MA203

Repeated trials

Repeated trials:
Given two experiments i.e., rolling a fair die and tossing a fair coin; find the probability that we get 2 on die and head on coin.

P(2 on die)=1/6
P(head on coin)=1/2
Assuming statistically independent, P(2 on die and head on coin)= P(2 on die)xP(head on coin)=1/12

Other way,
{1,2,3,4,5,6} {H,T}
Considering both as a single experiment, sample space is:
{1H,2H,3H,4H,5H,6H,1T,2T,3T,4T,5T,6T}
P(2 on die and head on coin)=1/12

Cartesian product: given 2 sets S1 and S2, product of the sets is a set S whose elements are all such pairs of elements of S1 and S2.

{H,T} {H,T} {HH,HT,TH,TT}

Bernoulli Trials:

Out of a set of n distinct objects, if k<n objects are taken out of n at a time, the total combinations are

$\frac{n!}{(n-k)!k!} = \binom{n}{k}$ (orders are not considered)

Consider an experiment S and an event A with $P(A)=p$, $P(A^c)=q$ and $p+q=1$

Repeat the experiment n times and the resulting product space is $S_n=S \times S \times \ldots \times S$ (n times)

Define $p_n(k)=P$(A occurs k times in any order)

P(A occurs k times in a specific order) $=p^k q^{n-k}$

$p_n(k)=\binom{n}{k}p^k q^{n-k}$ Fundamental theorem

Considering event A as success and A^c as failure, $p_n(k)$ gives prob of k successes in n independent trials.

Eg. 1) A pair of dice is rolled n times. i) Find the probability that 7 will not show at all. ii) Find the probability of double six at least once.

Ans. Sample space of single roll of 2 dice consists of 36 elements.

i) A={seven}={(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

P(A)=6/36=1/6

P(A^c)=5/6

P(7 will not show at all)=$\binom{n}{0}(1/6)^0(5/6)^{n-0}$

ii) X: double six at least once

P(X)=1-P(X^c)=1-$(35/36)^n$

Suppose we are interested in no. of throws required to ensure a 50% success of obtaining double six at least once.

So, 1-$(35/36)^n$ >0.5

or, n=24.605

Historical importance: one of the first problems solved by Pascal and correctly interpret gambler’s choice.

For a fixed n, consider $p_n(k)$ as a function of k

As k increases, $p_n(k)$ increases reaching a maximum for k=kmax

$$\frac{p_n(k-1)}{p_n(k)} = \frac{\binom{n}{k-1} p^{k-1} q^{n-k+1}}{\binom{n}{k} p^k q^{n-k}} = \frac{kq}{(n-k+1)p} < 1$$

if $k<(n+1)p$, $p_n(k-1)<p_n(k)$

If $k>(n+1)p$, then $p_n(k)$ is decreasing

Therefore, $p_n(k)$ is maximum for kmax=[(n+1)p]

If k1=(n+1)p is an integer, then $\frac{p_n(k-1)}{p_n(k)}=1$

So, $p_n(k)$ has maximum values for k=k1 and k=k1-1

Most likely number of success out of n independent trials

Example 2. n=10, p=1/3, calculate most likely number of successes

Example 3. In New York state lottery, the player picks 6 numbers from a sequence of 1 through 51. At a lottery drawing, 6 balls are drawn at random from a box of 51 balls numbered 1 through 51. What is the prob that a player has k matches?

P(k matches) = $\frac{\binom{6}{k}\binom{51-6}{6-k}}{\binom{51}{6}}$

For perfect match, k=6 and prob is 1/18009460

Random variable:
Finite, single-valued function which maps the function into sample space.
For every outcome a of sample space, we assign a number X(a) and this function is called RV.

Eg. S={HH,HT,TH,TT}
Function: number of heads

Outcomes of S (a)	**HH**	**HT**	**TH**	**TT**
RV X(a)	2	1	1	0

Consider, S={1, 2, 3, 4, 5, 6} of random experiment of rolling a fair die
Let, function is {(number of points on top less 3)^2}

Outcomes of S (a)	**1**	**2**	**3**	**4**	**5**	**6**
RV X(a)	4	1	0	1	4	9

Two types: continuous and discrete

Since, defined over a sample space of a random experiment, each value is associated with a probability.
{X<=x} means a subset of S consisting of all outcomes a such that X(a)<=x

