Classical Mechanics

Non-inertial frames: Pseudo forces

Dr V.S. Gayathri

Inertial frame of reference

- An inertial frame of reference is the frame, in which *law of inertia holds* and other *laws of physics are valid.*

- These frames are *unaccelerated frames* (at rest or moving with constant velocity) (acceleration of the frame, a_f=0).

- All frames which are moving with constant velocity w.r.t. an inertial frame are also inertial.

Inertial frame of reference

Train moving right with a

uniform velocity
V

Train moving with uniform velocity

Block returns to the same point from where it is thrown.

Non-inertial frame of reference

i. If a frame is *accelerated* w.r.t an inertial frame (road, train moving with constant velocity), that frame is called non-inertial frame.(acceleration of the frame, $a_f \neq 0$)

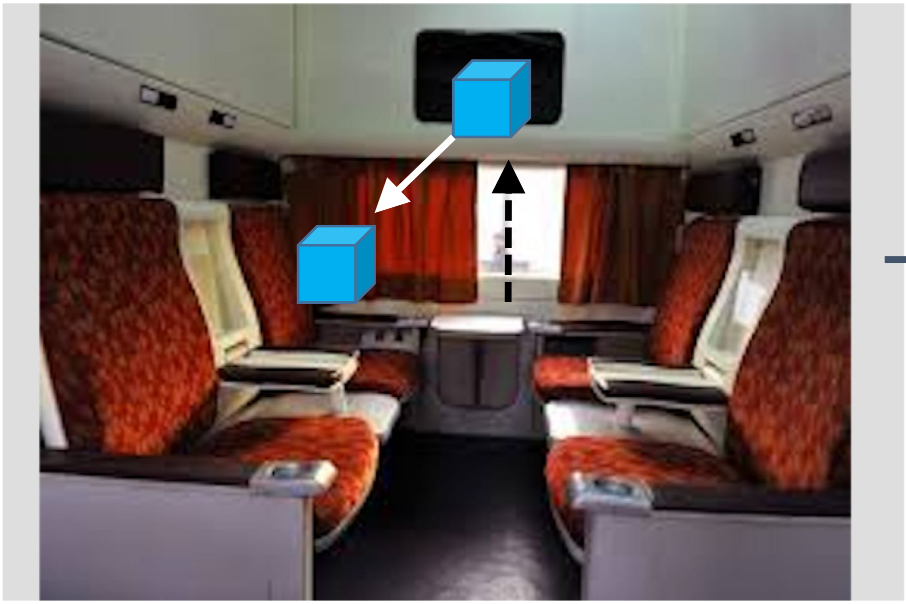
ii. In non-inertial frames, *newton's laws of motion are not valid*.

Frames can be non-inertial in two ways:

1. Uniformly accelerating systems: They can have linear acceleration w.r.t. an inertial frame. For eg. A car accelerating along a straight road.

