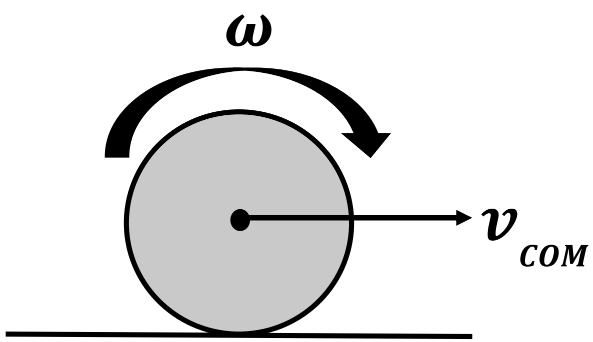
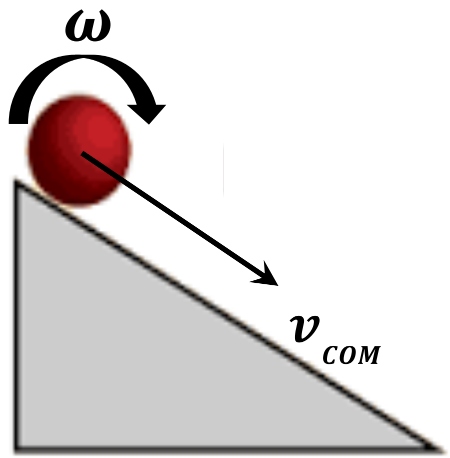
Classical Mechanics

Pure rolling motion

Dr VS Gayathri

Pure rolling motion



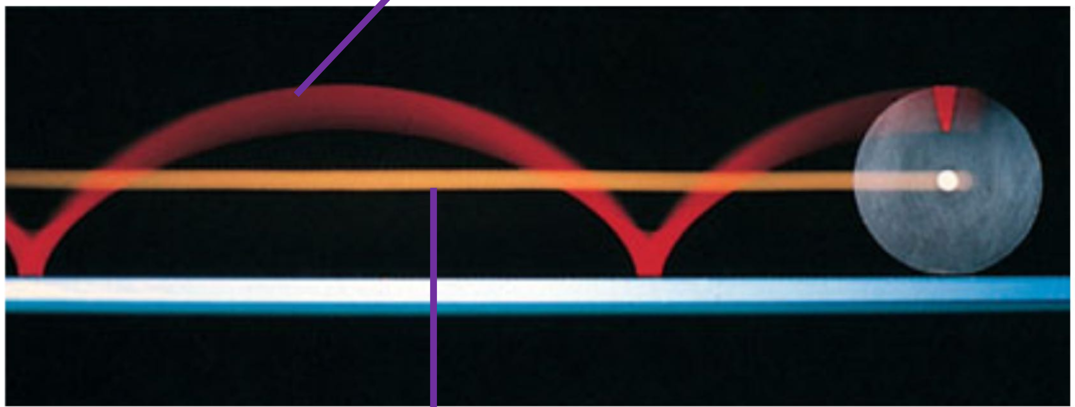


Translational and rotational motion in the same system

Rolling motion of a rigid body: Combination of translation of its center of mass and rotation about its center of mass.

