Unit 5: Graph Theory

Topic 3: Planar Graphs and Coloring

Outline

Planar Graphs

There are many types of graph modelings. Sometimes we may need a model in which the edges of the graph do not intersect so that the communication between the vertices become smooth and quick.

Definition (Planar Graph)

A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a **planar representation** of the graph.

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For example, C_n, W_n are always planar. K_4 is planar but K_5, $K_{3,3}$ are not planar.

Importance

Planarity of graphs plays an important role in the design of electronic circuits. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar. When this graph is not planar, we must turn to more expensive options.

The planarity of graphs is also useful in the design of road networks. We can built a road network without using underpasses or overpasses if the resulting graph is planar.

Properties

A planar representation of a graph splits the plane into **regions**, including an unbounded region. For any region, the number of edges on the boundary of this region is called **degree of the region**.

For $n \geq 3$ in each case, C_n has two regions with degree of each region n, W_n has $n+1$ regions with degree of unbounded region n and other regions 3, and K_4 has 4 regions with degree of each region 3.

Euler's formula

Theorem (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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2. If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.
3. If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Problem

1. Show that K_5 is non planar.

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2. Show that $K_{3,3}$ is non planar.

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2. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

Kuratowski's Theorem

Theorem (Kuratowski's Theorem)

A graph is non planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

More generally, if a graph contains a copy of a non planar graph as one of its subgraphs, then the graph is also non planar.

Outline

Graph Coloring

Problems related to the coloring of maps of regions, such as maps of parts of the world, have generated many results in graph theory.

Definition

A **coloring** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The problem of coloring a map is an interesting problem for many purposes. However, it is always expected that in such coloring minimal number of colors are used.

Chromatic number

Definition

The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph.

The chromatic number of a graph G is denoted by $\chi(G)$.

The chromatic number of C_n is 2 if n is even and 3 if n is odd. The chromatic number of any bipartite graph is 2. The chromatic number of K_n is n.

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Two things are required to show that the chromatic number of a graph is k. First, we must show that the graph can be colored with k colors. This can be done by constructing such a coloring. Second, we must show that the graph cannot be colored using fewer than k colors.

The Four Color Theorem

Theorem (The Four Color Theorem)

The chromatic number of a planar graph is no greater than four.

The Four Color Theorem is valid for planar graphs. However, a nonplanar graph can have arbitrary chromatic number.

In general, the problem of finding an approximation to the chromatic number of a graph is difficult.

Problem

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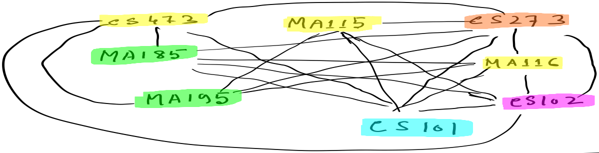
2. The number of edges of a k-colorable graph is at least $\binom{k}{2}$.

Problem

1. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other pair of courses.



MA115 — CS473, MA116, MA185
MA116 — CS473, MA115
MA185 — MA115, MA195
MA195 — CS101, CS102, MA185
CS101 — MA195,
CS102 — MA195
CS273 — 1
CS473 — MA115, MA116

Thank You

Any Question!!!