Algorithms 09

CS201

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Representation of Graphs

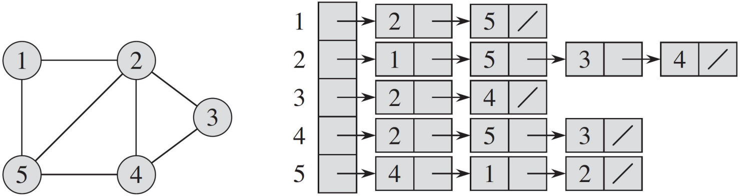
Two Ways of Representing a Graph $G = (V, E)$

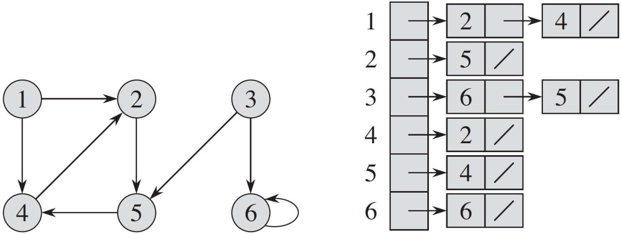
- **Adjacency-list:** Provides a compact way to represent sparse graphs—those for which E is much less than $|V|^2$.
- **Adjacency-matrix:** Preferred to represent dense graphs—those for which E is close to $|V|^2$ when we need to be able to tell quickly if there is an edge connecting two given vertices.

Adjacency-List

- The adjacency-list representation of a graph $G = (V, E)$ consists of an array *Adj* of $|V|$ lists, one for each vertex in V.
- For each $u \in V$, the adjacency list $Adj[u]$ contains all the vertices v such that there is an edge $(u, v) \in E$.
- $Adj[u]$ consists of all the vertices adjacent to $u \in G$. (Alternatively, it may contain pointers to these vertices.)
- Since the adjacency lists represent the edges of a graph, in pseudocode we treat the array *Adj* as an attribute of the graph, just as we treat the edge set E.
- If G is a directed graph, the sum of the lengths of all the adjacency lists is $|E|$. If G is an undirected graph, the sum of the lengths of all the adjacency lists is $2|E|$.
- For both directed and undirected graphs, the adjacency-list representation has the memory requirement of $\theta(V + E)$.

Adjacency-List





Adjacency-List

- For both directed and undirected graphs, the adjacency-list representation has the memory requirement of $\theta(V+E)$.
- We can readily adapt adjacency lists to represent weighted graphs, that is, graphs for which each edge has an associated weight, typically given by a weight function $w : E \rightarrow \mathbb{R}$.
- Provides no quicker way to determine whether a given edge (u, v) present in the graph than to search for v in the adjacency list $Adj[u]$.

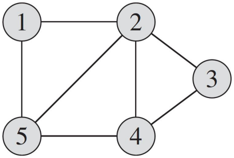
Adjacency-Matrix

- We assume that the vertices are numbered $1, 2, \ldots, |V|$ in some arbitrary manner. Then the adjacency-matrix representation of a graph G consists of a $|V| \times |V|$ matrix $A = (a_{ij})$ such that

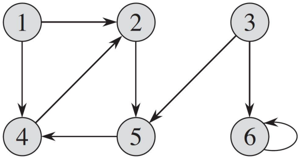
$$a_{ij} = \begin{cases} 1 \text{ if } (i,j) \in E, \\ 0 \text{ otherwise.} \end{cases}$$

- The adjacency matrix of a graph requires $|V|^2$ memory, independent of the number of edges in the graph.
- Since in an undirected graph, (u, v) and (v, u) represent the same edge, the adjacency matrix A of an undirected graph is its own transpose: $A = A^T$.
- We may store above the diagonal of the adjacency matrix, thereby cutting the memory needed to store the graph almost in half.