P{X<=x}=F_X(x) , distribution function or cumulative distribution function
let, tossing of 2 coins

Outcomes of S (a)	**HH**	**HT**	**TH**	**TT**
RV X(a)	2	1	1	0

F(x)=P(X<=x)

F(0)=P(X<=0)=
F(1)=P(X<=1)=
F(2)=P(X<=2)=

Properties:

1. $F(+\infty)=1$, $F(-\infty)=0$

$F(+\infty)=P\{X\le+\infty\}=P(S)=1$ $F(-\infty)=P\{X\le-\infty\}=0$

2. non-decreasing function of x i.e., if $x_1 < x_2$ then $F(x_1) \le F(x_2)$

$\{X\le x_1\}$ is a subset of $\{X\le x_2\}$ if $x_1 < x_2$
So $P\{X\le x_1\} \le P\{X\le x_2\}$ $F(x_1)\le F(x_2)$

3. If $F(x_0)=0$ then $F(x)=0$ for every $x\le x_0$

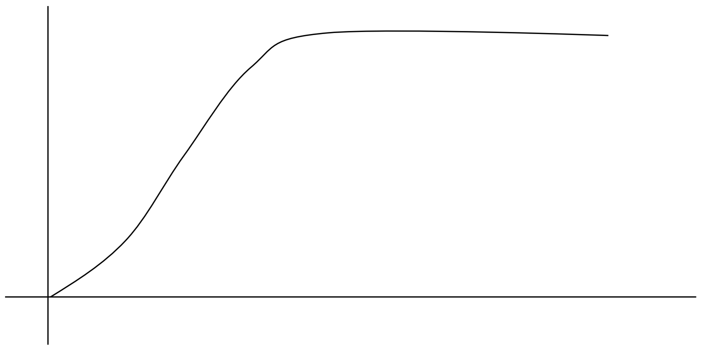
4. $P\{X>x\}=1-F(x)$

$\{X\le x\} \cup \{X>x\}=S$ and they are mutually exclusive So $P\{X>x\}=1-F(x)$

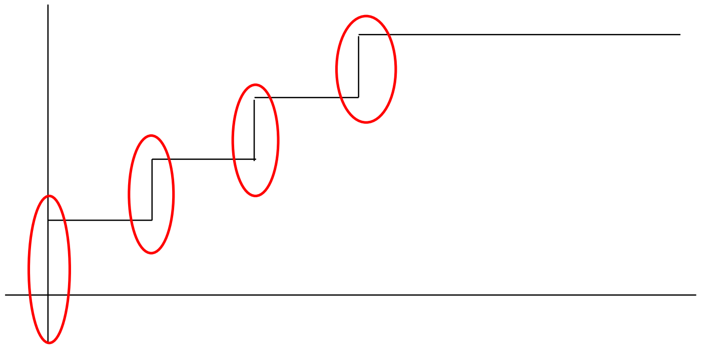
5. $P\{x_1<X\le x_2\}=F(x_2)-F(x_1)$

$\{X\le x_2\}=\{X\le x_1\} \cup \{x_1<X\le x_2\}$ [mutually exclusive]
$P\{X\le x_2\}=P\{X\le x_1\} + P\{x_1<X\le x_2\}$
$P\{x_1<X\le x_2\}=F(x_2)-F(x_1)$

Continuous RV : if F(x) is continuous



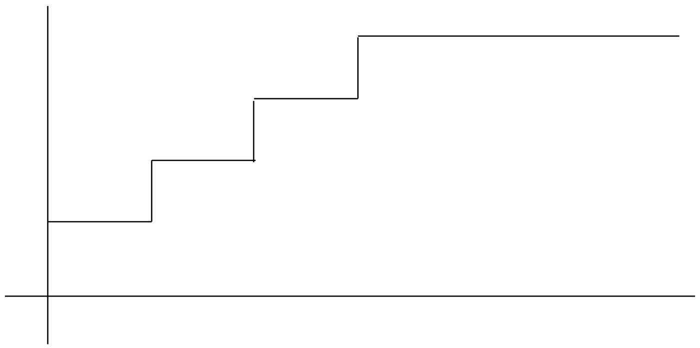
Discrete RV : if F(x) is constant except for a finite number of jump discontinuities (piecewise constant/step)

