Classical Mechanics

Dynamics of system of particles

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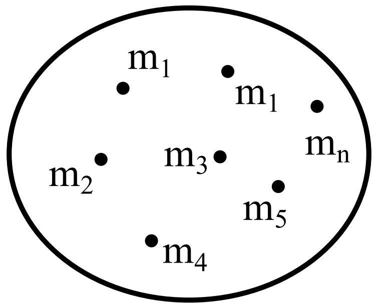
- Earlier, we have studied, the motion of single particle which is represented as a *point mass, m.*

- Now, we will extend our study to the dynamics of a *system of particles.*

What is a system of particles?

System of particles or extended objects : **A group of inter-related particles.**

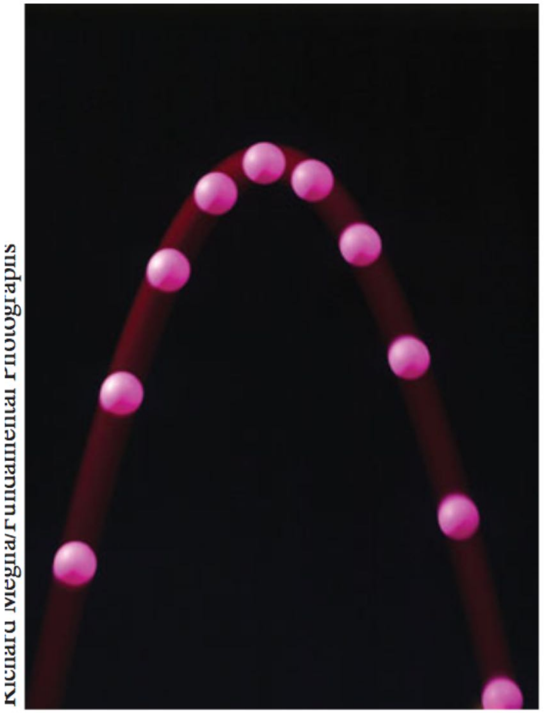
single particle

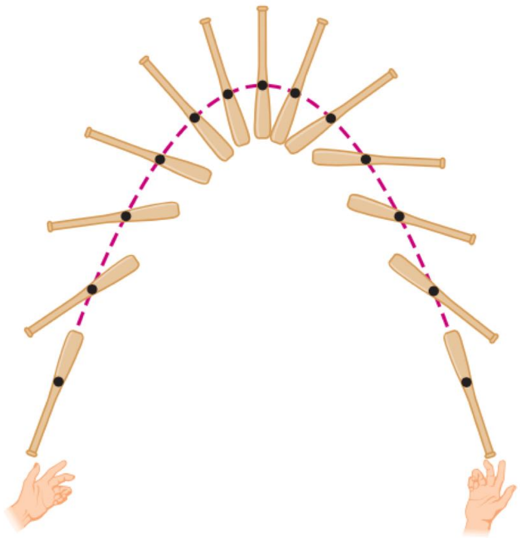


System of particles

System of Particles

System of particles or extended objects : A group of inter-related particles.





(*b*)

❖ **How to characterize the system of particles?**

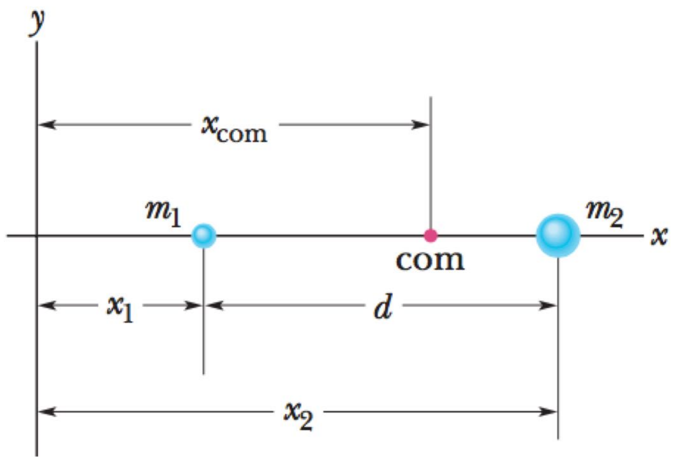
Fig:Fundamentals of Physics, Resnick et al.,

Center of Mass

Definition: The center of mass of a system of particles is the point that moves as though:

- all of the mass were concentrated there.
- all external forces were acting at that point.

- For a two particle system along x-axis:



$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$ (Mass–weighted mean of x_1 & x_2)

If $m_1 = m_2$,

$$x_{com} = \frac{x_1 + x_2}{2}$$ **(geometrical center)**

Center of Mass

- Generalizing to n-particles(along x-axis),

Total mass of the system $M = m_1 + m_2 + \cdots + m_n$

The location of center of mass is

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + \cdots \ldots + m_n x_n}{m_1 + m_2 + \ldots . + m_n}$$

$$x_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

Center of Mass

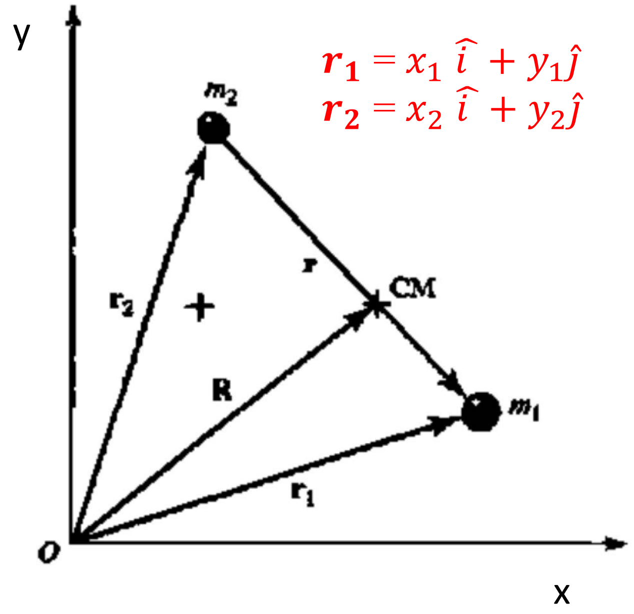
For particles distributed in 2D:

