MA203: IIDRVs, Correlation Matrix, Covariance Matrix and Multiple Jointly Gaussian RVs

$X_1 \; X_2 \; \cdots \; X_n$ are n RVs

$(X_1, X_2, \ldots, X_n)$

$$\underset{\sim}{\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}}$$

X: Random Vector

n-

X: n-dimensional Random Vector

Independent RVs: The RVs are called (mutually) independent if and only if

$$f_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)$$

$$= \prod_{i=1}^{n} f_{X_i}(x_i)$$

For example, if $X_1, X_2, \cdots, X_n$ are independent Gaussian RVs, then

$$f_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}}$$

where $\mu_i = E[X_i]$ and $\sigma_i^2 = E[(X_i - \mu_i)^2]$.

$X_1, X_2, \ldots, X_n$

$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n)$

$= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$

$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}\frac{(x_i-\mu_i)^2}{\sigma_i^2}}$

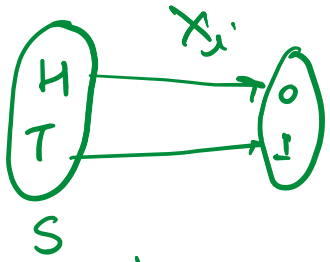
$X_i : \mu_i$

$: \sigma_i^2$

Identically Distributed RVs: The RVs $X_1, X_2, \cdots, X_n$ are called identically distributed if each RV has the same marginal distribution function, that is,

$$F_{X_1}(x_1) = F_{X_2}(x_2) = \cdots = F_{X_n}(x_n).$$

$X_1 \quad X_2 \quad \cdots \quad X_n$



$P_{X_i}(0) = \frac{1}{2}$

$P_{X_i}(1) = 1/2$

$P_{X_1, X_2, \dots, X_n}(1, 1, \dots, 1) = \left(\frac{1}{2}\right)^n$

Independent and Identically Distributed (IID) RVs:

The RVs $X_1, X_2, \cdots, X_n$ are called iid if $X_1, X_2, \cdots, X_n$ are mutually independent and each of $X_1, X_2, \cdots, X_n$ has the same marginal distribution function.

Mean Vector: The mean vector of X, denoted by μ_X, is defined as

$$\mu_X = E[\boldsymbol{X}] = \mathrm{E}[X_1 X_2 \ \dots \ X_n]^t$$

$$= \left[[E[X_1]\ E[X_2]\ \dots E[X_n]]\right]^t$$

$$= [\mu_{X_1}\ \mu_{X_2}\ \dots\ \mu_{X_n}]^t$$

$$E[X] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix} = \begin{bmatrix} \mu_{X_1} \\ \vdots \\ \mu_{X_n} \end{bmatrix} \Rightarrow$$

$x_1, x_2, \dots, x_n$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; \quad X^t = [x_1 \ x_2 \ \dots \ x_n]$$

Correlation Matrix: The correlation matrix of X is defined as $X^t = [x_1 \; x_2 \; \cdots \; x_n]$

$$R_{XX} = EXX^t$$

$$= E \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}} \underbrace{[x_1 \; x_2 \; \cdots \; x_n]_{1\times n}}$$

$$= E \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ \vdots & & & \\ x_nx_1 & x_nx_2 & \cdots & x_n^2 \end{bmatrix}$$

$$R_{xx} = \begin{bmatrix} E x_1^2 & E x_1 x_2 & \cdots & E x_1 x_n \\ E x_2 x_1 & E x_2^2 & \cdots & E x_2 x_n \\ \vdots & & & \\ E x_n x_1 & E x_n x_2 & \cdots & E x_n^2 \end{bmatrix} = E\, x\, x^t$$

$n = 2$

$$R_{xx} = \begin{bmatrix} E x_1^2 & E x_1 x_2 \\ E x_2 x_1 & E x_2^2 \end{bmatrix}$$

