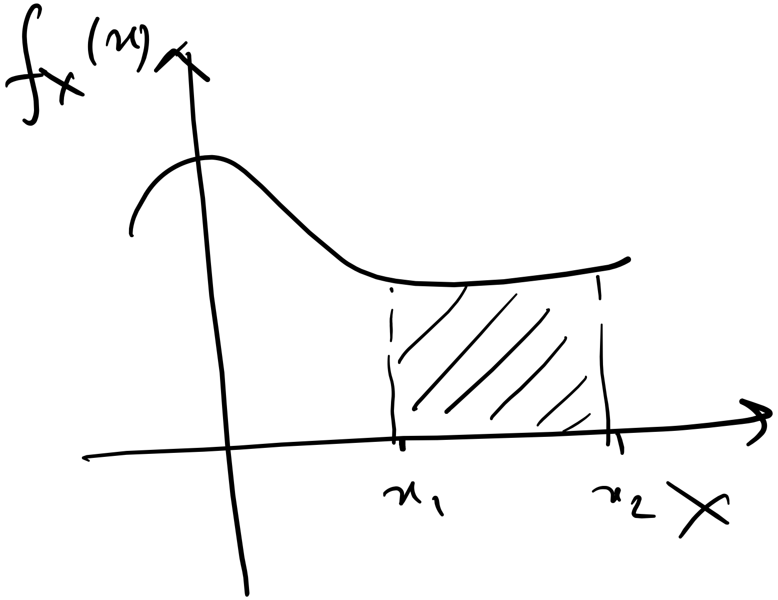
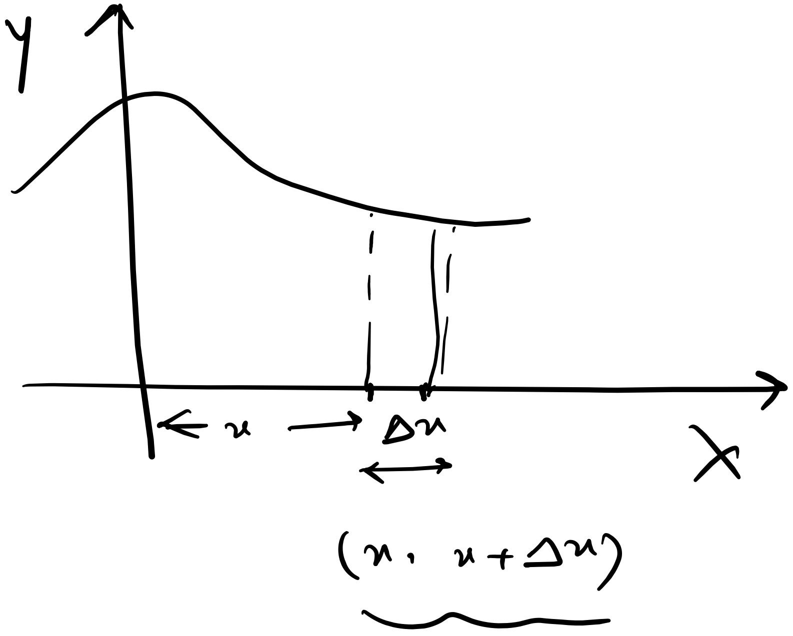
Conditional Distributions

Relationship Between Probability and Joint PDF

Let X be a continuous RV with PDF $f_X(x)$, then

1. $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx$
2. $P(x \leq X \leq x + \Delta x) \approx f_X(x)\Delta x$





Let (X,Y) be a joint RV with joint PDF $f_{X,Y}(x,y)$, then

1. $P(x_1 \le X \le x_2,\ y_1 \le Y \le y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dx\,dy$

2. $P(x_1 < X \le x_2,\ y \le Y \le y+\Delta y) = \int_{x_1}^{x_2} f_{X,Y}(x,y)\cdot\Delta y\cdot dx$

3. $P(x < X \le x+\Delta x,\ y_1 < Y \le y_2) = \int_{y_1}^{y_2} f_{X,Y}(x,y)\,\Delta x\,dy$

4. $P(x < X \le x+\Delta x,\ y < Y \le y+\Delta y) = f_{X,Y}(x,y)\cdot\Delta x\cdot\Delta y$

Case-1: When X and Y are discrete RVs

Conditional Probability Mass Function (PMF)

Suppose X and Y are two discrete jointly RV with joint PMF $p_{X,Y}(x,y)$. The conditional PMF of Y given $X = x$ is denoted by $p_{Y|X}(y|x)$ and defined as

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \text{ provided } p_X(x) \neq 0.$$

Proof:-

$$p_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$$= \frac{P(Y=y,\ X=x)}{P(X=x)}$$

$$= \frac{p_{X,Y}(x,y)}{p_X(x)}$$

A & B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Condition for Independence

$$\underbrace{p_{Y|X}(y|x)} = p_Y(y)$$

$$\Rightarrow \quad \frac{p_{X,Y}(x,y)}{p_X(x)} = p_Y(y)$$

$$\Rightarrow \quad \boxed{p_{X,Y}(x,y) = p_X(x) \cdot \times p_Y(y)}$$

A & B

$$P(A|B) = P(A)$$

Bayes' Rule

Given $p_X(x)$ and $p_{Y|X}(y|x)$, find $p_{X|Y}(x|y)$?

