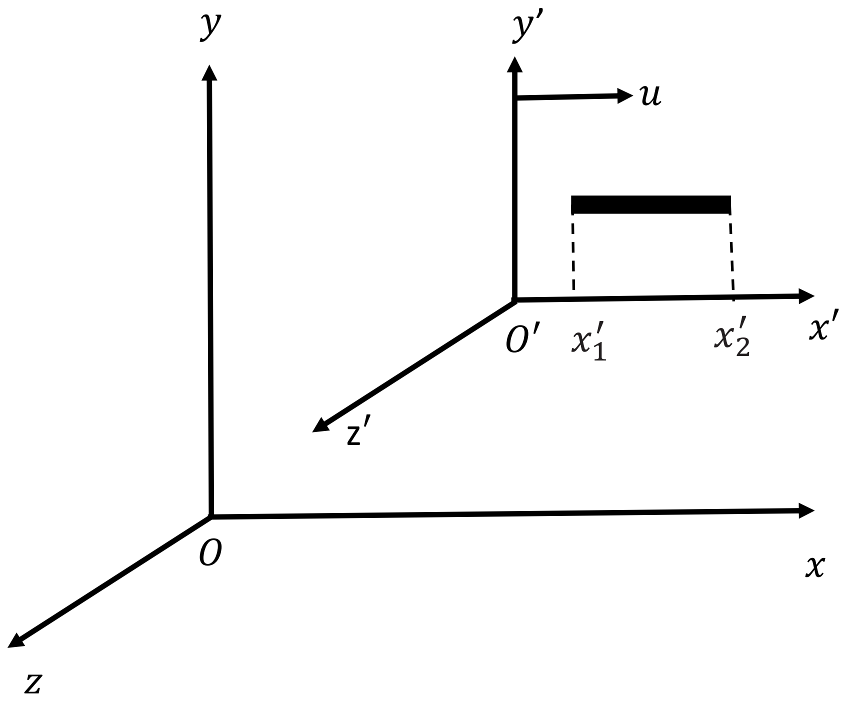
Derivation of relativistic length contraction using the Lorentz transformation

- A rod of length L_0 is at rest in the reference frame of observer O'. The rod extends along the x' axis from x_1' to x_2'.
  Observer O' measures the proper length of the rod,
$$L_0 = x_2' - x_1'.$$

- Observer O, relative to whom the rod is in motion, measures the ends of the rod to be at coordinates x_1 and x_2 and then the length is
$$L = x_2 - x_1.$$

- For O to determine the length of the moving rod, O must make a *simultaneous* determination of x_1 and x_2.



Suppose, the first event is O' setting off a flash bulb at one end of the rod at x_1' and t_1', which O observes at x_1 and t_1

The second event is O' setting off a flash bulb at the other end at x_2' and t_2', which O observes at x_2 and t_2.

From Lorentz transformation, we can relate these coordinates, specifically,

$$x_1' = \frac{x_1 - ut_1}{\sqrt{1 - u^2/c^2}} \qquad x_2' = \frac{x_2 - ut_2}{\sqrt{1 - u^2/c^2}}$$

Subtracting these equations, we obtain

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}} - \frac{u(t_2 - t_1)}{\sqrt{1 - u^2/c^2}}$$

If O observes the flashes to be simultaneous, then $t_2 = t_1$(They will *not* be simultaneous to O'

Now, we have

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - u^2/c^2}}$$

With $x_2' - x_1' = L_0$ and $x_2 - x_1 = L$, this become

$$\boldsymbol{L = L_0\sqrt{1 - u^2/c^2}}$$

This is identical to the equation which we derived earlier using Einstein's postulates. Here, we derived it using Lorentz transformation.

Derivation of the formula for time dilation using the inverse Lorentz transformation

Consider a clock (at rest) at point x' in the moving frame S'.

When observer O' in frame S' finds that the start of the time interval as t_1', an observer O in frame S will find it to be t_1.

Using inverse Lorentz transformation, we can write,

$$t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

After a time interval of Δt_0, the observer O' in the moving system finds that time is now t_2' according to his clock.

That is $\Delta t_0 = t_2' - t_1'$ (Proper time)

However, observer O in frame S measures the end of the same time interval to be:

$$t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

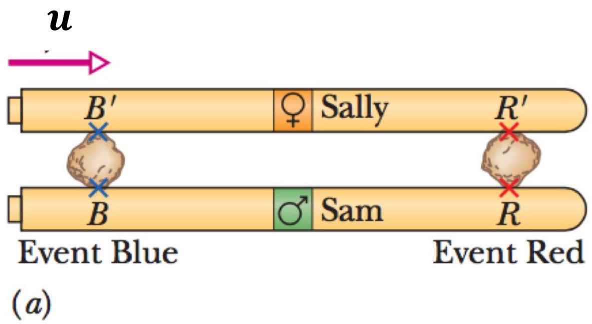
So, for observer O, duration of the time interval is:

$$\Delta t = t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

$$\boldsymbol{\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}}$$

This is same formula what we derived previously with help of a light pulse clock, considering Einstein's postulates. Here, we derived it using inverse Lorentz transformation.