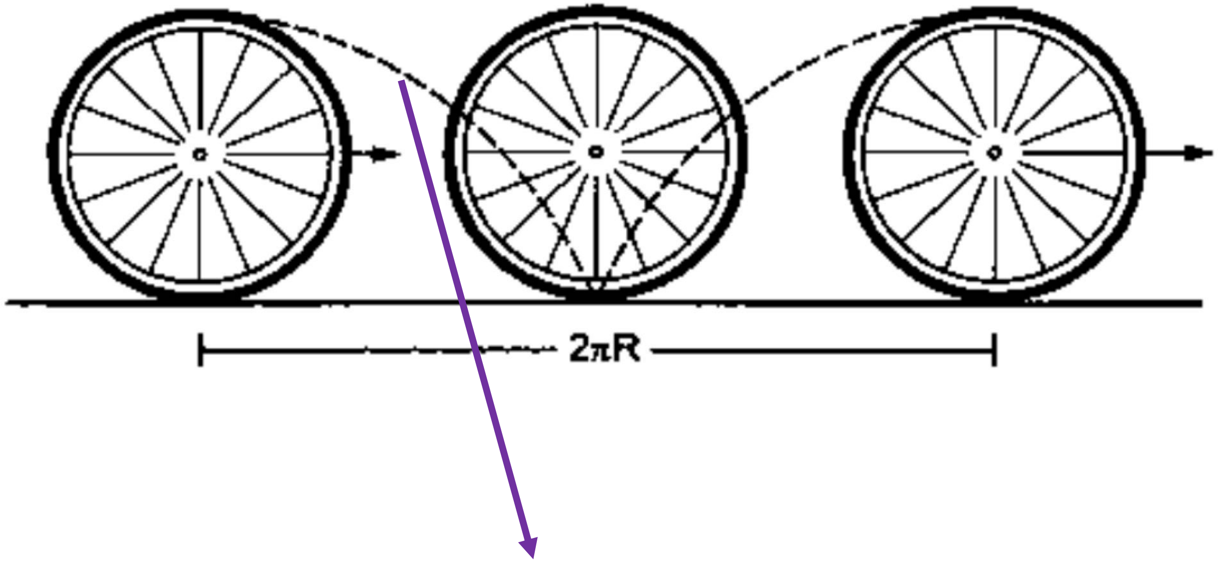
Rolling motion of a disk

“cycloid” curve: path traced by a **particle at the rim**



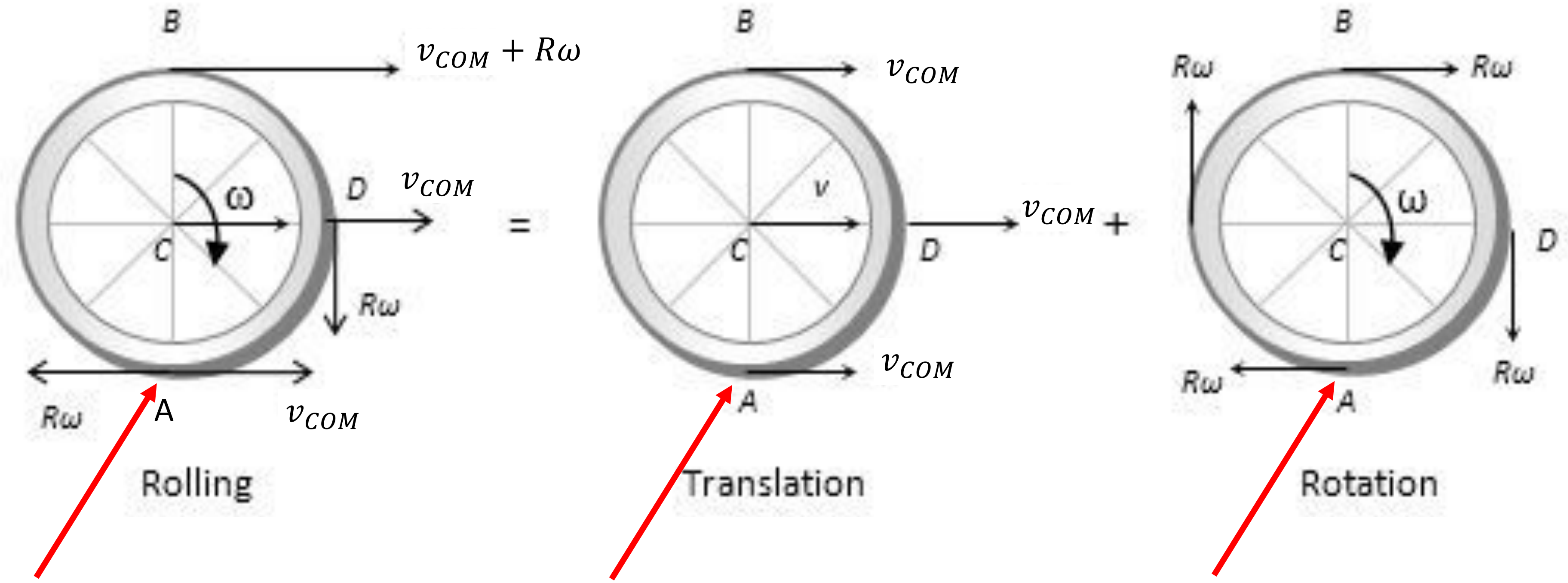
Straight line motion: **Path of center of mass**

