MA 203

PS-3:

1) Let $X \sim Bin(n,p)$, $n \to$ +ve integer , $p \in (0,1)$

$Y_1 = X^2$ and $Y_2 = \sqrt{X}$.

Find PMF's of Y_1 and Y_2.

PMF of $Y_1 = X^2$ is

$$f_{Y_1}(y) = \begin{cases} {}^nC_{\sqrt{y}}\, p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} & \text{if } y = 0,1,4,9 \dots n^2 \\ 0 & \text{o/w} \end{cases}$$

PMF of $Y_2 = \sqrt{X}$ is

$$f_{Y_2}(y) = \begin{cases} {}^nC_{y^2}\, p^{y^2} (1-p)^{n-y^2} & \text{if } y = 0, 1, \sqrt{2}, \sqrt{3} \dots \\ 0 & \text{o/w} \end{cases}$$

2) Let X be a r.v. with PMF

$$f_X(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 0, 1, 2 \dots \\ 0 & \text{otherwise} \end{cases}$$

Find CDF of $Y = \frac{X}{X+1}$ & hence determine PMF of Y.

Sol^n:

The CDF of $Y = \frac{X}{X+1}$ is

$$F_Y(y) = P\left(\frac{X}{X+1} \le y\right)$$

$$= P(X \le y + Xy) \quad \text{as} \quad P(X+1 > 0) = 1$$

$$= P(X(1-y) \le y)$$

$$= \begin{cases} P\left(X \le \frac{y}{1-y}\right) & \text{if } y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ P\left(X \le \frac{y}{1-y}\right) & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ \sum_{x=0}^{\left[\frac{y}{1-y}\right]} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ \frac{3}{2}\left(1 - \left(\frac{2}{3}\right)^{\left[\frac{y}{1-y}\right]+1}\right) & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

The PMF of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{y}{1-y}} & \text{if } y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \\ 0 & \underline{o.w} \end{cases}$$

3. If the MGF of a random variable X is

$$M_X(t) = \frac{1}{3t}\left(e^{t} - e^{-2t}\right) \text{ for } t \neq 0$$

Find PDF of $Y = X^2$. (Hint: Try to identify the distribution of X from's it's MGF.

uniform dist:

mean $= \frac{a+b}{2}$ variance $= \frac{(a-b)^2}{12}$

$$M_X(t) = \frac{1}{b-a}\left(\frac{e^{bt} - e^{at}}{t}\right)$$

Observe that the given MGF is the MGF of $U(-2, 1)$ RV. Thus

$P(Y \leq x) = P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x})$

where $X \sim U(-2, 1)$ and $x \geq 0$

4.

Here, we find CDF first and then differentiate it to find PMF

$P(Y \leq x) = 0$ for $x < 0$

Thus

$F_Y(x) = 0$ for $x < 0$

$= \frac{1}{3} \int_{-\sqrt{x}}^{\sqrt{x}} dt$ for $0 \leq x < 1$

$= \frac{2\sqrt{x}}{3}$ for $0 \leq x < 1$

$= \frac{1}{3} \int_{-\sqrt{x}}^{1} dt$ for $1 \leq x < 4$ $\frac{1}{3}$

$= \frac{1+\sqrt{x}}{3}$ for $1 \leq x < 4$

$= 1$ for $x \geq 4$

$\therefore$ PDF is $= 0$ for $x < 0$

$= \frac{1}{3\sqrt{x}}$ for $0 \leq x < 1$

$= \frac{1}{2\sqrt{x} \cdot 3}$ $1 \leq x < 4$

$= 1$ $x \geq 4$

4. Consider the following joint PMF of the random vector (X, Y)

x \ y	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

a) Find $P(X+Y < 8)$, $P(X+Y > 7)$, $P(XY \leq 14)$

b) Find Corr (X, Y)

