Classical Mechanics

Rotation about a fixed axis

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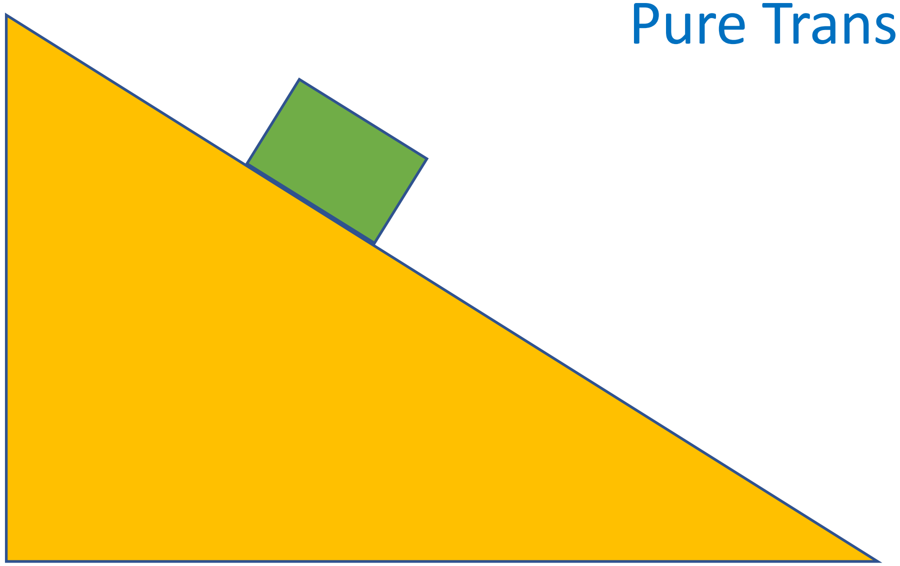
Rigid body

A rigid body is –

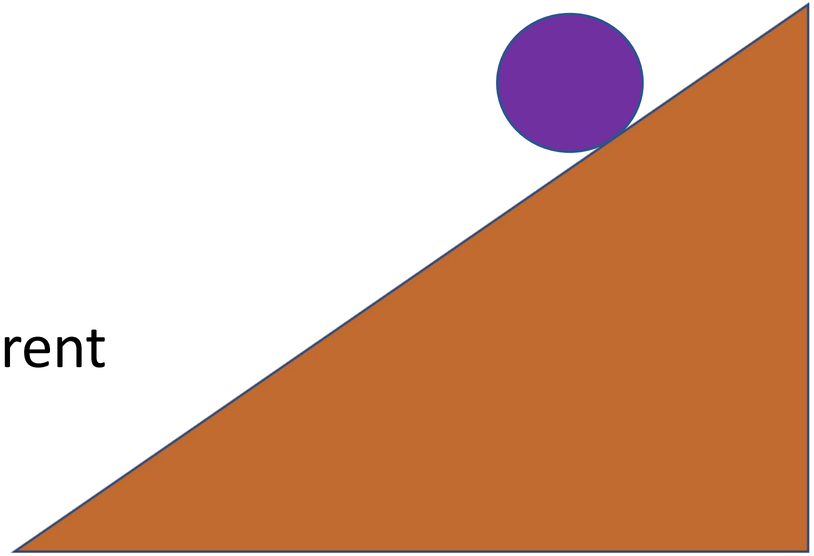
- A collection of particles whose relative distances are constrained to remain absolutely fixed
- an extended object with a fixed shape & size.
- does not deform under the action of applied force.

Motion of Rigid body-Not Pivoted

Pure Translation: All particles of the body have the same velocity at any instant



Translation + rotation: Particles move with different velocities.



Motion of Rigid body-Pivoted

Pure rotation(about a fixed axis): Every particle of the body moves in a circle, with same angular velocity.

Now, we shall study pure rotation about fixed axis.

