are an order of magnitude weaker is a good one, as discussed in class.

The gas range is not very efficient at transferring energy from the flame directly to the water. A lot of heat goes into heating the air around the flame. We can guess an efficiency of $\eta \sim 25\%$. An empirical estimate of the efficiency could be made by looking up the energy

[2] Asian ponytails average 10cm circumference. Furthermore, asian hair grows faster, about 6 inches per year, and as the figure shows, it also has much higher yield strength and breaking strength than caucasian hair. So an asian Rapunzel could get visitors into much higher towers. Perhaps asian witches know this, explaning why asian folklore doesn't seem to have any analog of this story, though there are plenty of analogs of, e.g., Cinderella.

consumption rate of a stove (electric and gas range efficiencies should be close if they are to be cost competitive) and comparing it with how long it takes to boil a pot of water. Using my 9000BTU/hr stove burner[3] it takes about an hour to boil off a liter of water. This gives an efficiency of $\eta \sim 1000\text{g/hr} \times 2200\text{J/g}/(9 \times 10^6\text{J/hr}) \sim 25\%$.

The amount of water produced by combustion is

$$N_c \sim \frac{2\mathrm{H_2O}}{1\mathrm{CH_4}} \times \frac{P}{\Delta H_{\mathrm{CH_4}}} \tag{2}$$

where P is the power of the gas range. The amount of water produced by boiling is

$$N_b = \frac{\eta P}{\Delta H_{\mathrm{H_2O}}}. \tag{3}$$

The ratio is then

$$\frac{N_c}{N_b} = \frac{2\Delta H_{\mathrm{H_2O}}}{\eta \Delta H_{\ \mathrm{CH_4}}} \sim \frac{2 \times 0.4\text{eV}}{0.25 \times 7\text{eV}} \approx 0.4. \tag{4}$$

Thus, $\boxed{\text{about 30\% of the steam is from combustion and 70\% is from boiling}}$. One could in principle also consider the power used in heating the water up to the boiling temperature, but this contribution is small. The specific heat of water is 1cal/g/K, so to heat from 20°C to 100°C requires 80cal/g which is much smaller than the heat of vaporization of 540cal/g.

This also explains why, when Sterl opens an empty but pre-heated gas oven to put in a pie, his glasses fog up.

3. **Areas of Countries (10pts: 4+4+2)** The land areas of countries or other political divisions given in atlases are generally calculated as the area of a map the country (with some fixed pixel resolution, usually about 0.3km to 3km), projected onto the "reference geoid" (the best-fit oblate spheroid representing the surface of constant gravitational potential corresponding to mean sea-level). This procedure introduces some errors and ambiguities, whose magnitude we should estimate.

   a) Boundary error: this is zero for boundaries defined by latitude and longitude lines (e.g., Colorado, Wyoming in the US), but not for boundaries defined by rivers (e.g., Kentucky, Missisippi in the US), mountain ridges and/or coastlines (e.g., Italy, the United Kingdom, Greece, Chile; the latter three are further complicated by their plethora of small islands, e.g. the Hebrides, the Cyclades, Chilean Patagonia). Making reasonable assumptions (please state and justify), estimate the uncertainty in the surface area of Italy introduced by defining its boundary at 1 km resolution instead of 1 m resolution.

   Solution:

   Although the coast is a fractal curve and thus has an infinitely long coastline[4], the *area* converges. An illustrative example is the Koch snowflake, which is constructed by taking an equilateral triangle and iteratively dividing each face into thirds to grow a

[3] 1 BTU=British Thermal Unit is the heat required to raise the temperature of one pound of water by 1 degree Farenheit, or $454\text{g} \times 1\text{cal g}^{-1}\text{C}^{-1} \times (5/9)\text{C} \times 4\text{J/cal} \simeq 10^3\text{J}$. 9000 BTU per hour thus is 2.5kW. Electric kettles are (limited by household fuses) about 1.2kW, but much more efficient in getting heat to the water.

[4] Louis Fry Richardson, who discovered the self-similar cascade in turbulence, and for whom the dimensionless Richardson number Ri in buoyant hydrodynamics is named, in a posthumous 1961 paper on the theory of wars, showed that the length of the coastline of Britain measured with a ruler length L scaled as $L^{-0.24}$, diverging as it was measured with ever smaller rulers. Reading this paper inspired Benoit Mandelbrot in 1975 to coin the term *fractal*, systematize their study, and interpret this as meaning that the coastline of Britain had a fractal dimension $D = 1.24$.

new equilateral triangle.[5] This has infinite length, and a dimension $D = \ln 4/\ln 3 = 1.26$, close to that Richardson measured for the coastline of Britain. After n iterations, $3/4 \cdot 4^n$ new triangles are added, each with fraction $1/9^n$ of the area of the original triangle. Hence, after n iterations the area is multiplied by factor $1 + \frac{1}{3}\sum_{k=0}^{n-1}(4/9)^k = [8 - 3(4/9)^n]/5$. Given that coastlines have fractal dimension roughly equal to the Koch snowflake (1.26), we can use the Koch snowflake as an estimate. Each iteration of the snowflake decreases the side length of a triangle by 1/3. Italy measures around 1000 km by 400 km, so a measuring stick of length 1 km corresponds to around the $n_1 = 6$ iteration of the Koch snowflake ($3^6 = 729$) while a resolution of 1 m corresponds to perhaps the $n_2 = 12$ iteration ($3^{12} = 531,441$). The ratio of areas is given by

$$\frac{8 - 3(4/9)^{n_2}}{8 - 3(4/9)^{n_1}} \approx 0.3\%, \tag{5}$$

so we conclude that changing the coastline resolution from 1 m to 1 km could give a discrepancy of about $\boxed{0.3\%}$. Indeed, an analysis of the UK coastline observed a change from 242,495 km^2 at 10 km resolution ($n_1 = 4$) to 245,415 km^2 at 0.1 km resolution ($n_2 = 8$), which is a change of about a percent.[6]. With those values of n_1 and n_2, equation 5 predicts an area change of 1.4%, in reasonable agreement (note, however, that unlike the Koch curve, which is all protrusions, actual coastlines may have indentations ("Koch anti-snowflake") instead of or as well as protrusions, leading to reduction of the area, or partial cancellation of the area changes).

b) Crinkle error: while some countries are very flat (e.g., the Netherlands), others are not (e.g. Switzerland, Tajikistan). Making reasonable assumptions (please state and justify), estimate the fractional error in the surface area of Switzerland introduced by projecting its map onto the smooth "flat" reference geoid instead of measuring the actual surface area of the mountain slopes and ravines.

Solution:

Mountains look fractal (smaller ridges on big peaks; boulders on the smaller peaks; rocks and sand on the boulders, etc). Each factor of 3 reduction in scale *multiplies* the surface area by a constant factor (for example, considering a mountain range as a straight extrusion with the cross-section of the top third of a Koch curve, the area would increase by a factor of (4/3) for every factor of 3 reduction in scale, so at resolution 3^{-n} of the largest scale, the area would be $(4/3)^n$, so the $\boxed{\text{area diverges as } n \to \infty}$. Thus we have to pick some "minimum reasonable scale" to get a finite answer. A $\boxed{\text{“minimum reasonable scale” of} \sim 100 \text{ m seems appropriate}}$ as the scale of plots of land people buy. When measuring the areas of such plots, they don't include the surface area of 10 m-scale boulders, trees or walls on their property.

We can attempt an order-of-magnitude estimate given a fixed resolution of measurement. We consider tiling a surface with plates a fixed size. For a two-dimensional grid (not a fractal), I can use a single plate ($N = 1$) with scale 1 ($\epsilon = 1$) to measure area $A = N\epsilon^2 = 1$. If I halve the plate size to obtain scale factor $\epsilon = 1/2$, then the same two-dimensional area will have $N = A/\epsilon^2 = 4$ plates. However, given a fractal dimension D, the number of plates used will go like A/ϵ^D, even though each plate still has area $1/\epsilon^2$. Thus, when I measure by counting plates, I observe an area enlarged by factor ϵ^{2-D}. In class, we saw that rivers and coastlines have fractal dimensions of around 1.2.

[5]https://en.wikipedia.org/wiki/Koch_snowflake
[6]See minute 5:50 of https://www.youtube.com/watch?v=PtKhbbcc1Rc

It turns out to be a good guess for mountains to pick a similar increase in dimension: Young mountains (like the Alps and the Rockies) have a fractal dimension of around 2.1–2.2.

Looking at a topographic map of Switzerland (Figure 2), we can estimate the smallest measuring stick for which we'll obtain the same surface area as if Switzerland were perfectly flat. Counting pixels (or using a ruler), geographic features seem to be around 5 km wide (this is reasonable: the mountains in Switzerland are 2-4.6 km high, and the angle of repose of rocks and gravel is about 30 degrees, so the largest peaks cannot have horizontal scales larger than 4-9 km. The 30 degree angle of repose (maximum stable sandpile slope) gives an area change of $1/\cos(30°) = 1.15$, not the 4/3 of the $D = 1.26$ Koch curve. This suggests that we ought to take $D \sim 2.1$, not $D \sim 2.2$. Taking plates larger than around 5 km by 5 km would likely result in the flat surface area of Switzerland. Moreover, we observe that around half the country is mountainous. Thus we expect an area increase at scale $\sim 5\epsilon$km over the flat projected area, of about $(\epsilon^{2-D} - 1)/2$, where $D \sim 2.1$ from our above analysis.

Figure 2: Topographic map of Switzerland (Wikipedia). The country measures around 220 km north-south and 350 km east-west.

Hence, at a resolution of 5 km or longer, we expect an area about equal to the projected 2D area of Switzerland. At a resolution 5ϵ km for $\epsilon \leq 1$, we expect an area larger by a factor of about $50(\epsilon^{-0.1} - 1)\%$. If we take 0.2km resolution, this corresponds to $\epsilon = 0.04$, and an $\boxed{18\%}$ increase in the land area of Switzerland. This is about double the actual computed values from topographic maps at 0.2 km resolution, which give Switzerland having 7% larger surface area than its flat projection area. Probably because even in mountainous areas, more than half of the area is occupied by eroded valleys, glacial moraines and alluvial fans that are smoother than mountains and have slopes much less than the 30 degree maximum angle of repose.

c) Geoid error: for the Netherlands, Qatar or the Maldives, the sea-level geoid is a much better approximation to the typical altitude than it is for Nepal or Tajikistan. Estimate the systematic fractional error in the area of Tajikistan introduced by the standard procedure of projecting its map onto the (mean sea-level) reference geoid instead of a geoid representative of its actual altitude.

Solution:

The mean altitude of both Tajikistan and Nepal are around 3.2 km, so the fractional geoid error is 3 km/$R_\oplus \approx 0.05\%$ in length, hence $\boxed{0.1\% \text{ in area}}$ (length squared). This is negligible compared to the coastline and crinkle errors.

4. **Geothermal heat (10 pts: 5+5)** The temperature in the Earth's crust increases at an average rate of about 20 K per kilometer of depth.[7]

   a) Estimate the total thermal power emerging from the earth.[8]
   Solution
   Purcell gives a typical thermal conductivity for an insulator (i.e. a non-metal) $k = 10^{-2}$cal s^{-1}cm^{-1}K^{-1} = 4W m^{-1}K^{-1}.[9] Thus the mean heat flux from the earth is $F = 20\text{K}/(1000\text{m}) \times 4\text{W m}^{-1}\text{K}^{-1} = 0.08\text{W m}^{-2}$. The Earth's total geothermal luminosity is then $\boxed{4\pi R_\oplus^2 F = 4 \times 10^{13}\text{W} = 40\text{TW}}$. This is in somewhat accidentally excellent agreement with the best geophysical integrations over the earth using temperature gradients and local rock conductivities from thousands of boreholes on all the continents and in the ocean, 46 ± 3TW.

   b) Could geothermal sources provide the human world's energy needs?
   Solution
   There are many ways to estimate world energy use, ranging e.g. from extrapolating from student experience driving cars to using world CO_2 production. *Car estimate* The USA has about 2 cars per household, driven on average 10,000 miles/y car at 20 mpg, so 500 gallons of gas used per year. A gallon is about 4 liters=4kg, so 2 tons oil/person/y in energy use for driving. That's typical for the USA, but most of the world population doesn't drive so much. However energy is also used for heating, industry, electricity generation, etc. So other western countries probably use a few times our gas estimate, but less developed countries use less. If we take average 2 tons oil/person/y times *half* the world population, $\sim 7 \times 10^9/2$, we get 7×10^7ton y^{-1} = 7×10^{15}g y^{-1} world oil consumption. $= 2 \times 10^8$g s^{-1}. The Purcell sheet gives a typical heat of combustion of hydrocarbon fuels like oil or coal as $H_c = 10^4$cal g^{-1} = 4×10^4J g^{-1}[10], so our estimate of the world thermal energy consumption corresponds to 10^{13}W=10TW. This is $\boxed{\text{about 25\% of the the 40 TW total thermal power coming from the earth}}$ (part a). However it's not very practical to pump the geothermal heat out of the oceans to where it is needed, and people might object if there were one geothermal plant on every square kilometer of land! Furthermore, the deepest oil wells ever drilled are only about 10 km deep, where the temperature 500 K would give a maximum Carnot efficiency of only $(1 - 300/500) = 0.4$. More practical much shallower wells would give much lower efficiency. So it is energetically possible, but probably impossible in practice.

   Alternative estimate using world carbon dioxide growth As widely reported, the CO_2 concentration in the atmosphere has been rising at about 2.5ppm per year for the past decade. The concentration is in number of molecules, so since CO_2 is (44/28) times the mass of the dominant nitrogen molecules, 2.5ppm by number is 2.5(44/28)=4ppm by

[7] Many gold mines extend $\sim 3-4$km below the earth's surface, and the rock temperatures there are $\sim 60-70°$C. The mines have to be cooled by pumping in ice to enable the miners to work. For obvious reasons, the deepest mines are also preferentially located in regions, e.g. South Africa, where the crustal temperature gradient is especially low.

[8] Everything you need is on the Purcell sheet. Rocks are insulators (like quartz), not pure metal. Finer points were discussed in Lecture 3.

[9] Tables of conductivity measured for various rock types give $k = 3-7$W m^{-1}K^{-1} depending on rock type. These bracket the typical insulator (i.e. non-metal) value given by Purcell.

[10] Differences between fossil fuels are small: methane is 52kJ/g, oil is 44kJ/g and coal is 49kJ/g. Dry wood averages about 14kJ/g.

mass, of which (12/44) is in carbon molecules. Thus the mass of carbon in the Earth's atmosphere is rising by about $\dot{M}_C = (12/44)(44/28) \times 2.5 \times 10^{-6} M_{\rm atm}$ per year, where $M_{\rm atm} \simeq 4\pi R_\oplus h \rho_{\rm atm}$ is the mass of the Earth's atmosphere, $h \sim 10$km is the scale height of the atmosphere, and $\rho \sim 1.2 \times 10^{-3}$g cm^{-3} is the atmospheric density a sea-level. This gives $\dot{M}_C \sim 7 \times 10^{15}$g y$^{-1} = 2 \times 10^8$g s^{-1}. Since the mass of hydrocarbon fuels is mostly from the carbon atoms, not the 0-4 hydrogen atoms attached, we can estimate the $\boxed{\text{thermal power from all fossil fuel burning to be } \dot{M}_c H_c \simeq 10^{13}\text{W} = 10\text{TW}}$. As of 2020, fossil fuels still supplied 85% of world energy use, so this should be a pretty good estimate. It agrees with our previous estimate from US driving, and is subject to many fewer uncertainties!

Further comments: Geothermal could be used for home steam heating with high efficiency (as it is in Iceland). At 30% generating efficiency, 46TW thermal could generate about 14TW electrical. World electrical consumption is about 3TW (about half the installed generating capacity: less than max capacity is used in spring, summer, nighttime).

5. **Blood flow I. (10pts: 2+3+3+2)**

a) Flow of a fluid in a horizontal pipe of length L and radius R occurs when there is a pressure difference Δp between the two ends of the pipe. Considering the kinematic viscosity ν (dimensions length^2/time) and other relevant variables, construct a complete set of Buckingham Pi dimensionless quantities.

Solution

Δp, R, L, ν, fluid velocity v and density ρ. Six variables in 3 units, so by Buckingham's Pi theorem, there are$6 - 3 = 3$ Π's. Following the class lecture and demo of marbles in corn syrup, we take $\Pi_1 = vR/\nu \equiv \text{Re}$, the Reynolds number. Another obvious one is $\Pi_2 = R/L$. To eliminate the mass unit in Δp and ρ, we take $\Pi_3 = \Delta p/(\rho v^2)$.

b) For low velocity flow in narrow pipes, solve these equations by making a similar argument as we made for the marbles in corn syrup: flow at a steady speed v requires a balance everywhere along the pipe between the driving pressure gradient $\Delta p/L$ and a drag force $\propto \nu$. Show that this gives a *unique* (up to a dimensionless constant) relation between v, the pressure gradient, and other variables in the problem. Use this to show that the mass per unit time of fluid flowing through the pipe $\dot{M} \propto R^4$.

