Unit 4: Combinatorics

Topic 3: Recurrence Relation

Outline

Introduction

There are many problems in combinatorics which can not be solved using the techniques discussed so far.

1. Number of bit strings of length 100 having two consecutive zeroes.
2. Number of moves required to complete the Tower of Hanoi puzzle.
3. Number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdots x_n$, to specify the order of multiplication.
4. Number of comparisons needed to find the maximum and minimum elements of the sequence with n elements.

Such problems can be modelled in terms of recurrence relation, which is relation between terms so that the nth term can be obtained from its previous terms.

Recurrence Relation

Definition (Recurrence Relation)

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

For example,

1. $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 1 = a_1$ is the Fibonacci sequence.
2. $a_n = 2a_{n-1} + 1$ with $a_1 = 1$ represents the Tower of Hanoi puzzle.
3. $a_n = 2a_{n/2} + 1$ with $a_1 = 0$ gives the number of comparisons to find maximum and minimum.

Solution of Recurrence Relation

Definition (Solution of a recurrence relation)

Any sequence satisfying a given recurrence relation is called solution of the recurrence relation.

For example,

1. $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ is a **solution** of the Fibonacci sequence.
2. $a_n = 2^n - 1$ is a **solution** of the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_1 = 1$.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of bit strings of length n containing no two consecutive zeroes.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of bit strings of length n containing no two consecutive zeroes.
   Ans: $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$ and $a_2 = 3$.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of bit strings of length n containing no two consecutive zeroes.
   Ans: $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$ and $a_2 = 3$.
2. There are three pegs mounted on a board with n disks of different sizes are placed on the first peg in order of size. One is allowed to move one disk at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The number of moves needed to move all n disks from one peg to another keeping the order same.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of bit strings of length n containing no two consecutive zeroes.
   Ans: $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$ and $a_2 = 3$.
2. There are three pegs mounted on a board with n disks of different sizes are placed on the first peg in order of size. One is allowed to move one disk at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The number of moves needed to move all n disks from one peg to another keeping the order same.
   Ans: $a_n = 2a_{n-1} + 1$ with $a_1 = 1$

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of ways to parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdots x_n$ to specify the order of multiplication.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of ways to parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdots x_n$ to specify the order of multiplication.
   Ans: $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-2} a_1 + a_{n-1} a_0$ with $a_0 = 0$ and $a_1 = 1$

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of ways to parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdots x_n$ to specify the order of multiplication.
   Ans: $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-2} a_1 + a_{n-1} a_0$ with $a_0 = 0$ and $a_1 = 1$
2. The number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

Problem

Ex: Construct a recurrence relation for each of the following with initial conditions:

1. The number of ways to parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdots x_n$ to specify the order of multiplication.
   Ans: $a_n = a_0a_{n-1} + a_1a_{n-2} + \cdots + a_{n-2}a_1 + a_{n-1}a_0$ with $a_0 = 0$ and $a_1 = 1$
2. The number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.
   Ans: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with $a_1 = 1, a_2 = 2$ and $a_3 = 4$

Outline

Linear Recurrence Relation

There are a wide variety of recurrence relations occur in different models of problems.

One of the easy methods to find solution of a recurrence relation is the iterative method.

However, there is no general method to solve all kind of recurrence relations.

Linear recurrence relation with constant coefficients: A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n), \tag{1}$$

where $c_1, c_2, \ldots, c_k$ are real numbers with $c_k \neq 0$, and $F(n)$ is constant or a function of n only.

1. $a_n = a_{n-1} + a_{n-2}$ is linear with constant coefficients of degree 2 (homogeneous).
2. $a_n = 2a_{n-1} + 1$ is linear with constant coefficients of degree 1 (non-homogeneous).
3. $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-2} a_1 + a_{n-1} a_0$ not linear.

Linear Recurrence Relation

If $F(n) = 0$, then the recurrence relation (1) is called **homogeneous linear recurrence relation of degree k with constant coefficients**.

On the other hand, if $F(n) \neq 0$, then the recurrence relation (1) is called **nonhomogeneous linear recurrence relation of degree k with constant coefficients**.

Linear Recurrence Relation

Theorem

If $\{a_n^{(p)}\}$ *is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, *where* $\{a_n^{(h)}\}$ *is a solution of the associated homogeneous recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Linear Recurrence Relation

Therefore, we first consider the solving method of homogeneous linear recurrence relation of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}. \tag{2}$$

In general, a recurrence relation of the form (2) is satisfied by infinitely many sequences $\{a_n\}$. If the recurrence relation (2) is prescribed with k initial conditions

$$a_0 = l_0, a_1 = l_1, \ldots, a_{k-1} = l_{l-1},$$

then the solution of (2) is unique.

Linear Recurrence Relation

The basic approach for solving linear homogeneous recurrence relation is to look for solutions of the form $a_n = r^n$, where r is constant. Thus $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0, \tag{3}$$

which is a polynomial equation of degree k in the variable r, called **characteristic equation** of the given linear homogeneous recurrence relation.

