Indian Institute of Information Technology Guwahati

MA203: Probability and Random Processes **Problem Sheet-VI**

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(1) If you wish to estimate the proportion of engineers who have studied probability theory and you wish your estimation to be correct within 2% with probability 95% or more, how large the sample you would take

  (a) if you have no idea what the true proportion is, [Ans: 12500]

  (b) if you are confident that the true proportion is less than 0.2. [Ans: 8000]

(2) Examine if the weak law of large numbers holds for the sequence of independent random variables defined as $P(X_k = \pm 2^k) = 2^{-(2k+1)}, P(X_k = 0) = 1 - 2^{-2k}$. [Ans: Yes]

(3) Let (X, Y) be bivariate normal such that $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Show that the two random variables $X + Y$ and $X - Y$ are independent.

(4) Let X_1 and X_2 be independent $N(0, 1)$ random variables and let $Y = X_1 + X_2$, $Z = X_1^2 + X_2^2$.

  (a) Show that the joint MFG of (Y, Z) is $M_{Y,Z}(t_1, t_2) = (1 - 2t^2)^{-1}e^{t_1^2 1-2t^2}$ if $t_1 \in \mathbb{R}$ and $t_2 < 1/2$.

  (b) Using (a), find $\mathrm{Corr}(Y, Z)$.

(5) Let (X, Y) be a random vector with PDF $f_{X,Y}(x, y) = 15e^{-(2x+3y)}$ if $0 < x < y < 1$ and 0 otherwise. Find $P(\{X \leq x\}|\{4.999 < Y \leq 5.001\})$.

(6) Let X_n be a sequence of discrete random variables such that $P(X_n = \frac{k}{2n}) = \frac{1}{2^n}$ for $k = 1, 2, \ldots, 2n$. Show that $X_n \to X$ in distribution, where $X \sim U(0, 1)$.

(7) Let $\{X_n\}$ be a sequence of identically distributed random variables with mean $\nu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, where $\sigma > 0$. Also assume that $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. Show that $X_n \to \nu$ in probability.

(8) Suppose that orders at a restaurant are iid random variables with mean Rs. 8 and standard deviation Rs. 2. Estimate the probability that the first 100 customers spent a total of more than Rs. 840. Estimate the probability that the first 10 customers spend a total between Rs 780 and Rs. 820. Determine the number of orders after which one can be 90% sure that the total spend by all customers is more than Rs. 1000.[Ans: 0.0228, 0.682, 129]

(9) A student uses pens whose lifetime is an exponential random variable with mean 1 week. Use central limit theorem to determine the minimum number of pens he should buy at the beginning of a 15 week long semester, so that with probability 0.99 he does not run out of pens during the semester.

(10) Let $X_i, i = 1, 2, \ldots, 50$ be independent random variables each being uniformly distributed over the interval $(0, 1)$. Find the approximate value of $P\{\sum_{i=1}^{50} X_i \geq 30\}$. You may use the fact that $\phi(6) = 0.9928$. [Ans: 0.0071.] (Hint: If $\{X_i\}$ is a sequence of uniformly distributed random variable with mean ν and variance σ^2, then $\sqrt{n}\frac{\sum_{i=1}^{n} X_i - \nu}{\sigma}$ follows standard normal distribution.]

(11) Let $X_1, X_2, \ldots, X_n$ be i.i.d poisson random variables with parameters λ. Estimate $P(120 \leq S_n \leq 160)$ where $S_n = \sum_{i=1}^{n} X_i$, $\lambda = 2$ and $n = 75$.