The Relativity of Simultaneity

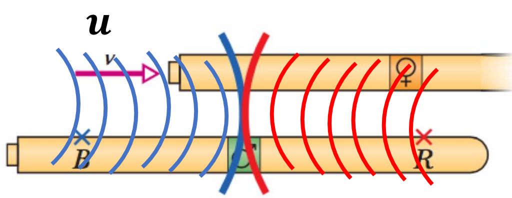


Consider two long spaceships of Sally and Sam (inertial reference frames). They both are stationed at the midpoint of their ships and both ships are momentarily aligned opposite each other. Sally's ship moves rightward with velocity $\boldsymbol{u}$ relative to Sam.

Events:

Two large meteorites strike the ships, one setting off the red flare (event Red) and the other blue flare (event Blue), not necessarily simultaneously. Each event leaves a permanent mark on each ship, at positions R, R' and B, B'.

The occurrences of events from Sam's view.



Sam detects both events

According to SAM:

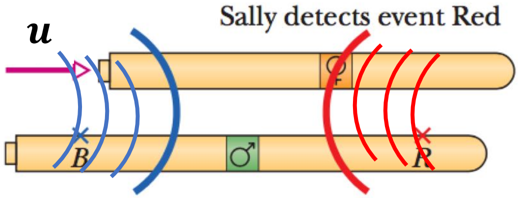
- From the marks on his spaceship, he finds that he was standing halfway between the two sources when the light from them reached him.

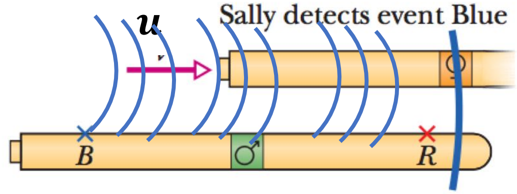
- Light from event Red and light from event Blue reaches him at the same time.

∴ The **two independent events**, Event Red and event Blue**, are simultaneous events**.

Will Sally also find that they occur at the same time?

The occurrences of events from Sally's view.





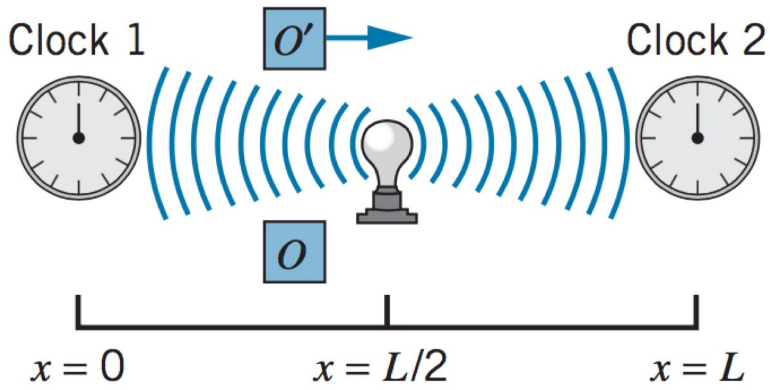
<u>According to Sally:</u>

➢ From the marks on her spaceship, she finds that she was standing halfway between the two sources.

➢ Light from event Red reached her before light from event Blue did.

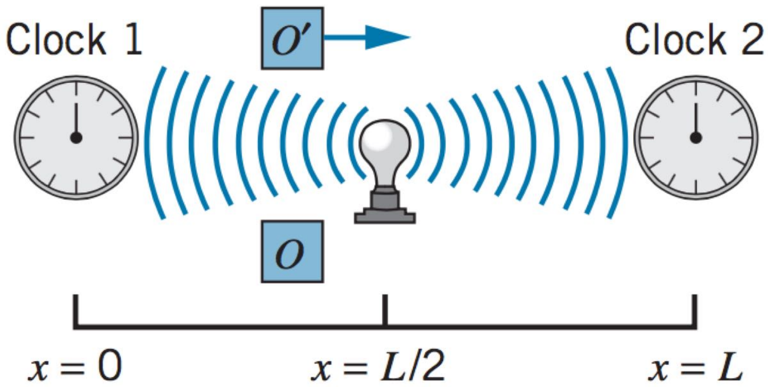
∴ **Event Red and event Blue are NOT simultaneous events**.

Simultaneity and Clock Synchronization



Consider two clocks located at $x = 0$ and $x = L$. A flash lamp is located at $x = L/2$ and the clocks are set running when they receive the flash of light from the lamp.

Simultaneity and Clock Synchronization



<u>According to observer *O*</u>: **"clocks are exactly synchronized"**

Flash of light emitted ⟶ clocks start together at a time $L/2c$

<u>According to observer *O′*</u>: clock 2 start ahead of clock 1

"clocks are not synchronized"

In the frame of reference of O, two events occur:

(i) Receipt of light signal by clock 1 (at $x_1 = 0$): $t_1 = L/2c$

(ii) Receipt of light signal by clock 2 (at $x_2 = L$): $t_2 = L/2c$

Since $t_1 = t_2$, **"clocks are exactly synchronized"**

Using $t' = \frac{t}{\sqrt{1-u^2/c^2}}$. (Lorentz transformation),

O' observes clock 1 to receive signal at: $t_1' = \frac{t_1-(u/c^2)x_1}{\sqrt{1-u^2/c^2}}$

$$= \frac{L/2c}{\sqrt{1-u^2/c^2}} \quad (\text{as } x_1 = 0)$$

Clock 2 receives its signal at:

$$t'_2 = \frac{t_2 - (u/c^2)x_2}{\sqrt{1-u^2/c^2}}$$

$$= \frac{L/2c-(u/c^2)L}{\sqrt{1-u^2/c^2}}. \qquad \text{(as } x_2 = L\text{)}$$

$$t'_2 < t'_1$$

Clock 2 appears to receive its signal earlier than clock 1,so that the clocks start at times that differ by

$$\Delta t' = t'_1 - t'_2 = \frac{uL/c^2}{\sqrt{1-u^2/c^2}}$$

If L = 0 (two events occur at same point in space), clocks are synchronized in all reference frames.