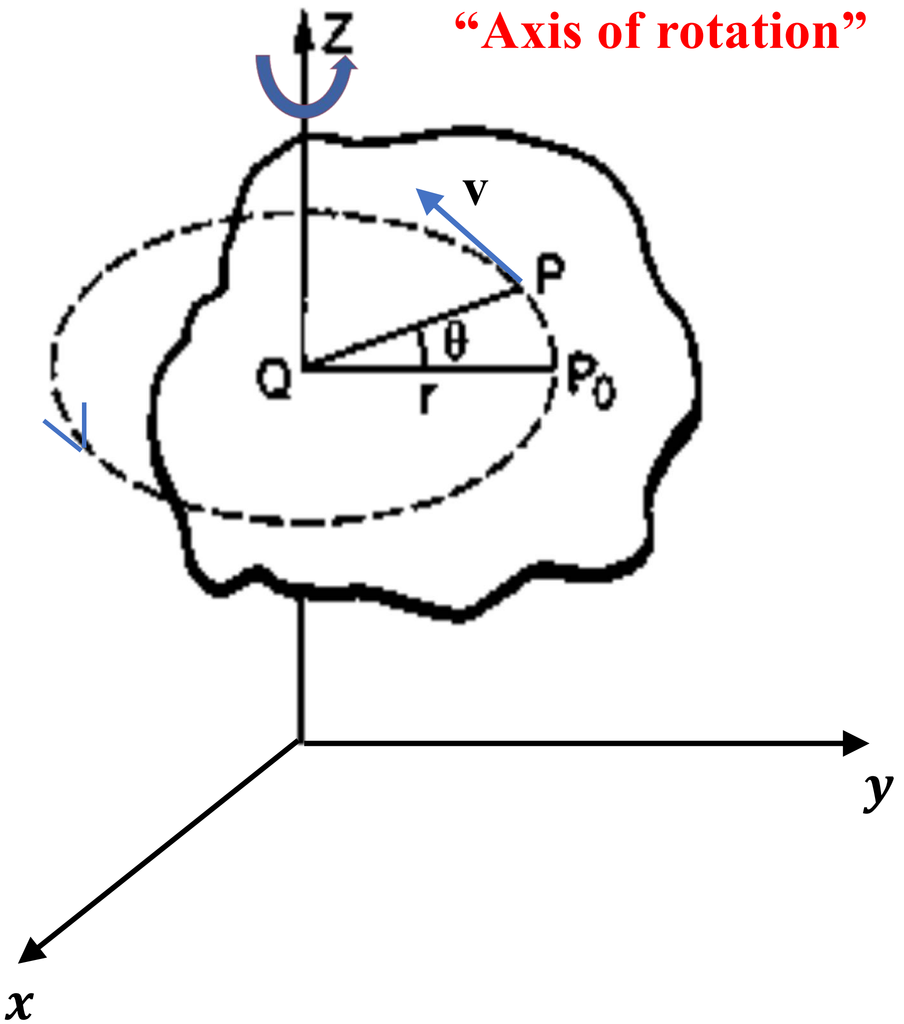
Rigid Body Rotation About a Fixed Axis

Definition: Displacement of a rigid body in which a given line
is held fixed,is called rotation of the rigid body about a fixed axis (*axis of rotation*).

For particles of the rigid body,

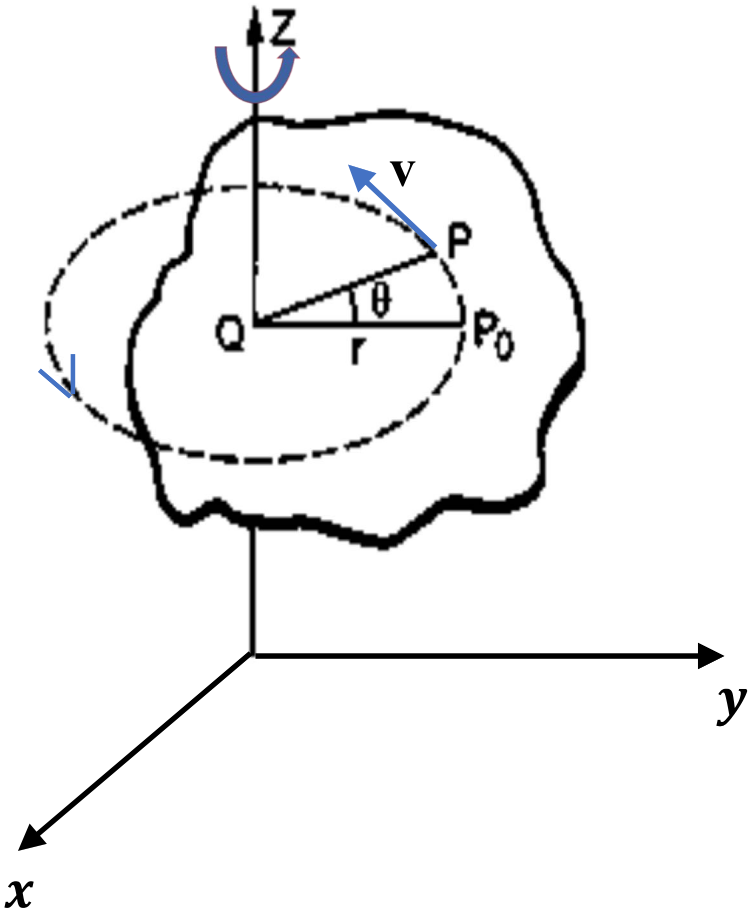
❖ The radius r is different.

❖ ω (angular velocity) and α (angular acceleration are the same for all the particles.



Angular Momentum of a Rotating Rigid Body

Consider a rigid body rotating about z-axis. Let The angular velocity of the body be ω and let us choose the origin to be at O, the center of the circle.



The linear velocity of i^{th} particle is

$$\boldsymbol{v}_i = \omega \times \boldsymbol{r}_i$$

$\boldsymbol{r}_i$: radius of the circle
$\boldsymbol{v}_i$: velocity of the particle

<u>Note:</u> $\boldsymbol{r}$ and $\boldsymbol{v}$ are perpendicular to each other.

Angular Momentum of a Rotating Rigid Body

Angular momentum of a i^{th} particle about point O is

$$\boldsymbol{l}\,(\boldsymbol{i}) = \boldsymbol{r}_i \times \boldsymbol{p}_i$$

The angular momentum of the particle about z-axis is:

$$l_z(i) = |\boldsymbol{r}_i \times \boldsymbol{p}_i|$$

$$= m_i\, v_i r_i$$

z-component of $\boldsymbol{l}$: $l_z(i) = m_i r_i^2 \omega.$ (sub $v_i = r_i\omega$)

Angular Momentum of a Rotating Rigid Body

The angular momentum of the rigid body is,

$$\boldsymbol{L} = \sum_{\boldsymbol{i}} l_{\boldsymbol{i}} = \sum \boldsymbol{r_i} \times \boldsymbol{p_i}$$

The angular momentum of the rigid body about z-axis,

$$L_{\boldsymbol{z}} = \sum_{\boldsymbol{i}} m_i r_i^2 \omega$$

sum over all particles

$$\boxed{L_{\boldsymbol{z}} = I\omega.}$$

(where $I = \sum_{\boldsymbol{i}} m_i r_i^2$)

Rotational inertia of a Rigid Body

Consider $$L_z = I\omega$$

In the above expression, $$I = \sum_i m_i r_i^2$$

I is a geometrical quantity called rotational inertia or moment of inertia of a rigid body about a particular axis.

