MA 203

Syllabus

Introduction to probability: mathematical background - sets, set operations, sigma and Borel fields; classical, relative-frequency and axiomatic definitions of probability; conditional probability, independence, total probability, Bayes rule; repeated trials;

Random variables: cumulative distribution function, continuous, discrete and mixed random variables, probability mass function, probability density functions; functions of a random variable; expectation - mean, variance and moments; characteristic and moment-generating functions; Chebyshev, Markov and Chernoff bounds; special random variables- Bernoulli, binomial, Poisson, uniform (discrete and continuous), Gaussian, exponential, geometric; joint distribution and density functions; Bayes rule for continuous and mixed random variables; joint moments, conditional expectation; covariance and correlation- independent, uncorrelated and orthogonal random variables; function of two random variables; sum of two independent random variables; random vector- mean vector and covariance matrix, multivariate Gaussian distribution; laws of large numbers, central limit theorem;

Random process: Discrete and continuous time (state) space; probabilistic structure of a random process; mean function, autocorrelation and autocovariance functions; Gaussian, Poisson and Markov processes, White noise, stationarity- strict-sense stationary and wide-sense stationary (WSS) processes, system with stochastic input and its applications in signal and system, cross-correlation functions, Convolution- time and frequency domain analyses, linear time-invariant systems with WSS process as an input, spectral representation of a real WSS process-power spectral density, power spectral density, time averages and ergodicity.

Basics of Queuing Theory, Characteristics of queuing systems.

Books

Texts:

- Papoulis and S.U. Pillai, Probability Random Variables and Stochastic Processes, 4/e, McGraw-Hill, 2002.
- A. Leon Garcia, Probability and Random Processes for Electrical Engineering, 2/e, Addison-Wesley, 1993.

References:

- H. Stark and J.W. Woods, Probability and Random Processes with Applications to Signal Processing, Prentice Hall, 2002.
- John J. Shynk, Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications, Wiley publications.

Evaluation details:

Midsem: 40
Endsem: 40

Classtest: 20

Basics: Set theory

Set: a collection (unordered) of objects called elements

e.g.: 1. vowels in the English alphabets {a,e,i,o,u}

2. First 7 prime numbers {2,3,5,7,11,13,17}

In general, $A=\{a_1,a_2,a_3,\ldots\ldots,a_n\}$

Representation: 1. by listing the elements

2. definition by property, using set builder notation

E={50,52,54,56,58,60}

E={x|50<=x<=60, x is an even integer}

$a_i \in A$ or $a_i \notin A$: is or is not an element

Empty or null set Φ

Universal set U (S)

Equal Sets: iff they have the same elements

e.g.: $A=\{1,2,3\}$, $B=\{2,1,3\}$

$A=B$

Subset: $B \subset A$ if every element of B is also in A

For any A, $\phi \subset A$, $A \subset A$, $A \subset S$

i) if $A \subseteq B$ and $A \neq B$, then $A \subset B$ (proper subset)

ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$

iii) if $A \subset B$ and $B \subset C$ then $A \subset C$

Cardinality: if set A contains exactly n elements where n is a non negative integer then A is finite and n is called cardinality of A

$|A|=n$

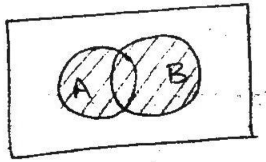
Power set: set of all subsets (P(A))

Venn diagram: geometric figures (circles, rectangles) to represent sets

Set operations:

1. Union: union of 2 sets A and B is the set that contains exactly all the elements that are in either A or B or in both.

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



eg. $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$

$A \cup B = B \cup A$ (commutative)

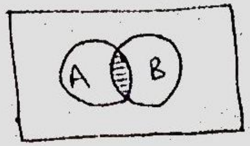
$(A \cup B) \cup C = A \cup (B \cup C)$ (associative)

if $A \subset B$, then $A \cup B = B$

$\therefore A \cup A = A$, $A \cup \phi = A$, $S \cup A = S$

2. Intersection: intersection of 2 sets A and B is the set that contains exactly all the elements that are in both A and B

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



eg. $\{1,2,3\} \cap \{3,4,5\} = \{3\}$

$A \cap B = B \cap A$, $(A \cap B) \cap C = A \cap (B \cap C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

if $A \subset B$ then $A \cap B = A$

$\therefore A \cap A = A$, $A \cap \{\phi\} = \{\phi\}$, $A \cap S = A$

3. Mutually exclusive sets (disjoint): if they have no common elements i.e., $A \cap B = \Phi$

4. Partition: A partition U of a set S is a collection of mutually exclusive subsets A_i of S whose union equals S.

$A_1 \cup A_2 \cup \cdots \cup A_n = S, A_i \cap A_j = \Phi$, $\mathrm{i} \neq j$