Note: In Lorentz transformation: $$t' = \underbrace{\frac{t}{\sqrt{1-u^2/c^2}}}_{\text{Time dilation}} - \underbrace{\frac{(u/c^2)x}{\sqrt{1-u^2/c^2}}}_{\text{Lack of synchronization}}$$

$\underline{\boldsymbol{O'\,observes}}$:

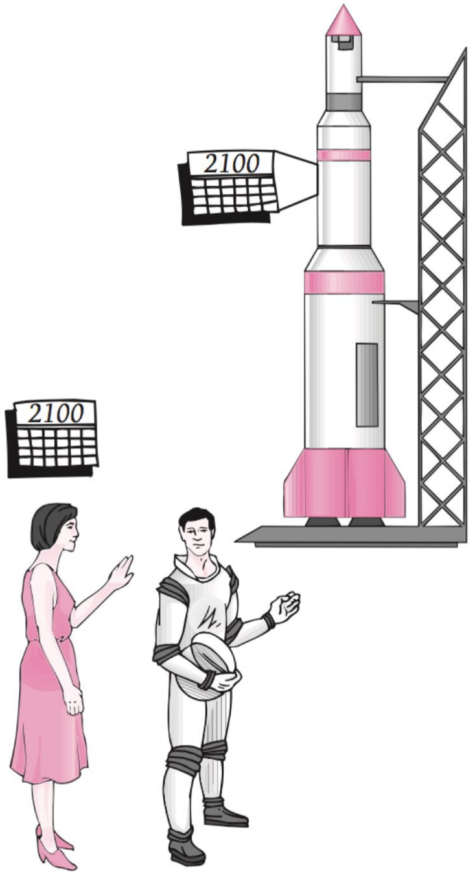
- Both clocks run slow, due to time dilation
- Clock 2 to be a bit ahead of clock 1.

Conclusion:

❖ Simultaneity is not an absolute concept, but a relative one, depending on the motion of the observer.

❖ If two observers are in relative motion, they will not, in general, agree as to whether two events are simultaneous. If one observer finds them to be simultaneous, the other generally will not, and conversely.

The Twin Paradox



Consider a pair of twins on Earth, Casper and Amelia. Casper is 20 years old when he takes off on a space voyage at a speed of 0.6c to a planet 12 light-years away.

According to Amelia:

Casper's clocks will run slow relative to his own (time dilation). So, he should be younger than her, when he returns to the Earth.

The pace of Casper's life (ex: heartbeat) is slower than her (time dilation) by a factor of

$$\sqrt{1-u^2/c^2} = \sqrt{1-(0.6c)^2/c^2}$$

$$= 80\%$$

So, Amelia expects Casper to be younger than her.

According to relativity:
For two observers in relative motion, **each** thinks the **other's** clocks are running slow.

According to Casper: Amelia on the Earth make a round-trip journey (at a relative speed of 0.6c) away from him and back again.

So, Casper should expect Amelia to be younger than her!

Twin Paradox: **"Each twin expects the other to be younger"**

Who is in real motion: Amelia or Casper?

After completing the voyage, which twin has aged less rapidly? Amelia or Casper?

Resolution of Twin Paradox

The two situations are NOT equivalent.

Onward journey of Casper:

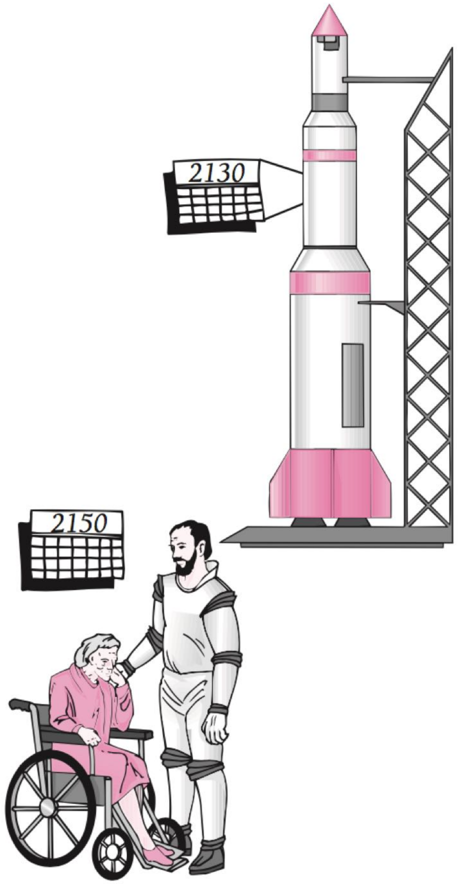
Casper spends all his time (Except acceleration of the rocket for a negligible amount of time) in a frame of reference moving at constant speed (Inertial frame) relative to Amelia.