2. Rotating coordinate systems: Frames itself can be rotating w.r.t. an inertial frame. For eg. Rotation of a disc.

Uniformly accelerating systems

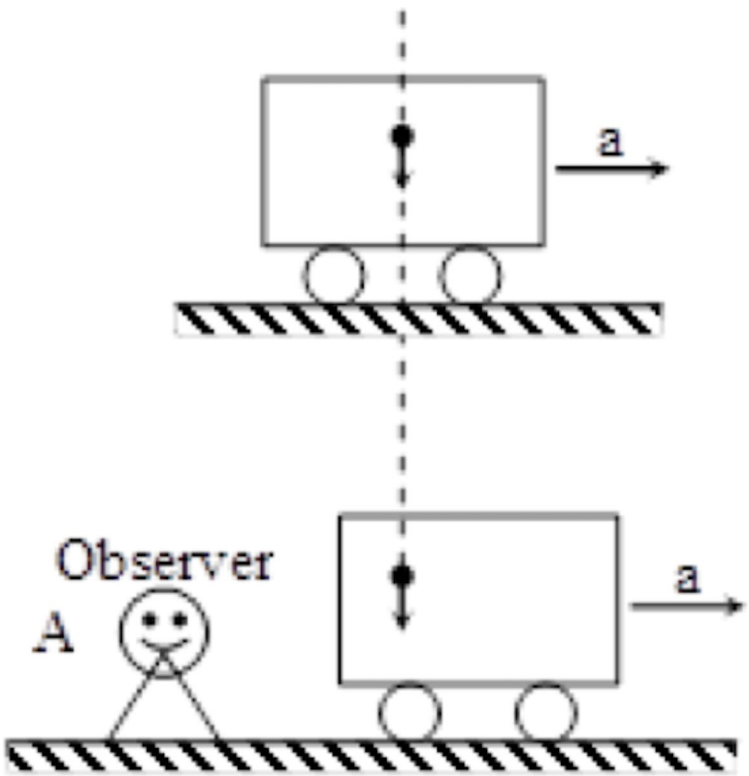


Train moving right
with acceleration

A

Train accelerating
Block lands backwards

Uniformly accelerating system



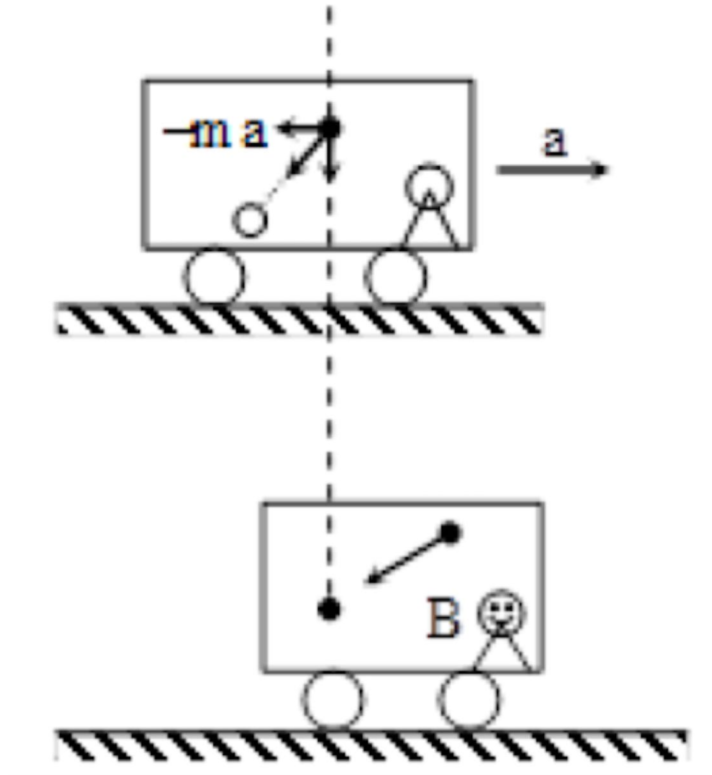
Suppose a ball is dropped inside a car moving to the right with acceleration **a**.

For a stationary observer on the ground:

(i) The ball is falling vertically downwards with an acceleration g.

(ii) The ball does not move horizontally. It is only the car that moves to the right.

Uniformly accelerating system

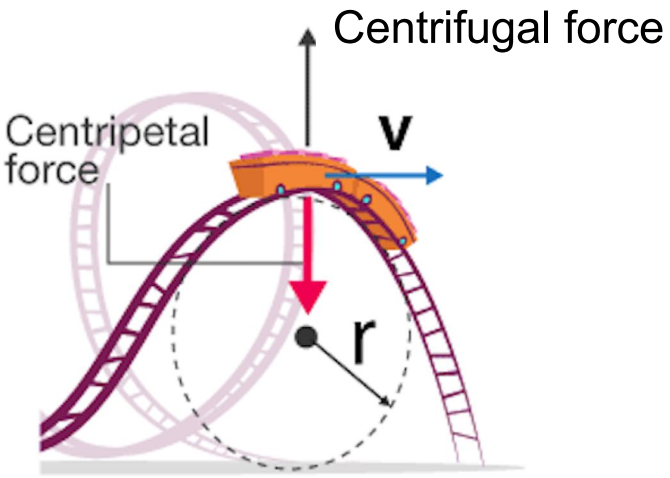


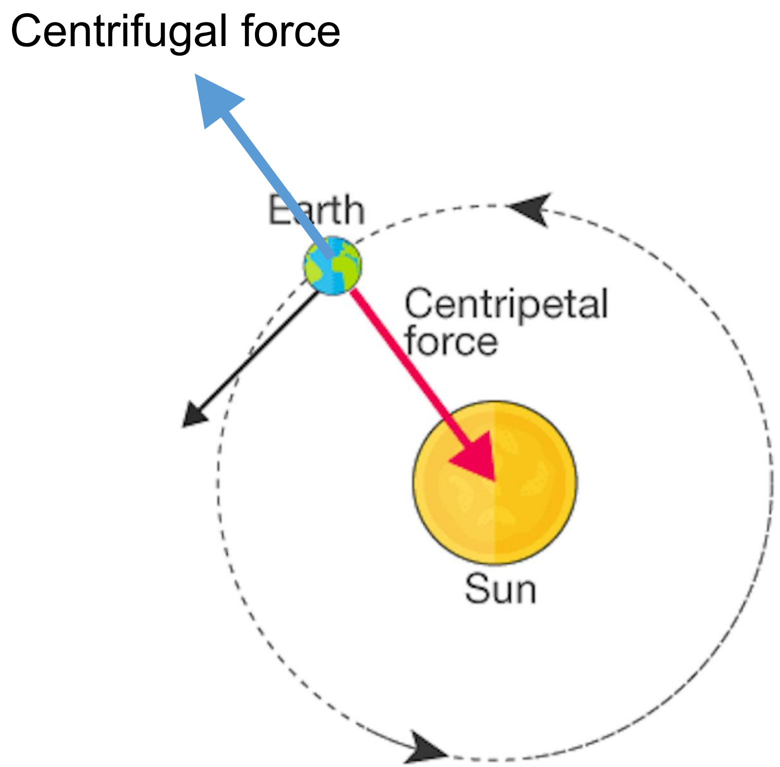
For an observer in an accelerated car:

(i) The ball moves down towards the left of the car.

(ii) The backward acceleration of the ball is caused by pseudo force.

