# 3QI3 Anyon Quantum Computing Report

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Anyons are 2-D quasi particles. Since it is a quasiparticle, it can be interpreted as an excitation of gauge fields rather than explicitly matter or energy [1]. Anyons are in a class of its own, not portraying exact properties to be identified as a boson or a fermion [2]. Rather an anyons wave function will determine whether it obeys Fermi-Dirac statistics or Bose-Einstein statistics [3]. Even though these statistics can be the case, Anyons can be identified into separate classifications to produce unique results, as such, classifications of Abelian and non-Abelian Anyons statistics have been formed [5]. Currently only abelian anyons can be produced through specific physical reactions. There are still many theoretical models on how to produce non-Abelian Anyons although mathematical proofs can assist the claim of their existence. Abelian Anyons have the property of being able to commute, thus operators can be reversed and easily defined through a phase shift. In terms of quantum computing, these 2-dimensional quasiparticles recreate standard quantum logic gates and operators through the interchanging of each respective position. The following figures will give an accurate representation of how the wave functions will differ after interchanging the positions of each Abelian Anyon.

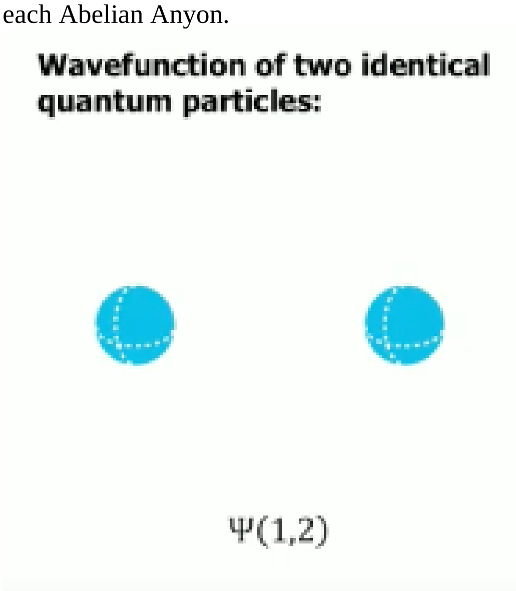



Figure 1 [4]

The initial state can be defined as the tensor product of each individual wave function to recreate the superposition of the 2 quasiparticles. In Dirac notation this can be written as:

$|\psi_1\psi_2\rangle$ [4]

After interchanging the position of each Anyon, the new wavefunction will look as such:

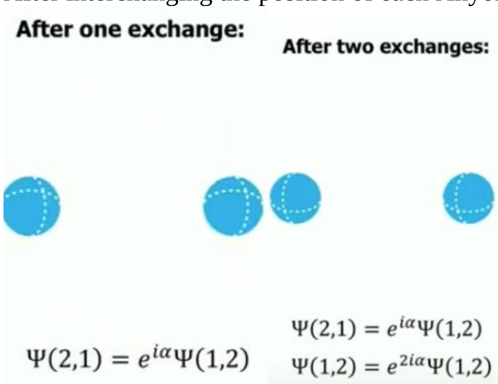



Figure 2

Where

$|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle \qquad 0 \leq \theta \leq 2\pi$

Where theta is a multiple of $\frac{\pi}{m}$

When theta is 0 or $2\pi$ the anyon portrays aspects of bosons whereas when theta is $\pi$ then the anyon will portray fermionic properties [5].
Another notation using the quantum spin number (s) is
$e^{i\theta} = e^{2i\pi s} = (-1)^{2s}$ so $|\psi_1\psi_2\rangle = (-1)^{2s}|\psi_2\psi_1\rangle$
The scenario shown above describes the case for Abelian Anyons. As such, the phase factor shows the fact that the quasiparticle is commutative [4]. Anyons also display a physical quantum phenomenon called the fractional quantum hall effect (FQHE) which is still a huge region of research within condensed matter physics. This brings upon fractional statistics and fractional charges. The most prominent interpretation of FQH has $\nu = \frac{1}{m}$ where m is an odd integer. $\nu$ is the ratio of electron density to magnetic flux. The fractional charge can be seen as $q = \frac{e}{m}$ and as stated before the fractional phase change when anyons are exchanged are interpreted as $\theta = \frac{\pi}{m}$ [5]. These fractional properties can be classified as Abelian Anyon statistics. Some implementations of the FQHE are topological computing as it brings upon the correct scenarios of topological order. There has been experimentation on the production of Abelian Anyons. Anyons can be formed by super cooling two dimensional semiconductors and applying a large magnetic field [4]. This creates a quantum liquid where fractional charges can be seen. Another example of anyon formation is with two-dimensional electron gasses collisions [6]. Basic operations can be done will Abelian Anyons but since the phase factor is quite determinate, there are less changes that can be applicable to the wave function. To create complex functioning quantum computers, Non-Abelian Anyons will be necessary [4].