Adjacency-Matrix



$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$



$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

Adjacency-Matrix

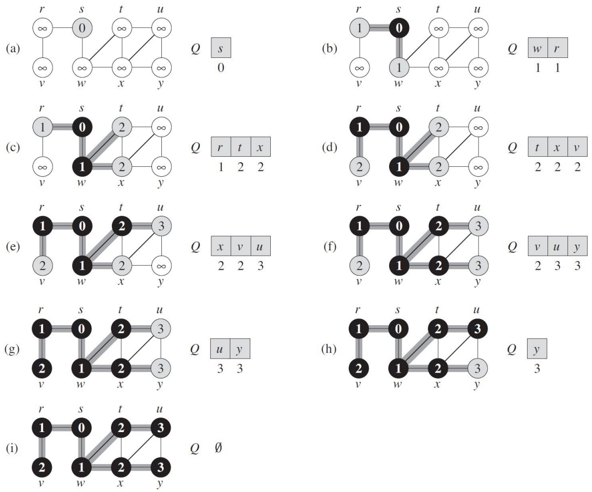
- An adjacency matrix can represent a weighted graph.
- We can store the weight $w(u,v)$ of the edge $(u,v) \in E$ as the entry in row u and column v of the adjacency matrix. If an edge does not exist, we can store a `NIL`, 0, or ∞.

Breadth-first Search

- Breadth-first search is one of the simplest algorithms for searching a graph and the archetype for many important graph algorithms.
- It produces a "breadth-first tree" with root s that contains all reachable vertices. For any vertex v reachable from source vertex s, the simple path in the breadth-first tree from s to v corresponds to a "shortest path" from s to v in G, that is, a path containing the smallest number of edges.
- Breadth-first search is so named because it expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier. The algorithm discovers all vertices at distance k from s before discovering any vertices at distance $k+1$.

Breadth-first Search

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```



Breadth-first Search

Printing Path

```
PRINT-PATH(G, s, v)
if v == s
    print s
elseif v.π == NIL
    print "no path from" s "to" v "exists"
else PRINT-PATH(G, s, v.π)
    print v
```

Breadth-first Search

Analysis

- The overhead for initialization is $O(|V|)$.
- After initialization, breadth-first search never whitens a vertex, and thus the test in line 13 ensures that each vertex is enqueued at most once, and hence dequeued at most once.
- The operations of enqueuing and dequeuing take $O(1)$ time, and so the total time devoted to queue operations is $O(|V|)$.
- Because the procedure scans the adjacency list of each vertex only when the vertex is dequeued, it scans each adjacency list at most once.
- Since the sum of the lengths of all the adjacency lists is $\Theta(|E|)$, the total time spent in scanning adjacency lists is $O(|E|)$.
- Thus the total running time of the BFS procedure is $O(|V|+|E|)$.

Depth-first search

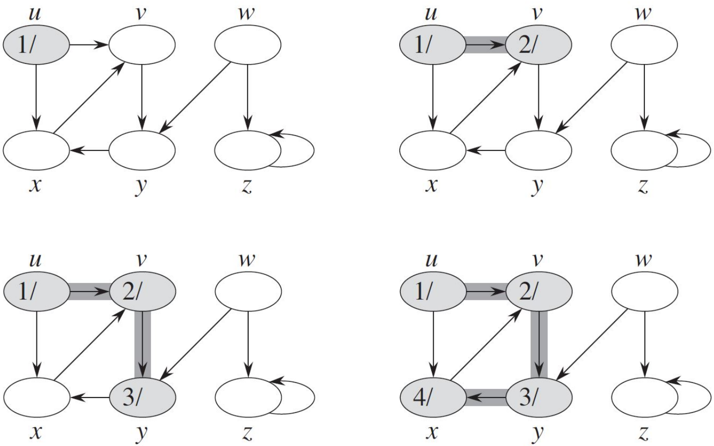
- The strategy followed by depth-first search (DFS) searches "deeper" in the graph whenever possible.
- DFS explores edges out of the most recently discovered vertex v that still has unexplored edges leaving it.
- Once all of v's edges have been explored, the search "backtracks" to explore edges leaving the vertex from which v was discovered.
- This process continues until we have discovered all the vertices that are reachable from the original source vertex.
- If any undiscovered vertices remain, then depth-first search selects one of them as a new source, and it repeats the search from that source. The algorithm repeats this entire process until it has discovered every vertex.