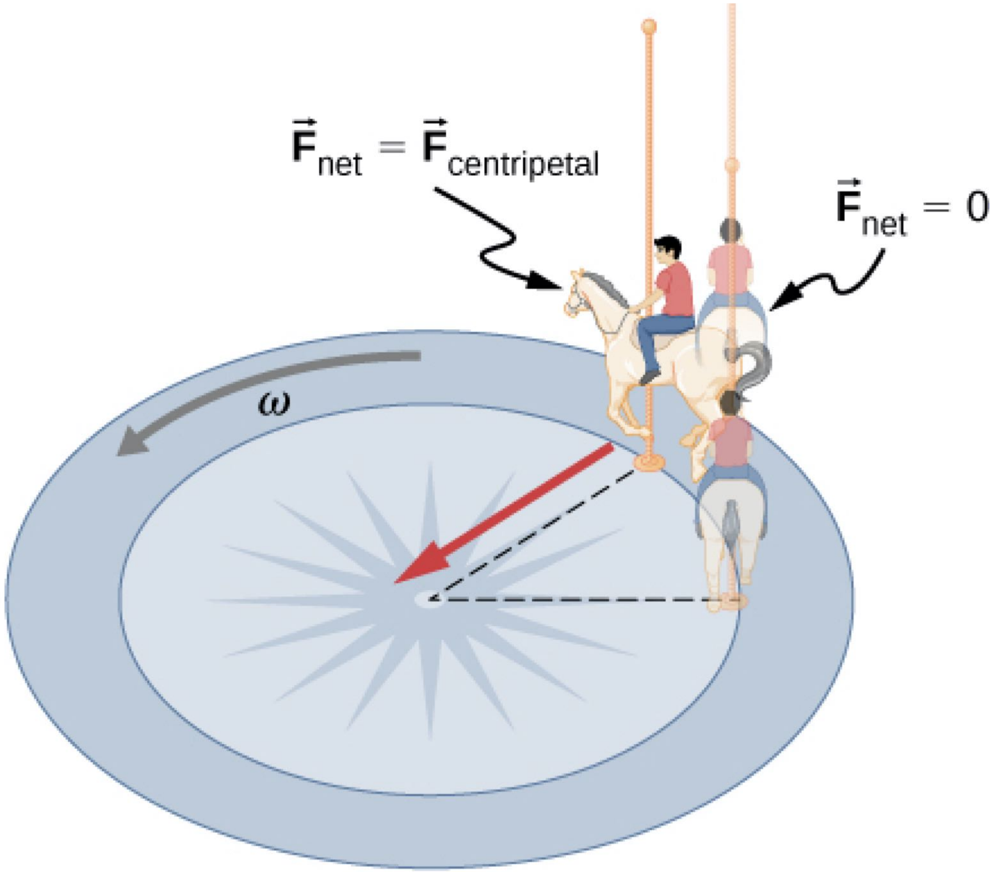
Rotating coordinate systems: Pseudo forces



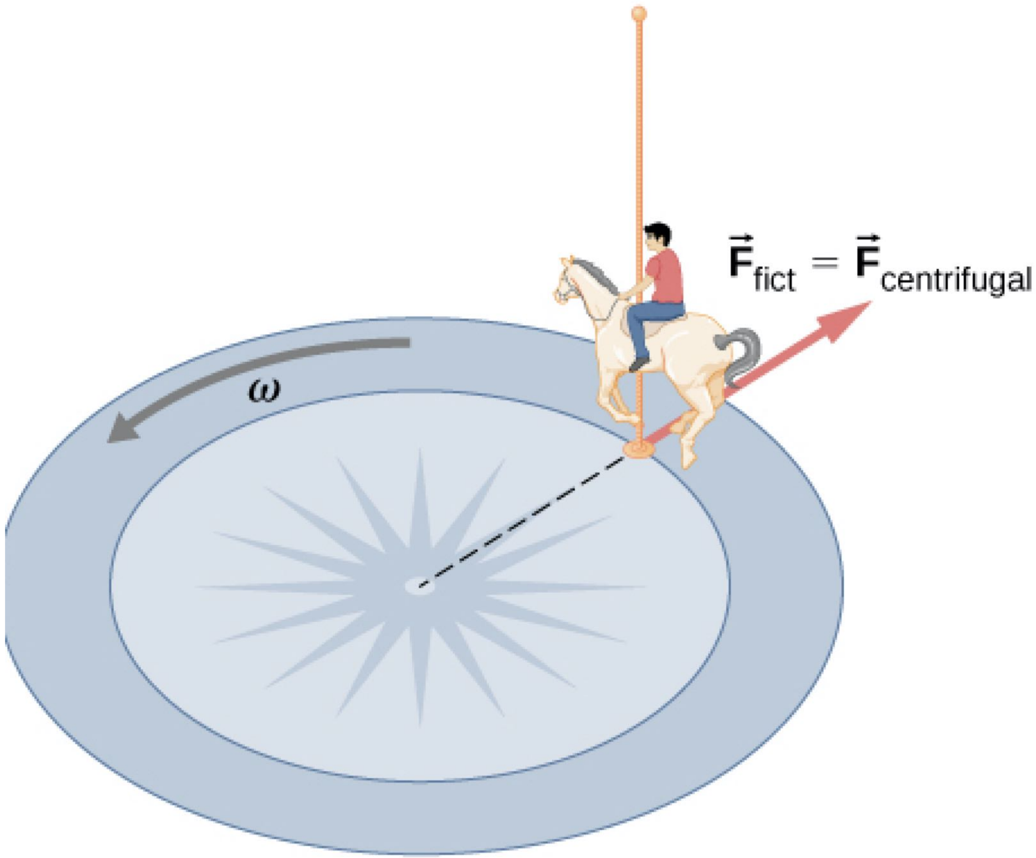


Pseudo ”centrifugal” force seems to push objects in circular path radially outward.

