MA 203: Convergence of Sequence of RVs

1. Convergence in Mean Square
2. Convergence in Distribution
3. Convergence in Almost Sure or with probability 1
4. Convergence in Probability

$A_1, A_2, \ldots, A_n, \ldots$

$$A_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} A_n \to 0 < \infty$$

$X_1, X_2, \ldots, X_n, \ldots$

$\boxed{X}$

1. Mean square sense
2. Probability
3. distribution
4. Almost sure or with probability

Convergence in Mean Square Sense: A random sequence $\{X_n\}_{n=1}^{\infty}$ is said to converge in the mean-square sense (m.s) to a random variable X if

$E[(X_n - X)^2] \to 0$ as n $\to \infty$.

$$\{X_n\}_{n=1}^{\infty} = X_1, X_2, \ldots, X_n, \ldots$$

$$\boxed{\lim_{n \to \infty} E\left[(X_n - X)^2\right] = 0}$$

Notation:

$$\{X_n\} \xrightarrow{m.s} X$$

Example 1: Suppose $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with

$$P(\{X_n = n\}) = \frac{1}{n^2}$$

and

$$P(\{X_n = 0\}) = 1 - \frac{1}{n^2}.$$

Examine if $\{X_n\}_{n=1}^{\infty}$ converges to $\{X = 0\}$ in the m.s sense.

$X_1, X_2, \ldots, X_n, \ldots$

$R_{X_n} = \{0, n\}$

$$E[X_n^2] = \sum_{x_i \in R_{X_n}} x_i^2 P(X_n = x_i)$$

$R_{X_1} = \{0, 1\}$

Sol:-

$$\boxed{\lim_{n\to\infty} E[(X_n - X)^2] = 0}$$

$$\Rightarrow \lim_{n\to\infty} E[(X_n - X)^2] = \lim_{n\to\infty} E[(X_n - 0)^2] = \lim_{n\to\infty} E[X_n^2]$$

$$= \lim_{n\to\infty} \left\{ 0^2 \times P\{X_n = 0\} + n^2 \times P\{X_n = 1\} \right\}$$

$$= \lim_{n\to\infty} \left\{ 0 \times 1 - \frac{1}{n^2} + n^2 \times \frac{1}{n^2} \right\}$$

$$= \lim_{n\to\infty} n^2 \times \frac{1}{n^2}$$

$$= \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} E[(X_n - X)^2] = 1$$

Therefore, $\{X_n\}$ is not converging into mean square sense.

Example 2: Suppose $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with

$R_{X_n} = \{0, 1\}$

$$P(\{X_n = 1\}) = \frac{1}{n}$$

and

$$P(\{X_n = 0\}) = 1 - \frac{1}{n}.$$

Examine if $\{X_n\}_{n=1}^{\infty}$ converges to $\{X = 0\}$ in the m.s sense.

Sol:-

$$\boxed{\lim_{n\to\infty} E[(X_n - X)^2] = 0}$$

$$\Rightarrow \lim_{n\to\infty} E[(X_n - X)^2] = \lim_{n\to\infty} E[(X_n - 0)^2] = \lim_{n\to\infty} E[X_n^2]$$

$$= \lim_{n\to\infty} \left[0^2 \times P\{X_n = 0\} + 1^2 \times P\{X_n = 1\}\right]$$

$$= \lim_{n\to\infty} \left[0 \times \left(1 - \frac{1}{n}\right) + 1^2 \times \frac{1}{n} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, $\{X_n\}$ is converging to RV $\{X=0\}$ in m.s sense.

Convergence in Distribution: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of RVs with CDF $F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n), \cdots$, respectively.

We say that $\{X_i\}_{i=1}^{\infty}$ converge in distribution to X if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for all x at which $F_X(x)$ is continuous.

$$\boxed{\lim_{n\to\infty} F_{X_n}(x) = F_X(x)}$$ $\forall x$ at which $F_X(x)$ is continuous.

Note two things:-

1. $\lim_{n\to\infty} F_{X_n}(x)$ may not be a distribution function.

