MA 203

Ergodic Random Process

1. Stationary R.P. $\Rightarrow$ Strict-Sense Stationary RP

$X(t)$: CDF is time invariant

$X(t_1) \quad X(t_2) \quad \dots \quad X(t_n)$

$$F_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n)$$

$$= F_{X(t_1+t_0), X(t_2+t_0), \dots, (t_n+t_0)}(x_1, x_2, \dots, x_n)$$

$\forall n$

2. WSS :-

1. $E[X(t)] = \mu_X$
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$

Ensemble mean $\Rightarrow E[x(t)] = \int_{-\infty}^{+\infty} A\cos(\omega_0 t+\phi)\, f_\phi(\phi)\, d\phi$

$x(t) = A\cos(\omega_0 t+\phi)$

ϕ ← R.V.

Ensemble autocorrelation. $\Rightarrow E[x(t)\,x(t+\tau)] = \int_{-\infty}^{+\infty} x(t)\cdot x(t+\tau)\, f_\phi(\phi)\, d\phi$

Time-Average:-

$X(t)$: RP

$X(t) = A\cos(\omega_0 t + \phi)$

$$\langle \mu_x \rangle_T = \frac{1}{2T}\int_{-T}^{+T} X(t)\, dt$$

$$\langle \mu_x(t) \rangle_T = g(\phi)$$

$$E[\langle \mu_x \rangle_T] = \mu_x$$

Time-Autocorrelation: –

Time-Averaged Autocorrelation

$$\langle R_x(\tau)\rangle_T = \frac{1}{2T}\int_{-T}^{+T} x(t)\cdot x(t+\tau)\,dt$$

Mean Ergodic RP:

A random process is known as mean ergodic RP if

$$\langle \mu_x\rangle_T \longrightarrow \mu_x \text{ as } T \longrightarrow \infty$$

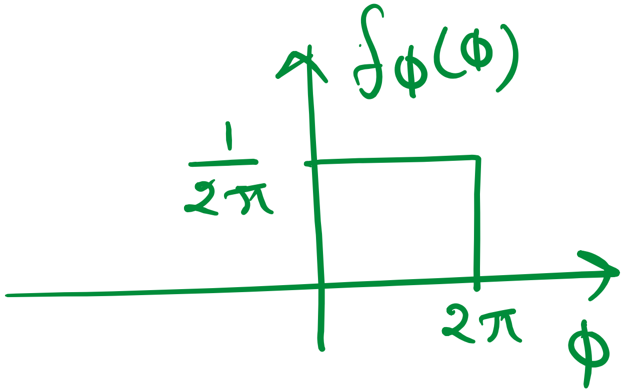
Autocorrelation Ergodic RP:-

$$\langle R_x(\tau)\rangle_T \longrightarrow R_x(\tau) \qquad T \to \infty$$

Ex:- $X(t) = A\cos(\omega_0 t + \phi)$;

where A & ω_0 are const.

$\phi \to$ RV



Sol:- $E[x(t)] = 0$

$$E[x(t)\,x(t+\tau)] = R_x(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

Time-Average Mean

$$\langle \mu_x \rangle_T = \frac{1}{2T} \int_{-T}^{+T} x(t)\, dt$$

$$= \frac{1}{2T} \int_{-T}^{+T} A \cos(\omega_0 t + \phi)\, dt$$

$$= \frac{A}{2T} \left\{ \frac{\sin(\omega_0 t + \phi)}{\omega_0} \right\} \Bigg|_{-T}^{+T}$$

$$= \frac{A}{T\omega_0} \sin \omega_0 t$$

$$\lim_{T \to \infty} \langle \mu_x \rangle_T = \lim_{T \to \infty} \frac{A}{T\omega_0} \sin \omega_0 t = 0$$

$\underbrace{\langle \mu_x \rangle_T} \longrightarrow \underbrace{E[x(t)] = \mu_x}$ as $T \to \infty$

$\Rightarrow$ X(t) is a mean Ergodic RP.