Return journey (to Earth) of Casper:

He must decelerate (only for a negligible amount of time) and reverse his motion.

∴ outward and return journeys of Casper occur in **different inertial frames.**

Resolution of Twin Paradox



Only Casper jumps to a new inertial frame to return back to Earth.

But Amelia remained in the same inertial frame during the whole voyage of Casper.

So, it is **Casper** who is **"really in motion"** **.** It is his clocks that are really running slow.

Time dilation formula is applicable only to Amelia's observation of Casper.

Casper is the younger twin on his return.

Quantitative Analysis of Twin Paradox

Assume that the distant planet is at rest relative to the Earth.

Distance of the planet from Earth = 12 light-years.
Speed at which Casper travels = 0.6c

According to Amelia:

time taken by Casper to reach the planet = $\frac{12\, light{-}years}{0.6c} = 20\ years$

time taken by Casper to return back = $20\ years$

Total round trip time = 40 years.

Quantitative Analysis of Twin Paradox

According to Casper:

Distance to the planet contracted by a factor = $\sqrt{1-(0.6c)^2/c^2}$= 0.8

Distance to the planet = 0.8 x 12 = 9.6 light-years

time taken by Casper to reach the planet = $\frac{9.6\ light\text{-}years}{0.6c}$ = 16 years

Total round trip time = 32 years

∴ Casper is younger on her return

“A longer life, but it will not seem longer”

Conclusion:

- The nonsymmetric aging of the twins has been verified by experiments, in which accurate clocks were taken on an airplane trip around the world and then compared with identical clocks that had been left behind.

- An observer who departs from an inertial system and then returns after moving relative to that system will always find his or her clocks slow compared with clocks that stayed in the system.

RELATIVISTIC DYNAMICS

Relativistic Momentum

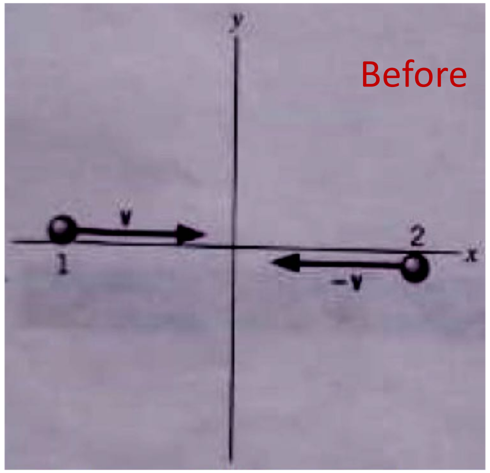
- Einstein's postulates have led to a new "relativity laws" of some absolute concepts as length and time and that the classical concept of absolute velocity is not valid any more.

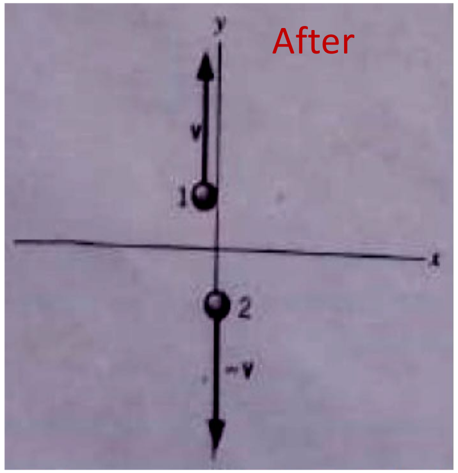
- As dynamical quantities, such as momentum and kinetic energy, depend on length, time, and velocity, therefore, there is a need of modification in these quantities also.

  - Are momentum, $p = mv$ and energy $\frac{1}{2}mv^2$ still valid?
  - Are the conservation laws of momentum and energy still applicable?

Consider the collision of two particles each of mass m from the frame of observer O.

- Particles are moving with equal and opposite velocities v and $-v$ along x –axis.





- The collision at the origin.

- After collision, the particles move along y –axis with equal and opposite final velocities.

- The collision is perfectly elastic , so that no Kinetic energy is lost. The final velocities must then be v and $-v$.

Using the classical formula $(p = mv)$, from the frame of observer O:

$$Initial: \rightarrow \quad \begin{aligned} p_{xi} &= mv + m(-v) = 0 \\ p_{yi} &= 0 \end{aligned}$$

$$Final: \rightarrow \quad \begin{aligned} p_{yf} &= mv + m(-v) = 0 \\ p_{xf} &= 0 \end{aligned}$$

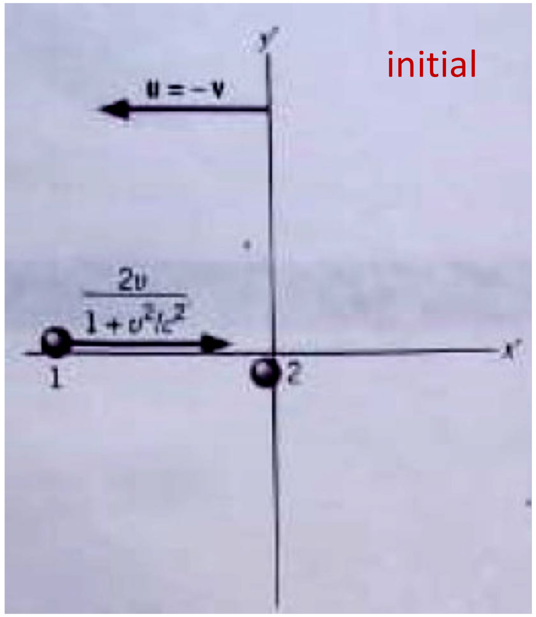
$$Therefore, \quad p_{xi} = p_{xf} \; and \; p_{yi} = p_{yf}$$