If the two particles are not along x-axis,

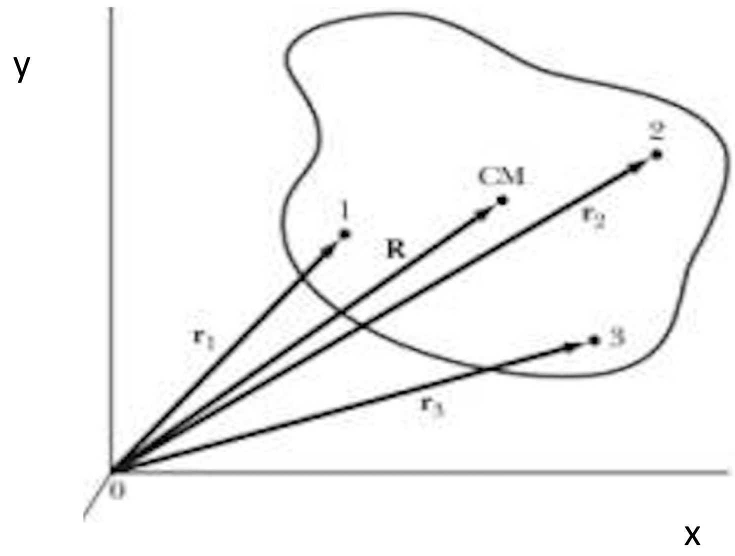
$$x_{com} = \frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2}$$

$$y_{com} = \frac{m_1\, y_1 + m_2\, y_2}{m_1 + m_2}$$

$$\boldsymbol{R} = x_{com}\hat{\imath} + y_{com}\hat{\jmath}$$



Center of Mass



❖ **3-particle system:**

$$x_{com} = \frac{m_1\, x_1 + m_2\, x_2\ + m_3 x_3}{m_1 + m_2 + m_3}$$

$$y_{com} = \frac{m_1\, y_1 + m_2\, y_2\ + m_3 y_3}{m_1 + m_2 + m_3}$$

$$\boldsymbol{R} = x_{com}\hat{\imath} + y_{com}\hat{\jmath}$$

Ref: Classical Dynamics of Particles & Systems, Marion et al.,

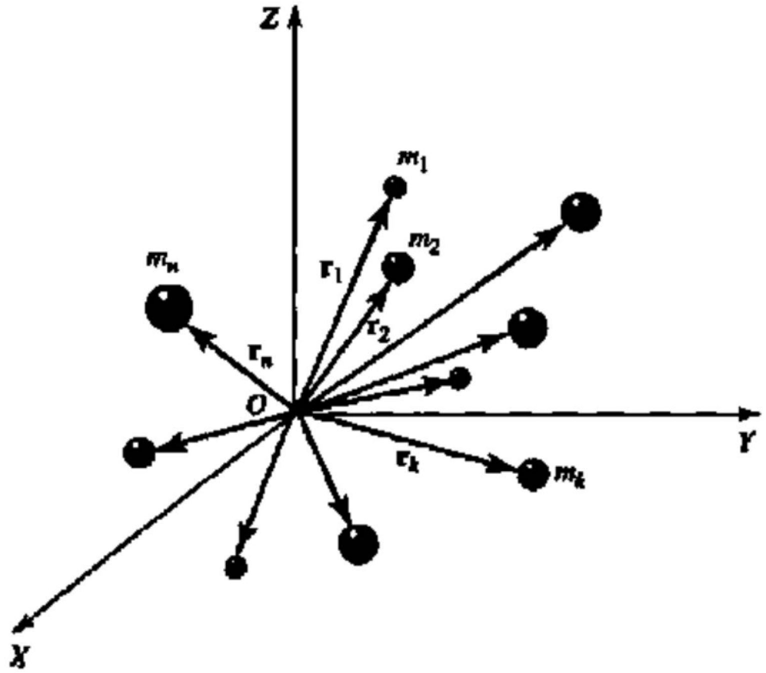
Center of Mass

❖ For an n-particle system(in 3D):

$$x_{com} = \frac{\Sigma\, m_i x_i}{M}$$

$$y_{com} = \frac{\Sigma\, m_i y_i}{M}$$

$$z_{com} = \frac{\Sigma\, m_i z_i}{M}$$



The position vector of the center of mass is:

$$\boldsymbol{R} = x_{com}\,\hat{\imath} + y_{com}\hat{\jmath} + z_{com}\,\hat{k}$$

$$\mathbf{R} = \frac{\Sigma\, m_i \boldsymbol{r}_i}{M}$$

(position vector of the i^{th} particle: $\boldsymbol{r}_i = x_i\,\hat{\imath} + y_i\hat{\jmath} + z_i\,\hat{k}$)

Ref: Classical Dynamics of Particles & Systems, Marion et al.,

Center of Mass

Note:

- The position of center of mass of a system of particles is independent of origin.