Rolling motion of a wheel



“cycloid” curve: path traced by a **particle at the rim**

Condition of Pure rolling motion without slipping



$\vec{\boldsymbol{v}}_{\boldsymbol{A}} = (v_{COM} - R\omega) = \mathbf{0}$

$\vec{\boldsymbol{v}}_{\boldsymbol{A}} = v_{COM}$

$\vec{\boldsymbol{v}}_{\boldsymbol{A}} = -R\omega$

Condition of Pure rolling motion without slipping

- If a body with circular symmetry (wheels, cylinder, sphere etc.) rolls on a plane surface without slipping, then, the condition for pure rolling motion without slipping can be obtained by the fact that the velocity of point A w.r.t the surface is zero.

$$\vec{v}_A = 0$$

$$(v_{COM} - R\omega) = 0$$

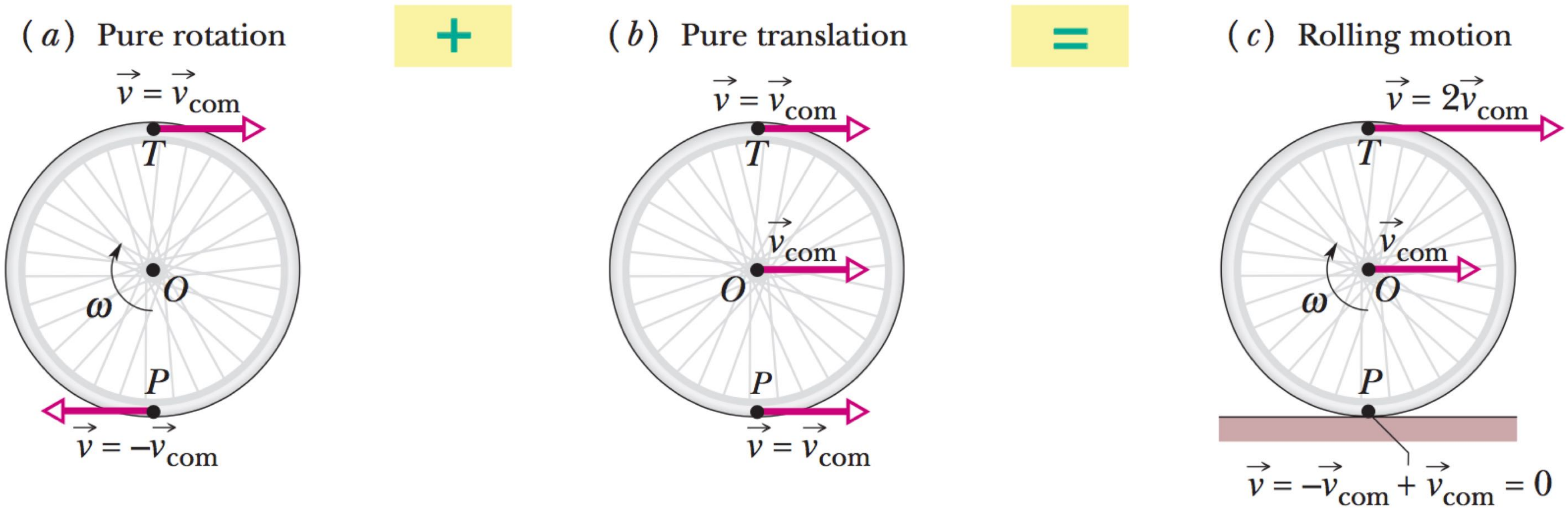
$$v_{COM} = R\omega \qquad \text{.............................1}$$

- If linear acceleration is a and angular acceleration is α, then on differentiating above eq. 1, we have

$$\frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$a = R\alpha \qquad \text{.............................2}$$

Pure rolling motion of a wheel without slipping



The wheel is moving faster near the top than near its bottom.