Solution

To get the pressure gradient as a variable, Δp and L must appear only in the combination

$$\Pi_2' = \Pi_2 \Pi_3 = \frac{\Delta p R}{L \rho v^2} = g(vR/\nu) \ ,$$

where $4g(vR/\nu) \equiv f(v2R/\nu)$, and $f(v2R/\nu)$ is known in the plumbing, HVAC and engineering industries as the Darcy-Weisbach "friction factor", and plotted, from experimental measurements, as a function of Re and the surface roughness of pipes in the famous "Moody chart". As noted above, at low velocities, $g(vR/\nu)$ must be proportional to the viscosity setting the drag, i.e. proportional to

$$\Pi_1^{-1} = \frac{\nu}{vR} \ .$$

Thus we must have

$$\boxed{\tfrac{\Delta p}{L} = \Pi \tfrac{\nu \rho v}{R^2} \quad .}$$

Derivation of the exact Poiseuille flow from the Navier-Stokes equations of fluid dynamics shows that the dimensionless constant $\Pi = 8$. [11] Since $\dot{M} = \pi R^2 \rho v$, this equation gives

$$\dot{M} = \frac{\pi}{8} \frac{\Delta p}{L} \frac{R^4}{\nu}$$

[The dimensionless factor of $\pi/8 = 0.4$ would become$\pi/128 = 0.025$ if we had used the conventional pipe diameter D instead of R.]

c) As the saying goes "blood is thicker than water": the viscosity of blood ν(blood) $\sim 0.05\mathrm{cm}^2\mathrm{s}^{-1}$ is several times higher than that of water ($\sim 0.01\mathrm{cm}^2\mathrm{s}^{-1}$), owing to the red blood cells and other plasma constituents. Normal resting human arterial blood pressure is 115 Torr (mm Hg) systolic (peak heart contraction) and 70 Torr (mm Hg) diastolic (resting heart between beats). In a crude approximation, treat your blood vessels as rigid pipes subject to a steady pressure differential. Your heart pumps about 5 liters per minute while you are at rest (and 3-6 times that during vigorous exercise). Your large arteries (supplying legs, brain, etc) are about $D = 2R = 1\mathrm{cm}$ internal diameter. Estimate the pressure drop (in Torr) across a 50 cm long artery of that diameter required to maintain blood flow while you are lying on your bed. What is the Reynolds number $\mathrm{Re} \equiv Rv/\nu$ of the flow, where R is artery radius and v the blood flow velocity?

Solution

Estimate that about 1/5 of the total 5 liters/minute resting blood flow goes through each of the several large arteries. Thus the blood velocity in the large artery must be given by

$$(\pi/4)D^2 v = \dot{V} = 1\mathrm{liter/min} = 17\mathrm{cm}^3\mathrm{s}^{-1}.$$

With $D = 1\mathrm{cm}$, we find $\boxed{v = 4\dot{V}/(\pi D^2) \sim 20\mathrm{cm\ s}^{-1}}$.

The pressure drop

$$\Delta p \simeq 32 L\nu\rho v/D^2 \simeq 32 \times 50\mathrm{cm} \times 0.05\mathrm{cm}^2\mathrm{s}^{-1} \times 1\mathrm{g\ cm}^{-3} \times 20\mathrm{cm\ s}^{-1}/(1\mathrm{cm})^2$$

$$\simeq 1.5 \times 10^3 \mathrm{dyn\ cm}^{-2} \simeq 1\mathrm{Torr}.$$

(where for the final result we used $1\mathrm{Torr} \equiv 1\mathrm{mm\ Hg} = 1\mathrm{atm}/760 = 10^6\mathrm{dyn\ cm}^{-2}/760 = 1.3 \times 10^3\mathrm{dyn\ cm}^{-2}$). This $\boxed{\Delta p \sim 1\mathrm{Torr}}$ is a small fraction of the overall pumping blood pressure (systolic minus diastolic $\sim$ 40mm Hg or 40 Torr.): most of the pressure drop occurs pushing blood through the tiny capillaries.

Using the arterial flow speed v estimated above, our

$$\boxed{\mathrm{Re} = Rv/\nu = 0.5\mathrm{cm}20\mathrm{cm\ s}^{-1}/(0.05\mathrm{cm}^2\mathrm{s}^{-1}) \simeq 200}$$

(or the conventional diameter-based Reynolds number $\boxed{\mathrm{Re} = DV/\nu \simeq 400}$). The resting blood flow is thus expected to be laminar. But during exercise (when the flow rate, and thus flow velocity is several times higher), and especially in people with clogged, narrow arteries ($v \propto \dot{V}/D^2$, so $\mathrm{Re} \propto \dot{V}/D$) the conventional Re can exceed the value at which

[11] The dimensionless constant becomes 32 if instead of the radius R across which the velocity goes from its maximum to zero, we had less sensibly used the pipe diameter $D = 2R$ by which plumbers label their pipes, and in terms of which the equation is usually given; with D instead of R everywhere, the Darcy-Weisbach friction factor for $\mathrm{Re} < 2000$ is $f = 64/\mathrm{Re}$ ($\Delta p \equiv f(L/D)\rho v^2/2$), where Re is also conventionally defined with diameter instead of radius: $\mathrm{Re} = vD/\nu = 2vRnu$.

turbulence begins (~ 2000 in long pipes, but ~ 500 in branched and valved pipes like arteries, c.f. the classic experiments of Stehbens 1958) The resulting enhanced shear stress can dislodge deposits in the vessels, leading to "endothelial dysfunction" and strokes. See, e.g. Montalvo et al 2022 or Saqr et al 2021

d) Explain why return flow in the veins from the legs is an issue in sitting or standing humans.

Solution
The hydrostatic pressure drop due to 50 cm of blood is $\Delta p_h = \rho g h = 1\text{g cm}^{-3} \times 10^3\text{cm s}^{-2} \times 50\text{cm} = 5 \times 10^4\text{dyn cm}^{-2}$ $\boxed{\sim 40\text{Torr}}$. This is an order unity change in the pressure, resulting in greatly lowered return flow in the veins, pooling of blood in the legs, and reduced flow to the heart and brain. The body has compensatory responses (vasoconstriction, increased heart-rate, etc.). Airline passengers and older folks benefit from compression socks. Military pilots making high-g maneuvers need "G-suits" to prevent black-outs.

Problem Set 5

Homework Problems (point values indicated; 50 points total):

1. **The 29 steps (12 pts: 3+4+5)** The stairs leading up to our classroom, 201 Bridge, are made of clay tiles that were originally 5 cm thick, but the front halves of the tread tiles have been worn down to 4 cm thickness by the racing feet of students in the ~ 34 years since the tiles were last replaced. The final flight of stairs has a width of 150 cm, and the rise height is 17 cm per step.

   a) Estimate the thickness of the layer of clay removed from beneath a shoe by a single average footstep on the tile.

   Solution
   The large classes in 201 Bridge, Ph 1, Ph 2 and Ph 12, involve essentially all Caltech freshmen and sophomores, i.e. about 500 students, though not all the Ph 1,2,12 students attend the 2-3 times weekly lectures. However, the stairs are also used by students attending the physics labs, and for the thursday physics colloquium, and by assorted office inhabitants. A reasonable estimate would therefore be $\boxed{\sim 300 \text{ people per day}}$, each making a trip up and down on $5 \times 30 = 150$ days per year. Thus in 34 years, the stairs have had $\boxed{N_s \sim 300 \times 2 \times 150 \times 34 \sim 3 \cdot 10^6 \text{ footsteps}}$. People normally put only one foot on each step when walking up or down stairs, so it is not appropriate to multiply by an additional factor of two for two feet. However a foot is only $\sim$ 10cm wide, while the stair is 150cm wide. If these were randomly placed, each step would have a probability of $10/150 = 1/15$ of hitting a given spot on the stair. But once the stairs start to wear, people will avoid stepping on the unstable high spots and stick to the lows (the centers of the soft clay tiles, avoiding the hard cement grout that does not wear), so perhaps a probability $\boxed{p \sim 1/7}$ of stepping on a wear area is more reasonable. Since $w \sim 1$cm has worn off in 34 years, we estimate **the wear per footstep** as $\boxed{\delta w \sim w/(pN_s) \sim 2 \cdot 10^{-6}\text{cm} \sim 20\text{nm}}$. Probably this is rather unevenly distributed: clean rubber sole shoes probably don't cause nearly this much wear, and most of the wear is probably caused by bits of sand and grit stuck to soles. These can make much deeper scratches in the tile (typically of order the thickness of a sheet of paper or two, $\sim 10^{-2}$cm). Further experiments would be interesting.

   b) To fire a clay tile requires heating it in a kiln to 1500 K. Clay tile is brittle, with a breaking strain in tension of $\epsilon_Y \sim 10^{-4}$. Estimate the minimum time you must take to heat a 5 cm thick clay tile from 300 K to 1500 K, in order to avoid cracking it while heating.

   Solution
   Heated tiles will expand due to thermal expansion. *Uniform* thermal expansion does not generate internal stress: as discussed in class, each atomic vibration just develops a larger average separation from its neighbor, due to the anharmonicity of the interatomic potential. However *nonuniform* heating does generate internal stress. This is easy to see:

if we heat the outer surface of a slab, the outer surface will thermally expand, and this expansion will stress the cool, unexpanded layer underneath, to which the expanding surface is attached. The result is elastic strain between the two layers: the hot layer trying to expand the cool one, and the cool layer trying to compress the hot layer.

A tile in a kiln whose temperature is rising will be some ΔT hotter at its surface than at its midplane, since heat is conducted inwards as the exterior temperature rises. To avoid cracking the tile, we require $\alpha\Delta T < \epsilon_Y$, where $\alpha \sim 2 \cdot 10^{-5}\mathrm{K}^{-1}$ is the coefficient of thermal expansion (from Purcell sheet or lecture, or just remember that $\alpha T \sim 1$ for $kT \sim U$, where $U \sim 10$ eV $\sim k(10^5\mathrm{K})$ is the binding energy of the solid). Thus we must keep at all times $\boxed{\Delta T < \Delta T_c \equiv \epsilon_Y/\alpha < 10^{-4}/2 \cdot 10^{-5}\mathrm{K}^{-1} = 5\mathrm{K}}$ **to avoid cracking the tile due to thermal expansion stresses**.

Method 1: To estimate the total heating time for a tile of total thickness, one approach is to calculate the heat flux into the tile. For a tile 5 cm thick, its midplane is $\Delta x \sim$ 2.5 cm from the surface, so the temperature gradient in the tile $\nabla T \sim \Delta T/\Delta x$. Therefore the heat flux into the middle of the tile is $K\Delta T/\Delta x$, where $K \sim 10^{-2}\mathrm{cal\ cm^{-1}s^{-1}K^{-1}}$ (from Purcell sheet or lecture). To avoid cracking, we have to keep $\Delta T < \Delta T_c = \epsilon_Y/\alpha$. The total heat that has to flow into a unit area of the tile to heat it from $T_{\mathrm{room}} \sim 300\mathrm{K}$ to the desired $T_h = 1500\mathrm{K}$ is $c_v(T_h - T_{\mathrm{room}})\Delta x$, where $c_v \sim 0.5\ \mathrm{cal\ cm^{-3}K^{-1}}$ (Purcell sheet, or just remember $3kT$ per atom). Therefore the **minimum heating time is the thermal energy per unit area that has to be added to reach the peak temperature, divided by the maximum heat flux possible without cracking:**

$$\boxed{t_h \lesssim \frac{c_v(T_h - T_{\mathrm{room}})\Delta x^2}{K}\frac{\alpha}{\epsilon_Y} \sim 10^5\mathrm{s}} \tag{1}$$

Cooling it back down slowly enough to avoid cracking during cooling will take another day, so tiles (or bricks) this thick will need to be in the kiln for a couple of days. This is just about the actual factory cycle time. Thinner ceramic tiles (like those used on walls or bathroom floors, $\Delta x \sim 0.5$ cm), can be made $\sim (2.5/0.5)^2 = 25$ times faster, or in about 2 hours, which is also just about their reported factory cycle times.

Method 2: An alternative way of deriving the heating time t_h is to consider a sequence of discrete 'heating events', each of which raises the temperature at the tile midplane by at most $\Delta T < \Delta T_c = \epsilon_Y/\alpha$. The energy per unit area required to raise the temperature of the tile by ΔT is $c_v\Delta T\Delta x$. Thus the time t_{diff} for the tile to heat through is

$$\boxed{t_{diff} \sim \frac{c_v\Delta T\Delta x}{K\Delta T/\Delta x} \sim \frac{c_v}{K}\Delta x^2 = \frac{\Delta x^2}{\kappa} \sim 300\ \mathrm{s}} \tag{2}$$

where the heat diffusivity is $\kappa = K/c_v \sim 0.02\ \mathrm{cm^2s^{-1}}$, and we could also have directly gotten the equality $t_{diff} \sim \Delta x^2/\kappa$ from the diffusion equation.

Therefore **to avoid cracking, we cannot raise the temperature faster than** $\boxed{dT/dt < \Delta T_c/t_{diff} \sim 2 \times 10^{-2}\mathrm{K\ s^{-1}}}$, and the **minimum time to heat the tile from 300 K to 1500 K is again given by equation 1**, $\boxed{\sim 10^5\mathrm{s} \text{ or about a day}}$.

[Aside: The numbers used were 'typical material' ones from the Purcell sheet. It is interesting to investigate what better values might be. Various sources on building insulation give typical brick as having $K \sim 0.6 - 1$W/m/K ($1/6 - 1/4$ of the generic Purcell value of 0.01cal/s/m/K = 4W/m/K). In the 1941, paper "Thermal Expansion of Clay Building Bricks" C.W. Ross (*Journal of Research of the NBS* **27**, 197-216) tested 137 bricks of various clay types and manufacturers, and found $\alpha \sim 5 - 11 \times 10^{-6}\mathrm{K}^{-1}$, with $\alpha \sim 6 \times 10^{-6}\mathrm{K}^{-1}$ typical, and about 1/3 of the generic Purcell value. Since only the ratio α/K appears in equation 1 using these figures instead of the Purcell ones would change t_h by a factor between 1 and 2. The special fireclay bricks used to make chimneys not surprisingly had lower thermal expansion coefficients $\alpha \sim 3 \times 10^{-6}\mathrm{K}^{-1}$. The $\epsilon_Y = 10^{-4}$ given in the problem statement was taken from sources on high-quality vitrified clay pipes (ultimate tensile stress 400psi, Young's modulus $4 \cdot 10^6$ psi). The clay flooring tiles don't look fully vitrified, so a somewhat different ϵ_Y might be appropriate. Clearly experimentation on the actual tiles is needed!]

c) Estimate the ratio of the energy required to make the tiles (150cm × 30cm × 5cm) for one step on the final flight of the 201 Bridge staircase, to the energy expended in going up that one step by all the students who have climbed it during the past 34 years. Hint: for estimating the tile-making energy, assume that the mass of the kiln is comparable to the mass of each batch of tiles fired in it, and that it is well enough insulated and efficient in combustion heating that losses are no more than about 50%.

Solution

The energy required to raise the tile temperature to the firing temperature is $\boxed{e \sim c_v(1500 - 300)\mathrm{K} \sim 0.5 \text{ cal cm}^{-3}\mathrm{K}^{-1} \times 1200\mathrm{K} \sim 600\text{cal cm}^{-3} \sim 2.4\text{kJ cm}^{-3}}$ (or, given that clay has a density $\rho \sim 2$ g cm^{-3}, about $e/\rho \sim 1.2$ kJg^{-1}). It would be more realistic to multiply by factor $\epsilon^{-1} \sim 3 - 4$ to allow for inefficiency of kiln combustion (heat goes up the chimney, and input air carrying oxygen has to be heated) and heat conduction losses through the walls of the kiln (generally made thick to save energy of course, but not so well insulated that it won't cool down in a day: this requires that the mass of the kiln be comparable to the mass of tile being fired in it, so heating the kiln walls takes about as much energy as goes into the tile, but is wasted). This gives $\epsilon^{-1}e/\rho \sim 4$kJ/g. The best modern industrial kilns in fact do manage $\sim 3 - 4$kJ/g. Primitive ones are much worse. Our estimate was pretty good.

