

Classical Mechanics

Dynamics of system of particles

Dr. Archana Kushwaha

Dr. A. Kushwaha

1

- Earlier, we have studied, the motion of single particle which is represented as a *point mass*, m .
- Now, we will extend our study to the dynamics of a *system of particles*.

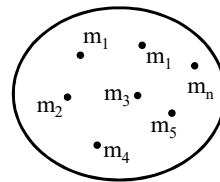
What is a system of particles?



System of particles or extended objects : **A group of inter-related particles.**

• m

single particle



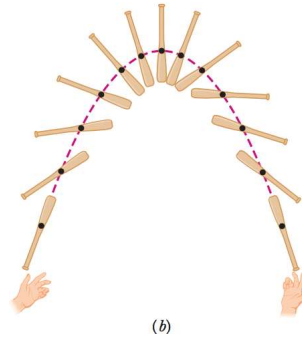
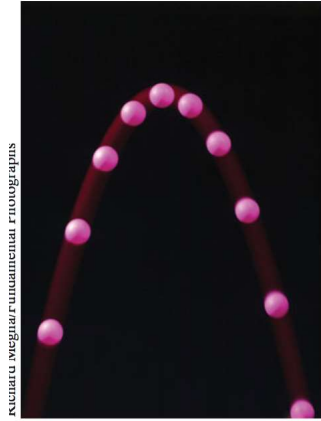
System of particles

Dr. A. Kushwaha

2

System of Particles

System of particles or extended objects : A group of inter-related particles.



❖ How to characterize the system of particles?

Dr. A. Kushwaha

Fig: 3 Fundamentals of Physics, Resnick et al., 3

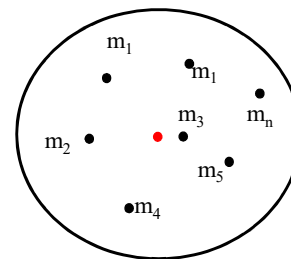
Center of Mass

The center of mass of a system of particles is the point that moves as though:

- all of the mass were concentrated there.
- all external forces were acting at that point.

In other words:

The centre of mass of a SOP is a point , where the total mass of the body is assumed to be concentrated virtually and all external forces are supposed to be applied over there.



Dr. A. Kushwaha

4

Center of Mass

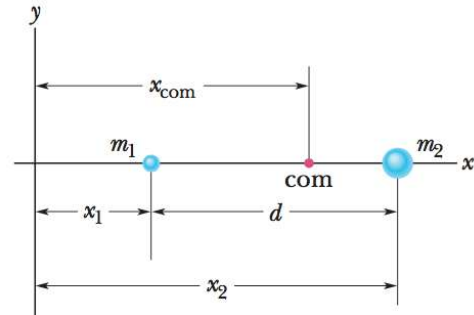
For a two particle system along x-axis:

The location of center of mass is:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

If $m_1 = m_2$,

$$x_{com} = \frac{x_1 + x_2}{2} \text{ (geometrical center)}$$



Dr. A. Kushwaha

5

Center of Mass

Generalizing to n-particles (along x-axis):

Total mass of the system $M = m_1 + m_2 + \dots + m_n$

The location of center of mass is

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$x_{com} = \frac{1}{M} \sum_{i=1}^n m_i x_i$$

Dr. A. Kushwaha

6

Center of Mass

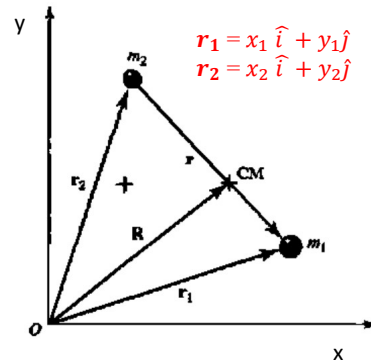
For particles distributed in 2D:

If the two particles are not along x-axis:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{com} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$\mathbf{R} = x_{com} \hat{i} + y_{com} \hat{j}$$



Dr. A. Kushwaha

7

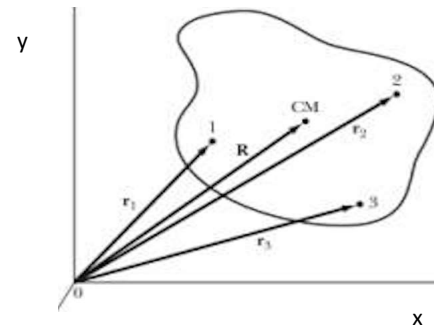
Center of Mass

3-particle system:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$y_{com} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$\mathbf{R} = x_{com} \hat{i} + y_{com} \hat{j}$$