Pure rolling motion of a wheel without slipping

Why are the Spokes near the top are more blur than those near the bottom?

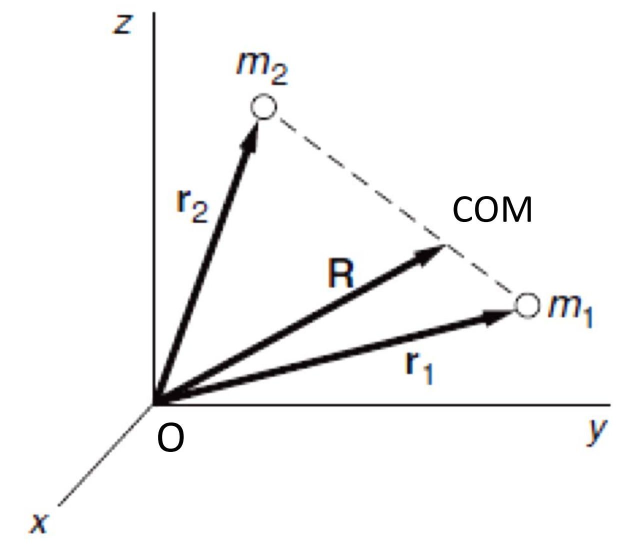
The wheel is moving faster near the top than near its bottom.

Center of Mass Coordinates

- The proper choice of coordinate system plays an important role to simplify a problem.

- The center of mass coordinate system, in which the origin lies at the center of mass, is particularly useful.

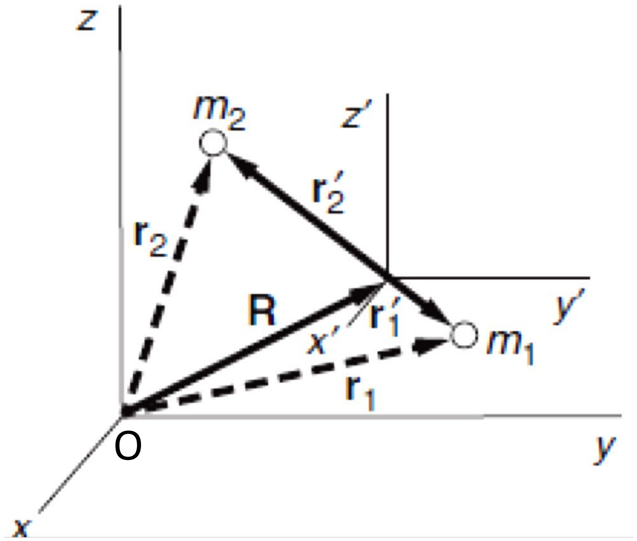
- Consider the case of a two-particle system with masses m_1 and m_2. In the initial coordinate system x, y, z, the particles are located at r_1 and r_2 and their center of mass is at:



$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

- Now, we will set up the center of mass coordinate system, x', y', z' with its origin at the center of mass.

- The origins of the old and new system are displaced by R. The center of mass coordinates of the two particles are



$$r_1' = r_1 - R$$
$$r_2' = r_2 - R$$

- Center of mass coordinates are the natural coordinates for an isolated two-body system. Such a system has no external forces, so the motion of the center of mass is trivial—it moves uniformly.

- Furthermore, $m_1 r_1' + m_2 r_2' = 0$ by the definition of center of mass, so that if the motion of one particle is known, the motion of the other particle follows directly.