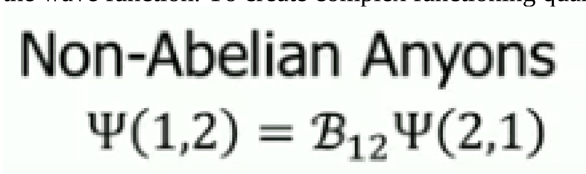


Figure 3 [4]

Non-Abelian Anyons do not have phase shifts like abelian anyons, rather each exchange of the anyon will be non-trivial unitary operators, that do not commute. When there is no degeneracy, which is the property of multiple ground states in the system (degeneracy), the particle will function in one-dimensional space, thus being commutative and having the phase factor as previously seen [4]. This is the effect of FQH on abelian anyons. Since non-abelian anyons are much more complex in the transformations and has properties of degeneracy, the subspace has higher order which allows the system the ability of not needing to commute with linear transformations. This is a similar property illustrated by matrices. Each unique exchange of a different anyon in a system will produce its own unque operator [4].

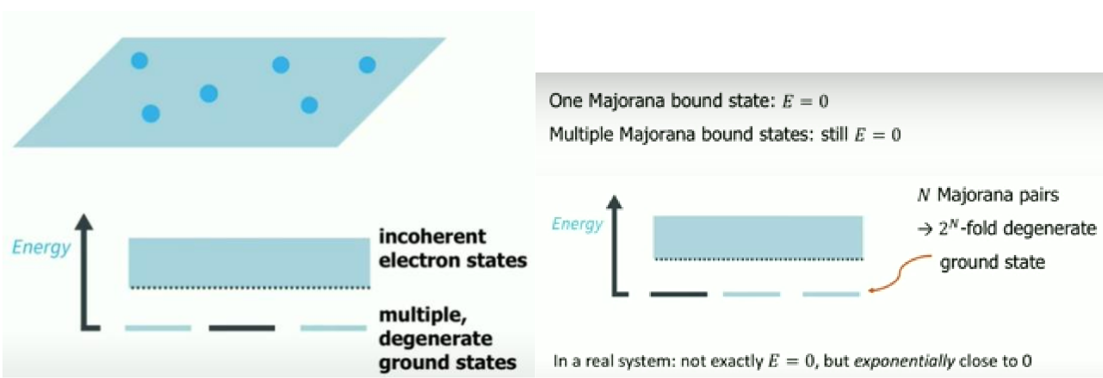


Figure 4 [4]

Non-Abelian anyons are produced at zero-point energy. Thus, when multiple of these anyons are produced, there are multiple degenerate ground states. Each of these quasiparticles can either be in its ground state or not. Consequently, the option of occupying the ground state brings the formula $2^N$ -fold degenerate ground states [4].
Non-Abelian anyons are majorana bound states, where a majorana fermion is bound to an anyon. This is also known as an Ising anyon. Majorana particles have the unique supersymmetry of a fermion being its' own antiparticle.

$$i\,\partial\!\!\!/\psi - m\,\psi_c = 0$$

The Majorana equation is defined as the following formula in Feynman notation to include gamma matrices. Gamma matrices ensures the anticommuting properties.

$$\mathrm{D_L}\,\psi_\mathrm{L} = 0$$

The majorana equation can also be described using the majorana operator $D_L$ as shown above.

Majorana fermions have the following identities to represent the supersymmetries where gamma is the annihilation operator and c is the creation operator.

$$\gamma_1 = 1/2\,(c + c^\dagger) \qquad c = (\gamma_1 + i\,\gamma_2)$$
$$\gamma_2 = i/2\,(c - c^\dagger) \qquad c^\dagger = (\gamma_1 - i\,\gamma_2)$$

Figure 5 [4]

There has been speculation that majorana fermions could be made from superconductors, using electron tunneling from a conductor creating a hole and a voltage. This interruption of electrons and holes interacting with a super conductor's cooper pair could potentially create a majorana fermion that would assist in the production of a majorana bound state [4].