A single step in Bridge requires a volume of tile $V_s \sim 150\mathrm{cm} \times 30\mathrm{cm} \times 5\mathrm{cm} \sim 2 \cdot 10^4\mathrm{cm}^3$. Thus **the energy to make a single tile step in Bridge** is $\boxed{\epsilon^{-1}E_s \sim eV_s \sim \epsilon^{-1}5 \cdot 10^7\mathrm{J} \sim 2 \cdot 10^8\mathrm{J}.}$

The **work done against gravity by all the** $m \sim 70$kg **students climbing the step of height** $h \sim 0.17$m was $\boxed{E_g \sim (N_s/2)mgh \sim 2 \cdot 10^8\mathrm{J}}$, **just about equal to the energy required to fire its tiles**. Since muscles have efficiency $\epsilon \sim 0.25$, the total caloric energy expended by the students in climbing the step was $\sim \epsilon^{-1} \sim 4$ times larger.

2. **Quantum liquids. (9 pts: 3+6)** The binding energies of atoms in the liquid phase are given by the heat of vaporisation (the binding energy in the solid state is given by the heat of sublimation, which is typically ~ 1.1 times larger).

Substance	He	H_2	Ne	Ar	N_2	Kr	H_2O	Hg	Pt
Heat of vaporisation (kJ/mole)	0.09	0.9	1.8	6	6	9	44	59	509

Note that 1 kJ/mole = 10^{-2}eV/(atom or molecule) = $k_B(120\mathrm{K})$.

a) Estimate the melting temperatures of the substances in the table. [hint: use Lindemann's rule as described in lecture].

Solution

The compounds should melt when the kinetic energy associated with the vibrations of the particles becomes comparable to the difference in binding energy between the solid and liquid state. As explained in the problem statement, the difference in binding energies is about 0.1 times the heat of vaporization. The kinetic energy associated with the vibrations should be approximated by kT. Then we have

$$kT \approx 0.1\Delta H_{vap}$$

$$T \approx 0.1\Delta H_{vap}/k = 0.1(120K)\frac{\Delta H_{vap}}{1\text{kJ/mol}} = (12K)\frac{\Delta H_{vap}}{1\text{kJ/mol}}.$$

Substance	True Melting Temp (K)	Predicted Melting Temp (K)
He	None	1
H_2	14	10
Ne	25	20
Ar	84	70
N_2	63	70
Kr	116	100
H_2O	273	500
Hg	234	700
Pt	2040	6000

Note that helium will not solidify at atmospheric pressure. Also, our estimates are clearly not very good for the last three entries in the table. This can be traced to the fact that the intermolecular bonds in water and metals have a substantially different character than the van der Waals bonds in the first six substances.

b) For each substance in the table, calculate the de Broglie wavelength of the atom or molecule at its melting temperature. Use this to explain why, of all elements and molecules, only He (and to a lesser extent H_2) show interesting quantum effects in their liquid state.

Solution

The de Broglie wavelength will be given by

$$\lambda = h/p = \frac{h}{\sqrt{2mkT}} = \frac{hc}{\sqrt{2mc^2kT}}.$$

We can see that the de Broglie wavelengths of the vast majority of materials at their melting points will be very small compared to the length scale set by the concentration of these materials in their liquid state (that is, their interatomic spacing). Helium, and perhaps hydrogen under the right circumstances, is the exception. This is why liquid helium becomes a quantum liquid at a temperature slightly below its boiling point (at atmospheric pressure) and does not freeze to solid form.

Substance	Predicted de Broglie Wavelength at T_m (Å)
He	15
H_2	6
Ne	1.5
Ar	0.6
N_2	0.7
Kr	0.3
H_2O	0.3
Hg	0.1
Pt	0.03

3. **Aluminum power line. (9 pts)** If the energy transmitted by a 100-kilovolt power line 100km long is used to refine aluminum (by electrolytic reduction of Bauxite ore), how long will it take to produce an amount of aluminum equal to that in the aluminum wires carrying the electricity? [hint for cgs Gaussian units users: 1 statvolt=300 volts].

Solution

The amount of energy per gram H required for electrolytic reduction should be roughly the same as the energy released in combustion (recall that combustion is oxidation, the opposite of reduction). The heat of combustion is given on the Purcell sheet as 10^4 cal/g. A calorie is a bit more than 4 J, so we'd guess that electrolytic reduction of aluminum oxide requires $H \sim 4 \times 10^{11}$erg (40kJ) per gram of Al produced (same as would be released in burning the aluminum). Alternatively we get about the same figure without consulting the Purcell sheet if we guess that the oxygen in Al_2O_3 is bonded to the aluminum with the usual several eV per oxygen bond. [1]

Let the powerline have length $L = 100$km, area A, and resisitivity $1/\sigma \sim 2 \times 10^{-6}$ohm cm (from Purcell sheet for copper, recalling from class that all metals are similar, and therefore resistance $R_{\text{line}} = L/(A\sigma)$, and mass $M = \rho_{\text{Al}} AL$, where $\rho_{\text{Al}} \sim 2.7\text{g cm}^{-3}$ (the simple estimate derived in class gives $\rho_{\text{Al}} \sim 27m_p/(3\text{Å})^3 = 2\text{g cm}^{-3}$). The energy required to produce the aluminum in the line is $E = HM$, where H is the energy required to reduce a gram of aluminum from Bauxite. If it carries current I, the power dissipated in the powerline is $P_{\text{line}} = I^2 R_{\text{line}}$. The power delivered to the load (the reduction factory) is $P_{\text{load}} = I^2 R_{\text{load}}$. It would be uneconomical for the power company to have $P_{\text{line}}/P_{\text{load}} > 1$, and a waste of effort to put up such thick powerlines that $P_{\text{line}}/P_{\text{load}} < 0.01$, so we guess that to order of magnitude $\boxed{P_{\text{line}}/P_{\text{load}} = R_{\text{line}}/R_{\text{load}} \sim 0.1 \equiv f}$ (the actual US average transmission loss is about 7%). Thus $R_{\text{line}} + R_{\text{load}} \sim (1 + 1/f)R_{\text{line}}$, and $I \sim V/(1 + 1/f)R_{\text{line}}$ and

$$P_{\text{load}} \sim \frac{V^2 f}{(1+f)^2 R_{\text{line}}}$$

The time t to deliver the energy E to reduce the aluminum in the powerline is thus

$$t \sim E/P_{\text{load}} = H\rho_{Al} AL \frac{L}{(A\sigma)V^2} \frac{(1+f)^2}{f}$$

[1] Actual numbers: the standard enthalpy of formation of Al_2O_3 is -1675kJ/mol, or 31kJ per gram of aluminum (atomic weight 27 grams/mol). Actual industrial reduction consumes a carbon anode by using the oxygen released to produce CO_2 ($H_f = -393$kJ/mol), so the net energy required in the reaction $3C + 2Al_2O_3 \rightarrow 4Al + 3CO_2$ is $(2 \times 1675 - 3 \times 393)/(4 \times 27) = 20\text{kJ g}^{-1}$ of Al produced). However heat is also required to keep the electrolyte bath molten at ~ 1250K, and the electrolysis isn't perfectly efficient. The actual total energy use in industrial electrolytic reduction is $5.3 \times 10^{11}\text{erg g}^{-1} = 53\text{kJ g}^{-1}$. This electrolytic reduction step accounts for 2/3 of the total energy used by the aluminum mining, refining and milling industry, according to DoE figures. So our crude estimate was good to a factor of 2.

Notice that $\boxed{\text{the cross-sectional area of the powerline } A \text{ cancels out!}}$

Our final expression is

$$t \sim \frac{H\rho_{Al}L^2}{V^2}\frac{1}{\sigma}\frac{(1+f)^2}{f}$$

Before putting in numbers, we have to be careful to look at the units: We have $1/\sigma$ in Ohm-cm, and $V = 10^5$Volt. Volt^2/Ohm = Joule/s, so we should use H in Joules per gram, not ergs per gram to get the units right.

$$t \sim \frac{4\times 10^4\text{J g}^{-1}\times 2.7\text{g cm}^{-3}\times(10^7\text{cm})^2\times 2\times 10^{-6}\text{ohm cm}}{(10^5\text{Volt})^2\times f} = 2\times 10^4\text{s}\frac{0.1}{f}$$

$\boxed{\text{About 6 hours for } f = 0.1.}$

4. **Electrical Conductivity of Water. (10 points: 7+3)**

   Estimate the electrical conductivity of:

   a) Pure water. This has a pH=7. That is, it has a concentration of about 10^{-7} moles of H^+ and OH^- ions per liter. Hint: Assume that each ion is enclosed in a cage of water molecules. Show that Stoke's drag (viscous drag at low Reynold's number, like the marbles in corn syrup demo) limits the movement of these assemblages in an applied electric field.

   Solution

   Model the moving ion as a sphere of mass M and radius R. Recall the discussion of viscous drag from lecture 3. Using the Buckingham Pi theorem, or (with not entirely negligible dimensionless factors given several times in class lectures), the Stokes' (viscous, low Reynold's number) drag force on the sphere is[2]

$$F_{drag} = M\frac{dv}{dt} = 6\pi\rho R\nu v = \frac{9}{2}Mv\frac{\nu}{R^2}, \tag{3}$$

   This force on the ion is balanced by the electromagnetic force due to the electric field E, so

$$eE \sim \frac{9}{2}M\nu\frac{v}{R^2}. \tag{4}$$

   when the terminal velocity v is reached. The current density for the ion pairs (with a single bare charge) is

$$J = N_\pm e(v_+ - v_-) = \frac{2}{9}\frac{N_\pm e^2}{\nu}\left(\left.\frac{R^2}{M}\right|_+ + \left.\frac{R^2}{M}\right|_-\right)\cdot E, \tag{5}$$

   with $N_\pm$ the number densities of the positive and negative ions and ν the kinematic viscosity of water. The conductivity is thus

$$\sigma \sim \frac{2}{9}\frac{N_\pm e^2}{\nu}\left(\left.\frac{R^2}{M}\right|_+ + \left.\frac{R^2}{M}\right|_-\right). \tag{6}$$

[2] Aside from the factor of 9/2, you can get the Stokes drag force using dimensional analysis informed by the physical consideration that the drag force should scale linearly with the viscosity. Low Reynold's number drag, including the factor of 9/2, was discussed in the class lecture.

Remember that the ions are surrounded by a cage of water molecules (due to the polarity of water molecules). If each ion carries with it a cage of 6 H_2O molecules, the mass of the charge carrier is dominated by the mass of the cage: $M \sim 6 \times 18 \times m_p$. Let the effective radius of the cage be $R \sim 6\text{Å}$. The viscosity of water is $10^{-2}\text{cm}^2\text{s}^{-1}$.
Since the pH of water is 7, (10^{-7} moles/liter), the ion density is $N = N_A \times 10^{-\text{pH}}/(10\text{cm})^3 \sim 6 \times 10^{13}$ particles cm^{-3}.
Inserting these numbers into our expression for the conductivity, we find[3] that the electrical conductivity of pure water is $\boxed{\sigma \sim 1.2 \times 10^4 \text{ s}^{-1} \sim 1.4 \times 10^{-8}/(\Omega \text{ cm})}$. (This is a factor of 4 smaller than the standard value for pure degassed water at 20C of $5 \times 10^{-8}/(\Omega\text{cm})$. [though there recently has been controversy about whether this truly applies to pure water, or an even higher value: https://pubs.acs.org/doi/pdf/10.1021/jp8037686
Note that distilled water in equilibrium with atmospheric air has pH=5.7 and a higher conductivity than this, because the equilibrium amount of CO_2 in the water (10^{-5} molar) creates ions of carbonic acid: HCO_3^- and H^+ at a molar concentration of 1.9×10^{-6}.)

b) Sea water. The mass of salt per unit mass of seawater is 0.035.

Solution
In this case, the salt is dissolved and almost completely ionized. The mean weights of Na and Cl are 22.99 and 35.45, respectively. Sea water has a density of 1.025 g cm^{-3}, giving a number density of order $N_\pm = 3 \times 10^{20}$ ions/cm^3. The mass of the charge carriers, however, is still dominated by the cage mass; similarly, the size of the ion plus the cage is about the same as for pure water. Therefore, the conductivity is about $\boxed{\sigma \sim 4 \times 10^{10}\text{s}^{-1} \sim 0.04\ /(\Omega \text{ cm})}$, which is very close to the actual conductivity of seawater=0.05/(Ω cm) at 20C.

5. **Electromagnets (10 points: 5+5)** Estimate the strongest magnetic field that one could make by passing an electric current though copper. Consider in turn limitations due to:

a) the yield strength of copper.

Solution:
Consider a straight cylindrical wire of radius R and length L carrying current density $\mathbf{J}$. By Ampère's law, there is a circular magnetic field $\mathbf{B}$ inside. There is a force density acting on the current due to this field $\mathbf{f} = \frac{1}{c}\mathbf{J} \times \mathbf{B}$, in analogy with the Lorentz force law. This force points radially inward and produces a stress $\sigma \sim f(\pi R^2 L)/(2\pi RL) \sim JBR/2c$. For a given field, the current density can be estimated from $\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$ to give $J \sim Bc/4\pi R$. In terms of field only, the stress is

$$\sigma \sim \left(\frac{Bc}{4\pi R}\right)\left(\frac{BR}{2c}\right) = \frac{B^2}{8\pi}.$$

At the yield stress σ_Y, the field must be $B \sim \sqrt{8\pi\sigma_Y}$. Or, said another way, the **components of the Maxwell stress tensor** $\sim B^2/4\pi$ **cannot exceed the yield stress of the material carrying the currents that produce them.** For **copper** with $\sigma_Y \sim 10^9$ dyn cm^{-2} ($\epsilon_Y \sim 10^{-3}$, $E \sim 10^{12}$ dyn cm^{-2}) this gives $\boxed{B \lesssim 150 \text{ kG}}$.

b) the ability of copper to conduct away the power dissipated by ohmic loss.

[3] Recall that the cgs units for conductivity are s^{-1}, with 1 s $= 9 \times 10^{11}\Omega$ cm.

Solution:
Ohmic dissipation in a cylinder of area $A = \pi R^2$ and length L for material with electrical conductivity σ_c carrying current density J is, in Gaussian units (c is the speed of light)

$$\left.\frac{dE}{dt}\right|_{\text{GAIN}} \sim \frac{J^2}{\sigma_c} \cdot LA \sim \frac{c^2}{16\pi\sigma_c} B^2 \cdot L.$$

A temperature gradient of $\Delta T/R$ enables heat conduction (thermal conductivity K_H) to conduct away the dissipated energy at a rate

$$\left.\frac{dE}{dt}\right|_{\text{LOSS}} \sim K_H \frac{\Delta T}{R} \cdot 2\pi RL \sim 2\pi K_H \Delta T \cdot L.$$

Equating the two expressions, one obtains a limiting field of

$$\boxed{B \lesssim \left(32\pi^2 \frac{K_H \Delta T \sigma_c}{c^2}\right)^{1/2}.} \tag{7}$$

The Purcell sheet gives $\boxed{K_H \sim 1 \text{ cal s}^{-1}\text{cm}^{-1}\text{K}^{-1} \sim 4 \times 10^7 \text{ erg s}^{-1}\text{cm}^{-1}\text{K}^{-1}}$, which is correct for copper at $0 - 100$ C. It also gives the electrical conductivity for copper at room temperature $\sigma_c = (2 \cdot 10^{-6}\text{ohm cm})^{-1}$. The conversion to Gaussian units is also conveniently on the Purcell sheet, $1 \text{ ohm}^{-1} = 9 \times 10^{11} \text{ cm s}^{-1}$ [or you could notice that σ in Gaussian units is replaced by $\sigma/4\pi\epsilon_0 = 9 \times 10^9 \sigma$ in SI units ((ohm-m)$^{-1}$). Either way, we get $\boxed{\sigma_c \sim 5 \times 10^{17}\text{s}^{-1}}$ for copper at room temperature. **The maximum ΔT we could have would be** $\boxed{\Delta T \sim T_m - T_{\text{room}} \sim 10^3 \text{ K}}$, where T_m is the melting temperature of copper. Inserting these values naïvely into equation 7, we find $B \lesssim 80\text{kG} = 8$ T. **However the thermal and electrical conductivities near T_m would both be reduced by a factor ~ 4 due to increased phonon scattering. So to maximize B, it is probably better to accept a smaller ΔT of ~ 300K and keep the conductivities within a factor of 2 of their room temperature value.** Accounting for this, the **maximum fields** would be $\boxed{\sim 40 \text{ kG} \sim 4 \text{ T}}$ for copper. In fact, as of 2015, the highest continous magnetic fields ever produced on earth are in the Bitter electromagnet at Radboud University in the Netherlands, which reaches 380kG (38 T). This manages to exceed our solid-copper estimate by forcing cooling water through numerous holes drilled in stacks of thin copper plates held together by high-strength steel posts. This eludes our solid-block heat conduction limit by making the scale of temperature gradients much smaller than the overall diameter of the current-carrying plates, and gets to the structural strength limit. It consumes 17 MW of electric power!