- If origin is chosen at the center of mass,

$$R = \frac{\Sigma\, m_i r_i}{M} = 0\,,$$

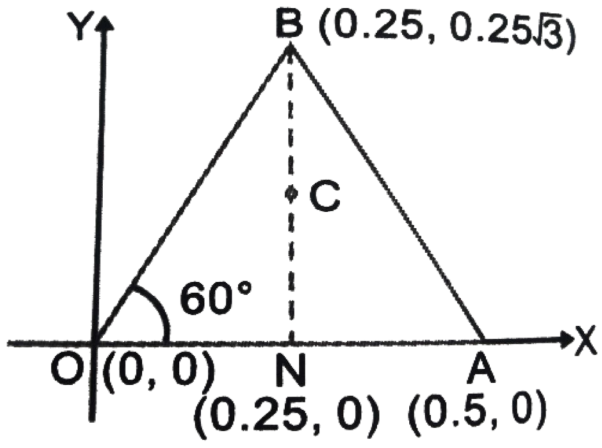
$$\Sigma\, m_i r_i = m_1 r_1 + m_2 r_2 + \cdots + m_n r_n = 0$$

$$\therefore\ sum\ of\ moments = 0$$

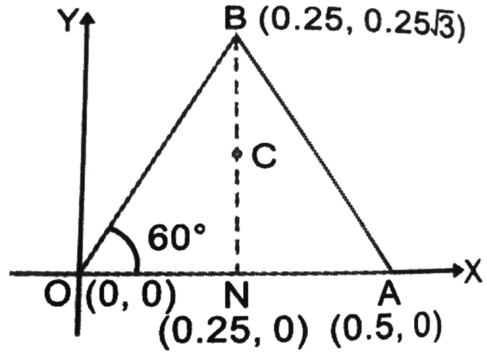
Moment of masses about the origin is zero.

Example-1

Find the center of mass of three particles at the vertices of an equilateral triangle. The masses of the particle are 100g,150g, and 200g respectively. Each side of the equilateral triangle is 0.5m along.



Particles	mass	x-coordinate	y-coordinate
O	100g	0	0
A	150g	0.5	0
B	200g	0.25	$0.25\sqrt{3}$



$$X = \frac{\sum_i m_i x_i}{m_i} = \frac{5}{18};$$

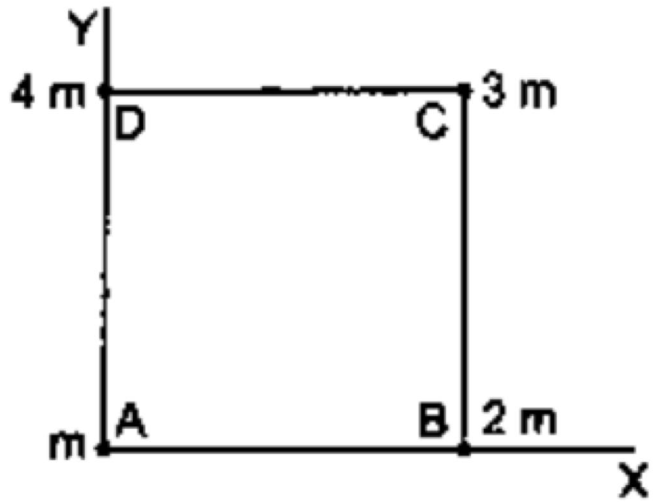
$$Y = \frac{\sum_i m_i y_i}{m_i} = \frac{1}{3\sqrt{3}}$$

Coordinates of COM, $R = \left(\frac{5}{18}, \frac{1}{3\sqrt{3}}\right).$

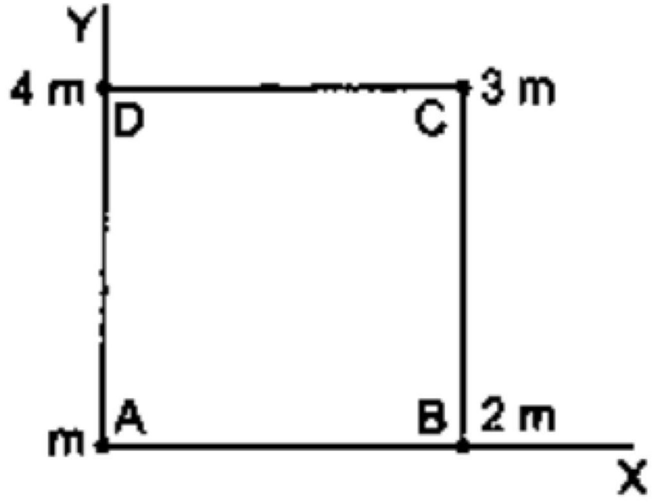
If "$m_1 = m_2 = m_3$, center-of-mass coincides with the centroid of the triangle (geometrical center)"

Example - 2

Four particles A, B, C and D having masses m, 2m, 3m and 4m respectively are placed in order at the corners of a square of side a. Locate the center of mass.



Particle	mass	x-coordinate	y-coordinate
A	m	0	0
B	2m	a	0
C	3m	a	a
D	4m	0	a



$$X = \frac{\sum_i m_i x_i}{m_i} = \frac{a}{2};$$

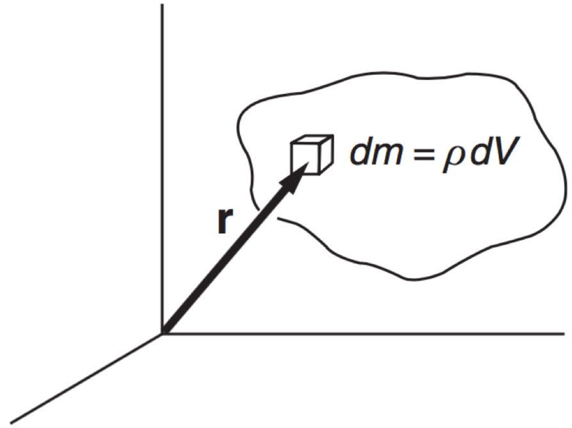
$$Y = \frac{\sum_i m_i y_i}{m_i} = \frac{7a}{10}$$

Coordinates of COM: $\boldsymbol{R} = \left(\frac{\boldsymbol{a}}{\mathbf{2}}, \frac{\mathbf{7}\boldsymbol{a}}{\mathbf{10}}\right)$.