Rotating coordinate systems: Merry-go-round

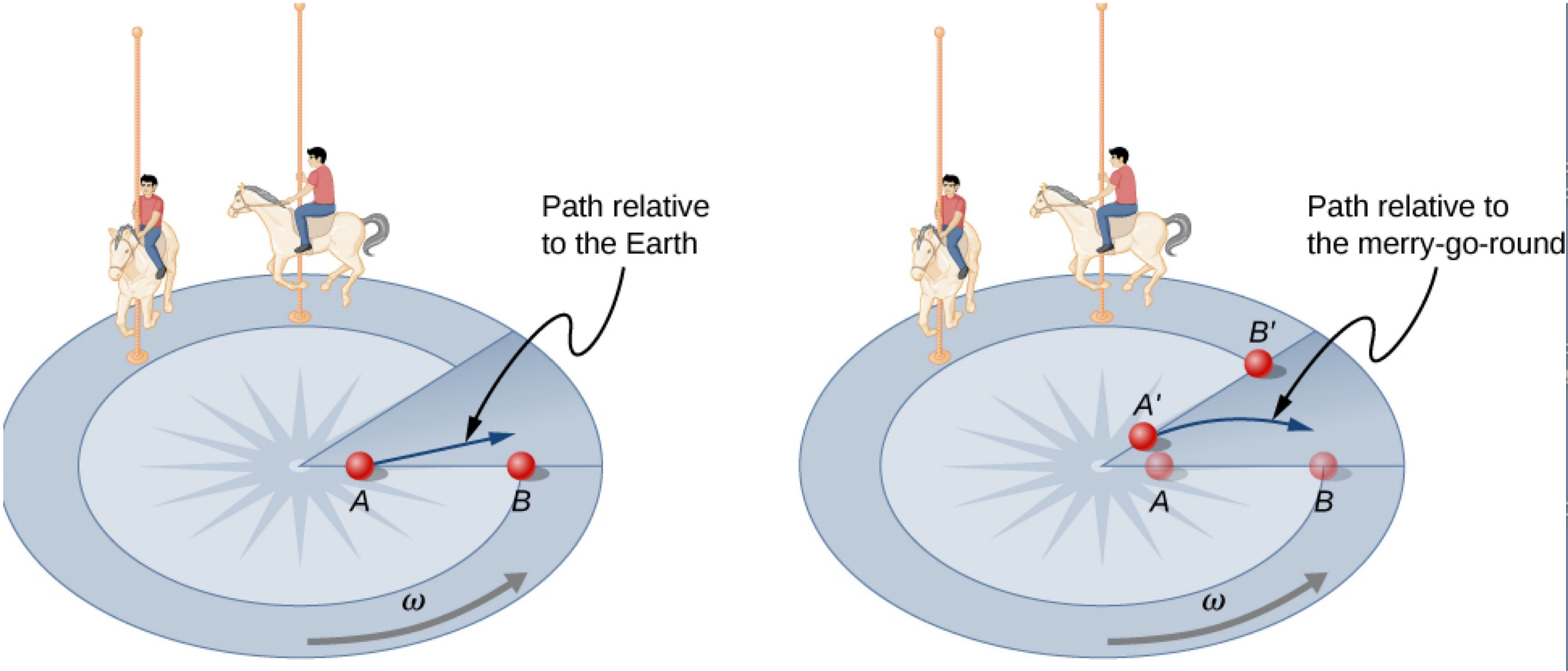


For a stationary observer:
Real "centripetal force"



For an observer in merry-go-round
Pseudo "centrifugal" force

Rotating coordinate systems: particle rolling on a Merry-go-round

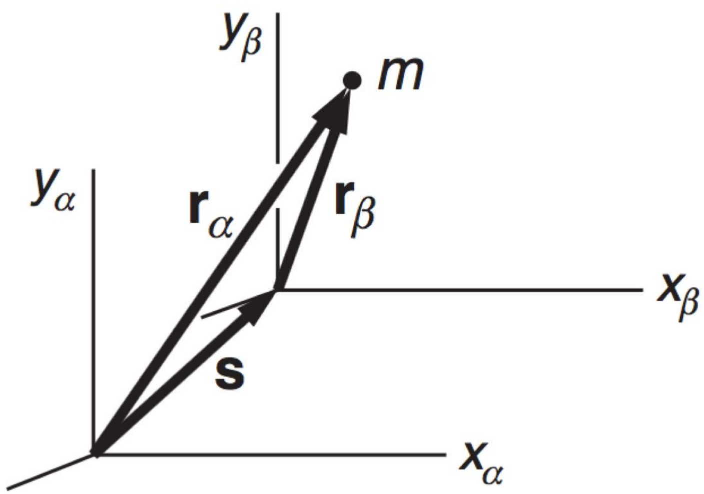


Pseudo "Coriolis" force seems to deflect the particle in rotating frame

Inertial frame of reference: Newton's II Law

Let α and β be the two observers measuring the force acting on mass m and its acceleration **a**. Let the coordinate of β be moving at uniform velocity $\boldsymbol{V}$ ($\ll c$), relative to the inertial frame of α. Assume that their corresponding axes are parallel. If the origins of the two systems are displaced by $\boldsymbol{S}$, then

$$\boldsymbol{r}_\beta = \boldsymbol{r}_\alpha - \boldsymbol{S}$$



$$\boldsymbol{F_\alpha} = m\boldsymbol{a_\alpha}$$
$$\boldsymbol{F_\beta} = m\boldsymbol{a_\beta}$$

(subscripts α and β refer to the measurements made by the respective observers)

What is the relation between $\boldsymbol{F_\alpha}$ and $\boldsymbol{F_\beta}$?

Inertial frame of reference: Newton's II Law

Successive differentiation of $\boldsymbol{r}_\beta = \boldsymbol{r}_\alpha - \boldsymbol{S}$ w.r.t time,

$$\boldsymbol{v}_{\boldsymbol{\beta}} = \boldsymbol{v}_{\boldsymbol{\alpha}} - \boldsymbol{V}$$

$$\boldsymbol{a}_{\boldsymbol{\beta}} = \boldsymbol{a}_{\boldsymbol{\alpha}} - \boldsymbol{A}$$

But if the relative motion is uniform, $\boldsymbol{V} = \dot{\boldsymbol{S}}$=constant

Hence $\boldsymbol{A} = \dot{\boldsymbol{V}} = \boldsymbol{0} \longrightarrow \boldsymbol{a}_{\boldsymbol{\alpha}} = \boldsymbol{a}_{\boldsymbol{\beta}}$

$$\boldsymbol{F}_{\boldsymbol{\alpha}} = \boldsymbol{F}_{\boldsymbol{\beta}}$$

All systems translating uniformly relative to an inertial system are inertial.