[The following is included for interest, not expected in your solutions:] To simplify the above expression and see how it relates to fundamental constants, one may make use of the Wiedemann-Franz Law relating the thermal and electrical conductivities, K_H and σ_c respectively:

$$K_H = \frac{\pi^2}{3} \frac{k^2 T}{e^2} \sigma_c,$$

so that equation7 becomes $B \lesssim 4\sqrt{6} e K_H / ck$, independent of temperature at high T.

To relate the answer to fundamental constants, one may estimate the mean free path of the electron as being limited by electron-phonon scattering ($\Rightarrow \lambda \sim aE_b/kT$, where a

is the atomic dimension and E_b is the atomic binding energy). Mobile electrons at the edge of the Fermi sea have an energy $E_f = \hbar^2/2m_e(3\pi^2 n)^{2/3}$ ($n \sim 1/a^3$ is the number density of atoms), and contribute a specific heat (per unit volume) $c_V \sim nk \cdot (kT/E_f)$. Hence, the thermal conductivity can be written

$$K_H \sim c_V \cdot \nu \sim nk\frac{kT}{E_f} \cdot \lambda v_f \propto \frac{m_e^2 e^6 k}{\hbar^5},$$

independent of temperature.

Using the above formula for the limiting field, one obtains a relation

$$\frac{B^2}{8\pi} \leq \frac{12}{\pi}\frac{e^2 K_H^2}{c^2 k^2} \sim \alpha^2 E\left(\frac{E_b}{\mathrm{Ry}}\right),$$

where $\alpha = e^2/\hbar c \sim 1/137$, E is the bulk modulus and $E_b/\mathrm{Ry} = O(1)$ for metals. Since the yield stress is typically $\sigma_Y \sim 10^{-3}B > \alpha^2 B$, cooling provides a stronger constraint than ultimate strength.

Problem Set 6

Due **in Canvas 11pm Wednesday, 17 May 2023**

Homework Problems: (point values indicated: 50 points total)

1. **Pencils. (8 points: 4+4)**

   a) How many atoms thick is the graphite layer left by a pencil writing on a piece of paper? [Depending on your background, you may answer this theoretically, or by performing an experiment, or both]

   Solution

   Performing an experiment I find that drawing 100 lines of 15 cm$\times \sim 1/2$ mm wears away 1 mm of pencil lead that is 1/2 mm in diameter. The volume of lead worn away is $\sim .2$ mm^3 and the area covered is ~ 7500 mm^2, so that the average thickness of the layer is $t = 0.2/7500 = 3 \times 10^{-5}$ mm. Graphite has density of about twice that of water so that graphite has number density approximately

$$2\text{g cm}^{-3} \times \frac{\text{nucleons}}{2 \times 10^{-24}\text{g}} \times \frac{\text{atoms}}{12\,\text{nucleons}} = 8 \times 10^{22}\text{atoms cm}^{-3}\,. \tag{1}$$

   Thus the mean separation between atoms is roughly $a = (8 \times 10^{-22}\text{ atoms cm}^{-3})^{-1/3} = 2 \times 10^{-8}\,\text{cm}$. Hence there are about $t/a \sim 100$ atoms per layer[1].

   Theoretical answer: Use the results given in class for the interaction of classical oscillators with radiation: The frequency-integrated cross-section for an oscillator coupled to electromagnetic radiation is

$$\int \sigma\, d\nu = \frac{\pi e^2}{m_e c} f\,,$$

   where $f = 1$ per electron classically, and is $\mathcal{O}(1)$ for strong quantum mechanical transitions. To look black, the graphite must absorb all colors at least over the visible spectrum of light, and likely a bit beyond, say $\Delta\nu \sim 1.5 \times 10^{15}$Hz. Since for graphite we have one of the weak out-of-plane π bond electrons per carbon atom, we expect $\boxed{\sigma \sim (\pi e^2/m_e c\Delta\nu) \sim 2 \times 10^{-17}\text{cm}^2.}$

   Graphite has density of about twice that of water so that carbon in graphite has number density approximately

$$n \sim \frac{2\text{ gm}}{\text{cm}^3} \times \frac{\text{nucleons}}{1.7 \times 10^{-24}\text{ gm}} \times \frac{\text{atoms}}{12\text{ nucleons}} = 10^{23}\text{ atoms cm}^{-3}\,. \tag{2}$$

   Thus the mean separation between atoms is roughly $a \sim n^{-1/3} \sim (10^{23}\text{atoms cm}^{-3})^{-1/3} = 2 \times 10^{-8}\,\text{cm}$.

[1] Minor correction: pencil leads are not pure graphite: they are a mixture of graphite, clay and wax. The common No. 2 HB pencil is in mass percentage, 68% graphite, 26% clay, and 5% wax, so our layer actually had 68 graphite atoms, not 100. Drawing enthusiasts know that there are 20 grades of drawing pencils, ranging from the "hardest" 9H: 41% graphite, 53% clay and 5% wax, to the "softest" and blackest 8B: 90% graphite, 5% clay and 5% wax. The common HB writing pencil is in the middle of the range of drawing pencil grades.

For the pencil mark to look gray if not black, we want an optical depth $n\sigma t$ of order 1 ($e^{-1} = 0.37$ transmission). Then $\boxed{t > 1/(n\sigma) \sim 5 \times 10^{-7}\,\text{cm},}$ so the layer must be $\boxed{t/a \gtrsim 25 \text{ atoms thick.}}$

Check: Since the atoms in graphite are coupled into a sheet, once an electron is excited, it becomes part of a conduction band, and can wander over the whole sheet, exciting vibrations (phonon coupling).

An elegant theorem for graphene (an isolated single, 2-D sheet of graphite), for which the band structure is very simple, gives the fraction of light[2] absorbed in going through a single layer to be $\pi\alpha = \pi/137 = 0.023$[3] ($\alpha = e^2/\hbar c$ is the dimensionless fine structure constant). Thus $1/0.023 = 43$ graphene layers are required to make an optical depth unity (reduce the transmitted power to $1/e$ of the incident power). We care about higher photon energies, and graphite does have some coupling between the layers, so this elegant result doesn't apply exactly to our optical transitions or to graphite, so we consult experiment: *Optical Properties of Graphite*, by E.A. Taft and H.R. Philipp 1964 *Phys Rev* 138, pp A197-A202 gives the complex dielectric constant $\epsilon \sim 4 + 8i$ in the optical region. Thus the index of refraction $n \sim 2.5 + 1.6i$, and the power $E^2 \sim \exp(-4\pi n_i z/\lambda_{vac})$, so the absorption length for e^{-1} power transmission is for green light $t = \lambda_{vac}/4\pi n_i = 5 \times 10^{-5}\text{cm}/(4\pi \times 1.6) \sim 2.5 \times 10^{-6}\text{cm}$, about 5 times larger than our simple classical oscillator estimate, and 2.8 times larger than the elegant theorem estimate.

b) How many words can a standard No. 2 pencil write (assume you don't waste lead by over-sharpening)?

Solution

A No. 2 pencil has diameter of lead ~ 2 mm and is $l \sim 15$ cm long which gives it a usable volume of lead of $\sim 500\ \text{mm}^3$. However people usually keep pencils sharpened to $d = 1/2$ mm points, so the usable volume of lead is only about $ld^2 = 40\,\text{mm}^3$. Thus a total area $ld^2/t \sim 10^6\,\text{mm}^2$ can be drawn. Assuming each word consists of a line $\sim 5\,\text{cm}$ long by 1/2 mm wide, each word has an area of $25\,\text{mm}^2$, so that $\sim 50{,}000$ words (one or two hundred pages) can be written if the whole pencil is used up and you don't unnecessarily reduce the length of the pencil by over-sharpening.

2. **Candles. (10points: 5+5)** Since 1979, the candela has been defined as a luminous intensity of 1/683 Watts per steradian at frequency of 5.4×10^{14}Hz (radiation at other frequencies is weighted according to the human eye sensitivity). This curious unit was chosen to match an earlier definition of the visible light intensity from the International Standard Candle [4], 25 cm long, 2.2 cm diameter, 75 g, made of wax from the head of sperm whales and a standardized wick.[5]

   a) If all the energy released by a burning candle were emitted as visible ($\lambda = 550\,\text{nm}$) photons, how long would a standard candle burn?

[2] at photon energies $\lesssim$ 1eV, where individual electronic transitions can be neglected and the bands dominate.

[3] Kuzmenko, A., van Heuen, E et al 2008 Phys Rev Lett 100, 117401.

[4] Defined by British Parliament in the Metropolitan Gas Act of 1860, and adopted as an international unit in 1883. Yes this was the first use of this term, now often used in astronomy!

[5] Spermaceti candles were advertised to "produce the cleanest and most beautiful flame of any substance that is known in nature, detering darkness and consequent robberies, burglaries and murders in your streets." The Ivy League Brown University was founded in 1764 by Obadiah Brown and family, endowed from profits from his spermaceti candle factory, which dominated the US market in the 1750's. Interesting articles on whaling and the spermaceti candle industry are here and here.

Solution

Wax is a hydrocarbon like gasoline or fat, so from class or the Purcell sheet, the heat of combustion of the candle is about 10^4cal/g$\approx 4 \times 10^4$J/g, so a 75 g standard candle will release about 3×10^6J. The visible light (at 555nm green light wavelength) luminosity of the candle is 4π candela-steradians (i.e. 4π lumens), so the visible light power output is $4\pi/683 = \frac{1}{55}$W. Therefore, a standard candle radiating only visible light would burn for 3×10^6J$/((1/55)$W$) \approx 2 \times 10^8$s $\approx$ 6years.

b) Infer the efficiency of a candle as a light source.

Solution

The "ideal" candle calculated in the previous part burns for $\sim 2 \times 10^8$s. As should have been familiar to you, a real candle only burns for several hours. The candle demonstrated in class burned down an inch (2.5 cm) in an hour, so a 10-inch standard candle would burn for 10 hours $\sim 4 \times 10^4$s. So the candle's radiative efficiency is only the ratio of those timescales, $\approx 2 \times 10^{-4}$.

Equivalently, the candle is releasing 3×10^6J in 10 hours, or a total power (mostly in heat) of 3×10^6J$/(10 \cdot 3600$s$) = 83$W, while producing a perceived luminosity corresponding to $4\pi \cdot (1/683)$W $= 1/55$W of green light, so the efficiency of a candle for visible illumination is only $(1/55)$W$/83$W $= 2 \times 10^{-4}$.

Figure 1: Chandeliers with candles. The three chandeliers together with a total of $\sim$ 100 candles provide about the same room illumination as a single 75-100 W incandescent bulb, or a single 6-8 W LED bulb. Be glad you don't have to replace and light each of those candles every day, and clean up their wax drippings. Or in summer, sweat under the 10 kW overhead heat source.

We can compare this to a typical 75 W incandescent bulb: my old box of them says they produce 1050 lumens, equivalent to $1050/4\pi = 84$ candles, but about the same total power use (75 W vs 83 W) as a single candle! The room shown in Figure 1 had about the same illumination as a single electric light bulb. So in terms of global warming, Edison's incandescent electric light was a factor ~ 100 improvement over candles! Still, the efficiency of the 75 W incandescent bulb is only $1050 \cdot (1/683)\mathrm{W}/(75\mathrm{W}) = 0.02$. My Phillips T8 LED tube lights (5000 K color temperature) use 10W and produce 1600 lumens, for an efficiency $1600 \cdot (1/683)\mathrm{W}/(10\mathrm{W}) = 0.23$. So they are another order of magnitude more efficient than incandescent bulbs.

3. **Tumbling cards. (10 pts: 7+3)** Try the following experiment: take a standard 3×5 inch index card[6]. Holding it carefully horizontally, drop it, without tipping. The card will always begin to wander and tumble as it falls. But four cards taped together always fall steadily downward, their plane remaining horizontal.

   a) Why? What dimensionless quantity or quantities determine whether the falling card tumbles or not? Test your hypotheses with pieces of foil or cardboard you may find or cut out of cereal boxes. [hint: there are several approaches to this problem, but it may be helpful to be aware of the von Kármán vortex shedding instability, illustrated here `https://en.wikipedia.org/wiki/Karman_vortex_street` and here `https://en.wikipedia.org/wiki/Vortex_shedding`. Unstable vortex shedding occurs only for Reynolds number $\mathrm{Re} \gtrsim 50$. Another example of a vortex street is in the first photograph of the oil slick `https://en.wikipedia.org/wiki/2010_Great_Barrier_Reef_oil_spill` behind a Chinese ship which in 2010 illegally entered the Great Barrier Reef Marine Park. After scraping a 3 km long scar in the coral, it got stuck, and began leaking fuel oil, which gave a very visible tracer of the flow of the ocean current past the ship.]

   Solution

   Method 1: very simple estimate: A card of density ρ_c, thickness t, surface density $\Sigma = \rho_c t$ and size L should tumble if the mass of air (density ρ_a) perturbed by the falling card, $\rho_a L^3$, exceeds the mass of the card, ΣL^2, i.e. if $\Sigma < \rho_a L$, i.e. if $t/L < \rho_a/\rho_c$ (or equivalently $\boxed{\text{tumbling will occur if both } Fr \equiv t\rho_c/(L\rho_a) \lesssim 1}$) $\boxed{\textit{and} \text{ if } \mathrm{Re} = Lv/\nu \gtrsim 50}$, so that the airflow past the card at the fall speed v (calculated below) is unstable and oscillates from side-to-side in a von Kármán vortex street. In this limit, the card is just a tracer of the unstable airflow. Thicker, heavier cards will have more mass than that of the air they are perturbing, so the unstable airflow will not be able to change their orientation or trajectory much. Since $\rho_a/\rho_c \sim 10^{-3}$, we conclude that tumbling in air will occur if $t \lesssim 0.01\mathrm{cm}(L/10\mathrm{cm})$. This is about half the actual condition determined from experiments on index cards and sheets of cardboard and paper. See below for check that $\mathrm{Re} \gg 50$ on earth. Note that for solids like glass, aluminum or flat pebbles (all $\rho_c \sim 3\mathrm{g\ cm}^{-3}$) falling in *water*, the $Fr \lesssim 1$ condition is $t \lesssim 0.3L$, so e.g. even a fairly thick cell phone dropped into water will tumble as it sinks.

   Method 2: estimate with more physical detail: ask —will the torque due to the different ram pressure on the two sides due to a vortex shed on one side be able to turn the card over before another random vortex shed from the opposite side reverses the sign of the torque? Consider a card of density ρ_c, thickness t, length L and width $W \sim L$ falling in air of density ρ_a. It's fall speed (terminal velocity) is given by

$$mg = (\rho_c L W t)g = (c_D/2)\rho_a L W v^2$$

[6] 3×5 inch means 7.5cm $\times$ 12.5cm. My index cards are 0.02 cm thick, twice the thickness of printer paper.

A flat rectangular plate perpendicular to flow has $c_D \sim 1.2$, so set $c_D/2 \sim 1$. Then we get $v^2 = (\rho_c/\rho_a)gt$ and $v \sim 10^2(t/0.02\text{cm})^{1/2}\text{cm s}^{-1}$. $\text{Re} = vL/\nu \sim 10^4$, easily in the vortex shedding regime. We used from the Purcell sheet, that the kinematic viscosity of air is $\nu = 0.2\text{cm}^2\text{s}^{-1}$ (his quoted dynamic viscosity divided by the density of air, same as the diffusion coefficient also given). The moment of inertia about axis along W direction is $I = (1/12)\rho L^3 W t$. The frequency f of vortex shedding is given by $\text{St} = fL/v \sim 0.2$ for $\text{Re} > 50$. So timescale between opposite vortex sheds $T_v \sim (1/2f) \sim 2.5L/v$. The net torque due to the difference of pressure between the two sides after a vortex shed will be about

$$\tau \sim (L/2)(\rho_a v^2/2)(LW/2)$$

which is also $\tau = I\dot{\Omega}$. The plate will turn over if $\dot{\Omega}(T_v)^2/2 > \pi/2$, i.e. if $\pi/2 \lesssim (\tau/I)T_v^2/2$. Using the values for τ, I and T_v already given, we find that the condition to tumble is approximately $\boxed{4 \lesssim (L/t)(\rho_a/\rho_c)}$ where the 4 is not to be taken more seriously than a factor of 3 either way.

Problem first studied by Maxwell 1853!
Willmarth, Hawk & Harvey 1964 Phys Fluids 7, 197 https://deepblue.lib.umich.edu/bitstream/handle/2027.42/71206/PFLDAS-7-2-197-1.pdf
Jones & Shelley 2005 J Fluid Mech 540, 393, doi:10.1017/S0022112005005859 https://www.math.nyu.edu/faculty/shelley/papers/JS2004.pdf
Pesavento 2006 PhD thesis! https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.962.9476&rep=rep1&type=pdf and associated journal paper: Andersen, Pesavento, and Wang [J. Fluid Mech., vol. 541, pp. 65-90 (2005)] https://dragonfly.tam.cornell.edu/publications/S0022112005005847a.pdf

b) What would be maximum thickness a slab of aluminum (e.g. a cell phone) could have to tumble while falling in water? What about for an index card falling on Mars $\rho_{\text{Mars}} \sim 10^{-2}\rho_\oplus$, $g_{\text{Mars}} \sim 0.5g_\oplus$?

