MA203: Function of Two Random Variables

Example 1: Let X and Y are continuous RVs. Find the probability density function of Z where $Z = \frac{X}{Y}$.

Sol:-

$$F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right)$$

$$= P\left(\frac{X}{Y} \le z,\ -\infty < Y \le \infty\right)$$

$$= P\left[\left\{\frac{X}{Y} \le z\right\}, \left(\left\{Y < 0\right\} \cup \left\{Y > 0\right\}\right)\right]$$

$$= P\left[\underbrace{\left\{\frac{X}{Y} \le z\right\}}_{A} \cap \left(\underbrace{\left\{Y < 0\right\}}_{B} \cup \underbrace{\left\{Y > 0\right\}}_{\bar{B}}\right)\right]$$

$$= P\left[A \cap (B \cup \bar{B})\right] = P(A \cap B) + P(A \cap \bar{B})$$

$$= P\left(\frac{X}{Y} \le z,\ Y < 0\right) + P\left(\frac{X}{Y} \le z,\ Y > 0\right)$$

$$= P(X \ge Yz,\ Y < 0) + P(X \le Yz,\ Y > 0)$$

$$\begin{cases} \underline{Y \ge 0} \\ X \le Yz \\ \underline{Y < 0} \\ X \ge Yz \end{cases}$$

$$\begin{cases} \underline{P(A, B)} \\ = P(A \cap B) \end{cases}$$

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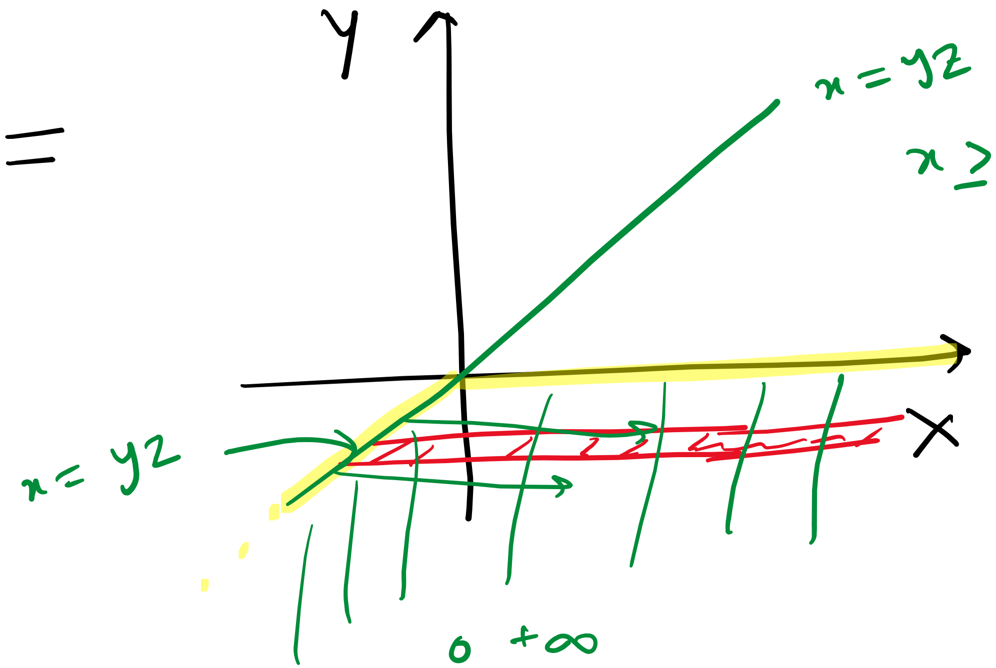

