MA 203

Joint Random Variable

Example 1: Consider two jointly distributed RVs X and Y with the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-y}); x \geq 0, y \geq 0 \\ 0; \quad Otherwise \end{cases}$$

(a) Find the marginal CDFs.

(b) Find the probability P(1 < X ≤ 2, 1 < Y ≤ 2).

Sol.: (a) We know that

$$F_X(x) = F_{X,Y}(x,\infty) = \lim_{y\to\infty} F_{X,Y}(x,y) = \begin{cases} \lim_{y\to\infty} (1-e^{-2x})(1-e^{-y}) \\ 0 \quad ; o.w. \end{cases}$$

$$= \begin{cases} (1-e^{-x}) \; ; x \geq 0 \\ 0 \quad ; o.w. \end{cases}$$

Similarly,

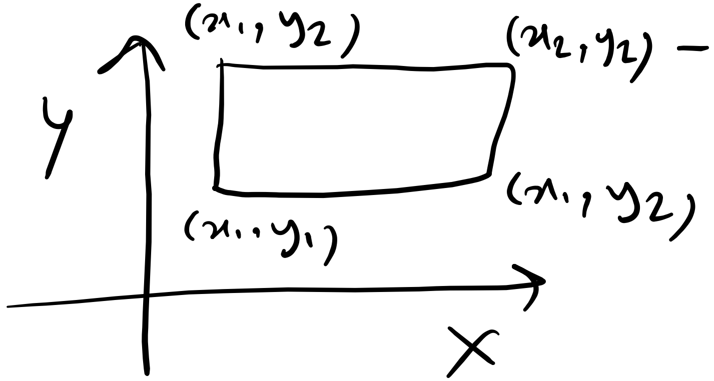
$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = \begin{cases} \lim\limits_{x \to \infty} (1-e^{-2x})(1-e^{-y}) & ; x \geq 0, y \geq 0 \\ 0 & ; o.w. \end{cases}$$

$$= \begin{cases} (1-e^{-y}) & ; y \geq 0 \\ 0 & ; o.w. \end{cases}$$

(b) $P(1 < X \leq 2,\ 1 < Y \leq 2)$?

$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2)$

$- F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1)$



$x_1 = y_1 = 1 \quad ; \quad x_2 = y_2 = 2$

$$P(1 < X \leq 2,\ 1 < Y \leq 2) = F_{X,Y}(2,2) - F_{X,Y}(1,2) - F_{X,Y}(2,1) + F_{X,Y}(1,1)$$

$$= (1-e^{-4})(1-e^{-2}) - (1-e^{-2})(1-e^{-2}) - (1-e^{-4})(1-e^{-1}) + (1-e^{-2})(1-e^{-1})$$

$$= \underline{0.0272}$$

Joint Probability Distribution Function (PDF)

If $F_{X,Y}(x,y)$ is continuous in both x and y, then joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$ provided it exists.

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

Properties of Joint PDF

1. $f_{X,Y}(x,y)$ is always a non-negative quantity. That is, $f_{X,Y}(x,y) \geq 0 \; \forall (x,y) \in R_X X R_Y$.

2. $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1$

Proof: $F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$

$$\underline{F_{X,Y}(\infty,\infty)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv\, du = 1$$

3. Marginal PDF of X, $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$

Proof: $f_X(x) = \frac{d}{dx} \underline{F_X}(x) = \frac{d}{dx} F_{X,Y}\underline{(x,\infty)}$

$$f_X(x) = \frac{d}{dx}\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,dy\,du = \frac{d}{dx}\int_{-\infty}^{x}\underbrace{\left\{\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,dy\right\}}_{g(u,y)} du$$

$$= \frac{d}{dx}\int_{-\infty}^{x} \underline{g(u,y)}\cdot \underline{du}$$

Using Leibniz Rule

$$= g(x,y)\times\frac{d(x)}{dx} - \cancel{g(-\infty,y)\cdot\frac{d(-\infty)}{dx}}^{\,0} + \cancel{\int_{-\infty}^{x}\frac{\partial}{\partial x} g(u,y)\,du}^{\,0}$$

$$f_X(x) = g(x,y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

Leibniz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} \underline{f(x,t)}\underline{dt} = f\big(x,b(x)\big)\times\frac{db(x)}{dx} - f\big(x,a(x)\big)\times\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt$

$$\underline{f_X(x)} = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

Similarly,

$$\underline{f_Y(y)} = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx$$

Example 2: The joint PDF of two RVs X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} cxy; 0 \le x \le 2, 0 \le y \le 2 \\ 0; Otherwise \end{cases}$$

(a) Find c

(b) Find $F_{X,Y}(x,y)$

(c) Find $f_X(x)$ and $f_Y(y)$

(d) What is the probability $P(0 < X \le 1, 0 < Y \le 1)$?