Angular momentum of a rigid body during rolling motion

- In rotation motion, the axis of rotation does not change, but in rolling motion, *the axis itself translate.*
- Let us discuss the angular momentum of a rigid body during rolling motion about z−axis(axis of rotation).
- Now, Lz can be written as sum of angular momentum $I_0\omega$ due to roatation of the body around its center of mass, and the angular momentum $(\boldsymbol{R}\times M\boldsymbol{V})_z$ due to motion of the center of mass w.r.t. the origin of the inertial coordinate system:

$$\boxed{L_z = I_0\omega + (\boldsymbol{R}\times M\,\boldsymbol{V})_z}$$

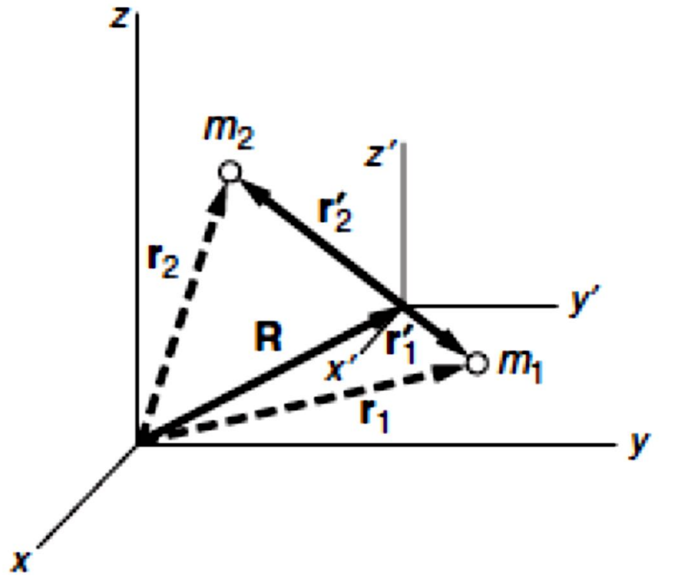
where, **R** is a vector from the origin to the center of mass ,
$\boldsymbol{V} = \dot{\boldsymbol{R}}, and\ I_0$ is the moment of inertia around the center of mass

Proof:

Angular momentum of a rigid body during rolling motion

Consider a rigid body made of *N* particles with masses m_i(*j* = 1, . . . , *N*), located at $\boldsymbol{r_j}$ with respect to the chosen origin in an inertial system. The angular momentum of the body is

$$\boldsymbol{L} = \sum_{j=1}^{N}\left(\boldsymbol{r}_j \times m_j\,\dot{\boldsymbol{r}}_{\boldsymbol{j}}\right)\text{..............................}1$$



The center of mass of the body has position vector R :

$$\boldsymbol{R} = \frac{\sum m_j \boldsymbol{r}_j}{M}\text{..............................}2$$

where *M* is the total mass and r_j is given as

$$\boldsymbol{r}_j = \boldsymbol{R} + \boldsymbol{r}_j'\text{..............................}3$$

Angular momentum of a rigid body during rolling motion

Combining Equation 1 and 2, we get

$$\boldsymbol{L} = \sum(\boldsymbol{R} + \boldsymbol{r}'_j)\times m_j(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}'_j)$$

$$\boldsymbol{L} = \boldsymbol{R}\times\sum m_j\dot{\boldsymbol{R}} + \sum m_j\boldsymbol{r}'_j\times\dot{\boldsymbol{R}} + \boldsymbol{R}\times\sum m_j\dot{\boldsymbol{r}}'_j + \sum m_j\boldsymbol{r}'_j\times\dot{\boldsymbol{r}}'_j \text{.....................4}$$

In this expression, there are four terms. The first and last terms have simple physical interpretations. However, we can show that the middle two terms are identically zero.

Starting with the second term, we have

$$\begin{aligned}\sum m_j\boldsymbol{r}'_j &= \sum m_j(\boldsymbol{r}_j - \boldsymbol{R})\\ &= \sum m_j\boldsymbol{r}_j - M\boldsymbol{R}\\ &= 0 \qquad\qquad (by\ equation\ 2)\end{aligned}$$

because $\sum m_j r'_j$ is identically zero, its time derivative vanishes.

∴ The third term is also zero.

Angular momentum of a rigid body during rolling motion

The first term is $$\boldsymbol{R}\times \sum m_j\dot{\boldsymbol{R}} = \boldsymbol{R}\times M\dot{\boldsymbol{R}}$$
$$= \boldsymbol{R}\times M\boldsymbol{V}$$

where $\boldsymbol{V} = \dot{\boldsymbol{R}}$ is the velocity of the center of mass with respect to the inertial system.

Now, Equation 4 becomes $$\boldsymbol{L} = \boldsymbol{R}\times M\boldsymbol{V} + \sum \boldsymbol{r}_j'\times m_j\dot{\boldsymbol{r}}_j' \quad \text{.........................5}$$

due to the center of mass motion

due to motion around the center of mass

The only way for the particles of a rigid body to move with respect to the center of mass is for the body as a whole to rotate.

Angular momentum of a rigid body during rolling motion

The Z-component of the total angular momentum given in equation 5, is

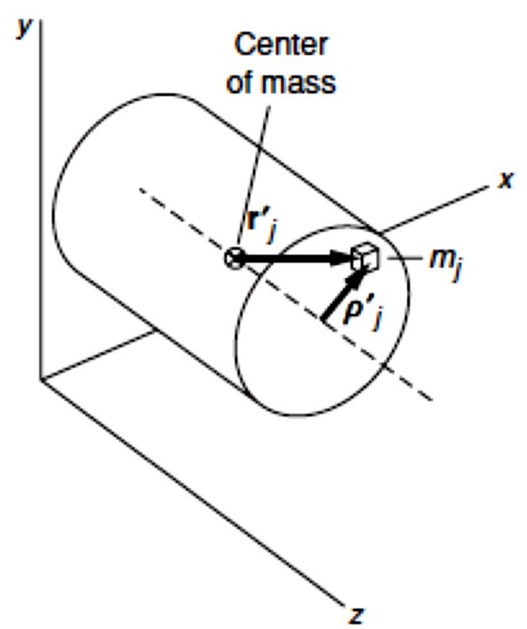
$$L_z = (\boldsymbol{R}\times M\boldsymbol{V})_z + \left(\sum \boldsymbol{r}_j'\times m_j\dot{\boldsymbol{r}}_j'\right)_z \text{……………………}6$$

The second term can be simplified. The body has angular speed ω around its center of mass, and because the origin of $\boldsymbol{r}_j'$ is the center of mass, the second term is identical in form to the case of pure rotation, we have done earlier.

$$\left(\sum \boldsymbol{r}_j'\times m_j\dot{\boldsymbol{r}}_j'\right)_z = \left(\sum m_j\boldsymbol{\rho}_j'\times\dot{\boldsymbol{\rho}}_j'\right)_z$$

$$= \sum m_j\rho_j'^2\,\omega$$

$$= I_0\omega$$



where $\boldsymbol{\rho}_j'$ is the vector to m_j perpendicular from a *z* axis through the center of mass, and $I_0 = \sum m_j\rho_j'^2$ is the moment of inertia of the body around this axis.

Angular momentum of a rigid body during rolling motion

Finally, the Z-component of the total angular momentum is given as:

$$L_z = I_0\omega + (\boldsymbol{R}\times M\boldsymbol{V})_z \ldots\ldots\ldots\ldots\ldots\ldots.7$$

So, we have proven that the angular momentum of a rigid object is the sum of the *angular momentum of rotation about its center of mass* and the *angular momentum of the center of mass about the origin.*

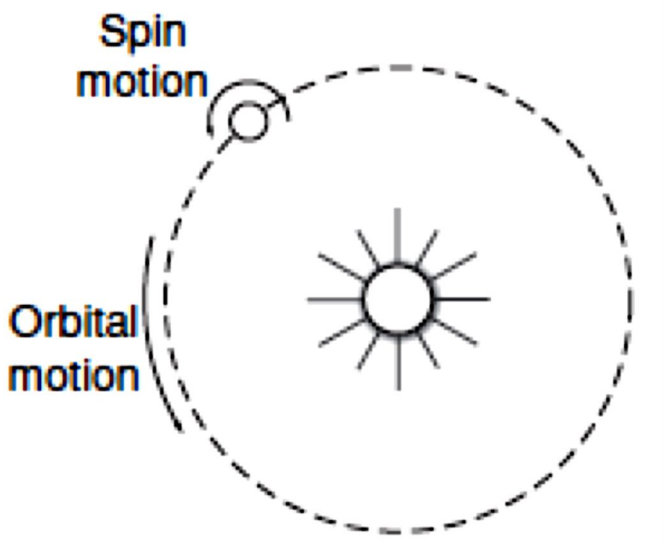
Equation 7 is valid even if the center of mass is accelerating, because L was calculated with respect to an inertial coordinate system.