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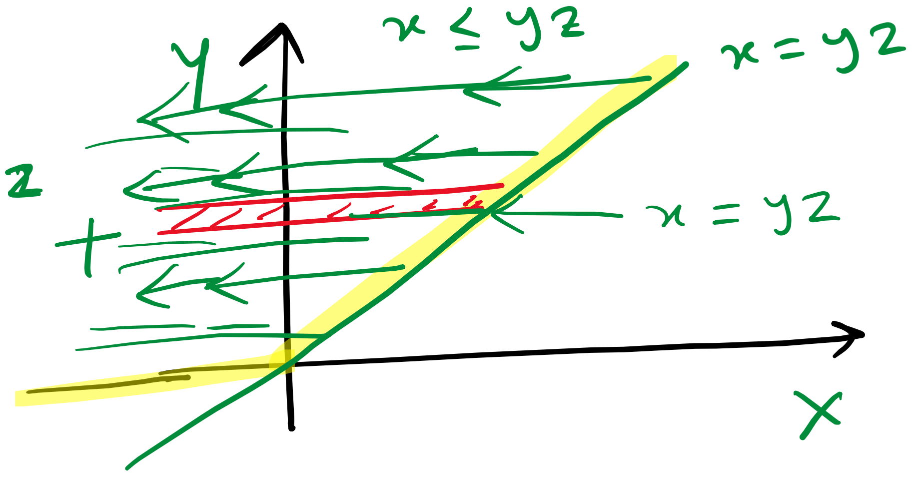


$$= \int_{-\infty}^{0} \int_{yz}^{+\infty} f_{X,Y}(x,y)\,dx\,dy + \int_{0}^{\infty} \int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{d}{dz}\int_{-\infty}^{0}\left\{\int_{yz}^{\infty} f_{X,Y}(x,y)dx\right\}dy + \frac{d}{dz}\int_{0}^{\infty}\left\{\int_{-\infty}^{yz} f_{X,Y}(x,y)dx\right\}dy$$

Using Leibniz Rule

$$f_Z(z) = \int_{-\infty}^{0}\frac{\partial}{\partial z}\left\{\int_{yz}^{\infty} f_{X,Y}(x,y)dx\right\}dy + \int_{0}^{\infty}\frac{\partial}{\partial z}\left\{\int_{-\infty}^{yz} f_{X,Y}(x,y)dx\right\}dy$$

Using Leibnitz Rule

$$f_Z(z) = \int_{-\infty}^{0}\left\{-\frac{\partial(yz)}{\partial z} f_{X,Y}(yz,y)\right\}dy + \int_{0}^{\infty}\left\{\frac{\partial(yz)}{\partial z} f_{X,Y}(yz,y)\right\}dy$$

$$f_Z(z) = \int_{-\infty}^{0}\{-y\}f_{X,Y}(yz,y)dy + \int_{0}^{\infty}\{y\}f_{X,Y}(yz,y)dy$$

$$= \int_{-\infty}^{0}|y|\, f_{X,Y}(yz,y)\,dy + \int_{0}^{\infty}|y|\, f_{X,Y}(yz,y)\,dy = \int_{-\infty}^{\infty}|y|\, f_{X,Y}(yz,y)\,dy$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}(x,b(x)) \times \frac{db(x)}{dx} - \mathrm{h}(x,a(x)) \times \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial h(x,y)}{\partial x}dy$

$$f_Z(z) = \int_{-\infty}^{0} |y| f_{X,Y}(yz, y) dy + \int_{0}^{\infty} |y| f_{X,Y}(yz, y) dy = \int_{-\infty}^{+\infty} |y| f_{X,Y}(yz, y)\, dy$$

If X and Y are independent:

$$\boxed{f_Z(z) = \int_{-\infty}^{+\infty} |y| f_X(yz) f_Y(y)\, dy}$$

Example 2: X and Y are independent zero mean Gaussian RVs with unity standard deviation. Find the probability density function of RV Z where Z = $\frac{X}{Y}$.

X $\mu_x = 0$ $\sigma_x = 1$ $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Y $\mu_y = 0$ $\sigma_y = 1$ $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$$f_Z(z) = \int_{-\infty}^{+\infty} |u| \, f_{X,Y}(uz, u) \, du = \int_{-\infty}^{+\infty} |u| \, \underbrace{f_X(uz)} \cdot f_Y(u) \, du$$

$$= \int_{-\infty}^{+\infty} |u| \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2 z^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{|u| \, e^{-(1+z^2)u^2/2}} \, du = \frac{2}{2\pi} \int_0^{\infty} u \cdot e^{-(1+z^2)u^2/2} \, du$$

put $\frac{(1+z^2)u^2}{2} = t$

$\frac{(1+z^2)\,2u\,du}{2} = dt$

$\Rightarrow (1+z^2)\,u\,du = dt$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-u} \frac{dt}{(1+z^2)}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{e^{-u}}{(1+z^2)} \, du = \frac{1}{\pi(1+z^2)} \left\{ -e^{-u} \right\} \Big|_0^{\infty}$$

$$= \frac{1}{\pi(1+z^2)} \Rightarrow \text{Cauchy Distribution}$$

Example 3: Let X and Y are continuous RVs. Find the probability density function of Z where Z = $max(X,Y)$.

