Unit 4: Combinatorics

Topic 1: Basic Principles of Counting

6 October, 2020.

Outline

1. Introduction
   - Topics to be covered
   - Applications and Objective
2. Basic Principles of Counting
   - The Product Rule
   - The Sum Rule
   - The Inclusion-Exclusion Rule
3. Pigeonhole Principle

Topics to be covered

Combinatorics deals with counting number of possible arrangements of objects. The need of combinatorics appeared first in problems related to gambling games. The arrangements may be ordered or unordered, may be with repetition or without repetition.

The main topics to be covered in this module are

1. Basic principles of counting
2. Pigeonhole principle
3. Permutations and Combinations
4. Recurrence relation and their solutions
5. Generating functions

Applications and Objective

There are many applications of combinatorics in computer science. For example,

1. Check complexity of algorithm
2. Sorting
3. IP/port/password availability
4. Game theory
5. Probability

The main objective of this module is to make you familiar with basic techniques of counting, and enable you to apply them.

Outline

The Product Rule

There are two basic counting rules in combinatorics:

(1) **The Product Rule:** Suppose a procedure can be broken down into a sequence of two tasks (no part of the two tasks is common). If the first task can be performed in n_1 ways and the second task in n_2 ways, then there are $n_1 \cdot n_2$ ways to complete the procedure.

Ex: There are 15 married couples in a party. Find the number of ways of choosing a woman and a man from the party such that the two are (i) married to each other (ii) not married to each other.

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(ii) We can choose a woman (or man) who is not wife (or husband) of the man (or husband) is 14. Thus a married couple can be chosen in $15 \times 14 = 210$ ways.

Ex: Find the number of bit strings of length 5.

Ex: Find the number of ways to arrange n distinct objects taking r at a time.

Ex: Find the number of many-one functions from a set of m elements to a set of n elements, $(n \geq m)$.

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Solution: The number of bit strings is

$$2 \times 2 \times 2 \times 2 \times 2 = 32.$$

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Ex: Find the number of many-one functions from a set of m elements to a set of n elements, $(n \geq m)$.
Solution: Total number of functions is

$$n^m.$$

The number of one-one functions is

$$n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!} = P(n,m).$$

Thus the number of many-one functions is

$$n^m - P(n,m).$$

The Sum Rule

(2) **The Sum Rule:** Suppose a task can be completed in any one of two distinct procedures. If the first procedure can be carried out in n_1 ways and the second one in n_2 ways, then the task can be done in $n_1 + n_2$ ways.

Ex: There are 26 ECE and 28 CSE students at a college. (i) How many ways two representatives can be picked so that one is from CSE and the other is from ECE? (ii) How many ways one representative can be picked from the college?

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(i) Two representatives can be picked so that one is from CSE and the other is from ECE in $26 \times 28 = 728$ ways.

(ii) One representative can be picked from the college in $26 + 28 = 54$ ways.

Ex: Find the number of different variables in BASIC if the name of a variable is a string of one or two alphanumeric (either 26 letters or 10 digits) begins with a letter such that it is different from the five strings of two characters that are reserved for programming use.

Ex: How many licence plates can be made using either two letters followed by four digits or four digits followed by two letters?

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Solution: The number of one alphanumeric variables is 26 as the first letter has to be alphabet. The number of two alphanumeric variables is $26 \times 36 - 5 = 931$. Thus total number of variables is $26 + 931 = 957$.

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Solution: The total number of plats is

$$26^2 \times 10^4 + 10^4 \times 26^2.$$

The Inclusion-Exclusion Rule

There is a third rule of counting, called **Inclusion-Exclusion Rule**, obtained by combining **The Product Rule** and **The Sum Rule**.

- **Inclusion-Exclusion Rule:** Let a task can be done in n_1 or in n_2 ways out of which n_3 ways are same in the both the procedure. Then the task can be done in $n_1 + n_2 - n_3$ ways.

Ex: How many bit strings of length eight either start with a 1 or end with the two bits 00?

Ex: How many positive integers between 100 and 999 inclusive are divisible by 3 or 4?

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Solution: The number of bit strings of length eight starting with a 1 is 2^7. The number of bit strings of length eight ending with the two bits 00 is 2^6. The number of bit strings of length eight starting with a 1 and ending with the two bits 00 is 2^5. Thus required number of bit strings is $2^7 + 2^6 - 2^5$.

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Ex: How many positive integers between 100 and 999 inclusive are divisible by 3 or 4?

