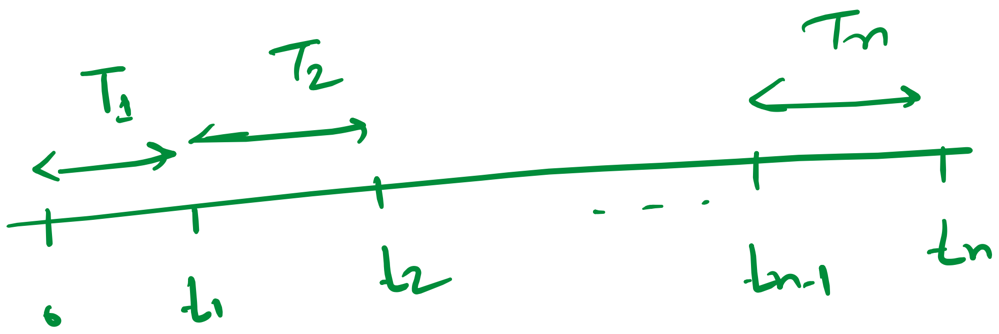
MA 203

Markov Process



$$f_{T_n}(t) = \lambda e^{-\lambda t}$$

Example 3: Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let $T_1, T_2, \ldots$ be the corresponding interarrival times.

(i) Find the probability that first arrival occurs after $t = 0.5$, i.e., $P(T_1 > 0.5)$.

(ii) Given that we have had no arrivals before $t = 1$, find $P(T_1 > 3)$.

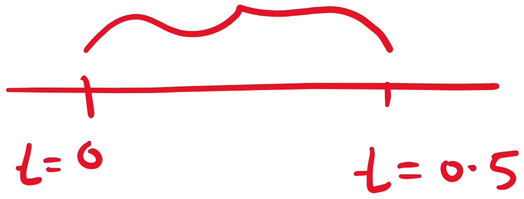
(iii) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrivals occurs after $t = 4$.

Sol:-

(i) $P\{T_1 > 0.5\} = e^{-\lambda t}$

$= e^{-2\times 0.5} = e^{-1}$

$\simeq 0.37$



(ii) $P\{T_1 > 3 \mid T_1 > 1\} = P\left[\{\text{no arrival in } (1,3] \mid \text{no arrival in } (0,1]\}\right]$

Using independ increment Prop.

$= P\{\text{no arrival in } (1,3]\}$

$= P\{T_1 > 2\}$

$= e^{-\lambda t} = e^{-2\times 2} = e^{-4}$

≈ 0.0183

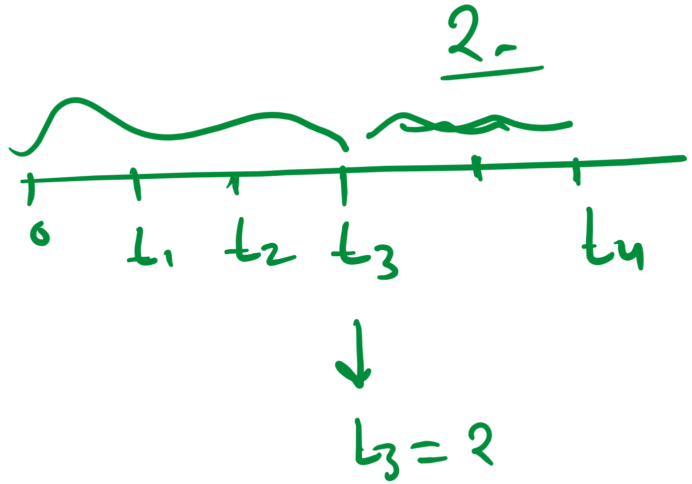
(iii) $P\{T_4 > 2 \mid T_1 + T_2 + T_3 = 2\}$

T_i's are independent

$= P\{T_4 > 2\}$

$= e^{-2\times 2} = e^{-2t}$

$\approx \underline{0.0183}$



Markov Process

$X(t)$: RP

$t_1 < t_2 < \cdots < t_{n-1} < t_n$

$$\begin{array}{ccccc} t_1 & t_2 & \cdots & t_{n-1} & t_n \end{array}$$

$$\begin{array}{ccccc} X(t_1) & X(t_2) & \cdots & X(t_{n-1}) & X(t_n) \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ R_1 & R_2 & & R_{n-1} & R_n \end{array}$$