Solution

In water on earth: Note that for solids like glass, aluminum or flat pebbles (all $\rho_c \sim 3\text{g cm}^{-3}$) falling in *water*, the $Fr \lesssim 1$ condition is $t \lesssim 0.3L$, so e.g. even a fairly thick cell phone dropped into water will tumble as it sinks.

In the atmosphere of Mars:
Since kinematic viscosity is approximately mean free path times molecular speed (roughly $\sqrt{kT/m}$) and the sizes of molecules in Mars' atmosphere (mainly carbon dioxide) are roughly the same size and cross-section σ as those on earth, the mean free path $\sim m/(\rho_a\sigma)$ of molecules there will be about 100 times larger than on Earth since the density is 100 times lower. Since the molecules have about the same mass as those in Earth's atmosphere, and the daytime temperature only a little colder, the viscosity of Mars' atmosphere will be about 100 times that of Earth's: i.e. $\nu \sim 20\text{cm}^2\text{s}^{-1}$ on Mars. Since Earth and Mars are made of similar rock, and $g \propto M/R^2 \propto \rho R$, and the radius of Mars is about half that of Earth, $g_{\text{mars}} \sim g_\oplus/2$. The tumbling criterion $\rho_a/\rho_c(L/t) \sim 1$, for $\rho_c \sim 1\text{g cm}^{-3}$, with the Martian $\rho_{a,\text{mars}} \sim 10^{-5}\text{g cm}^{-3}$, requires $t \sim 10^{-5}L \sim 10^{-4}\text{cm}$ for $L \sim 10\text{cm}$ cards, painfully thin (less than 1/10 the thickness of aluminum foil). Ordinary index cards on Mars will fall steadily, without tumbling.
What about the extraordinarily thin ones that the simple criterion suggests might tumble on Mars? For $t \sim 10^{-4}\text{cm}$, the terminal velocity $v_{\text{mars}} \sim (gt\rho_c/\rho_{a,\text{mars}})^{1/2} \sim 70\text{cm s}^{-1}$,

about the same as on earth. But because of the higher viscosity of Mars' atmosphere, $\mathrm{Re} = vL/\nu \sim 35$. If this OoM estimate is taken seriously, it is just below the $\mathrm{Re} \sim 50$ above which vortex shedding occurs. So the card might be in steady laminar flow and not tumble at all. However, the estimate for t is just OoM, so the factor of 1.4 in Re is not to be taken too seriously. They are close to the threshold for being able to tumble at all.

4. **IceCube. (10 points)** The IceCube high-energy neutrino detector in operation at the South Pole was constructed as follows: a high pressure jet of hot water was used to melt holes in the Antarctic ice (temperature ~ -50 C at depth). The holes were 2.5km deep and 0.5m in diameter. Then 60 photomultiplier tubes and computers were lowered into each hole. Estimate how long the physicists had to lower the equipment into the hot water in the hole before it froze again. [hint: this is a problem in heat conduction in the cylindrical radial direction, not the vertical one!]

Solution:

We know from the lectures that the characteristic time-scale, t, of heat transfer is given by $\kappa \sim L^2/t$ where for insulators $\kappa \sim 10^{-2}\mathrm{cm}^2/\mathrm{s}$. The appropriate L to use is here isn't as simple as the radius of the hole. The physical picture of the problem is that the hot water in the hole must first cool to 0C, and then lose the latent heat of fusion to turn it into ice. Thus the total thermal energy that must be transferred to the surrounding cold ice to refreeze the hole will be

$$Q = (c_{\mathrm{water}} T_{\mathrm{water}} + L_{\mathrm{fus}})\rho_{\mathrm{water}} \pi R^2 \ell \tag{3}$$

where T_{water} is the initial temperature of the hot water, c_{water} is the specific heat of water, R is the radius of our hole and ℓ is the length. If we assume that the hot water is at 50C then $c_{\mathrm{water}} T_{\mathrm{water}} \sim 50\mathrm{cal/g}$ while for water $L_{\mathrm{fus}} \sim 80\mathrm{cal/g}$ so the latent heat of fusion is about as important as heating the water. The characteristic radius in the ice from which this is supplied is the appropriate L for our diffusion timescale. This will roughly be given by equating the energy put into the surrounding ice to increase its temperature by 25C (halfway from the mean south pole temperature of –50C to freezing 0C) to equation 3

$$\pi L^2 \ell c_{\mathrm{ice}} \rho_{\mathrm{ice}} \tfrac{T_{\mathrm{ice}}}{2} \sim (c_{\mathrm{water}} T_{\mathrm{water}} + L_{\mathrm{fus}})\rho_{\mathrm{water}} \pi R^2 \ell$$
$$\rightarrow L \sim R\sqrt{\tfrac{2(c_{\mathrm{water}} T_{\mathrm{water}} + L_{\mathrm{fus}})}{c_{\mathrm{ice}} T_{\mathrm{ice}}}}$$

and so

$$t \sim \frac{R^2}{\kappa} \frac{2(c_{\mathrm{water}} T_{\mathrm{water}} + L_{\mathrm{fus}})}{c_{\mathrm{ice}} T_{\mathrm{ice}}}.$$

What temperature should we use for the surrounding ice? The diffusion length for 1 year in ice will be $\sqrt{\kappa t} \sim 560\mathrm{m}$ so for the majority of the length of the hole the appropriate temperature for the ice is roughly the mean temperature at the south pole, $\sim -50\mathrm{C}$. Calculating the timescale using a hot water temperature of 50C and $c_{\mathrm{ice}} \sim 0.5\mathrm{cal/g}$ gives

$$t \sim \frac{25^2}{10^{-2}} \frac{2(1 \times 50 + 80)}{0.5 \times 50} \sim 6.5 \times 10^5 \mathrm{s} \sim \boxed{7.5 \text{ days.}}$$

This agrees pretty well with the ~2 week freeze-in time quoted at https://icecube.wisc.edu/science/icecube/construction.

5. **Crossing streams. (12 points: 5+4+3)** In class, we showed that a fluid of density ρ flowing under gravity g down a slope $\alpha = dz/dx$ in a channel of depth d, with a flow velocity v of high enough Reynolds number $\mathrm{Re} = dv/\nu$ to be turbulent (ν is the kinematic viscosity, $\nu \sim 0.01\mathrm{cm}^2\mathrm{s}^{-1}$ for water), has an rms velocity given by

$$\rho g \alpha = f_D \frac{\rho v^2}{2d} \ , \tag{4}$$

where f_D is a dimensionless factor which would be of order unity for a "one–big-eddy" model of turbulence. More realistically, there is a turbulent shear layer in which eddies at a distance z above the bottom of the channel are limited to a scale $\sim z$, and if the channel bottom has a vertical roughness scale z_{rough}, the smallest relevant eddies have scale equal to the larger of z_{rough} or the Kolmogorov scale $\ell = d\ \mathrm{Re}^{-3/4}$ (at the cascade of turbulent eddies reaches $\mathrm{Re} \sim 1$).

$$f_D \simeq \max\left[\frac{1}{(\ln \mathrm{Re})^2} \ , \ \frac{1}{(\ln[d/z_{\mathrm{rough}}])^2}\right] \tag{5}$$

a) You are hiking along a stream-side trail of moderate slope (say 1 km elevation gain in 16 km), which crosses the stream. How deep a stream can you cross without being bowled over? Be sure to use a realistic coefficient of friction on the wet, and possibly leaf-covered or mossy rocks in the stream, and consider that the flow is turbulent (i.e. not steady).

Solution:
$v = \sqrt{4g\alpha d/f_D}$, with $f_D \sim 0.2$ for $z_{\mathrm{rough}} \sim 0.1d$, typical for a rocky stream bottom. Take $\alpha = 0.06$, a moderate stream-side trail slope (1km elevation gain in 16km). This gives $v = 180(d/30\mathrm{cm})^{1/2}\mathrm{cm\ s}^{-1}$. The horizontal drag force on two legs each of diameter $W =$ 15cm across water depth d is $F_D = (1/2)C_D\rho v^2 2Wd$. Take $C_D \sim 1$ (for a smooth cylinder at high Re, $C_D = 1.17$, which might be ok for bare legs. Legs wearing floppy pants will probably be higher). Thus $F_D = 1.5 \times 10^7(d/30\mathrm{cm})^{3/2}$dyne. Compare this to the frictional force of your feet on the rocky bottom, $F_F = \mu g m$, where m is your mass, and μ is the coefficient of friction ($\mu \sim 0.5$ for the static friction of rubber on wet concrete or rock, but $\mu \sim 0.03$ for the static friction of rubber on wet leaves or moss; the coefficients of sliding friction of course are somewhat less). $F_F = 6 \times 10^6(\mu/0.1)(m/60\mathrm{kg})$dyne. So you might be barely ok in 30 cm deep water with no moss, leaves or slippery mud on the rocks, but in big trouble if you hit a leaf or slippery patch. Remember also that once you are down, your cross-sectional area will be much larger than just that of two vertical legs, making it very difficult to get up again.

b) Consider a typical ravine in the San Gabriel mountains above Pasadena (e.g Eaton Canyon), draining a region a couple of km wide and several km long. Suppose it were steadily raining at a rate of 2 cm/hour. For a V-shaped stream bottom, what would be the steady-state depth of stream in such a rain? State any assumptions you make.

Solution:
It may take a while for ground to saturate and for small rivulets to reach the main stream, but once those conditions are met, the main stream must drain say 2km by 6km by 2cm per hour, or a volume flow of $\dot V \sim 7 \times 10^7\mathrm{cm}^3\mathrm{s}^{-1}$. In a not very realistic stream with a 45 degree V-shaped channel, depth d means width $2d$ and cross-sectional area d^2. Using the expression for flow velocity v from the previous part, and $\dot V = vd^2$, we have $\dot V = 1.6 \times 10^5(d/30\mathrm{cm})^{5/2}\mathrm{cm}^3\mathrm{s}^{-1}$, and thus $d \sim 350\mathrm{cm} \sim 3.5$m. More realistic channels will be V's of shallower angles, or eroded to more flat bottom over some area, and thus

wider than they are deep. During a rain storm, the river in Eaton Canyon is about 10m wide and 1-2m deep.

c) In that same ravine, estimate how long after the rain starts that it will take for the river to reach steady state (i.e. how long before the full steady-state flood reaches you).

Solution:

The flow time down the $L = 6$ km ravine in the 45 deg V model, with $d \sim 350$cm, hence $v \sim 6$m s^{-1}, and $t = L/v \sim 10^3$s. In actuality you may have somewhat more time, since 1) the channel is probably really wider and shallower, 2) the small slow rivulets have to reach the main channel, and 3) if the ground is dry, some water will have to percolate into the ground until it is saturated.

17 May 2023 **Ph 101** E.S. Phinney

Order of Magnitude Physics

Problem Set 7

Due **in Canvas 11pm Wednesday, 24 May 2023**

Homework Problems: (point values indicated: 50 points total)

1. **Sand ropes. (10 points: 3+4+3)** Although dry sand, and sand *saturated* with water both have negligible tensile strength, damp sand (only partially saturated) has finite tensile strength.

   a) Provide an explanation for this remarkable property of damp sand.
   Water wets sand, so it binds more strongly to sand than it does to itself. As a consequence of its surface tension, water located in the interstices between sand grains binds the grains together.

   If, on the other hand, the sand is not damp but saturated with water, there will be no water/air intersurface between sand grains - thus negligible tensile strength.

   b) Estimate the tensile strength of damp sand as a function of the radius of the sand grains. Consider two spherical sand grains of radius a bound by water in the gap between them. The water's surface tension pulls the grains together with a force $F \sim 2\pi r\gamma$, where γ is the surface tension coefficient and r the waist size of the water/air interface (see figure 1). For very big grains, the waist size r is significantly smaller than a (think of two basketball-sized sand "grains"). However for ordinary sand (a is roughly within the range of $0.1 \sim 1$mm) there is little difference between r and a. In the following we will assume $a \simeq r$.

   Suppose the sand grains are packed together, so across a planar section the number of grains per unit area is

$$N \sim \frac{1}{\pi a^2}$$

   The tensile strength is force/area, so we get

$$T \sim N \cdot F \sim \frac{2\gamma}{a} \sim 10^3 \text{dyn cm}^{-2} \left(\frac{a}{1\text{mm}}\right)^{-1}$$

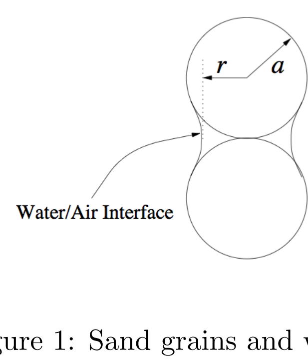

Figure 1: Sand grains and water.

c) How long a column of sand (from your favorite beach) could be hung vertically?
For a sand column to be hung vertically, the tensile strength must be large enough to counterbalance the gravity. Since per unit area, a column of length h weighs $w = \rho g h$, one has

$$\rho g h \sim T$$
$$\Rightarrow \quad h \sim \frac{T}{\rho g} \sim \frac{10^3 \text{dyn cm}^{-2}}{1\text{g cm}^{-3} \times 10^3 \text{dyn g}^{-1}} \left(\frac{a}{1\text{mm}}\right)^{-1} = 1\text{cm} \cdot \left(\frac{a}{1\text{mm}}\right)^{-1}$$

For $a = 0.1$mm, the column can be as long as 10cm.

Check: Personal experience making sand castles on trips to beaches with fine sand supports this estimate.

2. **Old water (10 points: 5+5).** The mass flux q of a viscous fluid like moving through a porous medium saturated with the fluid, in response to an applied pressure gradient dp/dx, depends linearly on dp/dx and may be written as

$$q = -K dp/dx$$

a) Derive an analytical expression for K. Assume that the porous medium can be entirely characterised by a typical pore size a some fraction of the typical grain size, assumed small (e.g. sand, silt or sandstone). Hints: Buckingham Pi theorem is your friend. Saturation means that the pore space is completely filled with fluid, so the total surface energy cannot change as the fluid flows.
Buckingham approach: $q/(dp/dx)$ (mass/area/time divided by energy/length^4) has units of time. To generate this, we may use the pore size a, the kinematic viscosity ν (units length^2/time) and the fluid density ρ. The only choice with the correct units is $\boxed{K = \Pi a^2/\nu}$, where Π is some dimensionless number, which might depend on the pore shapes, but nothing else. Since we were told to assume the pores are filled, surface tension is irrelevant (there is no change in the wetted area during flow through a saturated medium: however if the soil starts to dry out, removing water produces a large pore pressure resisting further flow out, as the surface tension energy of the increasing air-soil area increases).
Physical approach: we can think of a porous medium as being full of small tunnels of diameter a, with a fraction $f \lesssim 0.1$ of a any given cross-section being occupied by *connected* tunnels ("pores"). The viscous stress for a flow of speed u through such a tunnel is $\sigma \sim \rho\nu\partial u/\partial r \sim \rho\nu u/a$, and the viscous force per unit length retarding the flow is then $2\pi r\sigma \sim 2\pi\rho\nu u$. The force per unit length driving the flow is $\pi a^2 dp/dx$, and in steady state they should be equal, giving $u \sim -(dp/dx)a^2/\rho\nu$, and have a mass flux $q \sim \rho u$. The area-averaged mass flux (0 where there are no holes, and the above where there are), will thus be

$$\langle \rho u \rangle \sim -\frac{fa^2}{\nu}\frac{dp}{dx} \equiv \frac{\kappa}{\nu}\frac{dp}{dx} \equiv -K\frac{dp}{dx}$$

$\kappa \sim fa^2$ is called the permeability (units cm^2 or m^2), and $\boxed{K = \kappa/\nu \sim fa^2/\nu}$. In soil mechanics, $g\kappa/\nu = gK$ (where g is the usual gravitational acceleration on earth) is referred to as the hydraulic conductivity.

b) In the sandstone canyons of Utah and Colorado, there are springs in the lower levels of the canyons whence leaks out water which once fell on the plains 300 m above the canyon. The grain size of sandstone is that of fine silt, $\sim 3 \times 10^{-4}$cm, and pore spaces between the grains are about 1/10 this size. How long ago did the water now dripping from the springs fall on the plains? Hint: it is easier to consider a canyon with sloping walls than a narrow vertical slot canyon. You can also to the latter, but it will require considering rainfall as well.