Dr. A. Kushwaha

Ref: Classical Dynamics of Particles & Systems, Marion et al.,

8

Center of Mass

For an n-particle system(in 3D):

$$x_{com} = \frac{\sum m_i x_i}{M}$$

$$y_{com} = \frac{\sum m_i y_i}{M}$$

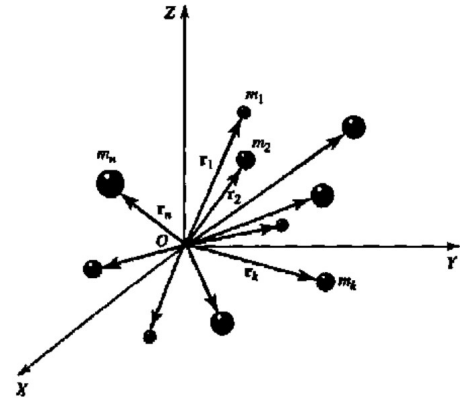
$$z_{com} = \frac{\sum m_i z_i}{M}$$

The position vector of the center of mass is:

$$\mathbf{R} = x_{com} \hat{i} + y_{com} \hat{j} + z_{com} \hat{k}$$

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M}$$

(position vector of the i^{th} particle: $\mathbf{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$)



Dr. A. Kushwaha

Ref: Classical Dynamics of Particles & Systems, Marion et al., 9

Center of Mass

Note:

- ❖ The position of center of mass of a system of particles is independent of origin.
- ❖ If origin is chosen at the center of mass,

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M} = 0,$$

$$\sum m_i \mathbf{r}_i = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \cdots + m_n \mathbf{r}_n = 0$$

$$\therefore \text{sum of moments} = 0$$

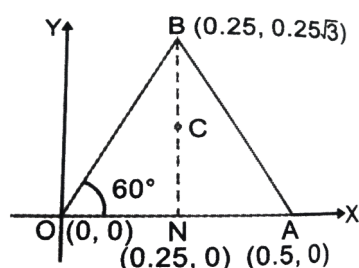
Moment of masses about the origin is zero.

Dr. A. Kushwaha

10

Example-1

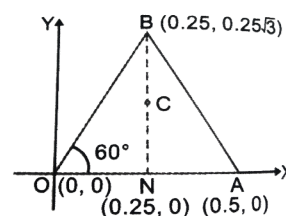
Find the center of mass of three particles at the vertices of an equilateral triangle. The masses of the particle are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m along.



Dr. A. Kushwaha

11

Particles	mass	x-coordinate	y-coordinate
O	100g	0	0
A	150g	0.5	0
B	200g	0.25	$0.25\sqrt{3}$



$$X = \frac{\sum_i m_i x_i}{m_i} = \frac{5}{18};$$

$$Y = \frac{\sum_i m_i y_i}{m_i} = \frac{1}{3\sqrt{3}}$$

Coordinates of COM, $R = \left(\frac{5}{18}, \frac{1}{3\sqrt{3}} \right)$.

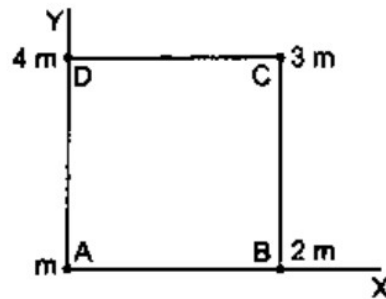
If " $m_1 = m_2 = m_3$, center-of-mass coincides with the centroid of the triangle (geometrical center)"

Dr. A. Kushwaha

12

Example - 2

Four particles A, B, C and D having masses m , $2m$, $3m$ and $4m$ respectively are placed in order at the corners of a square of side a . Locate the center of mass.



Dr. A. Kushwaha

13

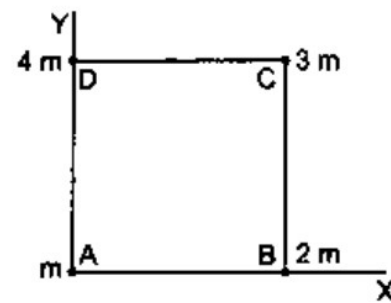
Particle	mass	x-coordinate	y-coordinate
A	m	0	0
B	$2m$	a	0
C	$3m$	a	a
D	$4m$	0	a

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{a}{2},$$

$$Y = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{7a}{10}$$

Coordinates of COM: $R = \left(\frac{a}{2}, \frac{7a}{10} \right).$

- If masses are not equal, the COM of the system will be shifted towards heavier mass, as above.
- If " $m_1 = m_2 = m_3 = m_4$ center-of-mass coincides with the geometrical center $\left(\frac{a}{2}, \frac{a}{2} \right)$."



