Unit 4: Combinatorics

Topic 2: Permutation and Combination

Outline

Distribution of objects into boxes

Many counting problems can be solved by enumerating the ways objects can be placed into boxes, where the order these objects are placed into the boxes does not matter. The objects can be either distinct (distinguishable) or identical (indistinguishable). In the same way, boxes can be distinct(distinguishable) or identical(indinguishable).

1. Distributing cards among four players.
2. Number of solutions of $x_1 + x_2 + x_3 = k$ in non-negative integers.
3. Allotment of similar types of jobs to four persons.
4. Placing of some copies of a book in five similar almirahs.

Distinguishable objects and distinguishable boxes

Distinguishable objects and distinguishable boxes: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Indistinguishable objects and distinguishable boxes

Indistinguishable objects and distinguishable boxes: There is a one-to-one correspondence between n-combinations from a set with k elements when repetition is allowed and the ways to place n indistinguishable balls into k distinguishable boxes.

To set up this correspondence, put a ball in the ith bin each time the ith element of the set is included in the n-combination.

The number of ways to distribute n indistinguishable objects into k distinguishable boxes is $C(k+n-1, n)$.

Distinguishable objects and indistinguishable boxes

Distinguishable objects and indistinguishable boxes: Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers

$$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$

are called **Stirling numbers of the second kind**.

The number of ways to distribute n distinguishable objects into k indistinguishable boxes (where the number of boxes that are nonempty equals $k, k-1, \ldots, 2$, or 1) equals

$$\sum_{j=1}^{k} S(n,j).$$

Indistinguishable objects and indistinguishable boxes

Indistinguishable objects and indistinguishable boxes: Distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in nonincreasing order.

If $a_1 + a_2 + \ldots + a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a partition of the positive integer n into j positive integers.

We see that if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes.

Problems

Ex: A professor packs her collection of 40 issues of a mathematics journal in four boxes. How many ways can she distribute the journals if

1. each box is numbered with 10 issues per box?
2. the boxes are identical with 10 issues per box?
3. the boxes are identical?

Problems

Ex: A professor packs her collection of 40 issues of a mathematics journal in four boxes. How many ways can she distribute the journals if

1. each box is numbered with 10 issues per box? **Ans:** $\frac{40!}{(10!)^4}$
2. the boxes are identical with 10 issues per box?
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3. the boxes are identical? **Ans:** $S(40,1) + S(40,2) + S(40,3) + S(40,4)$

Problems

Ex: How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

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Problems

Ex: In how many ways can a dozen books be placed on four distinguishable shelves

1. if the books are indistinguishable copies of the same title?

2. if no two books are the same, and the positions of the books on the shelves matter?
3. if no two books are the same, and the positions of the books on the shelves does not matter?

Problems

Ex: In how many ways can a dozen books be placed on four distinguishable shelves

1. if the books are indistinguishable copies of the same title?
   Ans: $C(4+12-1, 12)$
2. if no two books are the same, and the positions of the books on the shelves matter?
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Problems

Ex: How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

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Ans: $S(6,4) = \frac{1}{4!}\sum_{i=0}^{3}(-1)^i\binom{4}{i}(j-i)^6 = \frac{1}{12}(6^6 - 4\cdot 5^6 + 6\cdot 4^4 - 4\cdot 2^6)$

Problem

Ex: How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

1. both the balls and boxes are labeled?
2. the balls are labeled, but the boxes are unlabeled?
3. the balls are unlabeled, but the boxes are labeled?
4. both the balls and boxes are unlabeled?

Problem

Ex: How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

1. both the balls and boxes are labeled? **Ans:** $P(7,5)$
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4. both the balls and boxes are unlabeled? **Ans:** 1

Problem

Ex: How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

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2. the balls are labeled, but the boxes are unlabeled?
3. the balls are unlabeled, but the boxes are labeled?
4. both the balls and boxes are unlabeled?

Problem

Ex: How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

1. both the balls and boxes are labeled? **Ans:** $S(5,3) \cdot 3!$
2. the balls are labeled, but the boxes are unlabeled?
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3. the balls are unlabeled, but the boxes are labeled? **Ans:** $C(4,2)$
4. both the balls and boxes are unlabeled? **Ans:** $p_3(7) = 2$

Outline

Generating Permutation and Combination

In addition to finding number of possible ways to perform a work, it is necessary to generate all possible ways from one way to the remaining ones. There are many algorithms to generate all permutations and combinations of a set of n distinct elements without repetition. One of such methods is the **Lexicographic ordering** of the permutations and combinations.

Generating Permutation

Generating Permutation: For **Lexicographic ordering** of permutations, we first place a one-to-one correspondence between the set of n elements and the set $\{1, 2, \dots, n\}$.

Let $a_1 a_2 \dots a_{n-1} a_n$ be any permutation, then find two integers a_j and a_{j+1} with $a_j < a_{j+1}$ and $a_n < a_{n-1} < \cdots < a_{j+2} < a_{j+1}$.

Then the next lexicographic ordering larger permutation is obtained by putting the least element of $a_{j+1}, \dots, a_n$ in the jth position (i.e., in place of a_j) which is greater than a_j, and list the rest of $a_j, a_{j+1}, \dots, a_n$ in the increasing order in positions from $j+1$ to n.

Problems

Ex: Find the next larger permutation in lexicographic order of 54123 and *acfbde*.

Generating Combination

Generating Combination: For **Lexicographic ordering** of combinations, we first place a one-to-one correspondence between the set of n elements and the set $\{1, 2, \ldots, n\}$.

Let $a_1 a_2 \ldots a_{r-1} a_r$ be any r-combination, then find the last element a_i in the sequence such that $a_i \neq n - r + i$.

Then the next lexicographic ordering larger r-combination is obtained by replacing a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$.

Problem

Ex: Find the next larger 4-combination in lexicographic order of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 3, 4, 5\}$.

Outline

Combinatorial proof of identities

A **combinatorial proof** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called **double counting proofs** and **bijective proofs**, respectively.

Ex: Provide combinatorial proof of

1. $C(n,r) = C(n,n-r)$.
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Hint: (i) Construction of subsets of order r from a set of n elements.
Hint: (ii) Choosing $r + 1$ objects from $n + 1$ objects.

Problem

Ex: Give combinatorial proof of

1. $C(n,0) + C(n,1) + \ldots + C(n,n) = 2^n$.
2. $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

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Hint: (1) Number of subsets of a set with n elements.

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Hint: (1) Number of subsets of a set with n elements.
Hint: (2) Number of ways to select a subset with r elements from a set of n elements, and then selecting k elements from the set.

Problem

Ex: Give combinatorial proof of

1. $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.
2. $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

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Hint: (1) Number of ways to select r objects from $m+n$ objects with m objects of one kind and n objects are of another kind.

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Hint: (2) Number of ways to select 2 objects from a collection of two kinds of objects with n objects of each kind.

Thank You

Any Question!!!