SC 201-Physics I

Classical Mechanics

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Classical Mechanics

- Classical mechanics is the branch of Physics which deals with the motion of physical bodies at macroscopic level.

- Classical mechanics is based on Newton's laws of motion, so often called as Newtonian mechanics.

- To study the *motion of a physical body*, the most basic parameters are **space and time**. (both are continuous).

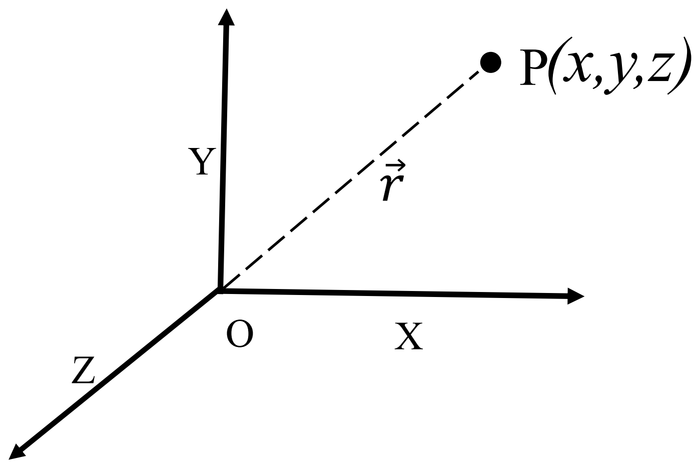
- So, to describe *the motion of a body* , one has to specify its *position in space as a function of time.*

- This needs a suitable choice of a **coordinate system**.

Frame of reference

- If we imagine a coordinate system attached to a rigid body and we describe the position of any particle w.r.t it, then such coordinate system is called as *frame of reference (FOR).*

- The simplest FOR is Cartesian coordinate system *(x,y,z)*



Types of frame of reference

1. Inertial frame of reference
2. Non-inertial frame of reference

1. Inertial frame of reference:

i. An inertial frames of reference are the frames, in which *law of inertia holds* and other *laws of physics are valid.*

ii. These frames are *unaccelerated frames (*at rest or moving with constant velocity) (acceleration of the frame, a_f=0).

iii. All frames which are moving with constant velocity w.r.t. an inertial frame are also inertial.

2. Non-inertial frame of reference:

i. If a frame is *accelerated* w.r.t an inertial frame (road, train moving with constant velocity), these frames are called non-inertial frames.(acceleration of the frame, $a_f \neq 0$)

ii. In non-inertial frames, *newton's laws of motion are not valid.*

iii. In non-inertial frames, $(ma \neq F)$

$$ma - F \neq 0$$

$$= F_p\text{, Pseudo force}$$

Therefore by adding an extra pseudo force with F, we are able to do all operations, like inertial frames.

Various Coordinate systems

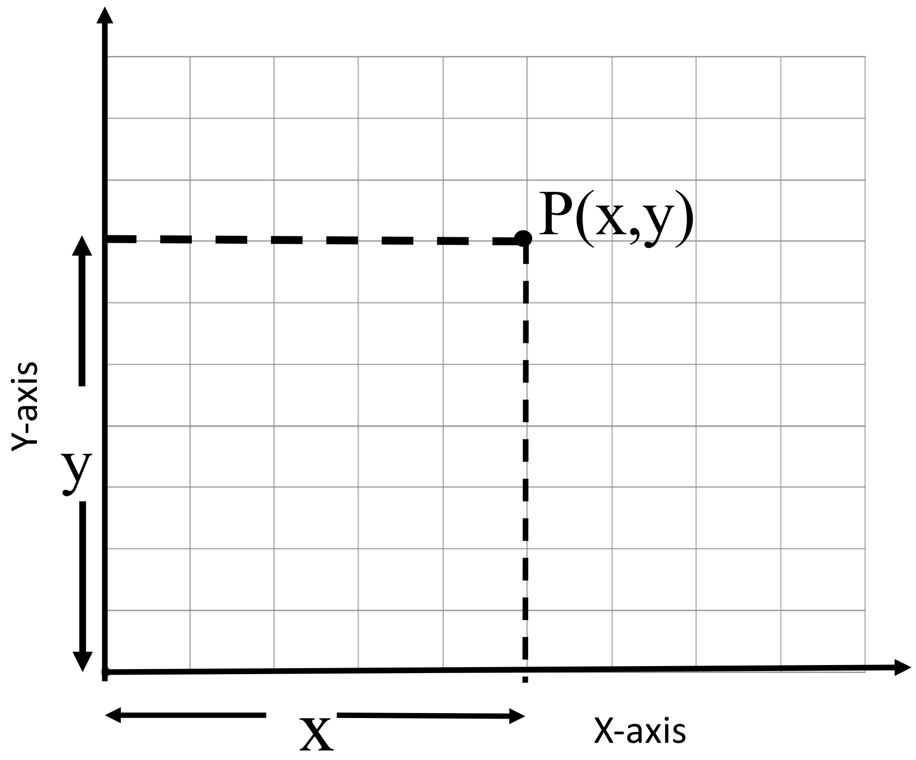
To specify the *position and motion of a body* in specific frame of reference, we have various coordinate systems.

i. *Cartesian or rectangular* coordinate system (x,y)
ii. *Plane polar* coordinate system (r,θ)
iii. *Cylindrical* coordinate system (ρ,ϕ,z)
iv. *Spherical polar* coordinate system (r,θ, ϕ)

- Frequently used 2D coordinate system *is* ***Cartesian or rectangular*** coordinate system.

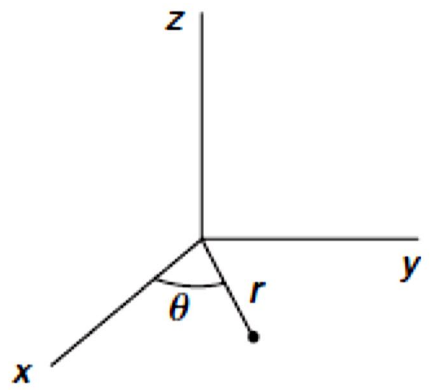
Cartesian or rectangular coordinate system

- To specify the position of any point in space , we need two perpendicular coordinates X-axis and Y-axis.
- We represent the position of the point P as P(x,y) in CC.
- It is a 2D coordinate system suitable for straight line motion.



