Modern Physics

Special theory of relativity

Dr VS Gayathri

Books:

1. Modern Physics by Kenneth S. Krane
2. Concept of Modern Physics by Arther Beiser

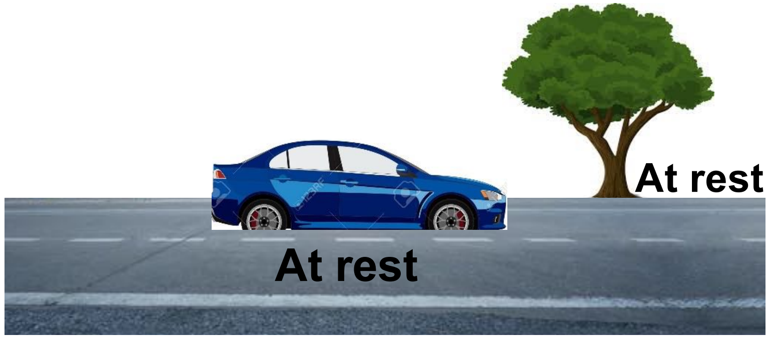
This 12-foot tall statue of Albert Einstein is located at the headquarters of the National Academy of Sciences in Washington DC. The page in his hand shows three equations that he discovered: the fundamental equation of general relativity, which revolutionized our understanding of gravity; the equation for the photoelectric effect, which opened the path to the development of quantum mechanics; and the equation for mass-energy equivalence, which is the cornerstone of his special theory of relativity.

Classical Relativity

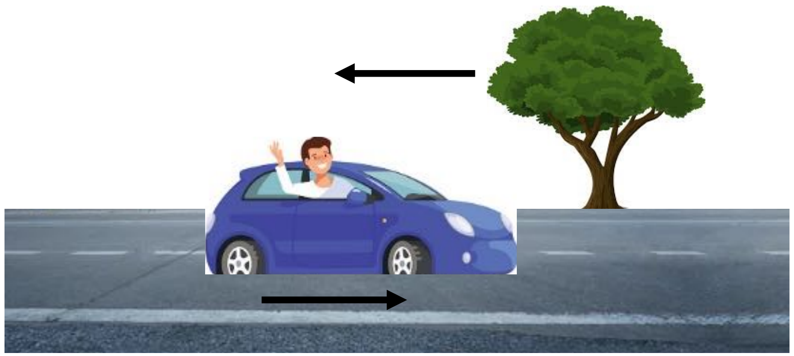
A "theory of relativity" is a way for observers in different frames of reference to compare the results of their observations.

Example:

- Consider *an observer in a car parked (at rest)* on the road near a large tree. To this observer, the tree is at rest.



- *Car started moving along the road*, now, observer sees the tree rush back as the car drives by. To this observer, the tree appears to be moving.



❖ A *theory of relativity* provides the conceptual framework and mathematical tools that enable the two observers to transform a statement:

- In one frame of reference, the statement is “Tree is at rest”.
- In another frame of reference, the statement “Tree is in motion”.

How to compare these two observations/statements?

❖ The mathematical basis for comparing the two descriptions is called a **transformation.**

❖ **Types of Transformation:**

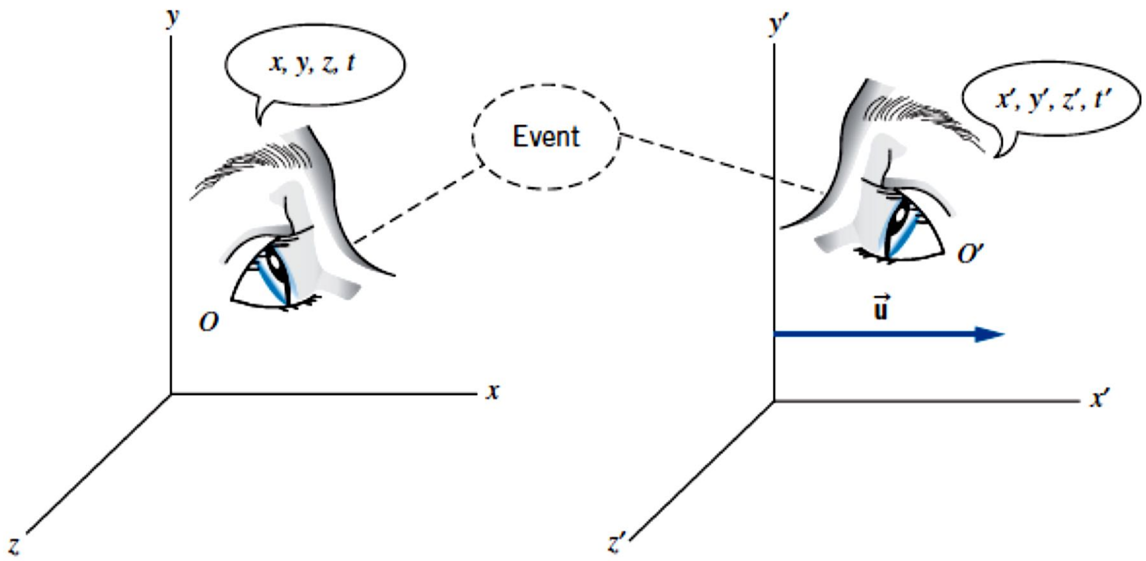
- Galilean transformation or Classical transformation
- Lorentz transformation

Classical or Galilean transformation

- Two observers O and O' are at rest in their own frames of reference (FOR), but move relative to each other with a constant velocity $\vec{u}$(<<c).

- $\vec{u}$ represents the velocity of O' measured by O.
-
- Thus, O' would measure velocity $-\vec{u}$ for O.

- They observe the same event, which happens at a particular point in space and a particular time.



Observations by two observers:

According to O, the space and time coordinates of the event :(x, y, z, t)
According to O', the space and time coordinates of the same event :(x', y', z', t')

How to relate these two observations?

As discussed earlier, to relate the coordinates (x, y, z, t) and (x', y', z', t'), we need a classical or *Galilean* transformation.

To derive a relation under *Galilean* transformation, we assume:

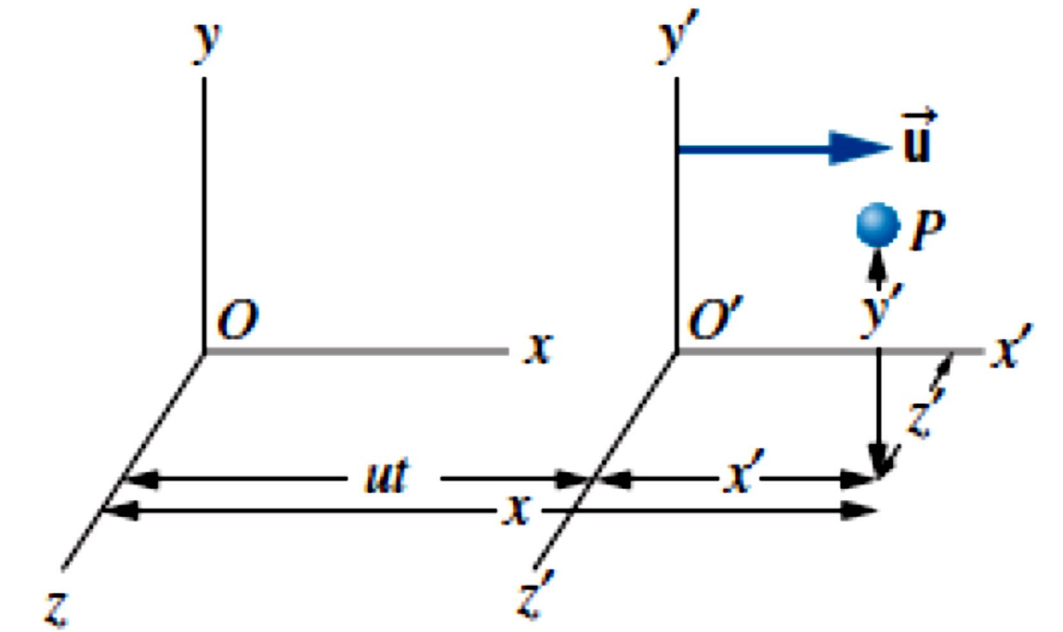
1. As a postulate of classical physics that $t = t'$ that is, time is the same for all observers.
2. The coordinate systems are chosen so that their origins coincide at $t = 0$.

***Galilean* coordinate transformation**

Now, Consider an object or a event P in O' in the coordinates x', y', z'.

According to *O*, the *y* and *z* coordinates are the same as those in O'

Along the *x* direction, *O* would observe the object at: $x = x' + ut$



We therefore have the *Galilean* coordinate transformation

$$\left.\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad 1$$

Galilean velocity transformation:

To find the velocities of the object as observed by $O \; and \; O'$, we take the derivatives of these expressions with respect to t' on the left and with respect to t on the right (as $t = t'$).

$$\left.\begin{aligned} v'_x &= v_x - u \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned}\right\} \quad 2$$

Galilean acceleration transformation:

In a similar manner, we can take the derivatives of Eq. 2 with respect to time to get acceleration equations.

$$\left.\begin{aligned} a'_x &= a_x \\ a'_y &= a_y \\ a'_z &= a_z \end{aligned}\right\} \quad 3$$

- Equation 3 shows that Newton's laws hold for both observers.

- As long as $\vec{u}$ is constant $(\frac{d\vec{u}}{dt} = 0)$, the observers measure identical accelerations and agree on the results of applying $\vec{F} = m\vec{a}$.

