Sets, Sequences and Functions

Department of Science and Mathematics

IIIT Guwahati.

August 26, 2020

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.
Examples of set.
$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.

Examples of set.

$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

$\{\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)\}$

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.
Examples of set.
$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.
$\{\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)\}$

Definition

Two sets are equal if and only if they have the same elements.

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.

Examples of set.

$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

$\{\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)\}$

Definition

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.

Examples of set.

$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

$\{\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)\}$

Definition

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

$\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal.

Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.

Examples of set.

$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

$\{\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)\}$

Definition

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

$\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal.

$\{1, 3, 5\}$ and $\{1, 1, 1, 3, 3, 3, 5, 5, 5\}$ are equal.

The set A is a subset of B if and only if every element of A is also an element of B.

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \to x \in B)$ is true.

Remark

To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Define A is a proper subset of B, denoted by $A \subset B$ using logical qantifiers.

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \to x \in B)$ is true.

Remark

To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Define A is a proper subset of B, denoted by $A \subset B$ using logical qantifiers.

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

A set is said to be infinite if it is not finite.

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

A set is said to be infinite if it is not finite.

Definition

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

What is the power set of the empty set?

What is the power set of the empty set? What is the power set of the set $\{\phi\}$?

What is the power set of the empty set? What is the power set of the set $\{\phi\}$?
A Universal Set is the set of all elements under consideration.

What is the power set of the empty set? What is the power set of the set $\{\phi\}$?

A Universal Set is the set of all elements under consideration.

Give definitions for:

Cartesian Product of two sets

Unioin of two sets

Intersection of two sets

Disjoint sets

Difference of two sets

Complement of A.

Show that $|A \cup B| = |A| + |B| - |A \cap B|$ and $A - B = A \cap B^c$.

Set Identities

$A \cup U = U, \ A \cup \phi = A$
$A \cap U = A, \ A \cap \phi = \phi$
$A \cup A = A, \ A \cap A = A$
$(A^c)^c = A$
$A \cup B = B \cup A, \ A \cap B = B \cap A$
$A \cup (B \cup C) = (A \cup B) \cup C, \ A \cap (B \cap C) = (A \cap B) \cap C$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$(A \cap B)^c = A^c \cup B^c, \ (A \cup B)^c = A^c \cap B^c$ (De Morgan's laws)
$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$
$A \cup A^c = U, \ A \cap A^c = \phi.$

1. Prove that $(A \cap B)^c = A^c \cup B^c$, $(A \cup B)^c = A^c \cap B^c$.

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. Prove that $(A \cup (B \cap C))^c = (C^c \cup B^c) \cap A^c$.

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. Prove that $(A \cup (B \cap C))^c = (C^c \cup B^c) \cap A^c$.

For $i = 1, 2, \ldots,$ define $A_i = \{i, i+1, i+2, \ldots\}$ find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

For $i = 1, 2, \ldots,$ define $A_i = \{i, i+1, i+2, \ldots\}$ find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

If I is any index set, $\bigcup_{i \in I} A_i = \{x | x \in A_i \text{ for some } i\}$ and
$\bigcap_{i \in I} A_i = \{x | x \in A_i \text{ for all } i\}$.

Let $A_i = \{1, 2, \ldots, i\}$ for $i = \{1, 2 \ldots\}$. Then find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

Computer Representation of Sets

- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).
- Specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.
- Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Computer Representation of Sets

- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).
- Specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.
- Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?
2. The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010, respectively. Use bit strings to find the union and intersection of these sets.

Functions

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Functions

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Determine whether f is a function from the set of all bit strings (of finite length) to the set of integers if

- f (S) is the position of a 0 bit in S.

Functions

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Determine whether f is a function from the set of all bit strings (of finite length) to the set of integers if

- f (S) is the position of a 0 bit in S.
- f (S) is the number of 1 bits in S.

Functions

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

Determine whether f is a function from the set of all bit strings (of finite length) to the set of integers if

- f (S) is the position of a 0 bit in S.
- f (S) is the number of 1 bits in S.
- f (S) is the smallest integer i such that the ith bit of S is 1 and f (S) $= 0$ when S is the empty string, the string with no bits.

Find the domain and range of these functions.