“**F**=m**a**” in a non-inertial frame

- We know that the Newton’s second law **F**=m**a** holds true only in inertial frames.

- Now, if we consider some non inertial frames eg. Elevators, merry-go-round, etc, is there any possible way to modify Newton’s laws, so that they hold in non-inertial frames also?

How to hold on to **F**=m**a** even in non-inertial frames?

Uniformly Accelerating Systems: Newton's II Law

Consider an observer in a system accelerating at rate $\boldsymbol{A}$ with respect to an inertial system. In this non-inertial system, we shall label quantities by primes.

$$\boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{A}$$

($\boldsymbol{A}$ **:** acceleration of the primed frame as measured in the inertial frame.)

In the accelerating system, the apparent force is

$$\begin{aligned}\boldsymbol{F}' &= m\boldsymbol{a}' \\ &= m(\boldsymbol{a} - \boldsymbol{A})\end{aligned}$$

$\boldsymbol{F} = m\boldsymbol{a}$: true force due to physical interactions.

Uniformly Accelerating Systems: Newton's II Law

Sub for **F**: $\boldsymbol{F}' = \boldsymbol{F} - m\boldsymbol{A}$

Let the fictitious force be: $\boldsymbol{F}_{fict} = -m\boldsymbol{A}$,

so rewriting the above equation:

$$\boldsymbol{F}' = \boldsymbol{F} + \boldsymbol{F}_{fict}$$

Therefore, by adding an extra pseudo force $\boldsymbol{F}_{fict}$, we can hold on to **F**=m**a** even in non-inertial frames.

Uniformly Accelerating Systems: Newton's II Law

- Fictitious forces originate in the acceleration of the coordinate system, not in the interaction between bodies.

- The fictitious force in a uniformly accelerating system behaves exactly like a constant gravitational force -- $\boldsymbol{F}_{fict} \propto mass$

- $\therefore$ The fictitious force on an extended body acts at the center of mass.

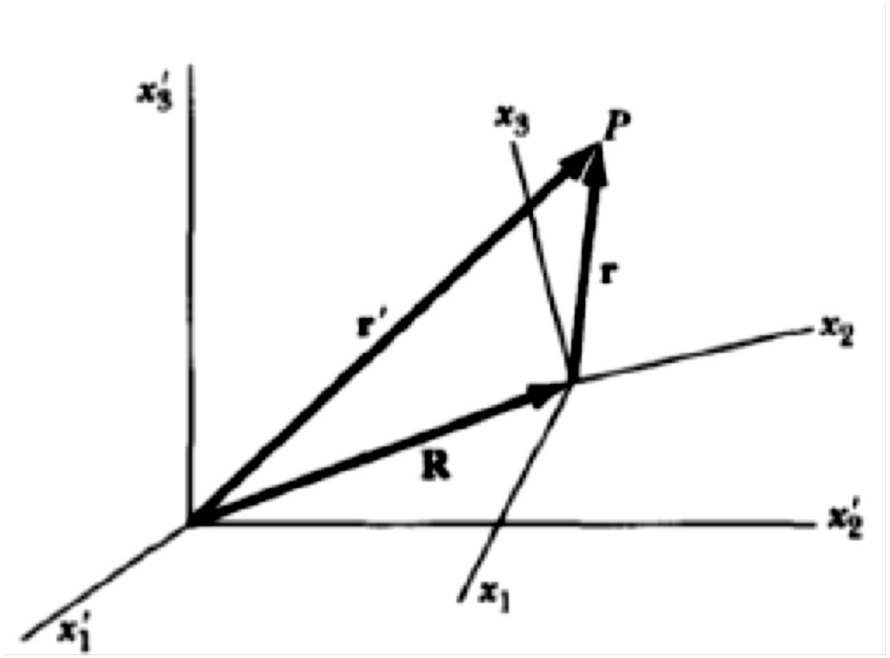
Rotating coordinate systems: Pseudo forces

- Let us consider two sets of coordinate axes, one for fixed or inertial axes (x_1', x_2', x_3') and one is rotating axes (x_1, x_2, x_3) w.r.t the inertial frame.

- If we choose a point P, we have

$$\boldsymbol{r'} = \boldsymbol{R} + \boldsymbol{r}$$

Where r' is the radius vector of P in the inertial system and **r** is the radius vector of P in the rotating system. The vector **R** locates the origin of the rotating system in the fixed system.



If rotating frame undergoes a infinitesimal rotation $\delta\boldsymbol{\theta}$ w.r.t the inertial frame, then for the infinitesimal displacement $\delta\boldsymbol{r}$, the position of P can be described as:

$$(\boldsymbol{dr})_{in} = d\boldsymbol{\theta}\times\boldsymbol{r}$$

Dividing by dt, the time interval during which the infinitesimal rotation takes place, we obtain the time rate of the change of **r** as measured in the inertial frame:

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{in} = \frac{d\boldsymbol{\theta}}{dt}\times\boldsymbol{r}$$

$$or \quad \left(\frac{d\boldsymbol{r}}{dt}\right)_{in} = \boldsymbol{\omega}\times\boldsymbol{r}$$

If point P has a velocity $\left(\frac{dr}{dt}\right)_{rotating}$ in the rotating frame (x_1, x_2, x_3), then this velocity must be added to $\boldsymbol{\omega}\times\boldsymbol{r}$ to obtain the time rate of change of r :

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{in} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{rot} + \boldsymbol{\omega}\times\boldsymbol{r}$$

Consider $\left(\frac{d\boldsymbol{r}}{dt}\right)_{in} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{rot} + \boldsymbol{\omega}\times\boldsymbol{r}$

we can write it as $\boldsymbol{v}_{in} = \boldsymbol{v}_{rot} + \boldsymbol{\omega}\times\boldsymbol{r}$