$x^t = [x_1 \; x_2]$

Covariance Matrix: The covariance matrix of X is defined as

$$C_X = E(X - \mu_X)(X - \mu_X)^t$$

$$= E\begin{bmatrix} (x_1 - \mu_{x_1}) \\ (x_2 - \mu_{x_2}) \\ \vdots \\ (x_n - \mu_{x_n}) \end{bmatrix} \begin{bmatrix} (x_1 - \mu_{x_1}) & (x_2 - \mu_{x_2}) & \cdots & (x_n - \mu_{x_n}) \end{bmatrix}$$

$$= E\begin{bmatrix} (x_1 - \mu_{x_1})^2 & (x_1 - \mu_{x_1})(x_2 - \mu_{x_2}) & \cdots & (x_1 - \mu_{x_1})(x_n - \mu_{x_n}) \\ (x_2 - \mu_{x_2})(x_1 - \mu_{x_1}) & (x_2 - \mu_{x_2})^2 & \cdots & (x_2 - \mu_{x_2})(x_n - \mu_{x_n} \\ (x_n - \mu_{x_n})(x_1 - \mu_{x_1}) & (x_n - \mu_{x_n})(x_2 - \mu_{x_2}) & \cdots & (x_n - \mu_{x_n})^2 \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{E(x_1-\mu_{x_1})^2} & \underbrace{E(x_1-\mu_{x_1})(x_2-\mu_{x_2})} & \cdots & \underbrace{E[x_1-\mu_{x_1})(x_n-\mu_{x_n})} \\ E[x_2-\mu_{x_2})(x_1-\mu_{x_1}) & \underbrace{E[x_2-\mu_{x_2})^2} & \cdots & E[x_2-\mu_{x_2})(x_2-\mu_{x_n}) \\ \vdots & \vdots & & \\ E[x_n-\mu_{x_n})(x_1-\mu_{x_1}) & E[x_n-\mu_{x_n})(x_2-\mu_{x_2}) & \cdots & E(x_n-\mu_{x_n})^2 \end{bmatrix}$$

$$C_x = \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) & \cdots & \underline{Cov(x_1, x_n)} \\ Cov(x_2, x_1) & Var(x_2) & \cdots & Cov(x_2, x_n) \\ \vdots & \vdots & & \vdots \\ \underline{Cov(x_n, x_1)} & Cov(x_n, x_2) & \cdots & Var(x_n) \end{bmatrix}$$

Properties of Covariance Matrix:

1. C_X is a symmetric matrix because $Cov(X_i, X_j) = Cov(X_j, X_i)$.

$$\underline{X} = [x_1 \quad x_2] \qquad \underline{n=2}$$

$$C_X = \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) \\ Cov(x_2, x_1) & Var(x_2) \end{bmatrix}$$

Uncorrelated RVs: n RVs $X_1, X_2, \cdots, X_n$ are called uncorrelated if for each (i, j), $i = 1, 2, \cdots, n$ and $\text{j} = 1, 2, \cdots, n$, $i \neq j$,

$$Cov(X_i, X_j) = 0.$$

X_1 X_2

$Cov(X_i, X_j) = 0$ $\quad Cov(X_1, X_2) = 0$

i, j

If $X_1, X_2, \cdots, X_n$ are uncorrelated, C_X will be a diagonal matrix.

$$C_X = \begin{bmatrix} Var(X_1) & 0 & \cdots & 0 \\ 0 & Var(X_2) & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & Var(X_n) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Example 1: Let $X = [X_1\ X_2]^t$ be a random vector with joint PDF

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{x_1}; 0 < x_1 < 1; 0 < x_2 < x_1 \\ 0; O.w. \end{cases}$$

Find Correlation matrix R_{XX} and Covariance Matrix C_X.

$n = 2$ $\quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\quad R_{XX} = \begin{bmatrix} E\, X_1^2 & E\, X_1 X_2 \\ E\, X_2 X_1 & E\, X_2^2 \end{bmatrix}$