Sol^n:-

a) $P(X+Y < 8) = P(X=4, Y=1) + P(X=4, Y=2) + P(X=4, Y=3)$
$+ P(X=5, Y=1) + P(X=5, Y=2) + P(X=6, Y=1)$

$= 0.08 + 0.11 + 0.09 + 0.04 + 0.12 + 0.09$

$= 0.53$

$P(X+Y \leq 8) = 0.53 + 0.03 + 0.21 + 0.06$
$= \underline{\underline{0.83}}$

b) $P(X+Y > 7) = 1 - 0.83 = \underline{\underline{0.17}}$
$(1 - P(X+Y \leq 8))$

c) $P(XY \leq 14) = 1 - P(X \geq 15)$
$= 1 - P(X=4, Y=4) - P(X=5, Y=3) - P(X=5, Y=$
$- P(X=6, Y=3) - P(X=6, Y=4)$

$=$ ______

$Corr(X, Y) = Cov(X, Y) / \sigma_x \sigma_y$

5. Three balls are randomly placed in three empty boxes B_1, B_2 and B_3. Let N denote the total number of boxes which are occupied and let X_i denote the number of balls in the box B_i, $i = 1, 2, 3$.

a) Find the joint PMF of (N, X_1)

b) Find the joint PMF of (X_1, X_2)

c) Find the marginal distribution of N and X_2.

d) Find the marginal PMF of X_1 from the joint PMF of (X_1, X_2).

The possible configurations are

3, 0, 0	N = 1
2, 1, 0	N = 2
1, 1, 1	N = 3
0, 3, 0	N = 1
0, 2, 1	N = 2
1, 2, 0	N = 2
0, 0, 3	N = 1
0, 1, 2	N = 2
1, 0, 2	N = 2
2, 0, 1	N = 2

$P(N, X_1) =$

$$P(N=1, X_1=0) = \frac{1}{5}$$

$$P(N=1, X_1=3) = \frac{1}{10}$$

	X_1	X_2	X_3
$N=1$	3	0	0
	0	3	0
	0	0	3
$N=2$	2	1	0
	0	2	1
	1	2	0
	0	1	2
	1	0	2
	2	0	1
$N=3$	1	1	1

a) PMF of (N, X_1)

$P(N=1, X_1=0) = \frac{1}{5}$

$P(N=1, X_1=3) = \frac{1}{10}$

$P(N=2, X_1=2) = \frac{1}{5}$

$P(N=2, X_1=0) = \frac{1}{5}$

$P(N=2, X_1=1) = \frac{1}{5}$

$P(N=3, X_1=1) = \frac{1}{10}$

b) $P(X_1=0, X_2=0) = \frac{1}{10}$

$P(X_1=0, X_2=1) = \frac{1}{10}$

$P(X_1=0, X_2=2) = \frac{1}{10}$

$P(X_1=0, X_2=3) = \frac{1}{10}$

$P(X_1=1, X_2=0) = \frac{1}{10}$

$P(X_1=1, X_2=1) = \frac{1}{10}$

$P(X_1=1, X_2=2) = \frac{1}{10}$

$P(X_1=2, X_2=0) = \frac{1}{10}$

$P(X_1=2, X_2=1) = \frac{1}{10}$

$P(X_1=3, X_2=0) = \frac{1}{10}$

c)

$P(N=1) = \frac{3}{10}$

$P(N=2) = \frac{8}{10}$

$P(N=3) = \frac{1}{10}$

$P(X_2 = 0) = \frac{4}{10}$

$P(X_2 = 1) = \frac{3}{10}$

$P(X_2 = 2) = \frac{2}{10}$

$P(X_2 = 3) = \frac{1}{10}$

d)

$$P(X_1 = 0) = \sum_{K=0}^{3} P(X_1 = 0, X_2 = K) = \frac{4}{10}$$

$$P(X_1 = 1) = \sum_{K=0}^{2} P(X_1 = 1, X_2 = K) = \frac{3}{10}$$

$$P(X_1 = 2) = \sum_{K=0}^{1} P(X_1 = 2, X_2 = K) = \frac{2}{10}$$

$$P(X_1 = 3) = P(X_1 = 3, X_2 = 0) = \frac{1}{10}$$

Joint distribution fn of X and Y

$F(x, y) =$ ....