Now, to get acceleration in inertial frame, we write

$$\boldsymbol{a}_{in} = \left[\frac{d(\boldsymbol{v}_{rot} + \boldsymbol{\omega}\times\boldsymbol{r})}{dt}\right]_{rot} + \{\boldsymbol{\omega}\times(\boldsymbol{v}_{rot} + \boldsymbol{\omega}\times\boldsymbol{r})\}$$

$$\boldsymbol{a}_{in} = \boldsymbol{a}_{rot} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r} + \boldsymbol{\omega}\times\boldsymbol{v}_{rot} + \boldsymbol{\omega}\times\boldsymbol{v}_{rot} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\}$$

Expressing Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}_{in}$, in terms of the quantities in the rotating frame, we get

$$m\boldsymbol{a}_{in} = m\boldsymbol{a}_{rot} + m\dot{\boldsymbol{\omega}}\times\boldsymbol{r} + m\boldsymbol{\omega}\times\boldsymbol{v}_{rot} + m\boldsymbol{\omega}\times\boldsymbol{v}_{rot} + \mathrm{m}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\}$$

$$or \qquad m\boldsymbol{a}_{rot} = \boldsymbol{F} - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r} - 2m\,\boldsymbol{\omega}\times\boldsymbol{v}_{rot} - \mathrm{m}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\}$$

As $\boldsymbol{\omega}$ *is constant*, $\dot{\boldsymbol{\omega}} = 0$. So, second term will be zero.

Now, we have $$\underbrace{m\boldsymbol{a}_{rot} = \boldsymbol{F}}_{\text{Real Force}} \underbrace{- 2m\,\boldsymbol{\omega}\times\boldsymbol{v}_{rot} - \mathrm{m}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\}}_{\text{Pseudo Forces}}$$

There are two terms in R.H.S. for two pseudo forces that arise because we are using coordinates in the non inertial rotating frame.

- The quantity $-2m\,\boldsymbol{\omega}\times\boldsymbol{v}_{rot}$ is called the **Coriolis** force term. It arises from the motion of particle in the rotating frame.
- The quantity $-\,\mathrm{m}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ is **centrifugal** force term. It is directed radially outward from the center of rotation.

The minus sign implies that the centrifugal force is directed radially outward from the center of rotation. The term $-\,\mathrm{m}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ reduces to $\mathrm{m}\omega^2\mathrm{r}$ for the case in which $\boldsymbol{\omega}$ is normal to the radius vector $\boldsymbol{r}$. Note that centrifugal force actually increases with distance, whereas real forces decrease with distance. Therefore it is a fictitious force.

Note that the Coriolis force does indeed arise from the motion of the particle, because the force is proportional to $\boldsymbol{v}_{rot}$ and hence vanishes if there is no motion.

Both centrifugal & Coriolis forces are pseudo forces; they arise from kinematics and are not due to physical interactions.

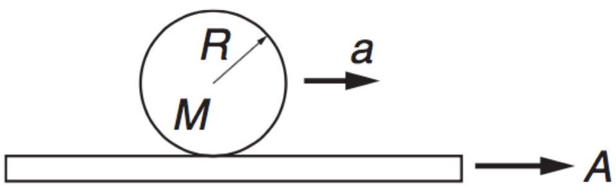
Therefore, finally we can write

$$ma_{rot} = F - \underbrace{2m\, \omega \times v_{rot}}_{\textbf{Coriolis}\text{ force}} - \underbrace{\mathrm{m}\omega \times (\omega \times \boldsymbol{r})\}}_{\textbf{Centrifugal}\text{ force}}$$

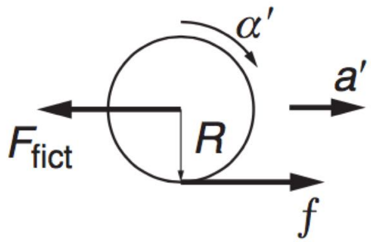
This is the equation of motion of a particle P of mass m in rotating frame (x_1, x_2, x_3).

Example1: Cylinder on an Accelerating Plank

A cylinder of mass M and radius R rolls without slipping on a plank which is accelerated at the rate **A**. Find the acceleration of the cylinder.



Force diagram



Horizontal force on the cylinder as viewed in the system accelerating with the plank

Solution:

$\boldsymbol{a'}$: acceleration of the cylinder as observed in a system fixed to the plank.

$\boldsymbol{f}$: friction force

$\boldsymbol{F_{fict}} = \boldsymbol{MA}$

Example1: Cylinder on an Accelerating Plank

The equation of motion in the system fixed to the accelerating plank,

$$\boldsymbol{f} - \boldsymbol{F}_{fict} = M\boldsymbol{a}' \text{……….1 (equation for translation)}$$

$$R\boldsymbol{f} = -I_0\boldsymbol{\alpha}' \text{……………2 (equation for rotation)}$$

The cylinder rolls on the plank without slipping, so

$$\boldsymbol{a}' = R\boldsymbol{\alpha}' \text{………..3}$$

Sub equations 2 and 3 in 1,

$$M\boldsymbol{a}' = -I_0\frac{\boldsymbol{a}'}{R^2} - \boldsymbol{F}_{fict}$$

Example1: Cylinder on an Accelerating Plank

$$\boldsymbol{a}' = -\frac{\boldsymbol{F}_{fict}}{M+\frac{I_0}{R^2}}$$

Since Rotational inertia $I_0 = {}^{MR^2}\!/_2$, $\boldsymbol{F}_{fict} = M\boldsymbol{A}$, we have

$$\boldsymbol{a}' = -\frac{2}{3}\boldsymbol{A}.$$ (acceleration of the cylinder in the plank)