(i) $E[X_1]$ (ii) $E[X_2]$ (iii) $E[X_1^2]$

(iv) $E[X_2^2]$ (v) $E[X_1 X_2]$

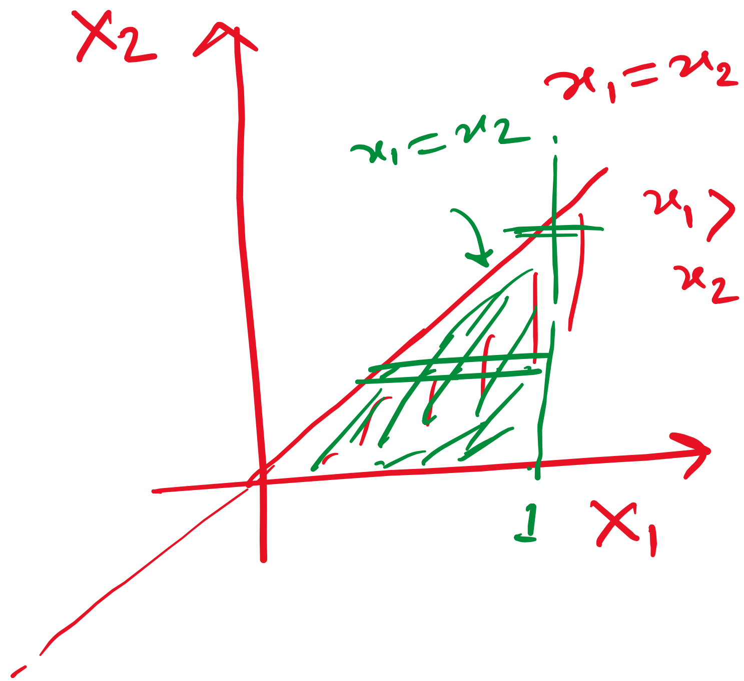
$C_X = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Var(X_2) \end{bmatrix}$

Sol:- $E[X_1^m X_2^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1^m x_2^n f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2$

$E[X_1] \Rightarrow m = 1, n = 0 \quad \Rightarrow \quad E[X_1^2] \Rightarrow m = 2, n = 0$

$$E[X_1] = \int_0^1 \int_{x_2}^1 x_1 \, f_{X_1,X_2}(x_1,x_2) \, dx_1 \, dx_2$$

$$= \int_0^1 \int_{x_2}^1 x_1 \times \frac{1}{x_1} \, dx_1 \, dx_2 = 1/2$$



$$\underline{E[X_2]} = \int_0^1 \int_0^{x_1} x_2 \times \frac{1}{x_1} \, dx_2 \, dx_1 = 1/4$$

$$E[X_1^2] = \int_0^1 \int_{x_2}^1 x_1^2 \, \frac{1}{x_1} \, dx_1 \, dx_2 = 1/3$$

$$E[X_2^2] = \int_0^1 \int_0^{x_1} x_2^2 \, \frac{1}{x_1} \, dx_2 \, dx_1 = 1/9$$

$$E[X_1 X_2] = \int_0^1 \int_0^{x_1} x_1 x_2 \frac{1}{x_1} \, dx_2 \, dx_1 = 1/6$$

$$R_{xx} = \begin{bmatrix} E X_1^2 & E X_1 X_2 \\ E X_2 X_1 & E X_2^2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/9 \end{bmatrix}$$

$$C_x = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Var(X_2) \end{bmatrix}$$

$$Var(X_1) = E[X_1^2] - (E[X_1])^2 = 1/3 - 1/4 = 1/12$$

$$Var(X_2) = E[X_2^2] - (E[X_2])^2 = 1/9 - 1/16 = 7/144$$

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

$$= 1/6 - 1/4 \times 1/2 = 1/6 - 1/8 = 1/24$$

$$C_x = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) \end{bmatrix} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 7/144 \end{bmatrix}$$

Multiple Jointly Gaussian RVs: For any positive integer n, $X_1, X_2, \cdots, X_n$ represent n jointly RVs. These n RVs define a random vector $X = [X_1, X_2, \cdots, X_n]^t$.

These RVs are called jointly Gaussian if the RVs $X_1, X_2, \cdots, X_n$ have joint PDF function given by

$$\boxed{f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = \frac{e^{-\frac{1}{2}(X-\mu_X)^t C_X^{-1}(X-\mu_X)}}{\left(\sqrt{2\pi}\right)^n \sqrt{det(C_X)}}}$$

where, $C_X = E(X-\mu_X)(X-\mu_X)^t$ is the covariance matrix and $\mu_X = \left[\mu_{X_1}\ \mu_{X_2}\ \ldots\ \mu_{X_n}\right]^t$ is the mean vector of X.

Properties:

1. If $X_1, X_2, \cdots, X_n$ are jointly Gaussian, then the marginal PDF of each of $X_1, X_2, \cdots, X_n$ is Gaussian.
2. If the jointly Gaussian RVs $X_1, X_2, \cdots, X_n$ are uncorrelated, then $X_1, X_2, \cdots, X_n$ are independent.