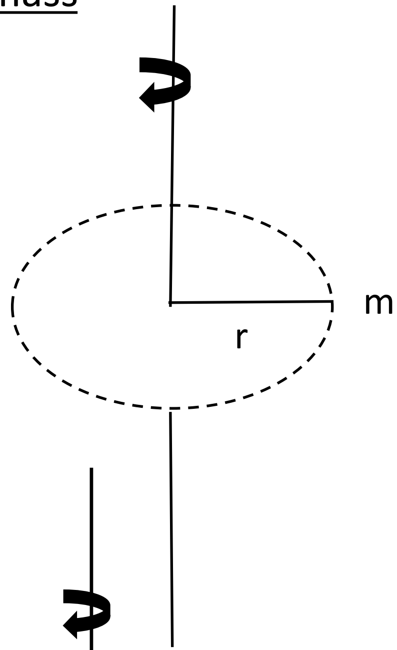
In the continuum limit, $$\sum_i m_i r_i^2 \longrightarrow \int r^2\, dm$$

dm: mass of small element of the rigid body.
r: perpendicular distance of the element from the axis.

$$I = \int r^2\, dm$$

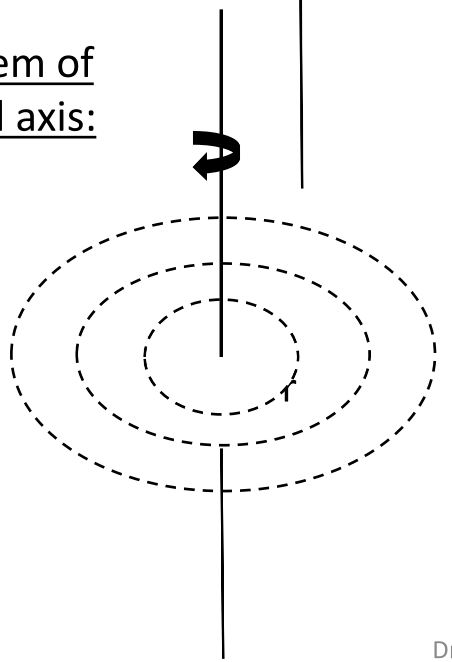
Moment of inertia, I for a particle of mass m rotating about a fixed axis:

$$I = mr^2$$

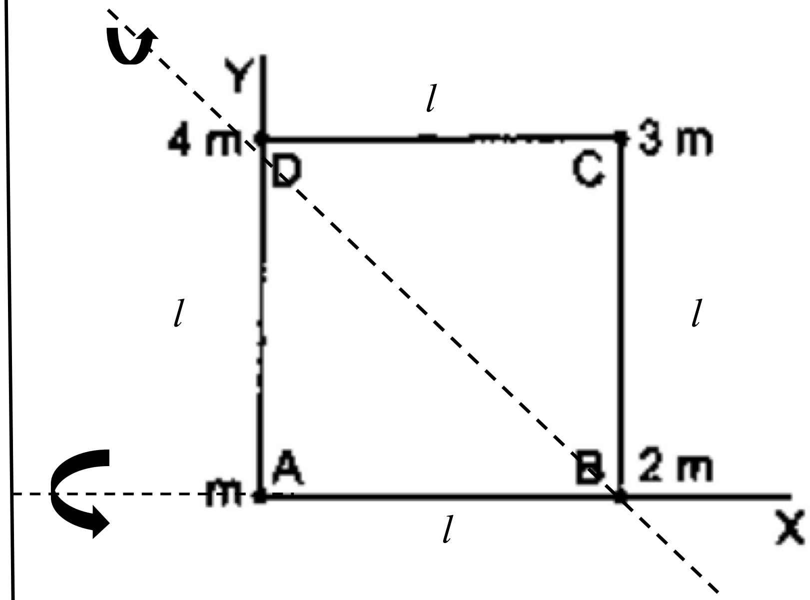


Moment of inertia, I for a system of particles rotating about a fixed axis:

$$I = \sum_i m_i r_i^2$$



Problem:

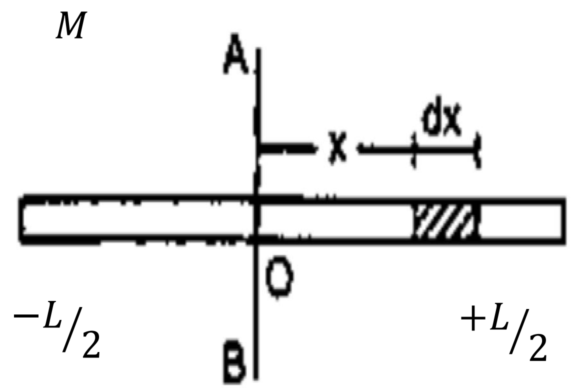


Find M.I. of SOP about an axis

i. line AB
ii. line BD
iii. Axis passing through A and perpendicular to the plane

<u>Example 1</u>

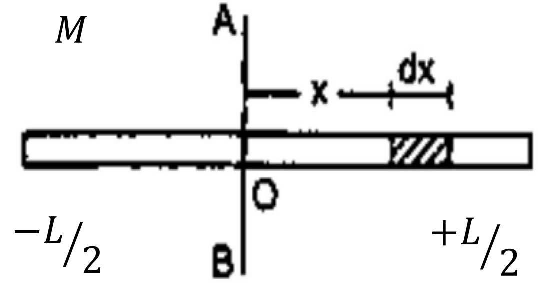
Find the moment of inertia of a uniform thin stick of mass M and length L, axis through the midpoint and perpendicular to the stick.



<u>Solution</u>:

Choose the origin at the middle point O of the rod. Consider the element of the rod between a distance x and $x + dx$ from the origin.