Sol:-

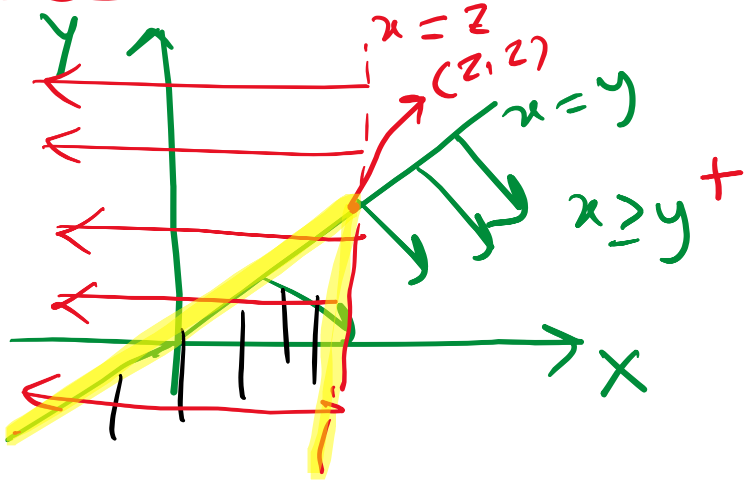
$$Z = \max(X,Y) = \begin{cases} X\,; & X \ge Y \\ Y\,; & X < Y \end{cases}$$

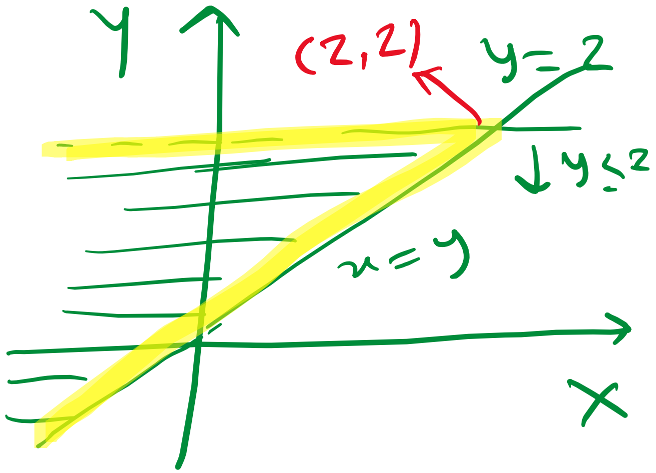
$$F_Z(z) = P(Z \le z) = P(\underbrace{Z \le z}_{A}, \underbrace{\{X \ge Y\} \cup \{X < Y\}}_{B})$$

$$= P(Z \le z, X \ge Y) + P(Z \le z, X < Y)$$

$$= P(X \le z, X \ge Y) + P(Y \le z, X < Y)$$

$$=$$





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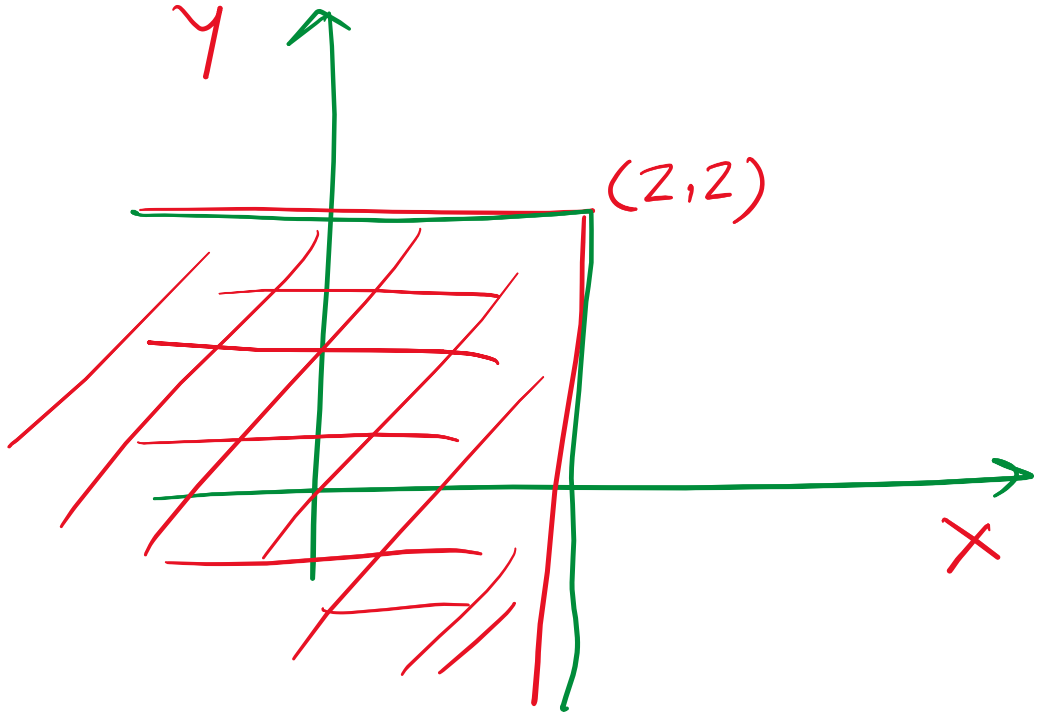