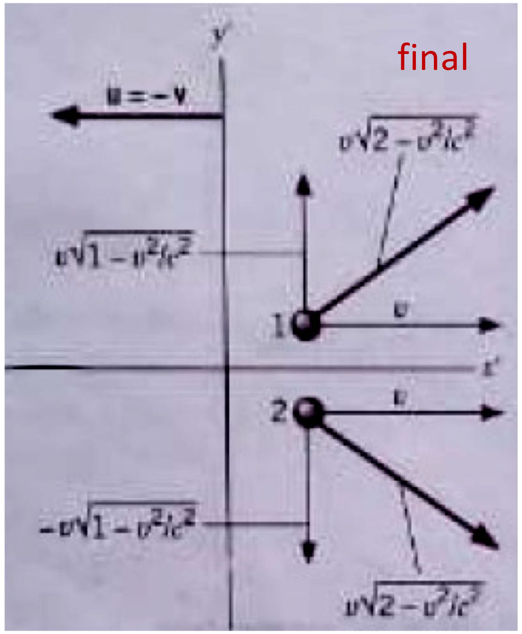
$$Initial\ momentum = final\ momentum$$

$$Momentum\ conserved\ in\ the\ reference\ frame\ of\ O.$$

From frame of observer O'

- Now, observation of same collision from O' which moves relative to frame O with speed $u = -v$.
- Particle 2 is at rest in frame O' before the collision.
- We use Lorentz velocity transformation, to find the transformed x' and y' components of Initial and final velocities, observed by O',





Now, we use these velocities to find the components of the momentum of the system in the frame O':

$$p'_{xi} = m\left(\frac{2v}{1+v^2/c^2}\right) + m(0) = \frac{2mv}{1+v^2/c^2}$$

$$p'_{xf} = mv + mv = 2mv$$

$$p'_{yi} = 0$$

$$p'_{yf} = mv\sqrt{1-v^2/c^2} + m(-v\sqrt{1-v^2/c^2}) = 0$$

$$here, \;\; p'_{xi} \neq p'_{xf},$$

So, observer O' concludes that momentum of the system is not conserved.

- It is clear from above calculations that the law of conservation of linear momentum, does not satisfy Einstein's first postulate (the law must be the same in all inertial frames) if we calculate momentum as $p = mv$.

- Therefore, If we retain this conservation of momentum as general law consistent with Einstein's first postulate, we must find a new definition of momentum.

- This new definition must have two properties:

    - ✓ It must yield a law of conservation of momentum that satisfies the principle of relativity.
    - ✓ At low speed, it reduces to classical momentum, $p = mv$

These requirements are satisfied by defining the relativistic momentum of a particle of mass m moving with velocity v as

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

Relativistic Momentum

In terms of components, we can write as

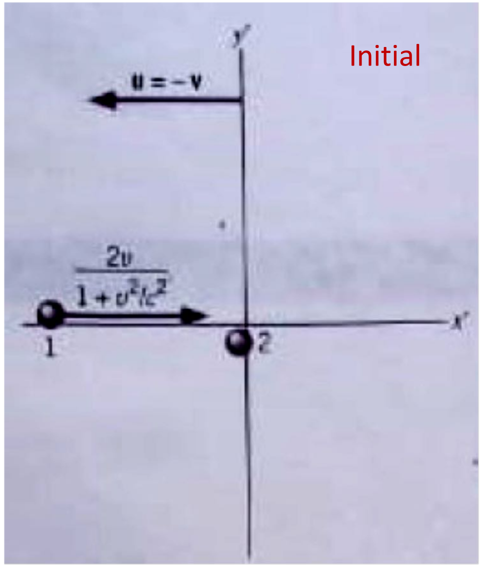
$$p_x = \frac{mv_x}{\sqrt{1 - v^2/c^2}} \qquad p_y = \frac{mv_y}{\sqrt{1 - v^2/c^2}}$$

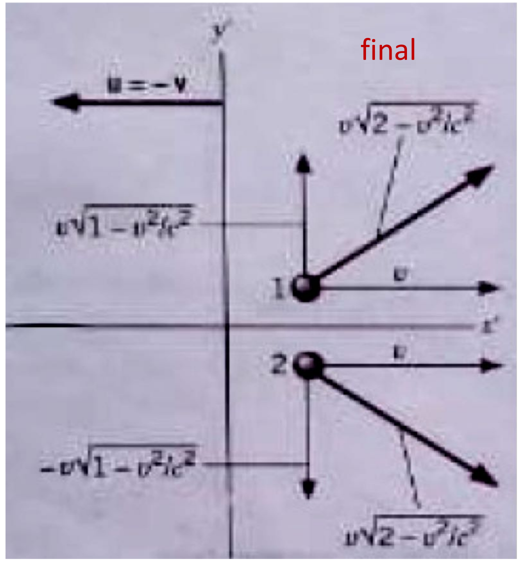
Note:

- The velocity v that appears in the denominator of these expressions is always the velocity of the particle as measured in a particular inertial frame. It is not the velocity of an inertial frame.
- The velocity in the numerator can be any of the components of the velocity vector.