Example 1



As the rod is uniform,

Mass per unit length of the rod = $\frac{M}{L}$

The mass of the element

$$dm = \frac{M}{L} dx$$

The perpendicular distance of the element from the line AB is x.

The moment of inertia of the element dx is:

$$dI = x^2 \frac{M}{L} dx$$

Example 1

The moment of inertia of the entire rod about the axis(AB) is given by

$$I = \int_{-L/2}^{+L/2} dI$$

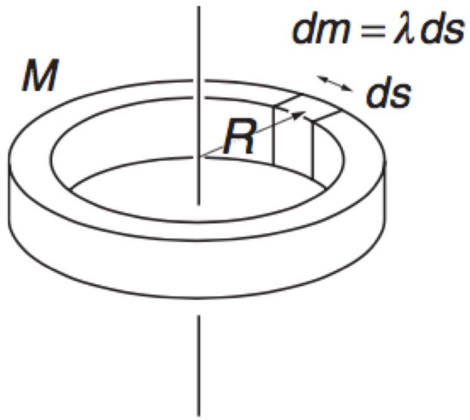
$$\mathrm{I} = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx$$

$$= \frac{M}{L}\frac{x^3}{3}\Bigg|_{-L/2}^{+L/2}$$

$$I = \frac{ML^2}{12}$$

Example 2

Find the moment of inertia of a uniform thin hoop of Mass M and radiusR,axis through the centre and perpendicular to the plane of the loop.

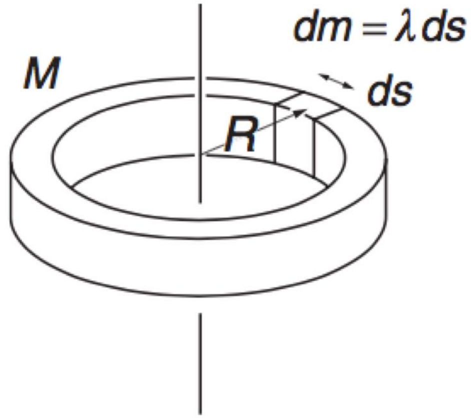


Solution:

The moment of inertia about the axis is given by

$$I = \int_0^{2\pi R} R^2 dm$$

Example 2



Since the hoop is thin,

$$dm = \lambda ds$$

mass per unit $length\ of\ hoop\ \ \lambda = \frac{M}{2\pi R}$

$$I = \int_0^{2\pi R} R^2 \lambda \, ds$$

$$= R^2 \left(\frac{M}{2\pi R}\right) s \Bigg|_0^{2\pi R} = MR^2$$

$$I = MR^2$$

Rotational Inertia

Moment of inertia :

It is a measure of how the mass of the rotating body is distributed about its axis of rotation.

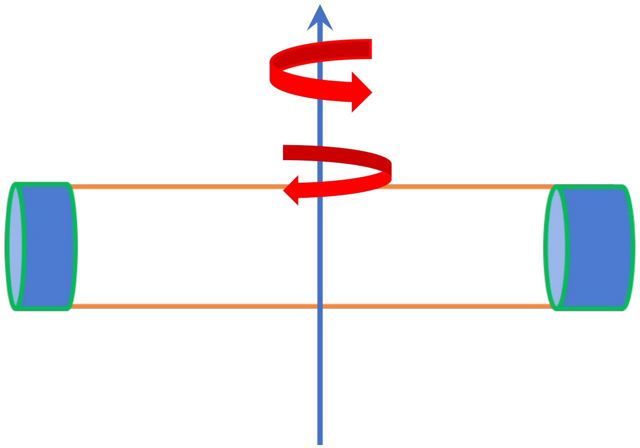
Example:

Dimensions,weights and mass of the rods are identical

Move rods back and forth rapidly (holding at the center)

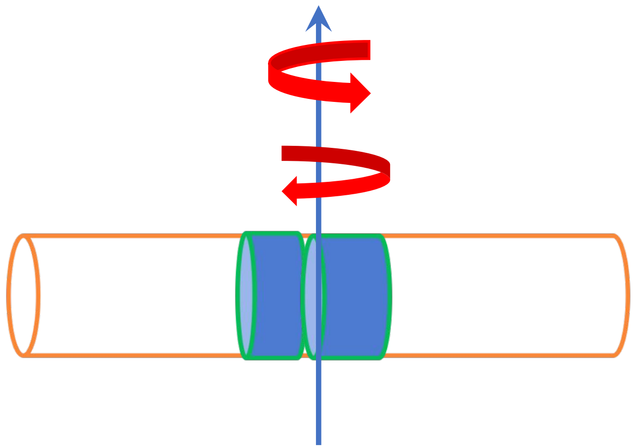
Rods are identical under translational motion

Now, twist those rods rapidly, back and forth in angular motion.



Internal weights one at each end

hard to twist



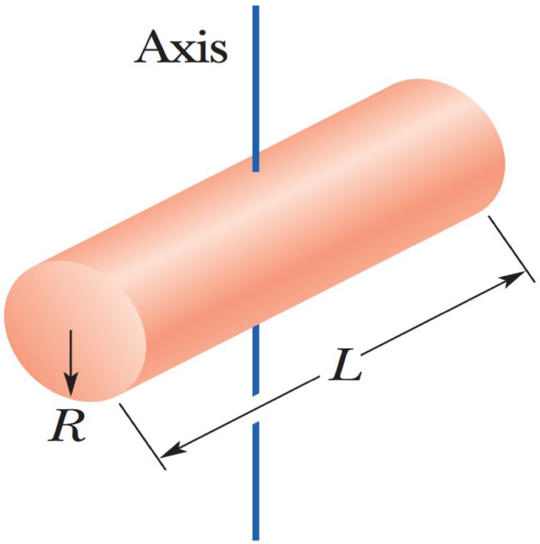
Internal weights near its center

Easy to twist

"Rotational Inertia Varies With the Distribution of Mass"