Dr. A. Kushwaha

14

Center of Mass of a System of Particles (extended body)

In the continuum limit, divide the body into N mass elements. If \mathbf{r}_j is the position of the j^{th} element, and m_j is its mass, then

$$\mathbf{R} = \frac{1}{M} \sum_{j=1}^N m_j \mathbf{r}_j$$

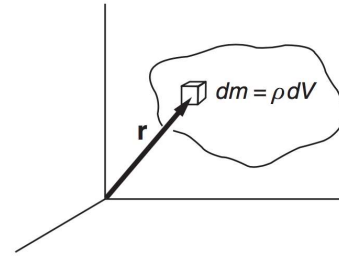
In the limit of $N \rightarrow \infty$,

$$\begin{aligned} \mathbf{R} &= \lim_{N \rightarrow \infty} \frac{1}{M} \sum_{j=1}^N m_j \mathbf{r}_j \\ &= \frac{1}{M} \int \mathbf{r} \, dm, \quad (dm : \text{differential mass element}) \end{aligned}$$

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm \quad \rightarrow \text{"mass integral"}$$

Dr. A. Kushwaha

15



Center of Mass of a System of Particles

$$x_{com} = \frac{1}{M} \int x \, dm;$$

$$y_{com} = \frac{1}{M} \int y \, dm;$$

$$z_{com} = \frac{1}{M} \int z \, dm$$

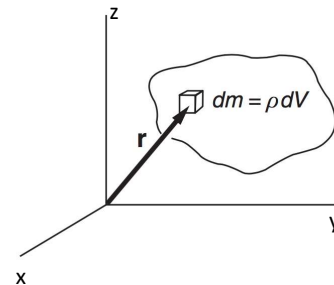
$$\mathbf{R} = x_{com} \hat{i} + y_{com} \hat{j} + z_{com} \hat{k}$$

Let dm be the mass in an element of volume dV located at position \mathbf{r} .
If the mass density at the element dm is ρ , then $dm = \rho \, dV$

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \rho \, dV \quad \rightarrow \text{"Volume integral"}$$

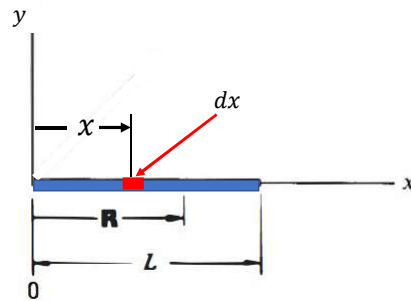
Dr. A. Kushwaha

16



Example 1: Center of Mass of a Nonuniform Rod

Consider a rod of length L with a nonuniform density λ . Mass/unit length of the rod varies as $\lambda = \lambda_0 \frac{x}{L}$ (x : distance from the end marked 0).



Solution: Let the rod lie along the x-axis.

Mass in an element of length dx , at distance x from the origin:

$$dm = \lambda dx = \lambda_0 x \frac{dx}{L}$$

Dr. A. Kushwaha

17

Example 1: Center of Mass of a Nonuniform Rod

Total mass:

$$\begin{aligned} M &= \int dm \\ &= \int_0^L \lambda dx = \int_0^L \frac{\lambda_0 x}{L} dx = \frac{\lambda_0 L}{2} \\ M &= \frac{\lambda_0 L}{2} \end{aligned}$$

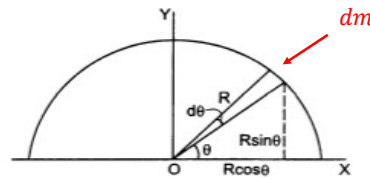
Position vector of center of mass:

$$\begin{aligned} \mathbf{R} &= \frac{1}{M} \int \mathbf{r} \lambda dx \\ &= \frac{2}{\lambda_0 L} \int_0^L (x\hat{i} + 0\hat{j} + 0\hat{k}) \frac{\lambda_0 x}{L} dx = \frac{2}{L^2} \frac{\hat{i}}{3} x^3 \\ \mathbf{R} &= \frac{2}{3} L \hat{i} \end{aligned}$$