Since silt and sandstone have grain sizes of order $\sim 3 \times 10^{-4}$cm, the pores are about 1/10 that size, and *connected* pores might occupy $f \sim 0.1$ of any given cross-section (lots of pores will be dead-ends, and thus not contribute to flow). So we estimate $\boxed{\kappa \sim fa^2 \sim 10^{-10}\text{cm}^2}$. From the Purcell sheet, water has $\rho\nu \sim 10^{-2}$dyn s cm^{-2}, and thus $\nu \sim 10^{-2}\text{cm}^2\text{s}^{-1}$.

The pressure squeezing the water to flow out of the lower canyon walls comes from the weight of the $h = 300$ m of rock above it (or, case 2: if the rock above the sandstone layer is more permeable, or cracked, just the weight of the water above it), which gives $p \sim \rho_{rock} gh \sim 10^8$dyn cm^{-2} (or in case 2: replace ρ_{rock} by $\rho_{water} \sim \rho_{rock}/3$). If this varies across a canyon with sides of slope angle $\alpha < 45°$, and thus half-width $h/\tan\alpha$ km, we have $dp/dx \sim 10^8\text{dyn cm}^{-2} \tan\alpha/3 \times 10^4\text{cm} \sim 3000 \tan\alpha \,\text{dyn cm}^{-3}$, and thus

$$v \sim \frac{\kappa}{\rho\nu}\frac{dp}{dx} \sim \frac{10^{-10}\text{cm}^2}{10^{-2}\text{dyn s cm}^{-2}} \times 3000 \tan\alpha \,\text{dyn cm}^{-3} \sim 3 \times 10^{-5} \tan\alpha \,\text{cm s}^{-1}$$

$\boxed{\text{So it will take the water} \sim 3 \times 10^4\text{cm}/(\tan\alpha)v \sim 10^9(\tan\alpha)^{-2}\text{s} \sim 30(\tan\alpha)^{-2}\text{y to cross}}$ the canyon. (or in case 2: about $300(\tan\alpha)^{-2}$ y). Typical values of $\alpha < 0.3$, so we are talking centuries to millenia. The case of slot canyons with vertical walls ($\alpha = \pi/2$) is trickier. In this case, the *initial* horizontal pressure gradient is infinite, so water near the canyon edge is quickly squirted out, leaving it non-saturated. The resulting enormous pore pressure due to surface tension (cf problem 1) then sucks water from farther from the edge. Recalling from the weather lectures that annual rainfall is typically 1 m/yr (or 0.2 m/yr in desert areas where famous canyons tend to be), filling up the say 30% pore space of a 300 m depth will take 100 y (typical rainfall) to 500 y (desert rainfall). Thus saturated pores in a canyon with vertical walls can only be maintained farther from the canyon edge than 300m (typical rainfall) to 1.5km (desert rainfall). Thus even vertical walled canyons behave more like ones with 45 degree (lush) to 11 degree (desert) walls. Real famous canyons like the Grand Canyon or Zion have many geologic layers of varying weather resistance and stability against crumbling, and are usually "stepped", i.e. a mix of vertical and sloped walls. **Check:** Radiocarbon dating of water coming out of seeps and springs in canyons of the west indicate ages of 300-13,000 years.

This problem is also relevant to oil recovery: the point of "fracking" is to use high pressure to crack the oil-bearing rocks, increasing their pore sizes, and also to pump down fluids to reduce the viscosity of the oil/fluid mixture. Both effects increase $K \propto a^2/\nu$, and thus allow far more oil to be extracted on a reasonable time.

3. **Room Acoustics (14 points: 4+6+4)** Sound waves in a room of volume V and surface area S, whose surfaces have an area-weighted mean acoustic absorption coefficient $\bar{\alpha}$, reverberate for a time T. The latter is conventionally expressed as T_{60}, the timescale for sound to decay by 60 dB (i.e. for the acoustic energy to drop by a factor of 10^6, say from a painfully loud 90 dB, to a faint 30 dB whisper). For 1 kHz waves, typical values of α (the fraction of

acoustic *power* incident on the surface, which is absorbed. The rest is reflected) for various materials are given in Table 3.

Material	1 kHz acoustic absorption α
stone, glass, concrete, plaster	0.05
wood	0.15
carpet	0.3
heavy curtains	0.5
acoustic ceiling tile	0.5
wood benches (occupied)	0.5
upholstered seats	0.88
fiberglass board	0.99
opening	1

a) Show (within a factor of two or so, and assuming $\bar{\alpha} < 0.5$), that

$$T_{60} \sim 0.16 \frac{V}{S\bar{\alpha}} \mathrm{s} \ ,$$

where V and S are in cubic meters and square meters respectively.
To approximate sound waves bouncing randomly in a rectangular room a room of width w, height h and length l, estimate the mean bounce frequency as the average of the frequency in each of the three principal directions,

$$\bar{\nu}_b = \frac{1}{3}\left(\frac{c}{w} + \frac{c}{h} + \frac{c}{l}\right) = \frac{c}{3}\frac{hl + wl + wh}{whl} = \frac{c}{6}\frac{S}{V} \ ,$$

where $c \sim 340\mathrm{m\ s}^{-1}$ is the speed of sound. [1] On each encounter, the magnitude of the acoustic flux will on average be reduced by a factor of $\bar{\alpha}$.
Consequently, the intensity of sound will decay as

$$I \propto \exp(-\bar{\alpha}\bar{\nu}_b t)$$

(Note: we assume each α is small enough that we can approximate $\ln[(1-\alpha)^N] \simeq -N\alpha$.)
Thus, defining $\exp(-\bar{\alpha}\bar{\nu}_b T_{60}) = 10^{-6}$, and using the expression above for $\bar{\nu}_b$, we find

$$T_{60} \sim \frac{6V\ln(10^6)}{S\bar{\alpha}c} = 36\ln(10)V/(c\bar{\alpha}S) \sim 0.25\frac{V}{\bar{\alpha}S}\mathrm{s/m} \ ,$$

This is just 1.5 times Wallace Sabine's famous formula given in the statement of the problem: $T_{60} = 4\ln(10^6)V/(c\bar{\alpha}S) = 0.16V/(\bar{\alpha}S)\mathrm{s\ m}^{-1}$. For meeting rooms and lecture halls, the ideal T_{60} reverberation time is 0.6 to 0.8 seconds (i.e. T_{20} of about 0.2 seconds); longer reverb times make speech hard to understand by overlapping syllables: a typical speech rate is 4-6 syllables per second (2–2.5 words per second). Notice that in a typical 5–10m meeting room, the kHz frequencies most important in human speech are reflected dozens of times, so all speakers sound similarly loud, and the sound level from all speakers is considerably amplified over what it would be outdoors. For concert

[1] Note that if one of w, h or l is substantially smaller than the others (e.g. a large room with a low ceiling, or a long, high cathedral with a narrow nave), then V/S is approximately one half the length of that smallest dimension L, which will dominate the bounce frequency

halls where classical music is played, reverberation times of about 2 seconds are optimal (Boston Symphony Hall, designed by Sabine, and widely considered the finest concert hall in the US, has $T_{60} = 1.9$s, as does Disney Hall in Los Angeles). Sacred choral and organ music (e.g. Tallis, Handel, Bach, etc) was designed for large stone cathedrals with even longer reverberation times. Rooms with reverberation times of less than 0.3 seconds are considered acoustically "dead" (The astronomy classroom, 219 Cahill, is an awful example).

b) At low frequencies a room has well-defined acoustic normal modes, with frequencies $f_{l,m,n}$ defined by integer 'quantum numbers' l, m, n, that satisfy the Helmholtz equation $\nabla^2 p + (2\pi f_{l,m,n}/c)^2 p = 0$ plus the boundary conditions at the walls. Above what frequency f_s do identifiable normal modes no longer exist? Hints: you will need to consider reverberation time, the bandwidth of a finite-duration wave train (e.g., the Fourier transform of a Gaussian wave packet), and the the 'mode counting in a box' arguments used in class to derive black body radiation and the specific heats of solids.

We think of the sound field as made of many Gaussian-modulated wave packets at frequency f and $1/e$ duration (4dB in power, i.e. $1/e^{1/2}$ in rms pressure) $T_4 = 1/\ln(10^6)T_{60} = 0.072T_{60}$. Defining the Fourier transform in frequency terms $\exp(-i2\pi ft)$ (not radian frequency $\exp(-i\omega t)$, the uncertainty principle relates the variance in frequency to that in time: $\sigma_f^2 = 1/(2\pi\sigma_t)^2$. More realistic (non-Gaussian) sound packets will have larger σ_f by a factor of order unity.

Thus each wave packet has a frequency bandwidth of *at least*

$$\sigma_f = 1/(2\pi T_4) = 2.2/T_{60} = \frac{2.2}{4\ln 10^6}\frac{c\bar{\alpha}S}{V} \ ,$$

where in the last equality we used Sabine's formula from the previous part for T_{60}. Counting modes (with zero boundary conditions) in a box of volume V, as we did in class (for specific heat of a crystal lattice, and black body radiation), the number of modes with wavenumber between k and $k + dk$ is

$$\frac{dN_{\rm mode}}{dk}\,dk = \frac{1}{2\pi^2}Vk^2\,dk \ .$$

In frequency terms, using $k = 2\pi f/c$,

$$\frac{dN_{\rm mode}}{df}\,df = 4\pi\frac{V}{c^3}f^2\,df \ .$$

When some modest number $\zeta \gtrsim 3$ of modes overlap in frequency, the sound spectrum will no longer have identifiable normal modes, but just become a random field of waves. Taking $df = \sigma_f$ estimated from the Sabine reverberation time, we have

$$\zeta = \frac{dN_{\rm mode}}{df}\,df = 4\pi\frac{V}{c^3}f^2\frac{2.2}{4\ln 10^6}\frac{c\bar{\alpha}S}{V} \sim 0.5\frac{f^2\bar{\alpha}S}{c^2} \ .$$

So the sound field will become random as modes overlap for $\boxed{f > f_c = (\zeta/3)^{1/2}c/(\sqrt{\bar{\alpha}S/6})}$, or equivalently, for wavelengths shorter than $\boxed{\lambda < \lambda_c = (3/\zeta)^{1/2}\sqrt{\bar{\alpha}S/6}}$.

This frequency is known as the *Schroeder frequency* (Manfred R. Schroeder 1954, "Die statistischen Parameter der Frequenzkurve von grossen Räumen," Acustica, 4, 594), often expressed numerically in terms of Sabine's reverberation time as $f_c = 2000\sqrt{T_{60}/V}$, where V is in cubic meters and T_{60} in seconds. It can also be shown that $\lambda_c = r_c$, where r_c is the distance beyond which the intensity of the direct sound from an isotropic source in the room falls below the intensity of the reverberant diffuse sound field of reflected waves from that same source (Schroeder 1996, https://doi.org/10.1121/1.414868).

c) Estimate T_{60} and f_s for (a) a typical cubical bedroom, 3m on a side, with carpeted floor, acoustic tile ceiling, 2 plaster walls, and 2 curtained walls, and (b) for King's College Chapel, Cambridge (Figure 2): 80m long, 24m high and 12m wide, all stone and glass except that much of the floor area is covered by wooden benches (pews).

Figure 2: The interior of King's College Chapel, Cambridge, England (from Wikimedia Commons).

(a) $V = 27\text{m}^3$, $S = 54\text{m}^2$, and from the table $\bar{\alpha} = (0.15 + 2 \times 0.05 + 2 \times 0.5 + 0.5)/6 = 0.3$, Sabine's formula gives $T_{60} = 0.3$s. Schroeder's equation gives $f_s = 2000(T_{60}/V)^{1/2} \sim 200$Hz. (b) $V = 2.3 \times 10^4\text{m}^3$, $S\bar{\alpha} = 0.05(80 \times 12) + 0.5(80 \times 12) + 0.05((80 + 12) \times 24) = 640\text{m}^2$. Sabine's formula gives $T_{60} = 6$s: quite close to the experimentally measured value for Kings College Chapel of 5 s. Schroeder's equation gives $f_s = 2000(T_{60}/V)^{1/2} \sim 30$Hz. Unlike, say, a tile shower stall (whose spectacular resonances you can find by humming at various frequencies, while moving around), a large stone church has essentially no acoustic resonances at any frequencies audible to humans, despite its very long reverberation time.

For an improved estimate of the Schroeder frequency, one should use acoustic absorption coefficients (or even better, the actual T_{60}) measured near f_s, not at 1 kHz. The frequency dependence of α varies significantly between materials: e.g., at 100 Hz, for glass and wood α is several times higher than at 1 kHz, while for carpet, curtains and acoustic tile, it is several times lower. For stone and concrete, it is not much changed.

4. **Boiling Water and Whistling Tea Kettles (16 points: 4(2+2)+12(2+2+3+5))**

a) It takes about 5 minutes to bring a liter of water to a boil on a kitchen stove.

i. How much power (in Watts) is being absorbed by the water?
Say the water starts out at 20 C. Raising the liter to 100 C takes energy $E \sim M c_p \Delta T$.

For water, by the definition of the calorie, $c_p = 1\text{cal g}^{-1}\text{C}^{-1}$ (for reasons discussed in class, it doesn't vary much with temperature for liquid water or even ice above the solid Debye temperature, and is always about $3k/m_a$), so

$$E \sim 1000\text{g} \times 1\text{cal g}{-}1\text{C}^{-1} \times 80\text{C} \sim 8 \times 10^4\text{cal} \sim 3 \times 10^{12}\text{erg}. \tag{1}$$

If this is dumped into the water in 5 minutes, the power is $P = E/300\text{s} \sim 10^{10}\text{erg s}^{-1} =$ $\boxed{1{,}000\text{ W.}}$

ii. Once the water is boiling, at what rate does the boiling water evaporate (express your answer in grams per second)? Assume no change to the stove setting.
Once the water gets to 100 C, further power addition does not heat the water, but instead goes into latent heat of evaporation $L_{vap} \sim 10^4\text{cal/mole} \sim 500\text{cal g}^{-1} \sim 2 \times 10^{10}\text{erg g}^{-1}$ (from the Purcell sheet; the exact value for water is 590cal g^{-1}). We assume the power input from the stove remains constant at $P = 1{,}000\text{W}$. Then the rate of evaporation of water is $\boxed{R = P/L_{vap} \sim 10^{10}\text{erg s}^{-1}/2 \times 10^{10}\text{erg g}^{-1} \sim 0.5\text{g s}^{-1}.}$
Note that this means that the liter of water will boil away in about a half hour, so keep an eye on the kettle.
The fact that it takes 7 times longer to boil away the water than to heat it to boiling is a consequence of the fact that the latent heat of evaporation is about 7 times the 80 cal it takes to heat the water to 100 C. The 7 is just a logarithm, resulting from the Boltzman factor which enters into the vapor pressure equation, and the fact that at air density much less than water density $kT_{vap} \sim m_a L_{vap}/\log(\text{a big number})$. Hence the heat content per gram at vaporization $\sim 3kT_{vap}/m_a \sim L_{vap}/10$ by Fermi's law that all logs are about 10, or 7 in this case.

b) Many tea kettles come with whistles, so you can tell when the water has boiled (see Figure 3. The basic whistle is a hole of radius $\approx 0.15\text{cm}$ through which water vapor can exit the kettle.