Ex:-

$$F_{X_n}(x) = \begin{cases} 1 & ; x > n \\ 0 & ; \text{o.w.} \end{cases}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 & ; x > \infty \\ 0 & ; \text{o.w.} \end{cases}$$

$$\boxed{F_X(\infty) = 1}$$

$$\Downarrow$$

$$\boxed{F_{X_n}(\infty) = 0}$$

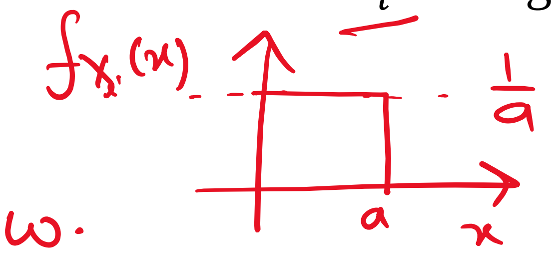
2. $\lim_{n\to\infty} F_{X_n}(x)$ converges to $F_X(x)$ for all x at which $F_X(x)$ is continuous distribution.

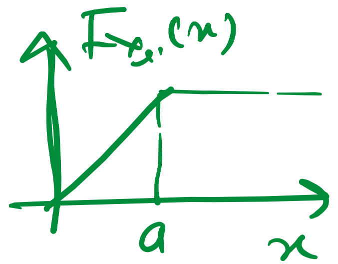
$$\boxed{\lim_{n\to\infty} F_{X_n}(x) = F_X(x)}$$

Example: Suppose $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent RVs with each RV X_i having the uniform density

$$F_X(x) = \int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^x \frac{1}{a}\,dx = \frac{x}{a};\ 0 \le x < a$$

$$f_{X_i}(x) = \begin{cases} \frac{1}{a}; & 0 \le x < a \\ 0; & \cancel{x \ge 0} \ \text{o.w.} \end{cases}$$





Define $Z_n = max(X_1, X_2, \cdots, X_n)$. Examine that $Z_1, Z_2, \cdots, Z_n, \cdots$ converges to RV Z in distribution where

$$f_Z(z) = \begin{cases} 0; z < a \\ 1; z \ge a \end{cases}.$$

$Z_1 = max(X_1) = X_1$

$Z_2 = max(X_1, X_2);\ Z_3 \cdots$

Sol:

