To show $C(x,\varepsilon)$ is closed we will show $X \setminus C(x,\varepsilon)$ is open ie.

For any $z \in X \setminus C(x,\varepsilon)$ there exist a $\delta > 0$ such that

$$B(z,\delta) \subseteq X \setminus C(x,\varepsilon)$$

Since $z \in X \setminus C(x,\varepsilon) \Rightarrow d(x,z) > \varepsilon$
$\Rightarrow d(x,z) - \varepsilon > 0$

Take $\delta = d(x,z) - \varepsilon > 0$

Consider $B(z,\delta)$

let $w \in B(z,\delta)$

We will show $w \in X \setminus C(x,\varepsilon)$

ie. $w \notin C(x,\varepsilon) \Rightarrow d(x,w) > \varepsilon$

$w \in B(z,\delta) \Rightarrow d(z,w) < \delta = d(x,z) - \varepsilon$
$\Rightarrow \varepsilon < d(x,z) - d(z,w)$ — (*)

observe, By Triangle Inequality

$$d(x,z) \leq d(x,w) + d(w,z)$$

$$\Rightarrow d(x,z) - d(w,z) \leq d(x,w)$$

from (*)

$\varepsilon < d(x,z) - d(z,w) \leq d(x,w)$ | As $d(z,w) = d(w,z)$

$\Rightarrow d(x,w) > \varepsilon$

$\Rightarrow w \notin C(x,\varepsilon)$

$\Rightarrow w \in X \setminus C(x,\varepsilon)$

$\Rightarrow B(z,\delta) \subseteq X \setminus C(x,\varepsilon)$

$\Rightarrow X \setminus C(x,\varepsilon)$ is open

$\Rightarrow C(x,\varepsilon)$ is closed.