Joint PDF $f(x,y) = \frac{\partial^2 f}{\partial x \partial y}$

Marginal PDF $f_Y(y) = \frac{\partial F_Y(y)}{y}$ $\qquad F_Y(y) = \lim_{x \to \infty} F(x,y)$

$f_X(x) = \frac{\partial F_X(x)}{x}$ $\qquad F_X(x) = \lim_{y \to \infty} F(x,y)$

$Cov(XY) = \mu'_{11} = E(XY) - E(X)E(Y)$

6. Let X be a random variable with MGF $M(t)$, $|t| < h$, for some $h > 0$
Prove that $P(X \geq a) \leq e^{-at} M(t)$ for $0 < t < h$ and
$P(X \leq a) \leq e^{-at} M(t)$ for $-h < t < 0$

Hint: Markov Inequality

Soln:

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \quad \text{for } 0 < t < h$$

Using Markov's inequality

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} M(t)$$

$$P(X \leq a) = P(e^{tX} \geq e^{ta}) \quad [\text{for } -h < t < 0 \;\; e^{tX} \text{ is decreasing}]$$

Using Markov's inequality

$$P(X \leq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E(e^{Xt})}{e^{ta}} = e^{-ta} M(t).$$

Markov's inequality

$P(X \geq 0) = 1$ and $E(X) = \mu$ exists the $\forall t > 0$

$$P(X \geq t) \leq \frac{E(X)}{t}$$

Chebyshev's inequality

$X \leftarrow$ R.V. with $E(X) = \mu$ and $Var(X) = \sigma^2$ for any $\varepsilon > 0$.

$$P(|X - c| \geq \varepsilon) \leq \frac{E\{(X-c)^2\}}{c^2}$$

If $c = \mu$

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

OR

$$P(|X - \mu| \geq \sigma K) \leq \frac{1}{K^2}$$

OR

$$P(|X - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

OR

$$P(|X - \mu| < \sigma K) > 1 - \frac{1}{K^2}$$

7. Let X be a random variable s.t. $P(X \le 0) = 0$ and $\nu = E(X)$ is finite show that $P(X \ge 2\nu) \le 0.5$

Sol^n:

$$P(X \ge 2\nu) = P(|X| \ge 2\nu) \qquad \because P(X \le 0) = 0$$

$$\le \frac{E(|X|)}{2\nu} = \frac{E(X)}{2\nu} = \frac{1}{2}$$

$$\left[\text{As } P(X \le 0) = 0 \quad E(X) = E(|X|)\right]$$

8. If X is a random variable s.t. $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$.

[Hint: Chebyshev inequality]

Sol^n:

$$P(-2 < X < 8) = P(-5 < X-3 < 5)$$

$$= P(|X-3| < 5)$$

using Chebyshev's inequality

$$P(|X-3| < 5) = 1 - P(|X-3| \ge 5)$$

$$1 - \frac{\sigma^2}{5^2} = 1 - \frac{13-9}{25} = \frac{21}{25}$$

Items produced in a factory during a week is a RV with mean 500. Find an upper bound on the probability that this weeks production will be atleast 1000? If the variance of a weeks production will be between 400 and 600.?

Let X be the R.V. that denotes the number of items produced by the factory in a week

Hence, $E(X) = 500$

Now, $P(X > 1000) = P(|X| > 1000) \leq \frac{E(|X|)}{1000} = \frac{1}{2}$ $\boxed{\text{using Markov's inequality}}$