5. Complements: set that contains exactly all the elements that are not in A (A^c)

6. Set difference: difference of set A and B is the set that contains exactly all elements of A but not of B

$$A - B = \{x | x \in A, x \notin B\}$$

7. De Morgan's law: $(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C$

8. Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$$

Proof: De Morgan's law

i) $x \in \overline{A \cup B}$

or, $x \notin A \cup B$

or, $x \notin A$ and $x \notin B$

or, $x \in \overline{A}$ and $x \in \overline{B}$ or $x \in \overline{A} \cap \overline{B}$

ii) $x \in \overline{A \cap B}$

or, $x \notin A \cap B$

or, $x \notin A$ or $x \notin B$

or, $x \in \overline{A}$ or $x \in \overline{B}$

or, $x \in \overline{A} \cup \overline{B}$

Example:

eg. 1) if A, B, C are sets, then $(\overline{A} - B) - C = (\overline{A} - C) - (B - C)$

$x \in (\overline{A} - B) - C$

or, $x \in (\overline{A} - B)$ and $x \notin C$

or, $x \in \overline{A}$, $x \notin B$, $x \notin C$

or, $x \in (\overline{A} - C)$ and $x \notin (B - C)$

or, $x \in (\overline{A} - C) - (B - C)$

2. Let U = { 1, 2, 3, 4, 5, 6, 7, 8 } T = {2, 5} V = {1, 2, 3, 7, 8}
Find (V∩T′)′

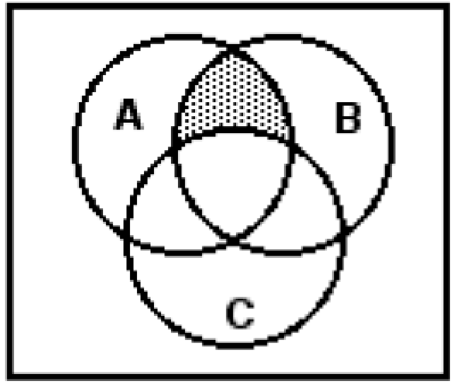
ans. {2, 4, 5, 6}

3. U = {1, 2, 3, 4, 5, 6, 7} S = {2, 4, 5} T = {3, 5, 7}

V = {2, 3, 4, 5, 7} W = {1, 2, 3, 4, 6} Find (S∩V)′ ∪ (W′∪ T)

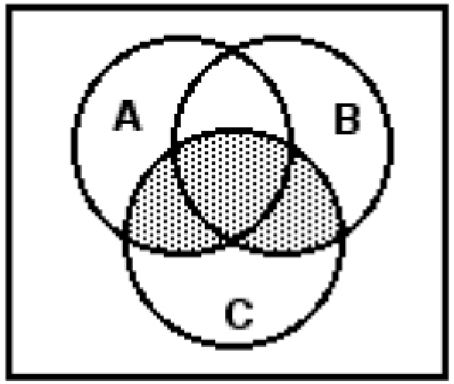
ans. {1, 3, 5, 6, 7}

4.



$A \cap (B \cap C^C)$

5.



$C \cap (A \cup B)$

PRACTICE EXERCISES

1 – 15: $U = \{b, c, d, e, f, g, h, i, j, k\}$ $S = \{b, c, d, h, i, k\}$
$T = \{b, d, e, f, h\}$ $V = \{b, d, e, f, g, i\}$

Find: **1.** S' **2.** T' **3.** V' **4.** $S \cap T$ **5.** $S \cup T$ **6.** $S \cap V$ **7.** $S \cup V$
8. $T \cap V$ **9.** $T \cup V$ **10.** $S' \cup V$ **11.** $S' \cap V$ **12.** $S' \cup T$ **13.** $S' \cap T$
14. $V' \cup T$ **15.** $V' \cap T$

16. Let $U = \{ b, c, d, e, f, g, h, i, j \}$ $V = \{e\}$ $W = \{c, f, g, j\}$ Find $(V' \cup W')'$

17. Let $U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$ $T = \{2, 3, 9\}$ $V = \{8, 9, 10\}$
Find $(T' \cup V)'$

18. Let $U = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$ $T = \{2, 5\}$ $V = \{1, 2, 3, 7, 8\}$
Find $(V \cap T')'$

