Unit 5: Graph Theory

Topic 2: Connectivity of Graphs

Outline

Introduction

Many problems can be modeled with paths formed by traveling along the edges of graphs.

For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model.

Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

Path

Definition

Let n be a nonnegative integer and G an undirected graph. A **path of length** n from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, 2, \ldots, n$, the endpoints x_{i-1} and x_i.

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When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$.

The path is a **circuit (cycle)** if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

The path or circuit is said to pass through the vertices $x_1, x_2, \dots, x_{n-1}$ or traverse the edges $e_1, e_2, \dots, e_n$.

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Paths and Circuits also can be used to find isomorphism or non-isomorphism of graphs.

Outline

Connected Graph

Definition

An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not connected is called **disconnected**.

Any two computers in a network can communicate if and only if he graph of the network is connected.

Disconnected Graph/Connected Component

We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph. Such vertices and edges are called **cut vertices (or articulation points)** and **cut edge (or bridge)**, respectively.

Definition

A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G.

A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Number of paths of definite length in a graph

Theorem

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of A^r.

Depending on the number of times a path or circuit crosses a vertex or passes through an edge, the paths or circuits are divided in two categories:

1. Euler Path and Euler Circuit:
2. Hamilton Path and Hamilton Circuit:

Outline

Euler Path

Definition

An **Euler path** in a graph G is a simple path containing every edge of G.

Theorem

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Using this, we can say that $K_n (n > 2)$, C_n never have an Euler Path. However $K_{m,n}$ has an Euler path if and only if $m = 2$ and n is odd or $n = 2$ and m is odd.

Euler circuit

Definition

An **Euler circuit** in a graph G is a simple circuit containing every edge of G.

Theorem

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

It is easy to verify that K_n has an Euler circuit if and only n is odd. C_n always has an Euler circuit. $K_{m,n}$ has an Euler circuit if and only if both m, n are even.

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Hamilton Path and Hamilton Circuit

Definition

A simple path in a graph G that passes through every vertex exactly once is called a **Hamilton path**.

Definition

A simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**.

Hamilton Path and Hamilton Circuit

There is no known necessary and sufficient condition for existence of Hamilton path and Hamilton Circuit in a graph. However, there are certain facts about them.

1. A graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit.
2. If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
3. **Dirac's theorem:** If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton circuit.
4. **Ore's theorem:** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Outline

Shortest Path Problem

Many problems can be modeled using graphs with weights assigned to their edges. Graphs that have a number assigned to each edge are called **weighted graphs**. Communications costs (such as the monthly cost of leasing a telephone line), the response times of the computers over these lines, or the distance between computers, can all be studied using weighted graphs.

Determining a path of least length between two vertices in a network is one such problem. To be more specific, the length of a path in a weighted graph be the sum of the weights of the edges of this path. There are several different algorithms that find a shortest path between two vertices in a weighted graph. One of such algorithms is the **Dijkstra's algorithm**.

Dijkstra's algorithm

It begins by labeling a with 0 and the other vertices with ∞. We use the notation $L_0(a) = 0$ and $L_0(v) = \infty$ for these labels before any iterations have taken place. Let S_k denote this set after k iterations of the labeling procedure. We begin with $S_0 = \phi$. The set S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with the smallest label. Once u is added to S_k, we update the labels of all vertices not in S_k, so that $L_k(v)$, the label of the vertex v at the kth stage, is the length of a shortest path from a to v that contains vertices only in S_k (that is, vertices that were already in the distinguished set together with u). Let v be a vertex not in S_k. To update the label of v, note that $L_k(v)$ is the length of a shortest path from a to v containing only vertices in S_k. The updating can be carried out efficiently when this observation is used: A shortest path from a to v containing only elements of S_k is either a shortest path from a to v that contains only elements of S_{k-1} (that is, the distinguished vertices not including u), or it is a shortest path from a to u at the $(k-1)$st stage with the edge $\{u, v\}$ added. In other words,

$$L_k(a, v) = \min\{L_{k-1}(a, v), L_{k-1}(a, u) + w(u, v)\},$$

where $w(u, v)$ is the length of the edge with u and v as endpoints. This procedure is iterated by successively adding vertices to the distinguished set until z is added. When z is added to the distinguished set, its label is the length of a shortest path from a to z.

Thank You

Any Question!!!