These two terms are often referred to as the *spin* and *orbital* terms, respectively.

➢ The Earth’s motion around the Sun illustrates the distinction nicely:

- The daily rotation of the Earth around its polar axis gives rise to the *Earth’s spin angular momentum*

- The Earth’s annual revolution around the Sun gives rise to its *orbital angular momentum.*



Torque on a rolling Body

Torque also naturally divides itself into two terms. The torque on a body is,

$$\boldsymbol{\tau} = \sum_j \boldsymbol{r_j} \times \boldsymbol{f_j}$$

$$= \sum_j (\boldsymbol{r'_j} + \boldsymbol{R}) \times \boldsymbol{f_j}$$

$$= \sum_j \left(\boldsymbol{r'_j} \times \boldsymbol{f_j}\right) + \boldsymbol{R} \times \boldsymbol{F}$$ ($\boldsymbol{F} = \sum \boldsymbol{f}_j$ is the total applied force)

The first term is the torque around the center of mass due to the various external forces, and the second term is the torque on the center of mass due to the total external force acting at the center of mass.

Torque on a rolling Body

For a fixed axis rotation $\omega = \omega\hat{\boldsymbol{k}}$, so the above equation becomes,

$$\tau_z = \tau_0 + (\boldsymbol{R}\times\boldsymbol{F})_z$$

τ_0: z-component of the torque about the center of mass.

Proof:

we can get torque by differentiating L_z w.r.t. time

$$\tau_z = \frac{dL_z}{dt}$$

$We\ know,$ $$L_z = I_0\omega + (\boldsymbol{R}\times M\boldsymbol{V})_z$$

$$\frac{dL_z}{dt} = I_0\frac{d\omega}{dt} + \frac{d(\boldsymbol{R}\times M\boldsymbol{V})_z}{dt}$$

Torque on a rolling Body

$$Using, \qquad \frac{d\boldsymbol{R}}{dt} = \boldsymbol{V}, \qquad \frac{md\boldsymbol{V}}{dt} = \boldsymbol{F} \;\; and \;\; \frac{d\omega}{dt} = \alpha$$

$$\frac{dL_z}{dt} = I_0\alpha + (\boldsymbol{R}\times\boldsymbol{F})_z$$

$$\tau_z = \tau_0 + (\boldsymbol{R}\times\boldsymbol{F})_z \qquad \text{where } \tau_0 = I_0\alpha$$

- $\tau_0 = I_0\alpha$ suggests that the rotational motion about the center of mass depends only on the torque about the center of mass, independent of translational motion. So this equation is valid even if the axis is accelerating.

- The result $\tau_0 = I_0\alpha$ closely resembles the equation of motion for translation in one dimension $F_z = ma_z$.

- It also demonstrates the close analogies between the equations of motion for translational and rotational motion even though the two modes of motion are totally independent.

Kinetic energy of a rigid body undergoing pure rolling motion

$$K = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M V^2$$

The first term corresponds to the kinetic energy of spin angular momentum, and the last term arises from the orbital center of mass motion.

In case of pure rolling, $V = R\omega$

$$K = \frac{1}{2}(I_0 + M\ R^2)\omega^2$$

Using parallel axes theorem, $I_0 + M\ R^2 = I.$

$$\boxed{K = \frac{1}{2} I\ \omega^2}$$

I: moment of inertia of the wheel about the line through the point of contact and parallel to the axis.

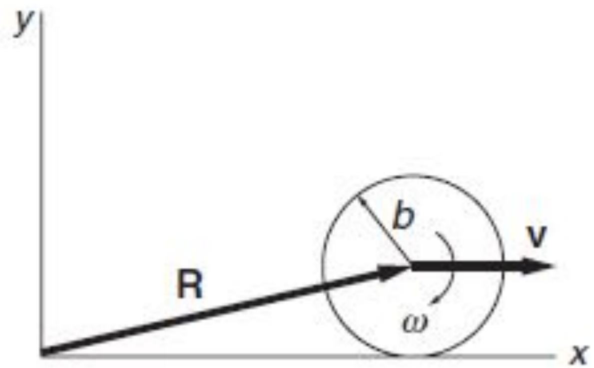
At any instant, rolling may be considered to be as pure rotation about an axis through the point of contact.

Problem-1 (Angular Momentum of a Rolling Wheel)

Calculate the angular momentum of a uniform wheel of mass *M* and radius *b* that rolls uniformly and without slipping. The moment of inertia of the wheel around its center of mass is $I_0 = \frac{1}{2}Mb^2$

Solution:

The wheel's angular momentum about the center of mass is



$$L_0 = -I_0\omega$$

$$= -\frac{1}{2}Mb^2\omega$$

L_0 is parallel to the *z* axis. The minus sign indicates that L_0 is directed into the paper, in the negative *z* direction.

As the wheel rolls without slipping, V= $b\omega$

Problem-1 (Angular Momentum of a Rolling Wheel)

The total angular momentum around the origin is then

$$L_z = I_0\omega + (R \times MV)_z$$

$$= -\frac{1}{2}Mb^2\omega - MbV$$

$$= -\frac{1}{2}Mb^2\omega - Mb^2\omega$$

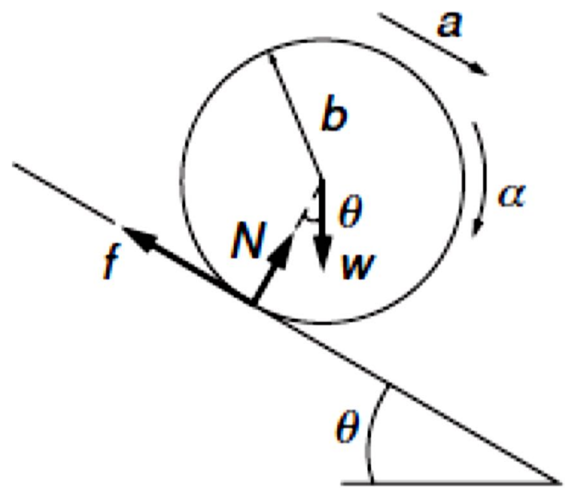
$$= -\frac{3}{2}Mb^2\omega$$

$$L_z = -\frac{3}{2}Mb^2\omega$$

Problem-2 Drum Rolling Down a Plane

A uniform drum of radius *b* and mass *M* rolls without slipping down a plane inclined at angle θ. The moment of inertia of the drum around its axis is $I_0 = \frac{1}{2}Mb^2$
Find the drum's acceleration along the plane.

Solution:



The forces on the drum are shown in the diagram. *f* is the force of friction. The translation of the center of mass along the plane is given by

$$W\sin\theta - f = Ma \qquad \text{.................... 1}$$

and the rotation around the center of mass obeys

$$bf = I_0\alpha \qquad \text{.................... 2}$$

For rolling without slipping, we have

$$a = b\alpha \qquad \text{.................... 3}$$

Problem-2 Drum Rolling Down a Plane

On putting the value of f from eq. 2 in eq. 1, we have

$$W sin\theta - \frac{I_0 \alpha}{b} = Ma$$

Using , $I_0 = \frac{1}{2}mb^2$, $\alpha = \frac{a}{b}$, and $W = mg$, we obtain

$$Mg sin\theta - \frac{Ma}{2} = Ma$$

or

$$a = \frac{2}{3} g sin\, \theta$$