19. $U = \{1, 2, 3, 4, 5, 6, 7\}$ $S = \{2, 4, 5\}$ $T = \{3, 5, 7\}$
$V = \{2, 3, 4, 5, 7\}$ $W = \{1, 2, 3, 4, 6\}$ Find $(S \cap V)' \cup (W' \cup T)$

20. $U = \{a, b, c, d, e, f, g\}$ $S = \{a, b, c, d, e, g\}$
$T = \{a, b, f, g\}$ $V = \{d\}$ $W = \{a, c, d, e, g\}$ Find $[(S \cup W) \cup T']'$

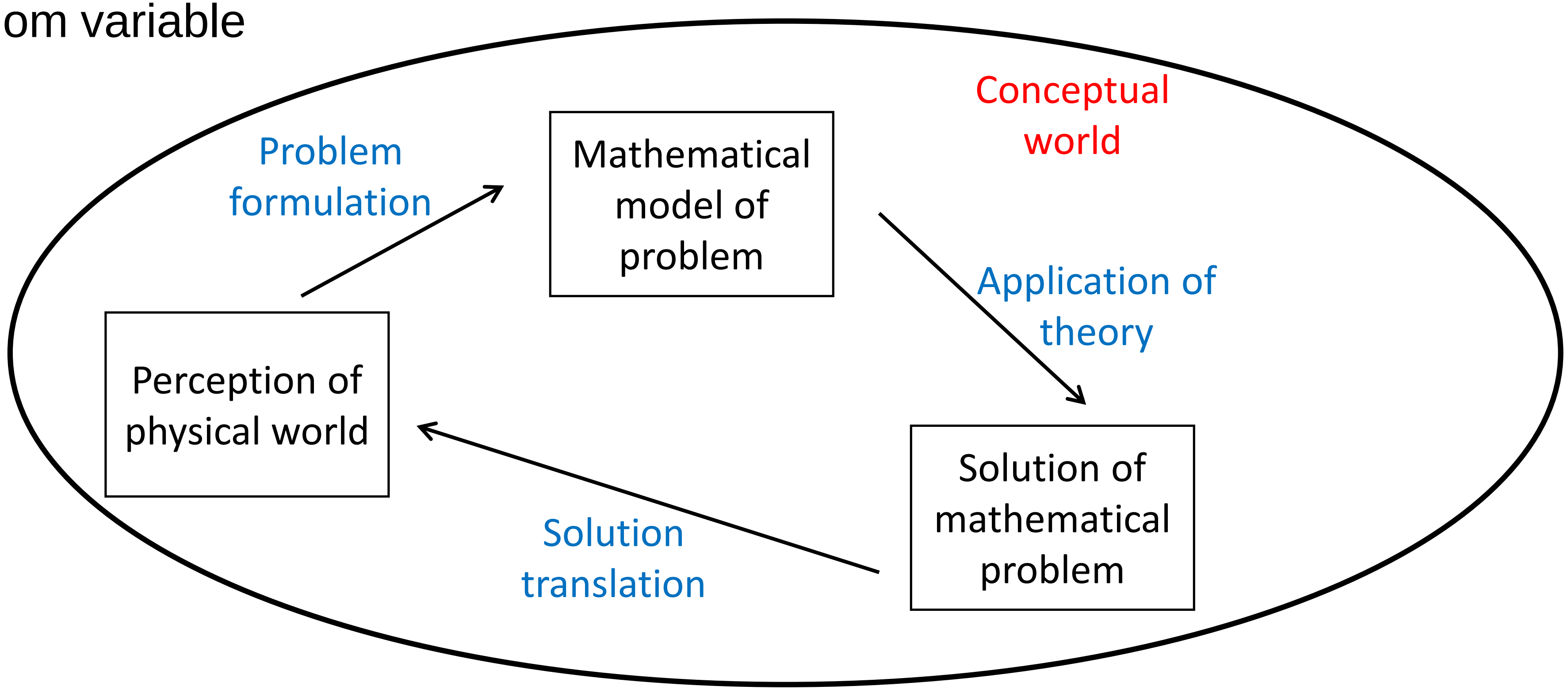
21. Let $U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$ $S = \{1, 2, 5, 6, 7\}$ $T = \{5, 6\}$
$V = \{1, 2, 3, 5, 6, 9\}$ Find $S' \cup (T \cap V)$

22 – 33: On a standard 3-circle Venn diagram, shade the region(s) corresponding to the set:

22. $(B \cup A) \cap C'$ **23.** $(C \cap B) \cup A$ **24.** $C \cup (A \cap B')$
25. $(A' \cap B) \cup (A \cap C)$ **26.** $(A' \cup B) \cup C'$ **27.** $(A \cup B)' \cap C$
28. $(A \cap B') \cup (B \cap C')$ **29.** $(C \cup A') \cap B'$ **30.** $(B \cap C') \cap A'$
31. $(B \cap C') \cup A$ **32.** $(A \cup C) \cap (A \cup B)$ **33.** $(A \cap C)' \cup B$

Probability: historical perspective-

- started with gambling and games of chance (middle of 17^{th} century)
- Works of Fermat, Pascal, Huygens – probability of stochastic events and expected value of random variable



Probability theory: 1. axiomatic definition (set theory, functions, field)

2. classical definition

3. relative frequency definition

Field: basics of probability theory

Suppose A_1, A_2, ….. A_n, …….. is an infinite sequence of sets.

Axiomatic foundations:

Definition 1.1 A collection of subsets of S is called Borel field, denoted by $\mathbb{B}$, it satisfies the following three properties:

a. $\emptyset \in \mathbb{B}$ *(the empty set is an element of* $\mathbb{B}$*).*

b. *If* $A \in \mathbb{B}$, *then* $A^c \in \mathbb{B}$ *(*$\mathbb{B}$ *is closed under complementation).*

c. *If* $A_1, A_2, \ldots \in \mathbb{B}$, *then* $\cup_{i=1}^{\infty} A_i \in \mathbb{B}$ *(*$\mathbb{B}$ *is closed under countable unions).*

If S is finite or countable, then for a given sample space S, Sigma algebra is a field that satisfies the conditions :

1. It contains complement of every subset present (considering S as universal set),
2. It contains countable number of unions of its subsets.

Definition 1.2 Given a sample space S and an associated sigma algebra ฿, a probability function is a function P with domain ฿ that satisfies

1. $P(A) \geq 0$ *for all* $A \in \mathbb{B}$.

2. $P(S) = 1$.

3. *If* $A_1, A_2, \ldots \in \mathbb{B}$ *are pairwise disjoint, then* $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

These are called the Axioms of Probability or Kolmogorov Axioms. Any function that satisfies the axioms is called a probability function.

Let, S={s_1, s_2, …, s_n} be a finite set; B be any sigma algebra of subsets of S; p_1, p_2,…., p_n be non-negative numbers that sum to 1. For any $A \in \mathcal{B}$, define $P(A) = \sum_{\{i: s_i \in A\}} p_i$, then P is a probability function. [this remains true if S is countably infinite set]

Proof: For finite S,

For any $A \in \mathbb{B}$, $P(A) = \sum_{i: s_i \in A} p_i \geq 0,$ [every p_i>=0]

Thus, axiom 1 is true.

$P(S) = \sum_{\{i:s_i \in S\}} p_i = \sum_{i=1}^{n} p_i$=1 (Axiom 2 is true)

Let, A_1, A_2, ….. A_k denote pairwise disjoint events.

$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{\{j:s_j \in \bigcup_{i=1}^{k} A_i\}} p_j = \sum_{i=1}^{k} \sum_{i\{j:s_j \in A_i\}} p_j = \sum_{i=1}^{k} P(A_i)$ (Axiom 3 is true)

[because of disjointedness of A_i's , p_j's will appear exactly once in the second equality]

So P(A) is probability function.

Defining probability:

1. game of dart (throwing the dart on the board and receiving a score)

P(scoring i points)=(area of region i)/(area of dart board)

2. A box contains m white and n black balls. Balls are drawn at random one at a time without replacement. Find the probability of drawing a white ball by k-th draw.

Calculus of probability:

Theorem 1: *If P is a probability function and A is any set in $\mathbb{B}$, then*

a. $P(\emptyset) = 0$, *where* $\emptyset$ *is the empty set.*

b. $P(A) \leq 1$.

c. $P(A^c) = 1 - P(A)$.

Theorem 2: *If P is a probability function and A and B are any sets in $\mathbb{B}$, then*

a. $P(B \cap A^c) = P(B) - P(A \cap B)$.

b. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

c. *If* $A \subset B$, *then* $P(A) \leq P(B)$.

From b, $P(A) + P(B) - P(A \cap B) \leq 1$

or, $P(A \cap B) \geq P(A) + P(B) - 1$

This is called special case of Bonferroni's inequality

Theorem 3: If P is a probability function, then

a) $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partition $C_1, C_2, \ldots$..

b) $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ for any sets $A_1, A_2, \ldots$.....

