MA203

Special RV

Special RV:

1. Bernoulli RV: X=0,1 and P(X=1)=p, P(X=0)=1-p; 0<p<1

   E(X)=p, Var (X)=p(1-p)

2. Binomial RV: n no. of independent Bernoulli trials and X is the total number of successes in n trials

$P(X=k)={}^{n}C_{k}\, p^{k}\, (1-p)^{n-k}$ k=0,1,2,….., n; 0<=p<=1

X~B(n,p)

E(X)=np, Var (X)=npq,

Mode: unimodal ([(n+1)p]) or bimodal ((n+1)p, (n+1)p-1)

Example: throwing a die and obtaining even number. X~B(1,1/2)

Theorem 1: Let, Xi (i=1, 2, ….,k) be independent binomial RVs with Xi~B(ni, p)

S=X1+X2+….Xk

S~B(ni+n2+…nk, p)

Example: X1~B(1,1/2), X2~B(1,1/2) both are tossing a coin and obtaining head

Consider, X=tossing a coin twice and obtaining head

X~B(2,1/2)

P(X=0)=1/4, P(X=1)=2/4, P(X=2)=1/4

Example 1: A die is thrown 600 times. Find the lower bound for the prob of getting 80 to 120 sixes.

$X \sim B(600,1/6)$

$P\{|X-\mu|<k\sigma\}>=1-1/k^2$

$\mu=600*1/6=100$

$Var(X)=600*1/6*5/6=500/6$

$K=2.19$

Lower bound of prob = 19/24

2. Use Chebyshev inequality to determine how many times a fair coin must be tossed in order that the prob will be at least 0.90 that the ratio of the observed no of heads to the no of tosses will lie between 0.4 and 0.6.

Consider, RV X/n, X: no of heads, n: total tosses

$X \sim B(n,1/2)$

$P\{|X/n-E(X/n)|<\varepsilon\}>= 1-(Var(X/n))/\varepsilon^2$

$E(X/n)=1/n.E(X)=1/n(n.1/2)=1/2$, $Var(X/n)=(1/n^2)Var(X)=1/(4n)$

$P\{|X/n-1/2|<k/(4n)\}>= 1-1/k^2$

$P\{|X/n-1/2|<k/(4n)\}>= 0.9$

So, $k=\sqrt{10}$

$N=250$

The coin must be tossed 250 times.

3. Poisson RV: X takes values 0, 1, 2,….. Inf with

P(X=k)=$e^{-\lambda}\lambda^k/k!$, λ>0 DF is $F(x) = \sum_{k=0}^{x} e^{-\lambda} \frac{\lambda^k}{k!}$

X~P(λ)

(To check pmf, $\sum_{k=0}^{\infty} P(X = k) = 1$)

Example: number of printing mistakes per page, number of goals in a football match etc..

$$E(X) = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda$$

Var(X)=E(X2)-(E(X))2 =λ^2+ λ- λ^2= λ

Mode: let, pk=P(X=k)

$$\frac{p_{k-1}}{p_k} = \frac{k}{\lambda}$$

If k< λ, then pk-1<pk

If k> λ, then pk-1>pk

If k= λ, then pk-1=pk

If λ is not an integer, then mode= [λ]

If λ is an integer, then mode = λ, λ-1

If λ<1, mode=0

MGF: $M(s)=e^{\lambda(\exp(s)-1)}$

Theorem 2: Let, $X1, X2, \ldots, Xn$ be independent Poisson RVS with $Xk \sim P(\lambda k)$, $k=1, 2, \ldots, n$.
Then $S \sim P(\lambda 1+ \lambda 2+\ldots \lambda k)$ where $S=X1+X2+\ldots+Xn$
(converse is true)

Theorem 3: Let X, Y be independent Poisson RVs with parameters $\lambda 1$, $\lambda 2$ respectively. Then
$(X|X+Y=n) \sim B$
$P\{X=m|X+Y=n\}=P\{X=m, Y=n-m\}/P\{X+Y=n\}$

$(X+Y) \sim P(\lambda 1+ \lambda 2)$
$(X|X+Y) \sim B(n, \lambda 1 /(\lambda 1+ \lambda 2))$

Theorem 4: if $X \sim P(\lambda)$ and $(Y|X=x) \sim B(x,p)$ then $Y \sim P(\lambda p)$

3. A firm receives on an average 2.5 telephone calls per day during 10 – 10:05 am. Find the probability that on a certain day, the firm receives no call; exactly 4 calls during the same time.
$X \sim P(2.5)$
$P(X=0)=0.0821$
$P(X=4)=0.1336$

4. Uniform RV (discrete) : RV X is said to be have uniform distribution on n points {x1, x2, …. Xn}, if its pmf is P{X=xi}=1/n, i=1,2,…,n
F(x)=x/n
E(X)=1/n(x1+x2+…+xn)=mean

$$Var(X) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (E(X))^2$$

If xi=i
E(X)= (n+1)/2
Var (X)= (n^2-1)/12

5. Geometric RV: Suppose, we are interested in the first success, i.e., how many trials are required to realize first success. Then no of trials needed is not fixed and its random numer.
X: no of trials needed to the first success
A: success event
P(X=k)=P(A^c A^c…… A^c A)={(1-p)^(k-1)}.p   k=1,2,3,…. Inf
X is said to be geometric RV with P(X=k)=p.q^(k-1),   k=1,2,3…… inf

$$P(X > m) = \sum_{k=m+1}^{\infty} q^{k-1}p = pq^m \frac{1}{1-q} = q^m$$

For integers m , n >1

$$P(X > m + n|X > m) = \frac{P(X > m + n)}{P(X > m)} = q^n$$

Given first m trial has no success, conditional prob that first success will appear after an additional n trials depend only on n and not on m – memoryless property

$$E(X) = \sum_{k=1}^{\infty} kq^{k-1}p = \frac{1}{p}$$

$$Var(X) = \sum_{k=1}^{\infty} k^2q^{k-1}p - {}^{1}/_{p^2} = \frac{1-p}{p^2}$$

DF is F(x)=P(X<=x)=$\sum_{u=1}^{x} q^u p = 1 - (1-p)^{k+1}$

MGF is $\mathrm{M}(s) = \sum_{k=1}^{\infty} e^{sk}(1-p)^{k-1}p = \frac{pe^s}{1-(1-p)e^s}$

Theorem-5. if X has a geometric distribution, then for any two non-negative integers m and n, P(X>m+n|X>m)=P(X>n)

- Memoryless distribution

Extend it to the number of trials needed for r successes.
Y: no of trials needed for r successes (Bernoulli trials)

P(Y=k)=P(r-1 successes in k-1 trials and success in k-th trial)

$P(Y=k)=\binom{k-1}{r-1}p^{r-1}(1-p)^{k-1-r+1}p=\binom{k-1}{r-1}p^{r}(1-p)^{k-r},$ k=r, r+1, r+2, ……. Inf

RV Y is called negative binomial RV.