Indian Institute of Information Technology Guwahati

MA203: Probability and Random Processes **Problem Sheet-III on Random Variable**

CDF: Commulative Distribution Function; PMF: Probability Mass Function; PDF: Probability Density Function; RV: Random Variable; CRV: Continuous Random Variable; DRV: Discrete Random Variable;

(1) A die is tossed two times. Let X be the absolute value of the difference in face values. What is S? What values do X assign to points of S? What are the events $\{X \leq 2.75\}, \{0.5 \leq X \leq 1.72\}$ and $\{X = 3\}$? Find the CDF and PMF/PDF of X.

(2) Let $S = [0, 1]$ and F be the Borel σ-field of subsets of S. Define X on S as follows:

$$X(\omega) = \begin{cases} \omega, & \text{if } 0 \leq \omega \leq \frac{1}{2}; \\ \omega - \frac{1}{2}, & \text{if } \frac{1}{2} < \omega \leq 1. \end{cases}$$

Is X an RV? If so, find the CDF and PMF/PDF. What is the event $\{\omega : X(\omega) \in (1/4, 1/2)\}$?

(3) Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ \frac{2}{3}, & \text{if } 0 \leq x < 1; \\ \frac{7-6c}{6}, & \text{if } 1 \leq x < 2; \\ \frac{4c^2-9c+6}{4}, & \text{if } 2 \leq x < 3; \\ 1, & \text{if } x \geq 3; \end{cases}$$

where c is a real constant.

(a) Find the value of constant c. (Ans: 1/4.)

(b) Using the distribution function, find $P(\{1 < X < 2\}), P(\{2 \leq X < 3\}), P(\{0 < X \leq 1\}), P(\{1 \leq X \leq 2\}), P(\{X \geq 3\})$, and $P(\{X = 2.5\})$. (Ans: 0, 1/12, 1/4, 1/3, 0, 0.)

(c) Find the conditional probabilities $P(\{X = 1\}|\{1 \leq X \leq 2\}), P(\{1 \leq X < 2\}|\{X > 1\})$, and $P(\{1 \leq X \leq 2\}|\{X = 1\})$. (Ans: 3/4,0,1.)

(d) Find the PMF/PDF of X .

(4) Let X be CRV with PDF $f_X(x) = \begin{cases} k - |x|, & \text{if } |x| < 0.5; \\ 0, & \text{otherwise}; \end{cases}$ where $k \in \mathbb{R}$.

(a) Find the value of constant k. (Ans: 5/4.)

(b) Using the PDF, evaluate $P(\{X < 0\}), P(\{X = 0\}), P(\{0 < X \leq 1/4\})$, and $P(\{-1/8 \leq X \leq 1/4\})$. (Ans: 1/2, 1/2, 9/32, 25/32.)

(c) Find the conditional probabilities $P(\{X > 1/4\}|\{|X| > 2/5\})$ and $P(1/10 < X < 1|1/10 < X < 1/5)$. (Ans: 1/2, 1.)

(d) Find the CDF of X.

(5) Let X be a random variable having CDF

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - (1-p)^{[x]}, & \text{if } x \geq 0. \end{cases}$$

Determine whether X is DRV or CRV. Find the PMF/PDF corresponding to the RV X.

(6) In a semiconductor manufacturing process, three wafers from a lot are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent. Determine the probability mass function of the number of wafers from a lot that pass the test.

(7) An experiment consists of three independent tosses of a fair coin. Let X = The numbers of heads, Y = the number of head runs, Z = the length of head runs, a head run being defined as consecutive occurrence of at least two heads, its length then being the number of heads occurring together in three tosses of the coin. Find the PMF of (i) X, (ii) Y, (iii) Z, (iv) $X + Y$, and (v) XY, and draw their probability charts.

(8) Let $p_k = p(1-p)^k$; $k = 0, 1, 2, \ldots$; $0 < p < 1$. Does p_k define the PMF of some RV? If X is an RV with PMF p_k, what is $P(n \leq X < N)$, where $n, N(N > n)$ are positive integers?

(9) Do the following functions define CDFs? If so, classify then and then find the corresponding PMF/PDF.

(a) $F(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } 0 \le x < \frac{1}{2}; \\ 1, & \text{if } x \ge \frac{1}{2}. \end{cases}$

(b) $F(x) = 0 \begin{cases} 0, & \text{if } x \le 1; \\ 1 - \frac{1}{x}, & \text{if } x > 1. \end{cases}$

(c) $F(x) = \begin{cases} 1 - e^{-x}, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0. \end{cases}$

(d) $F(x) = \begin{cases} 0, & \text{if } x < 1; \\ \frac{(x-1)^2}{8}, & \text{if } 1 \le x < 3; \\ 1, & \text{if } x \ge 3. \end{cases}$

(10) Check if the following functions are PMF/PDF correcsonding to a random variable.

(a) $f(x) = \begin{cases} x(2x-1), & \text{if } 0 < x < 2; \\ 0, & \text{otherwise.} \end{cases}$

(b) $f(x) = \begin{cases} \frac{1}{\lambda} e^{\frac{\theta - x}{\lambda}}, & \text{if } x > \theta; \\ 0, & \text{otherwise.} \end{cases}$ with $\theta > 0$

(c) $f(x) = \begin{cases} \frac{1}{2}(\frac{2}{3})^x, & \text{if } x = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$

(d) $f(x) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!}, & \text{if } k = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$

Remark: A function $F : S \to [0, 1]$ is said to be a CDF if (a) F is non-decreasing and has only jump discontinuities (b) $\lim_{x \to \infty^-} F(x) = 1, \lim_{x \to -\infty^+} F(x) = 0$ (c) F is right continuous, $\lim_{h \to 0^+} F(x + h) = F(x)$.

(11) Consider a population comprising of N (≥ 2) units out of which a ($< N$) are labeled as S (success) and rest $N - a$ are labeled as F (failure). Suppose that we are interested in drawing a sample of size n from this population, drawing one unit at a time. Let X denote the number of S in the drawn sample.

(a) Assuming that the draws are independent and sampling is with replacement, find the PMF of X.

(b) Assuming that sampling is without replacement, find the PMF of X. Are the draws independent?