$$\boxed{\lim_{n \to \infty} F_{Z_n}(z) = F_Z(z)}$$

$a > b$

$max(a, b) < c$

$\boxed{a < c}$

$$F_{Z_n}(z) = P(Z_n \le z) = P(max(X_1, X_2, \cdots, X_n) \le z)$$

$$= P(X_1 \le z, X_2 \le z, \cdots, X_n \le z)$$

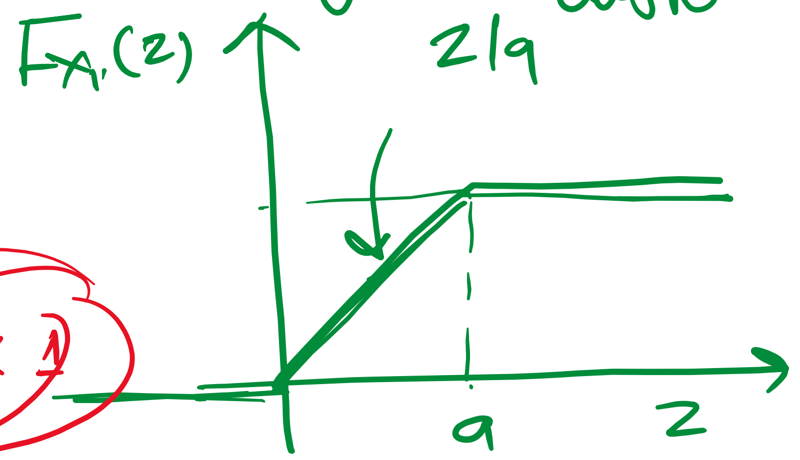
Using Independence

$$= P(X_1 \le z)\, P(X_2 \le z) \cdots P(X_n \le z)$$

$$= F_{X_1}(z) \cdot F_{X_2}(z) \cdots F_{X_n}(z)$$

using identically distⁿ

$$= \{F_{X_1}(z)\}^n$$



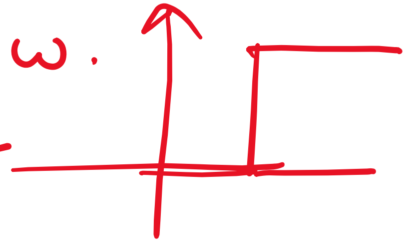
$$F_{Z_n}(z) = \begin{cases} 0 \;; & z < 0 \\ \left(\frac{z}{a}\right)^n ; & 0 < z < a \\ 1 \;; & z \ge a \end{cases}$$

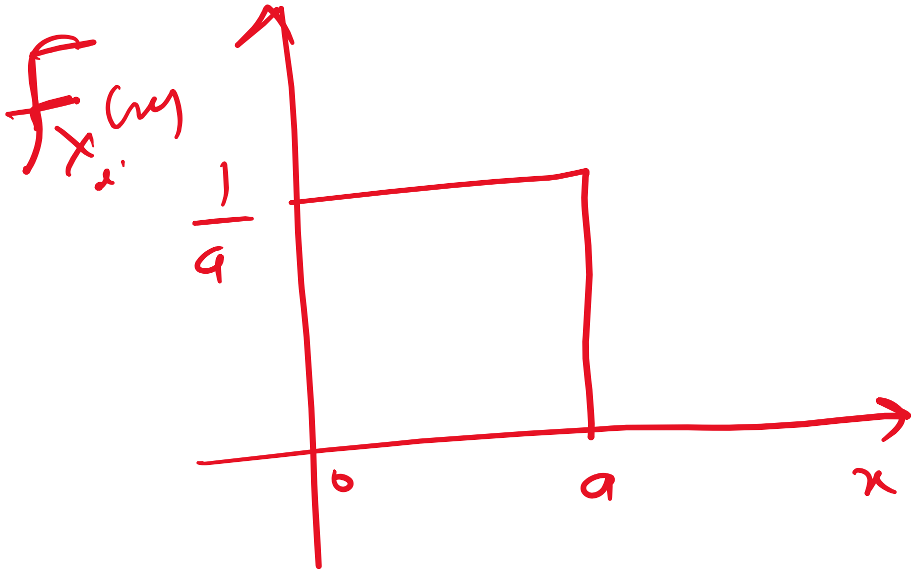
$\frac{z}{a} < 1$

$1 \quad z \ge a$

$\left(\frac{1}{2}\right)^n \to 0$

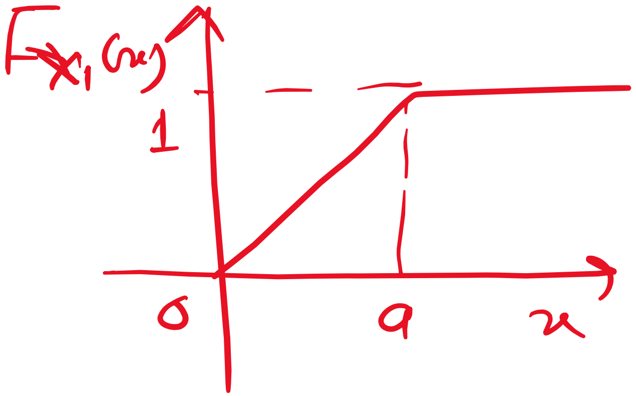
$$\lim_{n\to\infty} F_{Z_n}(z) = \begin{cases} 0 \;; & z < 0 \\ 0 \;; & 0 < z < a \\ 1 \;; & z \ge a \end{cases} = \begin{cases} 1 \;; & z \ge a \\ 0 \;; & \text{o.w.} \end{cases}$$





$$F_{X_1}(x) = \begin{cases} 0 ; & x < 0 \\ \frac{x}{a} ; & 0 \le x < a \\ 1 ; & x \ge a \end{cases}$$

$$F_X(x) = \int_0^x \frac{1}{a}\, du$$



$$= \frac{x}{a} ;$$

$$= \int_0^a \frac{1}{a}\, du$$

$$= 1$$