$$R_x(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)$$

$$\langle R_x(\tau) \rangle_T = \frac{1}{2T} \int_{-T}^{+T} \{A \cos(\omega_0 t + \phi)\}\{A \cos\{\omega_0 (t+\tau) + \phi\}\} \, dt$$

$$= \frac{1}{2T} \cdot \frac{A^2}{2} \int_{-T}^{T} \{2 \cos(\omega_0 t + \phi) \cos(\omega_0 (t+\tau) + \phi)\} \, dt$$

$$= \frac{A^2}{4T} \int_{-\infty}^{+\infty} \{\cos \omega_0 \tau + \cos(\omega_0 (2t+\tau) + 2\phi)\} \, dt$$

$$= \frac{A^2}{4T}\int_{-T}^{+T} \cos\omega_0\tau\, dt + \frac{A^2}{4T}\int_{-T}^{+T} \cos\{\omega_0(2t+\tau)+2\phi\}\, dt$$

$$= \frac{A^2}{2\cdot 2T}\{\cos\omega_0\tau\}\Big|_{-T}^{+T} + \frac{A^2}{4T}\cdot \frac{\sin(\omega_0(2t+\tau)+2\phi)}{2\omega_0}\Big|_{-T}^{+T}$$

$$= \frac{A^2\cos\omega_0\tau}{2} + \frac{A^2\sin(\omega_0(2T+\tau))}{4\omega_0 T}$$

$$\lim_{T\to\infty} \langle R_x(\tau)\rangle_T = \lim_{T\to\infty}\left\{\frac{A^2\cos\omega_0\tau}{2} + \frac{A^2\sin(\omega_0(2T+\tau))}{\underbrace{4\omega_0 T}}\right\}$$

$$= \frac{A^2\cos\omega_0\tau}{2} = \underline{\underline{R_x(\tau)}} = E[X(t)\,X(t+\tau)]$$

$X(t)$ is Ergodic in Autocorrelation.

So, $X(t)$ is Ergodic in mean

" in AutoCorrelation.

$\langle \mu_x \rangle_T$ $\qquad$ $X(t)$

$$\langle \mu_x \rangle_T = \frac{1}{2T} \int_{-T}^{+T} x(t)\, dt$$

$$E[\langle \mu_x \rangle_T] = \frac{1}{2T} \int_{-T}^{+T} E[\boxed{x(t)}]\, dt$$

$\left\{ \because E[x(t)] = \mu_x \right.$

$$= \frac{1}{2T} \int_{-T}^{+T} \mu_x\, dt = \mu_x$$

Variance of $\langle \mu_x \rangle_T$

$$E\left[\left(\langle \mu_x \rangle_T - \mu_x\right)^2\right] = E\left[\left\{\frac{1}{2T}\int_{-T}^{+T} x(t)\,dt - \mu_x\right\}^2\right]$$

$$= E\left[\left\{\frac{1}{2T}\int_{-T}^{+T}\left\{x(t)-\mu_x\right\}dt\right\}^2\right]$$

$$= \frac{1}{4T^2}\, E\left\{\int_{-T}^{+T}\left\{x(t)-\mu_x\right\}dt\right\}^2$$

$$\Downarrow$$

$$= \frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T} E\left[\left\{x(t_1)-\mu_x\right\}\left\{x(t_2)-\mu_x\right\}\right] dt_1\, dt_2$$

$$= \frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T} E\left\{\underbrace{(X(t_1)-\mu_x)(X(t_2)-\mu_x)}\right\} dt_1\, dt_2$$