Proof: a)

proof: since $C_1, C_2, \ldots$ form a partition,

$\therefore C_i \cap C_j = \phi \;\; \forall i \neq j$ and $S = \bigcup_{i=1}^{\infty} C_i$

$A = A \cap S = A \cap \left(\bigcup_{i=1}^{\infty} C_i\right) = \bigcup_{i=1}^{\infty} (A \cap C_i)$

since C_i's are disjoint, $A \cap C_i$ are also disjoint

$\therefore P(A) = P\left(\bigcup_{i=1}^{\infty} (A \cap C_i)\right) = P\left(\sum_{i=1}^{\infty} (A \cap C_i)\right)$

$= \sum_{i=1}^{\infty} P(A \cap C_i)$ (hence proved)

Proof: b) Consider a disjoint collection $A_1^*, A_2^*, \ldots$. with the property that

$$\bigcup_{i=1}^{\infty} A_i^* = \bigcup_{i=1}^{\infty} A_i$$

where $A_1^* = A_1$, $A_i^* = A_i \backslash \left(\sum_{j=1}^{i-1} A_j\right)$ i=2,3,……. ; [A\B=A-B]

So, $P(\bigcup_{i=1}^{\infty} A_i) = P(\bigcup_{i=1}^{\infty} A_i^*) = \sum_{i=1}^{\infty} P(A_i^*)$ (A_i^* are disjoint)

$$A_i^* \cap A_k^* = \{A_i \setminus (\bigcup_{j=1}^{i-1} A_j)\} \cap \{A_k \setminus (\bigcup_{j=1}^{k-1} A_j)\}$$

$$= \{A_i \cap (\bigcup_{j=1}^{i-1} A_j)^c\} \cap \{A_k \cap (\bigcup_{j=1}^{k-1} A_j)^c\}$$

$$= \{A_i \cap (\bigcap_{j=1}^{i-1} A_j^c)\} \cap \{A_k \cap (\bigcap_{j=1}^{k-1} A_j^c)\} \quad \text{[De Morgan's Law]}$$

if $i > k$, the first term is a subset of A_k^c whereas the second term is not a subset of A_k^c.

$\therefore A_i^* \cap A_k^* = \Phi$ [for $i < k$, similar argument holds true ($\not\subset A_i^c$ and $\subset A_i^c$)]

By construction $A_i^* \sqsubseteq A_i$

$$P(A_i^*) \leq P(A_i)$$

So, $\sum_{i=1}^{\infty} P(A_i^*) \leq \sum_{i=1}^{\infty} P(A_i)$

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i^*) \leq \sum_{i=1}^{\infty} P(A_i)$ **Boole's inequality**

If we apply Boole's inequality to A^c, then

$$P\left(\bigcup_{i=1}^{n} A_i^c\right) \leq \sum_{i=1}^{n} P(A_i^c)$$

or, $P\left(\bigcap_{i=1}^{n} A_i\right)^c \leq \sum_{i=1}^{n} P(A_i^c)$ [∵ De Morgan's law]

or, $1 - P\left(\bigcap_{i=1}^{n} A_i\right) \leq n - \sum_{i=1}^{n} P(A_i)$ [∵ $P(A^c) = 1 - P(A)$]

or, $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - n + 1$

or, $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$ —— generalized Bonferroni's inequality

Counting: used in finite sample spaces to construct probability assignment
If a job consists of k separate tasks, the i-th of which can be done in n_i ways (i=1,2,…..,k) then the entire job can be done in (n_1 x n_2 x ……. x n_k) ways

2 types of counting- with replacement, without replacement

Number of possible arrangements of size r from n objects:

	ordered	unordered
Without replacement	$n!/(n-r)!$	${}^{n}C_{r}$
With replacement	n^r	${}^{n+r-1}C_{r}$

Example 1. Four cards are drawn at random from a pack of 52 cards. Find the number of instances for

a) One king, one queen, one jack and one ace
b) Two kings, two queens
c) Two from hearts, two from spades

2. Sample space for random toss of 2 coins or rolling 2 dice.

[Cartesian product]

Classical definition:

1. Random experiment: an act whose results depend on chance and which can be repeated under some given condition

Like: tossing a coin, throwing a die, drawing cards from a pack etc..