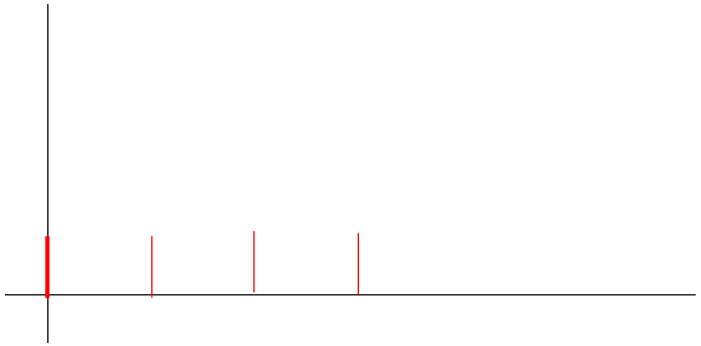


Probability density function:
Derivative of distribution function F(x)

$$f_X(x) \triangleq \frac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} \geq 0$$

As F(x) is monotonically increasing, so f(x)>=0 for all x
If X is continuous RV then f(x) will be continuous
If X is discrete RV, then pdf has the general form $f_X(x) = \sum_i p_i \delta(X - x_i)$, xi: jump discontinuities of F(x)
probability mass function (pmf)





pmf

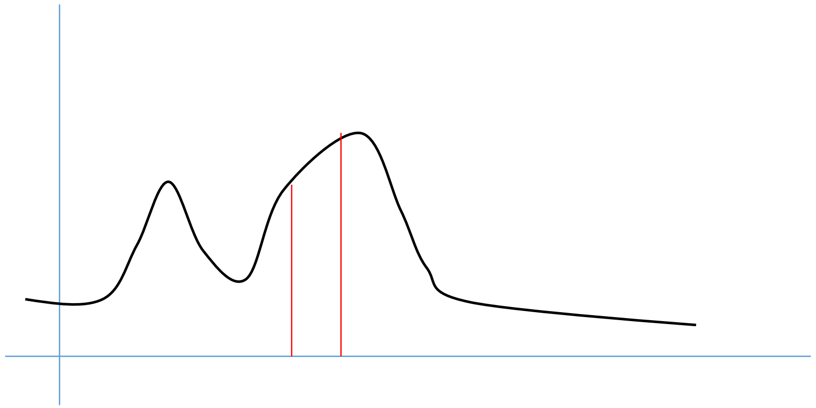
From definition,

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

Also, $\int_{-\infty}^{\infty} f_X(x)dx = 1$, so density function

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$$

Area under the curve in the interval



Probability that a continuous RV takes any specified value is 0

Example 1. X is an RV with distribution function

$$F(x)=\begin{cases} 0, & x<=0 \\ x, & 0<x<=1 \\ 1, & x>1 \end{cases}$$

$$f(x)=F'(x)=\begin{cases} 0, & x<0 \text{ or } x>1 \\ 1, & 0<x<1 \end{cases}$$

P{0.4<X<=0.6} =?

2. X have the triangular pdf

$$f(x)=\begin{cases} x, & 0<x<=1 \\ 2-x, & 1<=x<=2 \\ 0, & \text{otherwise} \end{cases}$$

Check f(x) is pdf

P{0.3<X<=1.5}=?

$$F(x) = 0 \quad , x \le 0$$

$$F(x) = \int_0^x t\,dt = \frac{x^2}{2} \quad , 0 < x \le 1$$

$$F(x) = \int_0^1 t\,dt + \int_1^x (2-t)\,dt \quad , 1 \le x \le 2$$

$$= \frac{1}{2} + 2x - 2 - \frac{x^2}{2} + \frac{1}{2} = 2x - \frac{x^2}{2} - 1$$

$$F(x) = \int_0^1 t\,dt + \int_1^2 (2-t)\,dt \quad , x \ge 2$$

$$= \frac{1}{2} + 2 - 2 + \frac{1}{2}$$

$$= 1$$

now,

$$P\{0.3 < X \le 1.5\} = P(X \le 1.5) - P(X \le 0.3)$$

$$= \int_{0.3}^{1.5} f(x)\,dx$$

$$= \int_{0.3}^{1} x\,dx + \int_1^{1.5} (2-x)\,dx$$

$$= \frac{1-0.09}{2} + 2(1.5-1) - \frac{2.25-1}{2}$$

$$= \cancel{0.8705}\ 0.83$$

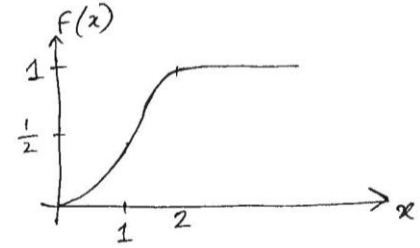
other way,

$$P\{0.3 < X \le 1.5\} = F(1.5) - F(0.3)$$

$$= 2.1.5 - \frac{1.5^2}{2} - 1 - \frac{0.3^2}{2}$$

$$= 0.83$$

If we plot F(x), it would be



For discrete RV, the collection of numbers {pi} satisfying P(X=xi)=pi >=0 for all i and $\sum_{i=1}^{\infty} p_i = 1$ is called pmf of RV X

