Classical Mechanics

Conservation Principles
for
System of Particles

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Conservation of Linear momentum for System of Particles

Consider the equation of motion of a system of particles, undergoing pure translation:

$$M\ddot{\boldsymbol{R}} = \frac{d}{dt}\mathbf{P}$$

$$= \boldsymbol{F}_{ext} \qquad \left(\ddot{\boldsymbol{R}} = \frac{d^2\boldsymbol{R}}{dt^2}\right)$$

For an isolated closed system, $\boldsymbol{F}_{ext} = 0.$

$$\frac{d}{dt}\mathbf{P} = 0 \qquad \therefore \boldsymbol{P} = \boldsymbol{constant} \text{ in time}$$

$$\boldsymbol{P}_{initial} = \boldsymbol{P}_{final}$$

If $\boldsymbol{P}(= M\boldsymbol{V}_{com})$ is constant,

$\boldsymbol{V}_{com} = constant$ (velocity of center of mass is constant)

$\boldsymbol{a}_{com} = 0$ (acceleration of center of mass must be zero)

Law of the conservation of linear momentum:

- If the net external forces acting on a system of particles is zero(isolated system) and that no particles enter or leave the system(closed system) , then the total linear momentum is conserved.

- If a component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

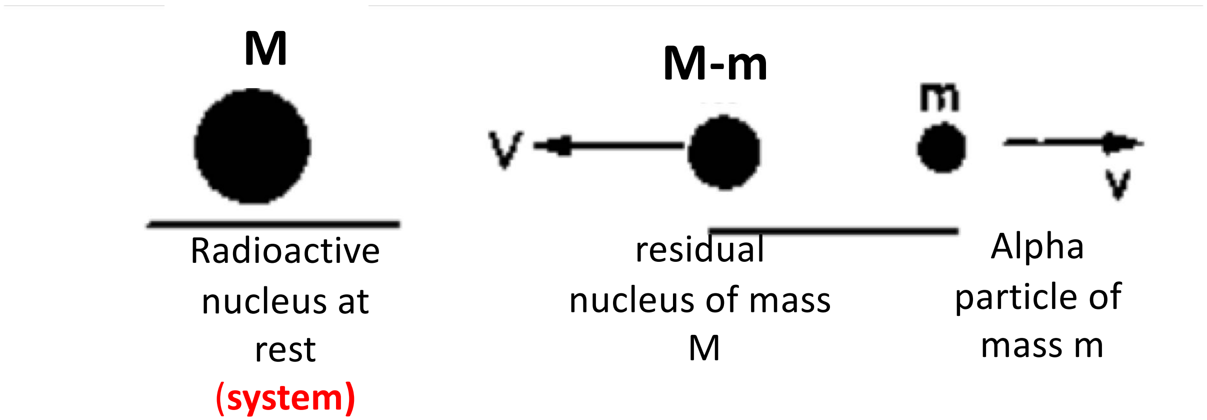
Examples using Conservation of Linear Momentum

Example1:

For a single particle of mass m moving with uniform velocity v, if no external force acts on it, then its linear momentum

$$\boldsymbol{P} = mv = \text{ constant in time}$$

Example 2: A system of two particles



Initial state

final state

Examples using Conservation of Linear Momentum

Since the radioactive nucleus is at rest initially,

$$P_{initial} = 0$$

Ejection of alpha particle is due to nuclear force **(internal force).**

Since External force = 0, $P_{initial} = P_{final}$. (Total P conserved)

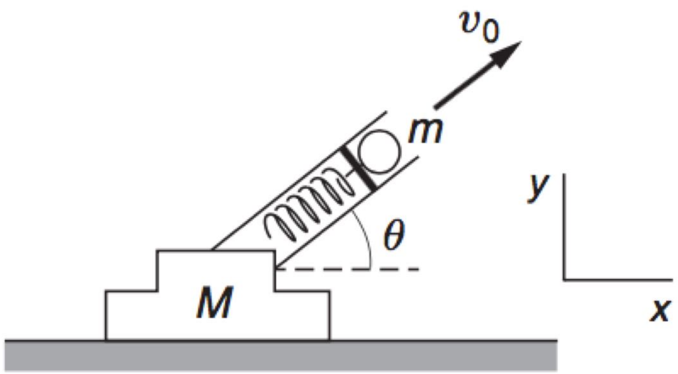
$$P_{final} = (M - m)V + mv = 0$$

Therefore, the final velocity of the residual nucleus is

$$\boldsymbol{V = \frac{-m}{(M-m)} v}$$

Example3: Spring Gun Recoil

A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angles of elevation θ. The mass of the gun is M, the mass of the marble is m and the muzzle velocity of the marble is v_0. What is the final motion of the gun?