- If masses are not equal, the COM of the system will be shifted towards heavier mass, as above.
- If "$m_1 = m_2 = m_3 = m_4$ center-of-mass coincides with the geometrical center $(\frac{a}{2}, \frac{a}{2})$"

Center of Mass of a System of Particles

In the continuum limit, divide the body into N mass elements. If $\boldsymbol{r}_j$ is the position of the j^{th}element, and m_j is its mass, then



$$\boldsymbol{R} = \frac{1}{M}\sum_{j=1}^{N} m_j \boldsymbol{r}_j$$

In the limit of $\mathrm{N} \to \infty$,

$$\boldsymbol{R} = \lim_{\mathrm{N}\to\infty} \frac{1}{M}\sum_{j=1}^{N} m_j \boldsymbol{r}_j$$

$$= \frac{1}{M}\int \boldsymbol{r}\, dm \text{ , } (dm : differential\ mass\ element)$$

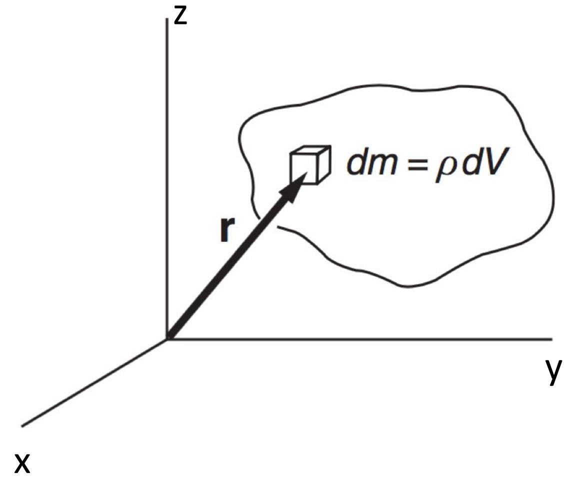
$$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\, dm$$ → "mass integral"

Center of Mass of a System of Particles

$$x_{com} = \frac{1}{M}\int x\, dm;$$

$$y_{com} = \frac{1}{M}\int y\, dm;$$

$$z_{com} = \frac{1}{M}\int z\, dm$$



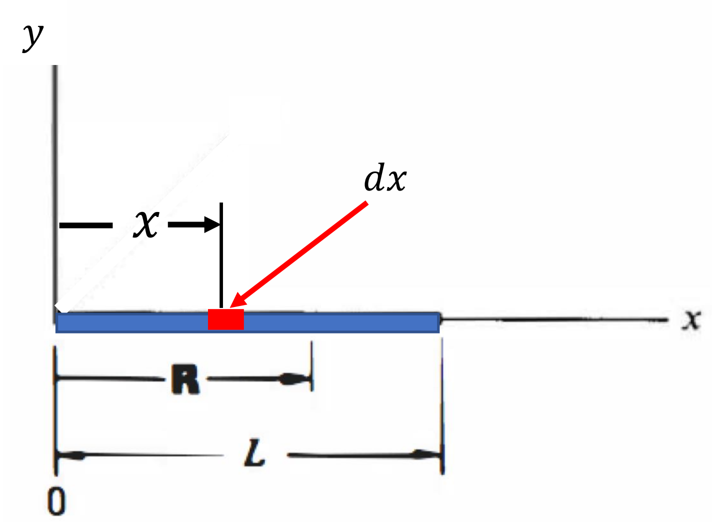
$$\boldsymbol{R} = x_{com}\,\hat{\imath} + y_{com}\hat{\jmath} + z_{com}\hat{k}$$

Let dm be the mass in an element of volume dV located at position $\boldsymbol{r}$.
If the mass density at the element dm is ρ, then $dm = \rho\, dV$

$$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\rho\, dV$$ → "Volume integral"

Example 1: Center of Mass of a Nonuniform Rod

Consider a rod of length L with a nonuniform density λ. Mass/unit length of the rod varies as $\lambda = \lambda_0 \frac{x}{L}$ (x: distance from the end marked 0).



Solution: Let the rod lie along the x-axis.

Mass in an element of length dx, at distance x from the origin:

$$dm = \lambda \, \mathrm{dx} = \lambda_0 x \frac{dx}{L}$$

Example 1: Center of Mass of a Nonuniform Rod

Total mass:

$$M = \int dm$$

$$= \int_0^L \lambda \, dx = \int_0^L \frac{\lambda_0 \, x \, dx}{L} = \frac{\lambda_0 \, L}{2}$$

$$M = \frac{\lambda_0 \, L}{2}$$

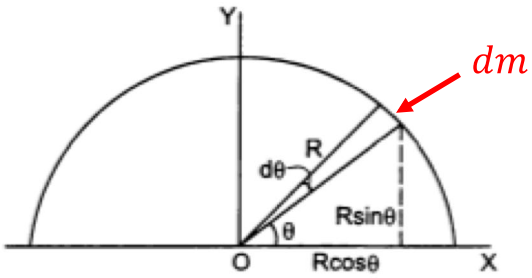
Position vector of center of mass:

$$\boldsymbol{R} = \frac{1}{M} \int \boldsymbol{r} \, \lambda \, dx$$

$$= \frac{2}{\lambda_0 L} \int_0^L (x\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\,\hat{\boldsymbol{k}}) \, \frac{\lambda_0 \, x \, dx}{L} = \frac{2}{L^2} \frac{\hat{\boldsymbol{i}}}{3} x^3$$

$$\mathbf{R} = \frac{2}{3} L \, \hat{\boldsymbol{i}}$$

Example 2: Center of Mass of Half of a Ring



Mass per unit length

$$\lambda = \frac{M}{\pi R}$$

As the wire is uniform, the mass per unit length of the wire is $\frac{M}{\pi R}$. The mass of the element is, therefore,

$$dm = \left(\frac{M}{\pi R}\right)(R\, d\theta) = \frac{M}{\pi}\, d\theta.$$

The coordinates of the centre of mass are

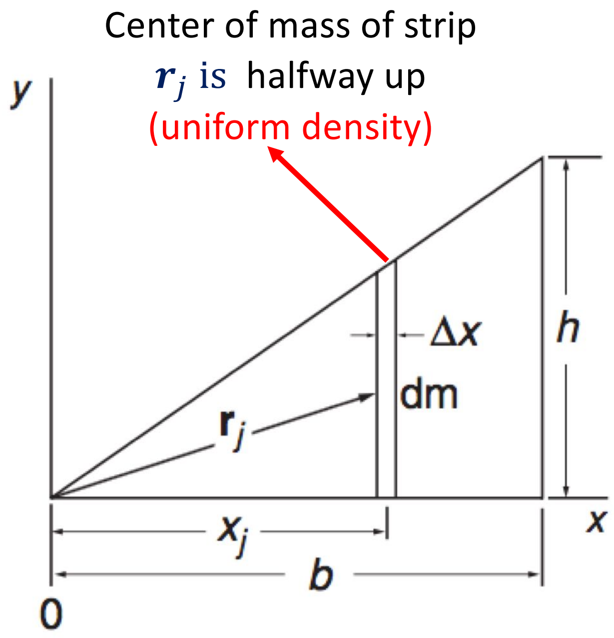
$$X = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^\pi (R \cos\theta)\left(\frac{M}{\pi}\right) d\theta = 0$$

and

$$Y = \frac{1}{M}\int y\, dm = \frac{1}{M}\int_0^\pi (R \sin\theta)\left(\frac{M}{\pi}\right) d\theta = \frac{2R}{\pi}$$

Example 3 : Center of Mass of a Triangular Plate

Consider the two-dimensional case of a uniform right triangular plate of mass M, base b,height h and small thickness t.



Solution:

Divide the triangle into strips of width Δx parallel to the y axis.

The total height of the j^{th} strip = $\frac{x_j h}{b}$. (Similar triangles)

Example 3: Center of Mass of a Triangular Plate

The position vector of the center of mass of j^{th} strip :

$$\boldsymbol{r}_j = x_j\hat{\boldsymbol{\imath}} + y_j\hat{\boldsymbol{\jmath}}$$

$$\boldsymbol{r}_j = x_j\hat{\boldsymbol{\imath}} + \frac{x_j h}{2b}\hat{\boldsymbol{\jmath}} \qquad \left(\text{as } y_j = \frac{\mathbf{1}}{\mathbf{2}}(\text{total height of the j}^{\text{th}}\text{strip})\right)$$

Center of mass of the plate :

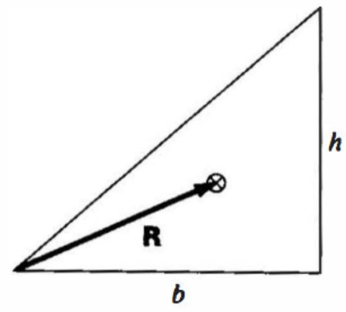
$$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\, dm \qquad \text{(limit of very narrow strips)}$$

Mass of the plate M = $\rho At = \frac{\rho tbh}{2}$

Mass element $dm = \rho tydx = \rho t\frac{xh}{b}dx$

Example 3: Center of Mass of a Triangular Plate

Sub for M and dm,



$$\boldsymbol{R} = \frac{2}{\rho t b h}\int \boldsymbol{r}\rho t \frac{xh}{b}\, dx$$