1. the function that assigns to each pair of positive integers the first integer of the pair
2. the function that assigns to each positive integer its largest decimal digit
3. the function that assigns to a bit string the number of ones minus the number of zeros in the string
4. the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
5. the function that assigns to a bit string the longest string of ones in the string

Express the definitions of one-one, surjective, increasing functions using logical operators and use them to write negations.

Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a. mobile phone number.
b. student identification number.
c. final grade in the class.
d. home town

Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

- one-to-one but not onto.
- onto but not one-to-one.
- both onto and one-to-one (but different from the identity function).
- neither one-to-one nor onto.

  Give an explicit formula for a function from the set of integers to the set of positive integers that is

- one-to-one, but not onto.
- onto, but not one-to-one.
- one-to-one and onto.
- neither one-to-one nor onto

1. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.
2. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = 1/f(x)$ is strictly decreasing.
3. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.
4. Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.
5. Give an example of an increasing function from $\mathbb{R}$ to itself that is not one-to-one.
6. Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one.
7. Give an example of a decreasing function from $\mathbb{R}$ to itself that is not one-to-one

Suppose that g is a function from A to B and f is a function from B to C.

1. Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
2. Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Summary

Suppose that $f : A \to B$.

- To show that f is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
- To show that f is not injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- To show that f is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- To show that f is not surjective: Find a particular $y \in B$ such that $f(x) \neq$ for all $x \in A$.

Definition

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective

Definition

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.

Cardinality of Sets

1. Show that if a set S has cardinality m, where m is a positive integer, then there is a one-to-one correspondence between S and the set $\{1, 2, ..., m\}$.
2. Show that if S and T are two sets each with m elements, where m is a positive integer, then there is a one-to-one correspondence between S and T.

Infinite Sets

- Which set has more elements, the set of all natural numbers or the set of even natural numbers?

Infinite Sets

- Which set has more elements, the set of all natural numbers or the set of even natural numbers?
- Which set has more elements, the set of all natural numbers or the set of integers?
- Which set has more elements, the set of all integers or the set of rational numbers?

Infinite Sets

- Which set has more elements, the set of all natural numbers or the set of even natural numbers?
- Which set has more elements, the set of all natural numbers or the set of integers?
- Which set has more elements, the set of all integers or the set of rational numbers?
- We defined the cardinality of a finite set as the number of elements in the set and used cardinality to tell us when they have the same size, or when one is bigger than the other.
- Can we extend this notion to infinite sets?
- Can we then use it to measure the relative sizes of infinite sets?

We showed that there is a one-to-one correspondence between any two finite sets with the same number of elements. We use this observation to extend the concept of cardinality to all sets, both finite and infinite.

Definition

The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

For infinite sets the definition of cardinality provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set. We can also define what it means for one set to have a smaller cardinality than another set.

Definition

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Countable Sets

We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers, $\mathbb{N}$ and those with a different cardinality.

Definition

A set that is either finite or has the same cardinality as the set of natural numbers, $\mathbb{N}$ is called countable. A set that is not countable is called uncountable.

When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (aleph null)

1. Show that the set of even positive integers is a countable set.
To show that the set of even positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers.

- Consider the function $f(n) = 2n$ from $\mathbb{N}$ to the set of odd positive integers. We show that f is a one-to-one correspondence by showing that it is both one-to-one and onto.
- To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, so $n = m$.
- To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = 2k = t$.

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

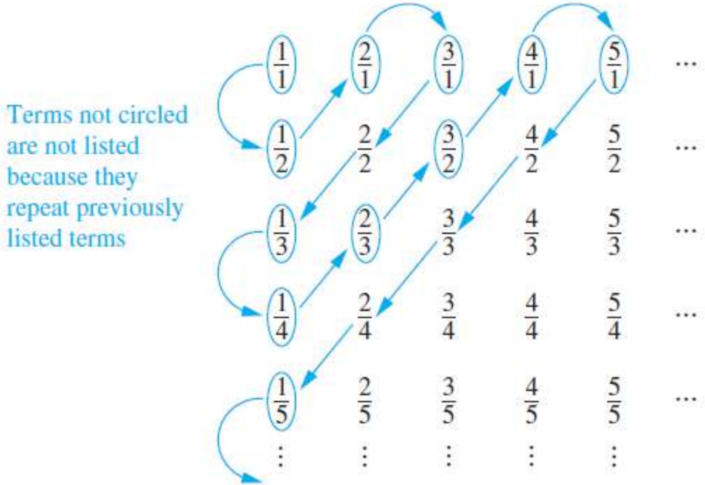
An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers). The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \cdots, a_n, \cdots$, where $a_1 = f(1), a_2 = f(2), \cdots, a_n = f(n), \ldots.$