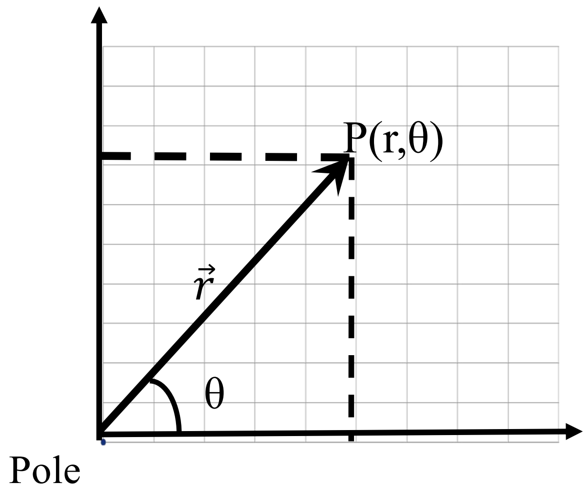
Plane polar coordinates

- To describe *circular motion*, we need plane polar coordinate system.



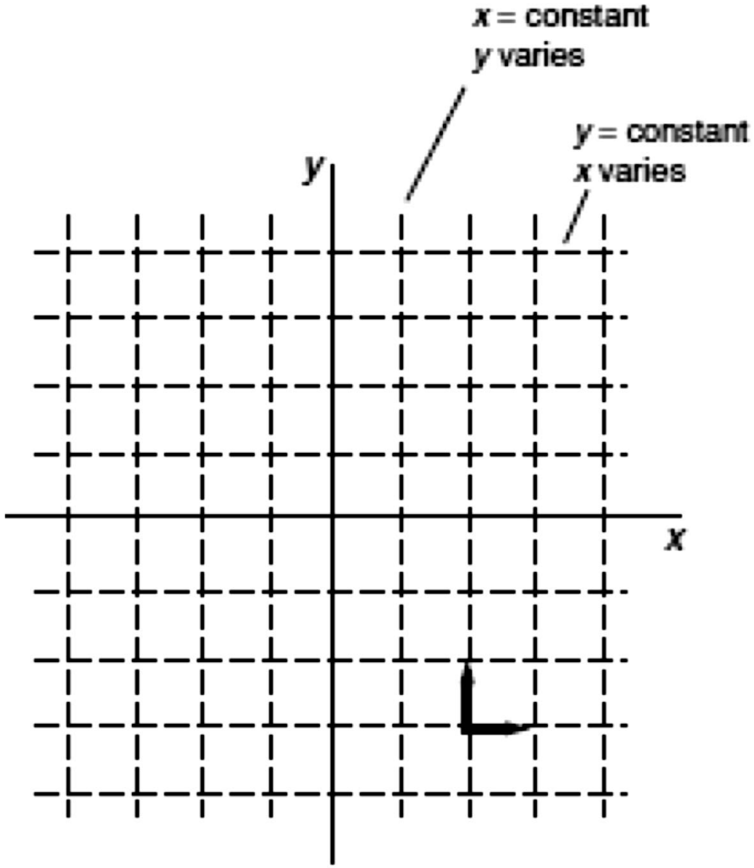
- This *two-dimensional coordinate system* is based on the three dimensional cylindrical coordinate system.

- In this coordinate system, position of particle P is represented by a ordered pair *(r, θ)*, where
  r—radial distance from origin /pole.
  θ---angle from a fixed direction(x-axis).

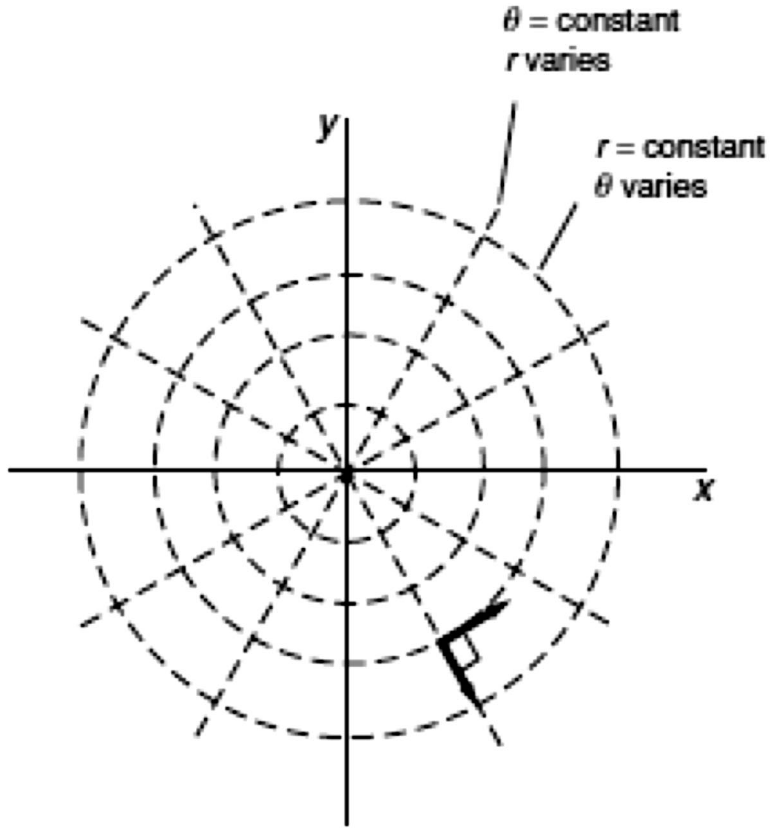


- The coordinates *r* and θ are called *plane polar* coordinates, as motion is restricted in x-y plane here.

Comparison of constant coordinate lines for the Cartesian coordinate system and for the plane polar coordinate system:



Cartesian



Plane polar

θ is +ve if we move counter clockwise from x-axis.

Relation Between Cartesian coordinates(CC) and plane polar coordinates(PPC).