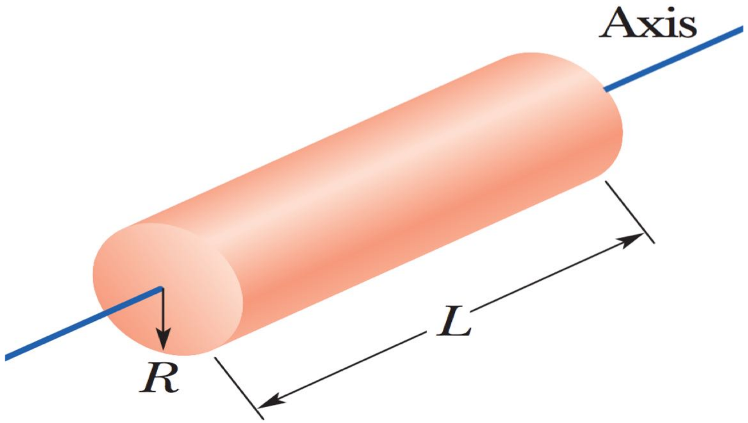
Rotational Inertia

Solid cylinder (or disk) about central diameter



$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

Solid cylinder(or disk) about central axis



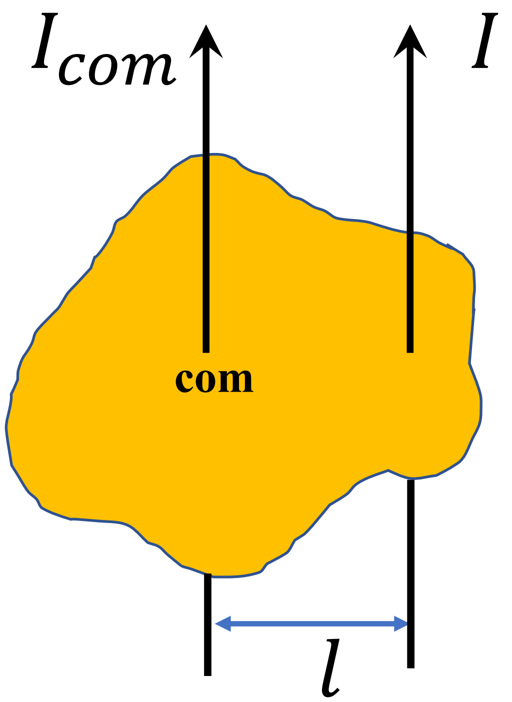
$$I = \frac{1}{2}MR^2$$

“Rotational Inertia Varies With the Axis of Rotation”

The parallel axis theorem

Statement:

The moment of inertia of a body about any axis (I) is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass (I_{com}) and the product of its mass M and the square of the distance between the two parallel axes l.



Let us consider a rigid body of mass M and the distance between the axes be l,
the theorem states that

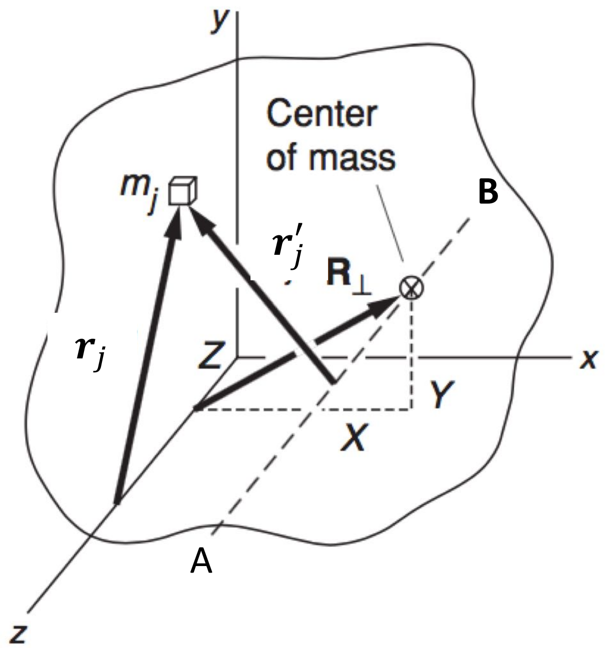
$$I = I_{com} + Ml^2$$

where, I_{com} : moment of inertia about the center of mass

I : moment of inertia about the parallel axis

The parallel axis theorem

Proof: Consider a rigid body rotating about z-axis with angular velocity ω.

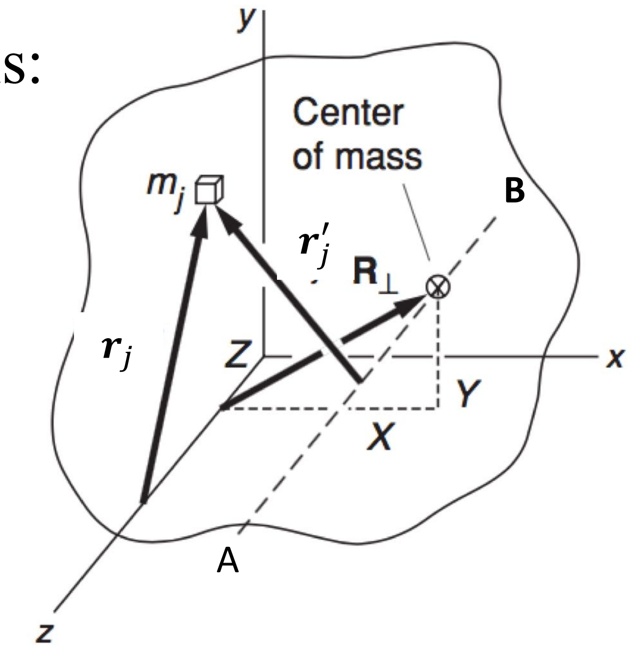


Consider a particle j with mass m_j and let $\boldsymbol{r}_j$ and $\boldsymbol{r}'_j$ be the perpendicular distances of the particle from z-axis and AB-axis passing through the center of mass, respectively.