Dr. A. Kushwaha

18

Example 2: Center of Mass of Half of a Ring



Mass per unit length

$$\lambda = \frac{M}{\pi R}$$

As the wire is uniform, the mass per unit length of the wire is $\frac{M}{\pi R}$. The mass of the element is, therefore,

$$dm = \left(\frac{M}{\pi R}\right)(R d\theta) = \frac{M}{\pi} d\theta.$$

The coordinates of the centre of mass are

$$X = \frac{1}{M} \int x dm = \frac{1}{M} \int_0^\pi (R \cos \theta) \left(\frac{M}{\pi}\right) d\theta = 0$$

and

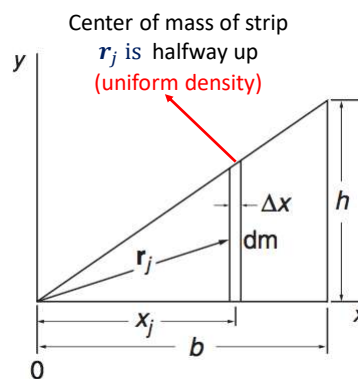
$$Y = \frac{1}{M} \int y dm = \frac{1}{M} \int_0^\pi (R \sin \theta) \left(\frac{M}{\pi}\right) d\theta = \frac{2R}{\pi}$$

Dr. A. Kushwaha

19

Example 3 : Center of Mass of a Triangular Plate

Consider the two-dimensional case of a uniform right triangular plate of mass M , base b , height h and small thickness t .



Solution:

Divide the triangle into strips of width Δx parallel to the y axis.

The total height of the j^{th} strip $= \frac{x_j h}{b}$. (Similar triangles)

Dr. A. Kushwaha

20

Example 3: Center of Mass of a Triangular Plate

The position vector of the center of mass of j^{th} strip :

$$\mathbf{r}_j = x_j \hat{\mathbf{i}} + y_j \hat{\mathbf{j}}$$

$$\mathbf{r}_j = x_j \hat{\mathbf{i}} + \frac{x_j h}{2b} \hat{\mathbf{j}} \quad (\text{as } y_j = \frac{1}{2} (\text{total height of the } j^{\text{th}} \text{ strip}))$$

Center of mass of the plate :

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm \quad (\text{limit of very narrow strips})$$

$$\text{Mass of the plate } M = \rho A t = \frac{\rho t b h}{2}$$

$$\text{Mass element } dm = \rho t y dx = \rho t \frac{x h}{b} dx$$

Dr. A. Kushwaha

21

Example 3: Center of Mass of a Triangular Plate

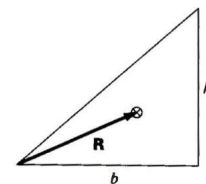
On Substituting M and dm ,

$$\begin{aligned} \mathbf{R} &= \frac{2}{\rho t b h} \int \mathbf{r} \rho t \frac{x h}{b} dx \\ &= \frac{2}{b^2} \int_0^b x \mathbf{r} \, dx \end{aligned}$$

On integrating w.r.t x ,

$$= \frac{2}{b^2} \int_0^b (x^2 \hat{\mathbf{i}} + \frac{x^2 h}{2b} \hat{\mathbf{j}}) dx$$

$$\mathbf{R} = \frac{2}{3} b \hat{\mathbf{i}} + \frac{h}{3} \hat{\mathbf{j}}$$



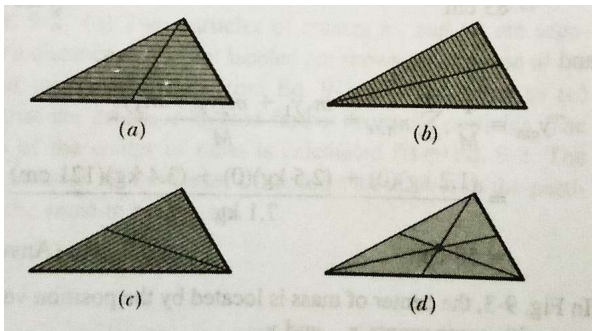
Note:

1. \mathbf{R} depends on b & h
2. But the position of center of mass w.r.t the triangular plate is independent of the choice of coordinate system

Dr. A. Kushwaha

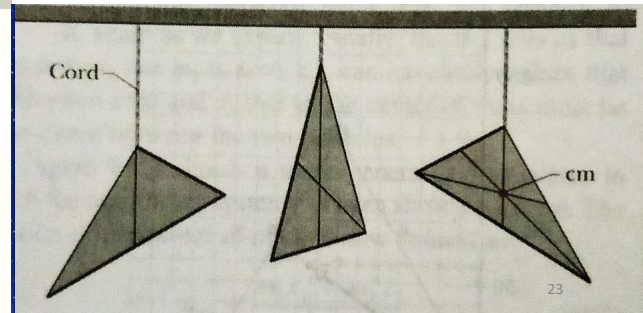
22

Location of Center of Mass of a Uniform Triangular Plate: By Symmetry



- center of mass lie along the bisecting line

- Locate center of mass by suspending from each vertex in turn



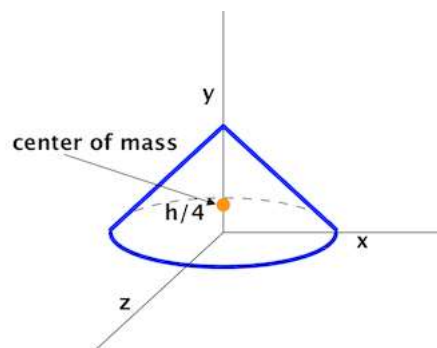
Dr. A. Kushwaha

23

Location of Center of Mass : By symmetry



Center: Point of symmetry



Axis: line of symmetry



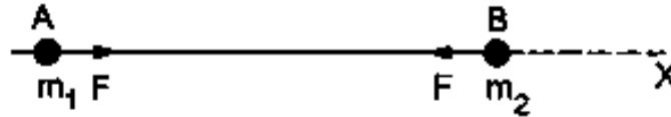
Center of mass

Dr. A. Kushwaha

24

Motion of the Center of Mass

Consider a two-particle system (A and B) of masses m_1 and m_2 respectively. Let x_1 and x_2 be the coordinates of the particles at time t .



- Assuming **no external force** on this two particle system. Only **internal forces** are present.
- Suppose the particles are initially at rest and exert attractive forces between them. They both move along the line AB.

Motion of the Center of Mass

The center of mass at time $t=0$ is situated at :

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (\text{center of mass moves along the X-axis.})$$

Velocity of the center of mass at time t is

$$v_{com} = \frac{dx_{com}}{dt} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Since initially particles are at rest at $t=0$,

$$v_1 = v_2 = 0$$

$$v_{com} = 0$$

Motion of the Center of Mass

The acceleration of the center of mass is

$$a_{com} = \frac{dV_{com}}{dt} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}$$

If external force = 0

Let F_{int} be the magnitude of the (internal) forces between the particles.

$$a_1 = F_{int}/m_1 \text{ and } a_2 = -F_{int}/m_2$$

$$\therefore a_{com} = \frac{m_1(F_{int}/m_1) + m_2(-F_{int}/m_2)}{m_1 + m_2} = 0 \text{ “Velocity of center of mass is constant”}$$

“If no external force acts on a two-particle system and its center of mass is at rest initially, it remains fixed even when the particles individually move and accelerate.”

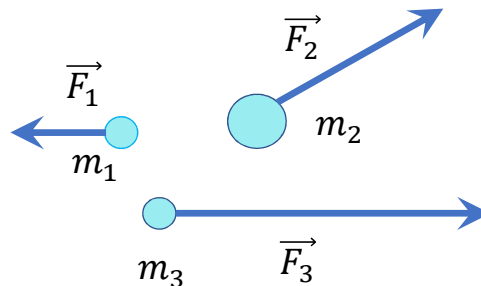
Dr. A. Kushwaha

27

Motion of the Center of Mass

If external force $\neq 0$

Consider a three particle system of masses m_1, m_2 and m_3 . Let \vec{F}_1, \vec{F}_2 and \vec{F}_3 be the external forces acting on masses m_1, m_2 and m_3 . The internal forces, F_{int} , between the masses cancel each other.



Dr. A. Kushwaha

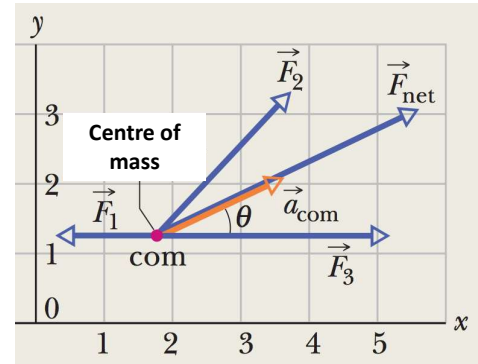
28

Motion of the Center of Mass

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M} = \frac{\vec{F}_{net}}{M}$$

❖ The acceleration \vec{a}_{com} of the center of mass of the system of particles, is in the same direction as the net external force \vec{F}_{net} .