It is easy to note that there k solutions to (3) with multiplicity, called **characteristic roots** of the given linear homogeneous recurrence relation.

Linear Recurrence Relation

If $r_1, r_2, \ldots, r_k$ be the k solutions to the characteristic equation (3), then the general solution of the linear homogeneous recurrence relation is given by

$$a_n = \begin{cases} \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n, & \text{if } r_1, r_2, \ldots, r_k \text{ are distinct;} \\ (\alpha_{11} + n\alpha_{12} + \cdots + n^{m_1-1}\alpha_{1m_1}) r_1^n + (\alpha_{21} + n\alpha_{22} + \cdots + n^{m_2-1}\alpha_{2m_2}) r_2^n & \\ + \cdots + (\alpha_{t1} + n\alpha_{t2} + \cdots + n^{m_t-1}\alpha_{tm_t}) r_t^n, & \text{if } m_i > 0 \text{ is multiplicity of } r_i, \end{cases}$$

where α's are constants, and those can be evaluated using the initial conditions. In the second case $r_1, r_2, \ldots, r_t$ are distinct and $m_1 + m_2 + \cdots + m_t = k$.

Problem

Ex: Solve the recurrence relations $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$.

Problem

Ex: Solve the recurrence relations $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$ with $a_0 = 3, a_1 = 6$.

Problem

Ex: Solve the recurrence relations $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5, a_1 = -9, a_2 = 15.$

Linear Recurrence Relation

Thus we now know to find $\{a_n^{(h)}\}$ for any nonhomogeneous linear recurrence relation of degree k with constant coefficients.

As a result, we are left to find a particular solution $\{a_n^{(p)}\}$ of the nonhomogeneous linear recurrence relation with constant coefficients to completely solve (or find general solution of) any nonhomogeneous linear recurrence relation of degree k with constant coefficients.

It is not easy to find $\{a_n^{(p)}\}$ for all possible forms of $F(n)$.

There is no general method to find particular solution of a nonhomogeneous linear recurrence relation that works for every $F(n)$ as there was for the homogenous part.

Linear recurrence relation

Theorem

Consider the linear nonhomogeneous recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

with real numbers $b_0, b_1, \ldots, b_t$ *and* s.
Case 1: *When* s *is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form*

$$(p_t n^t + p_{t-1} n^{t-1} \cdots + p_1 n + p_0) s^n.$$

Case 2: *When* s *is a root of this characteristic equation and its multiplicity is* m, *there is a particular solution of the form*

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Problems

Ex: Solve $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

1. $F(n) = n^2$
2. $F(n) = n^2 2^n$
3. $F(n) = (-2)^n$

Problem

Ex: Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$. Find all solutions of this recurrence relation. Find the solution of this recurrence relation with $a_0 = 4$.

Ex: A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n with initial conditions. Solve the recurrence relation.

Outline

1. Recurrence Relation
   - Basic definitions
   - Problems

2. Linear recurrence relations with constant coefficients
   - Problems

3. Divide and Conquer Recurrence Relation

Divide and Conquer Recurrence Relation

Divide and conquer is an algorithm that breaks (divide) a problem with given input into some smaller problems of similar kind, and then solving the individual problems, combine (conquer) them to obtain solution of the original problem.

Suppose that a recursive algorithm divides a problem of size n into A number of subproblems each of which is of size $\frac{n}{b}$. Suppose that a total of $g(n)$ extra operations are required in the conquer step of the algorithm to combine the solutions of the subproblems into a solution of the original problem. Then, if $f(n)$ represents the number of operations required to solve the problem of size n, it follows that f satisfies the recurrence relation

$$f(n) = Af\left(\frac{n}{b}\right) + g(n).$$

This is called a **divide and conquer recurrence relation**.

Divide and Conquer Recurrence Relation

It is easy to note that the divide and conquer recurrence relation can be converted to a linear recurrence relation with constant coefficients given by

$$a_k = Aa_{k-1} + g(b^k)$$

with the substitution $n = b^k$. We obtain the required solution by substituting back with $k = \log_b n$.

Ex: Let $f(n) = 5f(n/2) + 3$ and $f(1) = 7$. Find $f(2k)$, where k is a positive integer.

Divide and Conquer Recurrence Relation

Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, *b is an integer greater than 1, and c is a positive real number. Then*

$$f(n) \text{ is } = \begin{cases} O(n^{\log_b a}), & \text{if } a > 1; \\ O(\log n), & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ *and* $a \neq 1$, *where k is a positive integer,*

$$f(n) = C_1 n \log_b a + C_2,$$

where $C_1 = f(1) + c/(a-1)$ *and* $C_2 = -c/(a-1)$.

Divide and Conquer Recurrence Relation

Theorem (MASTER THEOREM)

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } = \begin{cases} O(n^d), & \text{if } a < bd; \\ O(n^d \log n), & \text{if } a = bd; \\ O(n^{\log_b a}), & \text{if } a > bd; \end{cases}$$

Thank You

Any Question!!!