Newton's laws remain valid in Galilean transformation from one inertial frame to other.

Whether other laws of physics are also invariant from one frame to another?

Example 1

Two cars are traveling at constant speed along a road in the same direction. Car A moves at 60 km/h and car B moves at 40 km/h, each measured relative to an observer on the ground. What is the speed of car A relative to car B?

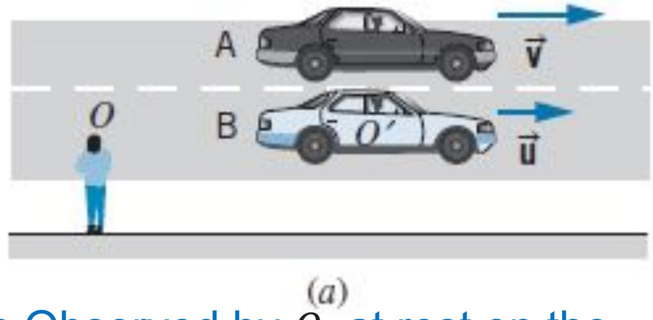
Solution:

Let O be the observer on the ground, who observes car A to move at $v_x = 60\ km/h$

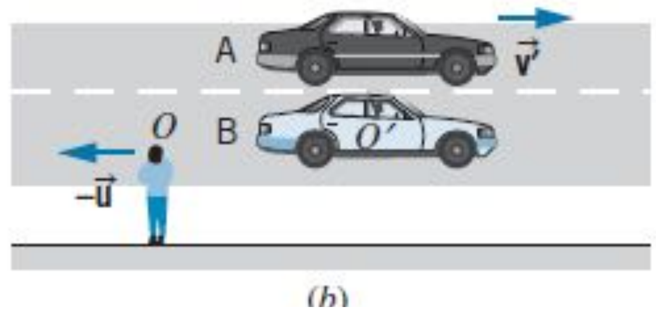
Assume O' to be moving with car B at $u = 40\ km/h$.

Then, the speed of car A relative to car B,

$$v'_x = v_x - u$$
$$= 60\ km/h - 40\ km/h$$
$$= 20\ km/h$$



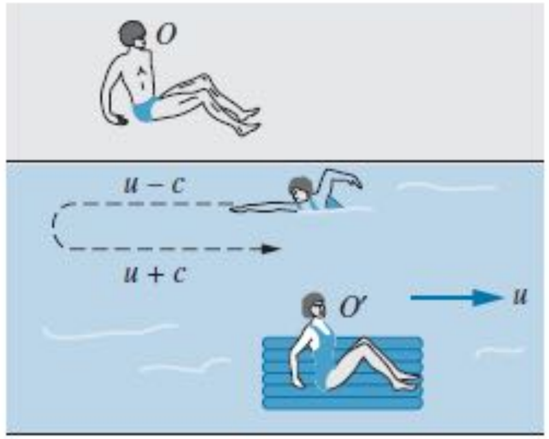
As Observed by O at rest on the ground

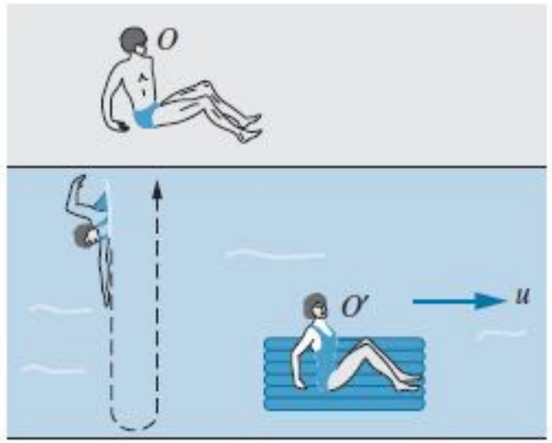


As Observed by O' in car B

Example 2

A swimmer capable of swimming at a speed c in still water is swimming in a stream in which the current is u ($\ll c$). Suppose the swimmer swims upstream a distance L and then returns downstream to the starting point. Find the time necessary to make the round trip, and compare it with the time to swim across the stream a distance L and return.





- The motion of a swimmer as seen by observer O at rest on the bank of the stream.
- Observer O' moves with the stream at speed u

Solution:

Let the FOR of O is ground and the FOR of O' is water, moving at speed u.

The velocity of swimmer relative to water, $v_x' = -c$ (for upstream swim).

Now, acc. To Galilean velocity transformation, $v_x' = v_x - u$

So, $v_x = v_x' + u = -c + u$, it is negative as $u < c$.

Thus, $|v_x| = c - u$

$$Therefore, for\ the\ upstream\ swim, \qquad t_{up} = L/(c-u)$$

$$For\ the\ downstream\ swim, \qquad t_{down} = L/(c+u)$$

$$So, \qquad the\ total\ time\ is\ , \qquad t = \frac{L}{c+u} + \frac{L}{c-u}$$

$$= \frac{L(c-u) + L(c+u)}{c^2 - u^2}$$

$$= \frac{2Lc}{c^2 - u^2}$$

$$\boxed{t = \frac{2L}{c}\frac{1}{1 - u^2/c^2}}$$

This is the total time taken by the swimmer to make the round trip (upstream a distance *L* and then returns downstream to the starting point).

Now, to swim directly across the stream, the swimmer's efforts must be directed somewhat upstream to counter the effect of the current.

This means, in the frame of reference of O, we have $v_x = 0$, which requires $v'_x = -u$ acc. to Galilean velocity transformations.

Since, the speed of the water is always c,

$$\sqrt{v_x'^2 + v_y'^2} = \mathrm{c}$$

$$thus, \qquad v'_y = \sqrt{c^2 - v_x'^2} = \sqrt{c^2 - u^2}$$

Thus, the round-trip time is

$$\boxed{t = 2t_{across} = \frac{2L}{\sqrt{c^2 - u^2}} = \frac{2L}{c} \frac{1}{\sqrt{1 - u^2/c^2}}}$$

We can compare it with upstream – downstream swim

Michelson-Morley Experiment: Ether hypothesis

- Is there any universal frame of reference, relative to which light propagates?
- Is motion "absolute" with respect to this universal frame of reference or relative to a specified frame?

Is motion absolute or relative?

Belief of the 19th century Physicists:

❖ Light wave require an invisible, massless medium called "ether", for its propagation.

❖ Speed of the beam of light (emitted by a source on Earth) depends on the magnitude and the direction of the ether.

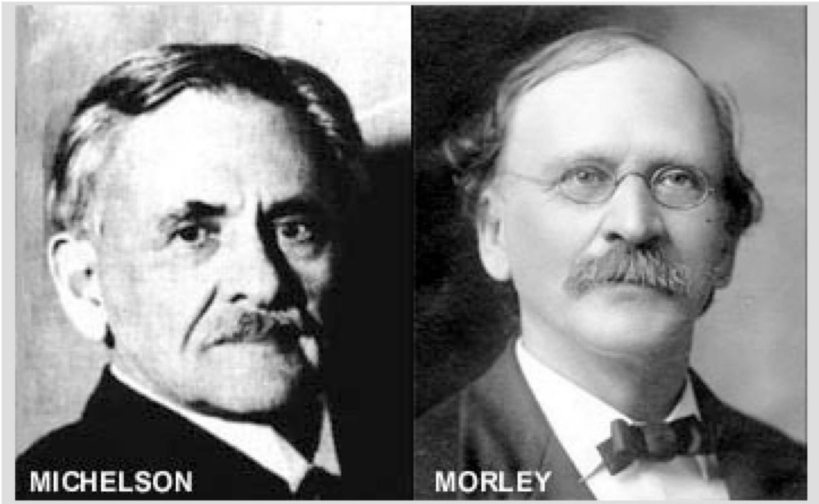
Challenges:

❖ How to prove the existence of ether ?

❖ How to measure the speed of ether relative to earth?

Michelson-Morley Experiment

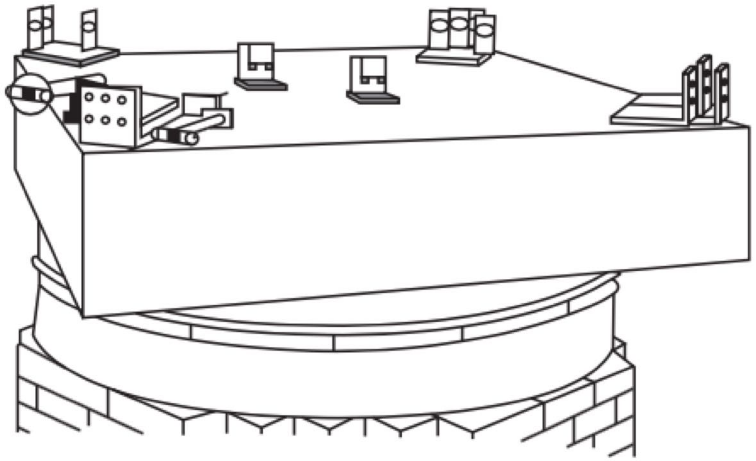
To test if there is a universal frame of reference (ether) and to measure the motion of the earth through the ether, Michelson-Morley performed an experiment in the year 1887.



