A random process $\{X(t), t \geq 0\}$ is called a Weiner process if
1. $X(t)$ has stationary independant increments
2. The increment $X(t) - X(s)$ $(t > s)$ is normally distributed.
3. $E(X(t)) = 0$
4. $X(0) = 0$.

Examples

PROBLEM SHEET 7

1. Families of random variables is called a random ~~pr~~ process.
$\{X(t, \omega) : t \in T, \omega \in S\}$ is called a random process with time space T and state space S.

(a) $\{W_K, K \in T\} \longrightarrow$ Discrete space Discrete time
(b) Discrete state continuous time.
(c) Discrete state discrete time.

2. (a) $X(t) = \cos(2\pi f t + \theta)$; $\theta \sim U[-\pi, \pi]$; $t \geq 0$.

$$\therefore \theta(x) = \begin{cases} \frac{1}{2\pi} , & -\pi \leq x \leq \pi \\ 0 ; & \text{otherwise} \end{cases}$$

$$\therefore \text{Mean} = E(X(t))$$
$$= \int_{-\infty}^{\infty} \theta \cos(2\pi f t + \theta) . d\theta$$
$$= \int_{-\infty}^{-\pi} 0 . \cos(2\pi f t + \theta) . d\theta + \int_{-\pi}^{\pi} \frac{1}{2\pi} . \cos(2\pi f t + \theta) . d\theta + \int_{\pi}^{\infty} 0 . \cos(2\pi f t + \theta) . d\theta$$
$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} . \cos(2\pi f t + \theta) . d\theta = \frac{1}{2\pi} \Big[\sin(2\pi f t + \theta) \Big]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi} \Big[\sin(\pi + 2\pi f t) + \sin(\pi - 2\pi f t) \Big]$$
$$= \frac{1}{2\pi} . \Big\{ (\sin \pi . \cos 2\pi f t + \sin 2\pi f t \cos \pi) + (\sin \pi . \cos 2\pi f t - \sin 2\pi f t \cos \pi) \Big]$$
$$= \frac{1}{2\pi} . 2 \sin \pi . \cos 2\pi f t = 0 .$$

Auto correlation $= R_X(t_1, t_2)$

$$= E(X(t_1)\, X(t_2))$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \cdot f(x_1, x_2) \cdot dx_1\, dx_2$$

$$= E\left[\cos(2\pi f t_1 + \theta) \cdot \cos(2\pi f t_2 + \theta)\right]$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \cdot \cos(2\pi f t_1 + \theta) \cdot \cos(2\pi f t_2 + \theta) \cdot d\theta$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{\cos (2\pi f (t_1 + t_2) + 2\theta) + \cos (2\pi f (t_1 - t_2))\right\} d\theta$$

$$= \frac{1}{4\pi} \left[\frac{1}{2} \cdot \sin\{2\pi f(t_1 + t_2) + 2\theta\} + \theta \cos 2\pi f(t_1 - t_2)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi} \cdot \left[\left\{\frac{1}{2}\left(\sin\{2\pi f(t_1 + t_2) + 2\pi\} - \sin\{2\pi f(t_1 + t_2) - 2\pi\}\right)\right\} + 2\pi \cdot \cos 2\pi f(t_1 - t_2)\right]$$

$$= \frac{1}{4\pi} \cdot 2\pi \cos 2\pi f(t_1 - t_2)$$

$$= \frac{1}{2} \cdot \cos 2\pi f(t_1 - t_2)$$

$\therefore$ Power $= R_X(t, t)$

$$= \frac{1}{2} \cdot \cos 2\pi f(t - t) = \frac{1}{2}.$$

$\therefore$ Auto-covariance $= E\left[\{X(t_1) - \mu_X(t_1)\} \cdot \{X(t_2) - \mu_X(t_2)\}\right]$

$$= E\left[X(t_1) \cdot X(t_2)\right]$$

$$= R_X(t_1, t_2)$$

$$= \frac{1}{2} \cdot \cos 2\pi f(t_1 - t_2).$$

Since $R_X(t_1, t_2)$ is a function of $(t_1 - t_2)$,

$\therefore$ $X(t)$ is a WSS.

Since, $\rho_X(t_1, t_2) \neq 0$, $X(t)$ is not a White noise.