Superconductors: particle-hole symmetry

$$\boxed{\gamma(E) = \gamma^\dagger(-E)}$$

At $E = 0$: Majorana bound states

$$\gamma = \gamma^\dagger$$

Figure 6 [4]

The figure above shows that the annihilation and hermetian conjugate annihilation operators are equal at the zero-energy state. Every non-abelian anyon will have this property and this is a method to confirm that the production of one is truly successful. There have been papers made describing the possibility of producing non-Abelian anyons in cold atom systems through auxiliary bosons [7].

The following figures will show practical results of testing and experimentation for majorana bound state production and detection.

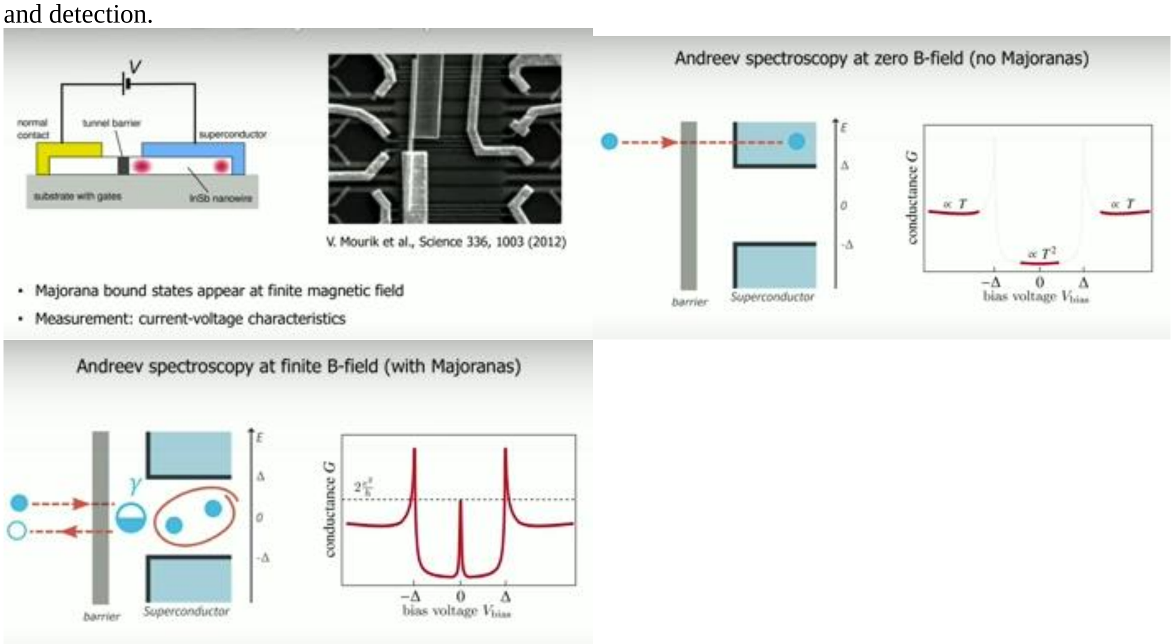



Figure 7 [4]

The first image describes the process of using a conductor, a tunnel and a superconductor as described previously. The appearance of these bound states only occur at definite magnetic fields and the conductance is proportional to the tunneling probability. Since majoranas are observed at zero-point energy, there would be a conductance peak at a value, quantized at $2\frac{e^2}{h}$[4]. This conductance quantization supports the FQHE. To produce these phenomena, it is imperative that a large magnetic field is applied and the surrounding temperature suffices for the needed parameters for the super conductor. The majorana bound state ensures that the zero-point energy state (ground state) anyon produced can essentially either be off or on by being in the ground state or incoherent.
To apply the concepts above into quantum computing, there a few standards that have to be fulfilled. This is known as the DiVincenzo criteria.

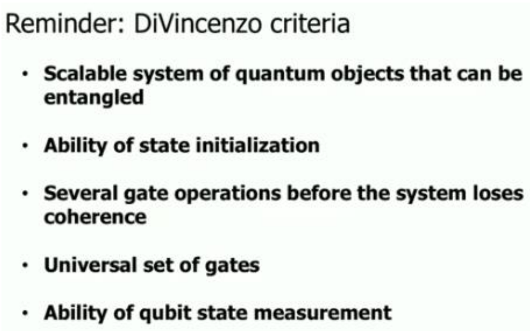


Figure 8 [4]

As of now, the following information will be presented to fulfill these standards. Anyons do not undergo entanglement through the standard notations we are familiar with rather, notations implementing entropy starts to become the standard practice. A gate called the S gate will help determine whether the qubits are entangled or not. Entanglement is generally formed through the braiding and construction, a unique operator of it’s own known as the entanglement operator [8].
Ising anyons are what form a topological qubit. The physical interpretation is presented through the following image.