$$\left\{ E[(x-\mu_x)(y-\mu_y)] \right. = C_x(x,y)$$

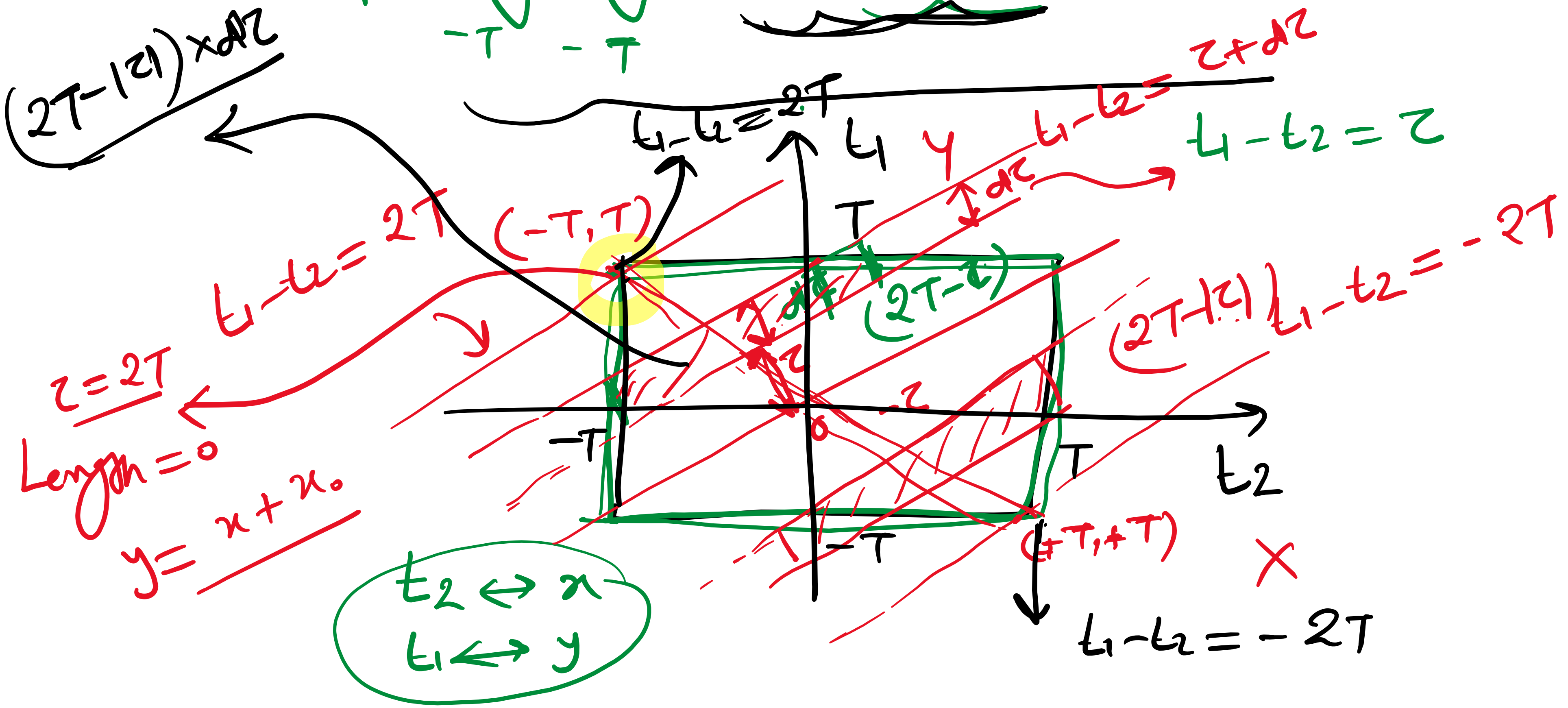
$$= \frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T} \underbrace{C_x(t_1-t_2)}\, dt_1\, dt_2$$

<u>In case of WSS</u>

$C_x(t_1,t_2) = C_x(t_1-t_2)$

$R_x(t_1,t_2) = R_x(t_1-t_2)$

Only for WSS RP



$$= \frac{1}{4T^2} \int_{-2T}^{+2T} C_x(\tau) \cdot (2T - |\tau|)\, d\tau$$

$$= \frac{1}{4T^2} \cdot 2T \cdot \int_{-2T}^{+2T} C_x(\tau) \left(1 - \frac{|\tau|}{2T}\right) d\tau$$

$$\boxed{\underline{\mathrm{Var}(\langle \mu_x \rangle_T)} = \frac{1}{2T} \int_{-2T}^{+2T} C_x(\tau) \cdot \left(1 - \frac{|\tau|}{2T}\right) d\tau}$$