(b) $X(t) = A_0 + A_1 t + A_2 t^2$

Given, $\mu_{A_0} = \mu_{A_1} = \mu_{A_2} = 0.$

$\sigma^2_{A_0} = \sigma^2_{A_1} = \sigma^2_{A_2} = 1.$

$$\therefore E(X(t)) = E(A_0 + A_1 t + A_2 t^2)$$
$$= E(A_0) + t \cdot E(A_1) + t^2 \cdot E(A_2)$$
$$= 0 + t \cdot 0 + t^2 \cdot 0 = 0$$

$$\therefore R_x(t_1, t_2) = E(X(t_1) \cdot X(t_2))$$
$$= E\left\{(A_0 + A_1 t_1 + A_2 t_1^2)(A_0 + A_1 t_2 + A_2 t_2^2)\right\}$$
$$= E\left\{A_0^2 + A_0 A_1 t_2 + A_0 A_2 t_2^2 + A_0 A_1 t_1 + A_1^2 t_1 t_2 + A_1 A_2 t_1 t_2^2 + A_0 A_2 t_1^2 + A_1 A_2 t_1^2 t_2 + A_2^2 t_1^2 t_2^2\right\}$$
$$= E(A_0^2) + t_2 \cdot E(A_0 A_1) + t_2^2 E(A_0 A_2) + t_1 E(A_0 A_1) + t_1 t_2 E(A_1^2) + t_1 t_2^2 E(A_1 A_2) + t_1^2 E(A_0 A_2) + t_1^2 t_2 E(A_1 A_2) + t_1^2 t_2^2 E(A_2^2).$$
$$= 1 + t_2 \cdot 0 + t_2^2 \cdot 0 + t_1 \cdot 0 + t_1 t_2 \cdot 1 + t_1 t_2^2 \cdot 0 + t_1^2 \cdot 0 + t_1^2 t_2 \cdot 0 + t_1^2 t_2^2 \cdot 1$$
$$= 1 + t_1 t_2 + t_1^2 t_2^2.$$

$$\therefore C_x(t_1, t_2) = E\left[\{X(t_1) - \mu_x(t_1)\} \cdot \{X(t_2) - \mu_x(t_2)\}\right]$$
$$= E\left[X(t_1) \cdot X(t_2)\right] = R_x(t_1, t_2) = 1 + t_1 t_2 + t_1^2 t_2^2.$$

$$\therefore \text{Power} = R_x(t, t) = 1 + t^2 + t^4.$$

$$\therefore \rho_x(t_1, t_2) = \frac{C_x(t_1, t_2)}{\sqrt{C_x(t_1, t_1)} \cdot \sqrt{C_x(t_2, t_2)}}$$
$$= \frac{1 + t_1 t_2 + t_1^2 t_2^2}{(1 + t_1^2 + t_1^4) \cdot (1 + t_2^2 + t_2^4)} \neq 0.$$

Since $R_x(t_1, t_2)$ is not a function of $(t_1 - t_2)$, therefore, it is not a WSS.
Also, $\rho_x(t_1, t_2) \neq 0$, therefore it is not white noise.

(c) ~~x(t) = 1 when they~~

$$X(t) = \begin{cases} 1 \ ; & \text{when even no. of failures} \\ -1 \ ; & \text{when odd no. of failures} \end{cases}$$

$$\therefore E(X(t)) = \cancel{\sum_{i=1}^{\infty} 1 \cdot 2i + \sum_{i=1}^{\infty} (-1) \cdot (2i-1)}$$

$$\cancel{\sum_{i=1}^{\infty}} = 1 \cdot \left(\frac{1}{2}\right) + (-1) \cdot \left(\frac{1}{2}\right) = 0 .$$

$$\therefore R_X(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\}$$

(d) $P(X(t)=n) = \dfrac{(at)^{n-1}}{(1+at)^{n+1}}$; $n = 1, 2, \ldots$ and $P(X(t)=0) = \dfrac{at}{1+at}$.

$$\therefore E(X(t)) = \sum_{n=1}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{at(1+at)} \cdot \sum_{n=1}^{\infty} n \left(\frac{at}{1+at}\right)^n .$$

$$= \frac{1}{at(1+at)} \cdot \sum_{n=1}^{\infty} n \cdot \left(1 - \frac{1}{1+at}\right)^n .$$

$$\cancel{= \frac{1}{at}}$$

Let $S_n = 1 \cdot \left(1 - \frac{1}{1+at}\right)^1 + 2 \cdot \left(1 - \frac{1}{1+at}\right)^2 + \cdots$

$\left(1 - \frac{1}{1+at}\right) S_n = \qquad 1 \cdot \left(1 - \frac{1}{1+at}\right)^2 + \cdots$

$$S_n \left(1 - 1 + \frac{1}{1+at}\right) = 1 - \frac{1}{1+at}$$

$$\Rightarrow S_n = (1+at)\left(1 - \frac{1}{1+at}\right) = 1 + at - 1 = at$$

$$\therefore E(X(t)) = \frac{1}{at(1+at)} \cdot at = \frac{1}{1+at}$$

$$R_X(t_1, t_2) = E(X(t_1) \cdot X(t_2))$$
$$= E($$

3. $Y_n = a_0 X_n + a_1 X_{n-1}$, $n = 1, 2, \ldots$

S.S.S. How? ~~X_i's are random~~ X_i's are i.i.d. r.vs.

