

MA203

Special RV

Special RV:

1. Bernoulli RV: $X=0,1$ and $P(X=1)=p$, $P(X=0)=1-p$; $0 < p < 1$
 $E(X)=p$, $\text{Var}(X)=p(1-p)$

2. Binomial RV: n no. of independent Bernoulli trials and X is the total number of successes in n trials

$$P(X=k) = {}^nC_k p^k (1-p)^{n-k} \quad k=0,1,2,\dots,n; 0 \leq p \leq 1$$

$$X \sim B(n,p)$$

$$E(X)=np, \text{Var}(X)=npq,$$

Mode: unimodal ($[(n+1)p]$) or bimodal ($(n+1)p, (n+1)p-1$)

Example: throwing a die and obtaining even number. $X \sim B(1,1/2)$

Theorem 1: Let, X_i ($i=1, 2, \dots, k$) be independent binomial RVs with $X_i \sim B(n_i, p)$

$$S = X_1 + X_2 + \dots + X_k$$

$$S \sim B(n_1 + n_2 + \dots + n_k, p)$$

Example: $X_1 \sim B(1,1/2)$, $X_2 \sim B(1,1/2)$ both are tossing a coin and obtaining head

Consider, X =tossing a coin twice and obtaining head

$$X \sim B(2,1/2)$$

$$P(X=0)=1/4, P(X=1)=2/4, P(X=2)=1/4$$

Example 1: A die is thrown 600 times. Find the lower bound for the prob of getting 80 to 120 sixes.

$$X \sim B(600, 1/6)$$

$$P\{|X - \mu| < k\sigma\} \geq 1 - 1/k^2$$

$$\mu = 600 \cdot 1/6 = 100$$

$$\text{Var}(X) = 600 \cdot 1/6 \cdot 5/6 = 500/6$$

$$K = 2.19$$

$$\text{Lower bound of prob} = 19/24$$

2. Use Chebyshev inequality to determine how many times a fair coin must be tossed in order that the prob will be at least 0.90 that the ratio of the observed no of heads to the no of tosses will lie between 0.4 and 0.6.

Consider, RV X/n , X : no of heads, n : total tosses

$$X \sim B(n, 1/2)$$

$$P\{|X/n - E(X/n)| < \varepsilon\} \geq 1 - (\text{Var}(X/n))/\varepsilon^2$$

$$E(X/n) = 1/n \cdot E(X) = 1/n(n \cdot 1/2) = 1/2, \quad \text{Var}(X/n) = (1/n^2)\text{Var}(X) = 1/(4n)$$

$$P\{|X/n - 1/2| < k/(4n)\} \geq 1 - 1/k^2$$

$$P\{|X/n - 1/2| < k/(4n)\} \geq 0.9$$

$$\text{So, } k = \sqrt{10}$$

$$N = 250$$

The coin must be tossed 250 times.

3. Poisson RV: X takes values 0, 1, 2,..... Inf with

$$P(X=k)=e^{-\lambda} \lambda^k/k! \quad , \lambda>0 \quad \text{DF is } F(x) = \sum_{k=0}^x e^{-\lambda} \frac{\lambda^k}{k!}$$

$$X \sim P(\lambda)$$

(To check pmf, $\sum_{k=0}^{\infty} P(X = k) = 1$)

Example: number of printing mistakes per page, number of goals in a football match etc..

$$E(X) = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Mode: let, $p_k = P(X=k)$

$$\frac{p_{k-1}}{p_k} = \frac{k}{\lambda}$$

If $k < \lambda$, then $p_{k-1} < p_k$

If $k > \lambda$, then $p_{k-1} > p_k$

If $k = \lambda$, then $p_{k-1} = p_k$

If λ is not an integer, then mode = $[\lambda]$

If λ is an integer, then mode = $\lambda, \lambda-1$

If $\lambda < 1$, mode = 0

MGF: $M(s) = e^{\lambda(\exp(s)-1)}$

Theorem 2: Let, X_1, X_2, \dots, X_n be independent Poisson RVS with $X_k \sim P(\lambda_k)$, $k=1, 2, \dots, n$.
Then $S \sim P(\lambda_1 + \lambda_2 + \dots + \lambda_n)$ where $S = X_1 + X_2 + \dots + X_n$
(converse is true)

Theorem 3: Let X, Y be independent Poisson RVs with parameters λ_1, λ_2 respectively. Then
 $(X|X+Y=n) \sim B$
 $P\{X=m|X+Y=n\} = P\{X=m, Y=n-m\} / P\{X+Y=n\}$

$(X+Y) \sim P(\lambda_1 + \lambda_2)$
 $(X|X+Y) \sim B(n, \lambda_1 / (\lambda_1 + \lambda_2))$

Theorem 4: if $X \sim P(\lambda)$ and $(Y|X=x) \sim B(x, p)$ then $Y \sim P(\lambda p)$

3. A firm receives on an average 2.5 telephone calls per day during 10 – 10:05 am. Find the probability that on a certain day, the firm receives no call; exactly 4 calls during the same time.

$X \sim P(2.5)$

$P(X=0) = 0.0821$

$P(X=4) = 0.1336$

4. Uniform RV (discrete) : RV X is said to be have uniform distribution on n points $\{x_1, x_2, \dots, x_n\}$, if its pmf is $P\{X=x_i\}=1/n, i=1,2,\dots,n$

$$F(x)=x/n$$

$$E(X)=1/n(x_1+x_2+\dots+x_n)=\text{mean}$$

$$Var(X) = \frac{1}{n} \sum_{i=1}^n x_i^2 - (E(X))^2$$

If $x_i=i$

$$E(X)= (n+1)/2$$

$$Var(X)= (n^2-1)/12$$

5. Geometric RV: