MA 203: Convergence of Sequence of RVs

1. Convergence in Mean Square
2. Convergence in Distribution
3. Convergence in Almost Sure
4. Convergence in Probability

Almost Sure (a. s.) Convergence or Convergence with Probability 1: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of RVs defined on the probability space (S, F, P).

The sequence $\{X_n\}_{n=1}^{\infty}$ is said to converge to RV X almost sure or with probability 1 if $P\left(\left\{s \,\middle|\, \lim_{n\to\infty} X_n(s) = X(s)\right\}\right) = 1.$

$\boxed{s \in S}$

Ex:-

$A = \left\{ s \,\middle|\, \lim_{n\to\infty} X_n(s) = X(s) \right\}$

$S = \{s_1, s_2, \ldots, s_m\}$

$A = \left\{ s_i \,\middle|\, \lim_{n\to\infty} X_n(s_i) = X(s_i) \right\}$

t

$\boxed{s_1, s_2, \ldots s_t}$

$P(A) = 1$

$\{m-t\}$

if $P(A) = 1$

$\{X_n\} \xrightarrow{P.} X$

$\boxed{A = \{ s_1, s_2, \ldots, s_t \}}$

$\boxed{t \leq m}$

Example 1: Suppose $S = \{s_1, s_2, s_3\}$ and $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with $X_n(s_1) = 1$, $X_n(s_2) = -1$ and $X_n(s_3) = n$.

Define a RV X such that $X(s_1) = 1$, $X(s_2) = -1$, and $X(s_3) = 1$.

Examine if $\{X_n\}$ converges to X with probability 1 (or almost sure).

Sol:-

$$A = \left\{ s \,\middle|\, \lim_{n\to\infty} X_n(s) = X(s) \right\} \qquad s \in S$$

$$\Rightarrow \quad X_n(s_1) = X(s_1) = 1$$

$$X_n(s_2) = X(s_2) = -1$$

$$X_n(s_3) = n \; ; \; X(s_3) = 1$$

$$X_n(s_3) \neq X(s_3)$$

$$A = \{s_1, s_2\} \; ; \quad \text{if } P(A) = 1$$

$$= P\{s_1, s_2\}$$

$$\boxed{P(s_3) = 0}$$

Example 2: Let $\{X_n\}$, n $= 1, 2, \cdots$, be a sequence of RVs defined on (Ω, F, P) s.t.

$$X_n(s) = 1 + \frac{1}{n}; n = 1, 2, \ldots$$

Examine if $\{X_n\}$ converges to $\{X = 1\}$ with probability 1 (or almost sure).

Sol:-

$$A = \left\{ s \,\middle|\, \lim_{s\to\infty} X_n(s) = X(1) \right\}$$

$$\lim_{n\to\infty} X_n(s) = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

$= S$

$A = S \Rightarrow P(S) = 1$

$\{X(1) = 1$

Given sequence converges ~~to~~ $\{X=1\}$ with probability 1,

$$\{X_n\} \xrightarrow{a.s.} \{X=1\}$$

$\{X_n\}$

Convergence in Probability

Let be a sequence of RVs defined on (Ω, F, P). We say that $\{X_n\}, n = 1, 2, \cdots$, converges in probability to X If, for any $\in > 0$, $\lim_{n\to\infty} P\{|X_n - X| > \in\} = 0$.

$\varepsilon > 0$

$$\lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0$$

Notation

$$\boxed{\{X_n\} \xrightarrow{P} \{X\}}$$

Example 3: Let $\{X_n\}$, n $= 1, 2, \cdots$, be a sequence of RVs defined on (Ω, F, P) s.t.
$P(X_n = 0) = 1 - \frac{1}{n}$ and $P(X_n = n) = \frac{1}{n}$.
Examine that $\{X_n\} \to \{X = 0\}$ in probability.

Sol:-

$\varepsilon > 0$

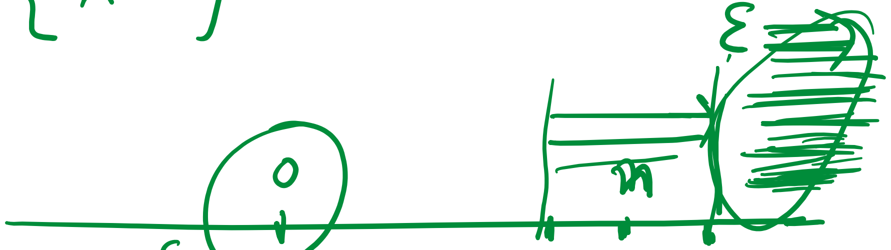
$$\lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0$$

$\boxed{\varepsilon > 0}$

$$\Rightarrow \quad P\{|X_n - 0|\} > \varepsilon\} = P\{|X_n| > \varepsilon\}$$

$$\{X_n\} \xrightarrow{P} \{X = 0\} \qquad |X_n| > \varepsilon \qquad = \begin{cases} \frac{1}{n} & ; \varepsilon < n \\ 0 & ; \varepsilon > n \end{cases}$$

$\{X_n > \varepsilon\}$



$$P\{|X_n| > \varepsilon\} = P\{X_n = n\} \qquad \lim_{n\to\infty} P\{|X| > \varepsilon\} = 0$$

Example 4: Suppose $\{X_n\}$ be a sequence of RVs with

$$P(X_n = 0) = 1 - \frac{1}{n^2}$$

and

$$P(X_n = n) = \frac{1}{n^2}$$

Examine if $\{X_n\}$ converges to $\{X_n = 0\}$ in probability.