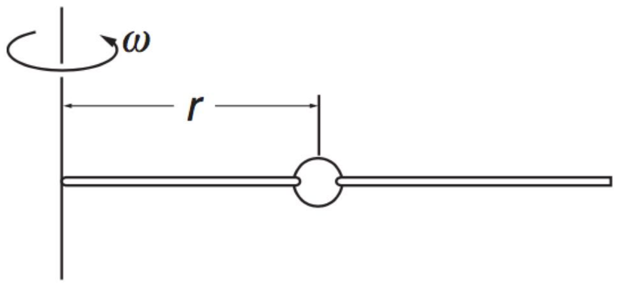
∴ acceleration of the cylinder in an inertial system is:

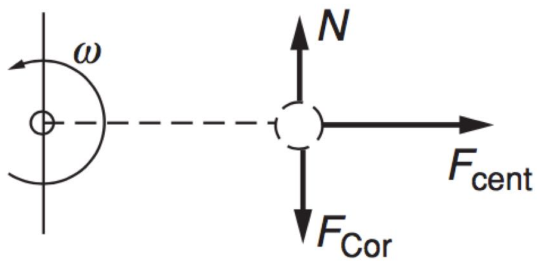
$$\boldsymbol{a} = \boldsymbol{A} + \boldsymbol{a}'$$

$$\boldsymbol{a} = \frac{1}{3}\boldsymbol{A}$$

Example2: Coriolis Force

A bead slides without friction on a rigid wire rotating at constant angular Speed ω. Find the force exerted by the wire on the bead. Neglect gravity.





Force diagram in the rotating system

In a coordinate system rotating with the wire, the motion is purely radial.

N: contact force is normal to the wire (as the wire is frictionless)
$\boldsymbol{F}_{cent}$: centrifugal force
$\boldsymbol{F}_{cor}$: Coriolis force

Example2: Coriolis Force

In the rotating system, the equations of motion are,

$$\boldsymbol{F}_{cent} = \mathrm{m}\ddot{\boldsymbol{r}} \ldots\ldots\ldots\ldots 1 \quad \text{(Radial equation of motion)}$$

Since there is no tangential acceleration in the rotating system,

$$\boldsymbol{N} - \boldsymbol{F}_{cor} = 0 \ldots\ldots\ldots 2 \quad \text{(Tangential equation of motion)}$$

Sub $F_{cent} = m\omega^2 r$ into equation 1,

$$\mathrm{m}\ddot{r} - m\omega^2 r = 0 \ldots\ldots\ldots 3$$

Solving equation 3,

$$r = Ae^{\omega t} + Be^{-\omega t}$$

Example2: Coriolis Force

From tangential equation we have,

$$\boldsymbol{N} = \boldsymbol{F}_{cor} = 2m\dot{\boldsymbol{r}}\omega$$

Differentiate r and substitute in the above equation,

$$\boldsymbol{F}_{cor} = 2m\omega^2(Ae^{\omega t} - Be^{-\omega t})$$

A complete solution requires initial conditions.

Does the spinning motion of Earth has any observable effect?

Earth as a rotating coordinate system

Counter clockwise rotation in northern hemisphere

Clockwise rotation in southern hemisphere

What Causes Hurricanes Spin?

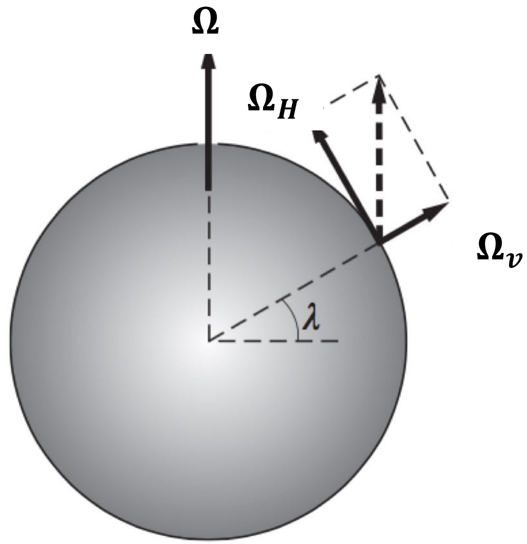
Why do they spin in different direction depending on their location?

Effect of Coriolis force due to Earth's spin

Consider a particle of mass m moving with velocity $\boldsymbol{v}$(tangential to the sphere) at latitude λ on the surface of a sphere. The sphere is rotating with angular velocity $\boldsymbol{\Omega}$. The Coriolis force is

$$F = -2m\boldsymbol{\Omega}\times\boldsymbol{v}$$

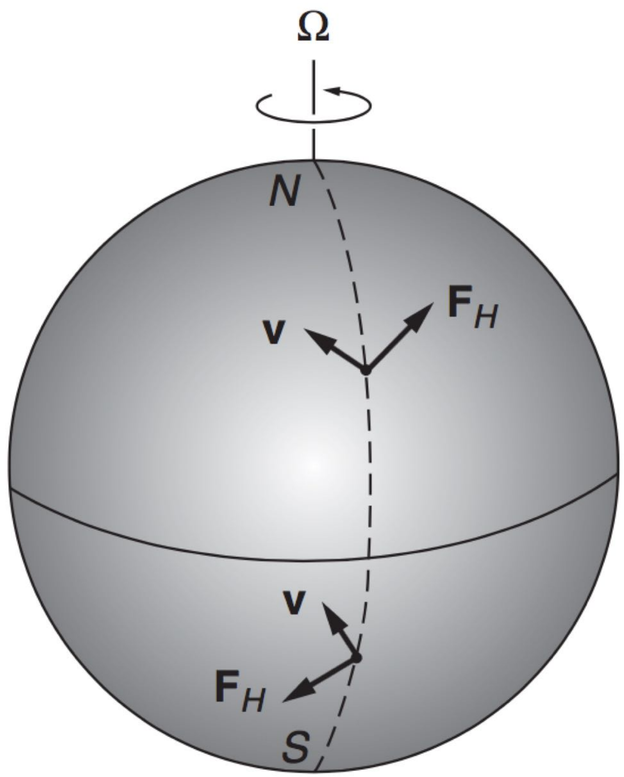
$$= -2m\,(\boldsymbol{\Omega_v}\times\boldsymbol{v} + \boldsymbol{\Omega_H}\times\boldsymbol{v})$$



($\boldsymbol{\Omega_v}$, $\boldsymbol{\Omega_H}$: vertical and horizontal parts of $\boldsymbol{\Omega}$.)