Sol:- (a) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{0}^{2}\int_{0}^{2} cxy\,dx\,dy = 1$$

$$\Rightarrow c \times 4 = 1 \Rightarrow c = 1/4$$

(b) $$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$$

$$= \int_{0}^{x}\int_{0}^{y} \frac{uv}{4}\,dv\,du = \frac{x^2 y^2}{16}$$

(c) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) \cdot dy$

$= \int_0^2 \frac{xy}{4} dy = \frac{x}{2}$; $0 \leq x \leq 2$

$f_Y(y) = \int_0^2 \frac{xy}{4} dx = \frac{y}{2}$; $0 \leq y \leq 2$

(d) $P(0 < X \leq 1, 0 < Y \leq 1) = F_{X,Y}(1,1) - F_{X,Y}(0,1) - F_{X,Y}(1,0) + F_{X,Y}(0,0)$

$= \frac{1}{16} - 0 - 0 + 0 = 1/16$

$$\underbrace{p_{X,Y}(x_1,y_1)}_{1/2} + \underbrace{p_{X,Y}(x_2,y_2)}_{0} + \underbrace{p_{X,Y}(x_1,y_2)}_{0} + \underbrace{p_{X,Y}(x_2,y_1)}_{1/2} = 1$$

$$\Rightarrow \quad \sum_{(x,y)\in R_X\times R_Y}\sum p_{X,Y}(x,y) = 1$$

$$\Rightarrow \quad (x,y) \notin R_X \times R_Y$$

$$p_{X,Y}(x,y) = 0$$

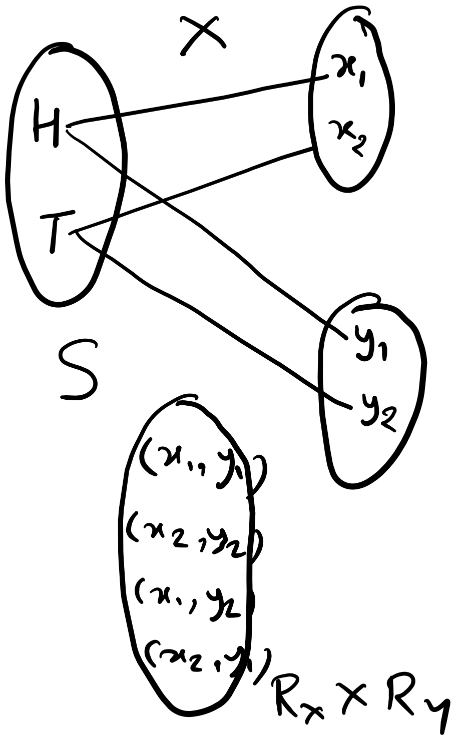
Case 2: When X and Y are discrete RVs

Joint probability mass function (PMF):

The joint PMF of joint RV (X,Y) is defined as

$$p_{X,Y}(x,y) = P(s|X(s) = x, Y(s) = y)\ \forall (x,y) \in R_X X R_Y$$

Ex:-



$R_x = \{x_1, x_2\}$; $R_y = \{y_1, y_2\}$

$R_x \times R_y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$

$p_{X,Y}(x_1, y_1) = P(H \mid X(H) = x_1, Y(H) = y_1)$

$= 1/2$; $p_{X,Y}(x_2, y_1) = 0$ $(x_1, y_1) \in R_x \times R_y$

$p_{X,Y}(x_1, y_2) = 0$; $p_{X,Y}(x_2, y_2) = 1/2$

Properties of joint PMF

1. $p_{X,Y}(x, y) = 0 \ \forall (x, y) \notin R_X X R_Y$

2. $$\sum_{(x,y) \in R_X \times R_Y} \sum p_{X,Y}(x, y) = 1$$

3. $$p_X(x) = \sum_{y \in R_Y} p_{X,Y}(x, y)$$

$R_X = \{x_1, x_2\}$ $R_Y = \{y_1, y_2\}$

$p_X(x_1) = p_{X,Y}(x_1, y_1)$

$+ p_{X,Y}(x_1, y_2)$

Example 3: Consider the RV X and Y with the joint PMF as tabulated in Table 1. Find marginal probabilities $p_X(x)$ and $p_Y(y)$**?**

$Y \backslash X$	0	1	2
0	$p_{X,Y}(0,0)$ 0.25	$p_{X,Y}(1,0)$ 0.10	$p_{X,Y}(2,0)$ 0.15
1	0.14	0.35	0.01

$p_Y(y)$	y
0.5	0
0.5	1

$R_X = \{0, 1, 2\}$

$R_Y = \{0, 1\}$

$p_X(x)$	x
0.39	0
0.45	1
0.16	2