2. Outcome: results of a random experiment

head, tail in tossing a coin; 1,2,3,4,5,6 in throwing a die; king, queen, ……. in drawing cards from a pack

3. Event: any phenomenon which occurs in a random experiment

i. elementary, ii. Composite

Elementary: obtaining 2 on the top surface of a die

Composite: obtaining an even number on the top surface of a die (i.e., 2 or 4 or 6)

4. Mutually exclusive events: none of them can occur simultaneously

Like head and tail, king and queen

5. Exhaustive set: if at least one of them must necessarily occur

Complete group of all possible elementary events of a random experiment

6. Trial: any particular performance of the random experiment

7. Equally likely: after considering all relevant evidence, none of the outcomes can be expected in preference to another

Classical definition: If a random experiment has n mutually exclusive, exhaustive and equally likely outcomes and m of these are favourable to event A, then probability of event A is

$$P(A)=m/n$$

$$0 \leq P(A) \leq 1$$

P(A)=0 means impossible event

P(A)=1 means certain or sure event

Limitations :

i) Based on the feasibility of subdividing the possible outcomes into mutually exclusive, exhaustive and equally likely events

ii) Circular in nature (equally likely means equally probable)

iii) Applicable for a limited class of problems

iv) Inappropriate for countably infinite possible outcomes

Examples: 1. Four cards are drawn at random from a full pack. What is the probability that they belong to a) 4 different suits, b) different suits and denominations? [drawn one by one without replacement]

(n=$^{52}C_4$, m = 52 x 39 x 26 x 13 for a); m= 52 x 36 x 22 x 10 for b))

2. One urn contains 2 white, 2 black balls; a second urn contains 2 white , 4 black balls.

i) If one ball chosen from each urn, what is the probability that they will be of the same colour?

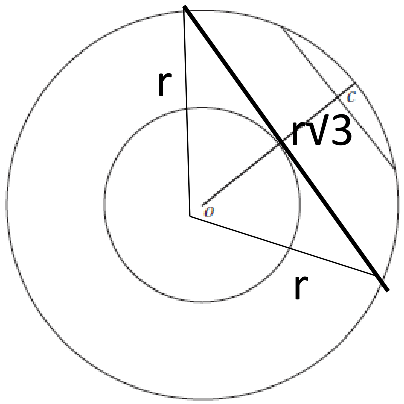
ii) If an urn is selected at random and one ball is drawn from it, what is the probability that it will be a white ball?

[i) either both white or both black, so (2/4 x 2/6) + (2/4 x 4/6) = ½
ii) (½ x 2/4) + (1/2 x 2/6) = 5/12]

Bertrand paradox:

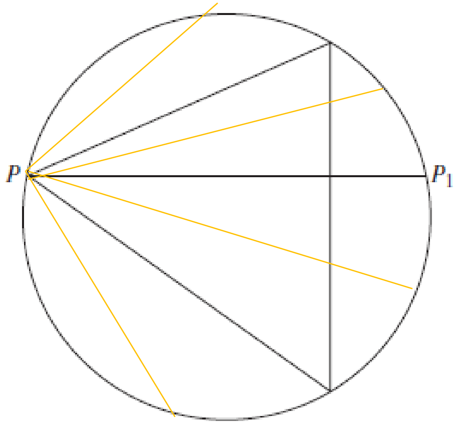
Given a circle C of radius r, calculate the probability p that the length l of a randomly selected chord AB is greater than the length r√3 of the inscribed equilateral triangle.
This problem can be solved in three ways depending on definition of random.

1.



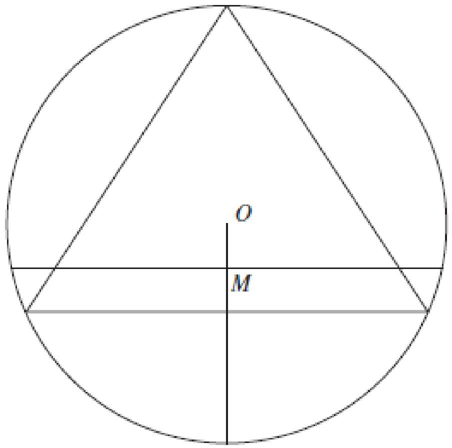
$P(A) = \pi(r/2)^2/\pi r^2 = 1/4$

2.



$P(A) = 1/3$

3.



$P(A) = 1/2$

Three different solutions for the same problem.
(demonstrate the ambiguities associated with classical definition)

Classical definition acts as working hypothesis
(equally likely: sometimes self-evident, sometimes assumed)

Relative Frequency definition:

Probability P(A) of an event A is the limit

$$P(A) = \lim_{n\to\infty}\left(\frac{n_A}{n}\right)$$

n_A: number of occurrences of event A, n: number of trials

Theorem of total probability:

If events A and B are mutually exclusive, then P(A+B)=P(A)+P(B)

Deductions from total prob. theorem:

1. $P(A_1+A_2+A_3+\ldots+A_k)=P(A_1)+P(A_2)+P(A_3)+\ldots+P(A_k)$
2. $P(A^c)=1-P(A)$
3. P(A+B)=P(A)+P(B)-P(AB)