Depth-first search

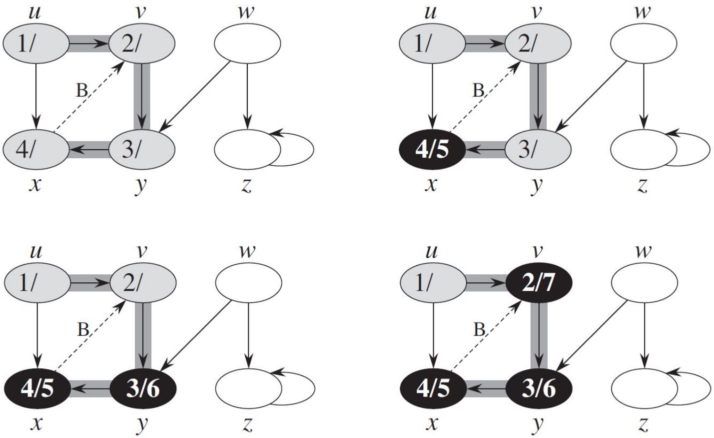
```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

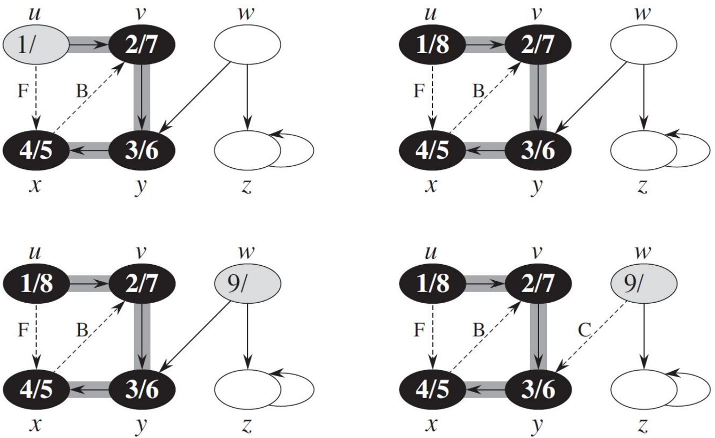
Depth-first search



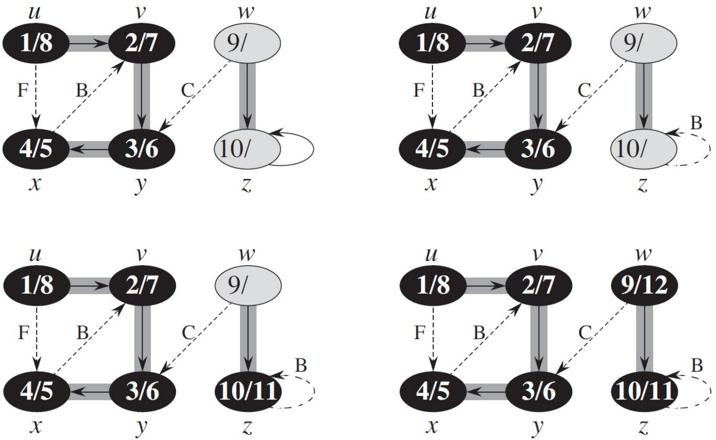
Depth-first search



Depth-first search



Depth-first search



Depth-first Search

Analysis

- The loops on lines 1–3 and lines 5–7 of DFS take time $\Theta(|V|)$, exclusive of the time to execute the calls to DFS-VISIT.
- The procedure DFS-VISIT is called exactly once for each vertex $v \in V$, since the vertex u on which DFS-VISIT is invoked must be white and the first thing DFS-VISIT does is paint vertex u gray.
- During an execution of DFS-VISIT (G, v), the loop on lines 4–7 executes $|Adj[v]|$ times.
- Since

$$\sum_{v \in V} |Adj[v]| = \Theta(|E|)$$

  the total cost of executing lines 4–7 of DFS-VISIT is $\Theta(|E|)$.
- The running time of DFS is therefore $\Theta(|V| + |E|)$.

BFS: Lemmas

Lemma 1

Let $G = (V, E)$ be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

$$\delta(s, v) \leq \delta(s, u) + 1$$

where $\delta(s, v)$ is the shortest-path distance from s to v defined as the minimum number of edges in any path from vertex s to vertex v; if there is no path from s to v, then $\delta(s, v) = \infty$.

BFS: Lemmas

Lemma 2

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$.

Lemma 3

Suppose that during the execution of BFS on a graph $G = (V, E)$, the queue Q contains the vertices $\langle v_1, v_2, \ldots, v_r \rangle$, where v_1 is the head of Q and v_r is the tail. Then, $v_r.d \leq v_1.d + 1$ and $v_i.d \leq v_{i+1}.d$ for $i = 1, 2, \ldots, r - 1$.

Lemma 4

Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \leq v_j.d$ at the time that v_j is enqueued.

Breadth-first Trees

- For a graph $G = (V, E)$ with source s, we define the predecessor subgraph of G as $G_\pi = (V_\pi, E_\pi)$, where

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$$

and

$$E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}$$