Let us see how this new definition of momentum at high speed restores the conservation of momentum law in collision example.

In frame O, the velocities before and after collision are equal and opposite and thus, relativistic momentum equation again gives zero for the initial and final momenta.





In frame O', we get,

$$p'_{yi} = 0$$

$$p'_{yf} = mv\sqrt{1-v^2/c^2} - mv\sqrt{1-v^2/c^2} = 0$$

$$p'_{xi} = \frac{m(\frac{2v}{1+v^2/c^2})}{\sqrt{1-(\frac{2v}{1+v^2/c^2})^2/c^2}} + m(0) = \frac{2mv}{1-v^2/c^2}$$

$$p'_{xf} = \frac{mv}{\sqrt{1-\frac{\left(v\sqrt{2-v^2/c^2}\right)^2}{c^2}}} + \frac{mv}{\sqrt{1-\frac{\left(v\sqrt{2-v^2/c^2}\right)^2}{c^2}}} = \frac{2mv}{1-v^2/c^2}$$

Thus, $p'_{yi} = p'_{yf} = 0$

$$p'_{xi} = p'_{xf} = \frac{2mv}{1 - v^2/c^2}$$

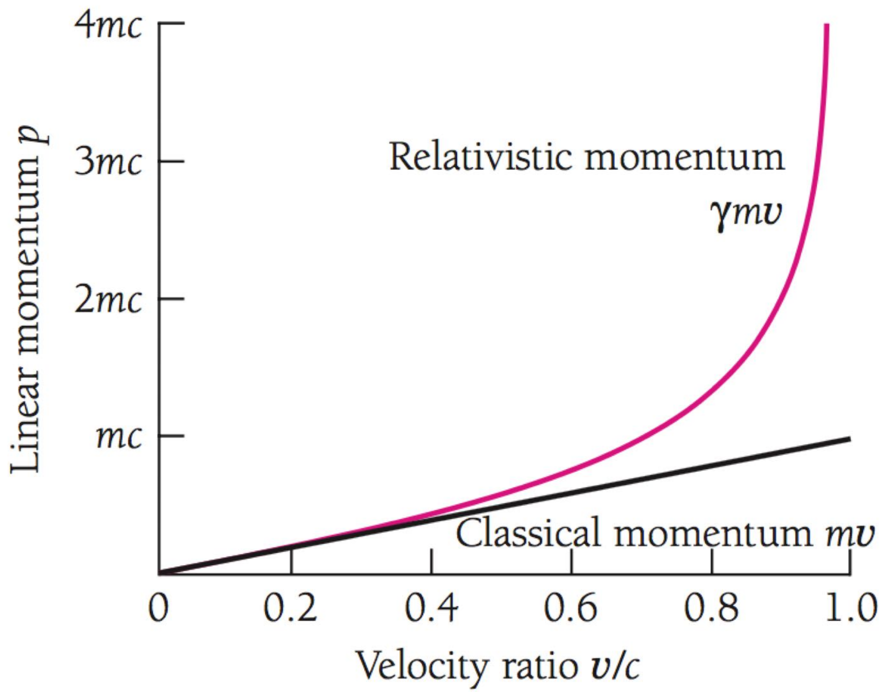
The initial and final momenta are equal in frame O' also.

The momentum is conserved in both O and O' frames now.

Conclusions:

- Definition of momentum given by $p = \frac{mv}{\sqrt{1-v^2/c^2}}$ satisfies conservation of momentum in all inertial frames (Einstein's first postulate) .
- At low speeds, it reduces to classical formula $p = mv$.

Comparison of classical & relativistic momentum



Let $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

Relativistic momentum $\mathrm{P} = \gamma m\boldsymbol{v}$ is always correct at all speeds.

Classical momentum $\mathrm{P} = m\boldsymbol{v}$ is valid for $v \ll c$.

Object's velocity v can never reach c because,

relativistic $P = \infty$ for $v = c$! (**impossible**)