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \frac{p_{X,Y}(x,y)}{\sum_{x \in R_X} p_{X,Y}(x,y)}$$

A & B

$P(B)$

$P(A|B)$

$P(B|A)$

$\Downarrow$

Example 1: Consider the RVs X and Y with the joint PMF as presented in the following table.

Find $p_{Y|X}(0|1)$ and $p_{X|Y}(0|1)$?

Y \ X	0	1	2
0	0.25	0.10	0.15
1	0.14	0.35	0.01

Sol:-

$$p_{Y|X}(0|1) = \frac{p_{X,Y}(1,0)}{p_X(1)} = \frac{0.10}{0.45} = 2/9$$

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{0.14}{0.50} = 7/25$$

Case-2: When X and Y are Continuous RVs

Conditional distribution Function

We can not define the conditional CDF of RV Y on the condition of the event $\{X = x\}$ by the relation

$$F_{Y|X}(y|x) = P(\{Y \leq y\}|\{X = x\})$$

$$= \frac{P(\{Y \leq y\},\{X = x\})}{P(\{X = x\})}$$

as P($\{X = x\}$) = 0 in the above expression.

X - Contn

$P(X = x_1) = 0$

The conditional distributed function is defined in the limiting case as follows:

$(\cancel{x + \Delta x})$

$(x, x+\Delta x)$

$$F_{Y|X}(y|x) = \lim_{\Delta x \to 0} P(Y \le y | x < X \le x + \Delta x)$$

$$= \lim_{\Delta x \to 0} \frac{P(Y \le y,\ x < X \le x + \Delta x)}{P(x < X \le x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{-\infty}^{y} f_{X,Y}(x, u) \cdot \Delta x \, du}{f_X(x) \cdot \Delta x}$$

$$= \frac{\int_{-\infty}^{y} f_{X,Y}(x, u) \, du}{f_X(x)}$$

Similarly,

$$F_{X|Y}(x|y) = \frac{\int_{-\infty}^{x} f_{X,Y}(u,y)\, du}{f_Y(y)}$$

Conditional Probability Density Function

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\left\{ \frac{d f(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right.$$

Proof:-

$$f_{Y|X}(y|x) = \frac{d}{dy} F_{Y|X}(y|x)$$

$$= \lim_{\Delta y \to 0} \frac{F_{Y|X}(y+\Delta y \mid X = x) - F_{Y|X}(y|x)}{\Delta y}$$

$$\boxed{\begin{aligned} &P(y < Y \leq y+\Delta y) \\ &= F_Y(y+\Delta y) \\ &\quad - F_Y(y) \end{aligned}}$$

$$= \lim_{\substack{\Delta x \to 0, \\ \Delta y \to 0}} \frac{F_{Y|X}(y+\Delta y \mid x < X \leq x+\Delta x) - F_{Y|X}(y \mid x < X \leq x+\Delta x)}{\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P(y < Y \leq y+\Delta y,\ x < X \leq x+\Delta x)}{\Delta y \cdot P(x < X \leq x+\Delta x)}$$

Condition for Independence

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f_{X,Y}(x,y)\,\Delta x\,\Delta y}{\Delta y \cdot \underline{f_X(x)\,\Delta y}}$$

$$\boxed{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

Independence:-

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\Rightarrow \boxed{f_{Y|X}(y|x) = f_Y(y)}$$

$$\frac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y) \Rightarrow \boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

Bayes' Rule for Continuous RVs

Given $f_X(x)$ and $f_{Y|X}(y|x)$, find $f_{X|Y}(x|y)$?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{+\infty} f_{X,Y}(x,y) \cdot dx}$$

$$\left\{ \begin{array}{l} f_X(x) \\ f_{Y|X}(y|x) \\ f_{X,Y}(x,y) \end{array} \right.$$

Example 2: For RVs X and Y, the joint PDF is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1+xy}{4}; & |x| \le 1, |y| \le 1 \\ 0; & Otherwise \end{cases}$$

Find $f_X(x), f_Y(y)$, and $f_{Y|X}(y|x)$. Are X and Y independent?

Sol:-

$$f_X(x) = \int_{-1}^{+1} f_{X,Y}(x,y)\,dy = \int_{-1}^{+1} \frac{1+xy}{4}\,dy = 1/2$$

$$f_Y(y) = \int_{-1}^{+1} \frac{1+xy}{4}\,dx = \frac{1}{4}\left\{ x + y\,x^2/2 \right\}\Big|_{-1}^{+1} = \frac{1}{4}\left\{ 1 + y \cdot \cancel{1/2} - (-1) - \cancel{y/2} \right\}$$

$$= 1/2$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{\frac{1+xy}{4}}{1/2} & ; |x| \le 1, \\ & |y| \le 1 \\ 0 & ; o.w. \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1+xy}{2} ; & |x| \le 1 , |y| \le 1 \\ 0 ; & o.w. \end{cases}$$

$$f_{X|Y}(x|y) = ?$$

$$\boxed{f_{Y|X}(y|x) \neq f_Y(y)}$$

X and Y are dependent RVs.