In Cartesian coordinates, position vector r is given as

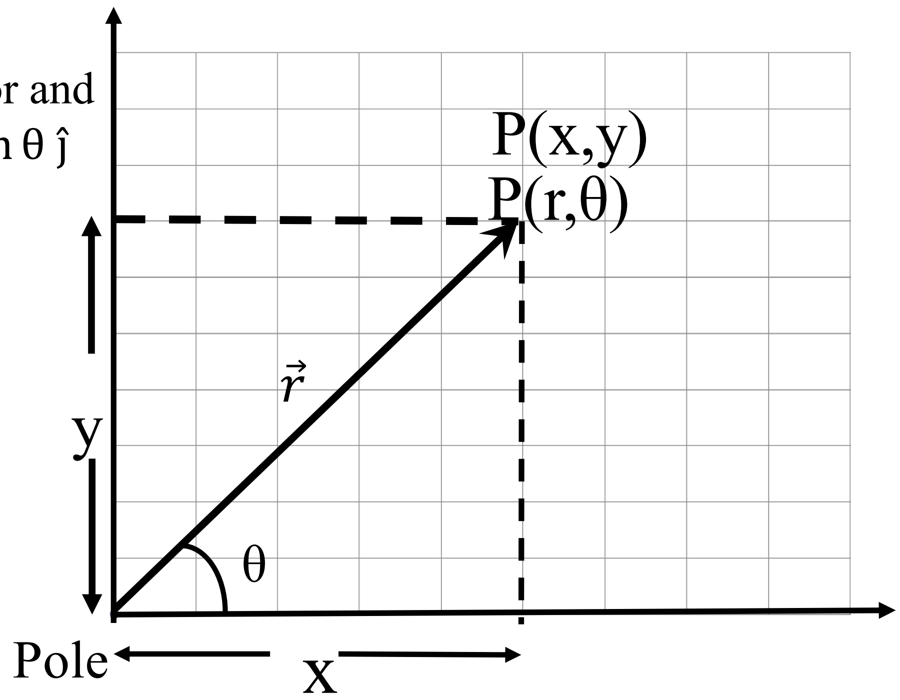
$$\vec{r}=x\hat{\imath}+y\hat{\jmath}$$

while in polar coordinates, we have

$$\vec{r}=r\,\hat{\boldsymbol{r}}$$

Here, $\hat{r}$ is radial unit vector and is given as $\hat{r}$ =cos θ î + sin θ ĵ

$$=r\,(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath}\,)$$



On comparing above equations, we have,

$$x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}=\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath}$$

$$\boldsymbol{x=r\cos\theta}$$

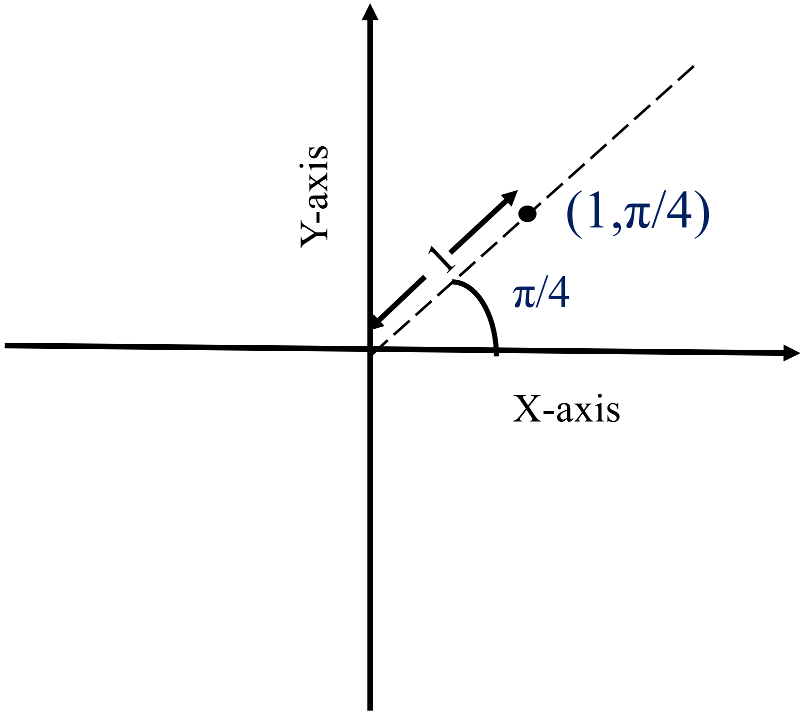
$$\boldsymbol{y=r\sin\theta}$$

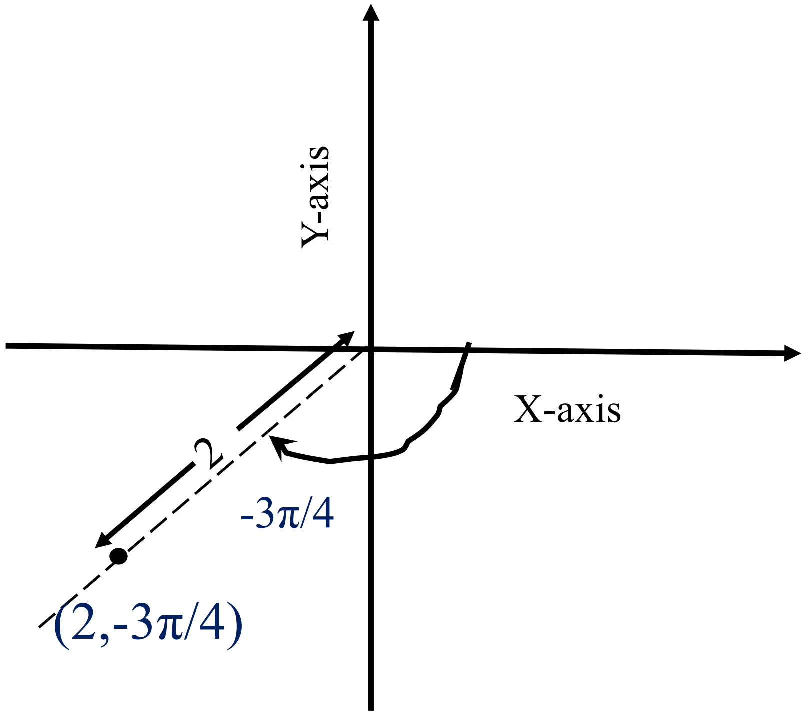
On squaring &adding and then taking ratio, we have

$$\boxed{\begin{array}{l}\boldsymbol{r=\sqrt{x^2+y^2}}\\ \boldsymbol{\theta=\tan^{-1}\frac{y}{x}}\end{array}}$$