Solution: Let gun+marble be the system.

Internal forces:

(i) marble's acceleration is due to force of the gun (firing of marble)

(i) Gun's recoil is due to the reaction force of the marble.

External forces: Gravity and normal force (of table) are only vertical

"Horizontal external forces =0"

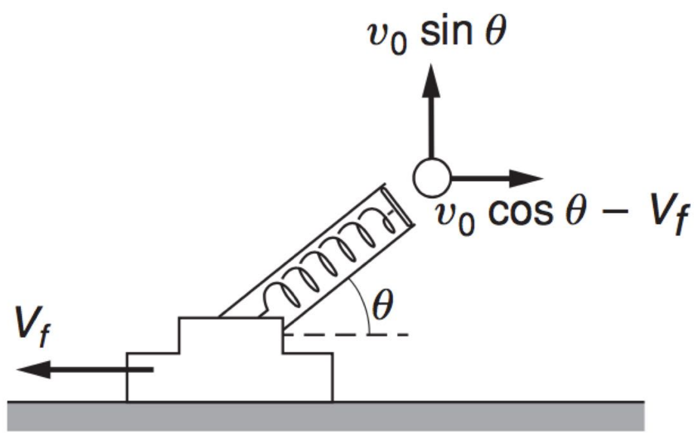
$\therefore$ The x-component of the vector equation $\boldsymbol{F} = {}^{d\boldsymbol{P}}/_{dt}$ is

$$\frac{dP_x}{dt} = 0$$

$$P_{x,initial} = P_{x,final}$$ ($only\ P_x$ is conserved)

The system is initially at rest, prior to firing: $P_{x,initial} = 0$

After firing,



The gun stops accelerating once the marble leaves the barrel. At the instant the marble and the gun part company,

Gun's final recoil speed $= V_f$ (moves to left)

$\therefore$ Gun's final horizontal momentum $= MV_f$

At the same instant the marble and the gun part company,

Speed of the marble relative to the gun $= v_0$

Final horizontal speed of the marble relative to the table $= v_0 \ cos\theta \ - V_f$

By conservation of horizontal momentum,

$$\therefore \mathrm{m}\left(v_0 \, cos\theta \ - V_f\right) - \mathrm{M}V_f = 0$$

Rearranging, we get the final speed of the gun:

$$V_f = \frac{mv_0 \, cos\theta}{m + M}$$

Conservation of Angular Momentum of a Rotating Body

Consider Newton's second law in angular form,

$$\boldsymbol{\tau}_{ext} = \frac{d\boldsymbol{L}}{dt}$$

If no net external torque acts on the system,

$$\frac{d\boldsymbol{L}}{dt} = 0. \qquad \therefore \boldsymbol{L} = constant \text{ in time}$$

$$\boldsymbol{L_i} = \boldsymbol{L_f}$$

$$I_i\omega_i = I_f\omega_f \qquad \text{(Since } L = \mathrm{I}\omega\text{),}$$

(subscript refers to the values of moment of inertia I and angular speed ω at the initial and final values)

Law of conservation of Angular momentum of a rotating rigid body

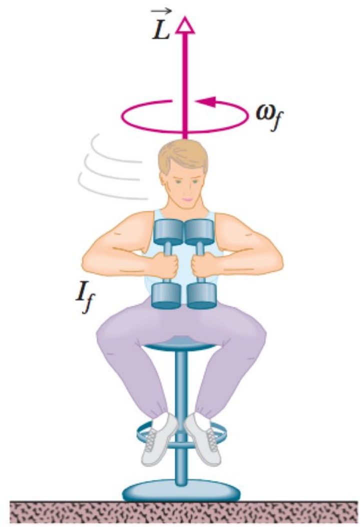
- If the net external torque acting on a system is zero, the angular momentum **L** of the system remains constant, no matter what changes take place within the system.