The parallel axis theorem

The perpendicular vector from the z-axis to particle j is:

$$\boldsymbol{r}_j = x_j\hat{\boldsymbol{\imath}} + y_j\hat{\boldsymbol{\jmath}}$$



The moment of inertia of j^{th} particle about z-axis is

$$I = \sum m_j r_j^2$$

Position vector of center of mass: $\mathbf{R} = X\,\hat{\boldsymbol{\imath}} + Y\,\hat{\boldsymbol{\jmath}} + Z\hat{\boldsymbol{k}}$

The vector perpendicular from the z-axis to the centre of mass is

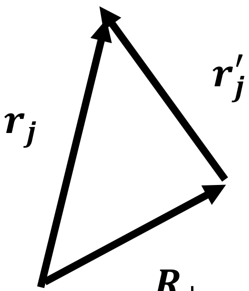
$$\boldsymbol{R}_\perp = X\,\hat{\boldsymbol{\imath}} + Y\,\hat{\boldsymbol{\jmath}}$$

The parallel axis theorem

Let the perpendicular vector from AB axis to the j^{th} particle be: $\boldsymbol{r}_{\boldsymbol{j}}'$
Then, The moment of inertia about the center of mass is

$$I_{com} = \sum m_j r_j'^2$$

From the figure, $\boldsymbol{r}_{\boldsymbol{j}} = \boldsymbol{r}_{\boldsymbol{j}}' + \boldsymbol{R}_{\perp}$



Consider $I = \sum m_j r_j^2$

$$= \sum m_j \left(\boldsymbol{r}_{\boldsymbol{j}}' + \boldsymbol{R}_{\perp}\right)^2$$

$$= \sum m_j \left(\boldsymbol{r}_{\boldsymbol{j}}'^2 + 2\boldsymbol{r}_{\boldsymbol{j}}'.\boldsymbol{R}_{\perp} + \boldsymbol{R}_{\perp}{}^2\right)$$

The parallel axis theorem

In the above expression, consider only the middle term:

$$= 2\sum m_j\, \boldsymbol{r}_j'.\boldsymbol{R}_\perp$$

$$= \mathbf{2}\sum m_j(\boldsymbol{r_j} - \boldsymbol{R}_\perp).\boldsymbol{R}_\perp \quad \text{(sub } \boldsymbol{r}_j' = \boldsymbol{r_j} - \boldsymbol{R}_\perp\text{)}$$

Rewriting $\boldsymbol{r_j}$ and $\boldsymbol{R}_\perp$ in terms of components, we get

$$= 2\sum m_j\,\{(x_j\hat{\boldsymbol{\imath}} + y_j\hat{\boldsymbol{\jmath}}) - (X\,\hat{\boldsymbol{\imath}}+\mathrm{Y}\,\hat{\boldsymbol{\jmath}})\}.(X\,\hat{\boldsymbol{\imath}}+\mathrm{Y}\,\hat{\boldsymbol{\jmath}}).$$

$$= 2\{\sum m_j\, x_j\hat{\boldsymbol{\imath}} + \sum m_j y_j\hat{\boldsymbol{\jmath}} - \sum m_j\ (X\,\hat{\boldsymbol{\imath}}+\mathrm{Y}\,\hat{\boldsymbol{\jmath}})\}.\ (X\,\hat{\boldsymbol{\imath}}+\mathrm{Y}\,\hat{\boldsymbol{\jmath}})$$

$$= 2\{\mathrm{M}\,(X\,\hat{\boldsymbol{\imath}} + \mathrm{Y}\,\hat{\boldsymbol{\jmath}}) - \mathrm{M}\,(X\,\hat{\boldsymbol{\imath}} + \mathrm{Y}\,\hat{\boldsymbol{\jmath}})\}.\ (X\,\hat{\boldsymbol{\imath}} + \mathrm{Y}\,\hat{\boldsymbol{\jmath}})$$

$$= 2\mathrm{M}(\boldsymbol{R}_\perp - \boldsymbol{R}_\perp).\boldsymbol{R}_\perp = 0 \quad \text{(the middle term vanishes)}$$

The parallel axis theorem

The first term $\sum m_j \boldsymbol{r}_{\boldsymbol{j}}'^2 = I_{com}$

The last term $\sum m_j {\boldsymbol{R}_\perp}^2 = Ml^2$ (magnitude of vector $\boldsymbol{R}_\perp$ be l)

Then the moment of inertia about z-axis is

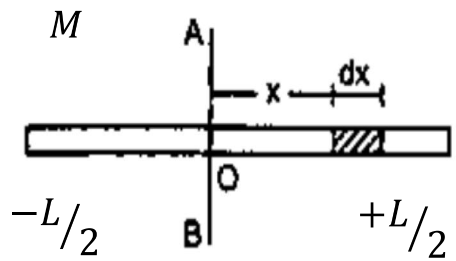
$$\boxed{I = I_{com} + Ml^2}$$

"This theorem is applicable to a body of any shape"

Example: Moment of inertia of a stick about its end point (using "parallel axis theorem")

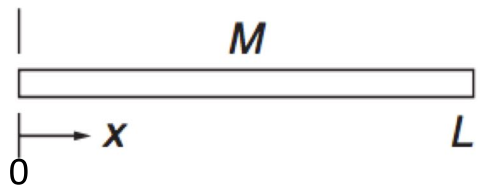
The moment of inertia of a stick about its midpoint: $I_{com} = \frac{1}{12} M L^2$