$E(X_i) = 0 \quad Var(X_i) = 2 \Rightarrow E(X_i^2) - \{E(X_i)\}^2 = 2$

$\Rightarrow E(X_i^2) = 2.$

~~Here, mean is constant.~~

$E(Y_n) = a_0 E(X_n) + a_1 E(X_{n-1}) = 0$ which is a constant.

$$R_X(t_1, t_2) = \cancel{E(X(t_1) \cdot X(t_2))}\ E(Y_i \cdot Y_j)$$

$$= E\{(a_0 X_i + a_1 X_{i-1}) \cdot (a_0 X_j + a_1 X_{j-1})\}$$

$$= E\{a_0^2 X_i X_j + a_0 a_1 X_i X_{j-1} + a_1 a_0 X_{i-1} X_j + a_1^2 X_{i-1} \cdot X_{j-1}\}$$

$$\cancel{= a_0^2 E(X_i) E(X_j) + a_0 a_1 E(X_i) E(X_{j-1}) + a_1 a_0 E(X_{i-1}) E(X_j) + a_1^2 E(X_{i-1}) \cdot E(X_{j-1})}$$

$$\cancel{= a_0^2 \, 0 = g(t_1 - t_2)}.$$

~~Hence, this is WSS.~~

$$= a_0^2 E(X_i \cdot X_j) + a_0 a_1 E(X_i \cdot X_{j-1}) + a_1 a_0 E(X_{i-1} \cdot X_j) + a_1^2 E(X_{i-1} X_{j-1})$$

$$= \begin{cases} 2a_0^2 + 2a_1^2 & ; \text{ if } i = j \\ a_1 a_0 & ; \text{ if } |j - i| = 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

5. $Y_n = \frac{1}{2}(X_n + X_{n-1})$ X_n is a Gaussian process with mean 0 and variance σ^2.

$\therefore E(Y_n) = \frac{1}{2}E(X_n) + \frac{1}{2}E(X_{n-1}) = 0$

$$R_X(i, j) = E\{Y_i \cdot Y_j\} = \frac{1}{4}E(\{X_i + X_{i-1}\}\cdot\{X_j + X_{j-1}\})$$

$$= \frac{1}{4}E(X_i \cdot X_j + X_i \cdot X_{j-1} + X_{i-1}\cdot X_j + X_{i-1}\cdot X_{j-1})$$

$$= \begin{cases} \frac{1}{2}\sigma^2 & ; \text{ if } i = j \\ \frac{1}{4}\sigma^2 & ; \text{ if } |j-i| = 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

$|j-i| = 1$

$j - i = 1 \Rightarrow j-1 = i$

or $i - j = 1 \Rightarrow i-1 = j$

$=$ Function of $(i-j)$ (as all are constants)

Now, $Y_{n+1} = \frac{1}{2}(X_{n+1} + X_n)$

$Y_n = \frac{1}{2}(X_n + X_{n-1})$

$\therefore Y_{n+1} - Y_n = \frac{1}{2}(X_{n+1} - X_{n-1})$

Now, $Y_{n-1} = \frac{1}{2}(X_{n-1} + X_{n-2})$.

6.

(b) $P(X(t+s) - X(s) = n) = \dfrac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}$

~~(a) $P(X(20) - X(10) = 10) = e^{-\lambda t}$~~

$\lambda = \dfrac{5 \text{ people}}{30 \text{ minutes}} = \dfrac{1}{6}$ people/min.

(a) $P(X(20) - X(10) = 10) = P(X(10) - X(0) = 10)$

$= \dfrac{e^{-\frac{1}{6}\cdot 10} + \left(\frac{10}{6}\right)^{10}}{10!} = P(X(10) = 10)$.