$$F_Z(z) = F_{X,Y}(z,z)$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} F_{X,Y}(z,z)$$

Example 4: Let X and Y are continuous RVs. Find the probability density function of R where R = $\sqrt{X^2 + Y^2}$.

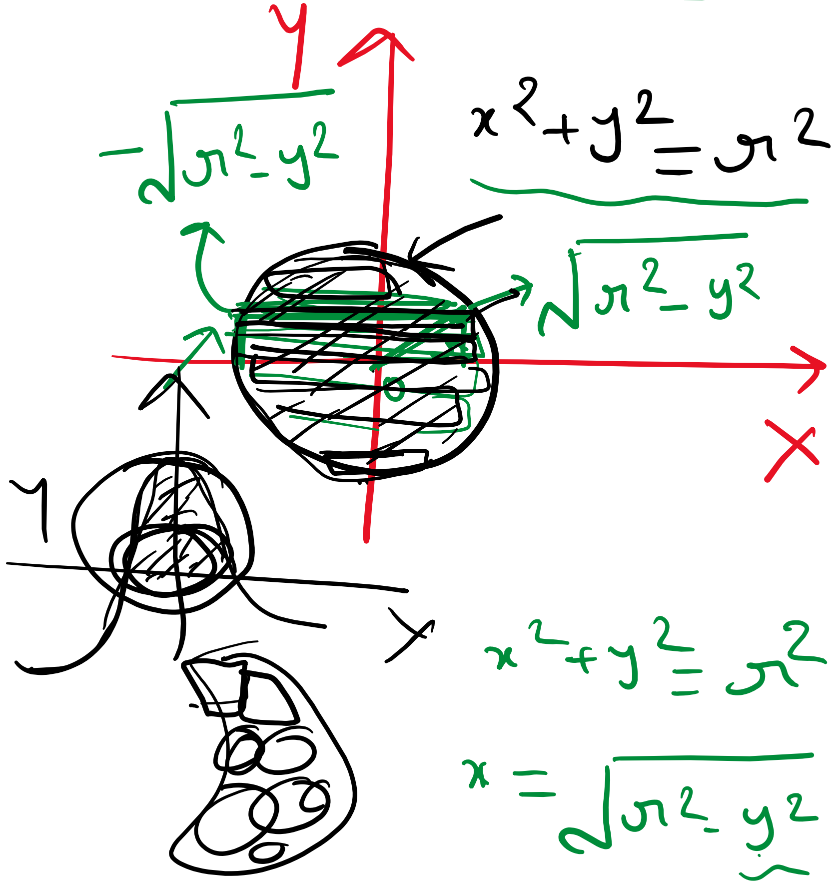
Sol:-

$$F_R(r) = P(R \le r) = P\left(\sqrt{x^2+y^2} \le r\right)$$

$$= P\left(x^2+y^2 \le r^2\right)$$

$$= \int_{-r}^{+r}\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} f_{x,y}(x,y)\,dx\,dy$$