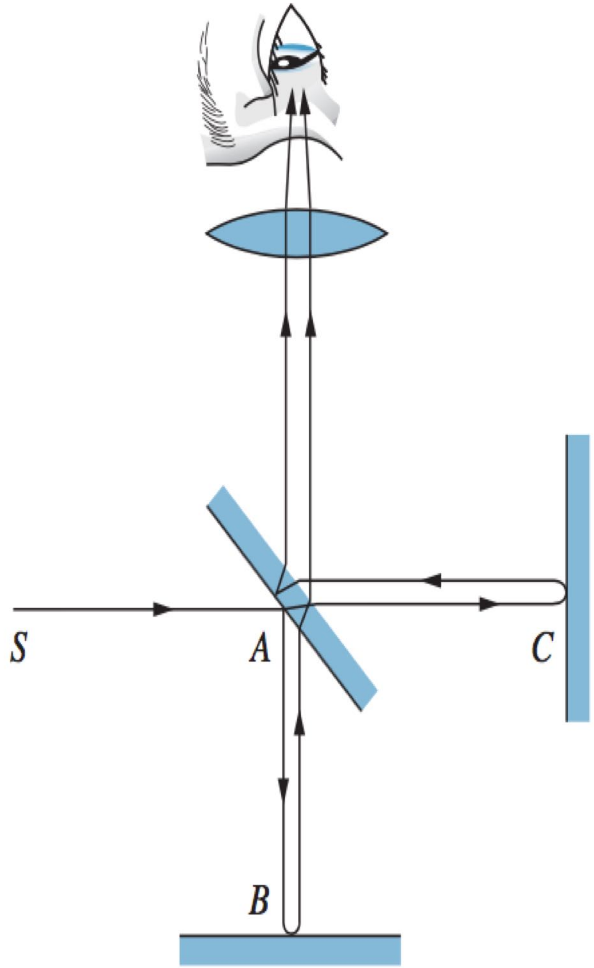
American physicist Albert A. Michelson and his associate E. W. Morley

Michelson-Morley Experimental set-up

Light from source *S* is split at *A* by the half-silvered mirror; one part is reflected by the mirror at *B* and the other is reflected at *C*. The beams are then recombined for observation of the interference.



Michelson's apparatus.



Michelson-Morley Experiment

Objective:

- To obtain the evidence for ether, by measuring the motion of the Earth moving through the ether

- Suppose the speed of light moving along x' direction, has a precise value of c for an observer O'. According to Galilean transformation, it would have a speed of $c + u$ relative to O. Is this true for light?

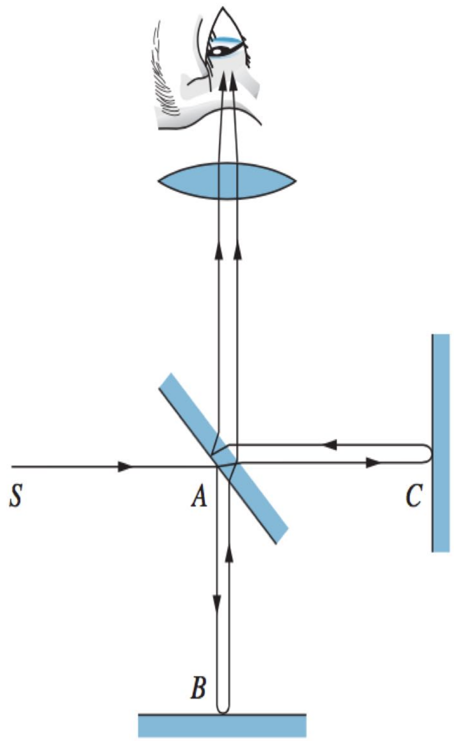
Consider the two beams:

(i) along a path parallel (upstream-downstream) to ether current (along the direction of Earth's motion)

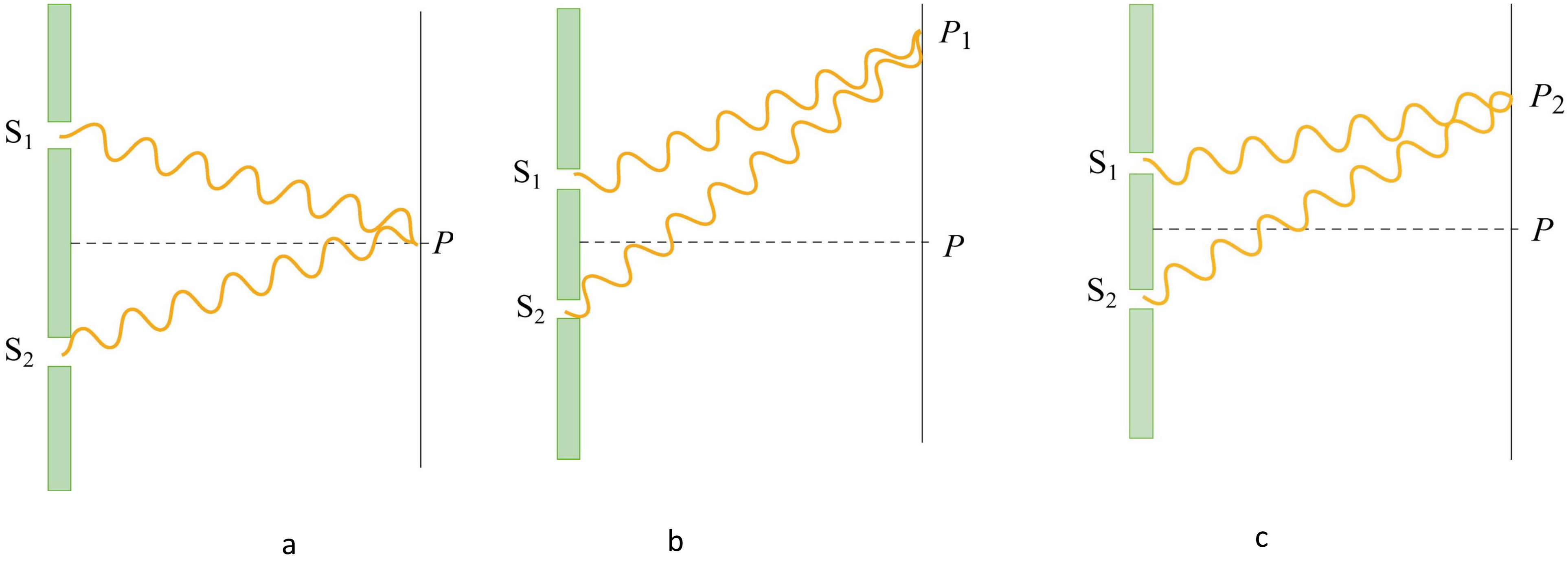
(ii) perpendicular to ether current(cross-stream)

Let Δt be the time difference between the two beams.

The upstream-downstream and the cross-stream beams superimpose on reaching the detector, resulting in interference.



Interference

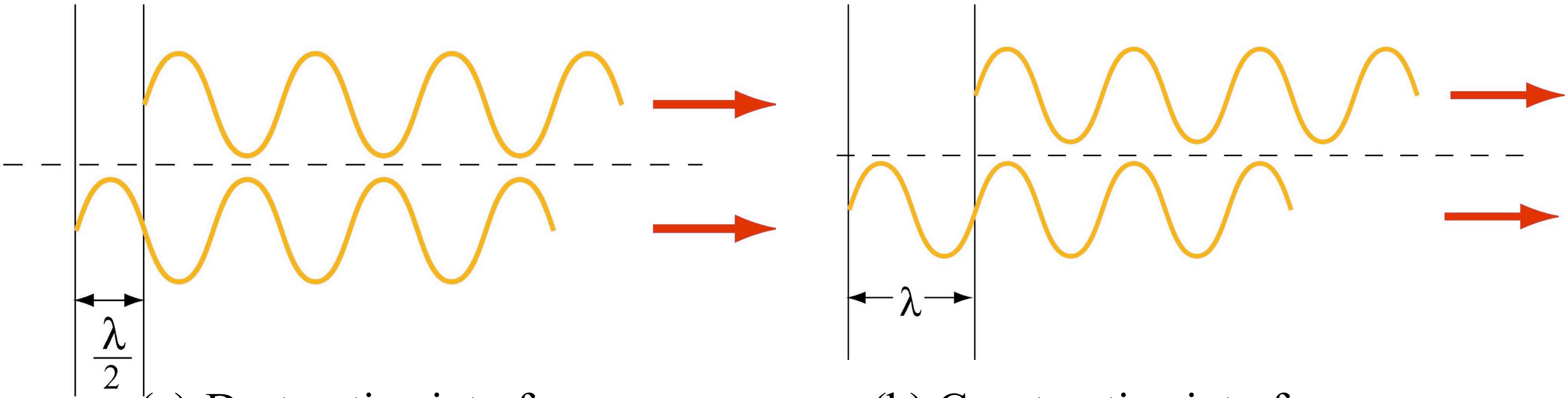


Constructive interference (a) at P , and (b) at P_1 . (c) Destructive interference at P_2 .

•Constructive interference –wave crest meet wave peaks/crest

•Destructive interference–wave peaks meet wave troughs

Interference and path difference



(a) Destructive interference (b) Constructive interference.