$$= \frac{2}{b^2}\int_0^b x\, \boldsymbol{r}\, dx$$

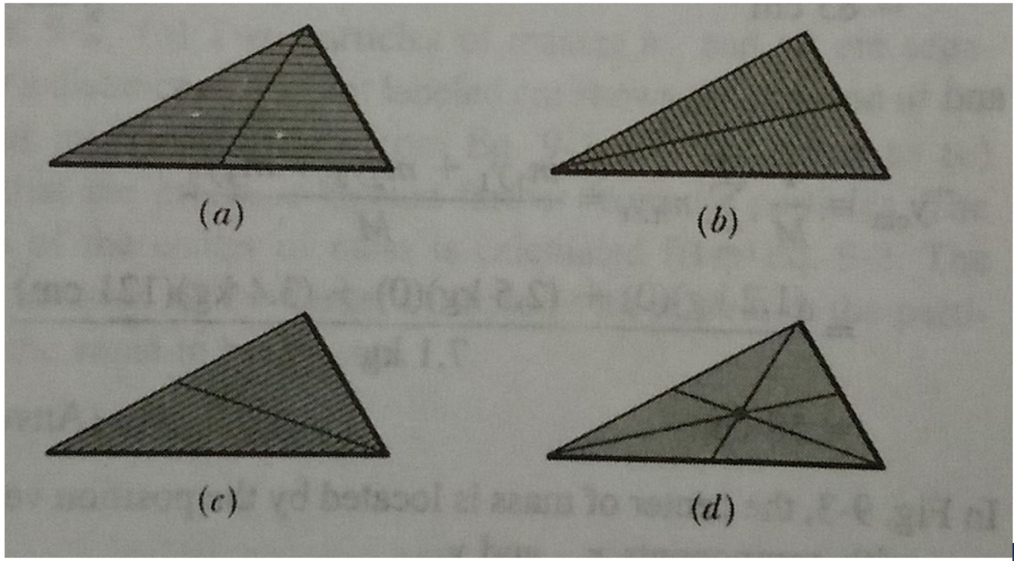
integrating w.r.t x,

$$= \frac{2}{b^2}\int_0^b (x^2\, \hat{\boldsymbol{\imath}} + \frac{x^2\, h}{2b}\hat{\boldsymbol{\jmath}})\mathrm{dx}$$

$$\boldsymbol{R} = \frac{2}{3}b\hat{\boldsymbol{\imath}} + \frac{h}{3}\,\hat{\boldsymbol{\jmath}}$$ ***R*** depends on ***b*** &***h***

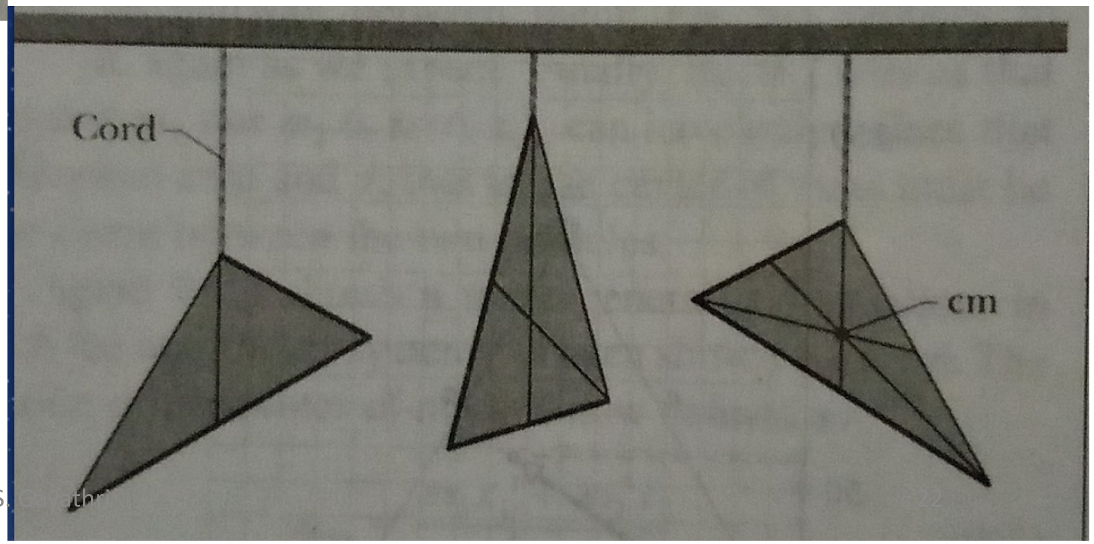
Note: But the position of center of mass w.r.t the triangular plate is independent of the choice of coordinate system

Location of Center of Mass of a Uniform Triangular Plate:By Symmetry

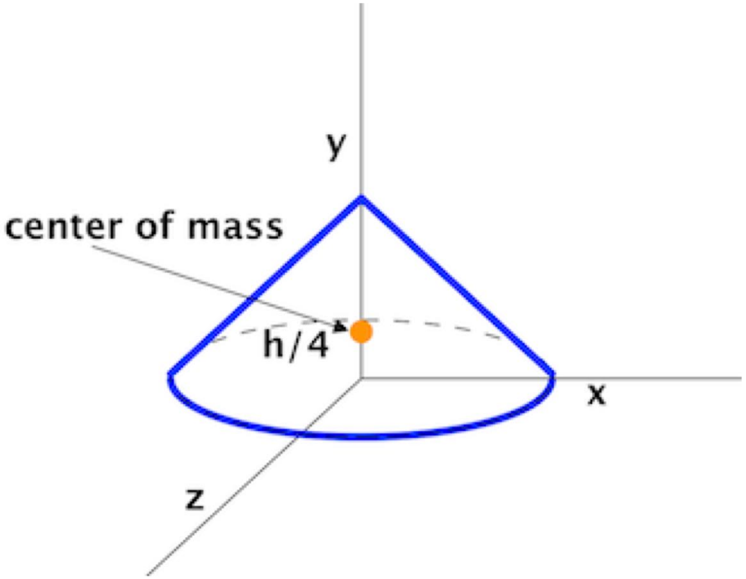


center of mass
lie along the bisecting line

Locate center of mass
by suspending from each
vertex in turn



Location of Center of Mass : By symmetry



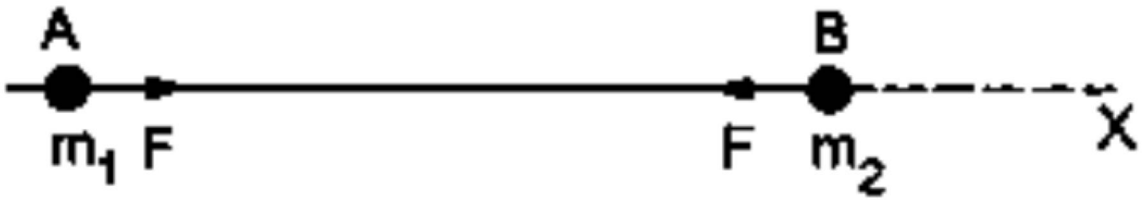
Center: Point of symmetry

Axis:line of symmetry

Center of mass

Motion of the Center of Mass

Consider a two-particle system(A &B) of masses m_1 and m_2 respectively. Let $x_1$and x_2 be the coordinates of the particles at time t.