As $P(X < 0) = 0$

The variance of X is $Var(X) = 100$

$$P(400 < X < 600) = P(-100 < X - 500 < 100)$$

$$= P(|X - 500| < 100)$$

$$= 1 - P(|X - 500| \geq 100)$$

Using Chebychev's inequality

$$\geq 1 - \frac{100}{100^2}$$

$$= 0.99$$

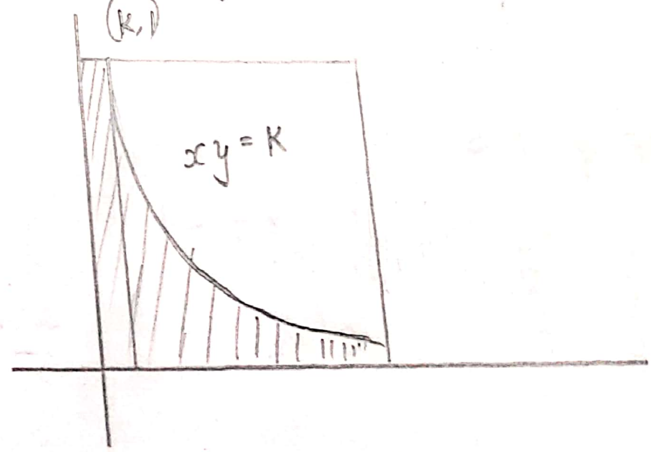
10. Q. Two numbers are independently chosen at random b/w 0 and 1. What is the probability that their product is less than a constant K $(0 < K < 1)$

Let the two numbers be X and Y

We are interested in calculating the probability of $P(XY < K)$

Now JPDF of (X, Y) is $f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } 0 < x, y < 1 \\ 0 & \text{o/w} \end{cases}$

Thus, $P(XY < K)$ = area of shaded region



$$= K + \int_K^1 \frac{K}{x}\,dx = K + \left[K \log(x)\right]_K^1$$

$$= K + K\log 1 - K\log K$$

$$= K(1 - \log K)$$

11. The joint PDF of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} 4xy & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal PDF's of X and Y.

b) joint CDF of (X, Y) and marginal CDF's of X and Y.

c) Joint MGF of (X, Y)

d) Find the co-variance Matrix and verify whether X and Y are independent.

e) Find $P(\{0 < X < 0.5, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$.

f) Find the marginal distributions of $V = \frac{X}{Y}$ and $V = XY$

The marginal PDF of X is

$$f_X(x) = \begin{cases} 4x \int_0^1 y\,dy & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{o/w} \end{cases}$$

The marginal PDF of Y is

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 < y < 1 \\ 0 & \text{o/w} \end{cases}$$

As $f_{XY}(x,y) = f_X(x) f_Y(y) \quad \forall x, y \in \mathbb{R}^2$

X, Y are independent.

$$P(0 < x < 0.5,\ 0.25 < Y < 1)$$

$$= P(0 < x < 0.5) P(0.25 < Y - 1)$$

$$= \left(\frac{1}{2}\right)^2 \times \left\{ 1 - \left(\frac{1}{4}\right)^2 \right\} = \frac{15}{64}$$

$$P(X+Y<1) = \iint_{x+y<1} f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 4xy\, dx\, dy$$

$$= 4 \int_0^1 y \int_0^{1-y} 4x\, dx\, dy$$

$$= \frac{4}{2} \int_0^1 y (1-y)^2\, dy$$

$$= \underline{\underline{\frac{1}{24}}}$$

Q13 ?? , Q14 ??

16. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$

$E(X^2) = E(Y^2) = 2$ and $\rho(X, Y) = \frac{1}{3}$.

Find $\rho\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)$.

<u>Sol^n:</u>

~~Corr~~ $\left(\frac{X}{}\right.$

Corr

$$\rho\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{Cov\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)}{\sqrt{Var\left(\frac{X}{3} + \frac{2Y}{3}\right) Var\left(\frac{2X}{3} + \frac{Y}{3}\right)}}$$

Now,

$$Cov\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{2}{9} Cov(X, X) + \frac{1}{9} Cov(X, Y) + \frac{4}{9} Cov(X, Y) + \frac{2}{9} Cov(Y, Y)$$