(c) $P(X(10) = 10 \mid X(20) = 15)$

$= \dfrac{P(X(10) = 10 \text{ and } X(20) = 15)}{P(X(20) = 15)}$

$= \cancel{\dfrac{P(X(20) = 15 \text{ and } X(10) = 10)}{P(X(20) = 15)}}$

$= \dfrac{P(X(10) = 10 \text{ and } X(20) - X(10) = 5)}{P(X(20) = 15)}$

$= \dfrac{P(X(10) = 10) \cdot P(X(20 - 10) = 5)}{P(X(20) = 15)}$

$= \dfrac{P(X(10) = 10) \cdot P(X(10) = 5)}{P(X(20) = 15)}$

$= \dfrac{\dfrac{e^{-\frac{1}{6}\cdot 10} \cdot \left(\frac{10}{6}\right)^{10}}{10!} \cdot \dfrac{e^{-\frac{1}{6}\cdot 10} \cdot \left(\frac{10}{6}\right)^{5}}{5!}}{\dfrac{e^{-\frac{1}{6}\cdot 20} \cdot \left(\frac{20}{6}\right)^{15}}{15!}}$

$= \dfrac{15!}{10!\,5!} \cdot e^{-\frac{10}{6} - \frac{10}{6} + \frac{20}{6}} \cdot \dfrac{\left(\frac{10}{6}\right)^{15}}{\left(\frac{20}{6}\right)^{15}}$

$= \dfrac{15!}{10!\,5!} \cdot \left(\dfrac{1}{2}\right)^{15}$.

6.

(b) $P(X(t+s) - X(s) = n) = \dfrac{e^{-\lambda t}\cdot(\lambda t)^n}{n!}$

(a) $\cancel{P(X(20) - X(10) = 10) = e^{-\lambda t}}$

$\lambda = \dfrac{5 \text{ people}}{30 \text{ minutes}} = \dfrac{1}{6}$ people/min.

(a) $P(X(20) - X(10) = 10) = P(X(10) - X(0) = 10)$

$= \dfrac{e^{-\frac{1}{6}\cdot 10} + \left(\frac{10}{6}\right)^{10}}{10!} = P(X(10) = 10)$.

(c) $P(X(10) = 10 \mid X(20) = 15)$

$= \dfrac{P(X(10) = 10 \text{ and } X(20) = 15)}{P(X(20) = 15)}$

$\cancel{= \dfrac{P(X(20) = 15 \text{ and } X(10) = 10)}{P(X(20) = 15)}}$

$= \dfrac{P(X(10) = 10 \text{ and } X(20) - X(10) = 5)}{P(X(20) = 15)}$

$= \dfrac{P(X(10) = 10)\cdot P(X(20 - 10) = 5)}{P(X(20) = 15)}$

$= \dfrac{P(X(10) = 10)\cdot P(X(10) = 5)}{P(X(20) = 15)}$

$= \dfrac{\dfrac{e^{-\frac{1}{6}\cdot 10}\cdot\left(\frac{10}{6}\right)^{10}}{10!}\cdot\dfrac{e^{-\frac{1}{6}\cdot 10}\cdot\left(\frac{10}{6}\right)^{5}}{5!}}{\dfrac{e^{-\frac{1}{6}\cdot 20}\cdot\left(\frac{20}{6}\right)^{15}}{15!}}$

$= \dfrac{15!}{10!\,5!}\cdot e^{-\frac{10}{6} - \frac{10}{6} + \frac{20}{6}}\cdot\dfrac{\left(\frac{10}{6}\right)^{15}}{\left(\frac{20}{6}\right)^{15}}$

$= \dfrac{15!}{10!\,5!}\cdot\left(\dfrac{1}{2}\right)^{15}$.

(d) $P(X(20) = 15 \mid X(10) = 10)$

$$= \frac{P(X(20) = 15 \text{ and } X(10) = 10)}{P(X(10) = 10)}$$

$$= \frac{P(X(20-10) = 15-10 \text{ and } X(10) = 10)}{P(X(10) = 10)}$$

$$= \frac{P(X(10) = 5) \cdot P(X(10) = 10)}{P(X(10) = 10)}$$

$$= \frac{\frac{e^{-\frac{1}{6}\cdot 10} \cdot \left(\frac{10}{6}\right)^5}{5!} \cdot \frac{e^{-\frac{1}{6}\cdot 10} \cdot \left(\frac{10}{6}\right)^{10}}{10!}}{\frac{e^{-\frac{1}{6}\cdot 10} \cdot \left(\frac{10}{6}\right)^{10}}{10!}} = \frac{e^{-\frac{1}{6}\cdot 10}}{5!} \cdot \left(\frac{10}{6}\right)^5$$