- If any component of the net external torque on a system is zero, then that component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

Spinning volunteer

(a)

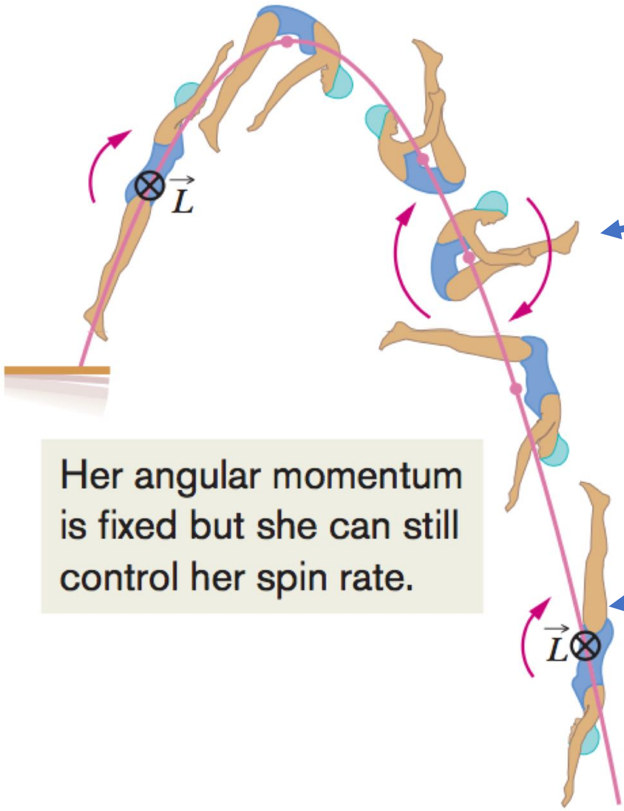
Large I and relatively
small ω
to maintain **L** constant



(b)

I reduces and hence
ω increases
to maintain **L** constant

The Springboard diver



Her angular momentum is fixed but she can still control her spin rate.

Tuck position

I reduces and ω increases to maintain L constant

Open layout position

I increases and ω decreases to maintain L constant

The Work-Energy Theorem for the translational motion of a System of Particles

Newton's II Law for system of particles, undergoing pure translation:

$$\boldsymbol{F} = \boldsymbol{M}\,\frac{d^2\boldsymbol{R}}{dt^2} \qquad \text{(equation of motion for the center of mass)}$$

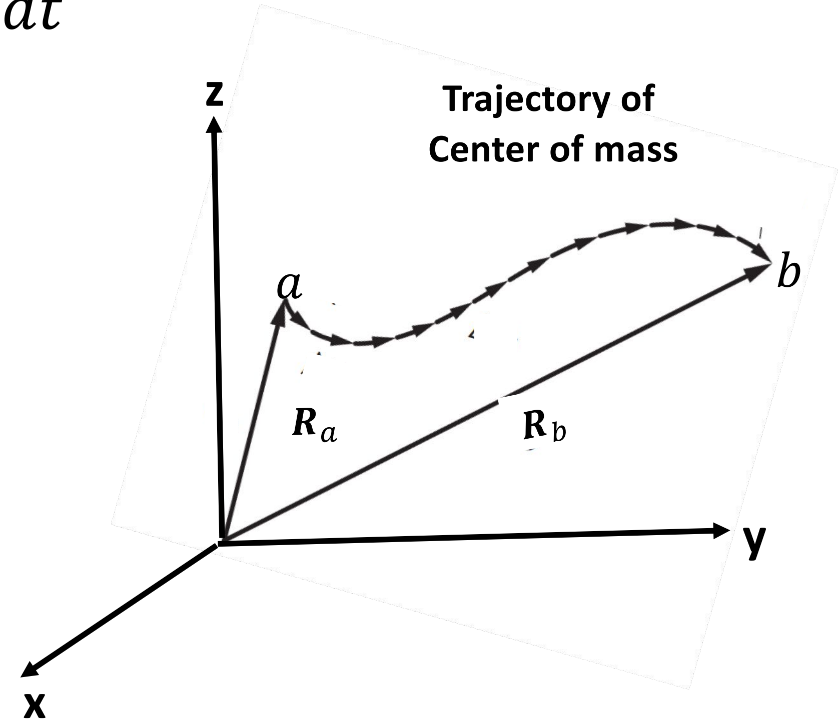
$$= \boldsymbol{M}\,\frac{d\boldsymbol{V}}{dt}, \qquad (\mathrm{V} = \dot{R} = \frac{dR}{dt} \text{ :velocity of center of mass})$$

