MA 205 Discrete Mathematics
Assignment 6

September 18, 2020

1. Suppose that a and b are integers, $a \equiv 4(\mod 13)$, and $b \equiv 9(\mod 13)$. Find the integer c with $0 \leq c \leq 12$ such that
   (a) $c \equiv 9a(\mod 13)$.
   (b) $c \equiv 11b(\mod 13)$.
   (c) $c \equiv a + b(\mod 13)$.
   (d) $c \equiv 2a + 3b(\mod 13)$.

2. Show that
   (a) if n is an integer then $n^2 \equiv 0$ or $1(\mod 4)$.
   (b) Use (a) to show that if m is a positive integer of the form $4k + 3$ for some nonnegative integer k, then m is not the sum of the squares of two integers.

3. Write out the addition and multiplication modulo 5 tables for $\mathbb{Z}_5$.

4. Determine whether each of the functions $f(a) = a \text{ div } d$ and $g(a) = a \mod d$, where d is a fixed positive integer, from the set of integers to the set of integers, is one-to-one, and determine whether each of these functions is onto.

5. The value of the Euler ϕ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n.
   (a) Find these values of the Euler ϕ-function.
   i) $\phi(4)$. ii) $\phi(10)$. iii) $\phi(13)$.
   (b) Show that n is prime if and only if $\phi(n) = n - 1$.
   (c) What is the value of $\phi(p^k)$ when p is prime and k is a positive integer?

6. Show that if a and b are positive integers, then $ab = \gcd(a, b) \cdot (a, b)$. [Hint: Use the prime factorizations of a and b and the formulae for $\gcd(a, b)$ and (a, b) in terms of these factorizations.]

7. Use the Euclidean algorithm to find $\gcd(1529, 14039)$ and write gcd as linear combination of 1529 and 14039.

8. Adapt the proof that there are infinitely many primes to prove that there are infinitely many primes of the form $4k + 3$, where k is a nonnegative integer. [Hint: Suppose that there are only finitely many such primes $q_1, q_2, \cdots, q_n$, and consider the number $4q_1q_2 \cdots q_n - 1$.]