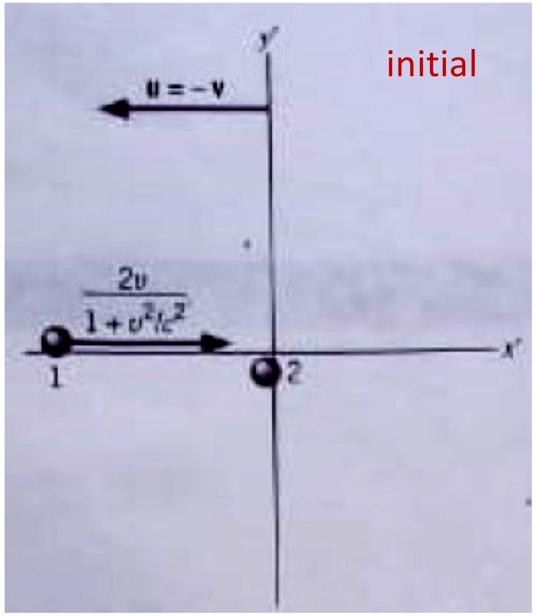
Problem on Relativistic momentum

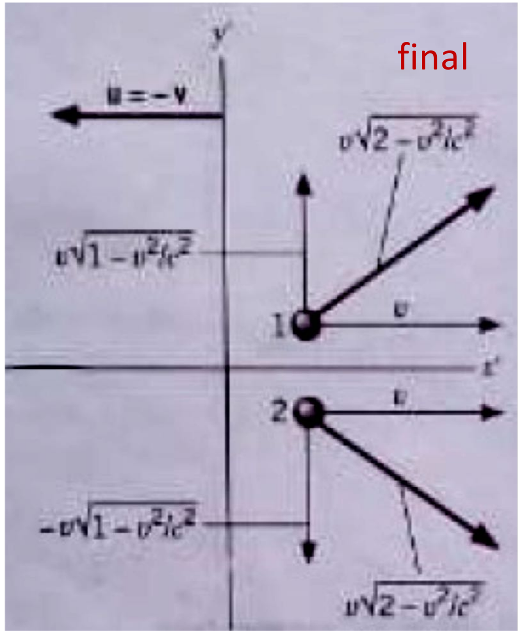
Inside a television, electrons move at a speed of $5\times10^{7}\ m/s$. If the mass of electron is $9.1\times10^{-31}Kg$, find its momentum.

Relativistic Kinetic Energy

Consider the collision event that we discussed for momentum. Using the classical expression for kinetic energy $K = \frac{1}{2}mv^2$,

Kinetic energy is conserved in O frame.





But for O' frame:

$$K_i' = \frac{2mv^2}{(1+v^2/c^2)^2}$$

$$K_f' = mv^2(2 - v^2/c^2)$$

$$K_i' \neq K_f'$$

The elastic collision does not conserve kinetic energy in the O' frame.

Elastic vs inelastic collision should depend

- on the properties of the colliding objects
- not on the particular reference frame

There is no limit (in either classical or relativistic dynamics) to the energy we can give to a particle. But if we allow K to increase without limit,

$$K = \frac{1}{2}mv^2$$

implies that the velocity must correspondingly increase without limit-

"violation of II postulate of relativity"

The **classical expression** for kinetic energy ($K = \frac{1}{2}mv^2$):

(i) depends on the choice of frame: **violates I postulate of relativity**

(i) Allows speeds in excess of speed of light: **violates II postulate of relativity**

we need a new definition of kinetic energy:

- To preserve the law of conservation of energy.
- which obeys both the postulates of relativity.
- to allow K to increase without limit, while its speed remains less than c.

Derivation of the relativistic formula for K

We know from work-energy theorem that:

$$\text{kinetic energy (K)} = \int_0^s F\, ds$$

Consider the relativistic form of II Law of motion:

$$F = \frac{dP}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1-v^2/c^2}}:$$

$$K = \int_0^s \frac{d}{dt}\frac{mv}{\sqrt{1-v^2/c^2}}\, ds \quad \text{(sub for F)}$$

changing variable from ds to dv

$$= \int_0^v \boldsymbol{v}\, d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right)$$

Integrating by parts,

$$K = \frac{mv^2}{\sqrt{1-v^2/c^2}} - m\int_0^v \frac{v\,dv}{\sqrt{1-v^2/c^2}}$$

On solving we get the relativistic expression for the kinetic energy of the particle to be:

$$K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$ "**Relativistic Kinetic Energy**"

From the first term of the above equation, K→ ∞ as $v = c$.

Thus we can **increase the kinetic energy** of a particle without limit, and its **speed will not exceed *c***.

on rearranging the relativistic kinetic energy,

$$\underbrace{\frac{mc^2}{\sqrt{1-v^2/c^2}}}_{\substack{\textbf{Relativistic}\\ \textbf{total energy}\\ \mathbf{E}}} = \underbrace{mc^2}_{\substack{\textbf{Rest}\\ \textbf{Energy}\\ \boldsymbol{E_0}}} + K$$

Consider the relativistic total energy $\mathrm{E} = E_0 + K$

(i) In an isolated system ($\Delta E = 0$), $\Delta K = -\Delta E_0$

Any **change in the kinetic energy** must be accompanied by a **change in rest energy** of the **opposite sign**.

(ii) The rest energy $\boldsymbol{E_0}$ is the relativistic total energy $\boldsymbol{E}$ of a particle measured in a frame of reference in which the particle is at rest.

So, if the particle is at rest(v=0) & K=0

Rest energy: $$\boxed{\boldsymbol{E_0} = mc^2}$$

The most famous relationship Einstein obtained from the postulates of special relativity.

Since mass and energy are not independent entities, their separate conservation principles are properly a single one– the principle of conservation of mass energy.

The Law of Conservation of Relativistic Mass Energy:

- In an isolated system of particles, the relativistic total energy remains constant.

- Mass can be created or destroyed, but when this happens, an equivalent amount of energy simultaneously vanishes or come into being and vice versa.

Conserved quantity vs invariant quantity:

Total relativistic energy E of an isolated system is a **conserved quantity** in a given reference frame–This means E may be different (for the same isolated system) as measured from another frame.

Rest mass ***m* is invariant** -- m has the same value in all inertial frames.