Figure 3: A tea kettle. The small hole in the spout at left allows steam to escape, making the whistling sound.

i. At what velocity (in cm or meters per second) does water vapor exit the hole when water is boiling inside the kettle?
Since $pV = NkT$, at temperature T, a mass M of water vapor of atomic mass $m_a = 18m_u = 3 \times 10^{-23}$g occupies a volume $V = (M/m_a)kT/p$. The pressure in the kettle will be a bit higher than 1atm $= 10^6$dyn cm^{-2} to drive the steam out (see below), but we'll neglect this, and assume $p =$ 1atm. Using this, we find that the $M = 0.5$g of vapor at $T = 373$K which goes through the whistle each second has a volume flow rate of $\dot{V} \sim 800\text{cm}^3\text{s}^{-1}$. The whistle opening has area $A = \pi r^2 = 0.07\text{cm}^2$, so the flow velocity must be

$$v = \frac{\dot{V}}{A} = \frac{800\text{cm}^3\text{s}^{-1}}{0.07\text{cm}^2} \sim \boxed{10^4\text{cm s}^{-1}}\,. \tag{2}$$

This is Mach 0.2 for steam at 373 K, so Bernouilli's equation ($h + v^2/2 = (5/2)p/\rho + v^2/2 = \text{const}$) implies that $(25/14)c^2_{s,kettle} = v^2/2 + (25/14)c^2_{s,atm}$. Thus $c^2_{s,kettle}/c^2_{s,atm} \sim 1 + (7/25)M^2 = 1.01 = (p_{kettle}/p_{atm})^{2/7}$, and finally $p_{kettle} = 1.04p_{atm}$. We were justified in neglecting the slight overpressure in the kettle in estimating $\dot{V}$.

ii. What is the Reynolds number of the flow near the hole?
The Reynolds number is $Re \sim rv/\nu$, where ν is the viscosity of steam, which we take as approximately that of air, $\nu \sim 0.2\text{cm}^2\text{s}^{-1}$. Then

$$\boxed{Re \sim \frac{0.15\text{cm} \times 10^4\text{cm s}^{-1}}{0.2\text{cm}^2\text{s}^{-1}} \sim 10^4} \tag{3}$$

Since $Re \gtrsim 10^3$, the flow will be turbulent. The fact that we are close to the critical $Re \sim 10^3$ for the transtion to turbulence means that vortices will be long-lived. You can see these yourself at $Re \sim 10^3 - 10^4$ by dragging a few centimeter diameter spoon or large stick through water ($\nu = 10^{-2}\text{cm}^2\text{s}^{-1}$ at $5 - 50$cm s^{-1}. The vortices are easier to see if the water has stuff floating on the top (dirt, small leaves, suds, pepper). Approximately one vortex is shed every time the spoon or stick moves through its own diameter. A better model of a whistle consists of making a dam where a small stream runs into a pond, with a few cm diameter hole near the top, so the stream makes a small jet into the pond. You will see this oscillate and make vortices.

iii. Why does the kettle whistle and what determines its frequency?
The flow of steam out of the whistle hole has zero velocity at the edge of the whistle, and maximum velocity in the middle of the hole. Thus the flow has vorticity ($\nabla \times \vec{v} \neq 0$) and a large shear. Shear flows at large Re are generically unstable to develop large side-to-side oscillations in the flow ("vortex shedding" and "Kelvin-Helmholtz" instability), and these modulate the airflow, producing sound at the oscillation frequency. The angular frequency of oscillation is roughly the time for the flow (oscillating at a speed which is a fraction of order unity of v) to cross the whistle hole, so $\omega \sim v/2r$, and $f \sim v/4\pi r \sim 10^4\text{cm s}^{-1}/(4\pi \times 0.15\text{cm}) \sim \boxed{7\text{kHz}}$. This is close to the highest note on a piano, which is about 8 kHz, and sounds about right. The whistles are designed to be attention-getting. Note that the wavelength (in air —we neglect the factor 1.2 difference in sound speed between 100 C steam and 20 C air) is about 4 cm, much larger than the hole.
[Note: the dimensionless number $\alpha = 2rf/v$ is known as the Strouhal number. In flow past a rigid cylinder, vortices are shed at with a frequency f such that

$\alpha \simeq 0.2$ for $10^2 < Re < 10^5$. Similar values apply to the dominant Kelvin-Helmoltz oscillations of jets emerging from apertures.
At higher Reynolds numbers, the flow becomes fully turbulent, with no special frequency.]

iv. Which multipole dominates the acoustic radiation? Estimate the acoustic power (in Watts). Using the espressions given in class, you can then estimate the decibel level 1 m from the kettle.

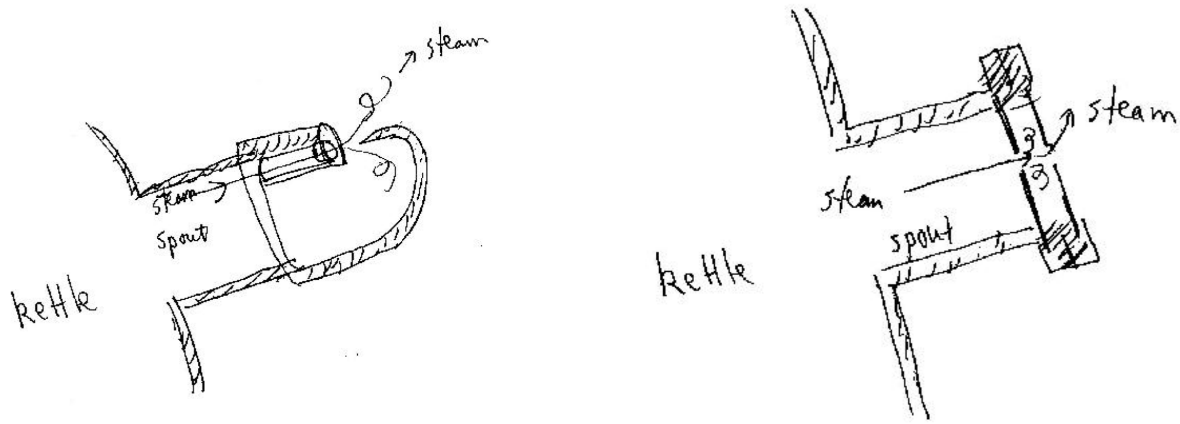

Figure 4: **Left**: a) Kettle 1 whistle: slot and edge **Right**: b) Kettle 2 whistle: two circular holes.

Because the instability leads to modulation of the mass flux out of the kettle hole, the whistle is an acoustic $\boxed{\text{monopole}}$. The detailed picture depends on the precise whistle design (see below for discussion of two examples), but does not matter for an OoM estimate of the power, as long as the whistle leads to an order unity fluctuation in the flow (pressure, velocity or mass flux).
There are many ways to estimate the acoustic power from a varying monopole source. Here is one simple one.
An order unity perturbation to the flow velocity requires a pressure perturbation $\Delta p \sim \rho v^2$ on the scale r of the whistle hole. At larger distances $D > r$, the pressure perturbation $\delta p(D) \sim \Delta p(r/D)$, and beyond the near zone, (size $d_{nz} \sim c_s/(2\pi f) \sim c_s 2r/v \sim$ 1cm for our $f \sim$ 7kHz) the perturbations become an acoustic wave. In an acoustic wave, the dimensionless acoustic pressure perturbation is of order the oscillation velocity in the wave over the sound speed: $\delta p/p \sim u/c_s$. The total energy density in the wave is twice the kinetic energy density: ρu^2. Thus the acoustic energy flux $F \sim c_s \rho u^2 \sim \delta p^2/(\rho c_s)$, and the radiated acoustic power is

$$P \sim 4\pi \frac{(\rho v^2 r)^2}{\rho c_s} \sim 4\pi \frac{v}{c_s} \rho v^3 r^2 \sim \boxed{3\text{W}} \tag{4}$$

Notice that this corresponds to radiating a fraction $v/c_s \equiv \mathcal{M}$ of the power density in the turbulent eddy $\rho v^3/r$ in the volume r^3, so the acoustic efficiency of a monopole eddy is the Mach number $\mathcal{M}$.
At a distance of 1 meter, the intensity is $I = 3\text{W}/4\pi\text{m}^2 = 0.25\text{W m}^{-2}$. Recalling the definition of decibel $dB = 10\log_{10}(I/10^{-12}\text{W m}^{-2})$, the kettle is about $\boxed{114 \text{ dB}}$.
The threshold of pain is 120 dB, and tea kettles are really painful to listen to, so

we are in the right ballpark. If you dropped the 4π, you'd get 0.3 W and 104 dB, which would be just as good an estimate at this crude level of whistle modelling.

The detailed picture depends on the whistle design, of which there are many. Sterl has two kettles. In one (figure 4a), the cap has a slot on one side, followed by a sharp edge. In this, the jet of steam emerging from the slot oscillates from side to side: in one half of the oscillation the steam emerges into the outside air, and on the other half of the oscillation, it goes into the interior of the cap. This cycle produces an oscillation in the emerging mass flux of steam from the kettle. [this "edge mode" phenomenon is also how recorders and some organ pipes produce sound, as the stream of air oscillates in and out of a resonant tube. Some fancy kettles have resonator tubes on the whistle, to produce more pure tones]. In Sterl's other kettle (figure 4b), the cap contains two parallel steel plates separated by about 0.5 cm, each with a hole in the center. In this, the jet of steam emerging from the first hole oscillates from side to side, so twice each cycle, the flow lines up with the outer disk and flows out, but at the left and right extremes of the oscillation cycle, doesn't line up and is at least partly blocked. Once again, the mass flux out is modulated, though probably at twice the jet oscillation frequency.

24 May 2023 **Ph 101** E.S. Phinney

Order of Magnitude Physics

Problem Set 8

Due **in Canvas 11pm Wednesday, 31 May 2023**

Homework Problems: (point values indicated: 50 points total)

1. **Walk before you run. (10 points: 5+5)** When you walk, one foot and leg acts as an inverted pendulum, while the other swings as a free pendulum.

   a) Considering the free pendulum leg, estimate the interval between footfalls you expect for your leg (try to do this carefully, considering realistic moments of inertia and center of mass positions for your leg). Time steps of your natural walk to check your answer (you should be able to get within 10 or 15% with elementary physics beyond OoM).

For a pendulum of mass M, length L, swinging at period P from its top end, about which the moment of inertia is $f_I ML^2$, and below which the center of mass lies $l_{cm} = f_{CM}L$, the kinetic energy is

$$E_{kin} = \frac{1}{2} f_I ML^2 \dot{\theta}^2 = \frac{1}{2} f_I ML^2 \left(\frac{2\pi}{P}\right)^2 \theta^2$$

and the potential energy is

$$E_{pot} = f_{cm} MLg(1 - \cos\theta) = f_{cm} MLg\frac{\theta^2}{2}$$

where θ is the angle from the vertical, assumed small in the right hand side of the last equation. Using the usual harmonic principle that the maxima of kinetic energy and potential energy are equal, we find

$$P = 2\pi\sqrt{\frac{f_I}{f_{cm}}\frac{L}{g}}\ .$$

If the leg were a uniform rod, $f_I = 1/3$, $f_{cm} = 1/2$. More realistically, the leg could be an isosceles triangle (pivot at the base, i.e. the wide thigh), $f_I = 1/6$, $f_{cm} = 1/3$, so $f_I/f_{cm} = 1/2$, or for those with very slender calves, a right circular cone (pivot at the base), $f_I = 1/10$ and $f_{cm} = 1/4$, so $f_I/f_{cm} = 1/2.5$.

For a leg of length $L = 1$m, the triangle and cone respectively give $P/2 = 0.7$s and $P/2 = 0.6$s. Sterl's $L = 1.0$m legs in a natural (unhurried) walk had 18 footfalls in 12 seconds, or $P/2 = 0.66$s, midway between the trianglar and conical leg estimates.

One can of course force a faster gait by using muscular force, rather than gravity to swing the leg, but this is rather a strain. Sterl's fast forced walk had 18 footfalls in 9 seconds, for $P/2 = 0.5$s.

   b) An upper limit to the walking speed is set by the inverted pendulum leg: what is the maximum speed at which you could walk without that leg being lifted off the ground by centrifugal acceleration? Compare this speed to one you estimate from the "natural pendulum" of part (a) and your natural step length. [humans have two gaits: walking, where only one foot is lifted at a time, and running, where both feet are sometimes off the ground. 4-legged animals like horses have more gaits, depending on whether 1, 2, 3 or 4 feet are off the ground: walk, trot, canter, gallop respectively.]

If $v^2/L > g$ the upwards centrifugal acceleration of the foot exceeds g, and the foot will leave the ground. Thus the transition from a walk to run should occur at about

$$v_t \sim \sqrt{gL} \sim 3\text{m s}^{-1} \sim 10\text{km s}^{-1} .$$

Olympic racewalkers can manage 12km h^{-1}, but the average man transitions to a jog at 8.5km h^{-1}, since the extreme leg extensions and forced pendulum frequency of fast walking are uncomfortable and energetically inefficient.

Recall that the dimensionless Froude number is $\text{Fr} = v/\sqrt{gL}$. So the walk/run transition occurs at $\text{Fr} \simeq 1$.

In unhurried walking, most people's natural step length is about 0.4 times their height, or about 0.75 of their leg length, corresponding to a leg swing of $\pm 22°$. The easy pendulum gait from part (a) thus correponds to a walking speed of

$$0.75L/(P/2) = \frac{0.75}{\pi}\sqrt{f_{cm}/f_I}\sqrt{gL} = 1.2(L/\text{m})^{1/2}\text{m s}^{-1}$$

(4km/h, or 2.7mph), where in the final numerical value we used the conical leg estimate $f_{cm}/f_I = 2.5$. In a faster walk, the swing angle increases (up to about 45°, and the leg pendulum is forced faster by muscles ($P/2 = 0.5$s vs 0.66s), allowing the walking speed to be adjusted up to a factor of about 2.5 times the "natural" walking speed.

2. **Exciting waves. (10 points: 5+3+2)**

   a) Why is it difficult to walk carrying a bowl of soup without spilling any? Be quantitative. [Hint: make use of the shallow water wave phase velocity derived in class, and the previous problem about walking.]

Recall from class that waves in shallow water of depth h have $\omega/k = \nu\lambda = \sqrt{gh}$. A rectangular bowl of length l in the direction of sloshing, and soup depth $h << l$ (shallow wave) has a fundamental sloshing mode of wavelength $\sim 2l$ (the sloshing mode has one side low while the other is high, i.e. π phase of a cosine wave). From the shallow wave phase velocity, the sloshing frequency is then$\nu_s = v_p/\lambda = \sqrt{gh}/(2l)$. This formula is exact for rectangular soup bowls of length l, but is approximate for more usual round ones, for which the factor 2 should be 1.7 if the depth is uniform, or $\pi/2$ if the bottom is parabolic. For a typical bowl of soup, $L = 15$ cm, $h = 2$ cm, and the frequency of the resonant sloshing mode is $\boxed{\nu \sim 1.4}$Hz. Walking at normal speeds provides footfall impulses at just this frequency (in the previous problem, we found step periods of $0.6-0.7$s, or $1.4-1.7$Hz), which resonantly drive the sloshing mode.

Remedies if your soup starts sloshing: walk more slowly (so the foot impulse driving is below the resonant frequency), or, if feeling brave and trusting physics: walk faster, putting the foot impulse frequency above resonance, and/or pour your self even more soup (raising the sloshing frequency)!

   b) Estimate the energy per unit area stored in ocean waves. Express your answer in terms of seconds of solar insolation.

Averaged over a wavelength, a wave with a crest height ξ is created by lifting a mass per unit area $\sim \rho\xi$ from the trough and putting it on the crest, increasing the potential energy per unit mass by $\sim g\xi$. Thus, the energy density is $\mathcal{E} \sim \rho g \xi^2 \sim 10^7$ erg cm^{-2} for a wave of crest height

$\xi \sim 1$ m. The sun overhead radiates a flux $F_\odot \equiv 1.4 \times 10^6 \text{erg cm}^{-2}\text{s}^{-1}$, so the waves have about $\boxed{\text{7 s of insolation.}}$

c) Estimate the energy per unit area in the winds that drive the ocean waves. Again express your answer in terms of seconds of solar insolation.

Typical non-storm wind speeds are around $v_w = 4\text{m s}^{-1}$ (10mph). The energy density in wind is thus $\rho v_w^2 h \sim 1.5 \times 10^8 \text{erg cm}^{-2}\text{s}^{-1}$ where $\rho \simeq 10^{-3}\text{g cm}^{-3}$ is the density of the earth's atmosphere, and $h \sim c_s^2/g \sim 10\text{km}$ is its scale height. Thus winds have about $\boxed{\text{100 s of insolation.}}$
Alternatively, recall that we estimated the wind velocity from turbulent dissipation of convection driven by the average solar insolation, $\rho v_w^3 \simeq \epsilon \langle F_\odot \rangle$, where $\langle F_\odot \rangle = (1/4)(1-a)F_\odot \simeq F_\odot/6$, with $\epsilon \sim 0.3$. Thus the timescale is roughly $(\epsilon/6)h/v_w \simeq 100\text{s}$, the large eddy turnover time, which is a rough estimate for the turbulence dissipation time. Both wind and waves require continous input of energy!

3. **Water bug propulsion and navigation. (16 points: 4+4+4+4)** First, a brief digression on the meaning of non-wettable: When a solid meets a liquid interface, there are three surface tensions: solid/air, solid/liquid and liquid/air. Equilibrium of forces at the 3-way interface line (see figure 1a) requires

$$\gamma_{la} \cos \theta_E = \gamma_{sa} - \gamma_{sl} \tag{1}$$

If $\gamma_{sa} > \gamma_{sl} + \gamma_{la}$, then θ_E is zero and the liquid spreads to an atomically thin layer to minimize its surface energy. This is known as "total wetting". If the inequality is the other way round, there is partial wetting. *wettable* means the angle of contact θ_E of the surface of the drop with the surface is nearer 0 than $\pi/2$; *non-wettable means the angle of contact is nearer* π. The consequences are shown in figure 1.