Assume external forces = 0

Internal force: Suppose the particles are initially at rest and exert attractive forces between them. They both move along the line AB.

Motion of the Center of Mass

The center of mass at time t=0 is situated at :

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

(center of mass moves along the X-axis.)

Velocity of the center of mass at time t is

$$v_{com} = \frac{dx_{com}}{dt} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Since initially particles are at rest at t=0,

$$v_1 = v_2 = 0$$

$$v_{com} = 0$$

Motion of the Center of Mass

The acceleration of the center of mass is

$$a_{com} = \frac{dV_{com}}{dt} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}$$

<u>if external force = 0</u>

Let F_{int} be the magnitude of the (internal) forces between the particles.

$$a_1 = {}^{F_{int}}\!/_{m_1} \text{ and } a_2 = {}^{-F_{int}}\!/_{m_2}$$

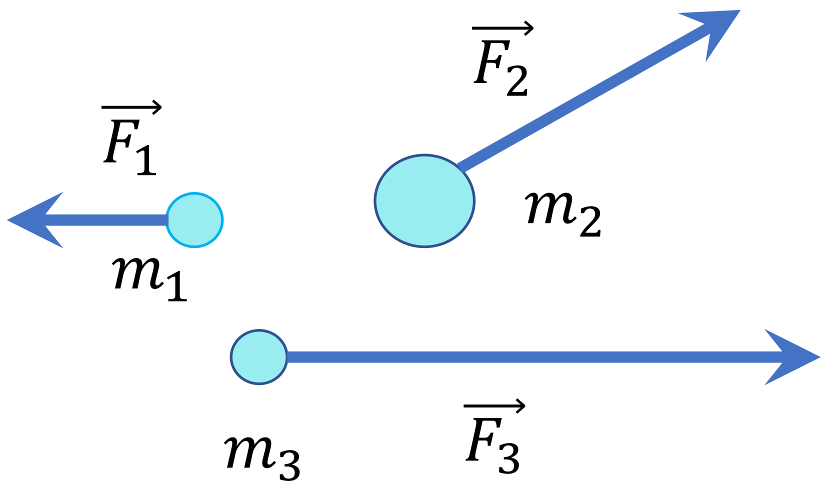
$$\therefore a_{com} = \frac{m_1\left({}^{F_{int}}\!/_{m_1}\right) + m_2\left({}^{-F_{int}}\!/_{m_2}\right)}{m_1 + m_2} = 0$$ *"Velocity of center of mass is constant"*

"if no external force acts on a two-particle system and its center of mass is at rest initially, it remains fixed even when the particles individually move and accelerate."

Motion of the Center of Mass

if external force ≠ 0

Consider a three particle system of masses m_1, m_2 and m_3. Let $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be the external forces acting on masses m_1, m_2 and m_3. The internal forces, F_{int}, between the masses cancel each other.

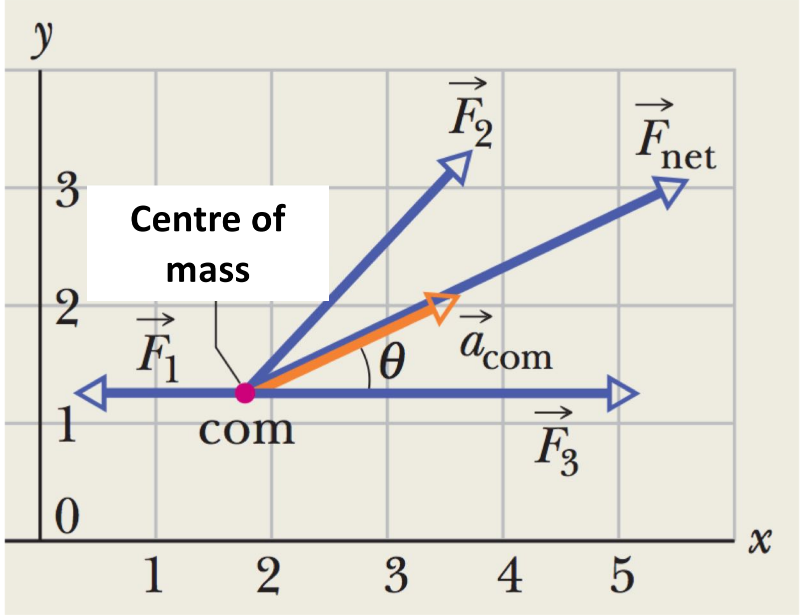


Motion of the Center of Mass

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M} = \frac{\vec{F}_{net}}{M}$$

❖ The acceleration $\vec{a}_{com}$ of the center of mass of the system of particles, is in the same direction as the net external force $\vec{F}_{net}$.

