MA203

Function of an RV

Conditional distribution: conditional distribution of RV X assuming M is
F(x|M) = P(X<=x|M) =P(X<=x, M)/P(M)

F(x|M) has same properties as F(x)
F(-inf|M)=0, F(inf|M)=1, P(x1<X<=x2|M)=F(x2|M)-F(x1|M)

Conditional density: f(x|M) is the derivative of F(x|M):   f(x|M)=dF(x|M)/dx

$$f(x|M) = \lim_{\Delta x \to 0} \frac{P(x \le X \le x + \Delta x|M)}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x|M) - F(x|M)}{\Delta x}$$

f(x|M) is non negative and area under the curve equals 1.

Example 1. let RV X(fi)=10i of fair die experiment where M={f2, f4, f6}. Find F(x|M)

If x>=60, {X<=x, M}=M; F(x|M)=1
If 40<=x<60, {X<=x, M}={f2, f4}; F(x|M)=(2/6)/(3/6)=2/3
If 20<=x<40, {X<=x, M}={f2}; F(x|M)=(1/6)/(3/6)=1/3
If x<20, {X<=x, M}=null set; F(x|M)=0

For F(x|M), underlying experiment knowledge is required unless M can be expressed in terms of X

Case 1:

1) An RV X and $M=\{X \leq a\}$, a is a number and $F(a) \neq 0$

$$F(x \mid X \leq a) = P\{X \leq x \mid X \leq a\}$$

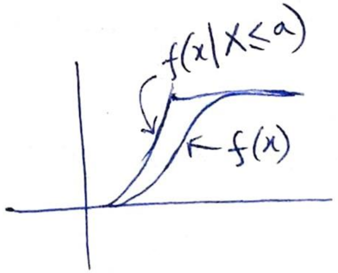
$$= \frac{P\{X \leq x,\ X \leq a\}}{P(X \leq a)}$$

if $x \geq a$, $\quad F(x \mid X \leq a) = \dfrac{P(X \leq a)}{P(X \leq a)} = 1$

if $x < a$, $\quad F(x \mid X \leq a) = \dfrac{P(X \leq x)}{P(X \leq a)} = \dfrac{F(x)}{F(a)}$

$$\therefore f(x \mid X \leq a) = \begin{cases} \dfrac{F'(x)}{F(a)}, & x < a \\ 0, & x \geq a \end{cases}$$

$F'(x) = f(x)$



Case 2:

2) Suppose, $M = (b < X \le a)$

$$F(x \mid b < X \le a) = \frac{P(X \le x,\ b < X \le a)}{P(b < X \le a)}$$

if $x \ge a$, $F(x \mid b < X \le a) = \dfrac{P(b < X \le a)}{P(b < X \le a)} = 1$

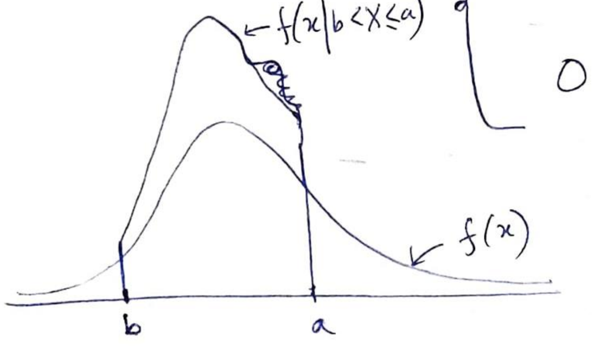
if $b \le x < a$, $F(x \mid b < X \le a) = \dfrac{P(b < X \le x)}{P(b < X \le a)}$

$$= \frac{F(x) - F(b)}{F(a) - F(b)}$$

if $x < b$, $F(x \mid b < X \le a) = 0$

corresponding density is given by,

$$f(x \mid b < X \le a) = \begin{cases} \dfrac{f(x)}{F(a) - F(b)}, & \text{for } b \le x < a \\ 0, & \text{otherwise} \end{cases}$$



Example 2. Determine the conditional density f(x | |X-μ| <=kσ) of an N(μ, σ^2) RV.

$$P(|X-\mu| \leq k\sigma) = P(\mu - k\sigma \leq X \leq \mu + k\sigma) = P(-k \leq Z \leq k) = \int_{-k}^{k} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} dz$$

$$f(x||X-\mu| \leq k\sigma) = \frac{\frac{1}{\sqrt{(2\pi)}\sigma} e^{-(x-\mu)^2/2\sigma^2}}{\int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz}$$

for |X-μ| <=kσ

Otherwise 0

This is called truncated normal distribution.