2. Show that the set of all integers, $\mathbb{Z}$ is countable.

2. Show that the set of all integers, $\mathbb{Z}$ is countable.
The function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \frac{n}{2}$ when n is even and $f(n) = -\frac{(n-1)}{2}$ when n is one-one and onto.

3. Show that the set of positive rational numbers, $\mathbb{Q}^+$ is countable.

3. Show that the set of positive rational numbers, $\mathbb{Q}^+$ is countable.



Hilbert's Grand Hotel

The famous mathematician David Hilbert invented the notion of the Grand Hotel, which has a countably infinite number of rooms, each occupied by a guest.

Hilbert's Grand Hotel

The famous mathematician David Hilbert invented the notion of the Grand Hotel, which has a countably infinite number of rooms, each occupied by a guest.

- Can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?

Hilbert's Grand Hotel

The famous mathematician David Hilbert invented the notion of the Grand Hotel, which has a countably infinite number of rooms, each occupied by a guest.

- Can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?
- Can we accommodate a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without removing any current guest?

Hilbert's Grand Hotel

The famous mathematician David Hilbert invented the notion of the Grand Hotel, which has a countably infinite number of rooms, each occupied by a guest.

- Can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?
- Can we accommodate a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without removing any current guest?
- Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Can all guests remain in the hotel?

Hilbert's Grand Hotel

Suppose that in an infinitely large bus countably infinite number of guests arrive at Hilbert's fully occupied Grand Hotel. Can all of them be given rooms without evicting any current guest?

Hilbert's Grand Hotel

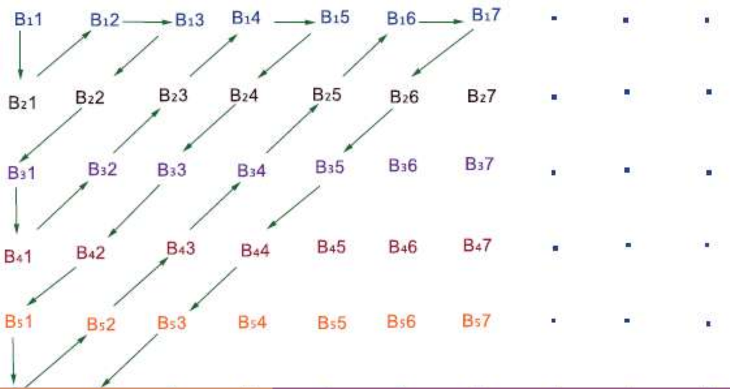
Suppose that in an infinitely large bus countably infinite number of guests arrive at Hilbert's fully occupied Grand Hotel. Can all of them be given rooms without evicting any current guest?

Hilbert's Grand Hotel

Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Can we accommodate all the arriving guests without evicting any current guest?

Hilbert's Grand Hotel

Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Can we accommodate all the arriving guests without evicting any current guest?



1. Show that subset of any countable set is countable.

1. Show that subset of any countable set is countable.
2. Show that countable union of countable sets is countable.

Uncountable Sets

Show that the set of real numbers, $\mathbb{R}$ is an uncountable set.

Uncountable Sets

Show that the set of real numbers, $\mathbb{R}$ is an uncountable set.

- Suppose that the set of real numbers is countable
- Then, the subset of all real numbers that fall between 0 and 1 would also be countable.
- Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r_1, r_2, r_3, ...$ Let the decimal representation of these real numbers be

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}\cdots d_{1n}\cdots \\
r_2 &= 0.d_{21}d_{22}d_{23}\cdots d_{2n}\cdots \\
&. \\
&. \\
r_n &= 0.d_{n1}d_{n2}d_{n3}\cdots d_{nn}\cdots
\end{aligned}
$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.