The moment of inertia about its end:



The end is $L/_2$ away from the centre of mass

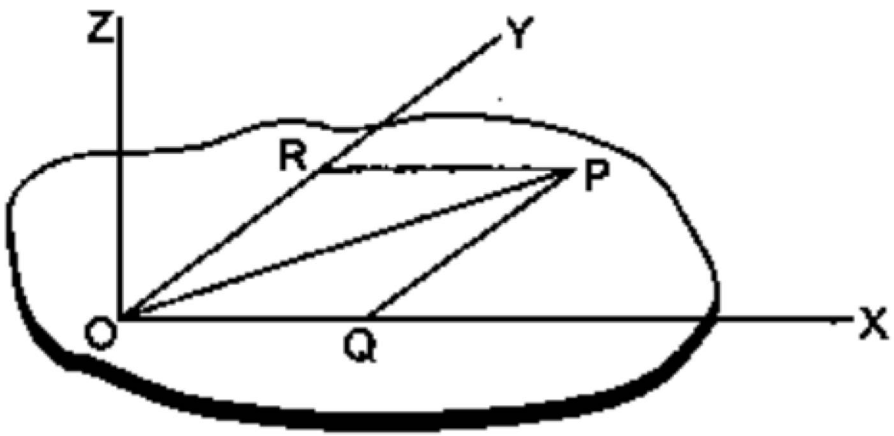
$$\therefore \quad I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$



$$I = \frac{ML^2}{3}$$

The perpendicular axes theorem

Consider a rigid body rotating about z-axis with angular velocity ω. Let X and Y-axes be chosen in the plane of the body and Z-axis perpendicular to this plane, three axes being mutually perpendicular.



The perpendicular axes theorem

Statement:

The theorem states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane I_z is equal to the sum of its moments of inertia about two perpendicular axes (I_x and I_y) concurrent with perpendicular axis and lying in the plane of the body.

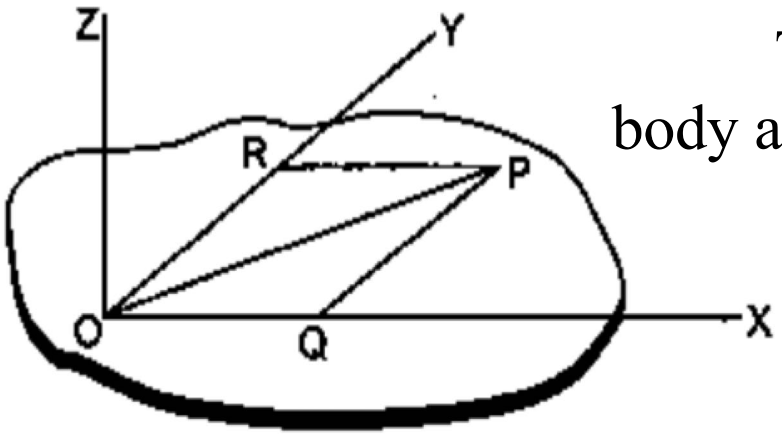
Thus the theorem states that,

$$I_z = I_x + I_y$$

The perpendicular axes theorem

Proof:

Consider an arbitrary particle P of the body. Let PQ and PR be the perpendiculars from P on the X and Y-axes respectively. Also PO is the perpendicular from P to the Z-axis.