$\left\{ r > 0 \right.$



$x^2+y^2=r^2$

$x = \sqrt{r^2-y^2}$

Step 2:-

$$f_R(r) = \frac{dF_R(r)}{dr} = \frac{d}{dr}\int_{-r}^{+r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\,dy$$

$$f_R(r) = \frac{d}{dr}\int_{-r}^{+r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

using Leibnitz Rule

$$f_R(r) = \underbrace{\frac{d(r)}{dr}\left\{\int_{-\sqrt{r^2-r^2}}^{\sqrt{r^2-r^2}} f_{X,Y}(x,r)\,dx\right\}}_{0} - \underbrace{\frac{d(-r)}{dr}\left\{\int_{-\sqrt{r^2-r^2}}^{\sqrt{r^2-r^2}} f_{X,Y}(x,-r)\,dx\right\}}_{0} + \int_{-r}^{+r}\frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

$$f_R(r) = \int_{-r}^{+r}\frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,dy = h(x,b(x)) \times \frac{db(x)}{dx} - h(x,a(x)) \times \frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial h(x,y)}{\partial x}\,dy$

$$f_R(r) = \int_{-r}^{+r} \frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\} dy$$

Using Leibnitz Rule

$$f_R(r) = \int_{-r}^{r}\left\{\frac{d\left(\sqrt{r^2-y^2}\right)}{dr} f_R\left(\sqrt{r^2-y^2}, y\right) - \frac{d\left(-\sqrt{r^2-y^2}\right)}{dr} f_R\left(-\sqrt{r^2-y^2}, y\right) + \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{\partial f_{X,Y}(x,y)}{\partial z}\,dx\right\} \mathrm{dy}$$

$$f_R(r) = \int_{-r}^{+r}\left\{\frac{r}{\sqrt{r^2-y^2}} f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + \frac{r}{\sqrt{r^2-y^2}} f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right)\right\} dy$$

$$\boxed{f_R(r) = \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}}\left\{f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right)\right\} dy}$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}\big(x, b(x)\big) \times \frac{db(x)}{dx} - \mathrm{h}\big(x, a(x)\big) \times \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial h(x,y)}{\partial x}\,dy$

Example 5: Let X and Y are independent zero mean Gaussian RVs with equal variance σ^2. Find the probability density function of R where $R = \sqrt{X^2 + Y^2}$.

Sol:- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$; $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2}$; $\underbrace{f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}}$

Given that X and Y are independent

Using Result of Example 4

$\Downarrow$

$$f_R(r) = \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} \left\{ f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right) \right\} dy$$

use $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

$$= \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} \left\{ \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2-\cancel{y^2}+\cancel{y^2})}{2\sigma^2}} + \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2-\cancel{y^2}+\cancel{y^2})}{2\sigma^2}} \right\} dy$$

$$= \frac{1}{2\pi\sigma^2} \int_{-r}^{+r} \left\{ \frac{2r}{\sqrt{r^2-y^2}} e^{-\frac{r^2}{2\sigma^2}} \right\} dy = \frac{2}{2\pi\sigma^2} \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} e^{-r^2/2\sigma^2} dy$$

$$f_R(r) = \frac{2}{2\pi\sigma^2} \int_{-r}^{+r} \frac{r}{\sqrt{r^2 - y^2}} e^{-r^2/2\sigma^2} \, dy$$

$$f_R(r) = \frac{re^{-r^2/2\sigma^2}}{\pi\sigma^2} \int_{-r}^{+r} \frac{1}{\sqrt{r^2 - y^2}} dy$$

$$f_R(r) = \frac{2re^{-r^2/2\sigma^2}}{\pi\sigma^2} \int_{0}^{+r} \frac{1}{\sqrt{r^2 - y^2}} dy$$

Put $y = rsin\theta, dy = rcos\theta d\theta$, and $\frac{dy}{\sqrt{r^2-y^2}} = \frac{rcos\theta d\theta}{rcos\theta} = d\theta$

$$f_R(r) = \frac{2re^{-r^2/2\sigma^2}}{\pi\sigma^2} \int_{0}^{\pi/2} d\theta$$

$$f_R(r) = \frac{2re^{-r^2/2\sigma^2}}{\pi\sigma^2} \left(\frac{\pi}{2} - 0\right) = \frac{re^{-r^2/2\sigma^2}}{\sigma^2}$$

$$f_R(r) = \frac{re^{-r^2/2\sigma^2}}{\sigma^2}; r \geq 0$$