The work done when the center of mass is displaced by **dR =** $\boldsymbol{V\,dt}$,

$$\boldsymbol{F}.d\boldsymbol{R} = M\,\frac{d\boldsymbol{V}}{dt}.\mathbf{V}\,\mathrm{dt} = \mathrm{d}(\,\frac{1}{2}MV^2)$$

Integrating $\boldsymbol{F}.d\boldsymbol{R}$ w.r.t position vector **R**,

$$W_{ba} = \int_{\boldsymbol{R}_a}^{\boldsymbol{R}_b} \underbrace{\boldsymbol{F}.d\boldsymbol{R}}_{\text{work integral}} = \int_{\boldsymbol{R}_a}^{\boldsymbol{R}_b} M\frac{d\boldsymbol{V}}{dt}.\mathbf{V}\,\mathrm{dt}$$



Changing the variable of integration from $d\boldsymbol{R}$ to dt,

$$\int_{\boldsymbol{R}_a}^{\boldsymbol{R}_b} \boldsymbol{F}.d\boldsymbol{R} = \int_{t_a}^{t_b} M\frac{d\boldsymbol{V}}{dt}.\boldsymbol{V}dt$$

$$= \int_{t_a}^{t_b} \frac{M}{2} \frac{d}{dt} \left(V^{2}\right) dt$$

$$= \frac{1}{2} M\, V_b^2 \; - \; \frac{1}{2}\, M\, V_a^2 \qquad \text{where } V_a = V(t_a)$$

$$\int_{R_a}^{R_b} \boldsymbol{F}.\, d\boldsymbol{R} = \frac{1}{2} M\, V_b^2 - \frac{1}{2} M\, V_a^2$$

$$\int_{R_a}^{R_b} \boldsymbol{F}.\, d\boldsymbol{R} = K_b - K_a$$

"Work-energy theorem" for the translational motion of the system of particles states that the change in the kinetic energy of a system is equal to the net work done on the system by the external forces, as the system moves from a to b.

This theorem depends on the path of the particle. **Is the theorem useless?**

Conservative Forces

If the work done by a force depends only on the initial and final states and not on the path taken, it is called conservative force.

If the work done by a force during a round trip of a system is always zero, then the force is said to be conservative.

eg: Force of gravity, force of spring, Coulomb force

Since the work done by a conservative force depends only on the end points,

Work integral $$\int_{\boldsymbol{R_a}}^{\boldsymbol{R_b}} \boldsymbol{F}.d\boldsymbol{R} = function\ of\ (\boldsymbol{R}_b) - \text{function of } (\boldsymbol{R}_a)$$

Conservative Forces

Only for a conservative force,

$$\int_{\boldsymbol{R_a}}^{\boldsymbol{R_b}} \boldsymbol{F}.\, d\boldsymbol{R} = \underbrace{-U(R_b) + U(R_a)}_{\text{difference in potential energy function}} = W_{ba}$$

From the work-energy theorem,

$$W_{ba} = \int_{R_a}^{R_b} \boldsymbol{F}.\, d\boldsymbol{R} = K_b - K_a$$

Combining the above two equations,

$$W_{ba} = K_b - K_a = U_a - U_b$$

Conservation of Mechanical Energy

On rearranging,

$$K_b - K_a = -(U_b - U_a)$$

$\therefore\ K_b + U_b = K_a + U_a = \mathrm{E}.$ ⟶ "**Total mechanical energy**" E of the system is constant

"The total energy is independent of the position of the particle"

Total mechanical energy of a system,

E = Kinetic energy(K) + Potential energy (U)

Conservation of Mechanical Energy

Principle of Conservation of Energy:

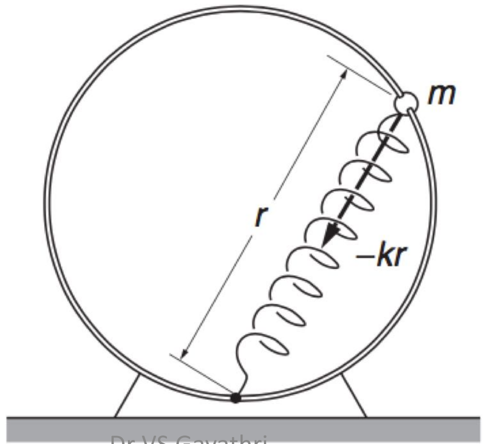
- The total mechanical energy of a system remains constant if the internal forces are conservative and the external forces do no work. This is called the principle of conservation of energy.