DF is F(x)=P(X<=x)=sum (pi) for xi<=x

Example: A box contains good and defective items. If an item drawn is good, we assign the number 1 to the drawing and otherwise the number is 0. Let probability of drawing a good item at random is p.

$$\left.\begin{array}{l} P(X=1)=p \\ P(X=0)=1-p \end{array}\right\} \text{pmf}$$

$$\left.F(x)=P(X\le x)=\begin{cases} 0 & , x<0 \\ 1-p & , 0\le x<1 \\ 1 & , x\ge 1 \end{cases}\right\} \text{DF}$$

Theoretical distribution: when a random experiment is theoretically assumed to serve as a model, the probability distribution of the RV associated with the random experiment is generally known as theoretical distribution.

Expectation – mean, variance, moments

Let, a discrete RV X assumes the values x_1, x_2, ……, x_n with probabilities p_1, p_2,……, p_n respectively. Expectation or expected value of X is

$E(X) = \sum_{i=1}^{n} p_i x_i$ provided $\sum_{i=1}^{n} p_i |x_i| < \infty$

Similarly, $E(X^2) = \sum_{i=1}^{n} p_i x_i^2$

Say, g(X) is a function of RV X

$$E[g(X)] = \sum_{i=1}^{n} p_i g(x_i)$$

Expectation of a constant k is the constant k itself.

$E(k)=\text{sum}(k.p_i) = k$ [since, $\text{sum}(p_i)=1$]

Mean – of a RV X is $E(X)=\mu$

Variance – $\sigma^2 = E(X-\mu)^2 = E(X^2-2X\mu+\mu^2) = E(X^2) - \mu^2$

Standard deviation (σ) is the positive square root of variance.

Moments:

r-th moment about A is

$$m_r' = E(X-A)^r = \sum_{i=1}^{n} p_i (x_i - A)^r$$

r-th raw moment is

$$\mu_r' = E(X)^r = \sum_{i=1}^{n} p_i {x_i}^r$$

r-th central moment is

$$\mu_r = E(X-\mu)^r = \sum_{i=1}^{n} p_i (x_i - \mu)^r$$

Where μ= E(X)

As per definition, $\mu_0' = \mu_0 = 1$, $\mu_1' = E(X) = \mu$, $\mu_1 = 0$, $\mu_2 = \sigma^2$

Central moments can be obtained from non-central moments as

$$\mu_2 = E(X-\mu)^2 = E(X)^2 - \left(E(X)\right)^2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_2' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

Expectation – mean, variance, moments

Let, a continuous RV X assumes the values between –inf to +inf with pdf f(x)

Expectation or expected value of X is

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$ provided $\int_{-\infty}^{\infty} |x|f(x)dx < \infty$

Mean – of a RV X is $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \mu$

Variance – $\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E(X)^2 - \mu^2$

Standard deviation (σ) is the positive square root of variance.

Moments:

r-th moment about A is

$$m_r' = E(X-A)^r = \int_{-\infty}^{\infty} (x-A)^r f(x)dx$$

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<u>Moment generating functions:</u>

Consider X is an RV.

M(s)=E(e^{sX}) exists provided $\int_{-\infty}^{\infty} |e^{sx}| f(x)dx < \infty$

M(s) is called moment generating function (MGF) of RV X

MGF uniquely determines the corresponding distribution function (DF) and if MGF exists, MGF is unique for an RV.

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f(x)dx$$

Differentiating by n times,

$$M^n(s) = \int_{-\infty}^{\infty} x^n e^{sx} f(x)dx = E[X^n e^{sX}]$$

For s=0,

$$M^n(0) = E[X^n]$$

Which is n-th order raw moment.