If the path difference (due to difference in path lengths) :

(i) Integral multiple of wavelength ($\lambda, 2\lambda\ldots$) then waves are in phase : constructive interference

(ii) Half-integral multiple of wavelength ($\frac{\lambda}{2}, \frac{3\lambda}{2}$ ….)the waves are π rad out of phase: Destructive interference

In Michelson-morley experiment, a monochromatic beam of light is split in two; the two beams travel different paths and are then recombined. Any phase difference between the combining beams causes bright and dark bands or “fringes” to appear, because of constructive and destructive Interference, respectively,

There are two contributions to the phase difference between the beams:

- Path difference $ACA - ABA$, if one of the beams travel a longer distance.
- The time difference between the upstream-downstream and cross-stream beams (**Proof of motion of Earth through ether**)

To isolate the second contribution, rotate the entire apparatus by 90°. Rotation does not change the first contribution to phase difference. But what was an upstream-downstream path before the rotation becomes a cross-stream path after the rotation

Expected deliverables from the experiment:

- If the transit times of the two beams are same and if they are in phase : Constructive interference

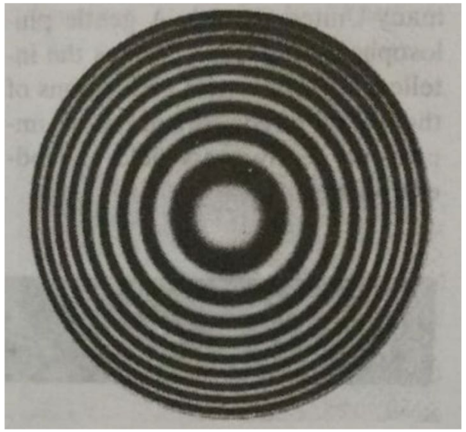
- If transit times are different because of the presence of ether current (parallel to one of the beams) : interference fringe pattern shifts.

- Using the value for Δt (difference in transit times), we can calculate the relative velocity $\boldsymbol{u}$ between O'(ether) and O(Earth's reference frame).

Result observed by Michelson-Morley

No change in the interference fringe pattern!

This implies $\Delta t = 0$, if path lengths are equal.



Interference fringes as observed with the Michelson interferometer

- They reasoned that perhaps the orbital motion of the Earth just happened to cancel out the overall motion through the ether.
- If it is true, six months later (when the Earth would be moving in its orbit in the opposite direction) the cancellation should not occur.
- When they repeated the experiment six months later, they again obtained a **NULL** result!

"Motion of the Earth through the ether could not be detected"

Michelson-Morley Experiment

Consequences of negative outcome:

- Ether does not exist. So, there is “no absolute motion ” relative to ether.
- All motion is relative to a specified frame, not to a universal one.

Postulates of the special theory of relativity

The postulates are proposed by Albert Einstein in 1905 and are as follows:

- **Postulate I:** The principle of relativity

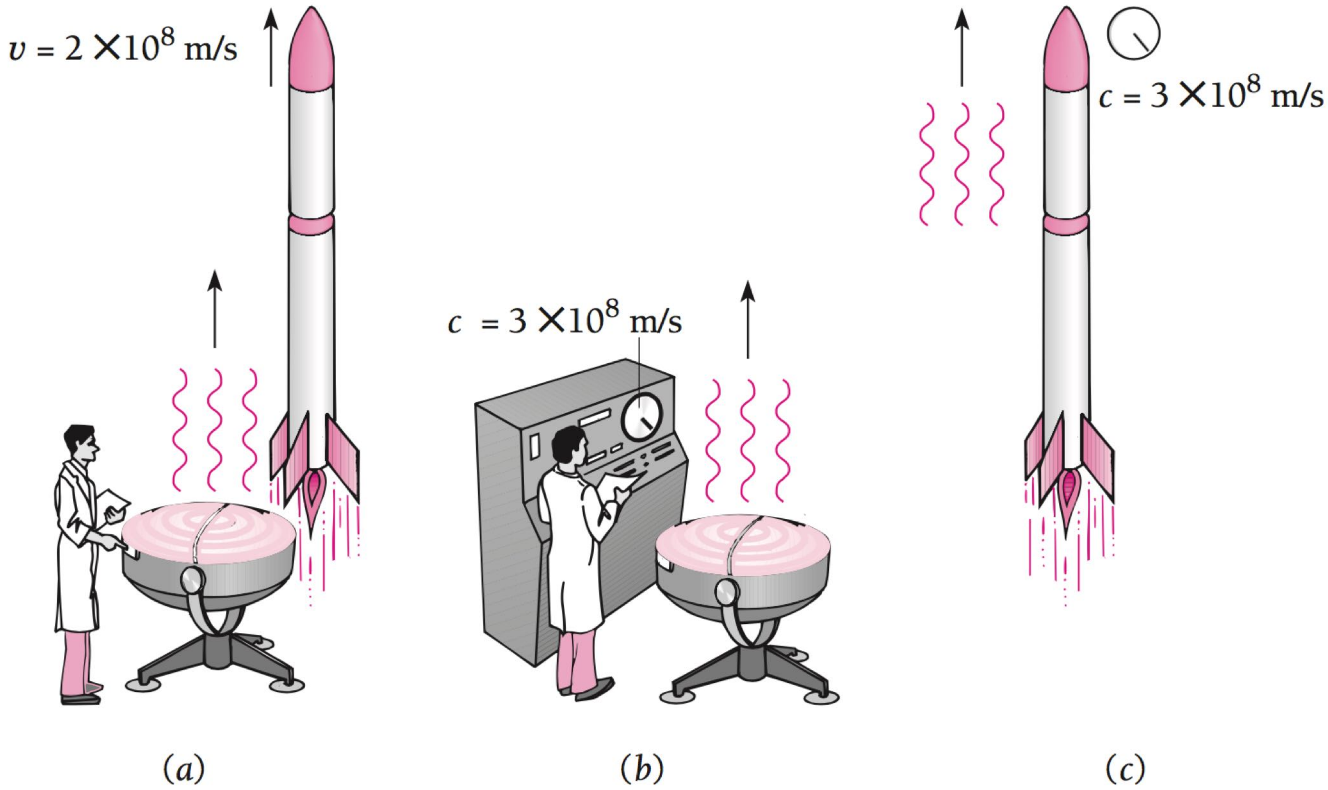
The laws of physics are the same in all inertial frames of reference.

(i) It follows from the absence of a universal frame of reference.
(ii) According to this postulate, laws of Physics are absolute, universal and same for all inertial observers.

- **Postulate II:** The principle of the constancy of the speed of light

The speed of light in free space has the same value c in all inertial reference frames.

Postulate II: Constancy of Speed of light



The measurement of space and time are not absolute but they are relative.

But speed of light in free space is same for all observers

(a) Stationary observer O turn on search light. Observer O' moving in space craft at $v = 2 \times 10^8\, m/s$

(b) Speed of light measured observer O : c= $3 \times 10^8\, m/s$

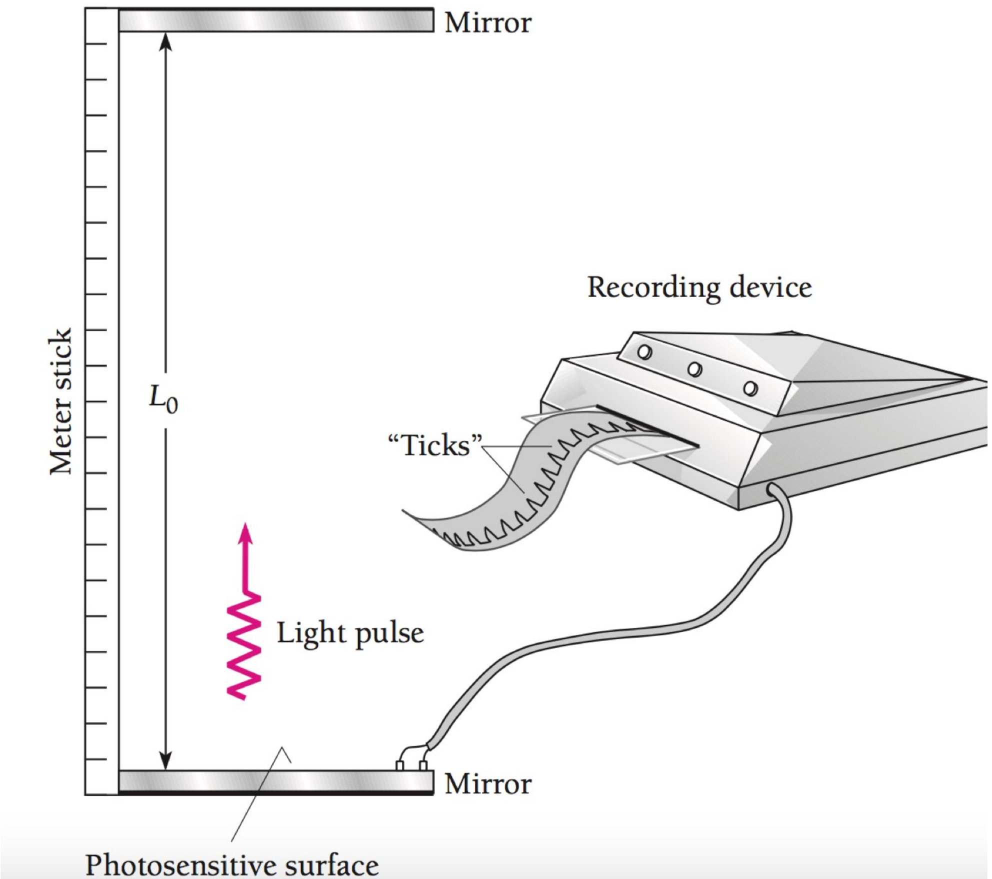
(a) Observer O' also measures speed of light c= $3 \times 10^8\, m/s$! (though he is moving parallel to the waves at $v = 2 \times 10^8\, m/s$)

Inferences from the special theory of relativity

- Ether hypothesis is discarded.
- There is no preferred frame of reference—as all inertial frames are equivalent.
- Eliminates the Galilean-Newtonian concept of absolute space and time. At higher speeds, Newtonian mechanics fails and must be replaced by the relativistic version.
- Einstein's postulates require a new consideration of the fundamental nature of time and space.