Problem 1

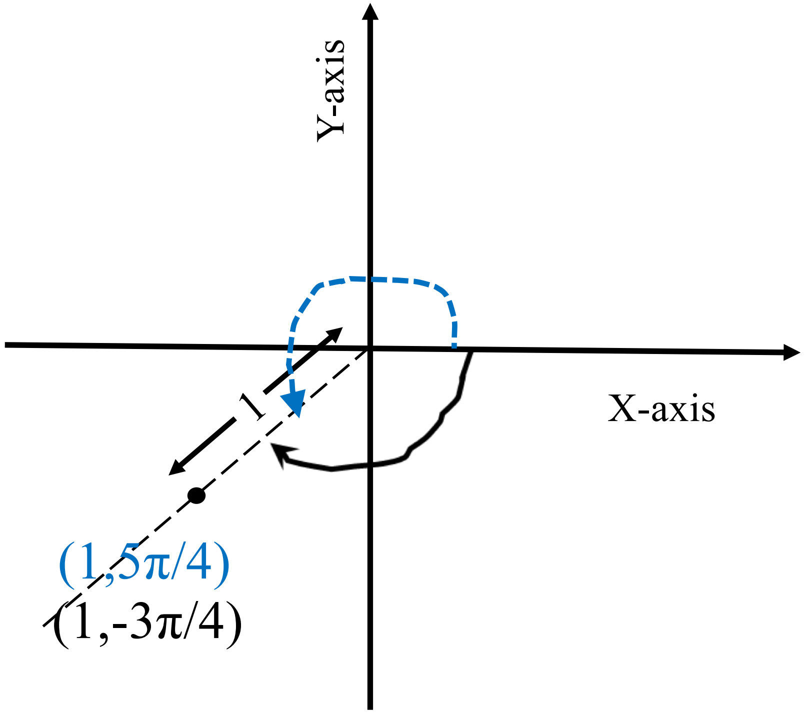
Plot the points whose polar coordinates are $(1, \pi/4)$ and $(2, -3\pi/4)$.





Problem 2

Plot the points having polar coordinates (1,5π/4) and (1,-3π/4)



In CC system, every point has only one representation, but in PPC system, each point may has many representation.

Problem 3

i. **Convert the point (2,π/3) in CC.**

x =r cos θ x=2cos π/3

$=2\times\frac{1}{2}$

$=1$

The answer is $(1, \sqrt{3}\ )$

y= r sin θ x=2sin π/3

$=2\times\frac{\sqrt{3}}{2}$

$=\sqrt{3}$

ii. **Represent the point(-3,4) in PPC.**

If we choose r positive, then $r = \sqrt{x^2 + y^2}$

$=\sqrt{(-3)^2+(4)^2}$

$=5$

$$\theta = \tan^{-1}\frac{(4)}{-3} = \tan^{-1}\left(\frac{-4}{3}\right) = 127^0$$

The answers is $(5, 127^0\ )$

<u>Problem 4</u>

Express the equation $x^2=4y$ in polar coordinates.

Put the value of $x = r\cos\theta$ & $y = r\sin\theta$ in the equation, we get

$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$$r\cos^2\theta = 4\sin\theta$$

$$r = 4\sin\theta/\cos^2\theta$$

$$r = 4(\sin\theta/\cos\theta)(1/\cos\theta)$$

$$r = 4\tan\theta\sec\theta$$

You can notice, now equation is having only polar coordinates r and θ.

Problem 5

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian coordinates.

$$r = \sin(2\theta)$$
$$r = 2 \sin\theta \cos\theta$$
$$r = 2.\, y/r.\, x/r$$
$$r = 2xy/ r^2$$
$$r^3 = 2xy$$
$$(x^2 + y^2)^{3/2} = 2xy$$

$x = r\cos\theta$ & $y = r\sin\theta$

Square both sides,

$$(x^2 + y^2)^3 = 4\, x^2 y^2$$

Problem 6

Express the equation $x^2+y^2= 6y$ in polar coordinates.

Put the value of $x = r\cos\theta$ & $y= r\sin\theta$ in the equation, we get

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 6\, r\sin\theta$$

$$r^2[\cos^2\theta + \sin^2\theta]=6\, r\sin\theta$$

As, $\cos^2\theta + \sin^2\theta=1$

$$r=6\sin\theta$$

Problem 7

Rewrite the polar equation $r = \frac{3}{1-2cos\theta}$ as a Cartesian equation.

$r(1\text{-}2\cos\theta) = 3$

$r\text{-}2r\cos\theta = 3$ Put $r\cos\theta = x$

$r\text{-}2x = 3$

$r = 3+2x$

$r^2 = (3+2x)^2$ Squaring both sides

$(x^2 + y^2) = (3+2x)^2$ Putting value of r^2

$(x^2 + y^2) = (9+4x^2+12x)$ Keep solving and arranging until you get a specific equation

$y^2\text{-} 3x^2\text{-}12x = 9$

$y^2\text{-} 3(x+2)^2 = \text{-}3$

$$\boxed{(x+2)^2\text{-}y^2/3=1}$$

Unit Vectors in Cartesian coordinates

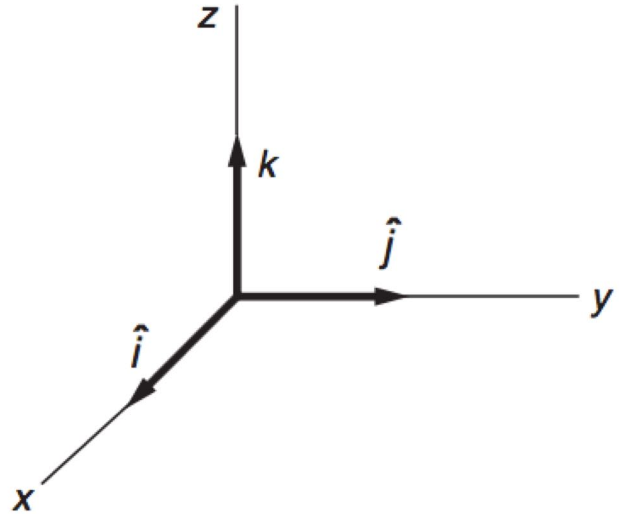
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

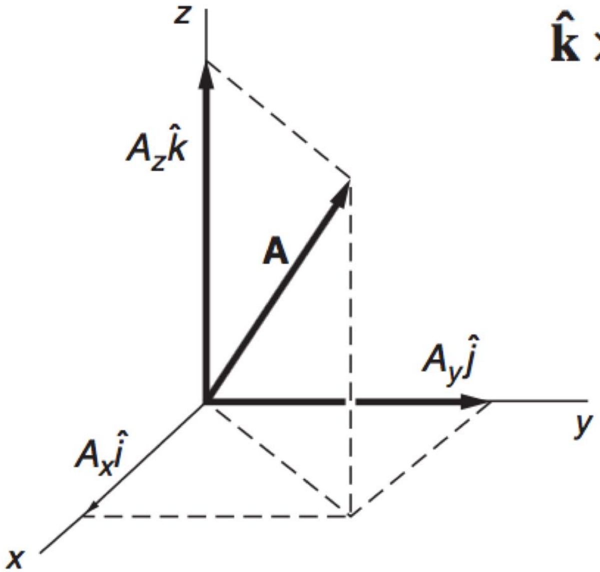
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}.$$





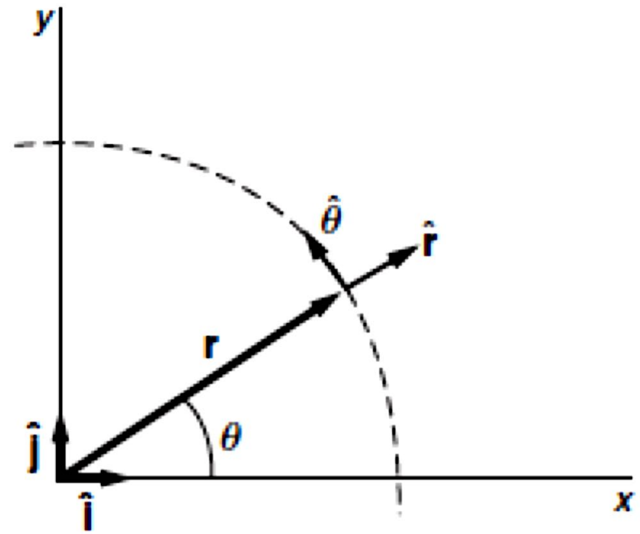
$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

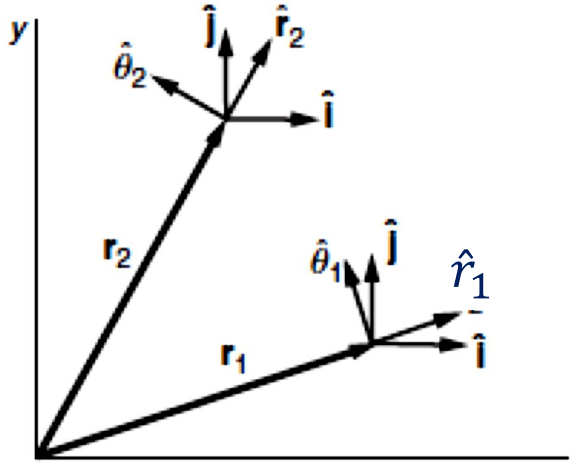
Unit Vectors in Polar coordinates

- In CC, there are base unit vectors $\hat{\imath}$ and $\hat{\jmath}$ where $\hat{\imath}$ indicates the direction of increasing x and $\hat{\jmath}$ indicates the direction of increasing y.

- In the same way, in PPC also, we have two base unit vectors, $\hat{r}$ and $\hat{\theta}$ that points in the direction of increasing r and increasing θ.

- The directions of $\hat{r}$ and $\hat{\theta}$ vary with position, whereas $\hat{\imath}$ and $\hat{\jmath}$ have fixed directions.

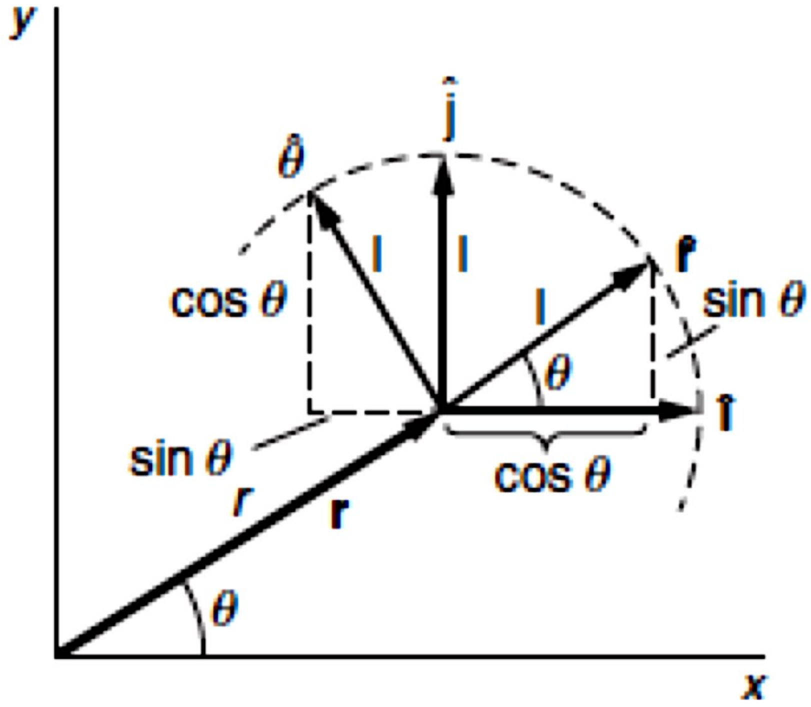




Unit vectors represention

$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}.$$



Properties of unit vectors of PCC

1. $|\hat{r}| = |\hat{\theta}| = 1$

2. $\hat{r} \, . \, \hat{\theta} = 0$

$\hat{r}$ and $\hat{\theta}$ are **orthogonal**

$Value\ of\ \frac{d\hat{r}}{dt}\ and\ \frac{d\hat{\theta}}{dt}$

$\underline{On\ differentiating\ \hat{r}\ and\ \hat{\theta}, we\ have}$

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos\theta)\,\hat{\mathbf{i}} + \frac{d}{dt}(\sin\theta)\,\hat{\mathbf{j}}$$
$$= -\sin\theta\,\dot{\theta}\,\hat{\mathbf{i}} + \cos\theta\,\dot{\theta}\,\hat{\mathbf{j}}$$
$$= (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\,\dot{\theta}.$$
$$= \dot{\theta}\,\hat{\boldsymbol{\theta}}.$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = (-\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}})\,\dot{\theta}$$
$$= -\dot{\theta}\,\hat{\mathbf{r}}.$$

Standard Notation

$$\frac{dr}{dt} = \dot{r} \qquad \frac{d^2r}{dt^2} = \ddot{r}$$

$$\frac{d\theta}{dt} = \dot{\theta} \qquad \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

$$\boxed{\frac{d\hat{\boldsymbol{r}}}{dt} = \dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\boldsymbol{\theta}}\hat{\boldsymbol{r}}}$$

Motion in Plane Polar Coordinates

Velocity in plane polar coordinates

The position vector $\vec{r}$ in polar coordinate is given by $\vec{r} = r\hat{r}$

The velocity in PPC is given by $\vec{v} = \frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(r\hat{r})$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$= radial\ velocity + tangential\ velocity$$

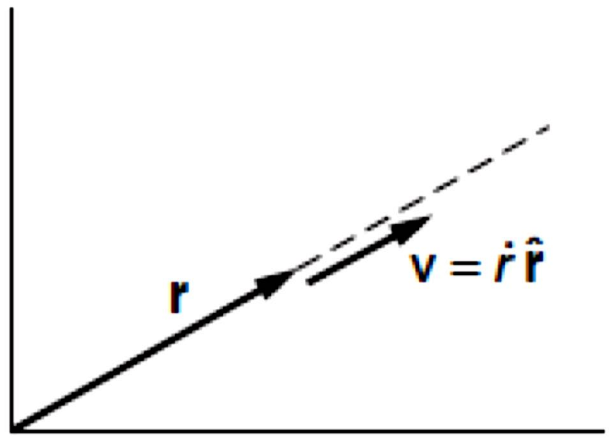
Therefore, the velocity in PPC is a combination of radial and tangential motion.

<u>Case 1: Radial velocity (θ = constant, *r* varies)</u>.

If, θ is a constant, $\dot{\theta}$= 0,

and, $\vec{v} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}}$

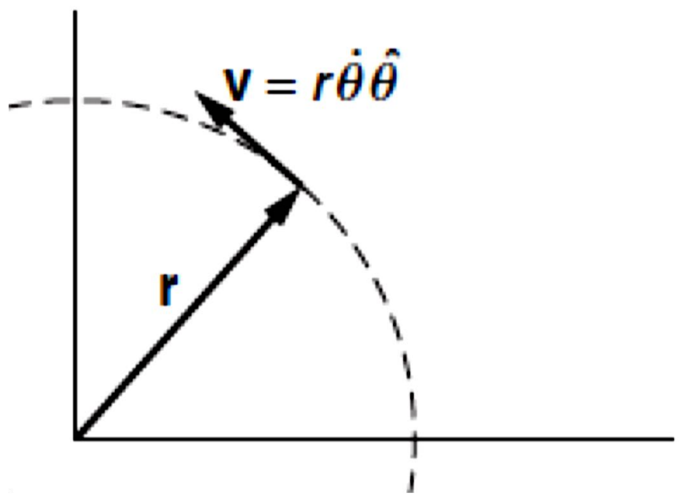
This implies one-dimensional motion in a fixed *radial* direction.



<u>Case 2: Tangential velocity (*r* = constant, θ varies)</u>.

In this case $\vec{v} = r\dot{\theta}\hat{\theta}$

Since *r* is fixed, the motion lies on the arc of a circle(in the *tangential* direction).



Acceleration in plane polar coordinates

Acceleration in plane polar coordinates, $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$= \frac{d}{dt}(\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}})$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\,\hat{\mathbf{r}}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}.$$

= radial acceleration + tangential acceleration

$$\vec{a} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta}$$

the Coriolis acceleration

the centripetal acceleration radially inward

the acceleration that arises from the changing tangential speed.

the acceleration due to a change in radial speed.

- The Coriolis acceleration we discussed here, is a real acceleration that is present whenever r and θ both change with time.

- Half of the Coriolis acceleration is due to the change in direction of the radial velocity, $dv_r/dt = v_r\,\dot{\theta}$.

- The other half arises, due to tangential speed $v_\theta = r\dot{\theta}$. If r changes by Δr, then v_θ changes by $\Delta v_\theta = \Delta r\dot{\theta}$, and the contribution to the tangential acceleration is therefore $\dot{r}\dot{\theta}$, the other half of the Coriolis acceleration.

<u>Newton's law in plane polar coordinates</u>

$$\vec{F} = m\vec{a}$$

$$= m[(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}]$$

In radial direction, $F_r = \mathrm{m}(\ddot{r} - r\dot{\theta}^2)$

In tangential direction, $F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Newton's law in polar coordinates do not follow its Cartesian form as,

$$F_r \neq m\ddot{r} \quad \text{or} \quad F_\theta \neq m\ddot{\theta}$$

It means, the form of Newton's law is different in different coordinate systems.

Problem-1

A particle moves in a circle of radius b with angular velocity $\dot{\theta} = \alpha t$, where α is a constant. (α has the units rad/s^2.) Describe the particle's velocity in polar coordinates. Find particle's position also.

Sol.