Hence, MGF

$$M'(0) = E(X) = \mu$$
$$M''(0) = E(X^2) = \sigma^2 + (M'(0))^2$$

Similarly, third order moment : measure of skewness

Fourth order moment : measure of kurtosis

Example-4) Let, X have the PDF as

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$\therefore$ MGF is $M(s) = \int_0^\infty e^{sx} \frac{1}{2} e^{-x/2} \, dx$

$$= \int_0^\infty \frac{1}{2} e^{(s-\frac{1}{2})x} \, dx = \left[\frac{1}{2} \frac{e^{(s-\frac{1}{2})x}}{s - \frac{1}{2}} \right]_0^\infty$$

$$= \frac{1}{1-2s} \quad , s < \frac{1}{2}$$

$$M'(s) = \frac{d}{ds}\left(\frac{1}{1-2s}\right) = \frac{-1(-2)}{(1-2s)^2} = \frac{2}{(1-2s)^2}$$

$$M''(s) = \frac{d^2}{ds^2}\left(\frac{1}{1-2s}\right) = \frac{-2.2(1-2s).(-2)}{(1-2s)^4} = \frac{8}{(1-2s)^3}$$

now, $M'(0) = E(X) = 2$ $\quad \therefore \mu = 2$

$M''(0) = E(X^2) = 8$ $\quad \sigma^2 = 8 - 2^2 = 4$

now, to check these results,

$$E(X) = \int_0^\infty x \cdot \frac{1}{2} e^{-x/2} \, dx = \frac{1}{2} \int_0^\infty x e^{-x/2} \, dx$$

$$= \frac{1}{2} \left[x \int e^{-x/2} dx - \int \left(1 . \int e^{-x/2} dx \right) dx \right]_0^\infty$$

$$= \frac{1}{2} \left[-2x e^{-x/2} - 4 e^{-x/2} \right]_0^\infty$$

$$= \frac{1}{2} \cdot 4 = 2$$

$$E(X^2) = \int_0^\infty x^2 \tfrac{1}{2} e^{-x/2}\, dx = \frac{1}{2}\left[x^2\left(-2e^{-x/2}\right) - \int 2x\left(-2e^{-x/2}\right) dx\right]_0^\infty$$

$$= \frac{1}{2}\left[-2x^2 e^{-x/2} + 4\int x e^{-x/2}\, dx\right]_0^\infty$$

(from $E(X)$ solution)

$$= \left[-x^2 e^{-x/2}\right]_0^\infty + 2 \cdot 4$$

$$= 8$$

$$\therefore \sigma^2 = 8 - 2^2 = 4$$ (hence verified)

values x_i with

Suppose, X is a discrete RV taking values xi with probability pi

$$M(s) = \sum_i p_i e^{sx_i}$$

However, if X takes only integer values, then Z transform is preferable to define MGF

$$\Gamma(z) = E(z^X) = \sum_{n=-\infty}^{\infty} P(X = n)z^n = \sum_{n=-\infty}^{\infty} p_n z^n$$

Differentiating it k times,

$$\Gamma^{(k)}(z) = E\{X(X-1)\ldots\ldots\ldots(X-k+1)z^{X-k}\}$$

With z=1,

$$\Gamma^{(k)}(1) = E\{X(X-1)\ldots\ldots\ldots(X-k+1)\}$$

So,

$$\Gamma'(1) = E(X)$$

$$\Gamma''(1) = E\big(X(X-1)\big) = E(X^2 - X) = E(X^2) - E(X)$$

Example 5. An RV X takes values 0 and 1 with P(X=1)=p and P(X=0)=q. Find MGF.

ans. $\Gamma(z) = E(z^X) = P(X=1)\cdot z^1 + P(X=0)\cdot z^0$

$= pz + q$

$\therefore \Gamma'(1) = \frac{d}{dz}(pz+q)\Big|_{z=1} = p$

$\therefore E(X) = p$

$\Gamma''(1) = \frac{d^2}{dz^2}(pz+q)\Big|_{z=1} = 0$

$\therefore E(X^2) - E(X) = 0$

& $E(X^2) = E(X) = p$

$\therefore$ variance $= \cancel{p}\, E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$

$= pq$

now to check,

$E(X) = 1\cdot p + 0\cdot q = p$

$E(X^2) = 1^2\cdot p + 0^2\cdot q = p$

$\therefore$ Variance $= E(X^2) - E^2(X) = p - p^2 = p(1-p) = pq$

(hence verified)