Introduction to Logic

Department of Science and Mathematics

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August 17, 2020

Definition

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

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Solution: Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent by proving $\forall x(P(x) \wedge Q(x))$ iff $\forall x P(x) \wedge \forall x Q(x)$.

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Solution: Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall xP(x) \wedge \forall xQ(x)$ are logically equivalent by proving $\forall x(P(x) \wedge Q(x))$ iff $\forall xP(x) \wedge \forall xQ(x)$. First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall xP(x) \wedge \forall xQ(x)$ is true. Second, we show that if $\forall xP(x) \wedge \forall xQ(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

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- $P(a)$ is true and $Q(a)$ is true
- $\forall x P(x)$ is true and $\forall x Q(x)$ is true.
- Thus, $\forall x P(x) \wedge \forall x Q(x)$ is true.

Therefore

- We can distribute a universal quantifier over a conjunction.

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- Is $\neg(\forall x \; P(x)) \equiv \exists \neg P(x)$?

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- $\neg(\forall x\ P(x))$ is true if and only if $\forall x\ P(x)$ is false.
- $\forall x\ P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false.

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- $\neg(\forall x\ P(x))$ is true if and only if $\forall x\ P(x)$ is false.
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- This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true.

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- This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true.
- Finally, there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x\ \neg P(x)$ is true.

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- This is true if and only if no x exists in the domain for which $Q(x)$ is true.

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- $\neg\exists x\ Q(x)$ is true if and only if $\exists x\ Q(x)$ is false.
- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- This is true if and only if $Q(x)$ is false for every x in the domain.

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- This holds if and only if $\forall x \neg Q(x)$ is true.
  We conclude that $\neg\exists x \; Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent.
- De Morgan's laws for quantifiers
  $\neg(\forall x \; P(x)) \equiv \exists \neg P(x)$
  $\neg(\exists \; P(x)) \equiv \forall x \neg P(x)$

When the domain has n elements $x_1, x_2, \cdots, x_n$, it follows that $\neg\forall x \ P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n))$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$. Similarly, $\neg\exists x P(x)$ is the same as $\neg(P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n))$, which by De Morgans laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n)$, and this is the same as $\forall x \neg P(x)$.

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- The negation is $\neg\exists x\ H(x) \equiv \forall x \neg\ H(x)$.
  This negation can be expressed as "Every politician is dishonest."
  Note: In English, the statement "All politicians are not honest" is ambiguous. In common usage, this statement often means "Not all politicians are honest." Consequently, we do not use this statement to express this negation.

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- Negation is $\neg\forall x \; C(x) \equiv \exists x \neg C(x)$.
  This negation can be expressed in several different ways, including "Some Indian does not drink tea" and "There is an Indian who does not drink tea."
- What are the negations of the statements
  $\forall x(x^2 > x)$
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- What are the negations of the statements
  $\forall x(x^2 > x)$
  $\exists x(x^2 = 2)$? (Truth values will depend on the domain.)

- Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

- Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent. (Hint: $p \to q \equiv \neg p \vee q$)

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  If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x\ (S(x) \rightarrow C(x))$.

What does $\forall x\ (S(x) \wedge C(x))$ represent?