❖ The motion of the center of mass of system is not affected by the internal forces.



What is the equation of motion for a system of N particles?

Dr. A. Kushwaha

Fig: Fundamentals of Physics, Resnick et al.,

Dynamics of System of Particles

Consider a system of N interacting particles.

masses : $m_1, m_2, m_3, \dots, m_n$.

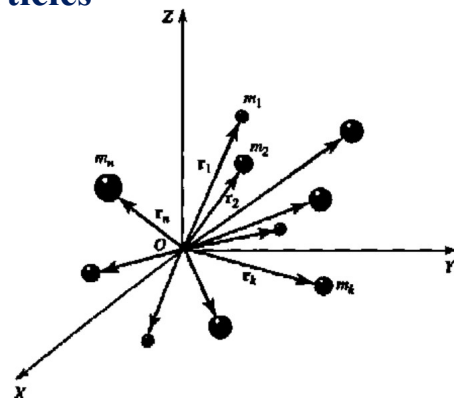
position vector of j^{th} particle: \mathbf{r}_j

momentum of j^{th} particle: $\mathbf{p}_j = m_j \dot{\mathbf{r}}_j$

Internal forces: \mathbf{f}_j^{int} (force on j due to all other particles in the system)

External forces: \mathbf{f}_j^{ext} (force due to sources outside the system)

Total force on j^{th} particle : $\mathbf{f}_j = \mathbf{f}_j^{int} + \mathbf{f}_j^{ext}$



Dr. A. Kushwaha

30

Dynamics of System of Particles

Newton's equation of motion for the j^{th} particle is

$$\mathbf{f}_j = \frac{d}{dt} \mathbf{p}_j \quad (\text{Newton's II Law})$$

$$\mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}} = \frac{d}{dt} \mathbf{p}_j$$

Therefore, equations of motion of all the particles in the systems:

$$\mathbf{f}_1^{\text{int}} + \mathbf{f}_1^{\text{ext}} = \frac{d}{dt} \mathbf{p}_1$$

.....

$$\mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}} = \frac{d}{dt} \mathbf{p}_j$$

.....

$$\mathbf{f}_N^{\text{int}} + \mathbf{f}_N^{\text{ext}} = \frac{d}{dt} \mathbf{p}_N$$

Dr. A. Kushwaha

31

Dynamics of System of Particles

Adding all these equations, we get:

$$\Sigma \mathbf{f}_j^{\text{int}} + \Sigma \mathbf{f}_j^{\text{ext}} = \Sigma \frac{d}{dt} \mathbf{p}_j \quad j = 1 \dots N$$

$$\Sigma \mathbf{f}_j^{\text{ext}} = \mathbf{F}_{\text{ext}} \quad \text{total external force acting on the system}$$

By Newton's III law: $\mathbf{f}_{ij}^{\text{int}} = -\mathbf{f}_{ji}^{\text{int}}$ “internal forces cancel in pairs”

$$\Sigma \mathbf{f}_j^{\text{int}} = 0$$

$$\mathbf{F}_{\text{ext}} = \Sigma \frac{d}{dt} \mathbf{p}_j = \frac{d}{dt} \Sigma \mathbf{p}_j = \frac{d}{dt} \mathbf{P}$$

Dr. A. Kushwaha

32

Dynamics of System of Particles

$$\mathbf{F}_{ext} = \frac{d}{dt} \mathbf{P}$$

Total external force = rate of change of the total momentum

This equation describes only “translation” of the center of mass of the body.

$$\mathbf{P} = M \mathbf{V}_{com}$$

$$\therefore \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{V}_{com}}{dt}$$

$$\mathbf{F}_{ext} = M \mathbf{a}_{com}$$

Dr. A. Kushwaha

33

Dynamics of System of Particles

Consider the equation of motion: $\mathbf{F}_{ext} = M \mathbf{a}_{com}$

If $\mathbf{F}_{ext} = 0$, then

$$\mathbf{a}_{com} = 0 \text{ (center of mass has no acceleration)}$$

$$\therefore \mathbf{V}_{com} = \text{constant}$$

“If there are **no external forces** acting on the system of particles, then

- If the **center of mass** was **at rest initially**, it will **continue to be at rest** (even if the individual particles moving and changing their position).
- If the **center of mass** was **moving at a speed \mathbf{V}_{com}** along a particular direction initially, it will **continue its motion** along the same straight line with the **same speed**.

Dr. A. Kushwaha

34