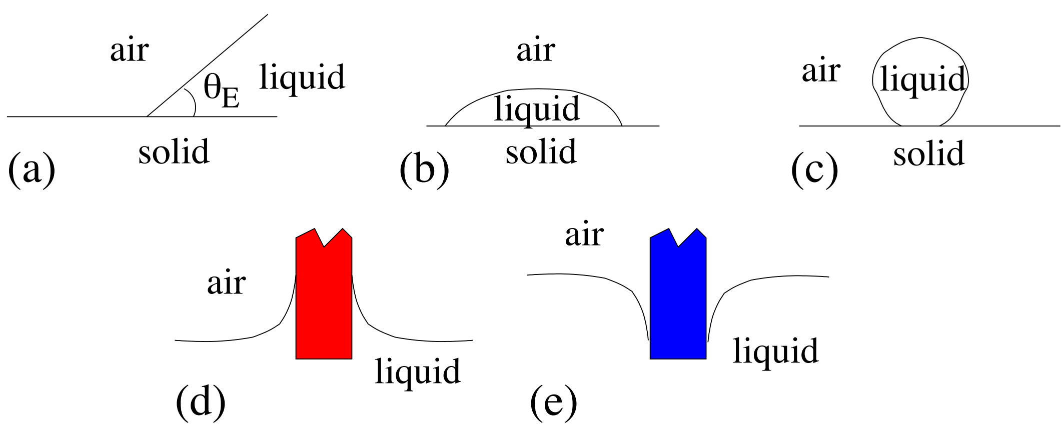

Figure 1: (a) Equilibrium between the three surface tensions of the three interfaces at the contact point sets the angle of contact θ_E. (b) A water drop spreads on a partly wettable surface like plastic or dirty glass, $\theta_E < \pi/2$. (c) A water drop balls up on a non-wettable surface like wax, $\theta_E > \pi/2$. (d) a wettable bug leg (red) is pulled *into* the water (watch bees and flies struggling, usually unsuccessfully, to get out of a pool), while (e) a non-wettable bug leg (blue) is pushed out of the water. In cases (d) and (e) the force is approximately γ_{la} times the circumference of the leg.

a) The marine water strider *Halobates* has a mass of about 10mg and is about 5mm long (with legs of 2mm in length). It can not only walk on water but is able to jump several cm into the air. It does so by using both its non-wettable legs and non-wettable special hairs on its legs. (These hairs are very long and can support some stress.) What is the length of contact (between parts of its anatomy including hairs, and the water surface) required at take-off to perform these jumps?

To jump on water, the marine water strider *Halobates* (see: http://www.jochemnet.de/fiu/Halobates.jpg) must use its non-wettable legs to push off of the water. By dimensional analysis or the discussion in the caption of figure 1, the surface tension force is given by: $F = \gamma L$ where L is the length of contact with the water. Note that unlike walking on land, the force is *not* proportional to the surface area of contact. We can solve for the length of contact by equating the work done to the energy needed to reach a height of several centimeters:

$$\gamma L d = mgh \tag{2}$$

where d is a few millimeters, i.e. the length of the bug's legs, which is roughly the distance over which the bug can exert force on the water by bending and straightening its "knees" (this is not a biology course, so we skip the jargon). Solving for L, we have:

$$L = \frac{mgh}{\gamma d} = \frac{0.01\text{g} \times 10^3\text{cm s}^{-2} \times 3\text{cm}}{50\text{erg cm}^{-2} \times 0.2\text{cm}} = 3\text{cm}\ . \tag{3}$$

b) The similar-sized beetle *Stenus* can float on water (i.e., like a boat, about one third submerged) and will, when necessary, propel itself at high speed by squirting a fluid out of its backside that lowers the surface tension dramatically. How does this provide propulsion? Estimate the peak velocity at which it can move.

The *Stenus* squirts behind it a fluid which lowers the surface tension dramatically. So we can simplify the problem to one where normal surface tension force on the front drags the *Stenus* forward, while there is no surface tension force on the behind. So the net surface tension force is $F = \gamma L$ where L is the size of the *Stenus* $\sim$ 5mm. The drag force exerted on the submerged part of the bug by the water is $\sim \frac{1}{2} C_D \rho_{water} v^2 L^2$. An upper limit is $C_D \sim 1$, which gives

$$\gamma L = \frac{1}{2} C_D \rho_w v^2 L^2 \tag{4}$$

$$v = \sqrt{\frac{2\gamma}{C_D \rho_w L}} = \sqrt{\frac{2 \times 50\text{erg cm}^{-2}}{1 \times 1\text{g cm}^{-3} \times 0.5\text{cm}}} = \boxed{14\text{cm s}^{-1}(1/C_D)^{1/2}} \tag{5}$$

If we'd taken $C_D \sim 0.1$ then $v \sim 44\text{cm s}^{-1}$ which is greater than the minimal phase velocity for water waves ($\sim 20\text{cm s}^{-1}$). Wave drag sets in when $v > v_{ph}$ and carries away even more momentum and energy from the poor little bug. So he's probably limited to somewhere between 14 and 20cm s^{-1}.

You can make your own model of this: build a soap-powered toy boat: https://sciencebob.com/build-a-soap-powered-model-boat/ or https://www.sciencebuddies.org/stem-activities/how-surface-tension-works

c) What is the smallest amount of material it needs to excrete per unit time to maintain this velocity? Compare the thrust with what would be possible if the same mass flux of material were ejected as a gas of density similar to air ("jet" propulsion) assuming the overpressure is of order atmospheric pressure and the gas has a sound speed not very different from air.

A molecular thickness (~ 5Å) of this material is enough to decrease the surface tension at the back. The mass of material that must be excreted per unit time is:

$$\dot{m} = \rho a L v \simeq 1\text{g cm}^{-3} \times (5 \times 10^{-8}) \times 0.5\text{cm} \times (14\text{cm s}^{-1}) = 3 \times 10^{-7}\text{g s}^{-1} , \tag{6}$$

where we have assumed that the surface-tension reducing material (like soap) has about the same water-like density as the rest of the insect. The thrust produced by this mechanism of propulsion is: $F = \gamma L = 25$dyn. Now, if the insect instead tried to use the jet propulsion by ejecting the same amount of material with an overpressure of order the atmospheric pressure, the ejection velocity would be of order the sound speed in air, c_s, which is $\sim 3 \times 10^4\text{cm s}^{-1}$. The thrust of such a jet would be:

$$F_{jet} = \dot{m} c_s = 3 \times 10^{-7}\text{g s}^{-1} \times 3 \times 10^4\text{cm s}^{-1} = 9 \times 10^{-3}\text{dyn} , \tag{7}$$

which is 0.0004 of the force the bug produces by surface tension reduction. Once again, evolution has crafted some pretty crafty creatures.

d) Whirligig beetles (length 10mm) are able to echolocate through reflection of their ripples from obstacles nearby. (They create these waves as they bounce their abdomens on the surface of the water.) They have been observed to move as fast as 0.4m/sec on water (with their abdomens remaining partly in contact with the water). Roughly what is the peak speed at which their echolocation system will function? [Note: Some of these bugs also use echolocation to communicate their sex to other bugs. The male adds a high frequency flourish to the wave train of the ripples it sends. But the notice won't reach her before he does if he is chasing the hot babe too fast!]

The whirligig beetles can't move faster than the group velocity of the ripples they make, otherwise the waves can't feed them information in time. The abdomen is smooth, so the smallest wavelength it can produce is roughly its size, i.e. $\lambda_{min} \sim 1$cm and $k_{max} = 2\pi/\lambda_{min} \sim 6\text{cm}^{-1}$. So for ripples we have from the disperson relation for capillary waves $\omega^2 = \gamma k^3/\rho$

$$V_{max} = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{\gamma k_{max}}{\rho}} \sim 25\text{cm s}^{-1} \tag{8}$$

which is comparable to the top speeds that these little creatures have been observed to move. In principle, they could move faster briefly but only if it's worthwhile to do so blindly (i.e., if being chased by a predator).

A fast whirligig would be like a supersonic jet trying to use sonar to map what is in front of it. The jet would overtake its own outgoing sound waves, rendering them useless.

4. **Čerenkov radiation. (14 points: 5+4+5)** A particle of speed $v = \beta c$ moving through a medium with index of refraction n emits Čerenkov radiation if $\beta > 1/n$, i.e. if it is moving faster than the phase-speed of light in the medium. The energy radiated in Čerenkov radiation at frequencies between ω_1 and ω_2, per unit length traversed by the particle, $dE(\omega_1, \omega_2)/dx$ depends on the electron charge e, speed of light c, ω_1, ω_2, n, β.

a) Use the Buckingham Pi theorem to find an expression for dE/dx.

The photons come out with some spectrum. Since ω_1 and ω_2 are arbitrary, let $\omega_1 = \omega$ and $\omega_2 = \omega + d\omega$. The contribution to dE/dx from photons in the frequency range $d\omega$ must be proportional to $d\omega$, so

$$\frac{d^2E}{dx d\omega} d\omega = F(e^2, c, \omega, n, \beta) d\omega, \tag{9}$$

where we used e^2 instead of e since e^2 has more convenient integer exponents of units. The variables n and β are already dimensionless. The remaining four variables have the following dimensions (in an *energy*, *velocity*, *length* system of units):

$$F \;:\; [E][V]^{-1} \tag{10}$$
$$c \;:\; [V] \tag{11}$$
$$\omega \;:\; [V][L]^{-1} \tag{12}$$
$$e^2 \;:\; [E][L]. \tag{13}$$

There is only one dimensionless variable, which we can take to be

$$\Pi_1 = \frac{Fc^2}{e^2\omega}. \tag{14}$$

From the Pi theorem, we have $g(\Pi_1, n, \beta) = 0$, or $\Pi_1 = \frac{Fc^2}{e^2\omega}$. Thus

$$\frac{dE}{dx} \sim \frac{e^2\omega}{c^2} f(n, \beta) d\omega = \frac{e^2\omega^2}{c^2} f(n, \beta) \frac{d\omega}{\omega} \tag{15}$$

with f some unknown function of the dimensionless variables. The right hand side is to be integrated between ω_1 and ω_2, but since we don't yet know f or how the index of refraction depends on frequency, we can't do the integral yet.

b) Recall from class that at frequency ω, the index of refraction for a medium composed of atoms whose electrons are approximated as harmonic oscillators of natural frequency ω_0 is given by $n^2 = 1 + 4\pi N e^2/[m_e(\omega^2 - \omega_0^2)]$, where N is the number of electrons per unit volume. For ultrarelativistic particles, the dE/dx of Čerenkov radiation is independent of β, and in a dilute medium such as air, with $n - 1 \ll 1$, it is clear that we must have $dE/dx \propto N$. Find an expression for dE/dx in this case ($\beta \to 1$, $n - 1 \ll 1$).

The index of refraction is

$$n^2 = 1 + \frac{4\pi N e^2}{m_e(\omega_o^2 - \omega^2)}. \tag{16}$$

For dilute media, where $n - 1 \ll 1$, this expression reduces to $n - 1 \propto N$. Since we are told that $dE/dx \propto N$ in this limit, and we are working in the ultrarelativistic limit (where β drops out), we know that $f(n, \beta) \sim n - 1$. Therefore,

$$\frac{dE(\omega)}{dx} \sim \frac{e^2\omega^2}{c^2}(n-1)\frac{d\omega}{\omega}. \tag{17}$$

c) Very high energy photons from astronomical sources such as quasars, pulsars and gamma-ray bursts create (via pair creation and bremsstrahlung) showers of electrons, positrons and photons in the earth's atmosphere. The showers develop conserving energy until the typical e^+e^- energy falls so low (below 50 MeV) that ionization losses begin to dominate over bremsstrahlung. At the peak of its shower development ($\sim$ 10km altitude) a typical high-energy photon has thus converted a few percent of its total energy into down-rushing $\sim$ 50MeV electrons. Estimate the number of visible wavelength (400nm-650nm) Čerenkov photons created by a 100 GeV photon incident on the earth's atmosphere.

To calculate the number of photons emitted, you first need to estimate how far a 50 MeV electron can travel in the atmosphere at 10 kilometers.

There are several ways you might estimate this:

- Use the Purcell sheet, which gives the radiation length (energy loss scale for relativistic particles) as 36g cm^{-2}. This gives a mean free path in sea-level air of 36g cm^{-2}/10^{-3}g cm^{-3} $\sim 4 \times 10^4$cm $\sim$ 0.4km. The air at 10 km altitude, one atmospheric scale height up, has about $1/e$ lower density than at sea level, so the mean free path there is about 1 km.
- Use scaling. 50 MeV is a not as high as energies probed by large particle accelerators like CERN (which can attain energies on the order of 1 TeV), but it is the amount of energy is reasonable to reach in a smaller experiment (such as in a laboratory experiment here at Caltech; Phys 7 labs look at radioactive beta decays with energies of a few MeV). Lead blocks a few centimeters thick are used for shielding. Lead has a density of about 12 g/cc, while air at 10 km has a density of about 3×10^{-4} g/cc. Assuming that the length to stop a high energy particle scales inversely with density, it would take about $(4 \times 10^4) \times 3$cm $\sim$ 2km.
- Use "known facts." By sea level, almost all primary cosmic rays are stopped by the earth's atmosphere. The atmosphere has a density of about 10^{-3}g/cm^3 and a characteristic height of 10km, which implies a total column density to sea level of about 10^{-3}g/cm$^3 *$10km = 1000g/cm^2. On the other hand, cosmic rays are easily studied in high-altitude balloons, and flying in airplanes increases your exposure considerably. This suggests that the stopping grammage in air is a few percent of the total atmospheric grammage, or around 30g/cm^2. The density of air at 10 km is about 3×10^{-4} g/cc, so a cosmic ray can travel 30g/cm^2/3×10^{-4}g/cc $\sim$ 1km. We might expect that 50 MeV electrons can penetrate the same amount of atmosphere.
- Use physics. At high energies, electrons mainly lose energy through bremsstrahlung (with a cross-section $\sim \alpha\sigma_T$, where $\alpha = 1/137$ is the fine structure constant) and at lower energies, mainly through ionisation (the other way to detect high-energy showers, via nitrogen fluorescence of the atmosphere). To estimate the ionization losses, note that if an electron with momentum $p = \gamma m v$ passes another electron (in an atom, treated as free) at distance r, the momentum it imparts is of order $(e^2/r^2)(r/v)$, and the energy imparted is of order $\Delta E \sim (e^2/(rv))^2/m_e$. Provided ΔE is a lot larger than the ionization energy (which justifies the free-electron approximation made earlier), the electron will be removed from the atom and acquire this energy from the passing relativistic electron. Thus we expect $dE/dx \sim (\Delta E)(\pi r^2)n_e \sim n_e[e^2/(m_e c^2)]^2[m_e c^2][c^2/v^2] \sim n_e \sigma_T m_e c^2 (c^2/v^2) * \log(\text{stuff})$. [The log arises because all impact parameters r contribute equally]. Since $n_e = (\rho/A m_p) * Z$, the stopping grammage for a relativistic electron ($v \sim c$) is thus the one for which $\gamma m_e c^2 = dE/dx * l$, so

$$\rho l = \rho\gamma/[n_e \sigma_T * \log(\text{stuff})] = \gamma(A/Z)(m_p/\sigma_T)/\log(\text{stuff}) = 5\text{g cm}^{-2}\gamma/\log(\text{stuff}). \tag{18}$$

For 50MeV electrons, $\gamma = 100$ and using Fermi's law that all logs are 10, we estimate the stopping grammage $\rho l = 50$g cm^{-2}. Knowing the density of air at 10km is about 3×10^{-4} g/cc gives a stopping length of about 2 km. For a more honest calculation, see Jackson (chapter 13 for ionization loss, chapter 15 for bremsstrahlung).

All four methods suggest that the electron can travel about 1 km through the atmosphere. The index of refraction of air is $n-1=3\times10^{-4}$ at standard atmospheric pressure, and it scales with density, to $n-1=10^{-4}$ at 10km. Blue light is about 0.4 microns wavelength, or about 3 eV energy, which (inserting into equation 17 written as $dE/dx \sim e^2(2\pi)^2\lambda^{-2}(n-1)d\omega/\omega$), and since the visible spectrum has a fractional range $d\omega/\omega \sim 0.5$, this leads to about 3×10^{-13}erg cm^{-1} radiated in visible Čerenkov light. So over 1km, the electrons lose about 3×10^{-8}erg $=0.02$ MeV in Čerenkov radiation, which is about 0.0004 of the 50 MeV end-point electron's energy. As stated in the problem, ~ 0.03 of the initial 100 GeV γ-ray energy ends up in 50MeV end-point electrons, so the net fraction of the initial 100 GeV $=10^5$MeV gamma-ray energy that ends up in blue Čerenkov photons is around $0.03\times0.02\text{MeV}/50\text{MeV}\sim1\times10^{-5}$, i.e. about 1MeV worth of $2-3$eV photons, or $\boxed{3\times10^5 \text{ photons}}$.

Check: this is fairly close to what one infers e.g. from plots in http://arxiv.org/abs/astro-ph/0108392, which show 100GeV photons producing about 20 photons/m^2 over a 100m radius, or 6×10^5 photons.