Conditional probability:

P(A|B) : probability of occurrence of event A when B has already occurred (when P(B)>0)

P(A|B)=P(AB)/P(B)

If B is a subset of A, then P(A|B) =1

If A is a subset of B, then P(A|B)>P(A)

Conditional probability is a probability function as
i) P(A|B)>=0
ii) P(S|B)=1 (B is a subset of S)
iii) $P(A \cup C|B) = \frac{P\left((A \cup C) \cap B\right)}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)} = P(A|B) + P(C|B)$

Theorem of compound probability:
P(AB)=P(A)P(B|A)

Proof: suppose, n: mutually exclusive, exhaustive, equally likely outcomes of a random experiment
Out of those, m_1:favourable to event A, m_2: favourable to event AB
P(A)=m_1/n
P(B|A)=m_2/m_1
P(AB)=m_2/n
P(A)P(B|A)=(m_1/n)(m_2/m_1)=m_2/n

can be extended to several events like P(ABC)=P(A)P(B|A)P(C|AB)

P(A)P(B|A)P(C|AB)=P(A)[P(AB)/P(A)][P(ABC)/P(AB)]=P(ABC)

Example: A box contains 3 white balls w1, w2, w3 and 2 red balls r1, r2. Draw 2 balls at random (one by one without replacement). Find probability of first ball is white and second is red?

$P(W1R2) = P(W1)P(R2|W1)=(3/5)\ (2/4)=3/10$

Deductions from compound prob theorem:

a) Event B may be associated with the occurrence or non-occurrence of another event A as

$B \cap A, B \cap A^c$ (mutually exclusive events)

$P(B)=P(AB)+P(A^cB)$

$P(B)=P(A)P(B|A)+P(A^c)P(B|A^c)$

b) $P(AB)=P(A)P(B|A)=P(B)P(A|B)$

Independent events: several events are considered to be statistically independent if the probability of occurrence of any of them remains unaffected by the supplementary knowledge regarding occurrence or non-occurrence of any number of remaining events.

$P(B)=P(B|A)=P(B|A^c)$

Now, $P(B|A)=P(AB)/P(A)$

or, $P(B)=P(AB)/P(A)$

$P(AB)=P(A)P(B)$

If A and B are independent, then A^c, B and A^c, B^c are also independent

$B=AB+ A^cB$

Or, $P(B)=P(AB)+P(A^cB)$

or, $P(A^cB)=P(B)-P(A)P(B) = P(B)[1-P(A)] = P(B)P(A^c)$

$(A^cB^c) + (A+B) =1$

$P(A^cB^c) +P(A+B)=1$

Or, $P(A^cB^c) + P(A) +P(B) +P(AB)=1$

Or, $P(A^cB^c) = 1- P(A) +P(B) +P(AB) = 1- P(A) - P(B) + P(A)P(B) = [1- P(A)] [1– P(B)] = P(A^c) P(B^c)$

Like tossing a coin twice and obtaining head in first toss and tail in second toss are independent events.

2 dice are thrown and the events six on die 1 and six on die 2 are independent.

Example: given $P(A)=3/8$, $P(B)=5/8$, $P(A+B)=3/4$. find $P(A|B)$, $P(B|A)$. Are A and B independent?

$P(A+B)=P(A)+P(B)-P(AB)$

$P(AB)=3/8+5/8-3/4=1/4$

$P(A|B)=(1/4)/(5/8)=2/5$

$P(B|A)=(1/4)/(3/8)=2/3$

$P(A)P(B)=15/64$

So, A, B are not statistically independent

Odds ratio:

Odds in favour of event A are a:b   P(A)=a/(a+b)

Odds against event A are a:b   P(A)=b/(a+b)

Total probability and Baye's theorem:

If U=[A_1, A_2, ……., A_n] is a partition of S and B is an arbitrary event, then

P(B)=P(B|A_1)P(A_1)+………+P(B|A_n)P(A_n)

$$B = B \cap S = B \cap (A_1 \cup \cdots \cup A_n) = (B \cap A_1) \cup \cdots \cup (B \cap A_n)$$

P(B)=P$(B \cap A_1) + \cdots + P(B \cap A_n)$

P(B)=P(B|A_1)P(A_1)+………+P(B|A_n)P(A_n)   [total prob]

Since, P(BA_i)=P(A_i|B)P(B)

So, P(A_i|B)=P(B|A_i)P(A_i)/P(B)

Substituting P(B),

likelihood   a-priori

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$$

posteriori   evidence

An event B can occur only if one of the mutually exclusive and exhaustive set of events A_1, A_2,…., A_n occurs. Then conditional prob of a specified event A_i when B has already occurred is given by Baye's theorem.