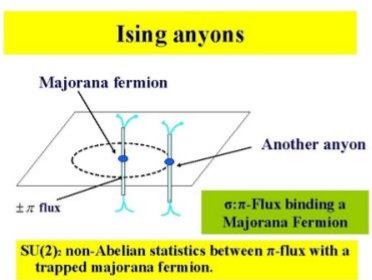


Figure 9 [8]

Thus, it is necessary to create a pair of Ising anyons which follows non-abelian statistics to create a qubit. The qubits can occupy a specific state but not a superposition. To create a superposition, there needs to be at least 2 pairs of Ising anyons. Therefore, there are 4 non-abelian anyons in this system to produce a superposition and 2 non-abelian anyons to produce a qubit in a specific state [9]. The following pictures will show how quantum operators and logic gates can be created through the exchange of the positions of each anyon. This process is also known as topological braiding. Through topology, when undergoing non-abelian statistics, the order of topology is denoted as $Z_2$ [1]. This will describe the topological order of the charge and the quantum vortex. A quantum vortex is a hole with a quantized flux around the axis of the hole and the hole can contain excitation of it’s own. The next image will show how braiding of anyons can perform operators utilizing the 2 spatial dimensions in respect to time.

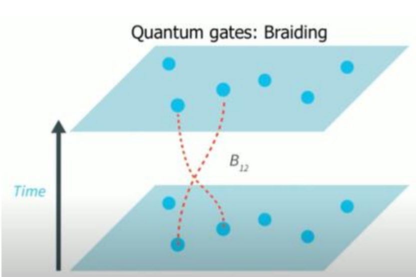


Figure 10 [4]

Anyons initial state can be denoted as quantum memory, operators can be applied with absolutely no unwanted decoherence and measurements of the state can be performed. Since all the requirements are satisfied, the following process can be used to make quantum computations.

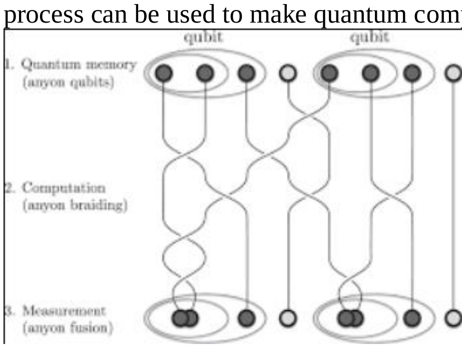


Figure 11

The energy of the anyons can be determined through the Hamiltonian operator. This gives insight onto the robustness of the topological winding of anyons, operators and occupation of zero energy state.

$$\hat{H}_{2\text{-anyon}} = -\frac{1}{2r^2}\left(\frac{\partial}{\partial\varphi} + i\alpha\right)^2 - \frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{r^2}{2}\,.$$

$$\hat{\hat{H}}_{N\text{-anyon}} = -\frac{1}{2}\sum_{q=1}^{N}\left(\nabla_q^2 + 2i\alpha \sum_{p(\neq q)=1}^{N} \frac{\epsilon^{ij}x_{pq}^j + ix_{pq}^i}{r_{pq}^2}\frac{\partial}{\partial x_q^i}\right).$$

Figure 12 [10]

Psi is a function dependant on its position in spherical or cylindrical coordinates and alpha is the phase factor. P and Q are the factors determining $\nu = \frac{p}{q}$ and x also determines position. This operator will effectively show the efficiency of the use of anyons in quantum computing [10].

To measure the state of the information, the anyons have to undergo through a process called anyon fusion. Anyon fusion is the process where anyons combine to create one composite anyon. Rather when two anyons are fused, it produces one fermion or vacuum (lack of fermion) [4]. This either produces a charge or lack of charge, thus arising a result from measurement which can be taken through a meter to detect electric potential. 2N anyons have $2^{N-1}$ fusion channels and can encode N-1 qubits [9]. Fusion can be described as