Functions of one RV
X is an RV, g(x) is a function
Y=g(X) is a new RV

$F_Y(y) = P(Y<=y) = P(g(X)<=y)$

g(X) to be an RV, g(X) must have
1. Domain must include the range of RV X
2. Must be a Borel function
3. Events {g(X)=inf or –inf} must have 0 prob

Distributions of g(X): $F_Y(y)$ is written in terms of $F_X(x)$

Distribution of $g(X)$ —

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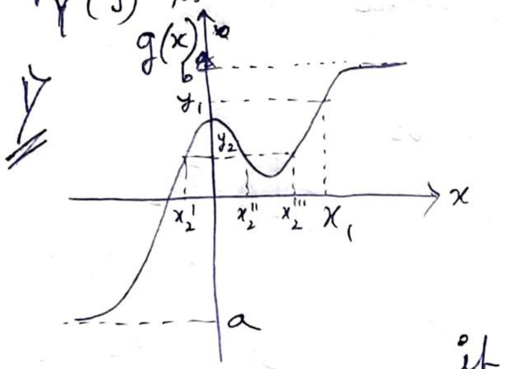

$g(x)$ is between a and b for any x.

$\therefore$ if $y \geq b$, then $g(x) \leq y$ for every x

$P(Y \leq y) = 1$, for $y \geq b$

if $y < a$, then no x for $g(x) \leq y$

$P(Y \leq y) = 0$, for $y < a$

with x_1 and $y_1 = g(x_1)$,

$g(x) \leq y_1$ for $x \leq x_1$

$= P(X \leq x_1) = F_X(x_1)$

$\therefore F_Y(y_1) = P(X \leq x_1) = F_X(x_1)$

$g(x) \leq y_2$ if $x \leq x_2'$ or $x_2'' \leq x \leq x_2'''$

$\therefore F_Y(y_2) = P\{X \leq x_2'\} + P\{x_2'' \leq X \leq x_2'''\}$

$= F_X(x_2') + F_X(x_2''') - F_X(x_2'')$ [$\because$ events are mutually exclusive]

2) Suppose, ~~g(x~~ $g(x)$ is constant in an interval (x_0, x_1).

$$g(x) = y_1 \quad , \; x_0 < x \le x_1$$

$$\therefore P(Y = y_1) = P\{x_0 < X \le x_1\} = F_X(x_1) - F_X(x_0)$$

Hence, $F_Y(y)$ is discontinuous at $Y = y_1$ and its discontinuity equals $F_X(x_1) - F_X(x_0)$.

3) $g(x)$ is a staircase function

$$g(x) = g(x_i) = y_i \quad , \; x_{i-1} < x \le x_i$$

Here, RV $Y = g(X)$ is of discrete type taking values y_i

$$P\{Y = y_i\} = P\{x_{i-1} < X \le x_i\} = F_X(x_i) - F_X(x_{i-1})$$

4) $g(x)$ is discontinuous at $x = x_0$ and

$g(x) < g(x_0^-)$ for $x < x_0$, $g(x) > g(x_0^+)$ for $x > x_0$

$\therefore$ if $g(x_0^-) \leq y \leq g(x_0^+)$, then $g(x) < y$ for $x \leq x_0$

$\therefore F_Y(y) = P\{X \leq x_0\} = F_X(x_0)$, $g(x_0^-) \leq y \leq g(x_0^+)$

5) X is of discrete type RV taking values x_k with prob p_k.

$Y = g(X)$ is also discrete RV taking values

$y_k = g(x_k)$

if $y_k = g(x)$ for only one ~~x_k~~ $x = x_k$, then

$P\{Y = y_k\} = P\{X = x_k\} = p_k$

if $y_k = g(x)$ for $x = x_k$ and $x = x_l$, then

$P\{Y = y_k\} = P\{X = x_k\} + P\{X = x_l\} = p_k + p_l$

Determination of density function ($f_Y(y)$)

Fundamental theorem: to find $f_Y(y)$ for a specific y, solve the equation y=g(x), denoting its real roots by xn, as y=g(x1)=g(x2)=g(x3)=……..=g(xn)=……

then,

$$f_Y(y) = \frac{f_X(x_1)}{|g\prime(x_1)|} + \frac{f_X(x_2)}{|g\prime(x_2)|} + \cdots$$

Where g'(x) is the derivative of x

1) $Y = aX + b \qquad g'(x) = a$

$y = ax + b \Rightarrow x = \frac{y-b}{a} \quad \forall y$

$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

special case — if X is uniform in the interval (x_1, x_2), then y is uniform in the interval $(ax_1+b, \; ax_2+b)$

2) $Y = \frac{1}{X} \quad , \quad g'(x) = -\frac{1}{x^2}$

$y = \frac{1}{x}$ has single solution $x = 1/y$

$\therefore f_Y(y) = \left(\frac{1}{1/y^2}\right)^{-1} f_X(1/y) = (1/y^2) f_X(1/y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right)$

3) if $Y = aX^2 \quad , \quad a > 0$

$g'(x) = 2ax$

if $y \le 0$, then $y = ax^2$ has no real solutions.