Example: 1. A certain test for a particular disease is known to be 95% accurate. A person submits to the test and the results are positive. Suppose the person came from a population of 100000 where 2000 people suffer from that disease. What can we conclude about the probability that the person under test has the particular disease?

T: test positive, D: disease, N: test negative, H: healthy

$P(T|D)=0.95$, $P(N|H)=0.95$, $P(D)=0.02$,

$P(N|D)=0.05$, $P(T|H)=0.05$, $P(H)=0.98$

$P(D|T)=[P(T|D)P(D)]/[P(T|D)P(D)+P(T|H)P(H)]=0.278$

2. Four boxes have components as follows.

Box 1: 2000 components, 5% defectives

Box 2: 500 components, 40% defectives

Box 3, 4: 1000 components each, 10% defectives

A box is selected randomly and a component is drawn. The selected component is defective. What is the probability that it came from box 1?

3. A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd?

Let, A_i=head appears at i-th toss for first time

$$\{\underbrace{T,T,T,\ldots\ldots..,T}_{i-1},H\}$$

$P(A_i)=P(T)P(T)\ldots\ldots P(T)P(H) = q^{i-1}p$

(each trial is independent, $P(H)=p$, $P(T)=q=1-p$)
P(head appears on an odd toss)=$P(A_1+A_3+A_5+\ldots\ldots\ldots)$

$$= \sum_{i=1}^{\infty} P(A_{2i+1}) = \sum_{i=1}^{\infty} q^{2i}p = p\sum_{i=1}^{\infty} q^{2i} = \frac{p}{1-q^2} = \frac{1}{1+q} = \frac{1}{2-p}$$

[for an unbiased coin, prob of obtaining first head on an odd toss is 2/3]

Theorem 1: If $A_1 \subset A_2 \subset \cdots$,is an increasing sequence of events, then $P(\bigcup_k A_k) = \lim_{n\to\infty} P(A_n)$

Proof: let, $B_1=A_1$, $B_2=A_2 - A_1 = A_2A_1^c$, $B_n=A_n-(B_1+B_2+\ldots+B_{n-1})$ are mutually exclusive.

$$\bigcup_{i=1}^{k} B_k = A_n$$

$$P\left(\bigcup_k A_k\right) = P\left(\bigcup_k B_k\right) = \sum_k P(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} B_k) = \lim_{n\to\infty} P(A_n)$$

Theorem 2: If $A_1 \supset A_2 \supset \cdots$,is a decreasing sequence of events, then $P(\cap\, k\; A_k) = \lim_{n\to\infty} P(A_n)$

Proof: consider the complementary events $\overline{A_1} \subset \overline{A_2} \subset \cdots$

$$P(\cap\, k\; A_k) = 1 - P\left(\bigcup_k \overline{A_k}\right) = 1 - \lim_{n\to\infty} P(\overline{A_n}) = \lim_{n\to\infty} [1 - P(\overline{A_n})] = \lim_{n\to\infty} P(A_n)$$

(De Morgan's law)

Theorem 3: in case of arbitrary events,
$P(\cup_k A_k) \le \sum_k P(A_k)$ (Boole's inequality)

Example: 4. The odds against student X solving a business statistics problem are 8 to 6; and the odds in favour of student Y solving the same problem are 14 to 16. What is the chance that the problem will be solved if they both try, independently of each other? What is the probability that neither solves the problem?
P(X solves the prob)=P(A)=6/14
P(Y solves the prob)=P(B)=14/30
P(A+B)=P(A)+P(B)-P(A)P(B)=73/105
$P(A^cB^c)=P(A+B)^c=1-P(A+B)=32/105$

Example: 5. An experiment succeeds twice as often as it fails. What is the prob that in the next 6 trials there will be at least 4 successes?

6. A box contains 7W and 5B balls. If 3 balls are drawn simultaneously at random, what is the probability that they are not all of the same colour? Also calculate the prob for drawing with replacement.

Either 2W1B or 1W2B

For 2W1B, prob = $({}^{7}C_2 \times {}^{5}C_1)/{}^{12}C_3$ =21/44

For 1W2B, prob = $({}^{7}C_1 \times {}^{5}C_2)/{}^{12}C_3$ =14/44

Total prob = 35/44

For 2W1B, prob = 3 x 7/12 x 7/12 x 5/12

For 1W2B, prob = 3 x 7/12 x 5/12 x 5/12

Total prob = 35/48