$\boldsymbol{\Omega_H}$ and $\boldsymbol{v}$ are horizontal and $\boldsymbol{\Omega_v}\times\boldsymbol{v}$ is vertical.

$$\therefore \underbrace{\boldsymbol{F}_H}_{\text{Horizontal Coriolis force}} = \boldsymbol{\Omega}_v \times \boldsymbol{v}$$



If $\hat{\boldsymbol{r}}$ is a unit vector perpendicular to the surface at latitude λ,

$$\boldsymbol{\Omega}_v = \Omega \sin \lambda \hat{\boldsymbol{r}}$$

So,

$$\boldsymbol{F}_H = -2m\Omega \sin \lambda \hat{\boldsymbol{r}} \times \boldsymbol{v}$$

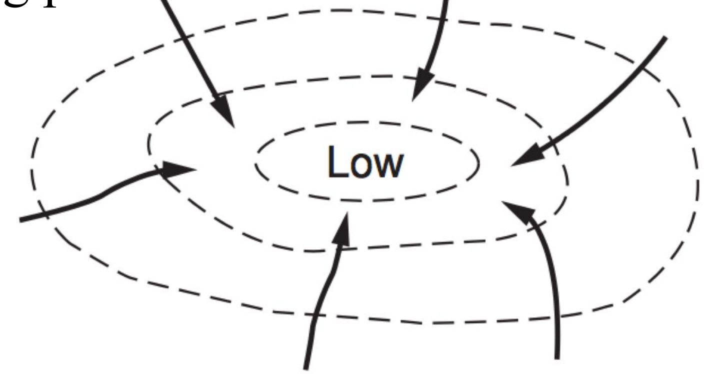
magnitude of $\boldsymbol{F}_H$: $F_H = 2mv\Omega \sin \lambda$

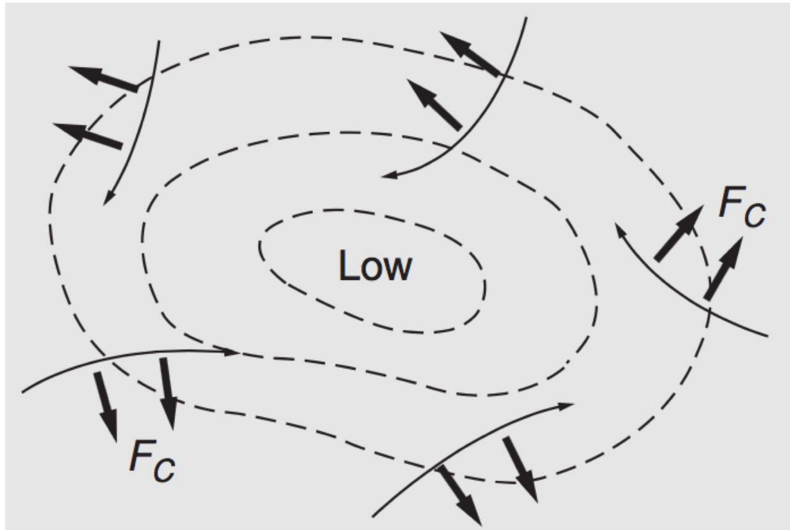
Coriolis force

- **depends on latitude**
- **turns straight line motion on a rotating sphere into circular motion**

Effect of Coriolis force on the weather system: Northern Hemisphere

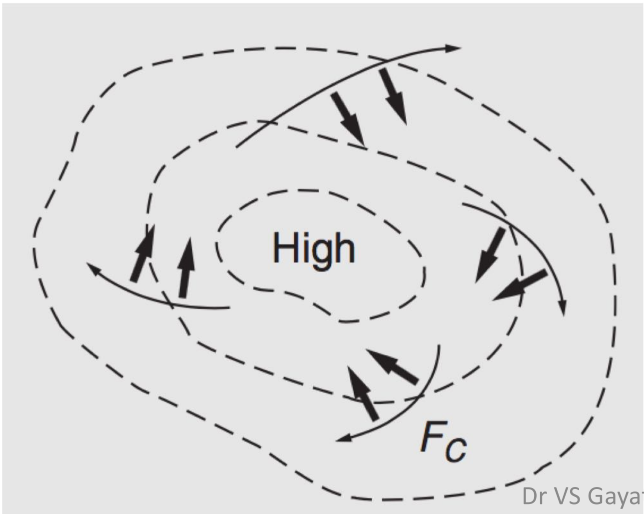
No force:winds blow inward equalizing pressure difference





Wind circulates counterclockwise about the low along the isobars

Wind circulates clockwise about regions of high pressure



Coriolis force constantly deflects air causing entire system to spin.

<u>In Northern Hemisphere:</u>

- Wind circulates counterclockwise about the low along the isobars.
- Wind circulates clockwise about regions of high pressure in the northern hemisphere.

The directions of rotations are reversed in Southern Hemisphere.

<u>Near Equator ($\lambda \approx 0°$):</u>

$$F_H \approx \mathbf{0}$$

∴circular weather systems cannot form there and weather tends to be uniform.