The predecessor subgraph G_π is a breadth-first tree if V_π consists of the vertices reachable from s and, for all $v \in V_\pi$, the subgraph G_π contains a unique simple path from s to v that is also a shortest path from s to v in G. A breadth-first tree is in fact a tree, since it is connected and $|E_\pi| = |V_\pi| - 1$. We call the edges in E_π tree edges.

Predecessor Subgraph of a Depth-first Search

- For a graph $G = (V, E)$, predecessor subgraph of a depth-first search can be defined as $G_\pi = (V, E_\pi)$ where

$$E_\pi = \{(v.\pi, v) : v \in V \text{ and } v.\pi \neq \text{NIL}\}$$

  The predecessor subgraph of a depth-first search forms a depth-first forest comprising several depth-first trees. The edges in E_π are tree edges.

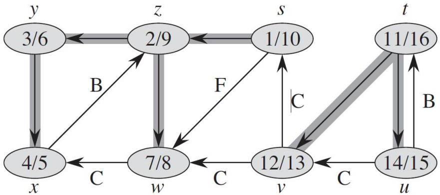
DFS: Theorems

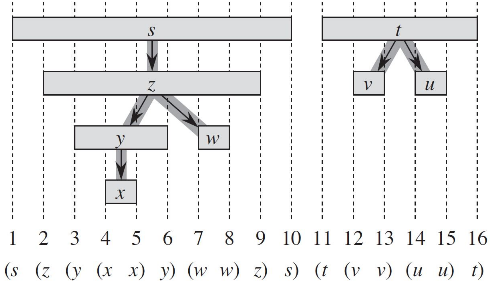
Parenthesis Theorem

In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

- the intervals $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest,
- the interval $[u.d, u.f]$ is contained entirely within the interval $[v.d, v.f]$, and u is a descendant of v in a depth-first tree, or
- the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$, and v is a descendant of u in a depth-first tree.

DFS: Theorems





DFS: Theorems and Corolaries

Corollary: Nesting of Descendants' Intervals

Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if $u.d < v.d < v.f < u.f$.

Theorem: White-path Theorem

In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

Classification of Edges

Tree edges

Tree edges are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if v was first discovered by exploring edge (u, v).

Back edges

Back edges are those edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree. We consider self-loops, which may occur in directed graphs, to be back edges.

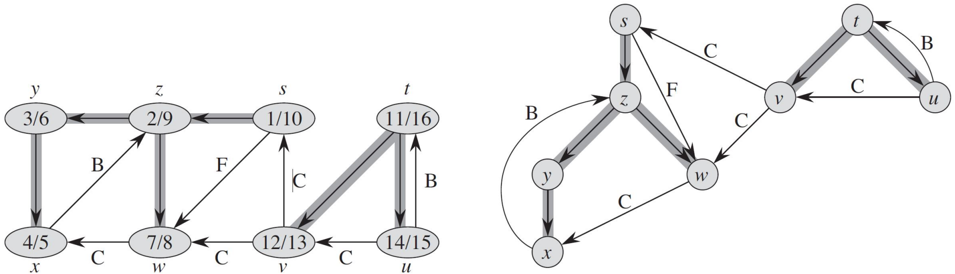
Forward edges

Forward edges are those nontree edges (u, v) connecting a vertex u to a descendant v in a depth-first tree.

Cross edges

Cross edges are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

