Unit 4: Combinatorics

Topic 4: Generating Functions

Outline

Introduction

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.
Generating functions can be used to solve many types of counting problems.
Generating functions are good tools to solve a recurrence relations.

Definition (Generating Functions)

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Introduction

The generating function for the sequence

1. $\{2^n\}$ is $1 + 2x + 4x^2 + 8x^3 \cdots = \frac{1}{1-2x}$.
2. $1, 1, 1, 1, 1$ is $1 + x + x^2 + x^3 + x^4 = \frac{x^5-1}{x-1}$.
3. $a_n = C(m, n)$ is
   $C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m = (1 + x)^m$.

We assume the following for the sequences for which the generating functions will be considered:

1. Questions about the convergence of these series are ignored.
2. All functions are convergent for at least certain values of x around $x = 0$. In this case, the expansions may be written in a closed form.
3. A function has a unique power series expansion around $x = 0$.

Properties of Generating Functions

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then we have the following two operations on $f(x)$ and $g(x)$:

1. $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$
2. $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$.

In this regard, we have the following definition of the **extended binomial coefficient** for real u and non-negative k

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)(u-2)\cdots(u-k+1)}{k!}, & \text{if } k > 0; \\ 1, & \text{if } k = 0. \end{cases}$$

and hence $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.

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Using the extended binomial coefficient, we have

$$\binom{n}{r} = \begin{cases} C(n, r), & \text{if } n \text{ is positive integer;} \\ (-1)^r C(-n+r-1, r), & \text{if } n \text{ is negative integer.} \end{cases}$$

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As a result, one can have

$$(1+x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$

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Note the following power series:

- $(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$
- $\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k$

Problem

Problem: Show that if n is a positive integer, then $\binom{-\frac{1}{2}}{n} = \binom{2n}{n}(-4)^{-n}$. Hence show that the coefficient of x^n in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\binom{2n}{n}$.

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Hint: It is easy to see that

$$\begin{aligned}
\binom{-\frac{1}{2}}{n} &= \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{-2n+1}{2}\right)}{n!} \\
&= \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} \\
&= \frac{(-1)^n \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1) \cdot (2n)}{2^n \cdot n! \cdot 2 \cdot 4 \cdots (2n)} \\
&= (-1)^n \frac{(2n)!}{4^n \cdot (n!)^2} \\
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Note that

$$(1-4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-4x)^n.$$

Outline

Counting Problems and Generating Functions

Generating function is used to solve many problems of combinatorics. In particular, the generating function can be used to find the r-combination of n distinct objects with repetition and other restrictions without using inclusion-exclusion rule.

Ex: Find the number of solutions of $x_1 + x_2 + x_3 = 17$, where x_1, x_2, x_3 are non negative integers with $2 \leq x_1 \leq 5, 3 \leq x_2 \leq 6, 4 \leq x_3 \leq 7$.

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$$G(x) = (x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7),$$

which is equal to the coefficient of x^8 in the expansion of

$$H(x) = (1 + x + x^2 + x^3)(1 + x + x^2 + x^3)(1 + x + x^2 + x^3) = \left(\frac{1 - x^4}{1 - x}\right)^3.$$

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The required solution is

$$C(3, 0)C(10, 8) - C(3, 1)C(6, 4) + C(3, 2)C(2, 0).$$

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Hint: Find coefficient of x^{25} in the expansion of the generating function

$$G(x) = (x^3 + x^4 + x^5 + x^6 + x^7)^4 = x^{14}(1 + x + x^2 + x^3 + x^4)^4.$$

Outline

Recurrence Relations and Generating Functions

Generating function is a good tool to solve many recurrence relation those can not be solved using the known methods. In this method, recurrence relation is translated into an equation involving a generating function involving terms of the sequence of the recurrence relation. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

Problem

Problem: Solve the following recurrence relations using generating functions:

1. $a_n = 8a_{n-1} + 10^{n-1}$ with $a_1 = 9$.
2. $a_n = a_{n-1} + 2a_{n-2} + 2^n$ with $a_0 = 4, a_1 = 12$.

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Solution: (i) Let the solution $\{a_n\}$ of the reccurence relation is given by the generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Thus the given reccurence relation can be written as

$$\sum_{n=1}^{\infty} a_n x^n = 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$

$$\Rightarrow G(x) - a_0 = xG(x) + \frac{x}{(1-10x)} \text{ note that } a_0 = 1$$

$$\Rightarrow G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right).$$

As a result, $a_n = \frac{1}{2}(8^n + 10^n)$.

(ii) Let the solution $\{a_n\}$ of the reccurence relation is given by the generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$\sum_{n=2}^{\infty} a_n x^n = x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \sum_{n=2}^{\infty} x^n 2^n$$

$$\Rightarrow G(x) - a_0 - a_1 x = G(x) - a_0 x + 2x^2 G(x) + \frac{4x^2}{1-2x}$$

$$\Rightarrow G(x) = \frac{4-12x^2}{(1-2x)^2(x+1)} = \frac{38}{9}\frac{1}{1-2x} + \frac{2}{3}\frac{1}{(1-2x)^2} - \frac{8}{9}\frac{1}{x+1}$$

Thus

$$a_n = \frac{38}{9}2^k + \frac{2}{3}(n+1)2^k - \frac{8}{9}(-1)^k.$$

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Ex: Solve the recurrence relations $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ with $C_0 = C_1 = 1$ using generating functions.

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Solution: Let the solution $\{C_n\}$ of the reccurence relation is given by

$$G(x) = \sum_{n=0}^{\infty} C_n x^n.$$

We first note that

$$G(x)^2 = \left(\sum_{i=0}^{\infty} C_i x^i\right)\left(\sum_{j=0}^{\infty} C_j x^j\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} C_j C_{i-j}\right) x^i = \sum_{i=0}^{\infty} C_{i+1} x^i = \frac{1}{x}(G(x) - C_0).$$

Hence

$$xG(x)^2 - G(x) + 1 = 0$$

$$\Rightarrow G(x) = \frac{1}{2x}(1 - \sqrt{1-4x})$$

$$\Rightarrow \frac{d}{dx}(xG(x)) = (1-4x)^{\frac{-1}{2}} \Rightarrow xG(x) = \int_{x=0}^{x} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4)^n x^n$$

$$\Rightarrow C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Thank You

Any Question!!!