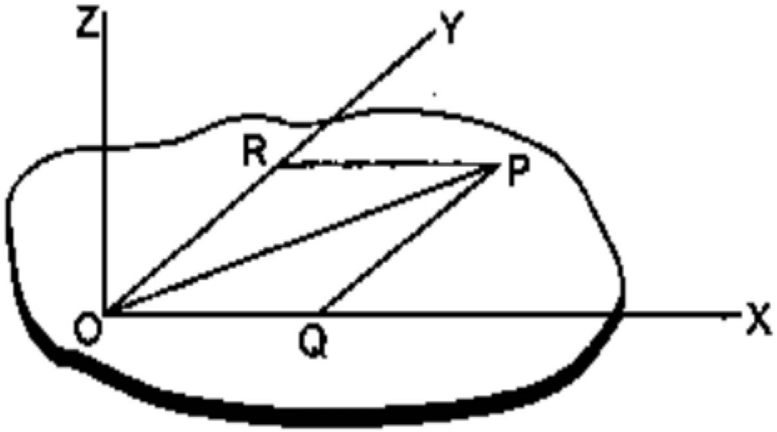


Thus, the moment of inertia of the body about the Z-axis is

$$I_z = \sum_i m_i (PO)^2$$

$$= \sum_i m_i (PQ^2 + OQ^2)$$

The perpendicular axes theorem



$$= \sum_i m_i(PQ^2 + PR^2)$$

$$= \sum_i m_i\,(PQ)^2 + \sum_i m_i\,(PR)^2$$

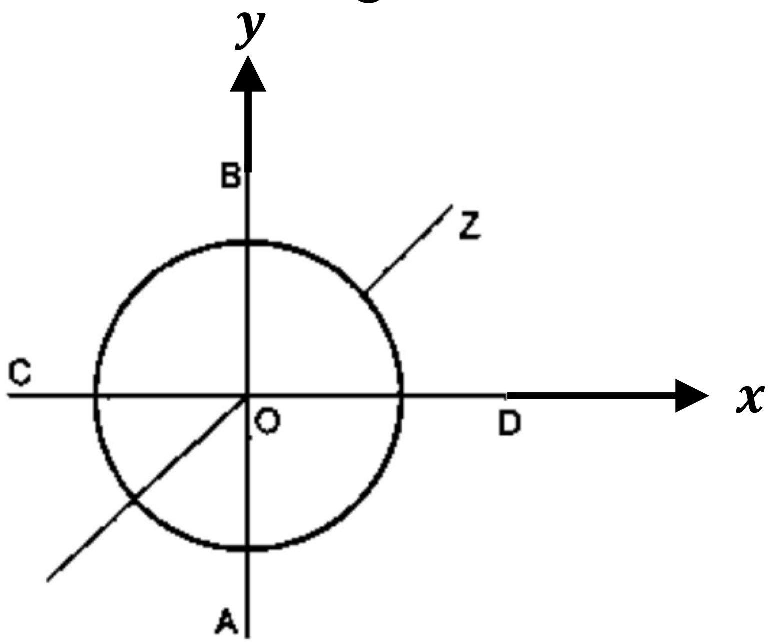
$$= I_x + I_y$$

$$I_z = I_x + I_y$$

Note: "This theorem is applicable only to the plane bodies".

Example:Using perpendicular axes theorem

Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.



Solution:

Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axes and the line perpendicular to the plane of the ring through the centre as the Z-axis.

Example: Using perpendicular axis theorem

The moment of inertia of the ring about Z-axis is

$$I_z = MR^2$$

As the ring is uniform, $I_x = I_y$ (the diameters are equivalent)

From perpendicular axes theorem,

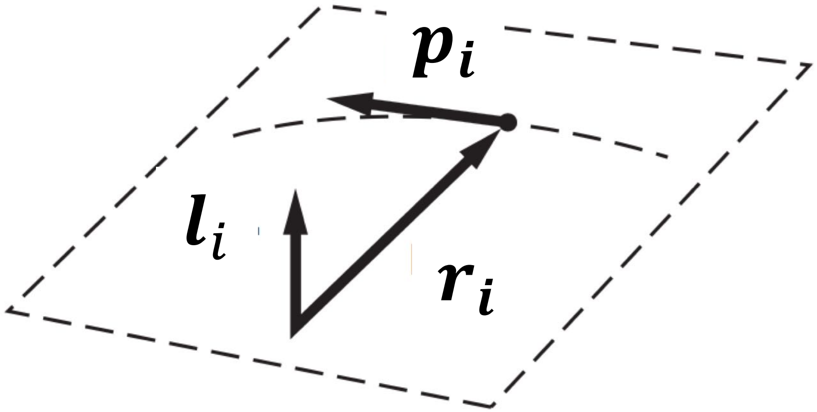
$$I_z = I_x + I_y$$

Hence, $I_x = \frac{I_z}{2} = \frac{MR^2}{2}$

What is the equation of motion of a rigid body rotating about a fixed axis?

Dynamics of Rotation About a Fixed Axis

Consider a body rotating (fixed axis rotation) with angular velocity ω about z-axis, under the action of total torque $\boldsymbol{\tau}_{tot}$. We are going to find out the equation of motion for a rigid body rotating about a fixed axis, under the action of $\boldsymbol{\tau}_{tot}$.



The angular momentum of the body about the axis of rotation is,

$$\boldsymbol{L} = \sum \boldsymbol{r_i} \times \boldsymbol{p_i}$$

Differentiating w.r.t time,

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}\sum \boldsymbol{r_i} \times \boldsymbol{p_i}$$

$$= \sum \left[\frac{d\boldsymbol{r_i}}{dt} \times \boldsymbol{p_i} + \boldsymbol{r_i} \times \frac{d\boldsymbol{p_i}}{dt}\right]$$

Dynamics of Rotation About a Fixed Axis

$$= \sum[\boldsymbol{v_i} \times m\boldsymbol{v_i} + \boldsymbol{r_i} \times \boldsymbol{f_i}]$$

$$= \sum_i (\boldsymbol{r_i} \times \boldsymbol{f_i})$$

$$= \boldsymbol{\tau}_{tot}$$

where

total torque on the rigid body: $\boldsymbol{\tau}_{tot}$

Total force on i^{th} particle: $\boldsymbol{f_i} = \boldsymbol{f}_{\mathrm{j}}^{\mathrm{int}} + \boldsymbol{f}_{\mathrm{j}}^{\mathrm{ext}}$

By Newton's III law: $\Sigma\, \boldsymbol{f}_j^{int} = 0$ **"internal forces cancel in pairs"**

Dynamics of Rotation About a Fixed Axis

Thus the equation of motion for a rigid body rotating about a fixed axis is,

$$\boxed{\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}_{ext}}$$

where

where the total external torque on the rigid body is $\boldsymbol{\tau}_{ext}$

Substituting the z-component of angular momentum is

$$L_z = I\omega$$

Dynamics of Rotation About a Fixed Axis

The z-component of torque: $\tau_z = \frac{d}{dt}(I\omega)$

$$= I\frac{d\omega}{dt}$$

$$\tau = I\alpha$$

Where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration.

Analogy between Linear and rotational motion:

Equation of motion for translation: $\boldsymbol{F} = m\,\boldsymbol{a}$

Equation of motion for rotation: $\tau = I\alpha$

Kinetic energy of a rigid body undergoing pure rotation

Consider a particle j moving with linear speed v_j.

The corresponding Kinetic energy of the particle is $k_j = \frac{1}{2} m_j v_j^2$

For a system of N particles in a rigid body, rotating with angular velocity ω,

The kinetic energy $\mathrm{K} = \sum_j \frac{1}{2} m_j v_j^2$

$$= \sum_j \frac{1}{2} m_j r_j^2 \, \omega^2. \qquad (\text{as } v_j = r_j \omega\ )$$

$$= \frac{1}{2} I \omega^2 \qquad (\text{as I} = \sum_j m_j r_j^2)$$

$$\mathrm{K} = \frac{1}{2} I \omega^2$$