Velocity of the particle is given as $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ 1

Since $r = b =$ constant,

Therefore, $\dot{r} = 0$

Also, $\dot{\theta} = \alpha t$

Putting these values in 1, we get

$\boxed{\vec{v} = b\alpha t\hat{\theta}}$ **v** is purely tangential

Position of the particle is given as

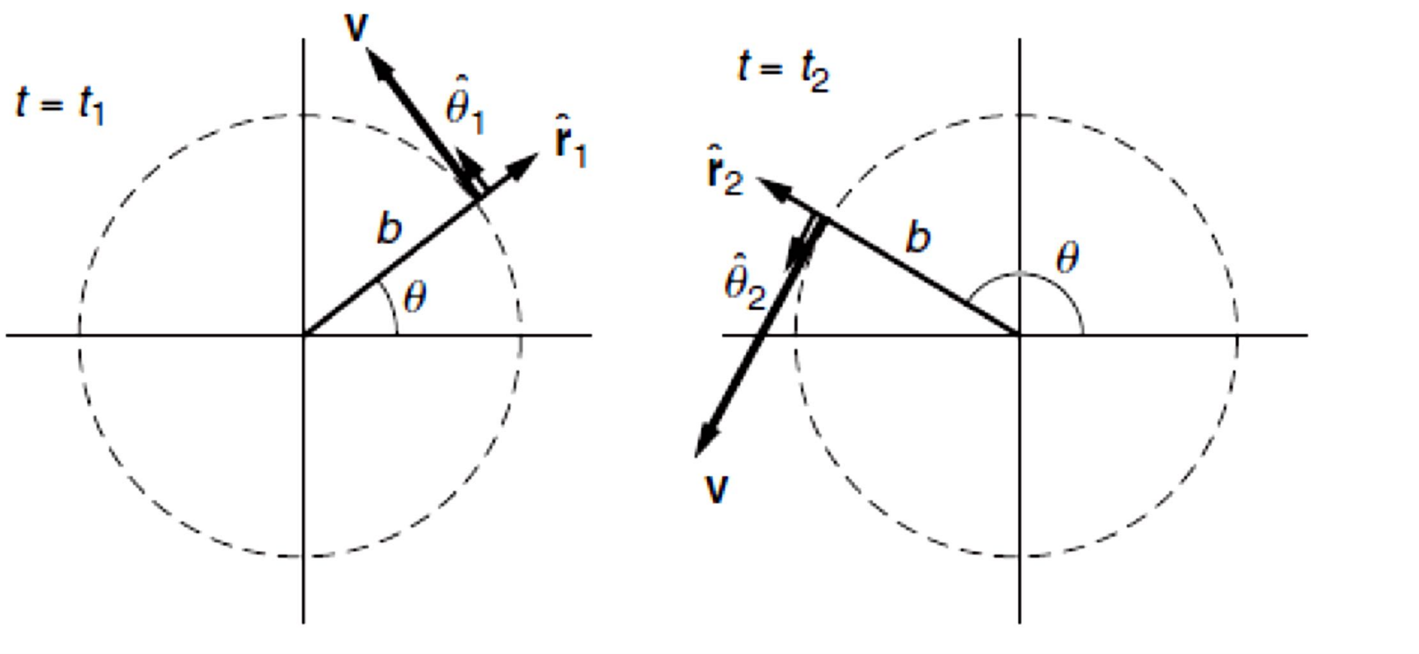
$$r = b, \quad \theta = \theta_0 + \int_0^t \dot{\theta} dt = \theta_0 + \tfrac{1}{2}\alpha t^2.$$

$$\boxed{\left(b, \frac{1}{2}\alpha t^2\right)}$$

If the particle is on the x axis at $t = 0$, then $\theta_0 = 0$.

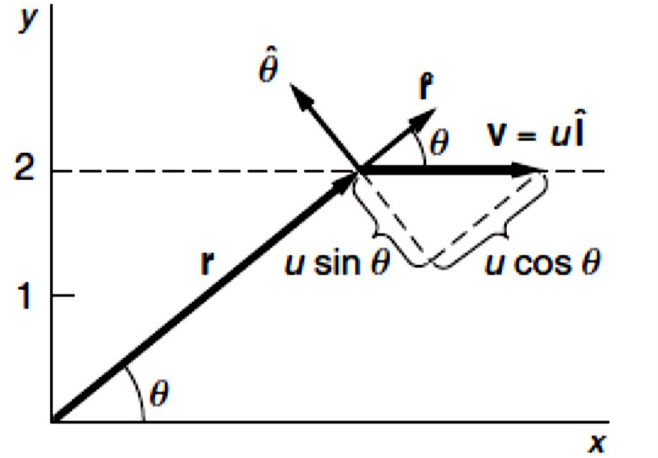
Discussion:

The particle's position vector is $\vec{r} = b\hat{r}$ but as diagram indicates, θ must be given to specify the direction of $\hat{r}$.



Problem-2

Consider a particle moving with constant velocity $\mathbf{v} = u\hat{\mathbf{i}}$ along the line $y = 2$. Describe $\mathbf{v}$ in polar coordinates:



Sol.

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$$

From the sketch,

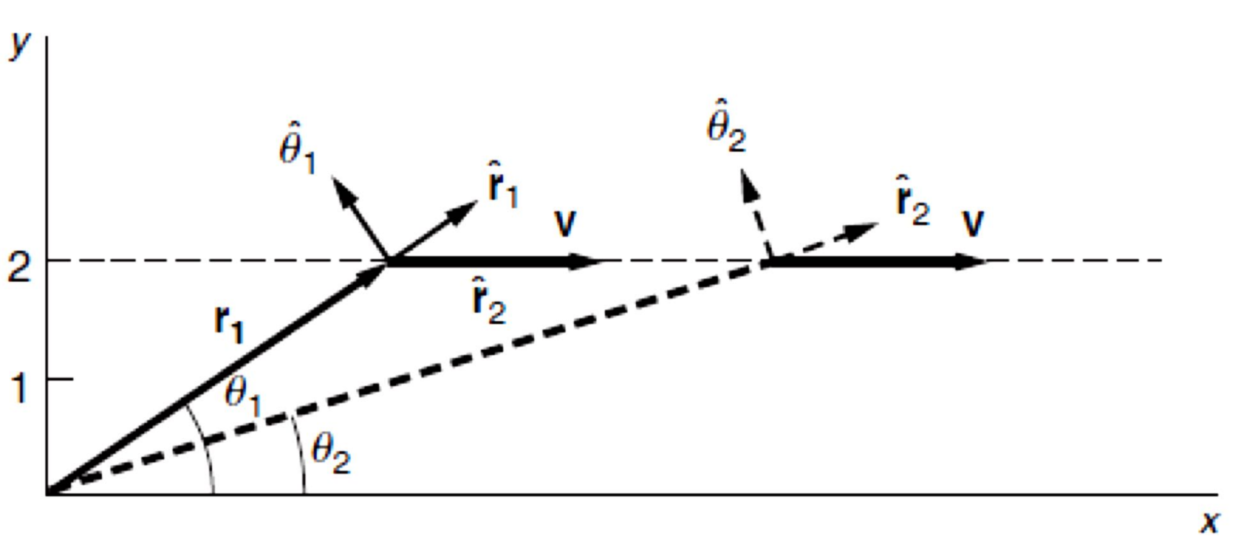
$$v_r = u\cos\theta$$

$$v_\theta = -u\sin\theta$$

$$\mathbf{v} = u\cos\theta\,\hat{\mathbf{r}} - u\sin\theta\,\hat{\boldsymbol{\theta}}.$$