Solution: The required number is

$$\left[\frac{900}{3}\right] + \left[\frac{900}{4}\right] - \left[\frac{900}{12}\right].$$

Ex: Find the number of onto function from a set of m elements to a set of n elements, where $n \leq m$

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Ans: $n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{m-1}C(n,m-1)$

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Ex: How many bits of length four do not have two consecutive 1s?
Ans: 8

Outline

Pigeonhole Principle

Pigeonhole principle is one of the simplest basic counting technique which is used in many practical applications.

Theorem (Pigeonhole Principle)

If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

As an application of Pigeonhole principle, we have the following results:

1. There is no one-to-one mapping from a set with n elements to a set with m elements, if $n \geq m + 1$.
2. At least two persons have same birthday among any group of 367 people.

Pigeonhole Principle

Theorem (Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Note the following:

- Among 23 cards of a standard deck of 52 cards, at least $\lceil \frac{23}{4} \rceil = 6$ are of same suit.
- We must select at least 85 persons to guarantee that at least 8 of them have same birth month.

Problems

Ex: There are 38 different periods during which classes at a college can be schedules. If there are 677 different classes, how many classrooms will be needed?

Ex: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in dark. How many socks must he take out to be sure that he has at least two socks of same color? How many socks he must take out to be sure that he has at least two black socks?

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Ans: 3 and 14

Ex: Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. What would happen if six integers are selected?

Ex: A man walked for 10 hours and covered a total of 45 miles. It is known that he walked 6 miles in the 1st hour and only 3 miles in the last hour. Show that he must have walked at least 9 miles within a certain period of two consecutive hours.

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Hint: The holes are $(1, 10)$, $(2, 9)$, $(3, 8)$, $(4, 7)$, and $(5, 6)$.

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Hint: Let a_i be the number of miles covered in the ith hour. Consider the holes of two consecutive hours $a_2 + a_3, a_4 + a_5, a_6 + a_7, a_8 + a_9$. As there are $45 - 9 - 3 = 36$ miles to be hiked in the four two consecutive hour slots, the person has at least one two consecutive hour slot which has atleast $\lceil \frac{36}{4} \rceil = 9$ miles hiked.

Ex: Given m integers $a_1, a_2, \ldots, a_m$, show that there exist $l, k \in \mathbb{Z}$ with $0 \leq l, k \leq m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m.

Ex: The total number of games played by a team in a 15-day season was 20. The rules required the team to play at least 1 game daily. Show that there was a period of consecutive days during which exactly 9 games were played.

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Hint: Consider the m numbers a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\ldots$, $a_1 + a_2 + \cdots + a_m$ as pigeons. If m divides any one one of these, say $a_1 + a_2 + \cdots + a_i$, then we are done with $k = 0$ and $l = i$. Otherwise when we divide these numbers by m, the possible reminders are $1, 2, \ldots, (m-1)$ as holes. Thus two of these numbers must have same remainder, and hence the result follows.

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Ex: The total number of games played by a team in a 15-day season was 20. The rules required the team to play at least 1 game daily. Show that there was a period of consecutive days during which exactly 9 games were played.
Hint: Let a_i be the number of games played by the team up to ith day. Then $1 \leq a_i \leq 20$ and $10 \leq a_i + 9 \leq 29$ for $i = 1, 2, \ldots, 15$. Since $i < j$ implies $a_i < a_j$ and $a_i + 9 < a_j + 9$, we must have $a_i = a_j + 9$ for some i, j. Thus the result follows.

Ramsey Theory

The generalized pigeonhole principle is used in an important part of combinatorics, called **Ramsey Theory**. For positive integers $m, n \geq 2$, the **Ramsey number** $R(m, n)$ denotes the minimum number of people in a party such that there are either m mutual friends or n mutual enemies.

Ex: Show that in a group of six people (any two people are either friends or enemies), there are either three mutual friends or three mutual enemies. Hence prove that $R(3, 3) = 6$.

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Solution: Let A has 5 relations with B, C, D, E, F. By pigeonhole principle, A has atleast $\lceil \frac{5}{2} \rceil = 3$ friends or enemies. W.L.O.G, let B, C, D be friends to A, and E, F enemies to A. If E, F are enemies to each other, then we are done with A, E, F as mutual enemies. Otherwise, atleast $\lceil \frac{3}{2} \rceil = 2$ are friends or enemies among B, C, D. If B, C are friends, we are done with A, B, C as mutual friends. If B, C are enemies, then we are done with A, B, D or A, C, D as mutual friends.

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From the first part, we have $R(3, 3) \leq 6$. Thus we need to show that $R(3, 3) > 5$.

Ex: Prove that $R(2, n) = n$.

Ex: Show that in a group of 10 people (any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, either three mutual enemies or four mutual friends.

Solution:

Thank You

Any Question!!!