$P\{X_n > 5\}$

$= P\{X_n = 10\}$

$\{X_n\} \xrightarrow{P} X = \cdots$

5

$n = 10$

$P\{|X_n| > 5\}$

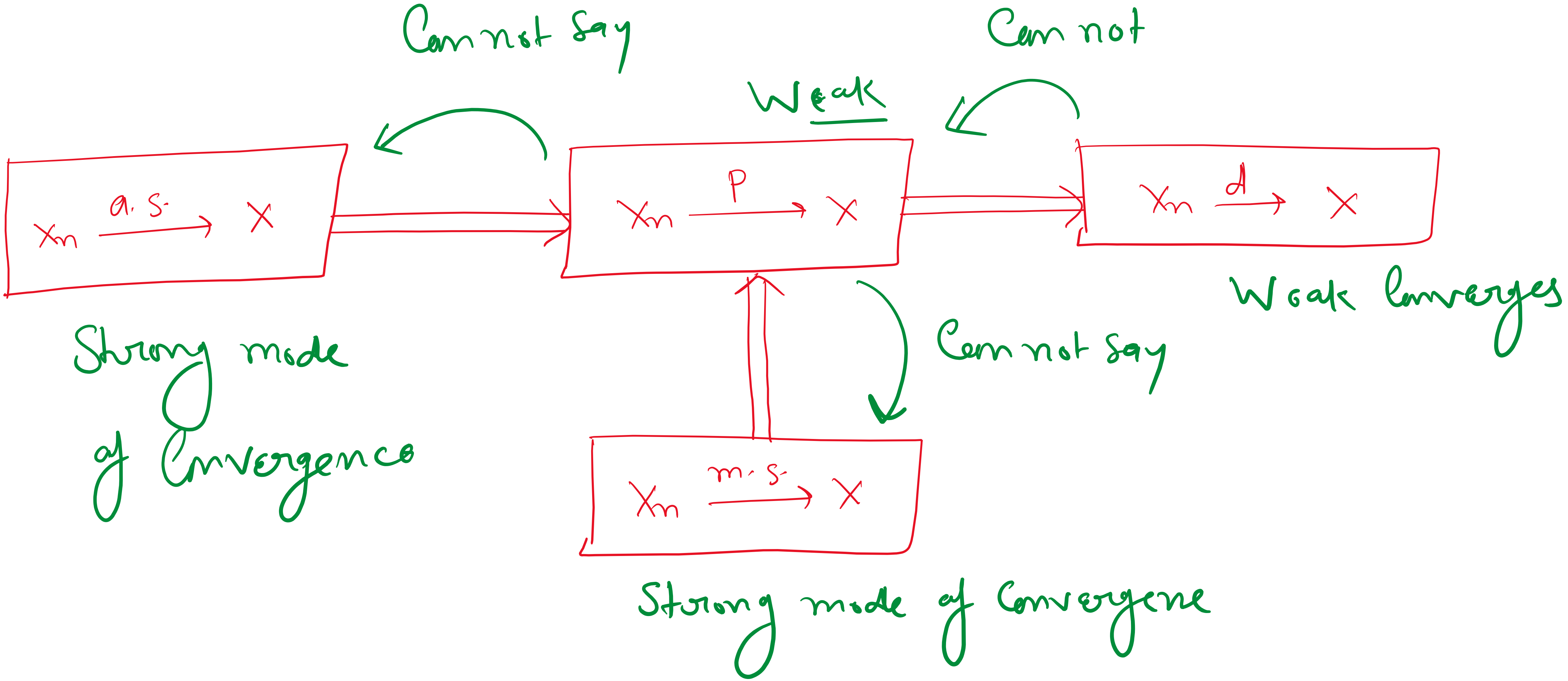
Sol: –

$$\lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0$$

$$\Rightarrow P\{|X_n - X| > \varepsilon\} = P\{|X_n| > \varepsilon\} = \begin{cases} \frac{1}{n^2} & ; \varepsilon \le n \\ 0 & ; \varepsilon > n \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} P\{|X_n| > \varepsilon\} = 0 \qquad \{X_n\} \xrightarrow{P} \{X\}$$

Relation Between Different Convergence Modes



Applications:

1. Law of Large Numbers
2. Central Limit Theorem

Sample Mean: Consider a sequence of RVs $\{X_i\}_{i=1}^{\infty}$ with $\mu_i = E[X_i]$.

M_n

$\boxed{n > M}$

The sample mean is defined by

$$\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$X_1 \quad X_2 \cdots X_n \cdots X_M$

Where $S_n = \sum_{i=1}^{n} X_i$.

Then,

$$E\left[\frac{S_n}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\mu_i$$

$X_1 + X_2 + \cdots + X_n$

$$\frac{0 + 1 + \cdots + 1}{n} = A$$

Sample mean

$$\frac{S_n}{n}$$

$$S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n$$

$$E\left[\frac{S_n}{n}\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mu_i \Rightarrow \text{True Mean}$$