Kinetic energy at low speeds($v \ll c$)

Consider $$\mathrm{K} = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

When the relative speed $v \ll c$, $\quad v^2/c^2 \ll 1$

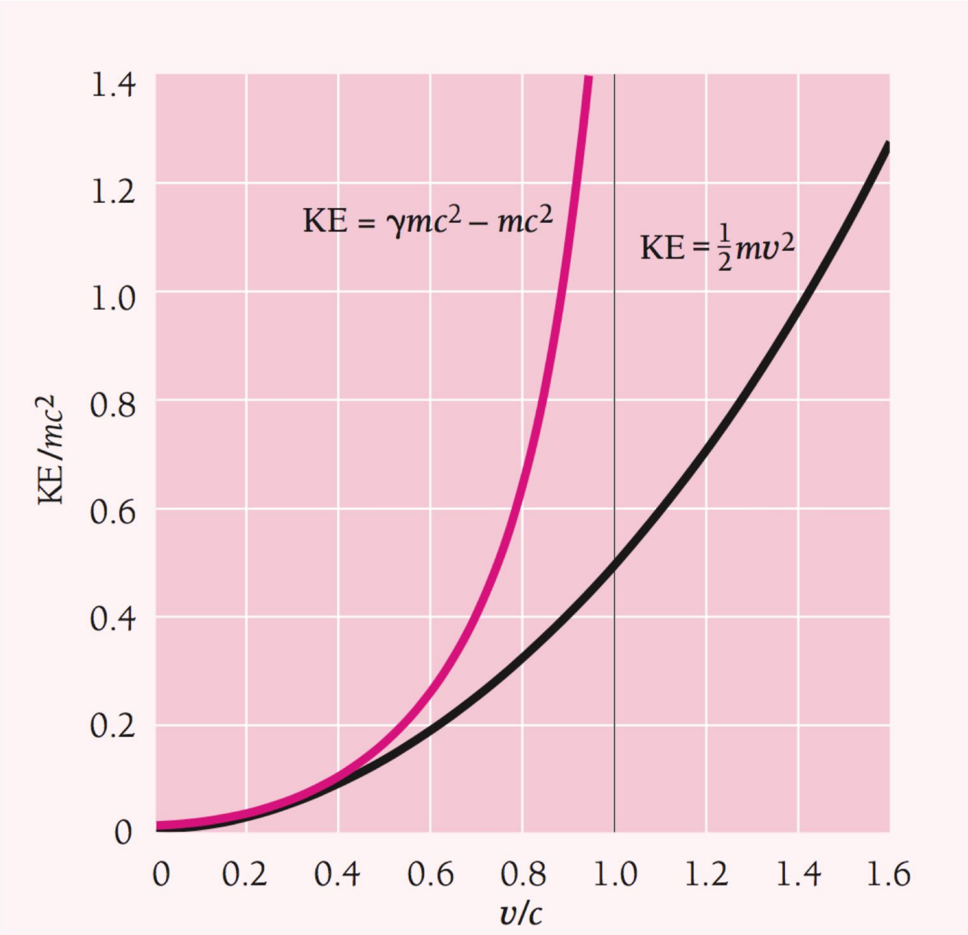
we can use the binomial approximation $(1+x)^n \approx 1 + nx$ (for $|x| \ll 1$), to obtain

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Thus we have, $$\mathrm{K} = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) mc^2 - mc^2$$

$$\mathrm{K} \approx \frac{1}{2}\, mv^2 \quad \text{(for } v \ll c\text{)}$$

Classical vs relativistic kinetic energy



For $v \ll c$:

The two formulas give the same result.

For $v \approx c$ Both diverge

For $v = c$:

Relativistic K = ∞

Classical K = $\frac{1}{2}$ (rest energy)

Relation between Energy and Momentum

In an **isolated system**,

- Total momentum and energy are **conserved.**
- Rest energy is **invariant**.

Consider $\quad E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

On squaring E and subtracting p^2c^2

$$E^2 - p^2c^2 = (mc^2)^2$$ "**Energy & momentum**"

Since mc^2 is invariant, $E^2 - p^2c^2$ is also invariant

"A particle has same value of $E^2 - p^2c^2$ in all frames of references"

Can massless particles exist?

For $\boldsymbol{m} = \mathbf{0}$ and $\boldsymbol{v} \ll \boldsymbol{c}$: $\quad \mathrm{E} = P = 0$

So, a massless particle with a speed less than that of light can have neither energy nor momentum.

For $\boldsymbol{m} = \mathbf{0}$ and $\boldsymbol{v} = \boldsymbol{c}$:

$\mathrm{E} = 0/0$ and $\mathrm{P} = 0/0$. **E and P can have any values**

$\mathrm{E} = pc$ **(for massless particles)**

∴ The equations for E and P are consistent with the existence of massless particles that possess E and P provided they travel with the speed of light.

"All massless particles travel at the speed of light"

Problem on Relativistic Energy

An electron ($m = 0.511\ MeV/c^2$) and a photon both have momenta of $2.000\ MeV/c$. Find the total energy of each.

Solution:

(a) The electron's total energy is

$$E = \sqrt{m^2c^4 + p^2c^2}$$
$$= \sqrt{(0.511\ MeV/c^2)^2 c^4 + (2.000\ \ MeV/c)^2 c^2}$$
$$= \sqrt{(0.511\ MeV)^2 + (2.000\ MeV)^2}$$

Electron's total energy = 2.064 MeV

(b) The photon's total energy is $E = pc$

$$= (2.000\ \ MeV/c)\, c$$

Photon's total energy $= \mathbf{2.000\ \ MeV/c}$