- When only a conservative force acts within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy E of the system does not change.

Example:Bead, Hoop and Spring

A bead of mass m slides without friction on a vertical hoop of radius R. The bead moves under the combined action of gravity and a spring attached to the bottom on the hoop. For simplicity, we assume that the equilibrium length of the spring is zero, so that the force due to the spring is $-kr$, where r is the instantaneous length of the spring, as shown.

The bead is released at the top of the hoop with negligible speed. How fast is the bead moving at the bottom of the hoop?



Example:Bead, Hoop and Spring

Solution:

Both gravitational and spring forces are conservative.

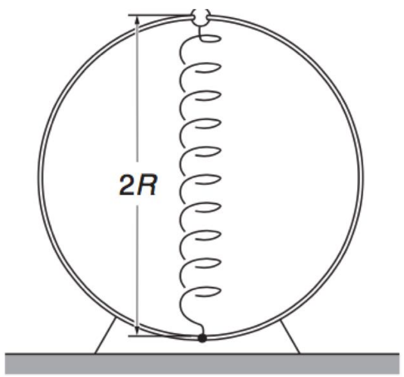
∴ Mechanical energy is conserved: $K_i + U_i = K_f + U_f$

gravitational potential energy of the bead = $mg(2R)$ (at the top of the hoop)

potential energy due to the spring = $\frac{1}{2}k(2R)^2 = 2\mathrm{K}R^2$

∴ initial potential energy $U_i = 2mgR + 2kR^2$

Potential energy at the bottom of the hoop $U_f = 0$

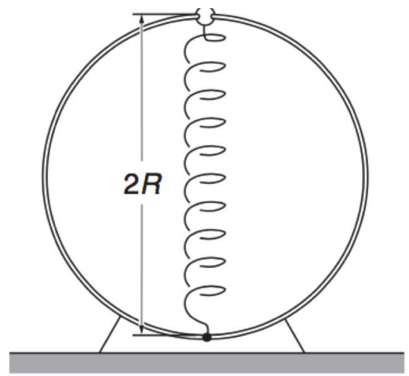


Example:Bead, Hoop and Spring

Initial kinetic energy $K_i = 0$

$$\therefore K_f = U_i - U_f$$

$$\frac{1}{2}mv_f^2 = 2mgR + 2kR^2$$



speed of the bead at the bottom of hoop:

$$v_f = 2\sqrt{gR + \frac{kR^2}{m}}$$

Work-energy theorem for a rigid body undergoing pure rotation

The equation of motion for fixed axis rotation about the centre of mass,

$$\tau_0 = I_0\alpha$$

$$= I_0\frac{d\omega}{dt}$$

The work done by the applied torque in rotating the rigid body by $d\theta = \omega\, dt$,

$$\tau_0 d\theta = I_0\frac{d\omega}{dt}\omega\, dt$$

$$= d(\frac{1}{2}I_0\omega^2)$$

Work-energy theorem

Integrating w.r.t θ,

$$W_{ab} = \int_{\theta_a}^{\theta_b} \tau_0 d\theta = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2$$

The general work-energy theorem for a rigid body is:

$$K_b - K_a = W_{ab}$$

where,

Total kinetic energy

$$K = \underbrace{\frac{1}{2} M V^2}_{\text{Translational energy}} + \underbrace{\frac{1}{2} I_0 \omega^2}_{\text{Rotational energy}};$$

W_{ab}: Total work done on the body as it moves from position a to b.

- Energy equation for translational motion: $\oint_a^b \boldsymbol{F}.dr = \frac{1}{2} M\, V_b^2 - \frac{1}{2} M\, V_a^2$

- Energy equation for rotational motion: $\int_{\theta_a}^{\theta_b} \tau_0 d\theta = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2$

- The work-energy theorem is true whether or not the forces are conservative.