How does these postulates affect the measurements of length and time intervals, at high speeds, by observers in different frames?

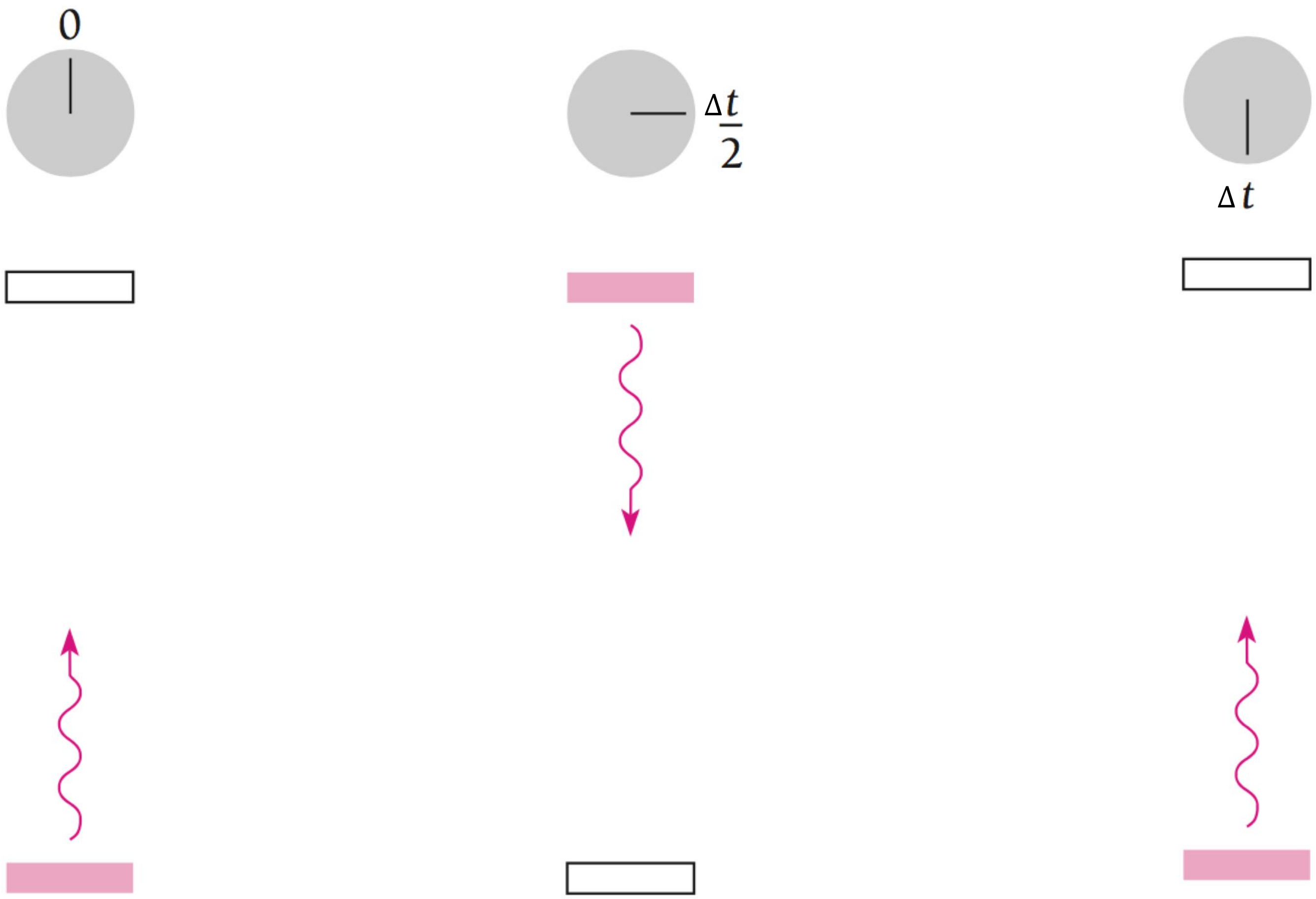
The relativity of Time:Time dilation



A simple light-pulse clock:

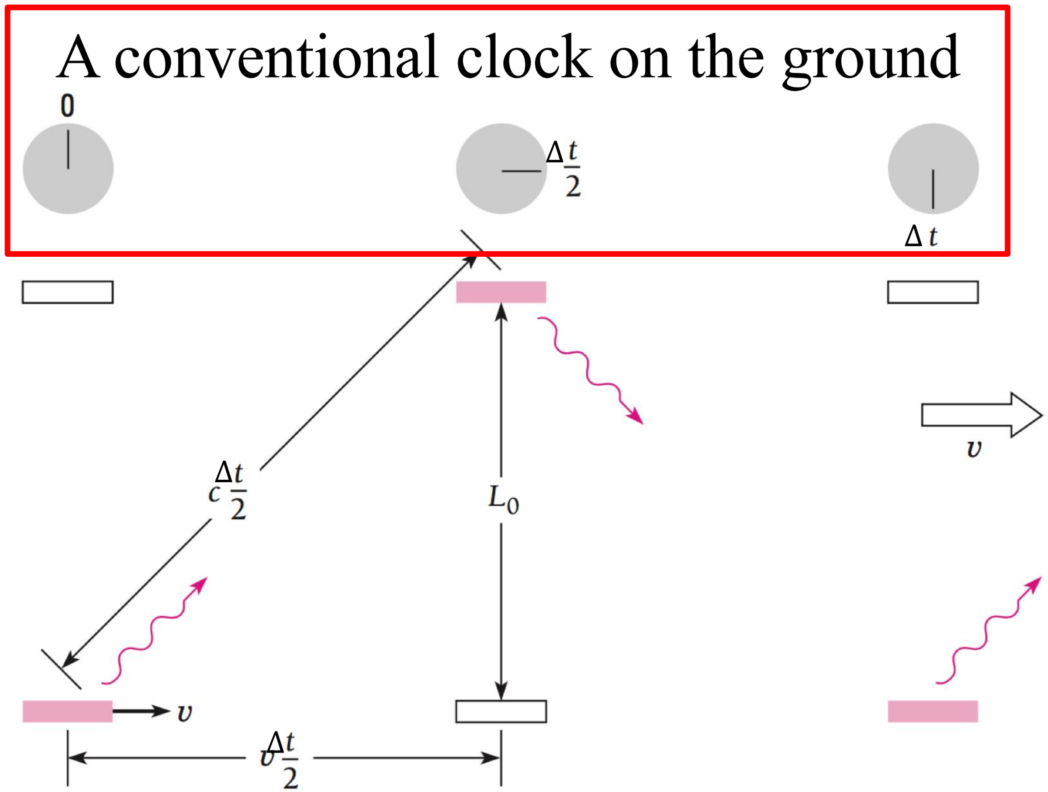
A pulse of light is reflected back & forth between two mirrors L_0 apart. Whenever light strikes the lower mirror, an electrical signal is produced that marks the recording tape. Each mark corresponds to the tick of an ordinary clock.

The relativity of Time:Time dilation



A light-pulse clock at rest on the ground as seen by an observer on the ground.

The relativity of Time: Time dilation



A light-pulse clock in a spacecraft as seen by an observer on the ground. The mirrors are parallel to the direction of motion of the spacecraft. Because the clock is moving, the light pulse, as seen from the ground, follows a zig-zag path.

Does an observer on the ground find both the clocks tick at the same rate?

In relativity, the frame in which the observed body is at rest is **proper frame/rest frame**. Consider an occurrence that has a duration Δt_0.

The quantity Δt_0, determined by events that occur at the same place(i.e. its beginning and end at the same point in space) in an observer's frame of reference, is called the proper time of the interval between the events.

Δt_0: time interval on clock at rest relative to an observer (proper time)

Δt: time interval on clock in motion relative to an observer

$\boldsymbol{v}$: speed of relative motion

c : speed of light

The time interval between ticks (proper time) for the laboratory clock. : Δt_0

The time taken by the light pulse(at the speed of light c) to travel between the mirrors :

$$\Delta t_0/2 = L_0/c$$

$$\Delta t_0 = 2L_0/c$$

The time interval between ticks for the moving clock: Δt

In time interval $\Delta t/2$, the pulse travels

(i) a horizontal distance = $\boldsymbol{v}\ \Delta t/2$
(ii) a total distance = $c\ \Delta t/2$

The relativity of Time

Since L_0 is the vertical distance between the mirrors

$$\left(\frac{c\Delta t}{2}\right)^2 = L_0^2 + \left(\frac{v\Delta t}{2}\right)^2$$ (Pythagoras' theorem)

On Rearranging,

$$\frac{\Delta t^2}{4}(c^2 - \boldsymbol{v}^2) = L_0^2$$

$$\Delta t^2 = \frac{4L_0^2}{(c^2 - \boldsymbol{v}^2)}$$

$$= \frac{(2L_0)^2}{c^2(1 - \boldsymbol{v}^2/c^2)}$$

We get,

$$\Delta t = \frac{2L_0/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

Since as $\Delta t_0 = 2L_0/c$, $\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}}$ **"Time dilation"**

- For a moving object, $\sqrt{1-\boldsymbol{v}^2/c^2} < 1$ always. So, $\Delta t > \Delta t_0$ always(irrespective of the magnitude or direction of $\boldsymbol{v}$).