$$a \times b = \sum_c N_{ab}^c c$$

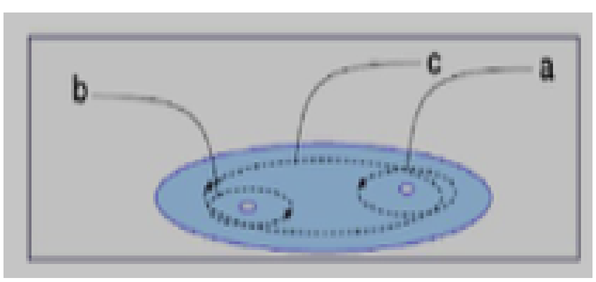


Figure 12 [8], [9]

$$\backslash\text{vac} \times \backslash\text{vac} = \backslash\text{vac} \qquad \backslash\text{vac} \times \psi = \psi \qquad \backslash\text{vac} \times \sigma = \sigma$$

$$\psi \times \psi = \backslash\text{vac} \qquad \psi \times \sigma = \sigma$$

$$\sigma \times \sigma = \backslash\text{vac} + \psi.$$

Where sigma is an anyon and psi is a fermion

Many quantum logic gates that we know of can be recreated through topological quantum computing.

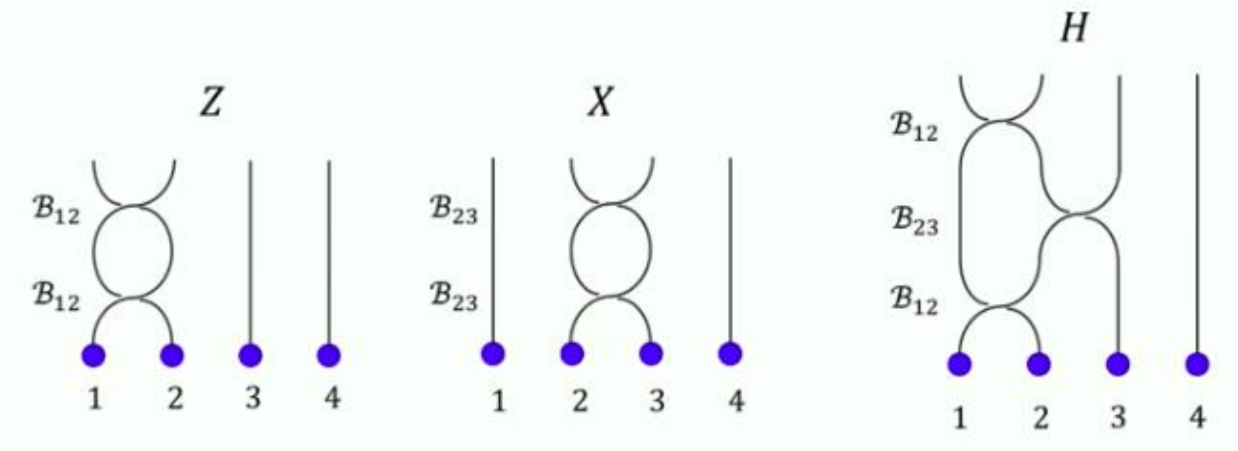



Figure 13

Through the interchanging of these anyons, the pauli Z, X and Hadamard gate can be realized [8]. To create these operations, the braids have to be very exact. With braiding operations, the entire bloch sphere cannot be obtained therefore some quantum gates required will be missing [4]. The operators can be replaced with topological operations on the qubit although it may not be exactly the same desired result. Using topology there is a type of operator called the string operator which can describe the interchanging of particle position also known as "particle hopping". This can essentially create pseudo-operators [8].

A few more matrices that are essential for topological quantum computing are the following, where psi represents a fermion and sigma represents an anyon.

**N matrices** $N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $N_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $N_\psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

**F matrices** $F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $F^{\sigma}_{\psi\sigma\psi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

**R matrices** $R^{1}_{\sigma\sigma} = e^{-\frac{\pi i}{8}}$ $R^{\psi}_{\sigma\sigma} = e^{\frac{3\pi i}{8}}$

Figure 14 [8]

The S matrix can be denoted as an $R_{12}=R_{34}$ matrix [8]. Using spin operators, it is possible to prove that entanglement is true. To create entanglement, the braiding essentially has to create an entangled knot through operations. Using the F and R operators provide 2 key axioms known as the Pentagon and Hexagon equation showing how applying these gates and changing basis a set number of times outputs the initial operating basis, similarly this can be used to reconstruct initial states [8].

