Sets, Sequences and Functions

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Introduction to Sets

A set is an unordered collection of objects, called elements or members of the set.

Examples of set.

$A = \{1, 2, 3, 4\}$, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [a, b], (a, b), [a, b)$.

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To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Define A is a proper subset of B, denoted by $A \subset B$ using logical qantifiers.

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Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

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Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

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Give definitions for:

Cartesian Product of two sets

Unioin of two sets

Intersection of two sets

Disjoint sets

Difference of two sets

Complement of A.

Show that $|A \cup B| = |A| + |B| - |A \cap B|$ and $A - B = A \cap B^c$.

Set Identities

$A \cup U = U, \; A \cup \phi = A$
$A \cap U = A, \; A \cap \phi = \phi$
$A \cup A = A, \; A \cap A = A$
$(A^c)^c = A$
$A \cup B = B \cup A, \; A \cap B = B \cap A$
$A \cup (B \cup C) = (A \cup B) \cup C, \; A \cap (B \cap C) = (A \cap B) \cap C$
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$(A \cap B)^c = A^c \cup B^c, \; (A \cup B)^c = A^c \cap B^c$ (De Morgan's laws)
$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$
$A \cup A^c = U, \; A \cap A^c = \phi.$

1. Prove that $(A \cap B)^c = A^c \cup B^c$, $(A \cup B)^c = A^c \cap B^c$.

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. Prove that $(A \cup (B \cap C))^c = (C^c \cup B^c) \cap A^c$.

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If I is any index set, $\bigcup\limits_{i \in I} A_i = \{x | x \in A_i \text{ for } \text{ some } i\}$ and

$\bigcap\limits_{i \in I} A_i = \{x | x \in A_i \text{ for all } i\}$.

Let $A_i = \{1, 2, \ldots, i\}$ for $i = \{1, 2 \ldots\}$. Then find $\bigcup\limits_{i=1}^{\infty} A_i$ and $\bigcap\limits_{i=1}^{\infty} A_i$.

Computer Representation of Sets

- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).
- Specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.
- Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

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1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?
2. The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010, respectively. Use bit strings to find the union and intersection of these sets.

Functions

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Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

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- f (S) is the position of a 0 bit in S.
- f (S) is the number of 1 bits in S.
- f (S) is the smallest integer i such that the ith bit of S is 1 and f (S) $= 0$ when S is the empty string, the string with no bits.

Find the domain and range of these functions.

1. the function that assigns to each pair of positive integers the first integer of the pair
2. the function that assigns to each positive integer its largest decimal digit
3. the function that assigns to a bit string the number of ones minus the number of zeros in the string
4. the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
5. the function that assigns to a bit string the longest string of ones in the string

Express the definitions of one-one, surjective, increasing functions using logical operators and use negations to define a function is not one-one, not surjective, not increasing.

Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a. mobile phone number.

b. student identification number.

c. final grade in the class.

d. home town

Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

- one-to-one but not onto.
- onto but not one-to-one.
- both onto and one-to-one (but different from the identity function).
- neither one-to-one nor onto.
  Give an explicit formula for a function from the set of integers to the set of positive integers that is
- one-to-one, but not onto.
- onto, but not one-to-one.
- one-to-one and onto.
- neither one-to-one nor onto

1. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.
2. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = 1/f(x)$ is strictly decreasing.
3. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.
4. Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.
5. Give an example of an increasing function from $\mathbb{R}$ to itself that is not one-to-one.
6. Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one.
7. Give an example of a decreasing function from $\mathbb{R}$ to itself that is not one-to-one

Suppose that g is a function from A to B and f is a function from B to C.

1. Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
2. Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Summary

Suppose that $f : A \to B$.

- To show that f is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
- To show that f is not injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- To show that f is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- To show that f is not surjective: Find a particular $y \in B$ such that $f(x) \neq$ for all $x \in A$.

Definition

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective

Definition

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.