"The duration of the interval between events Δt appears longer than the proper time Δt_0 —This effect is called **Time dilation**"

- The moving clock in the spacecraft appears to tick at a slower rate than the stationary one on the ground (for the observer on the ground).

- The same analysis holds for measurements of the clock on the ground by the pilots of the spacecraft.

Conclusions:

- ❖ A moving clock ticks more slowly than a clock at rest
- ❖ Every observer finds that clocks in motion relative to him tick more slowly than clocks at rest relative to him.
- ❖ All clocks run more slowly to an observer in relative motion.

For example Biological clocks: growth, aging and decay of living organisms are **slowed** by the **time dilation effect**.

Problem

A spaceship just returned to the earth from a ten years long voyage (acc. To observer on the earth). If the spaceship's velocity was 0.75c, how many years have elapsed for the crew on the ship?

Solution:

In the given question, $\Delta t = 10$ years

$$v = 0.75c$$

$$\Delta t_0 = ?$$

So, Using time dilation equation, $\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{u^2}{c^2}}}$, and after putting the values of Δt and v,

We get

$$\Delta t_0 = 10\times\left(1 - \frac{u^2}{c^2}\right)^{1/2}$$

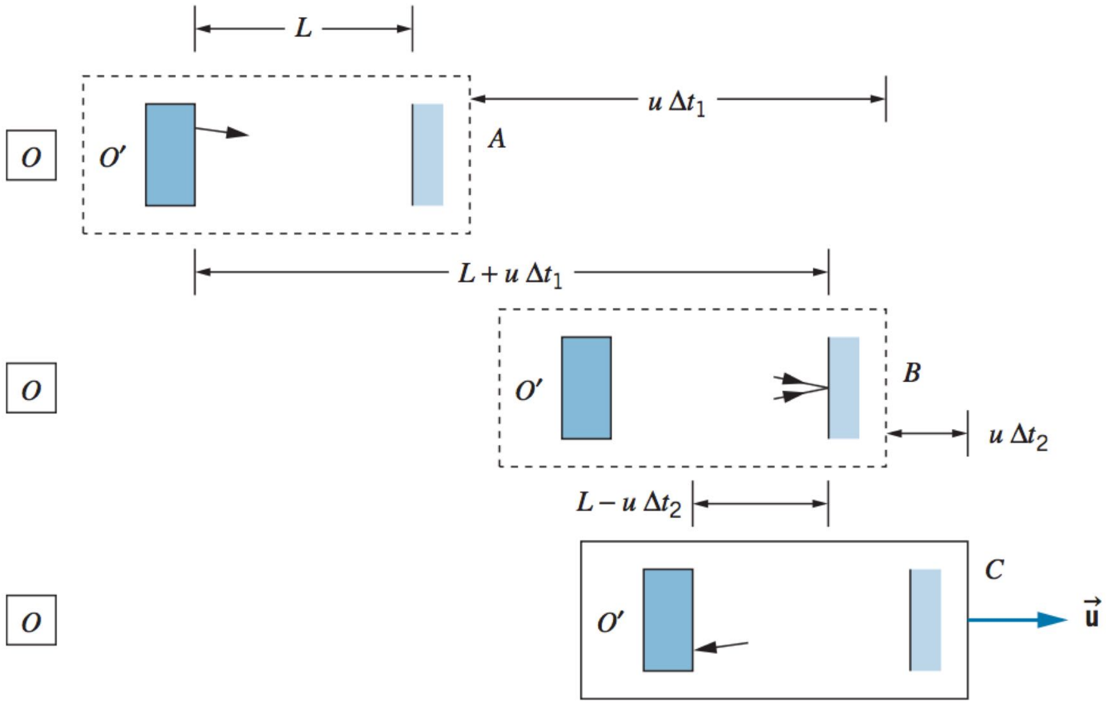
$$= 10\times\left(1 - \frac{(0.75c)^2}{c^2}\right)^{1/2}$$

$$= 10\times(1 - (0.75)^2)^{1/2}$$

$$= 6.6\ years$$

The Relativity of Length

Let us consider the timing device of O'. Turn it sideways so that the light travels parallel to the direction of motion of O'.



Clock carried by O' emits its light flash in the direction of motion.

The sequence of events that O observes for the moving clock.

The Relativity of Length

The length L_0 of the object as measured by the observer O', who is at rest with respect to the object, is called the proper length/rest length .

L_0: Length of the clock according to O' (Length of the clock in its rest frame)

L: Length of the clock according to O

u: speed of relative motion

c : speed of light

The Relativity of Length

The flash of light is emitted at A and reaches the mirror (position B) at time Δt_1 later.

$$c\Delta t_1 = L + u\Delta t_1 \ldots\ldots\ldots\ldots\ldots\ldots 1 \qquad \text{(distance travelled by the light in } \Delta t_1\text{)}$$

The flash of light travels from the mirror to the detector in a time Δt_2

$$c\Delta t_2 = L - u\Delta t_2 \ \ldots\ldots\ldots\ldots\ldots\ldots 2 \qquad \text{(distance travelled by the light in } \Delta t_2\text{)}$$

Solving equations 1 and 2, and adding them, we obtain:

$$\Delta t = \Delta t_1 + \Delta t_2 \text{ (total time interval)}$$

$$= \frac{L}{(c-u)} + \frac{L}{(c+u)} = \frac{2L}{c}\frac{1}{1-u^2/c^2} \ \ldots\ldots\ldots\ldots\ldots\ldots 3$$

The Relativity of Length

we know that,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}}. \qquad \text{(time dilation effect)}$$

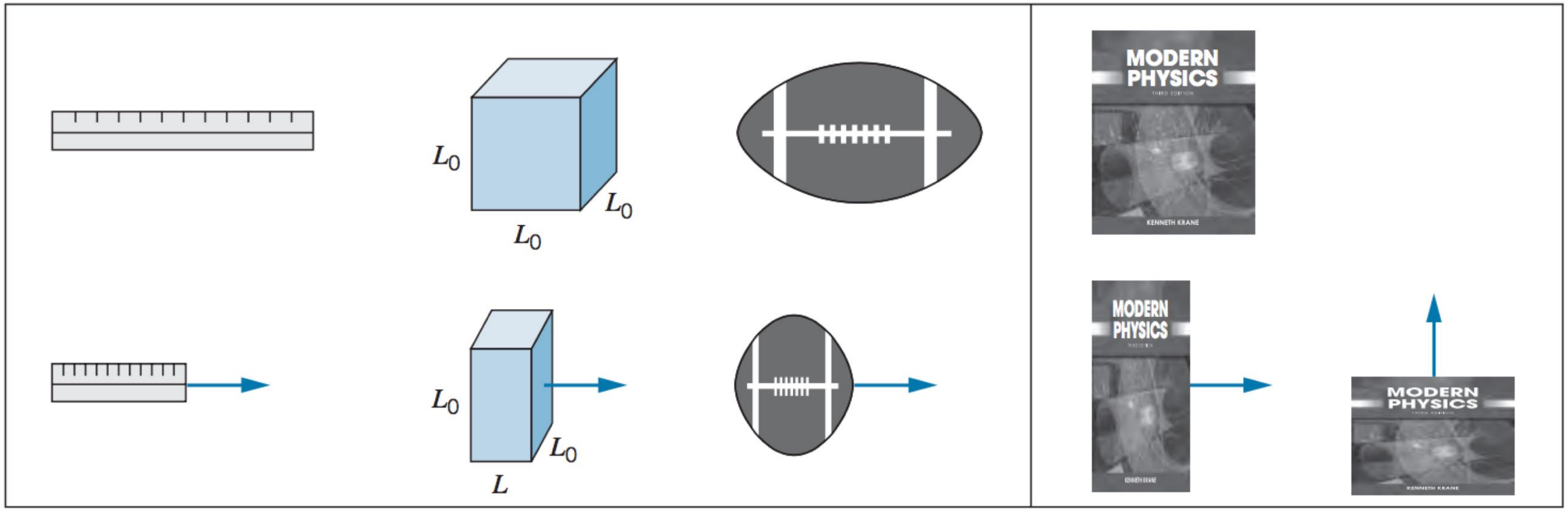
$$= \frac{2L_0}{c}\frac{1}{\sqrt{1-u^2/c^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ 4$$

Equating eqns 3 and 4, and solving:

$$\mathrm{L} = L_0\sqrt{1-u^2/c^2}$$ **"Length contraction"**

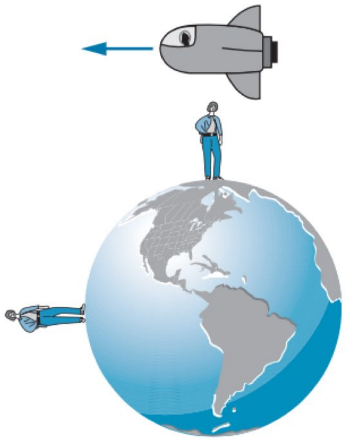
"Faster means shorter"

Length-contracted objects



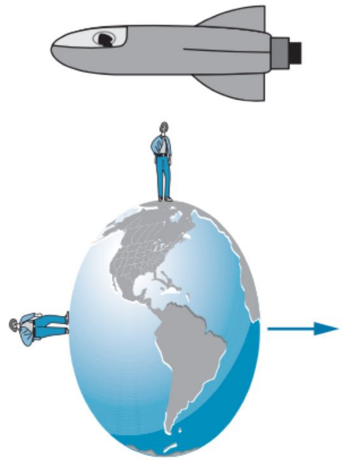
Shortening occurs only in the direction of relative motion

Measurement of length affected by relative motion



(a)

From the Earth's frame
the rocket appears contracted
(along the direction of motion)
not narrower.