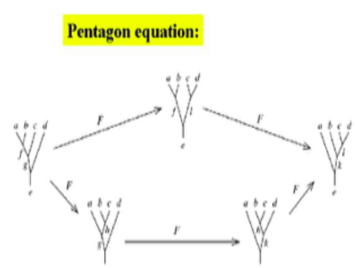
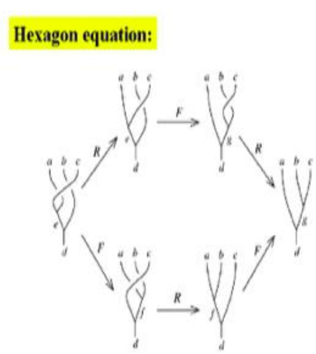



Figure 15 [8]



Operators that also have implementations through the Ising model include the control not gate and the Toffoli gate. The final implementation of anyons is known as Fibonacci anyons [8]. These anyons follow different fusion rules to output measurement. $\tau$ represents two anyons. The fusion rules with the Fibonacci implementation follow

$\tau \times \tau = 1 + \tau$

$\tau \times 1 = \tau = 1 \times \tau$

$1 \times 1 = 1$

Each anyon will encode for $\log_2(\tau)$ qubits. With Fibonacci anyons, the above matrices can be realized as the following image.

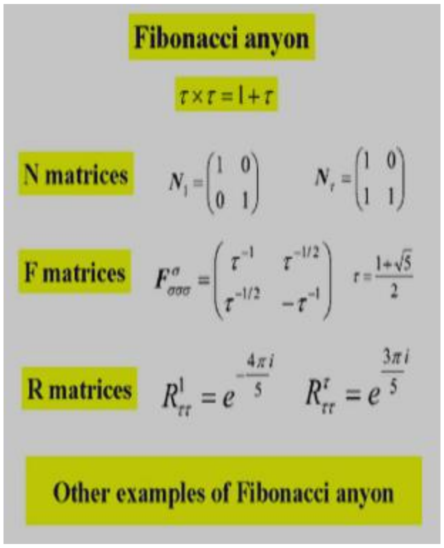


Figure 16

When comparing Fibonacci anyons with Ising anyons, Ising anyons have better implementations of gates, no decoherance or loss of information through braiding whereas Fibonacci anyons can produce a larger variety of gates with less braids but errors start to appear during the process of braiding. Ising anyons have a hard time making measurements as it can be hard to distinguish between a quantum vortex/vacuum charge and a fermion whereas using Fibonacci fermions, the measurement is much easier to extract data from since it is easier to distinguish from a vacuum charge, 2 anyons and the produced interference [8].

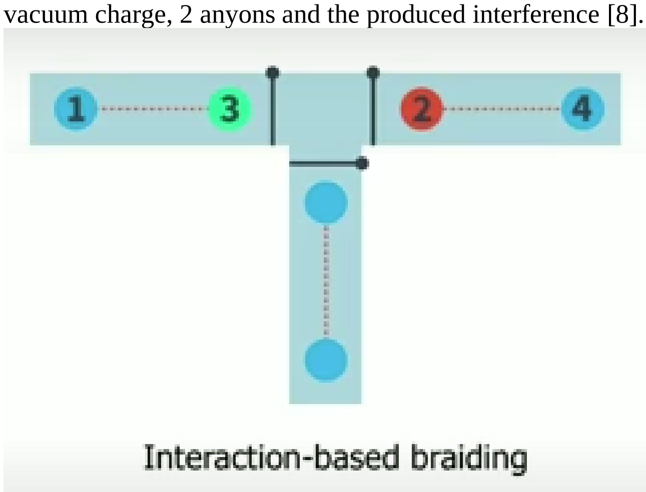


Figure 17 [4]

A physical example of how anyons would be able to create unique braids and operations is shown in the image above where each numbered particle is an anyon and the gates would open allowing for the movement of each anyon. By opening and closing the gates and the confined region of space the anyons can interact in, specific computations can be created [4].

Through the research of anyons, the conclusion made is that non-abelian anyons are crucial for the construction of an effective topological quantum computer. Many researchers are currently looking at large scale research to find and produce majorana particles but so far, the topic has not been completely proven to be reconstructed or found. The ideal aspects of the topological computers are the fact that noise and decoherence are no longer issues and extreme lab environments do not have to be reconstructed to house a quantum computer. There are many theoretical projections of how the production of non-abelian anyons can come to existence and the mathematics satisfies all theory. As such, there must be a physical representation. It is only a matter of time that top notch topological quantum computers are created and much more feasible for large scale institutionalization.

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Extra information


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