(e) $P(X(20) = 10 \mid X(19) = 8, X(18) = 6, X(17) = 4)$

$$= \frac{P(X(20) = 10, X(19) = 8, X(18) = 6, X(17) = 4)}{P(X(19) = 8, X(18) = 6, X(17) = 4)}$$

~~$= P(X(20) = 10, X(19) = 8, X($~~

$$= \frac{P(X(1) = 2) \cdot P(X(1) = 2) \cdot P(X(1) = 2) \cdot P(X(17) = 4)}{P(X(1) = 2) \cdot P(X(1) = 2) \cdot P(X(17) = 4)}$$

$$= P(X(1) = 2) = \frac{e^{-\frac{1}{6}\cdot 1} \cdot \left(\frac{1}{6}\right)^2}{2!}$$ Is this correct?

7. $P(X_0 = 0, X_1 = 1) = P(X_0 = 0, X_2 = 1) = P(X_1 = 1, X_2 = 1) = \frac{1}{3}$

$\therefore P(X_0 = 0, X_1 = 1, X_2 = 1)$

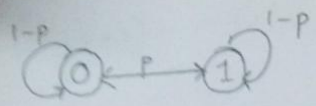
$= P(X_2 = 1 \mid X_1 = 1, X_0 = 0) \cdot P(X_1 = 1, X_0 = 0)$

$= P(X_2 = 1 \mid X_1 = 1) \cdot P(X_1 = 1, X_0 = 0)$

$= P(X_2 = 1, X_1 = 1) \cdot P(X_1 = 1) \cdot P(X_1 = 1, X_0 = 0)$

$= \frac{1}{9} \cdot P(X_1 = 1)$.

8.



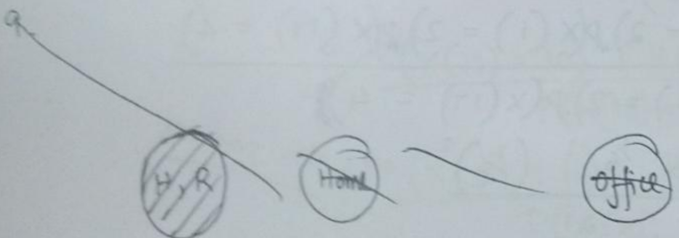
$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \end{array}$$

$$\therefore P^2 = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} = \cancel{\Big\{}$$

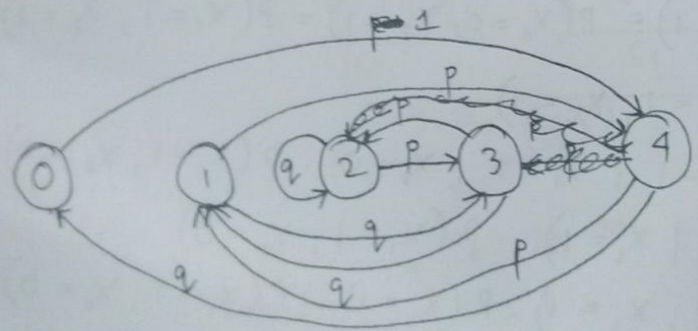
$$= \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \qquad\qquad 1 \end{array} \\ \begin{bmatrix} (1-p)^2 + p^2 & 2p(1-p) \\ 2p(1-p) & p^2 + (1-p)^2 \end{bmatrix} \end{array}$$

$\therefore$ Required probability $= p^2 + (1-p)^2$
$= p^2 + q^2$.

The matrix denote the 2 stage transitions. In (1,0), it should be initially at 1 & after two stages at 0. Why is it not taken?



9.



$$\begin{array}{c} \text{present} \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \text{future} \rightarrow \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & p \\ 0 & 0 & q & p & 0 \\ 0 & q & p & 0 & 0 \\ q & p & 0 & 0 & 0 \end{bmatrix} \end{array}$$

10. (a) $P\left(N(s+t) = j \mid N(s) = i\right)$

$\cancel{= P\left(N(s+t) - N(s) = j - i \mid N(s) = i\right)}$

$\cancel{= P\left(N(t) = j - i \mid N(s) = i\right)}$

$\cancel{= P(N} \quad = \dfrac{P\left(N(s+t) = j ,\ N(s) = i\right)}{P\left(N(s) = i\right)}$

$= \dfrac{P\left(N(t) = j - i ,\ N(s) = i\right)}{P\left(N(s) = i\right)}$

$= P\left(N(t) = j - i\right) =$