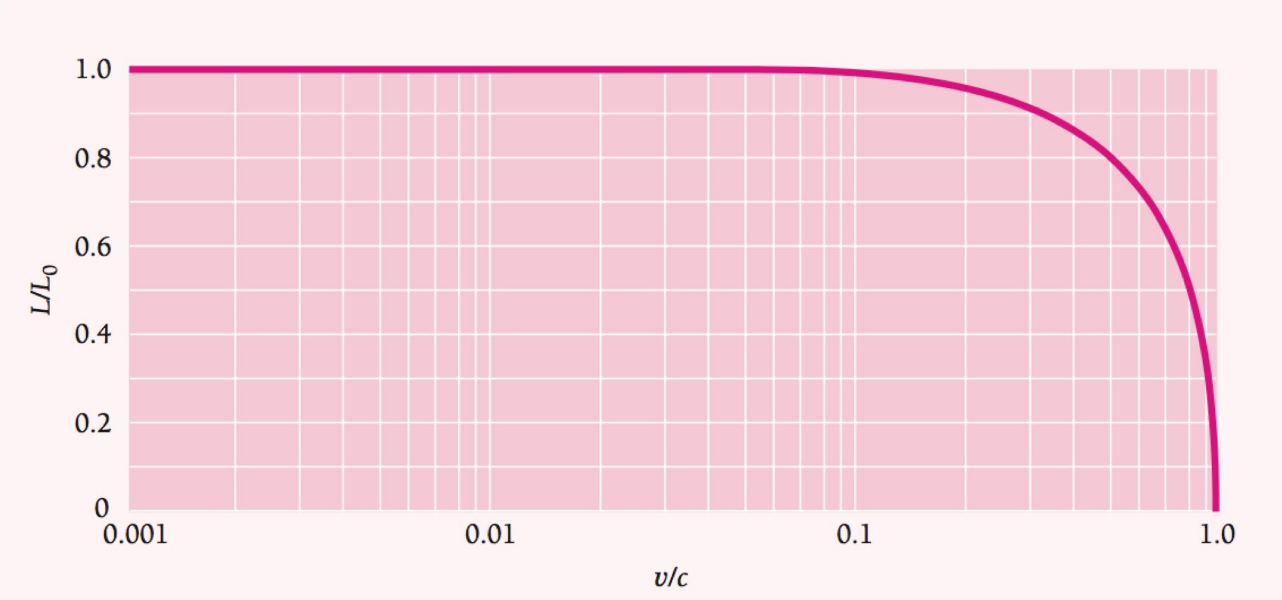


(b)

From rocket's frame, the passing
Earth appears contracted.

For $\boldsymbol{u} \ll c$, the effects of length contraction are too small.
Ex: if a rocket of length 100 m travelling at $\boldsymbol{u} = 11.2\ km/s$, for an observer on the Earth, the length contracts by only about 2 atomic diameters!

Relativistic length contraction



Length contraction is most significant at speeds $v \sim c$
(horizontal scale is logarithmic)

The Relativity of Length

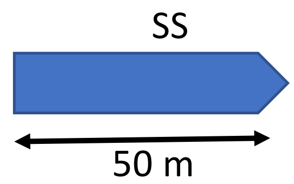
- All observers in motion relative to the the observer (who is at rest with respect to the object) measure a shorter length, but only along the direction of motion.

- Length measurements transverse to the direction of motion are unaltered.

- Proper length L_0 found in the rest frame is the maximum length any observer will measure.

Problem1 : Length contraction

Consider a spaceship(SS) that has a length of 50m while it was stationary on the Earth. Find the velocity that is required to observe the length of the spaceship contracted to 10 m.



Solution:

Proper length of the space ship, L_0=50 m
Contracted length of SS =10 m
Speed of Spaceship, $u = ?$

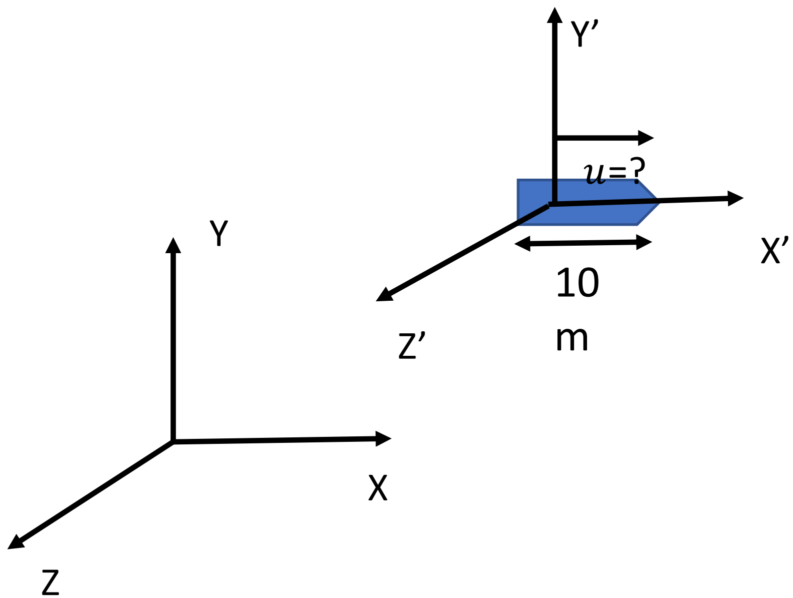
Therefore, from length contraction equation

$$L = L_0\sqrt{1 - u^2/c^2}$$

Putting the values, $$10 = 50\sqrt{1 - u^2/c^2}$$

On solving, $$u = 0.98c$$

Speed of Spaceship must be $0.98c$ to decrease the length of SS to 10m



Earth Frame: Stationary inertial frame

Problem 2: on Length contraction

- What will be the apparent length of a meter stick measured by an observer at rest, when the stick is moving along its length with a velocity equal to $c/_2$.

Lorentz transformation

Galilean transformation of coordinates, time, and velocity:

- It is not consistent with Einstein's postulates.
- It agrees with our "common-sense" experience *at low speeds* only.

Therefore, there was a need for a new set of transformation equations at high speeds:

- For replacement of the Galilean set of transformation.
- It must be capable of predicting such relativistic effects as time dilation, length contraction, velocity addition, and the Doppler shift.

- We seek a transformation that enables observers $O \ and \ O'$ in relative motion to compare their measurements of the space and time coordinates of the same event at high speeds $(u \sim c)$.

This new transformation consistent with special relativity is called the *Lorentz transformation*

Derivation of Lorentz transformation equations

The new transformation equations relate the measurements of O (namely, (x, y, z, t)) to those of O' (namely, (x', y', z', t')).

Let us assume that the velocity of O' relative to O is in the positive xx' direction.

Let us assume that, the modified relationship between x and x' is

$$x' = k(x - ut) \text{.............................} 1$$

Here k is a factor that does not depend upon either x or t, but may be a function of u.

The choice of equation 1 follows from several considerations:

1. It must be linear in x and x' so that a single event in frame S corresponds to an single event in frame S'(follows from homogeneity of space and time).
2. It is simple, and a simple solution to a problem should be explored first.
3. It must be consistent with Einstein's postulates.
4. It has the possibility of reducing to the Galilean transformation when the relative speed between $O \ and \ O'$ is small.

As the equations of physics must have the same form in both S and S', we need only change in sign of u (in order to take into account the diff. in the direction of relative motion) to write the corresponding equation for x in terms of x' and t':

$$x = k(x' + ut') \qquad \text{..........................2}$$

The factor k must be the same in both frames since there is no difference between S and S' other than in the sign of u.

As, in case of Galilean transformation, we again take

$$y' = y \text{.....................................} 3$$

$$z' = z \text{.....................................} 4$$

However, $t' \neq t$. We can see this by substituting the value of x' from eq. 1 into eq.2. This gives

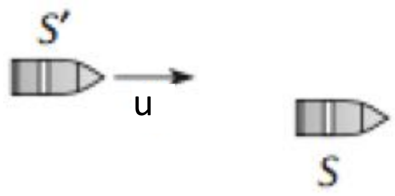
$$x = k^2(x - ut) + kut'$$

From the above equation, we have

$$t' = kt + \left(\frac{1-k^2}{ku}\right)x \text{........................} 5$$

Equations 1,3,4 and 5 constitute a coordinate transformation that satisfies the first postulate of special relativity.

- We can evaluate k using the second postulate of relativity.
- Let us do an experiment with two boats and flare light.
- Inertial frame S' is a boat moving at speed u in the x-direction relative to an another boat, which is inertial frame S.