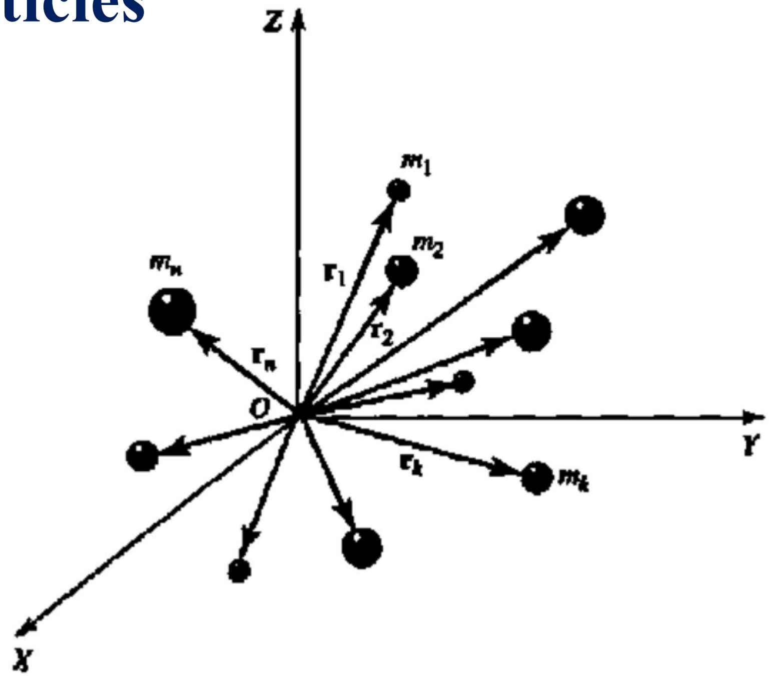
❖ The motion of the center of mass of system is not affected by the internal forces.



What is the equation of motion for a system of N particles?

Fig:Fundamentals of Physics, Resnick et al.,

Dynamics of System of Particles



Consider a system of N interacting particles.

masses : $m_1, m_2, m_3 \ldots\ldots\ldots m_n.$

position vector of *j^{th}* particle: $\mathbf{r}_j$

momentum of j^{th} particle: $\mathbf{p}_j = m_j\, \dot{\mathbf{r}}_j$

Internal forces: $\boldsymbol{f}_j^{\text{int}}$ (force on j due to all other particles in the system)

External forces: $\boldsymbol{f}_j^{\text{ext}}$ (force due to sources outside the system)

Total force on *j^{th}* particle : $\boxed{\boldsymbol{f}_j = \boldsymbol{f}_j^{\text{int}} + \boldsymbol{f}_j^{\text{ext}}}$

Dynamics of System of Particles

Newton's equation of motion for the j[th] particle is

$$\mathbf{f}_j = \frac{d}{dt}\mathbf{p}_j \qquad \text{(Newton's II Law)}$$

$$\mathbf{f}_j^{int} + \mathbf{f}_j^{ext} = \frac{d}{dt}\mathbf{p}_j$$

Therefore, equations of motion of all the particles in the systems:

$$\mathbf{f}_1^{int} + \mathbf{f}_1^{ext} = \frac{d}{dt}\mathbf{p}_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{f}_j^{int} + \mathbf{f}_j^{ext} = \frac{d}{dt}\mathbf{p}_j$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{f}_N^{int} + \mathbf{f}_N^{ext} = \frac{d}{dt}\mathbf{p}_N$$

Dynamics of System of Particles

Adding all these equations, we get:

$$\Sigma\, \boldsymbol{f}_j^{int} + \Sigma\, \boldsymbol{f}_j^{ext} = \Sigma\, \frac{d}{dt}\, \mathbf{p}_{\mathrm{j}} \qquad\qquad \mathrm{j} = 1 \ldots \mathrm{N}$$

$\Sigma\, \boldsymbol{f}_j^{ext} = \boldsymbol{F}_{\boldsymbol{ext}}$ **total external force** acting on the system

By Newton's III law: $\boldsymbol{f}_{ij}^{int} = -\boldsymbol{f}_{ji}^{int}$ **"internal forces cancel in pairs"**

$$\Sigma\, \boldsymbol{f}_j^{int} = 0$$

$$\boldsymbol{F}_{\boldsymbol{ext}} = \Sigma\, \frac{d}{dt}\, \mathbf{p}_{\mathrm{j}} = \frac{d}{dt}\, \Sigma\, \mathbf{p}_{\mathrm{j}} = \frac{d}{dt}\, \mathbf{P}$$

Dynamics of System of Particles

$$\boxed{\boldsymbol{F}_{ext} = \frac{d}{dt}\mathbf{P}}$$

Total external force = rate of change of the total momentum

This equation describes only "translation" of the center of mass of the body.

We know that, $\mathbf{P} = \mathrm{M}\boldsymbol{V}_{com}$ ($\boldsymbol{V}_{com}$:velocity of Center of mass)

$$\therefore \frac{d\boldsymbol{P}}{dt} = M\frac{d\boldsymbol{V}_{com}}{dt}$$

$$\boxed{\boldsymbol{F}_{ext} = \mathrm{M}\,\boldsymbol{a}_{com}}$$

Dynamics of System of Particles

Consider the equation of motion: $\boldsymbol{F_{ext}} = M\,\boldsymbol{a_{com}}$

If $\boldsymbol{F_{ext}} = 0$, then

$$\boldsymbol{a_{com}} = 0 \quad \text{(center of mass has no acceleration)}$$

$$\therefore \boldsymbol{V}_{com} = \text{constant}$$

"if there are **no external forces** acting on the system of particles, then

- ❖ If the center of mass was at rest initially, it will continue to be at rest(even if the individual particles go on a complicated path changing their position).
- ❖ If the center of mass was moving at a speed $\boldsymbol{V}_{com}$ along a particular direction initially, it will continue its motion along the same straight line with the same speed.