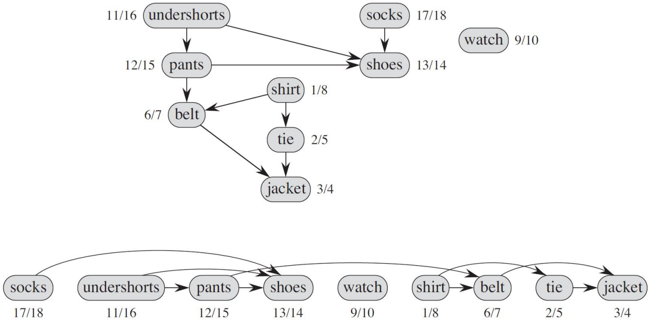
Classification of Edges



Topological Sort

- A topological sort of a dag $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.
- We can view a topological sort of a graph as an ordering of its vertices along a horizontal line so that all directed edges go from left to right.
- If the graph contains a cycle, then topological sorting is not possible.
- Many applications use directed acyclic graphs to indicate precedences among events.

Topological Sort



Topological Sort

Algorithm

TOPOLOGICAL-SORT(G)

call DFS(G) to compute finishing times $v.f$ for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices

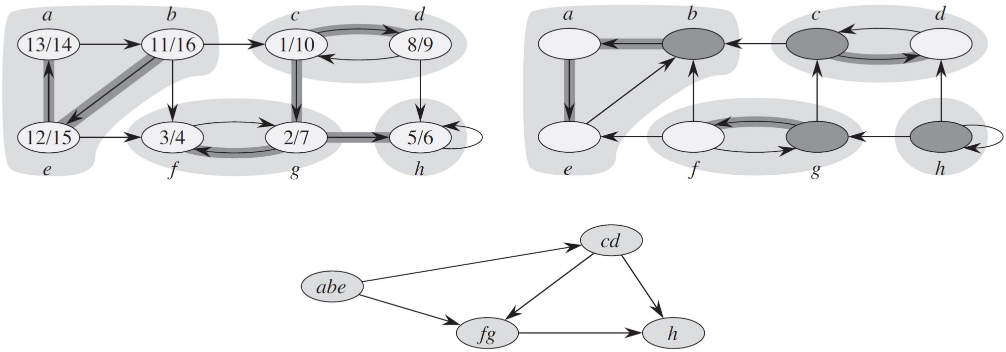
Analysis

- We can perform a topological sort in time $\Theta(|V|+|E|)$, since depth-first search takes $\Theta(|V|+|E|)$ time and it takes $O(1)$ time to insert each of the $|V|$ vertices onto the front of the linked list.
- A directed graph G is acyclic if and only if a depth-first search of G yields no back edges.

Strongly Connected Component

- A strongly connected component of a directed graph $G = (V, E)$ is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u and v in C, we have both $u \rightsquigarrow v$ and $v \rightsquigarrow u$, that is, vertices u and v are reachable from each other.
- The transpose of a graph $G = (V, E)$ is $G^T = (V, E^T)$, where $E^T = \{(u, v) : (v, u) \in E\}$.
- G and G^T have exactly the same strongly connected components: u and v are reachable from each other in G if and only if they are reachable from each other in G^T.

Strongly Connected Component



Strongly Connected Component

Algorithm

STRONGLY-CONNECTED-COMPONENTS(G)

1 call DFS(G) to compute finishing times $u.f$ for each vertex u
2 compute G^{T}
3 call DFS(G^{T}), but in the main loop of DFS, consider the vertices
    in order of decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the depth-first forest formed in line 3 as a
    separate strongly connected component

White Board

White Board