$\therefore f_Y(y) = 0$

if $y > 0$, then $y = ax^2$ has two solutions;

$x = \sqrt{\frac{y}{a}} \quad , \quad x = -\sqrt{\frac{y}{a}}$

$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{2a\sqrt{\frac{y}{a}}} + \frac{f_X\left(-\sqrt{\frac{y}{a}}\right)}{+2a\sqrt{\frac{y}{a}}}$

$= \frac{1}{2a\sqrt{\frac{y}{a}}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right] \quad , y > 0$

special case — if $a = 1$, i.e., $Y = X^2$ then

$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \quad , y > 0$

if $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\quad f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2}\right], y > 0$

$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad , y > 0$

4) $Y=\sqrt{X}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$

$y=\sqrt{x}$ has single solution $x=y^2$ for $y>0$ and no solution for $y<0$.

$$f_Y(y) = \frac{1}{\frac{1}{2y}} f_X(y^2) = 2y\, f_X(y^2)$$

5) $Y = X\,U(X)$, $g'(x) = U(x)$

for $y<0$, $f_Y(y)=0$, $F_Y(y)=0$

if $y>0$, $y = x\,U(x)$ has single solution $x_1 = y$

$F_Y(y) = F_X(y)$, $y>0$

$\therefore f_Y(y) = f_X(y)$

$\therefore F_Y(y)$ is discontinuous at $y=0$ with discontinuity

$F_Y(0^+) - F_Y(0^-) = F_X(0)$

$$\therefore \quad f_Y(y) = f_X(y)\,U(y) + F_X(0)\,\delta(y)$$

6) $Y = e^X$ $\qquad g'(x) = e^x$

$y=e^x$ has single solution $x = \ln y$.

if $y>0$,

$$f_Y(y) = \frac{f_X(\ln y)}{y} , \quad y>0$$

if $y<0$, then $f_Y(y) = 0$

7) $Y = a\sin(X+\theta)$, $a>0$

if $|y|>a$, then $y = a\sin(x+\theta)$ has no solution, hence

$$f_Y(y) = 0$$

if $|y|<a$, then it has infinitely many solutions

$$x_n = \sin^{-1}\frac{y}{a} - \theta \quad , n = -\cdots\cdots, -1, 0, 1, \cdots\cdots$$

$$g'(x) = a\cos(x+\theta)$$

$$\therefore g'(x_n) = a\cos(x_n+\theta) = \sqrt{a^2-y^2}$$

$$f_Y(y) = \frac{1}{\sqrt{a^2-y^2}} \sum_{n=-\infty}^{\infty} f_X(x_n) \quad , |y|<a$$

8) $Y = \tan X$

$y = \tan x$ has infinitely many solutions for any y.

$$x_n = \tan^{-1} y \quad , n = -\infty, \cdots, -1, 0, 1, \cdots$$

$$g'(x) = \frac{1}{\cos^2 x} = 1+y^2$$

$$\therefore f_Y(y) = \frac{1}{1+y^2} \sum_{n=-\infty}^{\infty} f_X(x_n)$$

Example 13) Suppose, resistance R is uniform between $900\,\Omega$ and $1100\,\Omega$. Determine the density of the corresponding conductance.

ans: $G = \frac{1}{R}$ $\qquad$ $g = \frac{1}{r}$ has one solution

$$f_R(r) = \frac{1}{1100-900} = \frac{1}{200} \qquad r = \frac{1}{g}$$

$$g'(r) = \frac{-1}{R^2}$$

$$\therefore f_G(g) = \frac{f_R\left(\frac{1}{g}\right)}{\frac{1}{\left(\frac{1}{g}\right)^2}} = \frac{1}{g^2} f_R\left(\frac{1}{g}\right) = \frac{1}{200g^2}$$

for $\frac{1}{1100} < g < \frac{1}{900}$

$$\therefore f_G(g) = \begin{cases} \frac{1}{200g^2} & , \frac{1}{1100} < g < \frac{1}{900} \\ 0 & , \text{elsewhere} \end{cases}$$

Example 16) If $X \sim N(\mu, \sigma^2)$ and $Y = e^X$

$Y = e^X \qquad \therefore g'(x) = e^x$

$y = e^x$ has single solution $x = \ln y$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma y} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}, \quad y > 0$$

and 0 otherwise.

– this density is called lognormal density.