$X(t)$ is a Markov Process if

$$P\{X(t_n)=i_n \mid X(t_1)=i_1,\ X(t_2)=i_2, \ldots, X(t_{n-1})=i_{n-1}\}$$

$$= P\{X(t_n)=i_n \mid X(t_{n-1})=i_{n-1}\}$$

where $i_1 \in R_1,\ i_2 \in R_2, \ldots, i_n \in R_n$

Ex:- $n=3$.

$$P\{\underset{=i_3}{X(t_3)} \mid X(t_1)=i_1,\ X(t_2)=i_2\} = P\{X(t_3)=i_3 \mid \underset{=i_2}{X(t_2)}\}$$

Memoryless Prop.

Markov chain: if $x(t)$ is a Markov process and its state space is finite / countably infinite, then $x(t)$ is known as Markov chain.

Ex:-

$X_1 \quad X_2 \quad X_3 \quad \cdots \quad X_n$

X_1: p H $\to 1$, T $1-p$ $\to 0$

$$S_n = X_1 + X_2 + \cdots + X_n$$

S_n: # heads in first n toss

$$S_{n+1} = S_n + X_{n+1}$$

$$P\{\underbrace{S_{n+1} = k+1}_{\text{\# heads in first } n+1 \text{ toss is } k+1} \mid \underbrace{S_n = k}_{\text{\# heads in first } n \text{ toss is } k}\}$$

heads in first n toss is k

heads in first n+1 toss is k+1

$$P\{S_{n+1} = k+1 \mid S_n = k\} = p$$

$$P\{S_{n+1} = k \mid S_n = k\} = 1-p$$

$$P\{S_{n+1} = k+1 \mid S_n = k-1\} = 0$$

S_n is a Markov Process $\left\{\text{This is a discrete-time discrete-space Markov Process}\right\}$

Types of Markov chain

→ Discrete Time Markov chain (DTMC) : T & ϕ both are finite / Countably infinite

→ Continuous Time Markov chain (CTMC) :

⇓

Parameter space Time space (T) is uncountable

State space (ϕ) is finite / Countably infinite

Discrete Time Markov Chain

$X[n]$: is a discrete RP.

X_n

- It satisfies Markov Prop.
- It has finite / countably infinite state space
- It has finite / countably infinite parameter space.

$$\underset{1}{X_1} \quad \underset{2}{X_2} \; \cdots \; \underset{m}{\cdots} \quad \underset{n}{X_n}$$

$$P_J = P\{X_n = J\} \qquad J \in R_n$$

$0 \le m \le n$

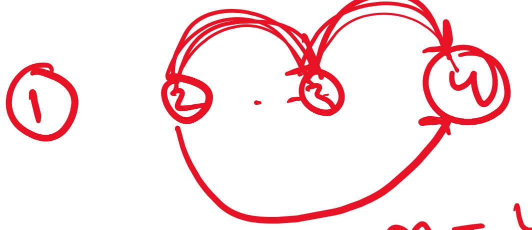
$$P_{iJ}(m, n) = P\{X_n = J \mid X_m = i\}$$

= Transition Probability from state i to J in $n-m$ steps

$n-m$

Homogeneous Markov Chain

$m = 2, \; n = 4$



$n = 4 - 2 = 2$

$$P_{ij}(m,n) = P\{X_n = j \mid X_m = i\}$$

$$n - m$$

$$n - m = m_0$$

$$P_{ij}(m,n) = P\{X_n = j \mid X_m = i\} = P\{X_{m+m_0} = j \mid X_m = i\}$$

$$= P_{ij}(m_0)$$

$= m_0$- steps transition probability

1- Step Probability

$n_0 = 1$

$$P_{ij}(1) = P\{X_{n+1} = j \mid X_n = i\}$$

0- Step transition probability

$$P_{jk}(0) = \begin{cases} 1 ; & j = k \\ 0 ; & o.w. \end{cases}$$

Transition Probability Matrix:-

$$P = [P_{ij}] = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$$

$i = 0, 1, 2$

$j = 0, 1, 2$

Prop. of Transition Probability Matrix

(i) $P_{ij} \geq 0 \quad \forall\, i, j \in \phi$

(ii) $\sum_{j \in \phi} P_{ij} = 1 \; ; \; i \in \phi$