$Var(X) = 2 \qquad Var(Y) = 2$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}}$$

$$\frac{1}{3} = \frac{Cov(X, Y)}{2} \qquad \therefore Cov(X, Y) = \frac{2}{3}$$

$$\therefore Cov\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right)$$

$$= \frac{4}{9} + \frac{2}{27} + \frac{8}{27} + \frac{4}{9}$$

$$= \frac{34}{27}$$

$$Var\left(\frac{X}{3}+\frac{2Y}{3}\right) = \frac{1}{9} Var(X) + \frac{4}{9} Cov(X,Y) + \frac{4}{9} Var(Y)$$

$$= \frac{2}{9} + \frac{8}{27} + \frac{8}{9}$$

$$= \frac{38}{27}$$

$$Var\left(\frac{2X}{3}+\frac{Y}{3}\right) = \frac{4}{9} Var(X) + \frac{4}{9} Cov(X,Y) + \frac{1}{9} Var(X,Y)$$

$$= \frac{8}{9} + \frac{18}{27} + \frac{2}{3}$$

$$= \frac{38}{27}$$

∴ Reqd correlation is $\frac{17}{19}$

Q. Suppose that the random vector (X,Y) is uniformly distributed over the region $A = \{(x,y) : 0 < x < y < 1\}$. Find $Cov(X,Y)$.

The JPDF of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = 2\int_0^1\int_0^y xy\,dx\,dy = \int_0^1 y^3 dy = \frac{1}{4}$$

$$E(X) = 2\int_0^1\int_0^y x\,dx\,dy = \frac{1}{3}, \quad E(Y) = 2\int_0^1\int_0^y y\,dx\,dy = \frac{2}{3}$$

$$Cov(X,Y) = 1/36$$

Q Let the joint distribution of X and Y be

$f(x,y) = 3e^{-x}e^{-3y}$ for $x, y \geq 0$. and 0 otherwise.

Find the distribution of X/Y.

Q3 Let the joint distribution of X and Y be

$f(x,y) = 3e^{-x}e^{-3y}$ for ~~x, y ≥ 0~~ $x, y \geq 0$ and 0 oth

Find the distribution of $\frac{X}{Y}$.

Q15: Let the joint probability density f^n. of X and Y be

$f(x,y) = 5(x+y-3xy^2)$ if $0 < x < y < 1$ and 0 otherwise

a) Calculate $E(X/Y=y)$ and $V(Y/X=x)$

b) Is $E(XY) = E(X)E(Y)$

c) Find $P(X+Y \leq 1)$

Q 13 Let

$f(x,y) = 3e^{-x}e^{-3y}$ $\quad x \geq 0$, $y \geq 0$

$\therefore U = \frac{X}{Y}, \quad V = Y$

$X = UY$

$X = UV$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$$

$|J| = V$

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= 3e^{-uv} e^{-3v} v$$

$$\therefore f_{UV}(u,v) = 3ve^{-(u+3)v}$$

$$f(u) = \int_0^\infty f(u,v)\,dv$$

$$= \int_0^\infty 3ve^{-(u+3)v}\,dv$$

$$= 3\int_0^\infty ve^{-(u+3)v}\,dv$$

$$= 3\left\{\left[v\int e^{-(u+3v)v}\,dv\right] - \int_0^\infty\left[\int_0^\infty e^{-(u+3)v}\,dv\right]dv\right\}$$

$$= 3\left\{\left[v\,\frac{e^{-(u+3v)}}{-(u+3)}\right]_0^\infty - \int_0^\infty \frac{e^{-(u+3)v}}{-(u+3)}\,dv\right.$$

$$= 3\left\{0 + \frac{1}{u+3}\left[\frac{e^{-(u+3)v}}{-(u+3)}\right]_0^\infty\right.$$

$$= \frac{3}{(u+3)^2}\left[e^{-(u+3)v}\right]_\infty^0$$

$$= \frac{3}{(u+3)^2}[1-0]$$

$$= \frac{3}{(u+3)^2} \qquad 0 < u < \infty$$

dist^n of

$$\frac{X}{Y} = \frac{3}{\left(\frac{X}{Y}+3\right)^2} \qquad 0 < \frac{X}{Y} < \infty$$