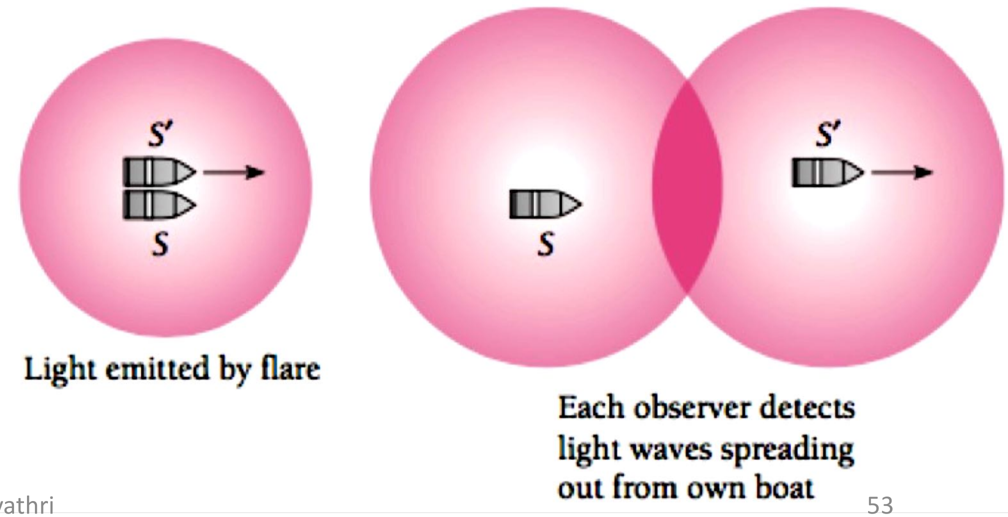


❖ **Situation (a)**

When $t = t_0 = 0$, S' is next to S and $x = x_0 = 0$.

At this moment, a flare is fired from one of boats.

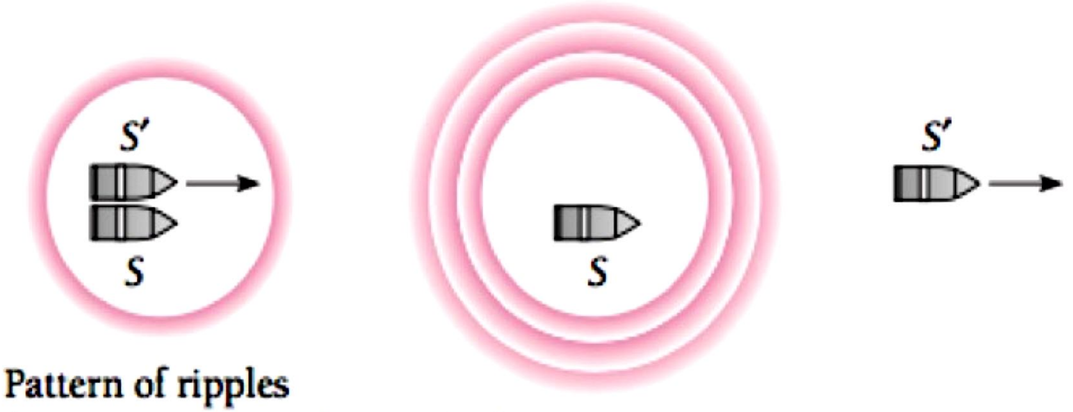
Observers on boats S and S' both detect light waves spreading out with speed c even though they are in relative motion.



❖ **Situation (b)**

At $t = t_0 = 0$, a stone was dropped in water,

Both observers will observe a pattern of ripples spreading out around S at different speeds relative to their boats



Pattern of ripples from stone dropped in water

Each observer sees pattern spreading from boat S

The difference between both situation :

Water, in which the ripples move, is **itself a frame of reference** whereas space, in which light moves , is not.

Both observers must find the same speed c (**for light wave**) which means that,

In the S frame $x = ct$……………………………….6

And in the S' frame $x' = ct'$………………………………...7

Substituting for x' and t' in eq. 7 from eq.1 and 5, we get

$$k(x - ut) = ckt + \left(\frac{1 - k^2}{ku}\right) cx$$

on solving for x, we get

$$x = ct\left[\frac{1 + \frac{u}{c}}{1 - \left(\frac{1}{k^2} - 1\right)\frac{c}{u}}\right]$$

This expression for x will be same as that given in eq.6, $x = ct$, provided that the quantity in the bracket equals 1, so

$$\frac{1+\frac{u}{c}}{1-\left(\frac{1}{k^2}-1\right)\frac{c}{u}} = 1$$

on solving, we get

$$k = \frac{1}{\sqrt{1-u^2/c^2}}$$

Finally, we put this value of k in eq.1 and eq.5, we have

$$x' = \frac{(x-ut)}{\sqrt{1-u^2/c^2}} \qquad t' = \frac{t-(u/c^2)x}{\sqrt{1-u^2/c^2}}$$

Lorentz coordinates transformation

$$\boldsymbol{x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}}$$

$$\boldsymbol{y' = y}$$

$$\boldsymbol{z' = z}$$

$$\boldsymbol{t' = \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}}}$$

<u>When $u \ll c$</u>

- The first three equations reduce directly to the Galilean transformation for space coordinates.
- The fourth equation, which links the time coordinates, reduces to $t' = t$, which is a fundamental postulate of the Galilean-Newtonian world.

Inverse Lorentz transformation

- In length contraction, $x' = \dfrac{x - ut}{\sqrt{1 - u^2/c^2}}$ can be easily used to find L in terms of L_0 and v.
- But if we want to examine time dilation, $t' = \dfrac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}}$ is not convenient.
- For this, it is easier to use the inverse Lorentz transformation.

$$x = \frac{x' + ut}{\sqrt{1 - u^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + (u/c^2)x'}{\sqrt{1 - u^2/c^2}}$$

Lorentz velocity transformation

If O observes a particle to travel with velocity v (components v_x, v_y, v_z), what velocity v' does O' observe for the same particle?.

The relationship between the velocities measured by $O \ and \ O'$ is given by the *Lorentz velocity transformation*:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}$$

$$v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$$

$$v'_z = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$$

Note:

- In the limit of low speeds $u \ll c$, the Lorentz velocity transformation reduces to the Galilean velocity transformation
- $v_y{}' \neq v_y$, even though $y' = y$. This occurs because of the way the Lorentz transformation handles the time coordinate $(t' \neq t)$.

Inverse velocity transformation

The set of inverse velocity transformation can be obtained by differentiating inverse Lorentz equations for coordinates x, y, z.

$$v_x = \frac{v'_x + u}{1 + v'_x u/c^2}$$

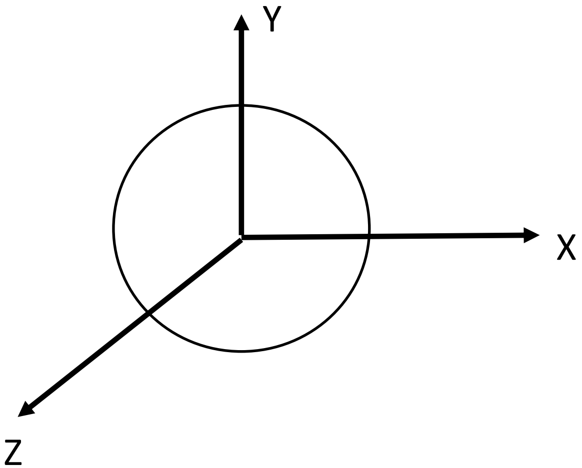
$$v_y = \frac{v'_y\sqrt{1 - u^2/c^2}}{1 - v'_x u/c^2}$$

$$v_z = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v'_x u/c^2}$$

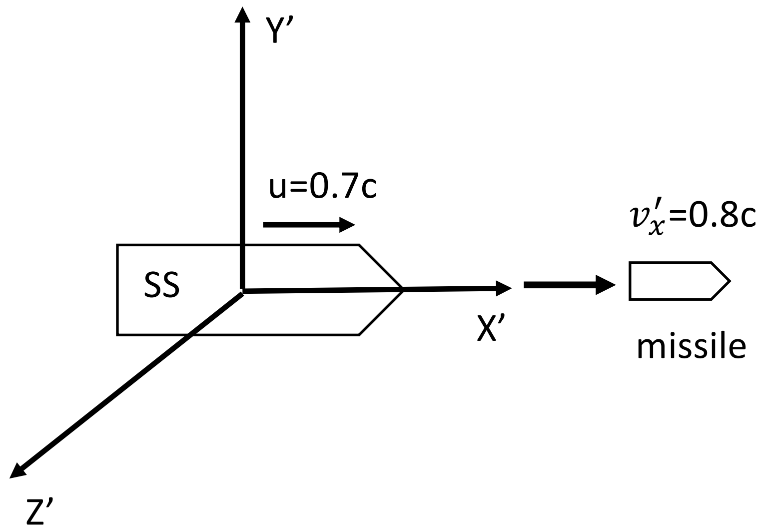
Problem

A spaceship takes off the Earth and Travels with a velocity of 0.7c along x-axis. It, then releases a missile straight ahead that moves with a velocity of 0.8c w.r.t. the spaceship. Calculate the speed of missile w.r.t. the Earth ?

Solution:



Earth Frame: Stationary inertial frame



Spaceship Frame: moving with velocity u=0.7c w.r.t. the Earth

We have to calculate speed of missile v_x w.r.t. the Earth.

According to Lorentz inverse velocity transformation,

$$v_x = \frac{v_x' + u}{1 + v_x' u / c^2}$$

Putting the values of v_x' and u, we get

$$v_x = \frac{0.8c + 0.7c}{1 + (0.8c \times 0.7c)/c^2}$$

$$v_x = \frac{0.15c}{1 + 0.56c^2/c^2}$$

$$v_x = 0.96\mathrm{c}$$

Therefore, speed of missile w.r.t. the Earth is $0.96c$