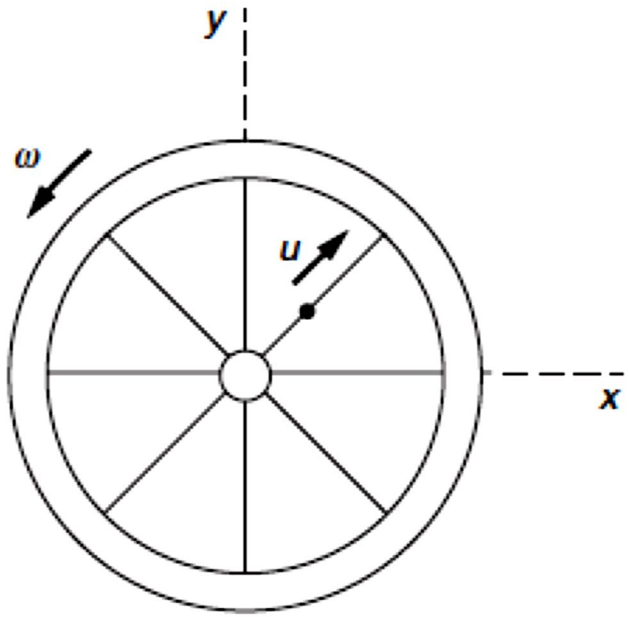
Discussion:

As the particle moves to the right, θ decreases and $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ change



Problem-3:

A bead moves along the spoke of a wheel at constant speed u meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis, and the bead is at the origin. Find the velocity at time t



Sol.

In polar coordinates, $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$. Hence

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = u\,\hat{\mathbf{r}} + u\omega t\,\hat{\boldsymbol{\theta}}.$$

At time t, the bead is at radius ut on the spoke, and the spoke makes angle ωt with the x axis.

Problem-4:

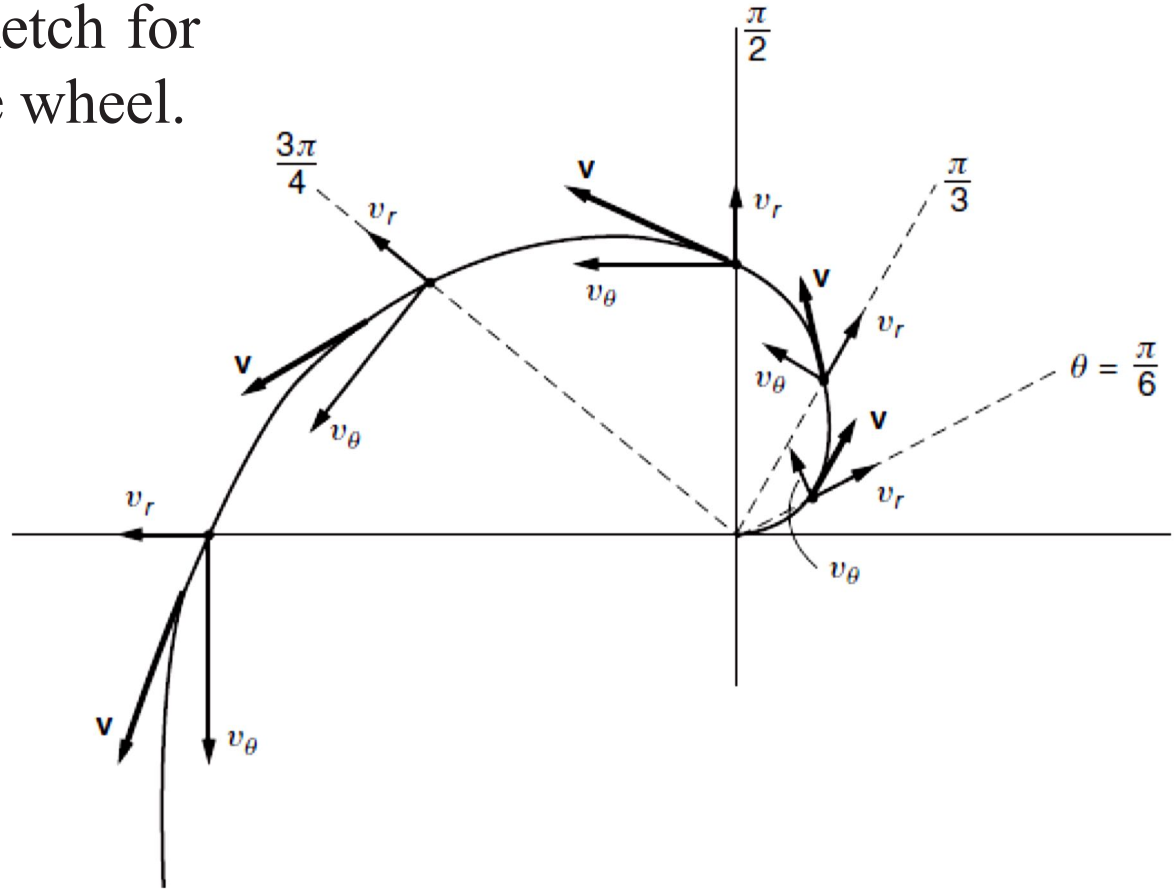
A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the acceleration.

The acceleration is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.\end{aligned}$$

Discussion:

The velocity is shown in the sketch for several different positions of the wheel.



Here, the radial velocity is constant.
The tangential acceleration is also constant